Regge Asymptotics of the Scattering Amplitude at Small Momentum Transfers in Ladder Approximation

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A method of calculation of the scattering amplitude for fermions and scalar bosons with exchanging of a scalar particle in ladder approximation is suggested. The Bethe-Salpeter ladder integral equations system for the imaginary part of the amplitude is costructed and solution in the Regge asymptotical form is found. The corrections to the amplitude due to the exit from mass shell are calculated and the real part of the amplitude is found.

1. Introduction

Many experimental and theoretical investigations of behavior of the scattering amplitude in the deeply inelastic region lead to a great interest for ladder models. The ladder approximation for the scattering amplitude in field theory models was early used for explanation of Regge behavior at high energies [1,2] and also for constructing of multiperiferic model [3]. As it is known the summation of the ladder diagrams leads always to the integral Bethe-Salpeter (BS) equations for the scattering amplitude. Exact solutions of the BS type ladder equations for forward scattering amplitudes in some scalar models were obtained by different methods and behavior of the amplitudes were investigated in the Regge and Bjorken regions [4-6]. The solution method of BS equation for an imaginary part of the scattering amplitude at small momentum transfers in $\lambda\varphi^3$-theory was considered in [7]. In Ref. [8] BS equation for the imaginary part of forward fermion-boson scattering amplitude has been investigated. The imaginary part of the forward scattering amplitude at small momentum transfers has been studied in quantum scalar electrodynamics in ladder approximation [9]. It is schown that high energy amplitudes have the Regge asymptotic. However, there are not investigations of fermion scattering amplitudes at nonzero momentum transfers in the ladder approximation.

In the present work we construct BS type integral equations system for the imaginary part of the fermion-boson scattering amplitude in the ladder approximation. The method of solution of such equations in the Regge energy region is presented. It is shown that exchanging mass impact signlificantly for the Regge asymptotic behavior at high energies.

2. BS equation for the imaginary part of the scattering amplitude

BS equation for the imaginary part of the scattering amplitude $F_{\alpha\beta}\left(p,p';k,k'\right)$ for the fermion (ψ) and scalar boson (ϕ) with exchanging scalar particle (φ)in the theory with $L_{int.} = g\left[\overline{\psi}\,\psi\right]\varphi +$ $+\lambda\varphi\phi^2$ is

$$\overline{\psi}\left(p'\right)F_{\alpha\beta}\left(s,t;p^2,p'^2;k^2,k'^2\right)\psi\left(p'\right) =$$
$$= \pi\lambda g\overline{\psi}\left(p'\right)\delta_+\left[\left(p+p'\right)^2-\mu^2\right]\theta\left(p_0+p_0'\right)\delta_{\alpha\beta}\psi\left(p'\right)+$$
$$+\frac{\pi\lambda g}{(2\pi)^4}\int\frac{\bar{\psi}(p')F_{\alpha\alpha'}\left(s',t;(p-q)^2,(k-q)^2\right)\psi\left(p'\right)\left(\widehat{p}-\widehat{q}+m\right)_{\alpha'\beta}}{\left[(p-q)^2-m^2\right]\left[(k-q)^2-m^2\right]}\times$$
$$\times\,\delta_{\alpha\beta}\delta_+\left(q^2-\mu^2\right)\theta\left(q_0\right)d^4q, \qquad (1)$$

where p,p' and k,k'- 4-momenta of the initial and final particles, respectively.

The amplitude $F_{\alpha\beta}$ is the function of the six invariants $p^2, p'^2, k^2, k'^2, t=(p-k)^2, s=(p+p')^2$. The amplitude $F_{\alpha\alpha'}$ is the function of the following invariants: $s'=(p+p'-q)^2, t, (p-q)^2, (k-q)^2$. μ is the mass of the exchanging particle, i.e., the mass on the ladders, and m is the mass of other propogators(the masses of the scalar (ϕ) and spinor (ψ) particles are taken equal to each other for simplicity).

In the case $p'^2=m^2, k'^2=m^2, k^2=m^2$ the amplitude $\overline{\psi}(p')F_{\alpha\beta}\psi(p')$ expand on Lorentz invariants scalars

$$\overline{\psi}(p')\left[f_1\widehat{p}+f_2\widehat{p}'+f_3\widehat{p}\widehat{p}'+f_4\right]\psi(p') \tag{2}$$

Using Dirac equation ($(\widehat{p}'-m)\psi(p')=0$) we can express the amplitude $F_{\alpha\beta}$ in terms of the two form-factors F_1 and F_2

$$\left[\widehat{p}F_1\left(s,t;p^2\right)+F_2\left(s,t;p^2\right)\right]\overline{\psi}(p')\psi(p'), \tag{3}$$

where F_1 and F_2 are linear combinations: $F_1=f_1+f_3m$, $F_2=f_2m+f_4$. The amplitude $F_{\alpha\alpha'}$ has the form

$$(\widehat{p}-\widehat{q})F_1\left(s',t;(p-q)^2,(k-q)^2\right)+F_2\left(s',t;(p-q)^2,(k-q)^2\right). \tag{4}$$

Taking into account Eq. (3) and (4) the Eq. (1) can be rewritten in the following form

$$\begin{aligned}&\left[\widehat{p}F_1\left(s,t;p^2\right)+F_2\left(s,t;p^2\right)\right]\overline{\psi}(p')\psi(p')=\\&=\pi\lambda g\delta_+\left(s-\mu^2\right)\theta\left(p_0+p_0'\right)\overline{\psi}(p')\psi(p')+\\&+\frac{\pi\lambda g}{(2\pi)^4}\int\frac{(\widehat{p}-\widehat{q}+m)\left[(\widehat{p}-\widehat{q})F_1\left(s',t;(k-q)^2\right)+F_2\left(s',t;(k-q)^2\right)\right]}{\left[(p-q)^2-m^2\right]\left[(k-q)^2-m^2\right]}\times\\&\times\bar{\psi}(p')\psi(p')\delta_+(q^2-\mu^2)\Theta(q_0)d^4q.\end{aligned} \tag{5}$$

Here the spinors $\psi(p')$ are normalized in the standard form

$$\sum_{n=1,2}\psi_\alpha^r(p')\overline{\psi}_\beta^r(p')=\frac{(\widehat{p}'+m)_{\alpha\beta}}{2m}.$$

Summing upon r we have

$$\begin{aligned}&\left[\widehat{p}F_1\left(s,t;p^2\right)+F_2\left(s,t;p^2\right)\right]\left(m+\widehat{p}'\right)=\pi\lambda g\delta_+\left(s-\mu^2\right)\left(m+\widehat{p}'\right)\theta\left(p_0+p_0'\right)+\\&+\frac{\pi\lambda g}{(2\pi)^4}\int\frac{(\widehat{p}-\widehat{q}+m)\left[(\widehat{p}-\widehat{q})F_1\left(s',t;(k-q)^2\right)+F_2\left(s',t;(k-q)^2\right)\right]}{\left[(p-q)^2-m^2\right]\left[(k-q)^2-m^2\right]}\times\\&\times(m+\widehat{p}')\delta_+\left(q^2-\mu^2\right)\theta\left(q_0\right)d^4q.\end{aligned} \tag{6}$$

Calculating the traces in Eq. (6) we have

$$\begin{aligned}&pp'F_1\left(s,t;p^2\right)+mF_2\left(s,t;p^2\right)=m\pi\lambda g\delta_+\left(s-\mu^2\right)\theta\left(p_0+p_0'\right)+\\&+\frac{\pi\lambda g}{(2\pi)^4}\int\frac{\delta_+\left(q^2-\mu^2\right)\theta\left(q_0\right)d^4q}{\left[(p-q)^2-m^2\right]\left[(k-q)^2-m^2\right]}\times\\&\times\Bigg[F_1\left(s',t;(k-q)^2\right)\left[m(p-q)^2+mp'(p-q)\right]+\end{aligned}$$

$$+F_2\left(s',t;(k-q)^2\right)\left[p'(p-q)+m^2\right]\Bigg], \tag{7}$$

$$mp^2F_1\left(s,t;p^2\right)+pp'F_2\left(s,t;p^2\right)=\pi\lambda g\delta_+\left(s-\mu^2\right)\theta\left(p_0+p_0'\right)pp'+$$

$$+\frac{\pi\lambda g}{(2\pi)^4}\int\frac{\delta_+\left(q^2-\mu^2\right)\theta\left(q_0\right)d^4q}{\left[(p-q)^2-m^2\right]\left[(k-q)^2-m^2\right]}\times$$

$$\times\Bigg[F_1\left(s',t;(k-q)^2\right)\left[pp'(p-q)^2+m^2p(p-q)\right]+$$

$$+F_2\left(s',t;(k-q)^2\right)\left[mp(p-q)+mpp'\right]\Bigg],$$

the second equation in system (7) is obtained by multiplying Eq. (6) on $\widehat{p}$ and further calculating of the traces.

3. Analises of the Equations

The equations (7) have quite complex kernels and their exact solutions are probably imposible. So analysing the system (7) at high energies in the kinematic region $s >> \mu^2, m^2$; $k^2 = m^2$ the form of equations can be simplified. Then Eqs. (7) takes the form

$$pp'F_1\left(s,t;p^2\right)=\frac{\pi\lambda g}{(2\pi)^4}\int\frac{\delta_+\left(q^2-\mu^2\right)\theta\left(q_0\right)d^4q}{\left[(p-q)^2-m^2\right]\left[(k-q)^2-m^2\right]}\times$$

$$\times\ \left[mp'(p-q)F_1\left(s',t;(k-q)^2\right)+p'(p-q)F_2\left(s',t;(k-q)^2\right)\right], \tag{8}$$

$$pp'F_2\left(s,t;p^2\right)=$$

$$=\frac{\pi\lambda g}{(2\pi)^4}\int\frac{pp'(p-q)^2F_1\left(s',t;(p-q)^2,(k-q)^2\right)\delta_+\left(q^2-\mu^2\right)\theta\left(q_0\right)d^4q}{\left[(p-q)^2-m^2\right]\left[(k-q)^2-m^2\right]}.$$

Let us divide both parts of the Eqs.(8) on relativistic invariant pp'. Then,

$$F_1\left(s,t;p^2\right)=$$

$$=\frac{\pi\lambda g}{(2\pi)^4}\int\frac{mF_1\left(s',t;(p-q)^2,(k-q)^2\right)+F_2\left(s',t;(p-q)^2,(k-q)^2\right)}{\left[(p-q)^2-m^2\right]\left[(k-q)^2-m^2\right]}\times$$

$$\times\left(1-\frac{p'q}{pp'}\right)\delta_+\left(q^2-\mu^2\right)\theta\left(q_0\right)\delta_+\left[(p+p'-q)^2-s'\right]\theta\left(p_0+p_0'-q_0\right)d^4q\,ds', \tag{9}$$

$$F_2\left(s,t;p^2\right)=\frac{\pi\lambda g}{(2\pi)^4}\int\frac{(p-q)^2F_1\left(s',t;(p-q)^2,(k-q)^2\right)}{\left[(p-q)^2-m^2\right]\left[(k-q)^2-m^2\right]}\times$$

$$\times\delta_+\left(q^2-\mu^2\right)\theta\left(q_0\right)\delta_+\left[(p+p'-q)^2-s'\right]\theta\left(p_0+p_0'-q_0\right)d^4q\,ds',$$

where the integrand is multiplied by

$$1=\int\delta_+\left[(p+p'-q)^2-s'\right]\theta\left(p_0+p_0'-q_0\right)ds'.$$

Note, that when particle with momentum p is on the mass surface ($p^2=m^2$) at $s\to\infty$, the Regge asymptotic $s^{\alpha(t)}$ is not the solution of the system (9).

4. Solutions of equations

Look for the solutions of Eqs. (9) in the Regge region in the form

$$F_1\left(s,t;p^2\right)\cong c_1\left(\frac{s'}{m^2}\right)^{\alpha(t)}\frac{1}{p^2},\;\; F_2\cong c_2m\left(\frac{s}{m^2}\right)^{\alpha(t)}, \tag{10}$$

where $c_1$and c_2 are constants. Correspondingly,

$$\begin{aligned} F_1\left(s',t;(p-q)^2\right) &\cong c_1\left(\frac{s'}{m^2}\right)^{\alpha(t)}\frac{1}{(p-q)^2}, \\ F_2\left(s',t;p^2\right) &\cong c_2m\left(\frac{s'}{m^2}\right)^{\alpha(t)}. \end{aligned} \tag{11}$$

Substituting Eqs.(10) and (11) into Eq.(9) we obtain

$$\begin{aligned} \frac{c_1}{p^2} &= \frac{\pi\lambda g}{(2\pi)^4}\int\frac{\delta_+\left(q^2-\mu^2\right)\theta\left(q_0\right)\delta_+\left[(p+p'-q)^2-s'\right]\theta\left(p_0+p_0'-q_0\right)}{\left[(p-q)^2-m^2\right]\left[(k-q)^2-m^2\right]}\times \\ &\times\left(1-\frac{p'q}{pp'}\right)\left[c_1m\left(\frac{s'}{s}\right)^{\alpha(t)}\frac{1}{(p-q)^2}+c_2m\left(\frac{s'}{s}\right)^{\alpha(t)}\right]ds'd^4q, \\ c_2m &= \frac{\pi\lambda g}{(2\pi)^4}\int\frac{\delta_+\left(q^2-\mu^2\right)\theta\left(q_0\right)\delta_+\left[(p+p'-q)^2-s'\right]\theta\left(p_0+p_0'-q_0\right)}{\left[(p-q)^2-m^2\right]\left[(k-q)^2-m^2\right]}\times \\ &\times c_1\left(\frac{s'}{s}\right)^{\alpha(t)}ds'd^4q. \end{aligned} \tag{12}$$

Then we get the particle with momentum k on the mass surface $k^2=m^2$ and the particle with momentum p near the mass surface $p\to m^2$, suppousing that corrections to the amplitude due to the exit for the mass surface is of the order $\sim\frac{p^2-m^2}{s},\frac{k^2-m^2}{s}$.

In the s.m.c.: $p=\left(p_0,\mathbf{p}\right),\,p'=\left(p_0',-\mathbf{p}\right)\;\; p+p'=\left(p_0+p_0',0\right)$ we determine s as total energy. Taking $q=\left(q_0,\mathbf{q}\right)$ it can be shown that the argument of the second δ-function in Eq. (12) has the form: $(p+p'-q)^2-s'=s-s'+m^2-2\sqrt{s}q_0$, because the first δ-function give $q^2=\mu^2$. So using the spherical coordinates $d^4q=|\mathbf{q}|^2\,d\,|\mathbf{q}|\,dq_0d\Omega$, where $d\Omega=\sin\theta d\theta d\varphi$, and integrating on $\mathbf{q}$, and q_0 using the $\delta-$functions, and further on φ we get

$$\begin{aligned} \frac{c_1}{m^2} &= \frac{\pi^2\lambda g}{8\,(2\pi)^4\,|\mathbf{p}^2|\sqrt{s}}\int\frac{\left(\frac{s'}{s}\right)^{\alpha(t)}\left(\frac{c_1}{m}+c_2m\right)}{\eta\,(\beta+z)\sqrt{\beta^2+2\beta z_0z+z^2+z_0^2-1}}\times \\ &\times\left(1-\frac{s-s'}{s\sqrt{s}}\left(\frac{\sqrt{s}}{2}+|\mathbf{p}|z\right)\right)dzds', \\ \frac{c_2}{c_1}m &= \frac{\pi^2\lambda g}{8\,(2\pi)^4\,|\mathbf{p}^2|\sqrt{s}}\int\frac{\left(\frac{s'}{s}\right)^{\alpha(t)}}{\eta\,(\beta+z)\sqrt{\beta^2+2\beta z_0z+z^2+z_0^2-1}}ds'dz, \end{aligned} \tag{13}$$

where $z=\cos\theta=\cos\left(\mathbf{p}\wedge\mathbf{q}\right),z_0=\cos\theta_0=\cos\left(\mathbf{p}\wedge\mathbf{k}\right)$, $\theta-$scattering angle, $|\mathbf{p}|=\sqrt{\frac{\mathrm{t}}{2(\mathrm{z}_0-1)}},\eta=\frac{s-s'}{2\sqrt{s}},\;\;\beta=\frac{s-s'}{4|\mathbf{p}|\eta}$.

In the case of the small momentum transfers scattering we can change $z_0 = 1+\epsilon$ ($\epsilon \ll 1$) under the integral and obtaine Eq. (13) in the form

$$\frac{1}{m^3} = \frac{G}{|\mathbf{p}|^2\sqrt{s}}\left(\frac{1}{m^2}+\frac{c_2}{c_1}\right)\int \frac{dzds'}{\eta\,(\beta+z)^2\sqrt{1+2\epsilon H}}\left(\frac{s'}{s}\right)^{\alpha(t)} \times$$

$$\times\left[1-\frac{s-s'}{s\sqrt{s}}\left(\frac{\sqrt{s}}{2}+\mathbf{p}z\right)\right], \tag{14}$$

$$\frac{c_2}{c_1}m = \frac{G}{|\mathbf{p}|^2\sqrt{s}}\int \frac{dzds'}{\eta\,(\beta+z)^2\sqrt{1+2\epsilon H}}\left(\frac{s'}{s}\right)^{\alpha(t)},$$

where $G=\frac{\lambda g}{128\pi^2}$, $H=\frac{1+\beta z}{(\beta+z)^2}$, and we ignore ϵ^2 in the "denominator" of the kernels in Eq. (14).

Expanding the kernel in Eq. (14) on ϵ and using the first two terms we get

$$\frac{1}{m^3} = \frac{G}{|\mathbf{p}|^2\sqrt{s}}\left(\frac{1}{m^2}+\frac{c_2}{c_1}\right)\int \frac{dzds'}{\eta\,(\beta+z)^2}\left(\frac{s'}{s}\right)^{\alpha(t)}\left(1-\epsilon\frac{\beta z+1}{(\beta+z)^2}\right) \times$$

$$\times\left[1-\frac{s-s'}{s\sqrt{s}}\left(\frac{\sqrt{s}}{2}+|\mathbf{p}|z\right)\right], \tag{15}$$

$$\frac{c_2}{c_1}m = \frac{G}{|\mathbf{p}|^2\sqrt{s}}\int \frac{dzds'}{\eta\,(\beta+z)^2\sqrt{1+2\epsilon H}}\left(\frac{s'}{s}\right)^{\alpha(t)}.$$

The integration limits on z and s' are determined from the kinematic condition $|\cos\theta| =$ $= \left|\frac{(\mathbf{pq})}{(\mathbf{p}^2\mathbf{q}^2)^{\frac{1}{2}}}\right| \leq 1$ and threshold condition: $s \gg \mu^2$: $-1\leq z\leq 1$, $0 \leq s' \leq s$, correspondingly.

Integrating on z (in this case we neglect $\frac{\ln s}{s}$–type term due to their very slow increasing at $s\to\infty$) we get

$$\frac{1}{m^3} = IG\left(\frac{1}{m^2}+\frac{c_2}{c_1}\right), \qquad \frac{c_2}{c_1}m = IG. \tag{16}$$

Here,

$$I = \frac{2}{|\mathbf{p}|^2\sqrt{s}}\int_0^s \frac{ds'}{\eta}\left(\frac{s'}{s}\right)^{\alpha(t)}\left[-\frac{1}{1-\beta^2}+\epsilon\frac{1}{3\left(1-\beta^2\right)^2}\right].$$

Expressing c_2 through c_1 we get (16) in the form

$$\frac{m\left(-1\pm\sqrt{5}\right)}{8G} = \int_0^1 \frac{y\,(1-y)^{\alpha(t)}}{\nu^2+y^2}dy \;+\; \epsilon\frac{|\mathbf{p}|^2}{3m^2}\int_0^1\frac{y^3\,(1-y)^{\alpha(t)}}{\left(\nu^2+y^2\right)^2}dy, \tag{17}$$

where $y = 1-\frac{s'}{s}$, $\nu^2=\frac{\mu^2}{m^2}$. The first integral in Eq. (17) is reduced to the sum of the two hypergeometric Gauss functions. The second integral is the special case[10], "Pikar integral" which can be transformed to the hypergeometric Appel function with the two variables. So we have,

$$\Lambda\left(\alpha(t)+1\right)\left(\alpha(t)+2\right) = {}_2F_1\left(1,2;\alpha(t)+3;-i\frac{m}{\mu}\right)+$$

$$+_2F_1\left(1,2;\alpha(t)+3;i\frac{m}{\mu}\right)+ \tag{18}$$

$$+\frac{\omega(\epsilon,t)}{(\alpha(t)+3)(\alpha(t)+4)}F_1\left(4,2,2;\alpha(t)+5;i\frac{m}{\mu};-i\frac{m}{\mu}\right),$$

where $\Lambda=\frac{32\pi^2\mu^2\left(-1\pm\sqrt{5}\right)}{\lambda gm}$, $\omega(\epsilon,t)=\epsilon\frac{|\mathbf{p}|^2}{\mu^2}$. Let us note that the expression (18) at $\omega=0$ coinsides with the result for the forward scattering [8].

The Appel function F_1 is more difficult for studing than the Gauss function $_2F_1$. But using the known representation of F_1 function through the degenerate hypergeometric function of the order $(3,2)$ with one variable [11], we can write Eq.(18) in the form

$$\Lambda(\alpha(t)+1)(\alpha(t)+2)={}_2F_1\left(1,2;\alpha(t)+3;-i\frac{m}{\mu}\right)+$$

$$+_2F_1\left(1,2;\alpha(t)+3;i\frac{m}{\mu}\right)+ \tag{19}$$

$$+\frac{\omega(\epsilon,t)}{(\alpha(t)+3)(\alpha(t)+4)}\,{}_3F_2\left(\begin{array}{c}2,\frac{5}{2},2;\\ \frac{\alpha(t)+5}{2},\frac{\alpha(t)+6}{2};-\frac{m^2}{\mu^2}\end{array}\right),$$

where,

$$\begin{aligned}{}_2F_1(\alpha,\beta;\gamma;z) &= \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n\,n!}z^n,\ \ \gamma\neq 0,-1,-2,...,\\ {}_3F_2\left(\begin{array}{c}\alpha_1,\alpha_2,\alpha_{3;}\\ \beta_1,\beta_2;z\end{array}\right) &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n(\alpha_3)_n}{(\beta_1)_n(\beta_2)_n\,n!}z^n,\ \ \beta_1,\beta_2\neq 0,-1,-2,...;\end{aligned} \tag{20}$$

$(\alpha)_n,(\beta)_n,(\gamma)_n$ are the Pokhhammer symbols. Using Eq.(20) we can rewrite Eq.(19) in the form

$$\Lambda(\alpha(t)+1)(\alpha(t)+2)=\sum_{n=0}^{\infty}\frac{(1)_n(2)_n}{(\alpha(t)+3)_n\,n!}\left[\left(i\frac{m}{\mu}\right)^n+\left(-i\frac{m}{\mu}\right)^n\right]+$$

$$+\frac{\omega(\epsilon,t)}{(\alpha(t)+3)(\alpha(t)+4)}\sum_{n=0}^{\infty}\frac{(2)_n\left(\frac{5}{2}\right)_n(2)_n}{\left(\frac{\alpha(t)+5}{2}\right)_n\left(\frac{\alpha(t)+6}{2}\right)_n n!}\left(-\frac{m^2}{\mu^2}\right)^n. \tag{21}$$

For determination of the explicit form of the Regge power $\alpha(t)$ we consider the three extremal cases:

1. The big exchanging masses $\frac{m^2}{\mu^2}<1$. In this case the series (21) is absolutely converted. Taking into account the first member ($n=0$) in Eq. (21) we get 4-th order algebraic equation, which allows us in principle to obtain the values of $\alpha(t)$ (see Appendix A). In the limit $\omega(\epsilon,t)=0$ we get the expression for α

$$\alpha=-\frac{3}{2}\pm\frac{1}{2}\left[1+\frac{m\lambda g}{4\pi^2\mu^2\left(-1\pm\sqrt{5}\right)}\right]^{\frac{1}{2}}, \tag{22}$$

that coinsides with the result of [8] for forward scattering.

2. The small exchanging masses $(\mu<m)$. As we have noted the generalized hypergeometric function of $(3,2)$ order is defined as the sum of generalized hypergeometric series (see Eqs.(20)) in its convergence and as analytic continuation of this series at $z\geq 1$. The analytical continuation

can be obtained, in the special case, using series near the special points $z = \infty$ (i.e. $\mu < m$) and $z = 1$ (i.e.$\mu = m$). Using the representation for ${}_3F_2$ [11]

$$
{}_3F_2\left(\begin{array}{c}\alpha_1, \alpha_2, \alpha_3;\\ \beta_1, \beta_2; z\end{array}\right) = \Gamma\left[\begin{array}{c}\beta_1, \beta_2\\ \alpha_1, \alpha_2, \alpha_3\end{array}\right] \sum_{n=1}^{3} \Gamma\left[\begin{array}{c}\alpha_n, (\alpha_3)' - \alpha_n\\ (\beta_2) - \alpha_n\end{array}\right] \times
$$

$$
\times \left(e^{\pi} z^{-1}\right)^{\alpha_n} {}_2F_1\left(1 + \alpha_n - (\beta_2), \alpha_n; 1 + \alpha_n - (\alpha_3)'; z^{-1}\right), \tag{23}
$$

(the prime means that $1 + \alpha_n - \alpha_l$ at $n = l$ is absent), where

$$
\Gamma\left[\begin{array}{c}\alpha_1, \alpha_2, \alpha_3\\ \beta_1, \beta_2\end{array}\right] = \frac{\prod\limits_{n=1}^{3} \Gamma(\alpha_n)}{\prod\limits_{l=1}^{3} \Gamma(\beta_l)},
$$

and Eqs. (20) and the analytic continuation of the hypergeometric Gauss function in the logarithmic case [10]

$$
{}_2F_1(\alpha, \alpha + m; \gamma; z) \frac{\Gamma(\alpha + m)}{\Gamma(\gamma)} = \frac{(-z)^{-\alpha - m}}{\Gamma(\gamma - \alpha)} \sum_{n=0}^{\infty} \frac{(\alpha)_{n+m} (1 - \gamma + \alpha)_{n+m}}{n! (m + n)!} z^{-n} \times
$$

$$
\times \left[\ln(-z) + h_n\right] +
$$

$$
+ (-z)^{-\alpha} \sum_{n=0}^{m-1} \frac{\Gamma(m - n) (\alpha)_n z^{-n}}{\Gamma(\gamma - \alpha - n) n!}, \tag{24}
$$

where $h_n = \psi(1 + m + n) + \psi(1 + n) - \psi(\alpha + m + n) - \psi(\gamma - \alpha - m - n)$, ψ is the logarithm derivative of Γ−function. So Eq. (19) can be rewritten in the following form

$$
\Lambda(\alpha(t) + 1) = -\frac{\mu^2}{m^2} \sum_{n=0}^{\infty} \frac{(-1 - \alpha(t))_{n+1}}{n! (n + 1)!} \times
$$

$$
\times \left[\left(-i\frac{m}{\mu}\right)^{-n} \left(\ln\left(i\frac{m}{\mu}\right) + \psi(1 + n) - \psi(\alpha(t) + 1 - n)\right) + \right.
$$

$$
\left. + \left(i\frac{m}{\mu}\right)^{-n} \left(\ln\left(-i\frac{m}{\mu}\right) + \psi(1 + n) - \psi(\alpha(t) + 1 - n)\right)\right] - \tag{25}
$$

$$
-\omega(\epsilon, t) \frac{e^{\frac{5}{2}\pi} (\alpha(t) + 5)(\alpha(t) + 6)}{16 (\alpha(t) + 1)(\alpha(t) + 2)(\alpha(t) + 3)(\alpha(t) + 4)} \times
$$

$$
\times \sum_{n=1}^{\infty} \frac{\left(\frac{1-\alpha(t)}{2}\right)_n \left(\frac{5}{2}\right)_n}{\left(\frac{3}{2}\right)_n n!} \left(-\frac{\mu^2}{m^2}\right)^{\frac{5}{2}n},
$$

and explicit form of the Regge power $\alpha(t)$ can be found in principle. Neglecting the first members in the sum in Eq.(25) (when $n = 0$) and using [10]

$$
\psi(\alpha + 1) = -\gamma + \sum_{n=1}^{\infty} \frac{\alpha}{n(\alpha + n)},
$$

($\gamma = 0,5772156649$...-Euler-Maccheroni constant) we get

$$
\Lambda = \frac{\mu^2}{m^2} \left[-\frac{1}{2} \ln \frac{\mu^2}{m^2} - \psi(\alpha(t) + 1)\right] -
$$

$$- \omega(\epsilon, t) \frac{(\alpha(t)+5)(\alpha(t)+6)}{16(\alpha(t)+1)^2(\alpha(t)+2)(\alpha(t)+3)(\alpha(t)+4)} \times$$
$$\times e^{\frac{5}{2}\pi}\left(-\frac{\mu^2}{m^2}\right)^{\frac{5}{2}}, \qquad (26)$$

which gives the principle possibility to obtain the values of $\alpha(t)$ (see Appendix A).

As it is known, the function $\psi(\alpha+1)$ has simple poles at points $\alpha = 0, -1, -2, -3, ...,$ and it changes the sign of the derivative when getting through the pole. At $\omega(\epsilon, t) \to 0$ and finite value coupling constant and the exchange mass (when $\mu \ll m$) we get the known expression of α (see Ref. [8])

$$\alpha \approx -n \pm \left[-\frac{16\pi^2 m\left(-1 \pm \sqrt{5}\right)}{\lambda g} - \frac{1}{2}\ln\frac{\mu^2}{m^2}\right], \quad n = 1, 2, 3, ..., \qquad (27)$$

$$\left|\frac{16\pi^2 m\left(-1 \pm \sqrt{5}\right)}{\lambda g} + \frac{1}{2}\ln\frac{\mu^2}{m^2}\right| \ll 1. \qquad (28)$$

It is seen from Eqs. (27) and (28) that in the case of the small exchanging masses the amplitude becomes non-analytical with respect to the coupling costant. It is nescesary to note that Eq.(28) leads to behavior of the amplitude that is in coincidence with Froissart restriction. Our results allow to state that only regular accounting of infrared singularities on the exchanging mass (μ) ($p^2 \neq m^2$, $k^2 = m^2$) can give the R6egge behavior of the scattering amplitude.

3. Let us consider the case when the masses of exchanging (μ) and external (m) particles are equal ($\mu = m$). In this case Eq. (21) has the form

$$\Lambda(\alpha(t)+1)(\alpha(t)+2) = \sum_{n=0}^{\infty}\frac{(1)_n(2)_n}{(\alpha(t)+3)_n n!}\left[(i)^n + (-i)^n\right] +$$
$$+ \frac{\omega(\epsilon, t)}{(\alpha(t)+3)(\alpha(t)+4)}\sum_{n=0}^{\infty}\frac{(2)_n\left(\frac{5}{2}\right)_n(2)_n}{\left(\frac{\alpha(t)+5}{2}\right)_n\left(\frac{\alpha(t)+6}{2}\right)_n n!}(-1)^n, \qquad (29)$$

that allows to find $\alpha(t)$ (see Appendix A). Using only the first members in Eq. (29) we obtain for the Regge power α in the limit $\omega(\epsilon, t) \to 0$:

$$\alpha = -\frac{3}{2} \pm \frac{1}{2}\sqrt{1 + \frac{\lambda g}{4\pi^2\mu\left(-1 \pm \sqrt{5}\right)}},$$

that is in occordance with previously obtained results [4-6, 8].

Appendix A

In order to check the validity of the obtained results for the Regge power $\alpha(t)$ the numerical calculations have been performed in the following cases:

1. The large values of the exchanged mass ($\mu > m$). Resticted ourselves by the first members of the series (21) we obtain the fourth order algebraic equation

$$\Lambda\alpha^4 + 10\Lambda\alpha^3 + (35\Lambda - 2)\alpha^2 + (50\Lambda - 14)\alpha + 24\Lambda - 24 - \omega = 0, \qquad (A.1)$$

where

$$\Lambda = \frac{32\pi^2\mu^2\left(-1 \pm \sqrt{5}\right)}{m\lambda g}, \quad \omega = \epsilon\frac{|\mathbf{p}|^2}{\mu^2},$$

the numerical solution of which at $\omega = 10^{-3};\ 10^{-2};\ 10^{-1}$ and $1), 2) \frac{\mu}{m} = 10;\ 100$ and $\frac{\mu}{\lambda g} = 10^{-2}; 10^{-1}; 1; 10; 10^2$ allows to find the mean value of $\alpha(t)$, which at the all values of ω, $\frac{\mu}{m}$, $\frac{\mu}{\lambda}$ is equal to $\approx -2,5$. This is in agreement with the conclusions found in the Regge theory and with the experimental data [12].

2. The small values of the exchanged mass ($\mu << m$). At $n = 0$ Eq. (25) can be rewritten as

$$\Lambda = 2\frac{\mu^2}{m^2}\left[\gamma - \frac{1}{2}\ln\frac{\mu^2}{m^2} - \frac{\alpha}{\alpha+1}\right] - \\ - \frac{\omega(\epsilon,t)}{16}\frac{(\alpha(t)+5)(\alpha(t)+6)}{(\alpha(t)+1)^2(\alpha(t)+2)(\alpha(t)+3)(\alpha(t)+4)}\times \\ \times e^{\frac{5}{2}\pi}\left(-\frac{\mu^2}{m^2}\right)^{\frac{5}{2}}. \tag{A.2}$$

Solving the Eq.(A.2) numerically at $\omega = 10^{-1}; 10^{-2}; 10^{-3}$: 1) $\frac{\mu}{m} < 3$; 2) $\frac{\mu}{m} < \frac{3}{2}$; 3) $\frac{\mu}{m} < 10^{-1}$ (let us note, that the values of $\frac{\mu}{m}$ are chosen from the convergence of the hypergeometric functions ${}_3F_2$ and ${}_2F_1$ in the case of analytical continuation $\left|\arg\frac{\mu}{m}\right| < \pi$ (see Eqs. (23)-(25)) and at values $\frac{m}{\lambda g} = 10^2; 10; 1; 10^{-1}; 10^{-2}$ we get the mean value of $\alpha(t) \approx -2,2$.

3. To find the numerical values of α in the case of equal masses ($\mu = m$) from Eq. (29) at $n = 0$, we find the 4-th order algebraic equation for $\alpha(t)$ in analogy with Eq. (A.1) with

$$\Lambda = \frac{32\pi^2\mu}{\lambda g}\left(-1 \pm \sqrt{5}\right), \quad \omega = \epsilon\frac{|\mathbf{p}|^2}{\mu^2};$$

the solution of which at $\omega = 10^{-1};\ 10^{-2};\ 10^{-3}$ and $\frac{\mu}{\lambda g} = 10^{-2};\ 10^{-1};\ 1;\ 10;\ 10^2$ for each value of ω and $\frac{\mu}{\lambda g}$ are equal to $\approx -2,5$.

The obtained numerical results allow to state, that the exchanged masses and coupling costants influence on the behavior of the scattering amplitude is very small, that indicates the importance of the scattering with light particles in our model. It is also worth to no that a correction (terms multiplied to ϵ) to the forward scattering amplitude, at small values of the momentum transfer does not impact on the behavior of the amplitude.

Appendix B

Let us represent the results of corrections to the amplitude $\delta F\ |_{p^2\neq m^2, k^2\neq m^2}$ due to the exit off the mass shell. Integrating the equations (8) we supposed that when $s >> \mu^2, m^2$ the corrections to the amplitude were of the following order $\sim (p^2 - m^2)/s$, $(k^2 - m^2)/s$.

It has been proved that the invariant amplitudes F_1 and F_2 have the Regge behaviour only in the case of taking into account the dependence on p^2 and k^2 in the amplitudes. Substituting the expressions (11) into Eq. (9) we have for δF_1 and δF_2

$$\delta F_1(s,t;p^2,k^2)\ |_{p^2\neq m^2,k^2\neq m^2} = \frac{\pi\lambda g}{(2\pi)^4}\int\frac{\left[m\frac{c_1}{p^2} + mc_2\right]\left(\frac{s'}{m^2}\right)^{\alpha(t)}}{\left[(p-q)^2 - m^2\right]\left[(k-q)^2 - m^2\right]}\times \\ \times\delta_+(q^2-\mu^2)\,\theta(q_0)\,\delta_+\left[(p+p'-q)^2 - s'\right]\theta(p_0+p'_0-q_0)\,d^4q\,ds', \tag{B.1}$$

$$\delta F_2(s,t;p^2,k^2)\ |_{p^2\neq m^2,k^2\neq m^2} = \frac{\pi\lambda g}{(2\pi)^4}\int\frac{c_1\left(\frac{s'}{m^2}\right)^{\alpha(t)}}{\left[(p-q)^2 - m^2\right]\left[(k-q)^2 - m^2\right]}\times$$

$$\times\delta_+\left(q^2-\mu^2\right)\theta\left(q_0\right)\delta_+\left[\left(p+p'-q\right)^2-s'\right]\theta\left(p_0+p_0'-q_0\right)d^4q ds'.$$

Let us solve the system (B.1) in s.m.c. where $\mathbf{p}+\mathbf{p}'=0$. Integrating with respect to momentum variables with $\delta-$ functions, and further on φ, we obtain ($s>>\mu^2$)

$$\delta F_1\left(s,t;p^2,k^2\right)|_{p^2\neq m^2,k^2\neq m^2}=$$

$$=\frac{\lambda g\left(\frac{c_1}{p^2}m+c_2m\right)}{128\pi^2\left|\mathbf{p}\right|\left|\mathbf{k}\right|\sqrt{s}}\int\frac{\left(\frac{s'}{m^2}\right)^{\alpha(t)}}{\eta\left(\alpha+z\right)\sqrt{\beta^2+2\beta z_0 z+z^2+z_0^2-1}}dzds',$$

$$\delta F_2\left(s,t;p^2,k^2\right)|_{p^2\neq m^2,k^2\neq m^2}= \tag{B.2}$$

$$=\frac{c_1\lambda g}{128\pi^2\left|\mathbf{p}\right|\left|\mathbf{k}\right|\sqrt{s}}\int\frac{\left(\frac{s'}{m^2}\right)^{\alpha(t)}}{\eta\left(\alpha+z\right)\sqrt{\beta^2+2\beta z_0 z+z^2+z_0^2-1}}ds'dz,$$

where

$$\alpha=\frac{p^2-m^2-2p_0\eta}{2\left|\mathbf{p}\right|\eta},\quad \beta=\frac{k^2-m^2-2k_0\eta}{2\left|\mathbf{k}\right|\eta},\quad \eta=\frac{s-s'}{2\sqrt{s}}.$$

In the case of small momenta transfer scattering we can change $z_0=1+\epsilon$ $(\epsilon\ll1)$ under the integrals and obtain (B.2) in the form

$$\delta F_1\left(s,t;p^2,k^2\right)|_{p^2\neq m^2,k^2\neq m^2}=$$

$$=\frac{\Lambda\left(\frac{c_1}{p^2}m+c_2m\right)}{\left|\mathbf{p}\right|\left|\mathbf{k}\right|\sqrt{s}}\int\frac{\left(\frac{s'}{m^2}\right)^{\alpha(t)}}{\eta\left(\alpha+z\right)\left(\beta+z\right)\sqrt{1+2\epsilon H}}dzds',$$

$$\delta F_2\left(s,t;p^2,k^2\right)|_{p^2\neq m^2,k^2\neq m^2}= \tag{B.3}$$

$$=\frac{c_1\Lambda}{\left|\mathbf{p}\right|\left|\mathbf{k}\right|\sqrt{s}}\int\frac{\left(\frac{s'}{m^2}\right)^{\alpha(t)}}{\eta\left(\alpha+z\right)\left(\beta+z\right)\sqrt{1+2\epsilon H}}ds'dz,$$

where

$$\Lambda=\frac{\lambda g}{128\pi^2},\, H=\frac{1+\beta z}{\left(\beta+z\right)^2}.$$

Expanding the kernel in (B.3) on ϵ and using the first two terms: $\frac{1}{\sqrt{1+2\epsilon H}}\approx1-\epsilon H$ we get:

$$\delta F_1\left(s,t;p^2,k^2\right)|_{p^2\neq m^2,k^2\neq m^2}=$$

$$=\frac{\Lambda\left(\frac{c_1}{p^2}m+c_2m\right)}{\left|\mathbf{p}\right|\left|\mathbf{k}\right|\sqrt{s}}\int\left(\frac{s'}{m^2}\right)^{\alpha(t)}\frac{ds'}{\eta}\frac{1}{\left(\alpha+z\right)\left(\beta+z\right)}\times$$

$$\times\left[1-\epsilon\frac{1+\beta z}{\left(\beta+z\right)^2}\right]dz, \tag{B.4}$$

$$\delta F_2\left(s,t;p^2,k^2\right)|_{p^2\neq m^2,k^2\neq m^2}=\frac{c_1\Lambda}{\left|\mathbf{p}\right|\left|\mathbf{k}\right|\sqrt{s}}\int\left(\frac{s'}{m^2}\right)^{\alpha(t)}\frac{ds'}{\eta}\frac{1}{\left(\alpha+z\right)\left(\beta+z\right)}$$

$$\left[1-\epsilon\frac{1+\beta z}{\left(\beta+z\right)^2}\right]dz.$$

Integrating over z in the limit $p^2 \simeq k^2,\ s >> \mu^2$ we obtain

$$\delta F_1\left(s,t;p^2,k^2\right)|_{p^2\neq m^2,k^2\neq m^2}= = -\epsilon\frac{2\left|\mathbf{p}\right|\left|\mathbf{k}\right|\Lambda}{\left(p_0\left|\mathbf{k}\right|-k_0\left|\mathbf{p}\right|\right)^2}\left(\frac{c_1}{p^2}m+c_2m\right)\left(\frac{s}{m^2}\right)^{\alpha(t)}I, \tag{B.5}$$

$$\delta F_2\left(s,t;p^2,k^2\right)|_{p^2\neq m^2,k^2\neq m^2}= -\epsilon\frac{2\left|\mathbf{p}\right|\left|\mathbf{k}\right|\Lambda c_1}{\left(p_0\left|\mathbf{k}\right|-k_0\left|\mathbf{p}\right|\right)^2}\left(\frac{s}{m^2}\right)^{\alpha(t)}I,$$

where

$$I=\int_0^1\frac{y\left(1-y\right)^{\alpha(t)}}{\left(z-y\right)^2}dy,$$

$$z=\frac{\left|\mathbf{k}\right|\left(p^2-m^2\right)-\left|\mathbf{p}\right|\left(k^2-m^2\right)}{\sqrt{s}\left(p_0\left|\mathbf{k}\right|-k_0\left|\mathbf{p}\right|\right)},y=1-\frac{s'}{s}. \tag{B.6}$$

The integral (B.6) can be represented in terms of hypergeometric Gauss function [10]. So,

$$\delta F_1\left(s,t;p^2,k^2\right)|_{p^2\neq m^2,k^2\neq m^2}=\Lambda\left(\frac{c_1}{p^2}m+c_2m\right)\left(\frac{s}{m^2}\right)^{\alpha(t)}K\left(s,t;;p^2,k^2\right),$$

$$\delta F_2\left(s,t;p^2,k^2\right)|_{p^2\neq m^2,k^2\neq m^2}=\Lambda c_1\left(\frac{s}{m^2}\right)^{\alpha(t)}K\left(s,t;;p^2,k^2\right), \tag{B.7}$$

where

$$K\left(s',t;;p^2,k^2\right)=-4\epsilon\frac{\left|\mathbf{p}\right|\left|\mathbf{k}\right|s}{\left(\left|\mathbf{k}\right|\left(p^2-m^2\right)-\left|\mathbf{p}\right|\left(k^2-m^2\right)\right)^2\left(\alpha\left(t\right)+1\right)\left(\alpha\left(t\right)+2\right)}\times$$

$$\times F\left(2,2;\alpha\left(t\right)+3;\frac{\left|\mathbf{k}\right|\left(p^2-m^2\right)-\left|\mathbf{p}\right|\left(k^2-m^2\right)}{\sqrt{s}\left(p_0\left|\mathbf{k}\right|-k_0\left|\mathbf{p}\right|\right)}\right).$$

Appendix C

Since the imaginary part of the amplitude has the correct analytical properties, let us write the one variable dispersion relation

$$ReF=\frac{1}{\pi}\int_0^\infty\frac{F\left(s',t\right)}{s'-s}ds' \tag{C.1}$$

Substituting Eq.(11) into Eq.(C.1) we get

$$ReF_1\left(s,t\right)=\frac{c_1}{\pi p^2}\left(\frac{s}{m^2}\right)^{\alpha(t)}\int_0^\infty\frac{x^{\alpha(t)}}{x-1}dx,$$

$$ReF_2\left(s,t\right)=\frac{c_2m}{\pi}\left(\frac{s}{m^2}\right)^{\alpha(t)}\int_0^\infty\frac{x^{\alpha(t)}}{x-1}dx,$$

where $x=\frac{s'}{s}$. The real part has the form

$$ReF_1\left(s,t\right)=-\frac{c_1}{p^2}\left(\frac{s}{m^2}\right)^{\alpha(t)}ctg\left[\left(\alpha\left(t\right)+1\right)\pi\right],$$

$$ReF_2\left(s,t\right)=-c_2m\left(\frac{s}{m^2}\right)^{\alpha(t)}ctg\left[\left(\alpha\left(t\right)+1\right)\pi\right].$$

From the obtained expressions for the real parts of the amplitudes, it is evident that they have also a power-like character in the Regge asymptotic form, which corresponds to the behavior of the amplitude at high energies.

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