OCTONION ELECTRODYNAMICS AND PHYSICS BEYOND STANDARD MODEL

(Particle Physics)

A THESIS SUBMITTED TO

KUMAUN UNIVERSITY NAINITAL

For the Award of the Degree of

DOCTOR OF PHILOSOPHY IN PHYSICS

by

BHUPESH CHANDRA CHANYAL

Under the supervision of

Prof. O. P. S. Negi

Department of Physics

Kumaun University, S. S. J. Campus

Almora-263601 (U. K.) India

(December 2012)

Dr. O. P. S. Negi

Professor

DEPARTMENT OF PHYSICS
KUMAUN UNIVERSITY
S. S. J. Campus, Almora
Pin–263601

Certificate

This is to certify that the entire work of the thesis entitled **"Octonion Electrodynamics and Physics Beyond Standard Model"** has been carried out my supervision. This study embodies the original contributions of the candidate to the best of my knowledge. Any part there of has not been submitted earlier in any other thesis in any university. I approve the submission of this thesis for the award of the degree of **"Doctor of Philosophy"** (Ph. D) in Physics. It is also certified that the candidate has put more than two hundred days attendance, in last three and half years, in the department of Physics, Kumaun University, Soban Singh Jeena campus, Almora for the completions of her work presented in this thesis.

Place: Almora
Date: 28/12/2012

Forwarded by:

Convener
RDC and Board of Studies
Department of Physics
Kumaun University, Nainital
(U.K.)

(O. P. S. Negi)
Supervisor
(Professor, Department of Physics)
K.U., S. S. J. Campus, Almora

<u>DECLARATION</u>

I declare that the thesis entitle **"Octonion Electrodynamics and Physics Beyond Standard Model"** (Particle Physics), is my own work conducted under supervision of Dr. O. P. S. Negi, Professor, Department of Physics, Kumaun university, S. S. J. Campus, Almora, Uttarakhand, approved by Research Degree Committee (RDC).

I further declare that to the best of my knowledge, the thesis does not contain any part of any work, which has been submitted for the award of any degree either in this university or in any other University / Deemed University without proper citation.

(Bhupesh Chandra Chanyal)

Department of Physics

Kumaun University

Soban Singh Jeena Campus,

Almora - 263601 (Uttarakhand)

Acknowledgements

I would like to express my sincere gratitude to my supervisor Prof. O. P. S. Negi for suggesting problems, valuable guidance and encouragement throughout the period of my research. Discussions with him have always inspired me to improve the quality of the manuscripts. My deepest thanks to him for his constant motivation and also for making a conducive work environment throughout the thesis work, without his guidance and support this work wouldn't have been possible.

I am highly grateful to Dr. P. S. Bisht for constant support and guidance, perpetual inspiration and encouragement throughout the progress of this work. His deep insight on the subject and vast experience has helped me a lot in learning the subject.

I would like to express my warmest thanks to Physics Department, Kumaun University Soban Singh Jeena Campus, Almora where I succeeded to complete *this work. I am grateful to Prof. M. C. Durgapal, Head of Physics Department (Kumaun University Nainital), and my teachers Prof. B. C. Joshi, Prof. K. L. Shah, and Dr. Pratibha Fuloria for their support and encouragement. I further extend my thanks to other non - teaching and technical staff of Physics Dept., Soban Singh Jeena Campus, Almora, Mr. R. S. Rayal, Mr. Bhuwan Negi, Mr. J. C. Upadhayay, Mr. Heera Singh Kharayat, Mr. P. Nailwal, Mr. Joga Ram, Mr. Rajendra Singh Rana Mr. Dan Singh and Mr. Bheem Singh. Friendly time to time co - operation from researchers Gaurav Karnatak, Pawan Kumar Joshi, Bhupendra C. S. Chauhan, Vishal Sharma and Dr. Pushpa of Physics Department, S. S. J. Campus Almora is greatly acknowledged.*

I am thankful to my elder brother Dr. S. K. Chanyal, Department of Mathematics, Kumaun University, D. S. B. Campus Nainital for his moral support and continuous encouragement received through out my Ph. D. work.

I have no words to express my heartiest regards and indebtedness to my Buaji (Mrs. Durga Dhauni) and my Fufaji (Mr. B. R. Dhauni) for their support and encouragement given throughout these years in which this work taken in hand.

The credit also goes to the inspiration that I had from my father Late R C Chanyal, my mother Late C Devi and relatives to whom I owe more than I can express.

In the last but not least, I am grateful to the authorities of Kumaun University for allowing me to carry out this research work at Kumaun University.

Place: **Almora**

Date: **28/12/2012**

(Bhupesh Chandra Chanyal)

Preface

The present thesis entitled **"Octonion Electrodynamics and Physics Beyond Standard Model"** (Particle Physics) Comprises the investigations carried by the author over the period of three and half years under the supervision of Prof. O. P. S. Negi. Department of Physics, Kumaun university, Soban Singh Jeena campus, Almora. The present thesis embodies the investigations towards the study of various field associated with dyons (particle carrying simultaneously electric and magnetic charges) and the electrodynamics, chiral media, quantum chromodynamics in octonion algebra.
The whole work is divided into Six chapters.
Chapter-1 is based on the historical development with reviewed literature of standard model, physics beyond the standard model, quaternions, octonions, monopoles and dyons.
Chapter-2 describes the solutions of wave equation and other field equations of monopoles and dyons interms of physical octonion variables. Starting with the definition of quaternions and octonions, we have discussed the asymmetry of Maxwell's equations, need of magnetic monopoles, 't Hooft-Polyakov monopoles and fields associated with dyons and it is emphasized that the BPS mass formula of dyons is universal and is also invariant under duality

and conformal transformations. The octonion electrodynamics has been analyzed from octonion wave equations of potential and currents to obtain the octonionic form of Generalized Dirac-Maxwell's equations and other quantum equations of dyons in simple, compact, consistent and manifestly covariant manner. It is shown that the octonion electrodynamics reproduces the dynamics of electric (magnetic) charge yielding to the usual form of Maxwell's equations in the absence of magnetic (electric charge) in compact, simpler and consistent way.

In **chapter-3**, we have made an attempt to investigate the generalized split octonion electrodynamics for dyons. Starting with the usual definition of split octonions along with their multiplication rules, we have established the inter relationship of split octonions with their convenient matrix realization in terms of 2×2 Zorn vector matrices in order to obtain the split octonions wave equation analogous to the potential wave equation of generalized electromagnetic fields of dyons. Consequently, the split octonion field equation in compact and simpler form has been developed and it is shown that the corresponding wave equation represents the generalized Dirac Maxwell's equations of dyons in the case of split octonion electrodynamics. Accordingly, we have made an attempt to investigate the work energy theorem or "Poynting Theorem", Maxwell stress tensor and Lorentz invariant for generalized fields of dyons in split octonion electrodynamics.

In **chapter-4**, we have discussed the octonion electrodynamics in homogeneous (isotropic) and chiral medium. Keeping in view the consequences of the present theory of dyons in isotropic medium, we have undertaken the study of the octonion analysis of time dependent Generalized Dirac - Maxwell's equations of dyons in chiral medium. Consequently, the octonionic forms of potential, field and current equation are developed in simple and compact manners in the case of homogeneous (isotropic) medium and it is emphasized that the corresponding quantum equations derived in terms of octonions are invariant under Lorentz and duality transformations. Accordingly, the generalized electrodynamics in chiral medium has been developed in terms of compact and simpler forms of octonion representations in presence of electric and magnetic charges of dyons.

In **chapter-5**, we have made an attempt to study the abelian and non-Abelian gauge theory of dyons has been made to discuss the $U(1)_e \times U(1)_m$ abelian gauge theory, $U(1) \times SU(2)$ electroweak gauge theory and also the $SU(2)_e \times SU(2)_m$ non-Abelian gauge theories of in term of 2×2 Zorn vector matrix realization of split octonions. It is shown that $SU(2)_e$ characterizes the usual theory of the Yang Mill's field (isospin or weak interactions) due to presence of electric charge while the gauge group $SU(2)_m$ predicts the existence of t-Hooft-Polyakov monopole in non-Abelian Gauge theory. Accordingly, we have established the relations between octonion basis elements and Gell-Mann λ matrices of $SU(3)$ symmetry on comparing the multiplication tables of these two. Consequently, the quantum chromodynamics (QCD) has been reformulated and it is shown that the theory of strong interactions could be explained better in terms of non-associative octonion algebra.

In **chapter-6**, we have made an attempt to discuss the role of octonions in physics beyond standard model. Thus, we have discussed the role of octonions in grand unified theories (GUTs) gauge group of which is describe is $SU(3) \times SU(2) \times U(1)$ followed by the role of octonions in supersymmetry. Further more, we have analyzed the role of octonions in gravity and dark matter where, we have described the octonion space as the combination of two quaternionic spaces namely gravitational G-space and electromagnetic EM-space. It is shown that octonionic hot dark matter contains the photon and graviton (i.e. massless particles) while the octonionic cold dark matter is associated with the $W^{\pm}, Z^{0}$ (massive) bosons. At last, we have described the role of octonion consistently in superstring theory (i.e. a theory of everything to describe the unification of all four types of forces namely gravitational, electromagnetic, weak and strong).

Contents

List of Figures

List of Tables

CHAPTER 1

General Introduction

ABSTRACT

Historical developments of standard model and physics beyond the standard model are summarized in this chapter to understand the behavior of monopoles and dyons in current grand unified theories and quark confinement problems relevant for their production and detection. On the other hand, the various roles of four division algebras (namely the algebras of real numbers, complex numbers, quaternions and octonions) in different branches of physics and mathematics are also summarized followed by the summery of the work done in different chapters of present thesis.

Chapter 1

General Introduction

1.1 Introduction

Physics is a natural science that involves [1] the study of matter and its motion through space and time, along with related concepts such as energy and force. The search for unity and simplicity has been the theme of physics ever since Newton first showed that celestial and terrestrial mechanics could be unified . The 20^{th} century has been a time for tremendous format and change in our understanding new phenomena or adding new features to the existing theories [2]. Particle physics [1,2] is the branch of physics which deals with the study of matter, energy, space and time. Its objectives are to identify the most simple objects out of which all matter is composed and to understand theï¿œforcesï¿œwhich cause them to interact and combine to make more complex things. Now a days particle physics is popularly known as high energy physics [1-3], which is the theory of basic structure of matter and its forces. The physics of elementary particles is currently described in terms of very successful theory called standard model [1,4]. It describes all known elementary particles and their interactions except gravitational interactions. The standard model accommodates the quarks and the leptons, which are constituents of matter, the vector particles that mediate the strong and electroweak forces and Higgs bosons, which is expected to account for the masses of particles. The standard model (SM) also describes [5] the unified picture of strong and electro-weak interactions within the framework of a $SU(3) \times SU(2) \times U(1)$ non-Abelian gauge theory.

Despite the full agreement with experimental data there are various motivations for believing that the SM cannot be a truly fundamental theory though it is a beautiful theory which describes well all known particle physics phenomena and their fundamental interactions up to energies of order 100 GeV. However, there are many open problems with the standard model particularly the theories beyond the energies above 100 GeV. Big Science experiments have unified three forces, but physicists now believe the energy required to unify gravity (the Grand Unification Energy) is beyond anything which could be produced on earth. Physics is now turning to Cosmology, hoping to see clues to the Grand Unification in high energy processes involving black holes, quasars or even the afterglow of the beginning of the universe. We have already seen in grand unified theories [6-12] that there should be a symmetry which relates quarks and leptons. How can we go even farther than that? We recall that the electroweak symmetry relates certain leptons to each other (such as electrons and electron neutrinos), and does the same for quarks (such as up and down quarks). The relevant equations involving these particles are invariant under this symmetry. That is, the equations are still valid if the related particles are interchanged. In the same way, grand unified theories, which unify the electroweak and strong forces, postulate an analogous symmetry between leptons and quarks. In theories of this sort, the equations are invariant under the symmetry even if leptons and quarks are interchanged. This is a kind of obvious thing to look for, but it has turned out that there are different forms this symmetry can take. There have been a number of difficulties, both experimental and theoretical, in establishing this kind of theory. Nevertheless, it's still an interesting possibility. Can we go even further? The answer seems to be yes, and in fact the resulting theory is in many ways easier to work with than what has been tried with grand unified theories. It comes from postulating a symmetry which can interchange fermions and bosons (i.e. the symmetry between forces and matter) is called supersymmetry [13-15]. In fact, there are several remarkable, physical reasons for believing that supersymmetry plays a fundamental role in particle physics:

- It is fully compatible with all the postulates of quantum field theory. There are many highly non-trivial constraints that any successful quantum field theory must satisfy (e.g., unitarity, Lorentz invariance, locality, etc.) and supersymmetry is compatible with all of them.

- It yields non-renormalization theorems, which make the theory much better behaved in the ultraviolet region. (This is because many divergences between fermion loops and bosonic loops cancel in Feynman diagrams.) These non-renormalization theorems work to all orders in perturbation theory. Moreover, they are easy to calculate, and show that supersymmetry is fundamentally intertwined with quantum field theory.

- It solves the hierarchy problem. In any grand unified field theory [16] of the strong, weak, and electromagnetic interactions, one encounters the fact that re-normalizations mix the low and high energy mass scales, making it impossible to maintain the hierarchy between these two mass scales. Supersymmetry, because of the non-renormalization theorems, prevents this mixing. Thus, without supersymmetry, GUT theories are inherently unphysical.

- It allows one to have a convergence of running coupling constants at the grand unified scale, so the coupling constant for the strong, weak, and electromagnetic interactions all meet at the GUT scale. (Without supersymmetry, the running coupling constants do not converge as well).

- It requires the presence of gravity when it is gauged. Although gravity is the most difficult force to describe in terms of quantum field theory (because quantum gravity is divergent by all power counting methods) we find that supersymmetry demands the existence of gravity once we make the symmetry a local one, and makes the theory much less divergent.

- Because it is a symmetry which mixes fermions and bosons, it provides a symmetry which may yield a truly unified field theory of all interactions. For example, all the elementary particles may lie within a single multiple of supersymmetry.

- It is easily generalized to the case of superstring and super membranes. (In fact, these extended objects require the existence of supersymmetry to make them finite or at least mathematically consistent).

- It accommodates the existence of non-perturbative dualities found in M-theory, which have revolutionized string theory. The existence of various D-brane states in different dimensions can also be seen if we reduce the algebra to lower dimensions.

Anyhow, if supersymmetry is correct, the universe must contain at least twice as many kinds of fundamental particles as those already known. Although this is rather a dramatic and daring prediction, it also means that supersymmetry is readily falsifiable, and hence much more than just an idle metaphysical speculation. Of course, the fact that no evidence has yet appeared for supersymmetry means it is also a theory in some peril of disproof. Since at least some of the super partners should be light enough to put them in a range accessible with accelerators that will be in operation within the next decade, we shouldn't have to wait very long to get some indications whether supersymmetry is a viable theory.

The asymmetry between electricity and magnetism became very clear at the end of the 20^{th} century with the formulation of Maxwell's equations for electromagnetism. Magnetic monopoles were advocated [17] to size the equations in a manifest way. But the precise checks of the consequences of Maxwell's equations, in the formulation without magnetic charges, deny any role of magnetic monopoles. In 1931, Dirac [17,18] gave the idea of magnetic monopoles as the natural generalization of usual electrodynamics. The idea is that magnetic monopoles stable particles, which carry magnetic charges, ought to exist has proved to be remarkably durable. The mere existence of particles with a magnetic charge (monopole) [17,18] implies that electric charge must be integer multiples of a fundamental units. Such a quantization of electric charges is actually observed in nature and no other explanation for this deep phenomenon was known. The numerous theoretical investigations following those of Dirac have confirmed the constancy of quantum physics with Dirac relation particularly in solving the difficulties associated with the string singularity introduced by Dirac [17-19]. Thus, monopole would symmeterize in terms of Maxwell's equations, but there would be a numerical asymmetry. These types of reasoning were the basis for the introduction of what we may now call the 'classical magnetic monopole'. In this formulation there was no prediction for the monopole mass. A kind of rule of thumb was established, assuming that the classical electron radius be equal to the classical monopole radius from which one has $M = {g_d^2 m_e}/{e^2} \approx 4700 m_e \approx 2.4 GeV$. This ingenious suggestion in connection with the existence of magnetic monopole gave rise to considerable literature [19-26] on the subject to predict the mass, size, spin and quantum properties of monopoles.

Eighth decades of twentieth century particularly last seven years after the report of Price et al [12] about the so-called experimental evidence of monopoles, witnessed the rapid development of group theory, gauge theory and quantum field theory to explain their group properties and symmetries. Due to the failure of number of attempts [27-29] to verify magnetic monopole experimentally sound compelling theoretical reasons were put forward against their existence. It was accepted as strong as that for any other undiscovered particles. But in view of lack of experimental evidence the literature partially turned to negative casting doubts on the existence of monopole in the attempts to construct a classical theory of electrodynamics in presence of both electric and magnetic charges. Rosenbaum [30], arguing against the existence of monopole, proved that it is impossible to formulate an action principle for a classical electrodynamics field [31] of such charges unless an extra restriction, contradicting Lorentz force law, is imposed on the path of magnetic charge. Similarly, it was suggested by Zwanziger [32] and Goldahaber [33] that the conjectured properties of relativistic S-matrix are violated for magnetic monopoles and Hagen [34] argued against the existence of these particles in view of the impossibility to formulate their Lorentz invariant field theory. Schwinger [35-38], as an exception to the argument against the existence of monopole, formulated a relativistically covariant quantum field theory of magnetic charges which maintained complete symmetry between electric and magnetic fields and sharpened Dirac's quantization condition by restricting the product of electric and magnetic charge to integer values. Schwinger [35-38] name the particles carrying simultaneous existence of electric and magnetic charges as Dyons. So, the theories of Schwinger [35-38] and Zwanziger [32] is the theory of particles named dyons. This quantization, though explains to some extent the negative experimental results in search of monopole, required to maintain the rotational symmetry which was violated due to existence of singular lines in the solution of the vector potential around a monopole. Peres [39] pointed out the controversial nature [40] of these singular lines [17-19,41] and derived the charged quantization +condition in purely group theoretical manner without using them. Attempts [42-44] were also made to develop the theories related to the possibility of formulating an action integral in presence of electric and magnetic charges.

Inspite of the enormous potential importance of monopole and the fact that these particles have been extensively studied, there has been presented no reliable theory which is conceptually transparent and practically tractable as the usual electrodynamics and the formalism necessary to describe them has been clumsy and not manifestly covariant. Schwinger's [35-38] gave relativistically invariant quantum field theory of spin half magnetic charges and its extension described by Nigerian [45-47] to the particle carrying electric and magnetic charges (namely dyons), though provided a natural generalization of electrodynamics, suffered due to the fact that the in the former the basic variable were not canonically conjugate while the latter admitted the duality of the number of variables to maintain locality.

On the other hand, the Lagrangian theory of monopole developed by Ezawa and Tze [48] in terms of non-Abelian gauge symmetry lacks in action principle in the presence of both electrically and magnetically charged fields. In the meantime, it become clear [45-48] that the monopole and dyons can be understood better in non-Abelian gauge theories and that the reasons for not seeing these particles so far with certainty lies elsewhere than in their inconsistencies with relativistic quantum field theory.

Fresh interest in the subject of monopoles has been enhanced by 't Hooft [49] and Polyakov [50] by demonstrating that the spontaneously broken gauge theories with compact U(1) gauge group guarantee the existence of smooth, topologically stable finite energy solutions with quantized magnetic charge. Such non-Abelian monopoles are known to arise as classical solutions in field theoretical models like the Georgi-Glassow model [51] and also in pure Yang-Mills theories where the role of fundamental Higgs scalars could eventually be played by some composite fields. In any case these non-Abelian monopoles can be understood, in the frame work of these models, as defects in space-time of U(1) gauge fields which arise once the unitary gauge is chosen [52,53]. It is notoriously difficult to describe such defects in terms of quantum fields. The 't Hooft-Polyakov monopole is not an elementary particle like that of Dirac but a complicated extended object having a definite mass and finite size inside of which massive fields play a role in providing a smooth structure and outside it they vanish rapidly leaving the field configuration identical to abelian Dirac monopole.The 't Hooft-Polyakov monopole was known numerically but there is simplified model introduced by Prasad and Sommerfield [54,55], which has an explicit stable monopole solutions. Such solution satisfying Bogomolny's condition [56] are named as Bogomolny-Prasad-Sommerfield (BPS) monopoles. Julia and Zee [57] extended the idea of 't Hooft-Polyakov and constructed classical solutions for non-Abelian dyons which arise as quantum mechanical excitations of fundamental monopoles. They come automatically from the semi classical quantization of global charge rotation degree of freedom of monopoles. In the attempts to explain CP-violation in weak interactions in terms of non-zero θ- angle of vacuum, Witten [58] showed that the non-Abelian monopoles are necessarily dyons with fractional electric charge and magnetic charge of one Dirac unit. Now it is widely recognized [59] that SU(5) grand unified model [51] is a gauge theory that contains monopole and dyon solutions. Consequently, monopoles and dyons have become intrinsic part of all current grand unified theories.

The existence of magnetic monopoles and dyons is a very general consequence of the unification of fundamental interactions but due to their heavy mass of the order of $10^{17} GeV$, there is no much hope of producing monopole and dyons by accelerators in the foreseeable future. The traditional searches for monopole have relied either on strong ionization power of a relativistic monopole [54-59], or on the assumption that it is trapped in the earth crust [60,61]. But a super heavy monopole may be expected to be slowly moving with such penetrating power. It need not ionize heavily or stop in the earth [62,63]. If a monopole is ever discovered and controlled, it will be momentous occasion with many fascinating implications confirming a fundamental prediction of grand unified theories, which could provide a unique window on new physics at incredibly short distances.

The two fundamental mathematical structures (division algebras) a physicist uses in his everyday life are the real $\mathbb{R}$ and the complex $\mathbb{C}$ numbers. As we well know, complex numbers can be treated as pairs of real numbers with a specific multiplication law. One can however go even further and build two other sets of numbers, known in mathematics as quaternions $\mathbb{H}$ [64-68] and octonions $\mathbb{O}$ [65,67]. The quaternions, formed as pairs of complex numbers are non-commutative whereas the octonions, formed as pairs of quaternion numbers are both non-commutative and non-associative. The four sets of numbers are mathematically known as division algebras. The octonions are the last division algebra, no further generalization being consistent with the laws of mathematics. There are exactly four normed division algebras [66]: the real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) [64,67], and octonions ($\mathbb{O}$) [65,68]. Octonions are a super set of quaternions in the same way that quaternions are a super set of complex numbers. We might expect this sequence to continue with an element consisting of 16 numbers, but such algebra does not exist, and the sequence ends with octonions. There are algebras, such as matrices and multi vectors, which can have more than 8 dimensions but these don't have the same properties that division always exists and norms preserved by multiplication. Soon after quaternions were discovered by Hamilton [64] then octonions were discovered separately by John Graves and Arthur Cayley [65,68]. Octonions are sometimes known as Cayley numbers. When we move from Complex numbers to Quaternions and then Octonions the system obeys fewer algebraic laws. When we go from Complex numbers to Quaternions we loose commutativity and when we go from Quaternions to Octonions we loose associatively. Hamilton's quaternions [64] play an important role in understanding the fundamental laws of physics. This number system was the very first examples of hyper complex numbers of algebra. Quaternions are extensively used in the connection of relativity [69], quantum mechanics [70], superluminal [71] and subluminal [72] Lorentz transformations and gauge theories [73]. Left-right handed Weinberg-Salam theory of electromagnetic interaction with gauge structure has been explained better in terms of quaternion [74].

On the other hand, “Physics beyond the standard model” [75,76] refers to the theoretical developments needed to explain the deficiencies of the Standard Model, such as the origin of mass, the strong CP problem, neutrino oscillations, matter–antimatter asymmetry, and the nature of dark matter and dark energy. Another problem lies within the mathematical framework of the Standard Model itself – the Standard Model is inconsistent with that of general relativity to the point that one or both theories break down in their descriptions under certain conditions (for example within known space-time singularities like the Big Bang and black hole event horizons). Theories that lie beyond the Standard Model include various extensions of the standard model through supersymmetry, such as the Minimal Supersymmetric Standard Model (MSSM) and Next-to-Minimal Supersymmetric Standard Model (NMSSM), or entirely novel explanations, such as string theory, M-theory and extra dimensions. As these theories tend to reproduce the entirety of current phenomena, the question of which theory is the right one, or at least the "best step" towards a “Theory of Everything”, can only be settled via experiments and is one of the most active areas of research in both theoretical and experimental physics.

As such, the grand unified theories (GUTs) [16] based on group later than SU(5) offer other possibilities for magnetic monopole charges and masses for their experimental detections [77-79], in particular one may have lighter monopoles ($m_M \approx 10^{10} GeV$) which may be multiply charges. If gravity is also brought into unifying picture, for instance in the form of Kaluza-Klein theories, then monopole could be much more massive ($m_M \geq 10^{19} GeV$). The smallest simple Lie group which contains the standard model, and upon which the first Grand Unified Theory was based, is $SU(5) \supset SU(3) \times SU(2) \times U(1)$ [16]. Such group symmetries allow the reinterpretation of several known particles as different states of a single particle field. However, it is not obvious that the simplest possible choices for the extended "Grand Unified" symmetry should yield the correct inventory of elementary particles. The fact that all matter particles fit nicely into three copies of the smallest group representations of $SU(5)$ and immediately carry the correct observed charges, is one of the first and most important reasons why people believe that a Grand Unified Theory might actually be realized in nature.Larger masses are also obtained in super symmetric theories. The simplest GUT theories yield too many monopoles while inflationary scenarios may lead to a very small number of monopoles. Magnetic monopoles of lower mass are expected to be stable since magnetic charge should be conserved like electric charge. Therefore, the original monopoles produced in the early universe should still be around as cosmic relics, whose kinetic energy has been strongly affected by their travel history through galactic magnetic fields [80-82].

Although dark matter [83] had historically been inferred by many astronomical observations, its composition long remained speculative. Early theories of dark matter concentrated on hidden heavy normal objects, such as black holes, neutron stars, faint old white dwarfs, as the possible candidates for dark matter. Furthermore, data from a number of lines of other evidence, including galaxy rotation curves, gravitational lensing, structure formation, and the fraction of baryons in clusters and the cluster abundance combined with independent evidence for the baryon density, indicated that 85-90% of the mass in the universe does not interact with the electromagnetic force. This nonbaryonic dark matter is evident through its gravitational effect. Consequently, the most commonly held view was that dark matter is primarily non-baryonic, made of one or more elementary particles other than the usual electrons, protons, neutrons and known neutrinos. Only about 4.6% of the mass-energy of the Universe is ordinary matter while about 23% is thought to be composed of dark matter [83]. The remaining 72% is thought to consist of dark energy, an even stranger component, distributed almost uniformly in space and with energy density non-evolving or slowly envoling with time.

In order to avoid the use of string singularities of monopoles and keeping in view, the formalism necessary to describe them has been clumsy and not manifestly covariant. Rajput et.al [84-92] have undertaken the study of second quantization and interaction of generalized electromagnetic fields associated with spin-1 and spin 1/2 particles carrying electric and magnetic charges. Throughout the work generalized fields, generalized charge, generalized current and generalized potential associated with these doubly charge particles have been taken as complex quantities with their electric and magnetic constituents real and imaginary parts. Undertaking the study of rotationally symmetric and gauge invariant angular momentum operators of dyons, it has already been shown that the presence of magnetic charge on dyons directly leads to a residual angular momentum and chirality dependent multiplicity in eigen values of third component of angular momentum operator [87]. The quaternionic formulation of generalized field equations, generalized potential, generalized current and Lorentz force equation of dyons has been investigated [88,89] in a unique, consistent and simple manner. Bi quaternion formulation of generalized field equation of dyons has been shown [90,91] in simple, unique, self consistent and manifestly covariant one. Instead of of real quaternions, complex quaternions (bi quaternions) method of description of generalized electromagnetic fields of dyons has been adopted and the corresponding physical quantities respectively. The generalized equations of dyons are described in simple and compact bi quaternion forms [92,93]. The Dirac - Maxwell equations, equation of motion, potential fields and Lagrangian density associated with generalized filed of dyons [94] and gravito dyons [95] for their manifestly covariant theory and self contained structure of dyons has been reformulated. Non - Abelian gauge theory of dyons and gravito dyons has been developed in terms of quaternions [96] and octonions [97,98] gauge group and it is shown that the gauge structure characterize Abelian $U(1)$ and non - Abelian $SU(2)$ gauge structures.

Accordingly, from dual symmetry, generalized electrodynamics and Maxwell's equation are discussed in the presence of monopole while the electric charge has been shown to be absent therein. Bisht et al [99] have also developed the quaternionic formulation of dual electrodynamics in simple, compact and consistent manner. Superluminal electromagnetic fields of dyons have been described in R^4 and T^4-spaces and their quaternion equivalents are analyzed accordingly. It is shown that on passing from subluminal to superluminal realm via quaternion, the theory of dyons becomes the tachyonic dyons. Correspondingly, the quaternionic field equations of bradyonic and tachyonic dyons are derived [100] in R^4 and T^4-spaces respectively. Generalized Dirac-Maxwell (GDM) equations have also been derived in presence of electric and magnetic sources in an isotropic (homogenous) medium. Bisht et al have also derived [101] other quantum equations of dyons in consistent and manifest covariant way. This theory has been shown to remain invariant under the duality transformations in isotropic homogeneous medium. Quaternion analysis of time dependent Maxwell's equations has been developed [102] in presence of electric and magnetic charges and the solution for the classical problem of moving charge (electric and magnetic) are obtained consistently. The time dependent generalized Dirac-Maxwell's (GDM) equations of dyons have also been discussed [103] in Chiral and inhomogeneous media and the solutions for the classical problem are obtained. The quaternion reformulation of generalized electromagnetic fields of dyons in chiral and inhomogeneous media has also been analyzed [104]. Bisht et al. have also discussed [105] the monochromatic fields of generalized electromagnetic fields of dyons in slowly changing media in a consistent manner. Application of quaternions and to Supersymmetry quantum mechanics are explored [106] while the behavior of tachyons in Supersymmetry has also been analyzed [107,108]. The quaternionic octonion gauge analyticity of dyons has also been studied [109] in simpler and consistent manner.

1.2 Summary of the present work

Keeping in view the recent updates of standard model, use of quaternions and octonions in various branches of physics and mathematics and also the fact that the formulation necessary to describe the theory relevant for the protection and detection of magnetic monopoles and dyons has been clumsy and manifestly non-covariant, in the present thesis we have made an attempt to developed the consistent octonion reformulation for generalized electrodynamics of dyons and their possible role in order to understand the physics beyond the standard model.
The entire work done in the present thesis is divided in six chapters.

Chapter-1 is based on the historical development with reviewed literature of standard model, physics beyond the standard model, quaternions, octonions, monopoles and dyons.

Chapter-2 describes the solutions of wave equation and other field equations of monopoles and dyons interms of physical octonion variables. In *section (2.2)*, we have reviewed the earlier literature on quaternion, octonion and

Fano plane. Here in *section (2.3),* the asymmetry of Maxwell's equations, need of magnetic monopoles, 't Hooft-Polyakov monopoles and fields associated with dyons are discussed in the context to the utility and advantages of monopoles and dyons. It is emphasized that the BPS mass formula of dyons is universal and is also invariant under duality and conformal transformations. In *section (2.4),* we have discussed the octonion wave equation from left and right regularity conditions of octonions. It is shown that the homogeneous octonion wave equation provides no place for electric and magnetic charges, while the inhomogeneous octonion wave equation deals with the charge and current source which may have important role in order to understand the existence of monopoles and dyons. The octonion wave equation thus can be interpreted as the classical wave (field) equation of physical variables. In *section (2.5),* the octonion electrodynamics has been analyzed interms of compact simple and manifestly covariant way of octonion wave equations of potential and currents. Here, we have obtained the octonionic form of Generalized Dirac-Maxwell's equations and other quantum equations of dyons in simple, compact and consistent way incorporating the non-associativity of octonion variables. *Section (2.6)* provides the discussion and conclusion of the whole work done in this chapter. It is concluded that the presents octonion reformulation of generalized fields of dyons represents well the invariance of field equations under the Lorentz and duality transformations. It also discussed the dynamics of electric (magnetic) charge yielding to the usual form of Maxwell's equations in the absence of magnetic (electric charge) in compact, simpler and consistent way.

In **chapter-3,** we have made an attempt to investigate the generalized split octonion electrodynamics for dyons. Starting with the usual definition of split octonions along with their multiplication rules, in *section (3.2)*, we have reconnected the split octonion with their convenient matrix realization in terms of 2×2 Zorn vector matrices. The multiplication rules and other properties of split octonion are analyzed in terms of 2×2 Zorn vector matrix realization and accordingly the differential operator has been rewritten in terms of 2×2 Zorn vector matrix realization of split octonions. Using the definitions of split octonions and their connection with Zorn vector matrix realization, we have developed the split octonionic form of generalized four potential of dyons (*section - 3.3*) and thus obtained the split octonion wave equation which is analogous to the potential wave equation giving rise to generalized electromagnetic fields of dyons. Consequently, the split octonion field equation in compact and simpler form has been developed and it is shown that the split octonion wave equation represents the generalized Dirac Maxwell's equations of dyons in the case of split octonion electrodynamics. Another quantum equations for generalized potential, fields, current and other physical variables are also developed in compact and similar form of split octonion electrodynamics in *section (3.4)*. As such, in *section (3.5)*, we have analyzed the laws associated with energy momentum conservation in split octonion electrodynamics. Accordingly, we have investigated the work energy theorem or "Poynting Theorem" to the case of generalized electromagnetic fields of dyons in split octonion formulation and their Zorn vector matrix realization in consistent manner. The Poynting theorem has been discussed for the conservation of energy associated with generalized fields of dyons in split octonion electrodynamics. Furthermore, the Maxwell stress tensor for generalized fields of dyons has also been reformulated for split octonion electrodynamics it is shown that the divergence of Maxwell's stress tensor represents the "generalized electromagnetic force" of dyons. More over it is shown that a part of Maxwell's stress tensor represents the generalized Dirac Maxwell's equations of dyons in split octonion electrodynamics. In *section (3.6)*, we have made an attempt to analyze the split octonion reformulation of Lorentz invariant of generalized split octonion electrodynamics of dyons and we have thus obtained the Lorentz invariants like $\left(\vec{E}^2 - \vec{H}^2\right), \left(\overrightarrow{E}.\overrightarrow{H}\right), \overrightarrow{\nabla}\left(\vec{E}^2 - \vec{H}^2\right)$, and $\overrightarrow{\nabla}\left(\overrightarrow{E}.\overrightarrow{H}\right)$ and it is shown that the reformulation of classical electrodynamics in terms of split octonion formulation in compact, simpler, manifestly covariant and consistent manner. Consequently, it is concluded that the split octonion electrodynamics reproduces the electrodynamics of electric (magnetic) charge in the absence of magnetic (electric) charge of dyons and vice-versa.

In **chapter-4**, we have discussed the octonion electrodynamics in homogeneous (isotropic) and chiral medium. In *section (4.2)*, we have discussed the definition of chiral medium. The chiral media are isotropic birefringent substances that responses to either electric or magnetic excitation with both electric and magnetic polarizations. In *section (4.3)*, we have obtained the generalized electromagnetic fields equations of dyons in isotropic medium. Thus, we have derived the generalized Dirac-Maxwell's equations and other various quantum equations in the homogeneous (isotropic) medium. It has been shown that the field equations of dyons remain invariant under the duality transformations in isotropic homogeneous medium and the equation of motion reproduces the rotationally symmetric gauge invariant angular momentum of dyons. Keeping in view the consequences of the present theory of dyons in isotropic medium, we have also undertaken the study of the octonion analysis of time dependent Maxwell's equations in chiral medium for dyons in presence of electric and magnetic charges (sources) are obtained in unique, simpler and consistent manner.Thus, in *section (4.4)*, we have discussed the generalized octonion Maxwell's equations in the case of isotropic medium. Accordingly, the octonionic forms of potential, field and current equation are developed in simple and compact manners in the case of homogeneous (isotropic) medium and it is emphasized that the corresponding quantum equations derived in terms of octonions are invariant under Lorentz and duality transformations. Accordingly, we have discussed the generalized octonion electrodynamics in chiral medium *(section 4.5)*. It provides the field equations, wave equations and other quantum equations of dyons in the case of chiral medium by means of octonionic eight dimensional representations. As such, we have described the chiral parameter and pairing constant in terms of octonionic representation of Drude-Born-Fedorov constitutive relations. Hence, we have derived the generalized theory of Dirac-Maxwell's equations in presence of electric and magnetic charges of dyons in the case of chiral media in simple, compact and consistent manner.

In **Chapter-5**, we have made an attempt to study the abelian and non-Abelian gauge theory of dyons with the application of split octonions and their Zorn vector matrix realization. In *section (5.2)* we have discussed the U(1)×U(1) abelian gauge theory of dyons from the invariance principles of Lagrangian formulation in order to obtain the dyonic field equations. It has been shown that this formalism provides better understanding to explain the duality conjunctive for the justification of existence of monopoles and dyons. In *section (5.3)*, we have discussed the U(1)×U(1) octonion gauge formulation in terms of 2×2 Zorn vector matrix realization of split octonion in compact and consistent manner. As such, we have developed the octonion covariant derivative for U(1)×U(1) gauge theory of dyons in terms of 2×2 Zorn matrix realization of split octonions. It is shown that the commutation relation between the octonion covariant derivative leads to two types of gauge field strength of generalized electromagnetic fields of dyons responsible for the simultaneous existence of electric charge and magnetic monopole. It is also shown that the generalized Dirac Maxwell's equations of dyons leads to two types of two photon in terms of two four currents associated with electric charge and magnetic monopole. In *section (5.4)*, we have discussed the octonion gauge fields as the combination of two quaternion gauge fields. The covariant derivative, abelian and non-Abelian gauge structure and the gauge current equation are described in split octonion formulation of gauge theory. As such, we have investigated the U(1)×SU(2) octonionic gauge formulation in simpler and compact manner. Our U(1)×SU(2) theory of weak interaction describes two fold symmetry of electroweak interactions. The first fold describes the gauge boson of standard electroweak theory while the second one has be investigated to describe the structure of alternative electroweak interaction to the presence of magnetic monopole. In *section (5.5)*, we have extended U(1)×SU(2) to the non-Abelian $SU(2)_e \times SU(2)_m$ gauge formulation in terms of 2×2 Zorn vector matrix of split octonions. Accordingly, the octonion gauge theory has been reconnected to the 't Hooft Polyakov magnetic monopole theory (*section 5.6*) in order to satisfy the existence of magnetic monopole in Grand Unified Theories (GUTs). In *section (5.7)*, we have discussed the SU(3) generators (Gell-Mann matrices) and their multiplication properties and accordingly the resemblance between the octonion basis elements and the SU(3) generators are discussed in *section (5.8)*, where a proper mapping between two has been investigated. Accordingly, the SU(3) symmetry as been developed interms of non-associativity of octonion basis elements which does not effect the invariance of SU(2) spin (i-spin) multiplets. Further more, it is concluded that the algebra of strong interactions correspond to SU(3) automorphism of octonion algebra and supports earlier results of Gï¿œnaydin [69,70]. In *section (5.9)*, we have discussed the relationship of octonions and the parameters of quantum chromodynamics (QCD). Consequently, the exact SU(3) symmetry of colors has been investigated in terms of octonion algebra in order to describe quantum chromodynamics (QCD). Hence, we have reformulated the theory of strong interaction (i.e. the quantum chromodynamics (QCD)) based on colors $SU(3)_c$ whose generators satisfy the non-associative algebra of octonions. It is shown that in this theory the gluonic field strength tensor of QCD behaves like to the electromagnetic field strength tensor of QED. More over, the SU(3) gauge theory of strong interactions and the invariant Lagrangian formulation has been suitably handled in terms of non-associativity of octonion in *section (5.10)*, where gauge transformations are octonionic, and the octonion affinity describes the Yang-Mill's field. It is concluded that octonionic colored quarks are dyons where the generalized field of dyons are discussed as the two fold gauge symmetries of SU(3) non-Abelian gauge group associated respectively with electric and magnetic charges.

Chapter-6 describes the role of octonions in physics beyond standard model. In *section (6.2)*, we have discussed the role of octonions in grand unified theory (GUT) gauge group of which is describes $SU(3) \times SU(2) \times U(1)$. Here we have extended $SU(2) \times U(1)$ (electroweak) gauge theory to the $SU(3) \times SU(2) \times U(1)$ gauge theory in terms of 2×2 Zorn vector matrix realization of split octonions. Thus, we have established the covariant derivative, gauge field strength and field equation for the case of grand unified theory in terms of 2×2 Zorn vector matrix realization of split octonion. As such, the octonionic formulation regardless a generalization of GUTs for the mixing of gauge current used for U(1), SU(2) and SU(3) sectors associated respectively with the electromagnetic, weak and strong interactions in presence of dyons. In *section (6.3)*, we have undertake the study of role of octonions in supersymmetry and features of octonions realization of supersymmetry. Accordingly, we have discussed the supersymmetry algebra and their properties in terms of 2×2 split octonionic valued matrices in simple, compact and consistent manner. In *section (6.4)*, we have analyzed the role of octonions in gravity and dark matter. Here, we have described the octonion space as the combination of two quaternionic spaces namely gravitational G-space and electromagnetic EM-space. As such, the octonionic differential operator, octonionic valued potential, octonionic field equation and other quantum equations have been reformulated in gravitational - electromagnetic space of octonion representation in simpler and consistent way. Consequently, we have discussed the radius vector, velocity representation and generalized charge and generalized mass of the particle in terms of octonion representations. It is shown that the gravitational - electromagnetic fields has been divided in terms of four type of sub-fields namely G-G, EM-G, EM-EM and G-EM subfields. Further more in subsection (6.4.1), we have reformulated the theory dark matter in terms of octonion variables. It is emphasized that the dark matter neither emits nor absorbs light or electromagnetic radiation at any significant level. Instead, its existence and properties have been analyzed from its gravitational effects on visible matter, radiation and large scale structure of the universe. Here the dark matter (nonbaryonic) has been investigated in terms of octonion hot-dark matter and octonion cold-matter. As such, we have derived the various quantum equations for octonionic hot dark matter and cold dark matter. It is shown that octonionic hot dark matter contains the photon and graviton (i.e. massless particles) while the octonionic cold dark matter is associated with the $W^{\pm}, Z^{0}$ (massive) bosons. At last in *section (6.5)*, we have discussed the role of octonion in superstring theory (i.e. a theory of everything to describe the unification of all four types of forces namely gravitational, electromagnetic, weak and strong). The octonionic differential operator, octonionic valued potential wave equation, octonionic field equation and other various quantum equations has been discussed the framework of superstring theory in simpler, compact and consistent manner. Consequently, the generalized Dirac-Maxwell's equations are studied with the preview of superstring theory by means of octonions.

Bibliography

[1] S. Weinberg, "**The quantum theory of fields**", Book, Cambridge University Press, vol **2** (1996).

[2] D. J. Griffiths, "**Introduction to Elementary Particles**", book, John Wiley & Sons (1987).

[3] M. E. Peskin and D. V. Schroeder, "**An introduction to quantum field theory**", Book, Harper Collins, (1995).

[4] S. F. Novaes, "**Standard Model: An Introduction**", arXiv: hep-ph/0001283, (2000).

[5] P. Langacker and A. K. Mann, **"The unification of electromagnetism with weak force"**, Physics Today, December (1989), 22.

[6] G. D. Coughtan and J. E. Dodd, **"The ideas of Particle Physics"**, Cambridge University Press, (1991).

[7] S. L. Glashow, **"Towards a unified theory: Threads in a Tapestry"**, Nobel Lecture, Reviews of Modern phys., **52** (1980), 539.

[8] G. t' Hooft, **"A property of electric and magnetic flux in non-Abelian gauge theories"**, Nucl. Phys., **B153** (1979), 141.

[9] N. Craigie, **"Theory and Detection of magnetic monopoles in Gauge Theories"**, World Scientific, (1986).

[10] Y. M. Cho, **"Restricted gauge theory"**, Phys. Rev., **D21** (1980), 1080.

[11] G. t' Hooft, **"Topology of the gauge condition and new confinement phases in non-abelian gauge theories"**, Nucl. Phys., **B190** (1981), 455.

[12] P. B. Price, E. K. Shirk, W. Z. Osborne and Pinsky, **"Evidence for the Detection of a Moving Magnetic Monopole"**, Phys. Rev., **D18** (1978), 1382.

[13] J. Wess and J. Bagger, **"Supersymmetry and Supergravity"**, Princeton University Press (1983).

[14] H. J. W. Muller Kisten and A. Wiedmann; **"Supersymmetry an Introduction with conceptual and calculational details",** World Scientific (USA) (1957).

[15] E. Witten, **"Introduction to Supersymmetry in the unity of fundamental Interactions",** Ed. A. Zichichi, Plenum Press, NY. 305 (1983).

[16] S. Raby, **"Grand Unified Theories"**, arXiv: hep-ph/0608183, (2006).

[17] P. A. M. Dirac, "**Quantized Singularities in the Electromagnetic Field**", Proc. Roy. Soc. London, **A133** (1931), 60.

[18] P. A. M. Dirac, **"The theory of magnetic poles",** Phy. Rev., **74** (1948), 817.

[19] P. A. M. Dirac, **"Generalized Hamiltonion Dynamics"**, Can. J. Math., **2** (1950), 129.

[20] J. Tamm, **"Die verallgemeinerten kugelfunktionen und die wellenfunktionen eines elektrons un felde eines magnetpoles"**, Z. Phys. <u>**71**</u> (1931), 141.

[21] B. O. Gronblom, **"ï¿œber singulï¿œre magnet-pole**", Ann. Phys., <u>**5**</u> (1938), 32.

[22] D. Sivers, **"Possible binding of a magnetic monopole to a particle with electric charge and a magnetic dipole moment"**, Phys. Rev., <u>**D2**</u> (1970), 2048.

[23] H. A. Wilson, **"Note on Dirac's Theory of Magnetic Poles"**, Phys. Rev., <u>**75**</u> (1949), 309.

[24] N. F. Ramsay, **"Time Reversal, Charge Conjugation, Magnetic Pole Conjugation, and Parity**", Phys. Rev., <u>**109**</u> (1958), 225.

[25] N. Cabibbo and E. Ferrari, "**Quantum electrodynamics with Dirac monopoles**", Nuovo Cim., <u>**23**</u> (1962), 1147.

[26] S. Weinberg, **"Photons and Gravitons in Perturbation Theory: Derivation of Maxwell's and Einstein's Equations"**, Phys. Rev., <u>**138B**</u> (1965), 988.

[27] E. M. Pureell, G. B. Collins, T. Fuggi, J. Harnbastel and F. Turkot, "**Search for the Dirac monopole with 30-Bev protons**", Phys. Rev., <u>**129**</u> (1963), 2326.

[28] M. Fidercaro, G. Finochiraro and G. Giacomelli, "**Search for magnetic monopoles**", Nuovo Cimento, <u>**22**</u> (1961), 657.

[29] E. Baver, "**The dual recurrence relation for multiplicative processes**", Proc. Cambridge Phil. Soc., <u>**47**</u> (1951), 777.

[30] D. Rosenbaum, "**Proof of the impossibility of a classical action princple for magnetic monopoles and charges without subsidiary conditions**", Phys. Rev., **147** (1966), 891.

[31] F. Rohruch, "**Classical Theory of Magnetic Monopoles**", Phys. Rev., **150** (1966), 1104.

[32] D. Zwanziger, "**Dirac Magnetic Poles Forbidden in S-Matrix Theory**", Phys. Rev., **137B** (1965), 647.

[33] A. S. Goldhaber, **"Spin and statistics connection for charge - monopole composites"**, Phys. Lett., **36** (1976), 1122.

[34] C. R. Hagen, **"Noncovariance of the Dirac monopole"**, Phys. Rev., **104B** (1965), 804.

[35] J. Schwinger, **"Dyons Versus Quarks"**, Science, **166** (1969), 690.

[36] J. Schwinger, **"Magnetic charge and Quantum Field Theory"**, Phys. Rev., **144** (1966), 1087.

[37] J. Schwinger, **"Sources and Magnetic charge,"** Phys. Rev., **173** (1968), 1536.

[38] J. Schwinger, "**A Magnetic Model of Matter**", Science, **165** (1969), 757.

[39] A. Peres, "**Rotational invariance of magnetic monopoles**", Phys. Rev., **167** (1968), 1449.

[40] A. Peres, "**Singular String of Magnetic Monopoles**", Phys. Rev. Lett., **18** (1967), 50.

[41] T. M. Van, "**Classical theory of magnetic charge**", Phys. Rev., **160** (1967), 1182.

[42] J. G. Taylor, "**Dispersion relations and Schwartz's distributions**" Ann. Phys., **5** (1958), 391.

[43] V. P. Martem'yanov and S. K. Khakimov, "**Slowing down of a Dirac monopole in metals and ferromagnetic substances**", Sov. Phys., JETP **35** (1972), 20.

[44] D. T. Miller, "**Comments on the classical theory of magnetic monopoles**", Proc. Cambridge Phil. Soc., **69** (1971), 499.

[45] D. Zwanziger, "**Local-Lagrangian Quantum Field Theory of Electric and Magnetic Charges**", Phys. Rev., **D3** (1971), 880.

[46] D. Zwanziger, "**Angular Distributions and a Selection Rule in Charge-Pole Reactions**", Phys. Rev., **D6** (1972), 458.

[47] D. Zwanziger, **"Quantum Field Theory of Particles with Both Electric and Magnetic Charges"**, Phys. Rev., **176** (1968), 1489.

[48] Z. F. Ezwa and H. C. Tze, "**Global signatures of gauge invariance, vortices and monopoles**", Phys. Rev., **D15** (1977), 1647.

[49] G. 't Hooft, **"Magnetic monopoles in unified gauge theory"**, Nucl. Phys., **B79** (1974), 276.

[50] A. M. Polyakov, "**Particle spectrum in quantum field theory**", JETP Lett., **20** (1974), 194.

[51] W. Georgi and S. L. Glashow, **"Unity of All Elementary-Particle Forces"**, Phys. Rev. Lett., **32** (1974), 438.

[52] M. Blagojavic and P. Sanjanevic, "**The quantum field theory of electric and magnetic charge**", Phys. Rep., **157** (1988), 233.

[53] A. Di Giacomo and M. Marthur, "**Magnetic monopoles, gauge invariant dynamical variables and Georgi Glashow model**". Phys. Lett., **B400** (1997), 129.

[54] M. K. Prasad, "**Yang-Mills-Higgs monopole solutions of arbitrary topological charge**", Comm. Math. Phys., **80** (1981), 137.

[55] M. K. Prasad and C. M. Sommerfield, "**Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon**", Phys. Rev. Lett., **35** (1975), 760.

[56] E. B. Bogomolny, "**Stability of Classical Solutions**", Sov. J. Nucl. Phys., **24** (1976), 449.

[57] B. Julia and A. Zee, **"Poles with both magnetic and electric charges in non-Abelian gauge theory"**, Phys. Rev., **D11** (1975), 2227.

[58] E. Witten, "**Dyons of charge** $e\vartheta/2\pi$", Phys. Lett., **B86** (1979), 283.

[59] C. Dokos and T. Tomaras, "**Monopoles and dyons in the SU(5) model**", Phys. Rev., **D21** (1980), 2940.

[60] H. H. Kolm et al. "**Search for magnetic monopoles**", Phys. Rev., **D4** (1971), 1285.

[61] P. H. Ebherard, "**Search for magnetic monopoles in lunar material**", Phys. Rev., **D4** (1971), 3260.

[62] J. Preskill, "**Cosmological Production of Superheavy Magnetic Monopoles**", Phys. Rev. Lett., **43** (1979), 1365.

[63] M. J. Longo, **"Massive Magnetic Monopoles: Indirect and Direct Limits on Their Number, Density, and Flux"**, Phys. Rev., **D25** (1982), 2399.

[64] W. R. Hamilton, **"Elements of Quaternions"** , Vol. I & II, Chelsea Publishing, New York (1969) 1185.

[65] A. Cayley, "**On certain results relating to quaternions**", Phil. Mag., **26** (1845), 210.

[66] R. Schafer, **"Introduction to Non-Associative Algebras"**, Dover, New York, (1995).

[67] P. G. Tait, **"An elementary treatise on quaternions"**, Oxford, (1867); 2^{nd} Ed.; Cambridge, (1873); 3^{rd}, (1890).

[68] J. Graves, **"On a connection between the general theory of normal couples and the theory of complete quadratic functions of two variables"**, Phil. Mag., **26** (1845), 320.

[69] H. T. Flint, **"Applications of quaternions to the theory of relativity**", Phil. Mag., **39** (1920), 439.

[70] D. Finkelstein, J. M. Jauch and D. Speiser, **"Notes on quaternion quantum mechanics**", I, II, III., CERN reports 597, 599, 5917. Published in: C.A. Hooker, (Reidel, Dordrecht, 1979) Vol. II 367.

[71] R. Mignani, "**Quaternionic form of superluminal Lorentz transformations**", Lett. Nuovo Cimento, **13** (1975), 134.

[72] K. Imaeda, "**On quaternionic form of superluminal transformations**", Lett. Nuovo Cimento., **15** (1976), 91.

[73] F. Gursey and H. C. Tze, **"Complex and quaternionic Analyticity in chiral and Gauge theories"**, Ann. of Phys., **128** (1980), 29.

[74] K. Morita, "**Quaternionic Variational Formalism for Poincare Gauge Theory and Supergravity**", Prog. Theor. Phys., **73** (1985), 4.

[75] Lykken, **"Beyond the Standard Model"**, arXiv: hep-ph/1005.1676 (2010).

[76] J. W. F. Valle, **"Physics Beyond the Standard Model"**, arXiv: hep-ph/9603307, (1996).

[77] G. Giacomelli, **"Magnetic Monopoles"**, 3^{rd} School on Non-Accelerator Particle Astrophysics, Trieste (1993).

[78] G. Giacomelli, "**Experimental status of magnetic monopole**", Proc. of workshop on magnetic monopole, Wingspread, (1982).

[79] R. A. Carrigan Jr. and W. P. Trower, "**Magnetic Monopoles**", Plenum, New York, (1983).

[80] D. Akers and D. O. Akers, "**Magnetic Monopole Spin Resonance**", Phys.Rev. **D29** (1984), 1026.

[81] D. Akers, "**Paschen-Back effect in dyonium**", Int. J. Theor. Phys., **26** (1987), 451.

[82] D. Akers, "**Mikhailov's experiments on detection of magnetic charge**", Int. J. Theor. Phys., **29** (1988), 1091.

[83] G. Bertone, D. Hooper and J. Silk, "**Particle darkmatter: evidence, candidates and constraints**", Physics Reports, **405** (2005), 279.

[84] P. S. Bisht, O. P. S. Negi and B. S. Rajput, **"Null-tetrad formulation of dyons"**, IL Nuovo Cimento **104A** (1991), 337.

[85] B. S. Rajput, **"Unification of generalized electromagnetic and gravitational fields"**, J.Math.Phys., **25** (1984), 351.

[86] D. S. Bhakuni, O. P. S. Negi and B. S. Rajput, **"Dyon Dyon Scattering In Born Approximation"**, Nuovo Cim. **A92** (1986), 72.

[87] S. Bisht, O. P. S. Negi and B. S. Rajput, **"Matrix bi quaternion formalism and tachyonic dyons"**, Ind. J. Pure Appl. Phys., **29** (1991), 457.

[88] D. S. Bhakuni, O. P. S. Negi and B. S. Rajput, **"Angular-momentum operators for dyons"**, **36** (1983), 499.

[89] B. S. Rajput and O. P. S. Negi, **"Monopoles Dyons and Tachyons"**, Proc. of Symp., Pragati Prakashan (1986), 220.

[90] B. S. Rajput, S. Kumar and O. P. S. Negi, **"Quaternionic formulation for generalized field equations in the presence of dyons"**, Lett. Nuovo Cim., **34** (1982), 180.

[91] P. S. Bisht, O. P. S. Negi and B. S. Rajput, **"Bi quaternionic formulation of gravito dyons"**, Ind. J. of Pure and Applied Phys., **28** (1990), 157.

[92] P. S. Bisht, O. P. S. Negi and B. S. Rajput, **"Null tetrad formulation of non Abelian dyons"**, Ind. J. of Pure and Applied Phys., **32** (1993), 2099.

[93] P. S. Bisht, Ph. D thesis entitled, **"Quaternion Octonion formalism for unified fields of dyons and gravito dyons"**, Kumaun University Nainital (1991) (unpublished).

[94] S. Dangwal, P. S. Bisht and O. P. S. Negi, **"Unified Angular Momentum of Dyons"**, Russ. J. Phys., **49** (2007), 1274.

[95] O. P. S. Negi, S. Bisht and P. S. Bisht, **"Revisiting quaternion formulation and electromagnetism"**, Nuovo Cim., **B113** (1998), 1449.

[96] J. Singh, Ph. D thesis entitled, **"Generalised Quaternionic Electrodynamics"**, Kumaun University Nainital (2007) (unpublished).

[97] P. S. Bisht, O. P. S. Negi, B. S. Rajput, **"Quaternion gauge theory of dyonic fields"**, Prog. Theor. Phys., **85** (1991), 157.

[98] S. Dangwal, P. S. Bisht, O. P. S. Negi, **"Octonionic Gauge Formulation for Dyonic Fields"**, arXiv:hep-th/0608061 9 Aug. 2006.

[99] P. S. Bisht and O. P. S. Negi, **"Revisiting Quaternionic Dual Electrodynamics"**, E-print-arXiv:0709.0088v1 [hep-th] 2 Sep 2007, Int. J. Theor. Phys., DOI 10.1007/s10773-008-9744-8.

[100] Jivan Singh, P. S. Bisht and O. P. S. Negi, **"Quaternionic Reformulation of Generalized superluminal electromagnetic fields of dyons",** E-print arXiv-hep-th/0607212, Proceedings of the UAE-CERN Workshop (AIP Conference Proceedings / High Energy Physics), **1006** (2008), 142

[101] Jivan Singh , P. S. Bisht and O. P. S. Negi, **"Generalized electromagnetic fields of dyons in isotropic medium"**, Communication in Physics, **17** (2007), 83.

[102] Jivan Singh, P. S. Bisht and O. P. S. Negi, **"Generalized electromagnetic fields in chiral media"** J. Phys. A: Math. and Theor., **40** (2007), 9137.

[103] Jivan Singh , P. S. Bisht and O. P. S. Negi, **"Quaternion Analyticity of Time-Harmonic Dyon Field Equations"**, arXiv:hep-th/0703107 (2007).

[104] Jivan Singh, P. S. Bisht and O. P. S. Negi, **"Quaternion Analysis for Generalized Electromagnetic Fields of Dyons in Isotropic Medium"**, J. Phys. A: Math. and Theor., **40** (2007), 11395.

[105] P. S. Bisht, Jivan Singh and O. P. S. Negi, **“Generalized Electromagnetic fields of dyons in inhomogenous media”**, E print-arXiv:070603010 [hep-th] 3 June (2007).

[106] S. Rawat, Ph. D thesis entitled, **“Quaternion Supersymmetric Quantum Mechanics”**, Kumaun University Nainital (2006) (unpublished).

[107] M. C. Pant, Ph. D thesis entitled, **“Supersymmetric behavior of Tachyons”**, Kumaun University Nainital (2007) (unplished).

[108] Meena Bisht, Ph. D thesis entitled, **“Supersymmetric Quantum Theory in higher dimensional space time”**, Kumaun University Nainital (2005) (unplished).

[109] Pushpa, Ph. D thesis entitled, **“Quaternionic octonion gauge analyticity”**, Kumaun University Nainital (2011) (unplished).

CHAPTER 2

Generalized Octonion Electrodynamics

A part of this chapter has been published in Int. J. Thoer. Phys., 49,(2010) 1333.

ABSTRACT

Starting with the definition of quaternions and octonions, we have discussed the asymmetry of Maxwell's equations, need of magnetic monopoles, 't Hooft-Polyakov monopoles and fields associated with dyons and it is emphasized that the BPS mass formula of dyons is universal and is also invariant under duality and conformal transformations. The octonion electrodynamics has been analyzed from octonion wave equations of potential and currents to obtain the octonionic form of Generalized Dirac-Maxwell's equations and other quantum equations of dyons in simple, compact, consistent and manifestly covariant manner. It is shown that the octonion electrodynamics reproduces the dynamics of electric (magnetic) charge yielding to the usual form of Maxwell's equations in the absence of magnetic (electric charge) in compact, simpler and consistent way.

Chapter 2

Generalized Octonion Electrodynamics

2.1 Introduction

The two fundamental mathematical structures (division algebras) a physicist uses in his everyday life are the real $\mathbb{R}$ and the complex $\mathbb{C}$ numbers. Complex numbers are described as pairs of real numbers with a specific multiplication laws. One can however go even further and build two other sets of numbers, known in mathematics as quaternions $\mathbb{H}$ [1] and octonions $\mathbb{O}$ [2]. The quaternions, formed as pairs of complex numbers are non-commutative whereas the octonions, formed as pairs of quaternion numbers are both non-commutative and non-associative. The four sets of numbers are mathematically known as division algebras. The octonions are the last division algebra, no further generalization being consistent with the laws of mathematics. So, there exists four normed division algebras [3]: the real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) [1, 4], and octonions ($\mathbb{O}$) [2, 5] . Thus octonions are regarded as a super-set of quaternions in the same way that quaternions are a super-set of complex numbers, i.e.

- Scalars are represented by 1 number.
- Complex numbers are represented by 2 numbers (1 real and 1 imaginary).
- Quaternions are represented by 4 numbers (1 real and 3 imaginary).
- Octonions are represented by 8 numbers (1 real and 7 imaginary).

We might expect this sequence to continue with an element consisting of 16 numbers, but such algebra does not exist as it losses the properties of division algebra and the sequence ends with octonions. Octonions are also known as Cayley numbers. When we move from Complex numbers to Quaternions and then Octonions the system obeys fewer algebraic laws. Going from Complex numbers to Quaternions we loose commutativity and from Quaternions to Octonions there is loss of associatively. Quaternions are extensively used in the connection of relativity [6], quantum mechanics [7], superluminal [8] and subluminal [9] Lorentz transformations and gauge theories [10]. Left-right handed Weinberg-Salam theory of electromagnetic interaction with gauge structure has been explained better in terms of quaternion [11].

In recent years, it has also drawn interests of many authors [12-15] towards the developments of wave equation [16] and octonion form of Maxwell's equations [17]. Octonion electrodynamics [17], dyonic field equation [18] and octonion gauge analyticity of dyons [19] have been further studied consistently in order to obtain the corresponding field equations (Maxwell's equations) and equation of motion [16-18] in compact and simpler notations. In 1961 Pais [20] pointed out a striking similarity between the algebra of interactions and the split octonion algebra. Some work [21, 22] has been done to relate octonions for extending (3+1) dimension of space time to eight dimensional theory to accommodate the ever increasing quantum numbers and internal symmetric assigned to elementary particles and gauge fields. The ingenious work regarding the octonion applications in high energy physics were done by Gunaydin et. al [23, 24] to formulate quark models and color gauge theory in terms of split octonion algebras and the related groups SO(8), SO(7) and SU(3). The SU(3) symmetry group appears as the automorphism group of octonion representation leaving the complex subspace and the scalar product invariant. The approach of Gunaydin et. al [23, 24] has been followed by Domokos et.al [25, 26] and Morita [27] to the algebraic color gauge theory and quarks confinement problem. Further octonions were used by Buoncristiani [28] in writing Yang-Mill's and the Maxwell's field equation in a simple form and showed that octonion algebras accommodates both space time symmetry. The extension of quaternionic matrices to octonions as interpreting non-Riemannian geometry has been described by Morques and Oliveira [29].

On the other hand magnetic monopoles were advocated [30, 31] to symmeterize Maxwell's equations in a manifest way that the existence of an isolated magnetic charge implies the quantization of electric charge. The fresh interests on monopoles have been enhanced by 't Hooft [32] and Polyakov [33] with the idea that the classical solutions having the properties of magnetic monopoles [30, 31] may be found in Yang - Mills gauge theories. Now, it has become clear that monopoles are better understood in grand unified theories [34] and supersymmetric gauge theories [35]. Magnetic monopoles are also predicted by some theories [36, 37] that seek to unify the electroweak and strong interactions. It was challenging aspect of grand unified theories that their conventional structure [38, 39] is just such that they possess classical particle like solutions with the interpretation of magnetic monopoles. However, the monopole masses that are predicted so called grand unified theories [34] are much too large abut 10^{16} Gev to be detected in experiments. Julia and Zee [40] extended the 't Hooft-Polyakov theory of monopoles and constructed the theory of non- Abelian dyons (particles carrying simultaneously electric and magnetic charges). The quantum mechanical excitation of fundamental monopoles include dyons [41] which are automatically arisen from the semi-classical quantization of global charge rotation degree of freedom of monopoles.

Despite of this much literature and the fact that the formalism necessary to describe magnetic monopoles (dyons) has been clumsy and manifestly non-covariant, in this chapter, we have made an attempt to obtain the solution of wave equation and other field equations of monopoles and dyons interms of physical octonion variables. In section (2.2), we have reviewed the earlier literature on quaternion, octonion and Fano plane. In section (2.3) the asymmetry of Maxwell's equations, need of magnetic monopoles, 't Hooft-Polyakov monopoles and fields associated with dyons are discussed in context to the utility and advantages of monopoles and dyons. It is emphasized that the BPS mass formula of dyons is universal and is also invariant under duality and conformal transformations. In section (2.4), we have discussed the octonion wave equation from left and right regularity conditions of octonions. It is shown that the homogeneous octonion wave equation provides no place for electric and magnetic charges, while the inhomogeneous octonion wave equation deals with the charge and current source which may have important role in order to understand the existence of monopoles and dyons. The octonion wave equation thus can be interpreted as the classical wave (field) equation of physical variables. In section (2.5) the octonion electrodynamics has been analyzed interms of compact simple and manifestly covariant way of octonion wave equations of potential and currents. Here we have obtained the octonionic form of Generalized Dirac-Maxwell's equations and other quantum equations of dyons in simple, compact and consistent way incorporating the non-associativity of octonion variables. Section (2.6) provides the discussion and conclusion of the whole work done in this chapter. It is concluded that the presents octonion reformulation of generalized fields of dyons represents well the invariance of field equations under the Lorentz and duality transformations. It also discussed the dynamics of electric (magnetic) charge yielding to the usual form of Maxwell's equations in the absence of magnetic (electric charge) in compact, simpler and consistent way.

2.2 Mathematical Preliminaries

2.2.1 Quaternions

The algebra of quaternion $\mathbb{H}$ is a four - dimensional algebra over the field of real numbers $\mathbb{R}$ and a quaternion ϕ is expressed in terms of its four base elements [1, 2] as

$$\phi = \phi_\mu e_\mu = \phi_0 + e_1\phi_1 + e_2\phi_2 + e_3\phi_3, \quad (\mu = 0, 1, 2, 3) \tag{2.1}$$

where ϕ_0, ϕ_1, ϕ_2, ϕ_3 are the real quarterate of a quaternion and e_0, e_1, e_2, e_3 are known as quaternion unit (basis elements). A quaternion is also expressed as the combination of scalar and vector parts i.e.

$$\phi = \left(\phi_0, \overrightarrow{\phi}\right); \tag{2.2}$$

here $\overrightarrow{\phi} = e_1\phi_1 + e_2\phi_2 + e_3\phi_3$ is vector part and ϕ_0 is scalar part. The quaternion units $e_A, (\forall A = 0, 1, 2, 3)$ satisfy the following relations

$$e_0 e_A = e_A e_0 = e_A;$$
$$e_A e_B = -\delta_{AB} e_0 + f_{ABC} e_C. (\forall A, B, C = 1, 2, 3) \tag{2.3}$$

Where δ_{AB} is the delta symbol and f_{ABC} is the Levi Civita three index symbol having value ($f_{ABC} = +1$) for cyclic permutation, ($f_{ABC} = -1$) for anti cyclic permutation and ($f_{ABC} = 0$) for any two repeated indices. As such we may write the following relations among quaternion basis elements

$$\begin{aligned} [e_A,\ e_B] &= 2\, f_{ABC}\, e_C; \\ \{e_A,\ e_B\} &= -2\, \delta_{AB} e_0; \\ e_A(\, e_B\, e_C) &= (e_A\, e_B\,)\, e_C. \end{aligned} \tag{2.4}$$

The brackets [,] and { , } are used respectively for commutation and the anti commutation relations while δ_{AB} is the usual Kronecker Dirac - Delta symbol. $\mathbb{H}$ is an associative but non commutative algebra. Alternatively, a quaternion is defined as a two dimensional algebra over the field of complex numbers $\mathbb{C}$ as

$$\phi = (\phi_0 + e_1\phi_1) + e_2\,(\phi_2 - e_1\phi_3) \tag{2.5}$$

The quaternion conjugate $\overline{\phi}$ is defined as

$$\overline{\phi} = \phi_\mu \overline{e_\mu} = \phi_0 - e_1\phi_1 - e_2\phi_2 - e_3\phi_3 \tag{2.6}$$

In practice ϕ is often represented as a 2×2 matrix where $e_0 = I$, $e_j = -i\sigma_j$ (j=1, 2, 3) and σ_j are the usual Pauli spin matrices. Hence a quaternion can be decomposed in terms of its scalar $(Sc(x))$ and vector $(Vec(x))$ parts as

$$\begin{aligned} Sc(\phi) &= \frac{1}{2}(\,\phi + \overline{\phi}\,); \\ Vec(x) &= \frac{1}{2}(\,\phi - \overline{\phi}\,). \end{aligned} \tag{2.7}$$

The norm of a quaternion is expressed as

$$N\left(\phi\right) = \phi\overline{\phi} = \overline{\phi}\phi = \mid \phi \mid^2 = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2. \tag{2.8}$$

Since there exists the norm of a quaternion, we have a division i.e. every ϕ has an inverse of a quaternion and is described as

$$\phi^{-1} = \frac{\overline{\phi}}{\mid \phi \mid}. \tag{2.9}$$

Rather the quaternion conjugation satisfies the following property

$$\overline{\phi_1\phi_2} = \overline{\phi_1}\,\overline{\phi_2}. \tag{2.10}$$

The norm of the quaternion is positive definite and obey the composition law

$$N\left(\phi_1\phi_2\right) = N\left(\phi_1\right)N\left(\phi_2\right). \tag{2.11}$$

The sum and product of two quaternions are described as

$$\begin{aligned} (\alpha_0, \overrightarrow{\alpha}) + \left(\beta_0, \overrightarrow{\beta}\right) &= \left(\alpha_0 + \beta_0, \overrightarrow{\alpha} + \overrightarrow{\beta}\right), \\ (\alpha_0, \overrightarrow{\alpha}) \,.\, \left(\beta_0, \overrightarrow{\beta}\right) &= \left(\alpha_0\beta_0 - \overrightarrow{\alpha}.\overrightarrow{\beta}, \alpha_0\overrightarrow{\beta} + \beta_0.\overrightarrow{\alpha}\right). \end{aligned} \tag{2.12}$$

Quaternion elements are non-Abelian in nature and thus represent a non commutative division ring.

2.2.2 Octonions

An octonion x is expressed [42, 43] as a set of eight real numbers

$$\begin{aligned} x =& e_0x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7 \\ =& e_0x_0 + \sum_{A=1}^{7} e_Ax_A \end{aligned} \tag{2.13}$$

where $e_A(A = 1, 2, ..., 7)$ are imaginary octonion units and e_0 is the multiplicative unit element. Set of octets $(e_0,\ e_1,\ e_2, e_3, e_4, e_5, e_6, e_7)$ are known as the octonion basis elements and satisfy the following multiplication rules

$$\begin{aligned} e_0 = 1;\ e_0e_A = e_Ae_0 = e_A; \\ e_Ae_B = -\delta_{AB}e_0 + f_{ABC}e_C.\ (A, B, C = 1, 2, ..., 7). \end{aligned} \tag{2.14}$$

The structure constants f_{ABC} is completely antisymmetric and takes the value 1 for following combinations,

$$f_{ABC} = +1; \forall(ABC) = (123), (471), (257), (165), (624), (543),(736). \tag{2.15}$$

It is to be noted that the summation convention is used for repeated indices. Here the octonion algebra $\mathcal{O}$ is described over the algebra of real numbers having the vector space of dimension 8. Octonion algebra is non associative and multiplication rules for its basis elements given by equations (2.14) and (2.15) are then generalized in the following table:

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	$-e_5$	e_4	e_7	$-e_6$
e_4	$-e_7$	$-e_6$	e_5	-1	$-e_3$	e_2	e_1
e_5	e_6	$-e_7$	$-e_4$	e_3	-1	$-e_1$	e_2
e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	-1	e_3
e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	-1

Table 2.1: Octonion Multiplication table

As such we may write the following relations among octonion basis elements

$$\begin{aligned} [e_A,\ e_B] &= 2\, f_{ABC}\, e_C; \\ \{e_A,\ e_B\} &= -2\,\delta_{AB} e_0; \\ e_A(\, e_B\, e_C) &\neq (e_A\, e_B\,)\, e_C. \end{aligned} \tag{2.16}$$

Octonion conjugate is defined as

$$\begin{aligned} \overline{x} =& e_0 x_0 - e_1 x_1 - e_2 x_2 - e_3 x_3 - e_4 x_4 - e_5 x_5 - e_6 x_6 - e_7 x_7 \\ =& e_0 x_0 - \sum_{A=1}^{7} e_A x_A \end{aligned} \tag{2.17}$$

where we have used the conjugates of basis elements as $\overline{e_0} = e_0$ and $\overline{e_A} = -e_A$. Hence an octonion can be decomposed in terms of its scalar ($Sc(x)$) and vector ($Vec(x)$) parts as

$$\begin{aligned} Sc(x) &= \frac{1}{2}(\, x\, +\, \overline{x}\,); \\ Vec(x) &= \frac{1}{2}(\, x\, -\, \overline{x}\,) = \sum_{A=1}^{7}\, e_A x_A. \end{aligned} \tag{2.18}$$

Conjugates of product of two octonions and its own are described as

$$\overline{(x\,y)} = \;\overline{y}\;\overline{x}; \qquad \overline{(\overline{x})} = x. \tag{2.19}$$

while the scalar product of two octonions is defined as

$$\langle x\,,\,y\rangle \;= \tfrac{1}{2}(x\,\overline{y}+y\,\overline{x}) = \tfrac{1}{2}(\overline{x}\,y+\overline{y}\,x) = \;\sum_{\alpha=0}^{7} x_{\alpha}\,y_{\alpha}. \tag{2.20}$$

The norm $N(x)$ and inverse x^{-1}(for a nonzero x) of an octonion are respectively defined as

$$N(x) = x\,\overline{x} = \overline{x}\,x = \sum_{\alpha=0}^{7} x_{\alpha}^{2}.e_0;$$
$$x^{-1} = \frac{\overline{x}}{N(x)} \;\Longrightarrow x\,x^{-1} = x^{-1}\,x = 1. \tag{2.21}$$

The norm $N(x)$ of an octonion x is zero if $x = 0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(x\,y) = \;N(x)\,N(y) = \;N(y)\,N(x). \tag{2.22}$$

Equation (2.16) shows that octonions are not associative in nature and thus do not form the group in their usual form. Non - associativity of octonion algebra $\mathcal{O}$ is provided by the associator

$$(x,y,z) = (xy)z - x(yz)\;\forall x,y,z \in \mathcal{O}$$

defined for any three octonions. If the associator is totally antisymmetric for exchanges of any three variables, i.e. $(x,y,z) = -(z,y,x) = -(y,x,z) = -(x,z,y)$, then the algebra is called alternative.

2.2.3 The Fano plane

Let us use the fig. (2.1) in order to remember how to multiply octonions:

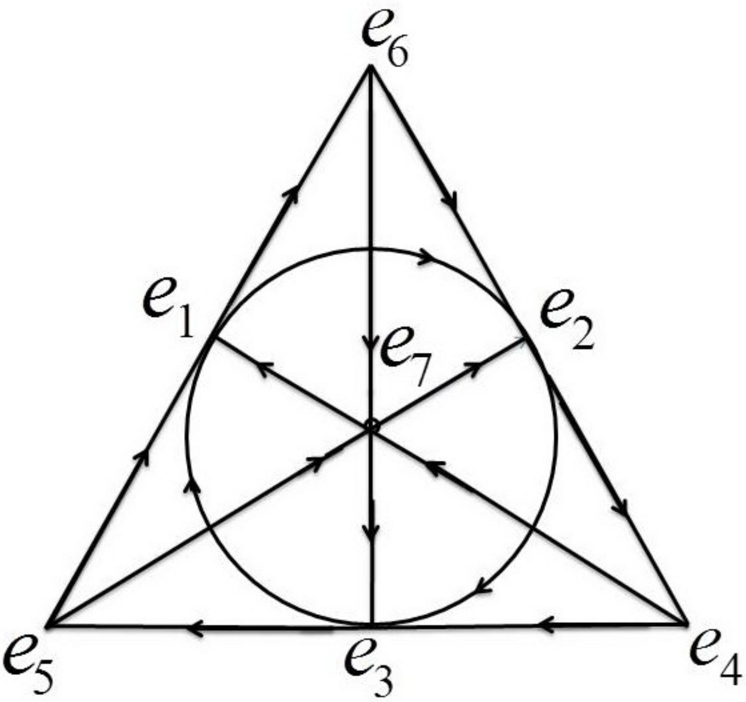

Figure 2.1: The Fano plane

This is the Fano plane [42], a little gadget with 7 points and 7 lines. The 'lines are the sides of the triangle, its altitudes, and the circle containing all the midpoints of the sides. Each pair of distinct points lies on a unique line. Each line contains three points, and each of these triples has has a cyclic ordering shown by the arrows. If e_i , e_j and e_k are cyclically ordered in this way then we get

$$e_i e_j = e_k, \quad e_j e_i = -e_k. \tag{2.23}$$

Together with these rules:

- 1 is the multiplicative identity.
- $e_1, e_2, ..., e_7$ are square roots of -1.

The Fano plane completely describes the algebra structure of the octonions. Each of the seven lines generates a sub algebra of $\mathbb{O}$ isomorphic to the quaternions $\mathbb{H}$.

2.3 Monopoles, Dyons and their Fields Equations

2.3.1 Asymmetry of Maxwell's equations

The asymmetry between electricity and magnetism is clear from the following differential form of the Maxwell's equation for electromagnetic fields as,

$$\begin{aligned} \overrightarrow{\nabla}.\overrightarrow{E} &= \rho_e; & \overrightarrow{\nabla}.\overrightarrow{H} &= 0; \\ \overrightarrow{\nabla}\times\overrightarrow{E} &= \frac{\partial\overrightarrow{H}}{\partial t}; & \overrightarrow{\nabla}\times\overrightarrow{H} &= \frac{\partial\overrightarrow{E}}{\partial t} + \overrightarrow{j_e}; \end{aligned} \tag{2.24}$$

where the vector fields $\overrightarrow{E}$ and $\overrightarrow{H}$, respectively, denote the electric and magnetic fields, ρ_e is the electric charge density while the electric current density has been denoted by $\overrightarrow{j_e}$ and use of natural units $c = \hbar = 1$ has been made throughout the notations. Maxwell's equations (2.24) in free space (i.e for $\rho = 0,\ j = 0$) are symmetric and dual invariant. Thus, in the presence of sources, Maxwell's equation (2.24) are neither symmetric nor dual invariant. The electric and magnetic fields in terms of the components of electric four - potential $\{A^{\mu}\} = \left\{\phi, \overrightarrow{A}\right\}$ are expressed in the following manner as ;

$$\begin{aligned}\overrightarrow{E} =& - \overrightarrow{\nabla}\phi - \frac{\partial \overrightarrow{A}}{\partial t};\\ \overrightarrow{H} =& \overrightarrow{\nabla} \times \overrightarrow{A}.\end{aligned} \tag{2.25}$$

In covariant notation, these electric and magnetic fields are the components of electromagnetic field tensor as;

$$F_{\mu\nu} = \partial_{\nu}A_{\mu} - \partial_{\nu}A_{\mu}; \quad \mu, \nu = 0, 1, 2, 3 \tag{2.26}$$

where $F_{0i} = E^{i}\,(\forall i = 1, 2, 3)$ and $F_{ij} = \epsilon_{ijk}H^{k}\,(\forall i, j, k = 1, 2, 3)$. Covariant forms of Maxwell's equations (2.24) may then be written as,

$$F_{\mu\nu,\nu} = \partial^{\nu}F_{\mu\nu} = j_{\mu}; \tag{2.27}$$

$$F^{d}_{\mu\nu,\nu} = \partial^{\nu}F^{d}_{\mu\nu} = 0. \tag{2.28}$$

Here $\{j^{\mu}\} = \left\{\rho_e, \overrightarrow{j}\right\}$ is a four current density. The dual part of electromagnetic field tensor $F^{d}_{\mu\nu}$ is defined as,

$$F^{d}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\lambda}F^{\sigma\lambda} \tag{2.29}$$

where $\epsilon_{\mu\nu\sigma\lambda}$ is completely antisymmetric Ricci tensor of rank four. Equations (2.27) and (2.28) are obviously asymmetrical in $F_{\mu\nu}$ and $F^{d}_{\mu\nu}$ and do not remain invariant under the duality transformation $F_{\mu\nu} \longmapsto F^{d}_{\mu\nu}$ and $F^{d}_{\mu\nu} \longmapsto -F_{\mu\nu}$.

The lack of symmetries in Maxwell's equations may be visualized in connection with the following points,

- There is no magnetic charge analogy of electric charge and current source densities.
- The magnetic field appears is produced by the motion of electric charge but there is no similar contribution of magnetic charge in producing electric field.
- In terms of four potential $\{A_\mu\}$, the equation (2.25) demands the different nature of relations, which show that the magnetic field is the effect for the rotation of the spatial part of potential while such interpretation cannot be given to the electric field.
- The symmetry is also explicitly revealed in the equation (2.26) for electromagnetic field tensor.
- Equation (2.24) gives no evidence for magnetic sources.

2.3.2 Dirac Magnetic Monopoles

Postulating the existence of magnetic monopoles, the generalized Dirac Maxwell's (GDM) equations [44, 45] are expressed (in SI units ($c = \hbar = 1$)) in the following differential form as

$$\overrightarrow{\nabla}.\overrightarrow{E} = \rho_e; \qquad \overrightarrow{\nabla}.\overrightarrow{H} = \rho_m;$$
$$\overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{H}}{\partial t} - \overrightarrow{j_m}; \qquad \overrightarrow{\nabla} \times \overrightarrow{H} = \frac{\partial \overrightarrow{E}}{\partial t} + \overrightarrow{j_e}; \tag{2.30}$$

where ρ_e and ρ_m are respectively the electric and magnetic charge densities, $\overrightarrow{j_e}$ and $\overrightarrow{j_m}$ are the corresponding current densities. GDM equations (2.30) are invariant not only under Lorentz and conformal transformations but also invariant under the following duality transformations [46, 47],

$$\overrightarrow{E} = \overrightarrow{E} \quad \cos\theta + \overrightarrow{H} \quad \sin\theta;$$
$$\overrightarrow{H} = -\overrightarrow{E} \quad \sin\theta + \overrightarrow{H} \quad \cos\theta. \tag{2.31}$$

For a particular value of $\theta=\frac{\pi}{2}$, equation (2.31) reduces to

$$\overrightarrow{E} \to \overrightarrow{H}, \quad \overrightarrow{H} \to - \quad \overrightarrow{E}; \tag{2.32}$$

which can be written as

$$\begin{pmatrix} \overrightarrow{E} \\ \overrightarrow{H} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{E} \\ \overrightarrow{H} \end{pmatrix}; \tag{2.33}$$

together with

$$j_e \to j_m,\, j_m \to -j_e \Longleftrightarrow \begin{pmatrix} j_e \\ j_m \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} j_e \\ j_m \end{pmatrix}. \tag{2.34}$$

Accordingly, the covariant form of GDM equations be written as

$$F_{\mu\nu,\nu} = \partial^{\nu} F_{\mu\nu} = j_{\mu};$$
$$F^{d}_{\mu\nu,\nu} = \partial^{\nu} F^{d}_{\mu\nu} = k_{\mu}. \tag{2.35}$$

Here $\{k^{\mu}\} = \left\{\rho_m,\ \overrightarrow{k}\right\}$ is the four magnetic current density due to the presence of magnetic monopole. Consequently, the Lorentz force equation of motion may be written in following form as

$$m\frac{d\overrightarrow{v}}{dt} = e\left(\overrightarrow{E} + \overrightarrow{u} \times \overrightarrow{H}\right) + g\left(\overrightarrow{H} - \overrightarrow{u} \times \overrightarrow{E}\right) \tag{2.36}$$

where m is the mass of the particle, e is the electric charge, $\{u^{\nu}\}$ is four - velocity of particle, space - time four vector is defined as $\{x^{\mu}\} \equiv \{t, \overrightarrow{x}\}$ and g is magnetic charge. Electric and magnetic four - current are related as $j^{\mu} = eu^{\mu}$ and $k^{\mu} = gu^{\mu}$. As such the duality invariance is an intrinsic property of Maxwell's Lorentz theory of electrodynamics in presence of monopole.

Thus, the existence of magnetic monopoles provides an explanation for the quantization of electric charge and maintains symmetries in the Maxwell's equations. Dirac gives an interesting result is that the product of a magnetic monopole charge (g) with the electron electric charge (e) must be quantized [45] i.e.,

$$eg = \frac{1}{2}n, \quad n = 1, 2, 3, ... \tag{2.37}$$

where e and g are respectively the electric and magnetic charges and n is the principle quantum number. This condition implies that in the presence of magnetic monopole, electric charge must be integral multiple of a fundamental unit. This quantization condition demands the existence of free magnetic pole having the pole strength;

$$g = \frac{e}{2\alpha}; \tag{2.38}$$

where α is fine structure constant. In deriving this condition it was assumed that a particle has either electric charge or magnetic charge (not both). This is also called Abelian magnetic monopole.

In spite of many good points, Dirac theory encounters with many difficulties. In this theory if the magnetic field,

$$\overrightarrow{H} = \frac{g\hat{r}}{r^2} \tag{2.39}$$

produced by magnetic charge g located at the origin is described by vector potential $\overrightarrow{A}(r)$, then obviously

$$\overrightarrow{H} \neq \overrightarrow{\nabla} \times \overrightarrow{A} \tag{2.40}$$

along the line going for monopole to infinity. Such a line may be curved or planer is referred as Dirac string in literature [41]. For the straight string, $S^{(n)}$ we may write

$$\begin{aligned} A^{(n)}(r) =& \frac{g}{r} \cdot \frac{\overrightarrow{r} \times \hat{n}}{r - (\overrightarrow{r}.\hat{n})} \\ =& \frac{g}{r} \cdot \frac{[\overrightarrow{r} \times \hat{n}\,(\overrightarrow{r}.\hat{n})]}{\left[r^2 - (\overrightarrow{r}.\hat{n})^2\right]}. \end{aligned} \tag{2.41}$$

For these vector potentials and $\overrightarrow{H} \neq \overrightarrow{\nabla} \times \overrightarrow{A}$ along the singular line $\overrightarrow{r} = c\hat{n}$,

$$\mid A^n \mid = \propto \tag{2.42}$$

and hence in Dirac theory,

- a string of arbitrary shape ends at each monopole.
- $\overrightarrow{A}$(r) is singular along string from monopole location to a infinity
- Charged particles can never pass through string.

2.3.3 'ʼt Hooft - Polyakov Monopoles

In contrast to Dirac's demonstration of the consistency of magnetic monopoles with quantum electrodynamics, the t' Hooft [32] and Polyakov [33] demonstrated the necessity of monopoles in grand unified gauge theories associated with the unification of three fundamental interactions namely. electromagnetic, weak and strong whereas the properties of the monopole are calculable therein. Thus, the 't Hooft – Polyakov monopole is free from singularities and is considered as a topological soliton. The 't Hooft – Polyakov was independently discovered that the bosonic part of the Georgi-Glashow model admits finite energy solutions that from far away look like Dirac monopoles. We will be concerned here only with the bosonic part of the model which consists of an $SO(3)$ Yang-Mills field theory coupled to a Higgs field in the adjoint representation. The Lagrangian density is given by [48],

$$L=-\frac{1}{4}G^{\mu\nu}\cdot G_{\mu\nu}+\frac{1}{2}D^{\mu}\overrightarrow{\phi}\cdot D_{\mu}\overrightarrow{\phi}-V\left(\overrightarrow{\phi}\right). \tag{2.43}$$

Here

- the gauge field-strength $G_{\mu\nu}$ is defined by

$$G_{\mu\nu}=\partial_{\mu}W_{\nu}-\partial_{\nu}W_{\mu}-eW_{\mu}\times W_{\nu} \tag{2.44}$$

  where W_{μ} are gauge potentials associated with the Lie algebra of $SO(3)$.

- the Higgs field $\overrightarrow{\phi}$ is a vector in the (three-dimensional) adjoint representation of $SO(3)$, with components $\phi_a=(\phi_1,\phi_2,\phi_3)$ which is minimally coupled to the gauge field; the gauge-covariant derivative is defined as

$$D_{\mu}\overrightarrow{\phi}=\partial_{\mu}\overrightarrow{\phi}-eW_{\mu}\times\overrightarrow{\phi} \tag{2.45}$$

- the Higgs potential $V\left(\overrightarrow{\phi}\right)$ is given by

$$V\left(\overrightarrow{\phi}\right) = \frac{\lambda}{4}\left(\phi^2 - a^2\right)^2 \tag{2.46}$$

where $\phi^2 = \overrightarrow{\phi} \cdot \overrightarrow{\phi}$ and λ is assumed to be non-negative.

The Lagrangian density (2.43) is invariant under the following $SO(3)$ gauge transformations

$$\begin{aligned} \overrightarrow{\phi} \mapsto \overrightarrow{\phi}' &= g(x)\overrightarrow{\phi}; \\ W_\mu \mapsto W'_\mu &= g(x)W_\mu g(x)^{-1} + \frac{1}{e}\partial_\mu g(x) g(x)^{-1}; \end{aligned} \tag{2.47}$$

where $g(x)$ is a possibly x- dependent 3×3 orthogonal matrix with unit determinant.

The classical dynamics of the fields W_μ and $\overrightarrow{\phi}$ are determined from the following equations of motion

$$\begin{aligned} D_\nu G^{\mu\nu} &= -e\overrightarrow{\phi} \times D^\mu \overrightarrow{\phi}, \\ D^\mu D_\mu \overrightarrow{\phi} &= -\lambda\left(\phi^2 - a^2\right)\overrightarrow{\phi} \end{aligned} \tag{2.48}$$

and by the Bianchi identity

$$D_\mu G^{d\mu\nu} = 0; \tag{2.49}$$

where $G^{d\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}G_{\lambda\rho}$.

The canonically conjugate momenta to the gauge field W_μ and the Higgs field $\overrightarrow{\phi}$ are described as

$$\vec{E}^i = -\,\vec{G}^{0i}, \qquad \vec{\Pi} = D_0\overrightarrow{\phi}; \tag{2.50}$$

$$\vec{G}_{ij} = -\,\epsilon_{ijk}\vec{B}^k; \tag{2.51}$$

The energy density may be written as

$$\mathcal{H} = \frac{1}{2}\vec{E}_i.\vec{E}_i + \frac{1}{2}\vec{\Pi}.\vec{\Pi} + \frac{1}{2}\vec{B}_i\cdot\vec{B}_i + \frac{1}{2}D_i\overrightarrow{\phi}\cdot D_i\overrightarrow{\phi} + V\left(\overrightarrow{\phi}\right). \tag{2.52}$$

Here, one can define a vacuum configuration to be one for which the energy density vanishes, i.e.

$$\vec{G}_{\mu\nu} = 0, \quad D^\mu\overrightarrow{\phi} = 0, \quad V\left(\overrightarrow{\phi}\right) = 0. \tag{2.53}$$

It should be noticed that the Higgs field obeys $\phi^2 = (\phi_1^2 + \phi_2^2 + \phi_3^2) = a^2$ in the Higgs vacuum. Such vacuum configuration is no more invariant under the transformations of $SO(3)$, but only under an $SO(2) \cong U(1)$ subgroup. Thus this model exhibits spontaneous symmetry breaking mechanism.

On the other hand, Olive [49] obtained the resultant masses of gauge particles as,

$$M(e,0) = a \mid e \mid; \tag{2.54}$$

where e is the eigen value of electric charge of a massive eigenstate and a specifies the the magnitude of the vacuum expectation value of scalar Higgs field. In ' t Hooft-Polyakov model, after symmetry breaking, we have the $U(1)$ gauge theory which has all the characteristics of Maxwell's electromagnetic theory. The 't Hooft - Polyakov monopole carries one Dirac unit of magnetic charge. These monopoles are not elementary particles like Dirac's monopoles but complicated extended objects having a definite size inside of which massive fields play a role in providing a smooth structure and outside they rapidly vanish leaving the field configuration identical to Dirac's monopoles. The 't Hooft -Polyakov monopole was known numerically but there is simplified model introduced by Prasad and Sommerfield [51] which has an explicit stable monopole solution. Such solution satisfying Bogomonly condition [50] are named as Bogomonly - Prasad - Sommerfield (BPS) monopoles. These static monopoles in R^3 - space have been extensively studied in recent years and it became clear that they have remarkable properties which are best understood as a special case of self-duality equations in four space for solutions independent of one of the variables. The mass of monopole solution with a smooth internal structure is calculable to have the following lower limit of Prasad and Sommerfield,

$$M(0, g) \geq a \mid g \mid . \tag{2.55}$$

Which is possible in Prasad-Sommerfield limit [51] , where the the ' t Hooft- Polyakov monopole solutions are generalized to vanish the self interaction of Higgs field.

2.3.4 Field Associated with dyons

The name 'dyons' was coined by Schwinger [31] for the particles carrying simultaneously the existence of electric and magnetic charges. Dyons are not strictly static, although they are stationary in certain gauges, and they have non - zero kinetic energy. A dyon with a zero electric charge is usually referred to as a magnetic monopole. Schwinger [31, 46] extended the Dirac quantization condition (2.37) to the dyon as

$$e_1 g_2 - e_2 g_1 = \frac{1}{2} n \tag{2.56}$$

for the interaction of two particles with electric and magnetic charges (e_1, g_1) and (e_2, g_2). In view of Witten effect [52] the monopole are automatically dyons when it carries electric charge in addition to magnetic charge. Like Schwinger [31, 46] and Zwanziger [47, 53, 54] , Julia and Zee [58] extended the 't Hooft Polyakov monopole for the dyons with extended structure to obeying the quantization condition (2.56). It has been pointed out that a dyon is energetically not allowed to decay into magnetic monopole emitting charge vector mesons. At classical level the charge of dyon is not quantized and the dyons are not much massive then magnetic monopole.

Now it has become clear that a theory, which describes electromagnetic fields in terms of single potential, can not avoid controversial Dirac string variables. String singularities are discarded by means of two four-potential [55, 56]. So the theory of dyons have been developed [60, 61, 69] on assuming the generalized charge, generalized current and generalized four-potential of dyons as a complex quantity with their real and imaginary parts. Let us define the generalized charge on dyons as,

$$q = e - ig \qquad (i = \sqrt{-1}), \tag{2.57}$$

where e and g are respectively electric and magnetic charges. Generalized four - potential $(\{V_\mu\} = \{\phi, \vec{V}\})$ associated with dyons is defined as,

$$\{V_\mu\} = \{A_\mu\} - i\{B_\mu\}. \tag{2.58}$$

Here $\{A_\mu\} = \{\phi_e, \vec{A}\}$ and $\{B_\mu\} = \{\phi_m, \vec{B}\}$ are respectively electric and magnetic four potentials, and generalized four - current $\{J_\mu\}$ of dyons may be written as

$$\{J_\mu\} = \{j_\mu\} - i\{k_\mu\} \tag{2.59}$$

where $\{j_\mu\} = (\rho_e, \vec{j})$ and $\{k_\mu\} = (\rho_m, \vec{k})$ are respectively electric and magnetic four current densities.

So the electric and magnetic fields associated with GDM equations (2.30) must be symmetric in terms of components of two- four potential and are thus described [60, 61] as

$$\overrightarrow{E} = -\overrightarrow{\nabla}\phi_e - \frac{\partial \overrightarrow{A}}{\partial t} - \overrightarrow{\nabla} \times \overrightarrow{B}; \tag{2.60}$$

$$\overrightarrow{H} = -\overrightarrow{\nabla}\phi_m - \frac{\partial \overrightarrow{B}}{\partial t} + \overrightarrow{\nabla} \times \overrightarrow{A}; \tag{2.61}$$

Accordingly the GDM equations (2.30) are considered as the field equation for dyons and the electric ($\vec{E}$) and magnetic field ($\vec{H}$) are described as the generalized electromagnetic fields of dyons. The relation between generalized field and the components of the generalized four - potential (2.58) of dyons as,

$$\overrightarrow{\psi} = \overrightarrow{E} - i\,\overrightarrow{H} = -\frac{\partial \overrightarrow{V}}{\partial t} - \overrightarrow{\nabla}\phi - i\overrightarrow{\nabla} \times \overrightarrow{V}. \tag{2.62}$$

The duality transformation [62] for two four - potentials respectively associated with electric and magnetic charges of dyons are discussed as

$$\overrightarrow{A}\rightarrow\overrightarrow{B},\,\overrightarrow{B}\rightarrow-\overrightarrow{A}\Rightarrow\left(\begin{array}{c}\overrightarrow{A}\\ \overrightarrow{B}\end{array}\right)=\left(\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right)\left(\begin{array}{c}\overrightarrow{A}\\ \overrightarrow{B}\end{array}\right);\tag{2.63}$$

$$\phi_{e}\rightarrow\phi_{m},\,\phi_{e}\rightarrow-\phi_{m}\Rightarrow\left(\begin{array}{c}\phi_{e}\\ \phi_{m}\end{array}\right)=\left(\begin{array}{cc}0 & 1\\ -1 & 0\end{array}\right)\left(\begin{array}{c}\phi_{e}\\ \phi_{m}\end{array}\right).\tag{2.64}$$

So, we may express the GDM equations (2.30) of dyon in the following covariant notation,

$$F_{\mu\nu,\nu}=\partial^{\nu}F_{\mu\nu}=j_{\mu}^{e};\tag{2.65}$$

$$F_{\mu\nu,\nu}^{d}=\partial^{\nu}F_{\mu\nu}^{d}=j_{\mu}^{m};\tag{2.66}$$

where

$$\begin{aligned}F_{\mu\nu} & =\left(A_{\mu,\nu}-A_{\nu,\mu}\right)-\frac{1}{2}\epsilon_{\mu\nu\sigma\rho}\left(B^{\rho,\sigma}-B^{\sigma,\rho}\right);\\ F_{\mu\nu}^{d} & =\left(B_{\mu,\nu}-B_{\nu,\mu}\right)+\frac{1}{2}\epsilon_{\mu\nu\sigma\rho}\left(A^{\rho,\sigma}-A^{\sigma,\rho}\right).\end{aligned}\tag{2.67}$$

Generalized fields of dyons given in equation (2.67) may directly be obtained from field tensors $F_{\mu\nu}$ and $F_{\mu\nu}^{d}$ as,

$$\begin{aligned}F_{0i} & =E^{i}; & F_{0i}^{d} & =\epsilon_{ijk}H^{k};\\ F_{0i}^{d} & =-H^{i}; & F_{ij}^{d} & =-\epsilon_{ijk}E^{k}.\end{aligned}\tag{2.68}$$

Accordingly, we obtain a new parameter $\overrightarrow{S}$ i.e.

$$\overrightarrow{S}=\square\overrightarrow{\psi}=-\frac{\partial\overrightarrow{J}}{\partial t}-\overrightarrow{\nabla}\rho-i(\overrightarrow{\nabla}\times\overrightarrow{J});\tag{2.69}$$

where $\Box$ is the D' Alembertian operator expressed as

$$\Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{\partial^2}{\partial t^2}. \tag{2.70}$$

Defining generalized field tensor as

$$G_{\mu\nu} = F_{\mu\nu} - iF^d_{\mu\nu}, \tag{2.71}$$

we can obtain the following generalized field equation of dyons

$$\begin{aligned} G_{\mu\nu,\nu} &= \partial^\nu G_{\mu\nu} = J_\mu; \\ G^d_{\mu\nu,\nu} &= \partial^\nu G^d_{\mu\nu} = 0; \end{aligned} \tag{2.72}$$

where $G_{\mu\nu} = V_{\mu,\nu} - V_{\nu,\mu}$ is called the generalized electromagnetic field tensor of dyons. Equation (2.72) may also written as follows like second order Klein - Gorden equation for dyon fields,

$$\Box V_\mu = J_\mu \;\; (Lorentz\ gauge) \tag{2.73}$$

Equation (2.65), (2.66) and (2.72) are invariant under duality transformation;

$$\left(F_{\mu\nu},\, F^d_{\mu\nu}\right) = \left(F_{\mu\nu}\cos\theta + F^d_{\mu\nu}\sin\theta\,,\, F_{\mu\nu}\sin\theta - F^d_{\mu\nu}\cos\theta\right); \tag{2.74}$$

$$(j_e,\, j_m) = (j_e\cos\theta + j_m\sin\theta\,,\, j_e\sin\theta - j_m\cos\theta)\,; \tag{2.75}$$

where

$$\frac{g}{e} = \frac{B_\mu}{A_\mu} = \frac{j_m}{j_e} = -\tan\theta \tag{2.76}$$

is described as constancy condition. Consequently, the generalized charge of dyon may be written as,

$$q = \mid q \mid e^{-i\theta}. \tag{2.77}$$

The suitable manifestly covariant Lagrangian density, which yields the field equation (2.72) under the variation of field parameters i.e. potential only without changing the trajectory of particle, may be written as follows;

$$L = m_0 - \frac{1}{4} G_{\mu\nu} G^{\star}_{\mu\nu} + V^{\star}_{\mu} J_{\mu}; \tag{2.78}$$

where m_0 is the rest mass of particle, $\star$ denotes the complex conjugate and

$$G_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\sigma\rho} G^{\rho\sigma}. \tag{2.79}$$

Lagrangian density given in equation (2.78) directly follows the following form of Lorentz four - force equation of motion (2.36) for dyons i.e.

$$f_{\mu} = m_0 \ddot{x}_{\mu} = \quad Re \quad q^{*} (G_{\mu\nu} u^{\nu}); \tag{2.80}$$

where 'Re' denotes the real part, $\{\ddot{x}_{\mu}\}$ is the four - acceleration and $\{u^{\nu}\}$ is the four - velocity of the particle.
So, according to Bogomolny bound [50] the mass of a dyon is expressed as

$$M(e, g) = a \mid q \mid = a \mid e - ig \mid = a \sqrt{(e^2 + g^2)}. \tag{2.81}$$

Which is known as BPS mass formula. This mass formula (2.56) does not distinguish between the fundamental quantum particles and the magnetic monopoles, being applicable to all of them, like meson-Solitons democracy in Sine Gorden Model [57]. The BPS mass formula is universal and is also invariant under electromagnetic duality transformations.

2.4 Octonion Wave Equation

Keeping in view the properties of octonions [63-67] and its eight dimensional connection, we may now write the octonion differential operator D [67, 68] as

$$\begin{aligned} D =& e_0 D_0 + e_1 D_1 + e_2 D_2 + e_3 D_3 + e_4 D_4 + e_5 D_5 + e_6 D_6 + e_7 D_7 \\ =& \sum_{\mu=0}^{7} e_\mu D_\mu, \qquad (\mu = 0, 1, 2, 3, ..., 7) \end{aligned} \tag{2.82}$$

where D_μ are described as the components of a differential operator in an eight dimensional representation. Here we may consider the eight dimensional space as the combination of two (external and internal) four dimensional spaces. As such, a function of an octonion variable may be described as

$$\mathcal{F}(X) \;\; = \textstyle\sum_{\mu=0}^{7} \;\; e_\mu f_\mu (X) = f_0 + e_1 f_1 + e_2 f_2 + ... + e_7 f_7, \tag{2.83}$$

where f_μ are scalar functions. Since octonions are neither commutative nor associative, one has to be very careful to multiply the octonion either from left or from right in terms of regularity conditions [65]. As such, a function $\mathcal{F}(X)$ of an octonion variable $X = \sum_{\mu=0}^{7} e_\mu X_\mu$ is left regular at X if and only if $\mathcal{F}(X)$ satisfies the condition

$$D\mathcal{F}(X) \;\; = \;\; 0. \tag{2.84}$$

Similarly, a function $G(X)$ is a right regular if and only if

$$G(X)D = 0, \tag{2.85}$$

where $G(X) = g_0 + g_1e_1 + g_2e_2 + ... + g_7e_7$. Then we get

$$D\mathcal{F} = I = I_0 + I_1e_1 + I_2e_2 + I_3e_3 + I_4e_4 + I_5e_5 + I_6e_6 + I_7e_7, \tag{2.86}$$

where

$$\begin{aligned}
I_0 &= \partial_0 f_0 - \partial_1 f_1 - \partial_2 f_2 - \partial_3 f_3 - \partial_4 f_4 - \partial_5 f_5 - \partial_6 f_6 - \partial_7 f_7; \\
I_1 &= \partial_0 f_1 + \partial_1 f_0 + \partial_2 f_3 - \partial_3 f_2 + \partial_6 f_5 - \partial_5 f_6 - \partial_7 f_4 + \partial_4 f_7; \\
I_2 &= \partial_0 f_2 + \partial_2 f_0 + \partial_3 f_1 - \partial_1 f_3 + \partial_4 f_6 - \partial_6 f_4 - \partial_7 f_5 + \partial_5 f_7; \\
I_3 &= \partial_0 f_3 + \partial_3 f_0 + \partial_1 f_2 - \partial_2 f_1 + \partial_6 f_7 - \partial_7 f_6 + \partial_5 f_4 - \partial_4 f_5; \\
I_4 &= \partial_0 f_4 + \partial_4 f_0 + \partial_3 f_5 - \partial_5 f_3 - \partial_2 f_6 + \partial_6 f_2 - \partial_1 f_7 + \partial_7 f_1; \\
I_5 &= \partial_0 f_5 + \partial_5 f_0 + \partial_1 f_6 - \partial_6 f_1 + \partial_7 f_2 - \partial_2 f_7 - \partial_3 f_4 + \partial_4 f_3; \\
I_6 &= \partial_0 f_6 + \partial_6 f_0 - \partial_1 f_5 + \partial_5 f_1 + \partial_2 f_4 - \partial_4 f_2 - \partial_3 f_7 + \partial_7 f_3; \\
I_7 &= \partial_0 f_7 + \partial_7 f_0 + \partial_1 f_4 - \partial_4 f_1 + \partial_2 f_5 - \partial_5 f_2 - \partial_6 f_3 + \partial_3 f_6.
\end{aligned} \tag{2.87}$$

The regularity condition (2.84) may now be considered as a homogeneous octonion wave equation for octonion variables without sources. On the other hand, equation (2.86) is considered as the inhomogeneous wave equation as

$$D\mathcal{F} = I. \tag{2.88}$$

where I is also an octonion. Similarly, we may also write the homogeneous as well as inhomogeneous octonion wave equations on using the right regularity condition (2.85). We may now interpret these octonion wave equations as the classical wave (field) equations of physical variables. Thus, one dimensional octonion representation is identical to eight dimensional spaces over the field of real numbers. It is isomorphic to four-dimensional space representation over the field of complex variables which is equivalent to two-dimensional space representation over quaternion field variables. Similarly, one dimensional quaternion space is isomorphic to four-dimensional space over the field of real numbers which is identical to two-dimensional space over the field of complex numbers.

2.5 Octonion Electrodynamics

In order to consider the generalized electromagnetic fields of dyon, we may write the various quantum equations of dyons in octonion formulation. Thus the octonion valued potential, in eight dimensional formalism as the combinations of two four dimensional spaces, is defined as

$$\mathbb{V} = e_0V_0 + e_1V_1 + e_2V_2 + e_3V_3 + e_4V_4 + e_5V_5 + e_6V_6 + e_7V_7. \quad (2.89)$$

We may now identify the components of generalized potential of dyons as

$$(V_0, V_1, V_2, V_3, V_4, V_5, V_6, V_7) \Longrightarrow (\varphi,\, A_x,\, A_y, A_z,\, iB_x,\, iB_y,\, iB_z, i\phi) \;\; (i = \sqrt{-1}) \quad (2.90)$$

where $(\phi,\, A_x,\, A_y,\, A_z) = (\phi, \overrightarrow{A}\,) \equiv \{A^\mu\}$ and $(\varphi,\, B_x,\, B_y,\, B_z) = (\varphi, \overrightarrow{B}\,) \equiv \{B^\mu\}$ are respectively described as the components of electric $\{A_\mu\}$ and magnetic $\{B_\mu\}$ four potentials of dyons. Equation (2.89) may then be written as

$$\mathbb{V} = e_1(A_x + ie_7B_x) + e_2(A_y + ie_7B_y) + e_3(A_z + ie_7B_z) + (\varphi + ie_7\phi), \quad (2.91)$$

which may be reduced in term of quaternionic potential as

$$\mathbb{V} = e_1 \mathrm{V_x} + e_2 \mathrm{V_y} + e_3 \mathrm{V_z} + ie_7 \emptyset, \tag{2.92}$$

where $(\emptyset, \mathrm{V_x}, \mathrm{V_y}, \mathrm{V_z}) = (\emptyset, \overrightarrow{\mathrm{V}}) = \{\mathrm{V}_\mu\}$ are then be described as the components of generalized four potential $\{\mathrm{V}_\mu\}$ associated with generalized charge $(q = e - i\,g)$ of dyons [69]. Here we have used the procedure to make a octonion as the order pair of quaternions. In order to obtain the generalized field equations of dyons in four dimensional space time, we may identify differential operator (2.82) to be four dimensional, so that the differential operator (2.82) be written as

$$D \longmapsto \boxdot = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + e_3 \frac{\partial}{\partial z} - ie_7 \frac{\partial}{\partial t}, \tag{2.93}$$

where we have replaced $\partial_7 = -i\frac{\partial}{\partial t}$ $(i = \sqrt{-1})$, $\partial_j = \frac{\partial}{\partial x_j}$ $(j = 1, 2, 3)$ in equation (2.82), other components may be taken vanishing as we are concerned with classical electrodynamics of dyons in four dimensional space-time world. Accordingly we have taken other components like $\partial_0, \partial_4, \partial_5, \partial_6$ of equation (2.93) vanishing though some authors [63-68] explored the possibility of taking the rest components of differential operator (2.82) in terms of quaternion values mass, i.e. mass has also four-dimensional structure, but we have ignored this possibility.

Octonion conjugate of equation (2.93) may then be written as

$$\overline{\boxdot} = -e_1 \frac{\partial}{\partial x} - e_2 \frac{\partial}{\partial y} - e_3 \frac{\partial}{\partial z} + ie_7 \frac{\partial}{\partial t}. \tag{2.94}$$

Now operating $\overline{\boxdot}$ given by equation (2.94) to octonion potential $\mathbb{V}$ of equation (2.91) for the octonionic potential wave equations, we get

$$\begin{aligned}\overline{\boxdot}\,\mathbb{V} = &-e_0(\overrightarrow{\nabla}\cdot\overrightarrow{A}+\frac{\partial\phi}{\partial t})\\ &+e_1(-\frac{\partial\varphi}{\partial x}+\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}-\frac{\partial B_x}{\partial t})\\ &+e_2(-\frac{\partial\varphi}{\partial y}+\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial zx}-\frac{\partial B_y}{\partial t})\\ &+e_3(-\frac{\partial\varphi}{\partial z}+\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}-\frac{\partial B_z}{\partial t})\\ &-ie_4(-\frac{\partial\phi}{\partial x}-\frac{\partial B_z}{\partial y}+\frac{\partial B_y}{\partial z}-\frac{\partial A_x}{\partial t})\\ &-ie_5(-\frac{\partial\phi}{\partial y}-\frac{\partial B_x}{\partial z}+\frac{\partial B_z}{\partial x}-\frac{\partial A_y}{\partial t})\\ &-ie_6(-\frac{\partial\phi}{\partial z}-\frac{\partial B_y}{\partial x}+\frac{\partial B_x}{\partial y}-\frac{\partial A_z}{\partial t})\\ &+ie_7(\overrightarrow{\nabla}\cdot\overrightarrow{B}+\frac{\partial\varphi}{\partial t}).\end{aligned} \tag{2.95}$$

On applying the Lorentz gauge conditions, respectively for the dynamics of electric and magnetic charges of dyons as

$$\overrightarrow{\nabla}\cdot\overrightarrow{A}+\frac{\partial\phi}{\partial t}=0; \qquad \overrightarrow{\nabla}\cdot\overrightarrow{B}+\frac{\partial\varphi}{\partial t}=0, \tag{2.96}$$

we [70] get the following octonionic form of equation (2.95) i.e.

$$\overline{\boxdot}\,\mathbb{V} \;=\; \mathbb{F} \tag{2.97}$$

where $\mathbb{F}$ is again an octonion reproduces the generalized electromagnetic fields of dyons. Thus the generalized electromagnetic field of dyons is described as

$$\mathbb{F} = e_0 F_0 + e_1 F_1 + e_2 F_2 + e_3 F_3 + e_4 F_4 + e_5 F_5 + e_6 F_6 + e_7 F_7 \tag{2.98}$$

where

$$\begin{aligned}
F_0 &= -(\overrightarrow{\nabla} \cdot \overrightarrow{A} + \frac{\partial \phi}{\partial t}) = 0; \\
F_1 &= (-\frac{\partial \varphi}{\partial x} + \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} - \frac{\partial B_x}{\partial t}) = H_x; \\
F_2 &= (-\frac{\partial \varphi}{\partial y} + \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} - \frac{\partial B_y}{\partial t}) = H_y; \\
F_3 &= (-\frac{\partial \varphi}{\partial z} + \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} - \frac{\partial B_z}{\partial t}) = H_z; \\
F_4 &= -i\,(-\frac{\partial \phi}{\partial x} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} - \frac{\partial A_x}{\partial t}) = -i\,E_x; \\
F_5 &= -i\,(-\frac{\partial \phi}{\partial y} - \frac{\partial B_x}{\partial z} + \frac{\partial B_z}{\partial x} - \frac{\partial A_y}{\partial t}) = -i\,E_y; \\
F_6 &= -i\,(-\frac{\partial \phi}{\partial z} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} - \frac{\partial A_z}{\partial t}) = -i\,E_z; \\
F_7 &= i\,(\overrightarrow{\nabla} \cdot \overrightarrow{B} + \frac{\partial \varphi}{\partial t}) = 0.
\end{aligned} \tag{2.99}$$

where the generalized electric ($\overrightarrow{E}$) and magnetic ($\overrightarrow{H}$) fields of dyons, in terms of components of electric and magnetic four potentials, are already defined in the equations (2.53) and (2.54).

We may now write equation (2.98) as

$$\mathbb{F} = e_1(H_x + ie_7 E_x) + e_2(H_y + ie_7 E_y) + e_3(H_z + ie_7 E_z), \tag{2.100}$$

were can also be written to the following quaternionic form

$$\mathbb{F} = \sum_{\mu=1}^{3} e_\mu (H_\mu + ie_7 E_\mu), \tag{2.101}$$

where $\{H_\mu\} = (0, \vec{H})$ and $\{E_\mu\} = (0, \vec{E})$, so we may write the equation (2.100) as

$$\mathbb{F} = e_1 \Psi_x + e_2 \Psi_y + e_3 \Psi_z, \tag{2.102}$$

where

$$\overrightarrow{\Psi} = \overrightarrow{H} + i\, e_7 \overrightarrow{E}, \tag{2.103}$$

which is expressed in terms of the components of generalized four-potential $\{\mathrm{V}_\mu\}$ as

$$\overrightarrow{\Psi} = -\frac{\partial \overrightarrow{\mathrm{V}}}{\partial t} - \nabla \emptyset - i \left(\nabla \times \overrightarrow{\mathrm{V}} \right). \tag{2.104}$$

Now applying the differential operator (2.93) to equation (2.100), we get

$$\begin{aligned}
\boxdot\, \mathbb{F} = & - e_0 \left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} \right) \\
& + e_1 \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \frac{\partial E_x}{\partial t} \right) \\
& + e_2 \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \frac{\partial E_y}{\partial t} \right) \\
& + e_3 \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \frac{\partial E_z}{\partial t} \right) \\
& + i e_4 \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - \frac{\partial H_x}{\partial t} \right) \\
& + i e_5 \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \frac{\partial H_y}{\partial t} \right) \\
& + i e_6 \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - \frac{\partial H_z}{\partial t} \right) \\
& + i e_7 \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right).
\end{aligned} \tag{2.105}$$

which reduces to

$$\begin{aligned}
\boxdot\mathbb{F} = & - e_0(\overrightarrow{\nabla}.\overrightarrow{H}) \\
& + e_1[(\overrightarrow{\nabla}\times\overrightarrow{H})_x - \frac{\partial E_x}{\partial t}] \\
& + e_2[(\overrightarrow{\nabla}\times\overrightarrow{H})_y - \frac{\partial E_y}{\partial t}] \\
& + e_3[(\overrightarrow{\nabla}\times\overrightarrow{H})_z - \frac{\partial E_z}{\partial t}] \\
& + ie_4[(\overrightarrow{\nabla}\times\overrightarrow{E})_x - \frac{\partial H_x}{\partial t}] \\
& + ie_5[(\overrightarrow{\nabla}\times\overrightarrow{E})_y - \frac{\partial H_y}{\partial t}] \\
& + ie_6[(\overrightarrow{\nabla}\times\overrightarrow{E})_z - \frac{\partial H_z}{\partial t}] \\
& + ie_7(\overrightarrow{\nabla}.\overrightarrow{E}). \qquad (2.106)
\end{aligned}$$

Equations (2.105) and (2.106) may then be written [70] in following compact notation in terms of an octonion i.e.

$$\boxdot\,\mathbb{F} \;=\; \mathbb{J}; \qquad (2.107)$$

where $\mathbb{J}$ is also an octonion described the octonion form of generalized current given by

$$\begin{aligned}
\mathbb{J} = & - e_0\varrho + \mathrm{e_1j_x + e_2j_y + e_3j_z - ie_4k_x - ie_5k_y - ie_6k_z + ie_7}\rho \\
= & (\mathrm{e_1j_x + e_2j_y + e_3j_z - e_0}\varrho) + i(\mathrm{e_1k_x + e_2k_y + e_3k_z} - \rho)\mathrm{e_7} \\
= & \mathrm{e_1(j_x} + i\mathrm{e_7k_x) + e_2(j_y + ie_7k_z) + e_3(j_z + k_z)} - (\rho + \mathrm{ie_7}\varrho) \\
= & \mathrm{e_1J_x + e_2J_y + e_3J_z + e_0J_0}. \qquad (2.108)
\end{aligned}$$

Here $(\rho, \overrightarrow{j}) = \{\mathrm{j}_\mu\}$, $(\varrho, \overrightarrow{j}) = \{\mathrm{k}_\mu\}$ and $(\mathrm{J_0}, \overrightarrow{J}) = \{\mathrm{J}_\mu\}$ are respectively the four currents associated with electric charge, magnetic monopole and generalized fields of dyons.

2.5.1 Octonion form of Generalized Dirac-Maxwell's (GDM) equations

Generalized Dirac-Maxwell's equations represent one of the most elegant and concise ways to state the fundamentals of electricity and magnetism. Thus, equations (2.107) and (2.108) thus lead to following differential equations,

$$
\begin{aligned}
(\overrightarrow{\nabla} \cdot \overrightarrow{H}) &= \varrho; \\
(\overrightarrow{\nabla} \times \overrightarrow{H})_x &= \frac{\partial E_x}{\partial t} + \mathrm{j_x}; \\
(\overrightarrow{\nabla} \times \overrightarrow{H})_y &= \frac{\partial E_y}{\partial t} + \mathrm{j_y}; \\
(\overrightarrow{\nabla} \times \overrightarrow{H})_z &= \frac{\partial E_z}{\partial t} + \mathrm{j_z}; \\
(\overrightarrow{\nabla} \times \overrightarrow{E})_x &= -\frac{\partial H_x}{\partial t} - \mathrm{k_x}; \\
(\overrightarrow{\nabla} \times \overrightarrow{E})_y &= -\frac{\partial H_y}{\partial t} - \mathrm{k_y}; \\
(\overrightarrow{\nabla} \times \overrightarrow{E})_z &= -\frac{\partial H_z}{\partial t} - \mathrm{k_z}; \\
(\overrightarrow{\nabla} \cdot \overrightarrow{E}) &= \rho.
\end{aligned} \tag{2.109}
$$

Equation (2.109) may then be written in simplest form as

$$
\begin{aligned}
(\overrightarrow{\nabla} \cdot \overrightarrow{E}) &= \rho; \\
(\overrightarrow{\nabla} \times \overrightarrow{E}) &= -\frac{\partial \overrightarrow{H}}{\partial t} - \overrightarrow{k}; \\
(\overrightarrow{\nabla} \times \overrightarrow{H}) &= \frac{\partial \overrightarrow{E}}{\partial t} + \overrightarrow{j}; \\
(\overrightarrow{\nabla} \cdot \overrightarrow{H}) &= \varrho;
\end{aligned} \tag{2.110}
$$

which are the generalized Dirac-Maxwell's (GDM) equations of generalized fields of dyons. Like quaternion formulation of generalized electromagnetic fields of dyons. Octonion formulation is compact and simpler. Since e_7 is coupling two quaternions into one octonion and also reverses its sign in its combination with quaternion units.

2.5.2 Generalized Octonion Potential wave Equation

In the case of generalized octonion potential wave equation, we operate $\boxdot\bar{\boxdot}$ to the octonion potential $\mathbb{V}$ given by equation (2.91) as,

$$\begin{aligned}\boxdot\bar{\boxdot}\mathbb{V} =&\mathbb{J}\\ =&e_0\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}+\frac{\partial^2 B_x}{\partial x\partial t}+\frac{\partial^2 B_y}{\partial y\partial t}+\frac{\partial^2 B_z}{\partial z\partial t}\right)\\ &-e_1\left(\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2}-\frac{\partial^2 A_x}{\partial t^2}-\frac{\partial^2 A_z}{\partial x\partial z}-\frac{\partial^2 A_y}{\partial y\partial x}-\frac{\partial^2\phi}{\partial x\partial t}\right)\\ &-e_2\left(\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial z^2}-\frac{\partial^2 A_y}{\partial t^2}-\frac{\partial^2 A_x}{\partial y\partial x}-\frac{\partial^2 A_z}{\partial y\partial z}-\frac{\partial^2\phi}{\partial y\partial t}\right)\\ &-e_3\left(\frac{\partial^2 A_z}{\partial z^2}+\frac{\partial^2 A_z}{\partial y^2}-\frac{\partial^2 A_z}{\partial t^2}-\frac{\partial^2 A_y}{\partial z\partial y}-\frac{\partial^2 A_x}{\partial z\partial x}-\frac{\partial^2\phi}{\partial z\partial t}\right)\\ &+ie_4\left(-\frac{\partial^2 B_x}{\partial y^2}-\frac{\partial^2 B_x}{\partial z^2}+\frac{\partial^2 B_x}{\partial t^2}+\frac{\partial^2 B_y}{\partial x\partial y}+\frac{\partial^2 B_z}{\partial z\partial x}+\frac{\partial^2\varphi}{\partial x\partial t}\right)\\ &+ie_5\left(-\frac{\partial^2 B_y}{\partial x^2}-\frac{\partial^2 B_y}{\partial z^2}+\frac{\partial^2 B_y}{\partial t^2}+\frac{\partial^2 B_z}{\partial z\partial y}+\frac{\partial^2 B_x}{\partial y\partial x}+\frac{\partial^2\varphi}{\partial y\partial t}\right)\\ &+ie_6\left(-\frac{\partial^2 B_z}{\partial x^2}-\frac{\partial^2 B_z}{\partial y^2}+\frac{\partial^2 B_z}{\partial t^2}+\frac{\partial^2 B_x}{\partial x\partial z}+\frac{\partial^2 B_y}{\partial z\partial y}+\frac{\partial^2\varphi}{\partial z\partial t}\right)\\ &-ie_7\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2 A_x}{\partial x\partial t}+\frac{\partial^2 A_y}{\partial t\partial y}+\frac{\partial^2 A_z}{\partial z\partial t}\right).\end{aligned} \tag{2.111}$$

Equation (2.111) then reduces to

$$\boxdot\,\bar{\boxdot}\,\mathbb{V}\;=\bar{\boxdot}\,\boxdot\,\mathbb{V}=\;\mathbb{J}, \tag{2.112}$$

where $\mathbb{J}$ is described as the octonion form of generalized current associated with dyon is given by equation (2.108). Equation (2.112) may also be written as

$$\begin{aligned}\boxdot\,\bar{\boxdot}\,\mathbb{V}=\bar{\boxdot}\,\boxdot\,\mathbb{V}=&e_0[\nabla^2\varphi+\frac{\partial}{\partial t}(\overrightarrow{\nabla}\cdot\overrightarrow{B})]\\&-e_1[\Box A_x-\frac{\partial}{\partial x}(\overrightarrow{\nabla}\cdot\overrightarrow{A}+\frac{\partial\phi}{\partial t})]\\&-e_2[\Box A_y-\frac{\partial}{\partial y}(\overrightarrow{\nabla}\cdot\overrightarrow{A}+\frac{\partial\phi}{\partial t})]\\&-e_3[\Box A_z-\frac{\partial}{\partial z}(\overrightarrow{\nabla}\cdot\overrightarrow{A}+\frac{\partial\phi}{\partial t})]\\&-ie_4[\Box B_x-\frac{\partial}{\partial x}(\overrightarrow{\nabla}\cdot\overrightarrow{B}+\frac{\partial\varphi}{\partial t})]\\&-ie_5[\Box B_y-\frac{\partial}{\partial y}(\overrightarrow{\nabla}\cdot\overrightarrow{B}+\frac{\partial\varphi}{\partial t})]\\&-ie_6[\Box B_z-\frac{\partial}{\partial z}(\overrightarrow{\nabla}\cdot\overrightarrow{B}+\frac{\partial\varphi}{\partial t})]\\&-ie_7[\nabla^2\phi+\frac{\partial}{\partial t}(\overrightarrow{\nabla}\cdot\overrightarrow{A})],\end{aligned}\tag{2.113}$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},\tag{2.114}$$

and

$$\bar{\boxdot}\boxdot=\boxdot\bar{\boxdot}=\Box=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2}=\nabla^2-\frac{\partial^2}{\partial t^2}.\tag{2.115}$$

Here $\Box$ is D' Alembertian operator. Using the Lorentz gauge conditions (2.96) and the definition of octonion valued generalized current of dyon given by equation (2.108), we get

$$\Box\phi=\rho;\qquad\Box\varphi=\varrho;\qquad\Box\mathrm{A}_\mu=\quad\mathrm{j}_\mu;\qquad\Box B_\mu=k_\mu.\tag{2.116}$$

As such, we have obtained consistently the generalized Dirac Maxwell's (GDM) equations from octonion wave equations on considering the non associativity of octonion variables. The advantages of present formalism are discussed in terms of compact and simpler notations of octonion valued potential, field and currents of dyons despite of non associativity of octonions. The present octonion reformulation of generalized fields of dyons represents well the invariance of field equations under Lorentz and duality transformations. It also reproduces the dynamics of electric (magnetic) charge yielding to the usual form of Maxwell's equations in the absence of magnetic (electric charge) in compact, simpler and consistent way.

2.6 Discussion and Conclusion

The mathematical background of quaternions and octonions has been given in sections (2.2). In section (2.3), we have taken the Maxwell's equations and their asymmetry. Postulating the existence of magnetic monopole, we have discussed the Dirac Maxwell's equations to maintain the electromagnetic duality and Lorentz invariance. The necessity of magnetic monopole in grand unified theory associated with the unification of three fundamental interactions namely electromagnetic, weak and strong has also been explored. In subsection (2.3.4), we have extended the theory of monopoles to the case of dyons and accordingly obtained the field equation and equation of motion therein.

In section (2.4), octonion wave equation has been discussed from the octonion differential operator in terms of eight dimensional representation given by equation (2.82). It consists of an eight dimensional space as the combination of two four- dimensional spaces associated with quaternions. As such, a function of octonions variable has been defined by equation (2.83). Accordingly, the left and right regularity homogeneous wave equation respectively given by equations (2.84) and (2.85) are described. On the other hand, the in-homogeneous wave equation for octonion variables in presence of sources has been explored by equation (2.86), whose components are provided by equation (2.87). The difference between compact equations (2.84) and (2.87) is that the former is homogeneous octonion wave equation without source while the later describes the in-homogeneity in terms of sources which are also the octonion variables. We interpret these octonion wave equation as the classical wave (field) equations of physical variables where the one dimensional octonion representation may be visualized to eight dimensional spaces over the field of real numbers. It is isomorphic to four-dimensional space representation over the field of complex variables which is equivalent to two-dimensional space representation over quaternion field variables. Similarly, one dimensional quaternion space is isomorphic to four-dimensional space over the field of real numbers which is identical to two-dimensional space over the field of complex numbers.

We have used the usual electrodynamics by means of octonions in eight dimensions by maintaining the usual form of Euclidean four space-time in section (2.5). In this section, equation (2.89) describes the octonion valued potential in eight dimensional formulation, which is the combination of four dimensional external space followed by four - dimensional internal space. The electric four potential A^{μ} and magnetic four potentials B^{μ} are respectively associated with the electric and magnetic charges in external and internal spaces. As such, we have considered the electric four-potential in usual four-dimensional external space while the magnetic four-potential has been considered in internal four dimensional space. So, the magnetic charge in internal space plays the role of electric charge in external space or vice - versa. The octonionic potential of dyons in terms of electric and magnetic four potential is described by equation (2.91). Equation (2.92) describes the reduced form of dyonic potential in terms of quaternions where the procedure to make a octonion as the order pair of quaternions has been made. In order to obtain the generalized field equations of dyons in four dimensional space time, we have replaced the differential operator (2.82) by the equation (2.93). It is shown that the octonion unit e_7 playing the role of imaginary quantity is no more invariant scalar as it does not commute with other quaternion units e_1, e_2, e_3. So, it has not been taken as a ordinary imaginary quantity $i = \sqrt{1}$. Other four components of equation (2.82) (i.e. $\partial_0, \partial_4, \partial_5, \partial_6$) have been taken vanishing as we are concerned only with classical electrodynamics of dyons in four dimensional space-time world. Accordingly, the components $\partial_0, \partial_4, \partial_5, \partial_6$ of equation (2.93) are also vanishing. Equation (2.94) represent the conjugate of octonion differential operator (2.93). Equation (2.95) describes the octonion potential wave equation in terms of the components of octonion valued potential. The Lorentz gauge conditions (2.96) thus represents the dynamics of electric and magnetic potential of dyons. As such, the equation (2.97) may be visualized as the octonion wave equation for dyons (particle carrying simultaneously electric and magnetic charge) in compact form. It should be noticed that the octonion wave equation (2.97) in not only compact but is also simpler, manifestly covariant and consistent. It also reproduced the dynamics of electric (magnetic) energy in the absence of magnetic (electric) charge of dyons or vice - versa. The generalized electromagnetic fields of dyons in the eight dimensional octonionic form is described by equation (2.98), where the components $F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7$ are discussed by

equation (2.99), in terms of components of generalized electromagnetic fields of dyons.
Since octonion violates the associativity, we have decomposed them in terms of two quaternions, each of which is associated with physical four dimensional space time. Accordingly, the quaternionic nature of generalized octonion electromagnetic fields of dyons in terms of generalized electric and magnetic fields of dyons has been described by equation (2.100) and (2.101). Similarly, equation (2.102) describes the quaternionic form of generalized electromagnetic field in terms of generalized electromagnetic fields vector ($\vec{\psi}$) given by equation (2.103). Equation (2.104) represents the generalized electromagnetic field in terms of the components of generalized four potential whereas equation (2.105) and (2.106) provide the components of octonion variables in terms of generalized electromagnetic fields of dyons. Hence we have obtained equation (2.107) in compact notation of octonionic current in eight dimensional representation whose components are described as generalized current given by equation (2.108). Equation (2.107) is the compact form of generalized Dirac-Maxwell's (GDM) equations of dyons. As such, the beauty of octonion is that instead of writing the eight different differential Maxwell's types equations. We have obtained one compact equation which is understood to be manifestly covariant, dual invariant, simpler and consistent. It reproduced the dynamics of electric (magnetic) energy in the absence of magnetic (electric) charge of dyons or vice - versa.
In section (2.51), we have obtained the components of octonion wave equation (2.108) responsible for the generalized Dirac-Maxwell's (GDM) equations. Equation (2.109) represent the eight different GDM equations associated with electric and magnetic charge charges which are compactified interms of equation (2.110). It is most elegant and concise ways to state the fundamentals

of the dynamics of electric and magnetic fields. So, the octonion formulation is compact and simpler. Accordingly, the octonionic potential wave equation of dyon are described by equations (2.111) and (2.112). Hence, the equation (2.113) shows the expression of octonionic wave equation in terms of octonion variables for electric and magnetic four potential. Equation (2.116) thus represents the generalized Dirac-Maxwell's (GDM) equations of dyons in terms of generalized four current.

As such, we have obtained consistently the generalized Dirac Maxwell's (GDM) equations from octonion wave equations on considering the non associativity of octonion variables. The advantage of presents formalism are discussed in terms of compact and simpler notations of octonion valued potentials, fields and currents of dyons despite of non associativity of octonions. The presents octonion reformulation of generalized fields of dyons represents well the invariance of field equations under the Lorentz and duality transformations. It also discussed the dynamics of electric (magnetic) charge yielding to the usual form of Maxwell's equations in the absence of magnetic (electric charge) in compact, simpler and consistent way.

Bibliography

[1] W. R. Hamilton, **"Elements of Quaternions"** , Vol. I & II, Chelsea Publishing, New York (1969) 1185.

[2] A. Cayley, "**On certain results relating to quaternions**", Phil. Mag., **26** (1845), 210.

[3] R. Schafer, **"Introduction to Non-Associative Algebras"**, Dover, New York, (1995).

[4] P. G. Tait, **"An elementary treatise on quaternions"**, Oxford, (1867); 2^{nd} Ed.; Cambridge, (1873); 3^{rd}, (1890).

[5] J. Graves, **"On a connection between the general theory of normal couples and the theory of complete quadratic functions of two variables"**, Phil. Mag., **26** (1845), 320.

[6] H. T. Flint, **"Applications of quaternions to the theory of relativity**", Phil. Mag., **39** (1920), 439.

[7] D. Finkelstein, J. M. Jauch and D. Speiser, **"Notes on quaternion quantum mechanics**", I, II, III., CERN reports 597, 599, 5917. Published in: C.A. Hooker, (Reidel, Dordrecht, 1979) Vol. II 367.

[8] R. Mignani, "**Quaternionic form of superluminal Lorentz transformations**", Lett. Nuovo Cimento, **13** (1975), 134.

[9] K. Imaeda, "**On quaternionic form of superluminal transformations**", Lett. Nuovo Cimento., **15** (1976), 91.

[10] F. Gursey and H. C. Tze, **"Complex and quaternionic Analyticity in chiral and Gauge theories"**, Ann. of Phys., **128** (1980), 29.

[11] K. Morita, "**Quaternionic Variational Formalism for Poincare Gauge Theory and Supergravity**", Prog. Theor. Phys., **73** (1985), 4.

[12] F. G*ü*rsey, "**Non-associative Algebras in Quantum Mechanics and Particle Physics**", U. Virginia, (1977).

[13] R. Foot and G. C. Joshi, "**Space-time symmetries of superstring and Jordan algebras**", Int. J. Theor. Phys., **28** (1989), 1449.

[14] R. Foot and G. C. Joshi, **"String theories and the Jordan algebras"**, Phy. Lett., **B199** (1987), 203.

[15] T. Tolan, K. *Ö*zdas and M. Tansili, **"Reformulation of electromagnetism with octonions",** Nuovo Cimento, **B121** (2006), 43.

[16] A. Gamba, "**Maxwell's equations in Octonion form**", Nuovo Cimento., **111A** (1998), 3.

[17] M. Gogberashvili, **"Octonionic electrodynamics"**, J. Phys. A: Math. Gen., **39** (2006), 7099.

[18] K. Lechner, **"A Quantum field theory of dyons",** arXiv:hep-th/0003003v1 (2000).

[19] P. S. Bisht and O. P. S. Negi, "**Quaternion-Octonion Analyticity for Abelian and Non-Abelian Gauge Theories of Dyons**", Int. J. Theor. Phys., **47** (2008), 1497.

[20] A. Pais, **"Remark on the Algebra of Interactions"**, Phys. Rev. Lett., **7** (1961), 291.

[21] A. Gamba, **"Peculiarities of the Eight Dimensional Space",** J. Math. Phys., **8** (1967), 775.

[22] J. Soucek, **"Quaternion quantum mechanics as a true 3+1 dimensional theory of tachyons"**, J. Phys. A. Math., **14** (1981), 1629.

[23] M. Gunaydin and F. Gursey, **"Quark structure and octonions",** J. Math. Phy., **14** (1973), 1651.

[24] M. Gunaydin and F. Gursey, **"Quark statistics and octonions",** Phys. Rev., **D9** (1974), 3387.

[25] G. Domokos and S. Kovesi-Domokos, **"The algebra of color ",** J. Math. Phy. **19** (1978), 1477.

[26] G. Domokos and S. Kovesi-Domokos, **"Towards an algebraic quantum chromodynamics",** Phys., Rev., **D19** (1979), 3984.

[27] K. Morita, **"Gauge Theories over Quaternions and Weinberg-Salam Theory",** Prog. Theor. Phys., **65** (1981), 2027.

[28] A. M. Buoncristiani, "**An algebra of the Yang Mills field**", J. Math. Phys. **14** (1973), 849.

[29] S. Morques and C. G. Oliveria, **"An extension of quternionic matrics to octonions",** J. Math. Phys., **26** (1985), 3131.

[30] P. A. M. Dirac, **"The theory of magnetic poles"**, Phy. Rev., **74** (1948), 817.

[31] J. Schwinger, "**Dyons Versus Quarks"**, Science, **166** (1969), 690.

[32] G. 't Hooft, **"Magnetic monopoles in unified gauge theories"**, Nucl. Phys., **B79** (1974), 276.

[33] A. M. Polyakov, **"Particle spectrum in quantum field theory"**, JETP Lett., **20** (1974), 194.

[34] G. Ross, **"Grand Unified Theories"**, Westview Press, (1984).

[35] K. Intriligator and N. Seiberg, **"Lectures on supersymmetric gauge theories and electric-magnetic duality"**, Nucl. Phys., **B45** (1996), 1.

[36] D. M. Scott, **"Monopoles in grand unified theories based on $SU(5)$"**, Nucl. Phys., **B171** (1980), 95.

[37] P. Godddard, **"Magnetic monopoles in grand unified theories"**, Phil. Trans. R. Soc. Lond., **A304** (1982), 87.

[38] H. Georgi, **"Lie Algebras in Particle Physics: from isospin to unified theories"**, Reading Mass., (1982).

[39] H. Georgi, **"Unified Gauge Theories"**, Theories and Experiments in High Energy Physics, New York (1975), 329.

[40] B. Julia and A. Zee , **"Poles with both magnetic and electric charges in non-Abelian gauge theory"**, Phys. Rev., **D11** (1975), 2227.

[41] J. Preskill, **"Magnetic monopoles"**, Ann. Rev. Nucl. Sci., **34** (1984), 461.

[42] J. C. Baez, **"The Octonions"**, arXiv: math. RA/0105155v4, (2002).

[43] G. M. Dixon, **"Division Algebras:: Octonions Quaternions Complex Numbers and the Algebraic Design of Physics"**, book, Springer (1994).

[44] P. A. M. Dirac, **"Generalised Hamiltonion Dynamics"**, Can. J. Math., **2** (1950), 129.

[45] P. A. M. Dirac, **"Quantized singularities in the electromagnetic field"**, Proc. Roy. Soc. London, **A133** (1931), 60.

[46] J. Schwinger, **"A Magnetic Model of Matter",** Phys. Rev. **144** (1966), 1087.

[47] D. Zwanziger, "**Quantum field theory of particles with Both clectric and magnetic charges**", Phys. Rev. **176** (1968), 5.

[48] J. M. Figueroa and O. Farrill, **"Electromagnetic Duality for Children",** www.maths.ed.ac.uk, Teaching-Lectures, (1998).

[49] D. I. Olive, **"Exact Electromagnetic Duality"**, E-print, hep-th/9508089, (1995).

[50] E. B. Bogomolny, "**Stability of Classical Solutions**", Sov. J. Nucl. Phys., **24** (1976), 449.

[51] M. K. Prasad and C. M. Sommerfield, "**Exact classical solution for the 't Hooft monopole and the Julia-Zee dyon**", Phys. Rev. Lett., **35** (1975), 760.

[52] E. Witten, **"Dyons of charge $e\vartheta/2\pi$"**, Phys. Lett., **B86** (1979), 283.

[53] D. Zwanziger, **"Local-Lagrangian Quantum Field Theory of Electric and Magnetic Charges"**, Phys. Rev., **D3** (1971), 880.

[54] D. Zwanziger, **"Angular Distributions and a Selection Rule in Charge-Pole Reactions**", Phys. Rev., **D6** (1972), 458.

[55] W. R. Hamilton, **"On quaternions"**, Proc. Roy. Irish Acad., **2** (1843), 423.

[56] W. R. Hamilton, **"On quaternions, or on a new system of imaginaries in algebra"**, Phil. Mag., **25** (1844), 489.

[57] S. V. Ketov, **"Solitons, monopoles and duality: from sine-Gordon to Seiberg-Witten",** Fortsch. Phys., **45** (1997), 237.

[58] B. Julia and A. Zee, **"Poles with both magnetic and electric charges in non-Abelian gauge theory",** Phys. Rev., **D11** (1975), 2227.

[59] J. P. Gauntlett, **"Duality and Supersymmetric Monopoles"**, hep-th /9705025, (1997).

[60] B. S. Rajput, S. R. Kumar and O. P. S. Negi, **"Quaternionic formulation for dyons"**, Lett. Nuovo Cimento, **36** (1983), 75.

[61] O. P. S. Negi and B. S. Rajput, **"Quaternionic formulation for electromagnetic-field equations"**, Lett. Nuovo Cimento, **37** (1983), 325.

[62] H. Dehnen and O. P. S. Negi, **"Electromagnetic Duality, Quaternion and Supersymmetric Gauge Theories of Dyons",** arXiv: hep-th/0608164v1, (2006).

[63] R. Penny, **"Octonions and isospin",** Nuovo Cimento, **B3** (1971), 95.

[64] R. Penny, **"Octonions and Dirac equation",** Amer. J. Phys., **36** (1968), 871.

[65] K. Imaeda, H. Tachibaba and M. Imaeda, **"Octonions, superstrings and ten-dimensional spinors",** Nuovo Cim., **100B** (1987), 53.

[66] S. Okubo, **“Introduction to Octonion and Other Non-Associative Algebras in Physics”**, Cambridge University Press, Cambridge, (1995).

[67] K. Morita, **“Octonions, Quarks and QCD”**, Prog. Theor. Phys., **65** (1981), 787.

[68] P. S. Bisht, B. Pandey and O. P. S. Negi, “**Octonion wave Equation and Generalized fields of Dyons**”, Proc. National Symp. Mathematical Sciences, Nagpur, **12** (2001), 137.

[69] D. S. Bhakuni, O. P. S. Negi and B. S. Rajput, **“Generalized fields associated with dyons”,** Lett. Nuovo Cim., **36** (1983), 479.

[70] B. C. Chanyal, P. S. Bisht and O. P. S. Negi, **“Generalized Octonion Electrodynamics”,** Int. J. Theo. Phys, **49** (2010), 1333.

CHAPTER 3

Generalized Split-Octonion Electrodynamics

A part of this chapter has been published in Int. J. Thoer. Phys., 50, (2011) 1919.

ABSTRACT

Starting with the usual definition of split octonions along with their multiplication rules, we have established the inter relationship of split octonions with their convenient matrix realization in terms of 2×2 Zorn vector matrices in order to obtain the split octonions wave equation analogous to the potential wave equation of generalized electromagnetic fields of dyons. Consequently, the split octonion field equation in compact and simpler form has been developed and it is shown that the corresponding wave equation represents the generalized Dirac Maxwell's equations of dyons in the case of split octonion electrodynamics. Accordingly, we have made an attempt to investigate the work energy theorem or "Poynting Theorem", Maxwell stress tensor and Lorentz invariant for generalized fields of dyons in split octonion electrodynamics.

Chapter 3

Generalized Split-Octonion Electrodynamics

3.1 Introduction

The relationship between mathematics and physics has long been an area of interest and speculation. Magnetic monopoles [1] where advocated to symmeterize Maxwell's equations in a manifest way that the mere existence of an isolated magnetic charge implies the quantization of electric charge and accordingly the considerable literature [2-7] has come in force. The fresh interests are enhanced with idea of 't Hooft [8] and Polyakov [9] that the classical solutions having the properties of magnetic monopoles may be found in Yang-Mills gauge theories. Julia and Zee [10] extended it to construct the theory of non-Abelian dyons (particles [2, 3] carrying simultaneously electric and magnetic charge). In view of the explanation of CP-violation in terms of non-zero vacuum angle of world [11], the monopole are necessary dyons and Dirac quantization condition permits dyons to have analogous electric charge. The quantum mechanical excitation of fundamental monopoles include dyons which are automatically arisen [5, 7] from the semi-classical quantization of global charge rotation degree of freedom of monopoles. Accordingly, the self-consistent and manifestly covariant theory of generalized electromagnetic fields associated with dyons (particle carrying electric and magnetic charge) has been discussed [12, 13].

The close analogy between Newton's gravitation law and Coulomb's law of electricity led many authors to investigate further similarities, such as the possibility that the motion of mass-charge could generate the analogous of a magnetic field which is produced by the motion of electric-charge, i.e. the electric current. So, there should be the mass current would produced a magnetic type field namely 'gravitomagnetic' field. Maxwell [14] in one of his fundamental works on electromagnetism , turned his attention to the possibility of formulating the theory of gravitation in a form corresponding to the electromagnetic equations. In 1893 Heaviside [15] investigated the analogy between gravitation and electromagnetism where he explained the propagation of energy in a gravitational field, in terms of a gravito-electromagnetic Poynting vector, even though he (just as Maxwell did) considered the nature of gravitational energy a mystery. The analogy has also been explored by Einstein [16], in the framework of General Relativity, and then by Thirring [17] and Lense and Thirring [18], that a rotating mass generates a gravito magnetic field causing a precession of planetary orbits. Exploding the basics of the gravito electromagnetic form of the Einstein equations, theory of gravitomagnetism has also been reviewed by Ruggiero-Tartaglia [19].

Decomposition of four algebras, in view of celebrated Hurwitz theorem, has been characterized from Cayley Dickson process over the field of real numbers of dimensions N=1, N=2, N=4 and N=8 respectively for real, complex, quaternion and octonion algebras. Split octonion electrodynamics [20] has been developed interms of Zorn's vector matrix realization and corresponding field equation. Split octonion formulation of dyon field has has been carried out to reformulate the generalized four-potential, current equations, field equations, and electro-magnetic fields of dyons. So, there has been a revival in the formulation of natural laws so that there exists [21] four-division algebras consisting the algebra of real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$) and Octonions ($\mathcal{O}$). All four algebra's are alternative with totally anti symmetric associators. Quaternions [22, 23] were very first example of hyper complex numbers have been widely used [24-30] to the various applications of mathematics and physics. Since octonions [31] share with complex numbers and quaternions, many attractive mathematical properties, one might except that they would be equally as useful as others. Octonion [31] analysis has been widely discussed by Baez [32]. It has also played an important role in the context of various physical problems [33-36] of higher dimensional supersymmetry, super gravity and super strings etc. In recent years, it has also drawn interests of many [37-40] towards the developments of wave equation and octonion form of Maxwell's equations. Bisht et al. [41, 42] have also studied octonion electrodynamics, dyonic field equation and octonion gauge analyticity of dyons consistently and obtained the corresponding field equations (Maxwell's equations) and equation of motion in compact and simpler formulation.

Keeping in view the recent interests on the existence of monopoles and dyons at one end and quaternion-octonion formulation of generalized Dirac-Maxwell's (GDM) equations at other end, the generalized Dirac-Maxwell's of dyons have been reformulated [43] by means of octonion variables in compact and consistent manner. So, in this chapter we have made an attempt to investigate the generalized split octonion electrodynamics for dyons. Starting with the usual definition of split octonions along with their multiplication rules, in section (3.2), we have reconnected the split octonion with their convenient matrix realization in terms of 2×2 Zorn vector matrices. The multiplication rules and other properties of split octonion are analyzed in terms of 2×2 Zorn vector matrix realization and accordingly the differential operator has been rewritten in terms of 2×2 Zorn vector matrix realization of split octonions. Using the definitions of split octonions and their connection with Zorn vector matrix realization, we have developed the split octonionic form of generalized four potential of dyons (section - 3.3) and thus obtained the split octonion wave equation which is analogous to the potential wave equation giving rise to generalized electromagnetic fields of dyons. Consequently, the split octonion field equation in compact and simpler form has been developed and it is shown that the split octonion wave equation represents the generalized Dirac Maxwell's equations of dyons in the case of split octonion electrodynamics. Another quantum equations for generalized potential, fields, current and other physical variables are also developed in compact and similar form of split octonion electrodynamics in section (3.4). In section (3.5), we have analyzed the laws associated with energy momentum conservation in split octonion electrodynamics. Accordingly, we have investigated the work energy theorem or "Poynting Theorem" to the case of generalized electromagnetic fields of dyons in split octonion formulation and their Zorn vector matrix realization in consistent manner. The Poynting theorem has been discussed for the conservation of energy associated with generalized fields of dyons in split octonion electrodynamics. Furthermore, the Maxwell stress tensor for generalized fields of dyons has also been reformulated for split octonion electrodynamics it is shown that the divergence of Maxwell's stress tensor represents the "generalized electromagnetic force" of dyons. More over it is shown that a part of Maxwell's stress tensor represents the generalized Dirac Maxwell's equations of dyons in split octonion electrodynamics. In section (3.6), we have made an attempt to analyze the split octonion reformulation of Lorentz invariant of generalized split octonion electrodynamics of dyons and we have thus obtained the Lorentz invariants like $\left(\vec{E}^2-\vec{H}^2\right), \left(\vec{E}.\vec{H}\right), \vec{\nabla}\left(\vec{E}^2-\vec{H}^2\right)$, and $\vec{\nabla}\left(\vec{E}.\vec{H}\right)$ and it is shown that the reformulation of classical electrodynamics in terms of split octonion formulation in compact, simpler, manifestly covariant and consistent manner. Consequently, it is concluded that the split octonion electrodynamics reproduces the electrodynamics of electric (magnetic) charge in the absence of magnetic (electric) charge of dyons and vice-versa.

3.2 Split-Octonions Definitions

The split octonions are the non associative extension of quaternions (or the split quaternions). They differ form the octonion in the signature of quadratic form. The split octonions have a signature (4,4) whereas the octonions have positive signature (8,0). The Caylay algebra of octonions over the field of complex numbers $\mathbb{C}_{\mathbb{C}}=\mathbb{C}\otimes C$ is visualized as the algebra of split octonions with its following basis elements.

$$\begin{aligned}
u_0 &= \frac{1}{2}\left(e_0+ie_7\right), \;\; u_0^{*}=\frac{1}{2}\left(e_0-ie_7\right); \\
u_1 &= \frac{1}{2}\left(e_1+ie_4\right), \;\; u_1^{*}=\frac{1}{2}\left(e_1-ie_4\right); \\
u_2 &= \frac{1}{2}\left(e_2+ie_5\right), \;\; u_2^{*}=\frac{1}{2}\left(e_2-ie_5\right); \\
u_3 &= \frac{1}{2}\left(e_3+ie_6\right), \;\; u_3^{*}=\frac{1}{2}\left(e_3-ie_6\right);
\end{aligned} \tag{3.1}$$

where $(\star)$ is used for complex conjugation and $(i=\sqrt{1})$ commutes with all seven octonion imaginary unit e_A $(A=1,2,......,7)$ whose properties are illustrated in chapter-2. The automorphism group of the octonion algebra is the 14-parameter exceptional group G_2. The imaginary octonion units e_A $(A = 1, 2, ..., 7)$ fall into its 7-dimensional representation. Under the $SU(3)_c$ subgroup of G_2 that leaves e_7 invariant, u_0 and u_0^* transform like singlets, while u_j and u_j^* $(\forall\, j = 1, 2, 3)$ transform like a triplet and anti-triplet respectively.

The split octonion basis element satisfy the following multiplication rule

$$\begin{aligned} u_i u_j &= \epsilon_{ijk} u_k^*,\; u_i u_j^* = -\delta_{ij} u_0,\; u_i u_0 = 0, \\ u_i^* u_j &= -\delta_{ij} u_0,\; u_i u_0^* = u_0,\; u_i^* u_0^* = 0, \\ u_0 u_i &= u_i,\; u_i^* u_0 = u_i^*,\; u_0^* u_i^* = u_i, \\ u_0^2 &= u_0,\; u_0^{*2} = u_0^*, u_0 u_0^* = u_0^* u_0 = 0.\; (\forall i, j, k = 1, 2, 3) \end{aligned} \tag{3.2}$$

The multiplication table [33] can now be written in a manifestly $SU(3)_c$ invariant manner as

$\cdot$	u_0^*	u_1^*	u_2^*	u_3^*	u_0	u_1	u_2	u_3
u_0^*	u_0^*	u_1^*	u_2^*	u_3^*	0	0	0	0
u_1^*	0	0	u_3	$-u_2$	u_1^*	$-u_0^*$	0	0
u_2^*	0	$-u_3$	0	u_1	u_2^*	0	$-u_0^*$	0
u_3^*	0	u_2	$-u_1$	0	u_3^*	0	0	$-u_0^*$
u_0	0	0	0	0	u_0	u_1	u_2	u_3
u_1	u_1	$-u_0$	0	0	0	0	u_3^*	$-u_2^*$
u_2	u_2	0	$-u_0$	0	0	$-u_3^*$	0	u_1^*
u_3	u_3	0	0	$-u_0$	0	u_2^*	$-u_1^*$	0

Table 3.1: Split-Octonion Multiplication Table

From the multiplication rules (3.2), we may obtain

$$(u_i u_j)u_k = -\,\epsilon_{ijk}u_0^*, \tag{3.3}$$

so that, we may put together the compactified multiplication table for the split octonion units as [33]

.	u_0	u_0^*	u_k	u_k^*
u_0	u_0	0	u_k	0
u_0^*	0	u_0^*	0	u_k^*
u_j	0	u_j	$\epsilon_{jki}u_i^*$	$-\delta_{jk}u_0$
u_j^*	u_j^*	0	$-\delta_{jk}u_0^*$	$\epsilon_{jki}u_i$

Table 3.2: Compactified Split-Octonion Multiplication Table

Thus, one can relate u_j and u_j^* with fermionic annihilation and creation operators as

$$\{u_i, u_j\} = \{u_i^*, u_j^*\} = 0, \quad \{u_i, u_k^*\} = -\delta_{ij}. \tag{3.4}$$

This fermionic Heisenberg algebra shows the three split unit u_i to be Grassmann numbers. Being non-associative, these split units give rise to an exceptional Grassmann algebra. Operators u_i, unlike ordinary fermion operators, are non associative. We also have

$$\frac{1}{2}[u_i, u_j] = \epsilon_{ijk}u_k^*. \tag{3.5}$$

The Jacobi identity does not hold since

$$[u_i, [u_i, u_k]] = -\,ie_7 \neq 0; \tag{3.6}$$

where e_7, anti commute with u_i and u_i^*. It is to be noticed that, like the imaginary units e_A, the split units cannot be represented by matrices. Unlike the octonion algebra, the split octonion algebra contains zero divisors and is therefore not a division algebra.

The associators of split octonion units are given below:

$$\begin{aligned}
[u_i, u_j, u_k] =& \epsilon_{ijk}(u_0^* - u_0), \\
\left[u_i^*, u_j^*, u_k^*\right] =& \epsilon_{ijk}(u_0 - u_0^*), \\
[u_i, u_j, u_0] =& - \epsilon_{ijk} u_k^*, \\
[u_i, u_j, u_0^*] =& \epsilon_{ijk} u_k^*, \\
[u_i, u_j, u_k^*] =& \delta_{jk} u_i - \delta_{ik} u_j, \\
\left[u_i, u_j^*, u_k^*\right] =& \delta_{ik} u_j^* - \delta_{ij} u_k^*, \\
\left[u_i^*, u_j^*, u_0\right] =& \epsilon_{ijk} u_k, \\
\left[u_i^*, u_j^*, u_0^*\right] =& - \epsilon_{ijk} u_k, \\
\left[u_i, u_j^*, u_0\right] =& 0, \\
\left[u_i, u_j^*, u_0^*\right] =& 0.
\end{aligned} \tag{3.7}$$

The Hermitian conjugation for split octonion basis elements can now be defined in terms of both complex and octonion conjugation as

$$u_i^\dagger = \bar{u}_i^* = -u_i^*, \qquad u_0^\dagger = \bar{u}_0^* = u_0. \tag{3.8}$$

Following new definition for split octonions may also be made as

$$u_{\mu\nu} = \frac{1}{2}(u_\mu^\dagger u_\nu - u_\nu^\dagger u_\mu), \tag{3.9}$$

and

$$u'_{\mu\nu} = \frac{1}{2}(u_\mu u_\nu^\dagger - u_\nu u_\mu^\dagger), \tag{3.10}$$

while leads the left handed product [33];

$$u'_{\mu\nu} = 0 \tag{3.11}$$

and in the component form the right handed product $u_{\mu\nu}$ survives only as $u_{0i} = \frac{1}{2}e_i$, i.e.

$$u_{ij} = 0; \tag{3.12}$$

with

$$u_{0i} = \frac{1}{2}(u_i + u_i^*) = \frac{1}{2}e_i \tag{3.13}$$

thereby reducing the octonions to purely vectorial quaternions.

So, the convenient realization for the basis elements (u_0, u_j, u_0^*, u_j^*) in term of Pauli spin matrices may now be introduced as

$$\begin{aligned} u_0 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \qquad u_0^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \\ u_j &= \begin{bmatrix} 0 & 0 \\ e_j & 0 \end{bmatrix}; \qquad u_j^* = \begin{bmatrix} 0 & -e_j \\ 0 & 0 \end{bmatrix}; \qquad (\forall j = 1, 2, 3) \end{aligned} \tag{3.14}$$

The split Cayley (octonion) algebra is thus expressed in terms of 2×2 Zorn's vector matrices components of which are scalar and vector parts of a quaternion i.e.

$$\mathcal{O} = \{ \begin{pmatrix} m & \vec{p} \\ \vec{q} & n \end{pmatrix}; \quad m, n \in Sc(H); \quad \& \, \vec{p}, \vec{q} \in Vec(H) \ \}. \tag{3.15}$$

As such, we may write an arbitrary split octonion A in terms of following 2×2 Zorn's vector matrix realization as

$$A = au_0^* + bu_0 + x_i u_i^* + y_i u_i = \begin{pmatrix} a & -\overrightarrow{x} \\ \overrightarrow{y} & b \end{pmatrix}, \tag{3.16}$$

where a and b are scalars and $\vec{x}$ and $\vec{y}$ are three vectors. Thus the product of two octonions in terms of following 2×2 Zorn's vector matrix realization is expressed as

$$\begin{pmatrix} a & \overrightarrow{x} \\ \overrightarrow{y} & b \end{pmatrix} * \begin{pmatrix} c & \overrightarrow{u} \\ \vec{v} & d \end{pmatrix} = \begin{pmatrix} ac + (\overrightarrow{x} . \overrightarrow{v}) & a\vec{u} + d\vec{x} + (\overrightarrow{y} \times \vec{v}) \\ c\overrightarrow{y} + b\vec{v} - (\vec{x} \times \overrightarrow{u}) & bd + (\overrightarrow{y} . \overrightarrow{u}) \end{pmatrix} \tag{3.17}$$

where $(\times)$ denotes the usual vector product, e_j $(j = 1, 2, 3)$ with $e_j \times e_k = \epsilon_{jkl} e_l$ and $e_j e_k = -\delta_{jk}$.
Octonion conjugate of equation (3.16) in terms of 2×2 Zorn's vector matrix realizations is now defined as

$$\overline{A} \ = au_0 + bu_0^\star - x_i u_i^\star - y_i u_i \ = \begin{pmatrix} b & \overrightarrow{x} \\ -\overrightarrow{y} & a \end{pmatrix}. \tag{3.18}$$

The norm of A is defined as

$$N(A) = \overline{A}A = A\overline{A} = (ab + \overrightarrow{x} . \overrightarrow{y})\hat{1} = n(A)\vec{1}, \tag{3.19}$$

where $\hat{1}$ is the identity elements of matrix order 2×2, and the expression $n(A) = (ab + \overrightarrow{x}.\overrightarrow{y})$ defines the quadratic form which admits the composition as

$$n(\vec{A} \cdot \vec{B}) = n(\vec{A})\, n(\vec{B}), \quad (\forall\, \vec{A}, \vec{B} \in \mathcal{O}) \tag{3.20}$$

As such, we may easily express the Euclidean or Minkowski four vector in split octonion formulation in terms of 2×2 Zorn's vector matrix realizations. So, any four - vector A_μ (complex or real) can equivalently be written in terms of the following Zorn matrix realization as

$$Z(A) \;=\; \begin{pmatrix} x_4 & -\overrightarrow{x} \\ \overrightarrow{y} & y_4 \end{pmatrix}; \;\; Z(\overline{A}) = \begin{pmatrix} x_4 & \overrightarrow{x} \\ -\overrightarrow{y} & y_4 \end{pmatrix}. \tag{3.21}$$

Hence, we may define the split octonion equivalent of space - time four differential operator $\boxdot$ may be written in terms of 2×2 Zorn's vector matrix as [44]

$$\begin{aligned} \boxdot =& \partial_t u_0^* - \partial_t u_0 + \overrightarrow{\nabla} u_i^* + \overrightarrow{\nabla} u_i \\ \cong& \begin{pmatrix} \partial_t & -\overrightarrow{\nabla} \\ \overrightarrow{\nabla} & -\partial_t \end{pmatrix}; \end{aligned} \tag{3.22}$$

where $\partial_t = \frac{\partial}{\partial t}$, we have taken other components like $\partial_0, \partial_4, \partial_5, \partial_6$ of equation vanishing. Accordingly, split octonion conjugate $\overline{\boxdot}$ of four differential operator may be written in terms of 2×2 Zorn's vector matrix as

$$\begin{aligned} \overline{\boxdot} =& -\partial_t u_0^* + \partial_t u_0 - \overrightarrow{\nabla} u_i^* - \overrightarrow{\nabla} u_i \\ \cong& \begin{pmatrix} -\partial_t & \overrightarrow{\nabla} \\ -\overrightarrow{\nabla} & \partial_t \end{pmatrix}; \end{aligned} \tag{3.23}$$

As such , we get

$$\boxdot\overline{\boxdot} = \overline{\boxdot}\boxdot = \begin{pmatrix} \nabla^2 - \frac{\partial^2}{\partial t^2} & 0 \\ 0 & \nabla^2 - \frac{\partial^2}{\partial t^2} \end{pmatrix}, \tag{3.24}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and $\Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{\partial^2}{\partial t^2}$ (d' Alembert operator).

3.3 Generalized Split Octonion Electrodynamics

In order to write the various quantum equations of dyons in split octonion formulation, we start with octonion form of generalized potential [42] of dyons, which is given in chapter-2. Using the definitions of split octonions and their connection with Zorn's vector matrix realization, it is easy to write the split octonion form of generalized four potential given by equations (2.89 - 2.92) as [44],

$$\mathbb{V} = \begin{pmatrix} \text{ï¿œ}_- & -\overrightarrow{\text{ï¿œ}_+} \\ \overrightarrow{\text{ï¿œ}_-} & \text{ï¿œ}_+ \end{pmatrix} = \begin{pmatrix} (\varphi - \phi) & -\left(\overrightarrow{A} + \overrightarrow{B}\right) \\ \left(\overrightarrow{A} - \overrightarrow{B}\right) & (\varphi + \phi) \end{pmatrix}. \tag{3.25}$$

Here $\left[\text{ï¿œ}_- \to (\varphi - \phi)\,,\ \text{ï¿œ}_+ \to (\varphi + \phi)\,,\ \overrightarrow{\text{ï¿œ}_-} \to \left(\overrightarrow{A} - \overrightarrow{B}\right)\ \overrightarrow{\text{ï¿œ}_+} \to \left(\overrightarrow{A} + \overrightarrow{B}\right)\right]$. Now operating $\overline{\boxdot}$ given by the equation (3.23) to octonion potential $\mathbb{V}$ (3.25), we get [44];

$$\overline{\boxdot}\,\mathbb{V} = \begin{pmatrix} \frac{\partial}{\partial t} & -\overrightarrow{\nabla} \\ \overrightarrow{\nabla} & -\frac{\partial}{\partial t} \end{pmatrix} * \begin{pmatrix} (\varphi - \phi) & -\left(\overrightarrow{A} + \overrightarrow{B}\right) \\ \left(\overrightarrow{A} - \overrightarrow{B}\right) & (\varphi + \phi) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\partial\varphi}{\partial t}+\frac{\partial\phi}{\partial t}+\overrightarrow{\nabla}\cdot\overrightarrow{A}-\overrightarrow{\nabla}\cdot\overrightarrow{B} & \frac{\partial\overrightarrow{A}}{\partial t}+\frac{\partial\overrightarrow{B}}{\partial t}+\overrightarrow{\nabla}\varphi+\overrightarrow{\nabla}\phi \\ & -\overrightarrow{\nabla}\times\overrightarrow{A}+\overrightarrow{\nabla}\times\overrightarrow{B} \\ -\overrightarrow{\nabla}\varphi+\overrightarrow{\nabla}\phi+\frac{\partial\overrightarrow{A}}{\partial t}-\frac{\partial\overrightarrow{B}}{\partial t} & \\ +\overrightarrow{\nabla}\times\overrightarrow{A}+\overrightarrow{\nabla}\times\overrightarrow{B} & \overrightarrow{\nabla}\cdot\overrightarrow{A}+\overrightarrow{\nabla}\cdot\overrightarrow{B}+\frac{\partial\varphi}{\partial t}+\frac{\partial\phi}{\partial t} \end{pmatrix}. \tag{3.26}$$

It is to be noted that we have used S.I. system of natural units ($c = \hbar = 1$) through out the text. On applying the Lorentz gauge conditions, respectively for the dynamics of electric and magnetic charges of dyons as

$$\overrightarrow{\nabla}\cdot\overrightarrow{A}+\frac{\partial\phi}{\partial t}=0, \qquad \overrightarrow{\nabla}\cdot\overrightarrow{B}+\frac{\partial\varphi}{\partial t}=0; \tag{3.27}$$

We get the reduced form of equation (3.26) as

$$\overline{\boxdot}\,\mathbb{V} = \mathbb{F}; \tag{3.28}$$

where $\mathbb{F}$ is also an octonion describing the generalized electromagnetic fields of dyons given by equations (2.100 - 2.102) whereas the split octonion equivalent in terms of 2×2 Zorn's vector matrix realization may be written as [44]

$$\mathbb{F} = \begin{pmatrix} 0 & -\overrightarrow{F}_{+} \\ \overrightarrow{F}_{-} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\left(\overrightarrow{F}_{g}+\overrightarrow{F}_{e}\right) \\ \left(\overrightarrow{F}_{g}-\overrightarrow{F}_{e}\right) & 0 \end{pmatrix}, \tag{3.29}$$

where $\overrightarrow{F}_{+}\rightarrow\overrightarrow{F}_{g}+\overrightarrow{F}_{e}$, $\overrightarrow{F}_{-}\rightarrow\overrightarrow{F}_{g}-\overrightarrow{F}_{e}$; and

$$\begin{aligned} \overrightarrow{F}_{g} &= -\frac{\partial\overrightarrow{B}}{\partial t}-\overrightarrow{\nabla}\varphi+\overrightarrow{\nabla}\times\overrightarrow{A}. \longrightarrow \overrightarrow{H}; \\ \overrightarrow{F}_{e} &= -\frac{\partial\overrightarrow{A}}{\partial t}-\overrightarrow{\nabla}\phi-\overrightarrow{\nabla}\times\overrightarrow{B}; \longrightarrow \overrightarrow{E}. \end{aligned} \tag{3.30}$$

Here $\overrightarrow{E}$ and $\overrightarrow{H}$ are respectively denoted as generalized electric and magnetic fields of dyons described in chapter-2. As such, we may write the generalized electromagnetic field vector $\mathbb{F}$ of dyons in term of following split octonionic representation as

$$\mathbb{F}=\begin{pmatrix} 0 & -\left(\overrightarrow{H}+\overrightarrow{E}\right) \\ \overrightarrow{H}-\overrightarrow{E} & 0 \end{pmatrix}=\begin{pmatrix} 0 & -\overrightarrow{\psi_{+}} \\ \overrightarrow{\psi_{-}} & 0 \end{pmatrix}, \tag{3.31}$$

where $\overrightarrow{\psi_{+}}=\overrightarrow{H}+\overrightarrow{E}$ and $\overrightarrow{\psi_{-}}=\overrightarrow{H}-\overrightarrow{E}$ are the generalized electromagnetic vector fields of dyons. Now applying the differential operator $\boxdot$ given by (3.22) to equation (3.29) for generalized fields of dyon, we get

$$\begin{aligned}
\boxdot\,\mathbb{F} &= \begin{pmatrix} \partial_t & -\overrightarrow{\nabla} \\ \overrightarrow{\nabla} & -\partial_t \end{pmatrix} * \begin{pmatrix} 0 & -\overrightarrow{F_{+}} \\ \overrightarrow{F_{-}} & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial}{\partial t} & -\overrightarrow{\nabla} \\ \overrightarrow{\nabla} & -\frac{\partial}{\partial t} \end{pmatrix} * \begin{pmatrix} 0 & -\left(\overrightarrow{F_g}+\overrightarrow{F_e}\right) \\ \left(\overrightarrow{F_g}-\overrightarrow{F_e}\right) & 0 \end{pmatrix} \\
&= \begin{pmatrix} \overrightarrow{\nabla}\cdot\overrightarrow{F_g}-\overrightarrow{\nabla}\cdot\overrightarrow{F_e} & \frac{\partial\overrightarrow{F_g}}{\partial t}+\frac{\partial\overrightarrow{F_e}}{\partial t}-\overrightarrow{\nabla}\times\overrightarrow{F_g}+\overrightarrow{\nabla}\times\overrightarrow{F_e} \\ \frac{\partial\overrightarrow{F_g}}{\partial t}-\frac{\partial\overrightarrow{F_e}}{\partial t}+\overrightarrow{\nabla}\times\overrightarrow{F_g}+\overrightarrow{\nabla}\times\overrightarrow{F_e} & \overrightarrow{\nabla}\cdot\overrightarrow{F_g}+\overrightarrow{\nabla}\cdot\overrightarrow{F_e} \end{pmatrix},
\end{aligned} \tag{3.32}$$

which is further reduced to the following wave equation in split octonion form as

$$\boxdot\mathbb{F}=-\mathbb{J}. \tag{3.33}$$

Here $\mathbb{J}$ is the split octonion equivalent of generalized four current of dyons and may be written in terms of 2×2 Zorn's vector matrix realization as [44],

$$\mathbb{J}=\begin{pmatrix} (\varrho-\rho) & -\left(\overrightarrow{j}+\overrightarrow{k}\right) \\ \left(\overrightarrow{j}-\overrightarrow{k}\right) & (\varrho+\rho) \end{pmatrix}=\begin{pmatrix} \jmath_{-} & -\vec{\jmath}_{+} \\ \vec{\jmath}_{-} & \jmath_{+} \end{pmatrix}; \tag{3.34}$$

where $\jmath_{-} \rightarrow (\varrho-\rho)$, $\vec{\jmath}_{-} \rightarrow \left(\overrightarrow{j}-\overrightarrow{k}\right)$, $\jmath_{+} \rightarrow (\varrho+\rho)$, $\vec{\jmath}_{+} \rightarrow \left(\overrightarrow{j}+\overrightarrow{k}\right)$. Here $(\rho,\ \overrightarrow{j})=\{\mathrm{j}_{\mu}\}$, $(\varrho,\ \overrightarrow{j})=\{\mathrm{k}_{\mu}\}$ and $(\mathrm{J}_0,\ \overrightarrow{J})=\{\mathrm{J}_{\mu}\}$ are respectively the four currents associated with electric charge, magnetic monopole and generalized fields of dyons. Equations (3.33) contains the following differential equations

$$\begin{aligned}
(\overrightarrow{\nabla}\cdot\overrightarrow{F_e}) &= \rho; \\
(\overrightarrow{\nabla}\times\overrightarrow{F_e}) &= -\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{k}; \\
(\overrightarrow{\nabla}\times\overrightarrow{F_g}) &= \frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{j}; \\
(\overrightarrow{\nabla}\cdot\overrightarrow{F_g}) &= \varrho.
\end{aligned} \tag{3.35}$$

Replacing $\overrightarrow{F_e} \longrightarrow \overrightarrow{E}$, $\overrightarrow{F_g} \longrightarrow \overrightarrow{H}$, equation (3.35) is changed to following form of Maxwell's equations

$$\begin{aligned}
(\overrightarrow{\nabla}\cdot\overrightarrow{E}) &= \rho; \\
(\overrightarrow{\nabla}\times\overrightarrow{E}) &= -\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{k}; \\
(\overrightarrow{\nabla}\times\overrightarrow{H}) &= \frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{j}; \\
(\overrightarrow{\nabla}\cdot\overrightarrow{H}) &= \varrho.
\end{aligned} \tag{3.36}$$

Which are the Generalized Dirac-Maxwell's (GDM) equations of dyons discussed in chapter-2.

3.4 Generalized Split Octonion Wave Equations

In this section we consider the case of generalized electromagnetic fields of dyons (particles carrying simultaneous existence of electric and magnetic charge) for which we may now define generalized octonion valued potential wave equations, generalized octonion current wave equations and generalized octonion fields wave equations in eight dimensional formalism as the combinations of two four-dimensional spaces. Let us divide this section to following subsections.

3.4.1 Generalized Split-Octonion Potential Wave Equations

In order to write the split-octonion form of potential wave equations, let us use the equations (3.24) and (3.25) and we get

$$\boxdot\overline{\boxdot}\mathbb{V}=\overline{\boxdot}\boxdot\mathbb{V}=-\mathbb{J}; \tag{3.37}$$

which can be visualized in term of 2×2 Zorn's vector matrix as

$$\begin{aligned}\boxdot\overline{\boxdot}\mathbb{V}&=\begin{pmatrix}\frac{\partial}{\partial t} & -\overrightarrow{\nabla}\\ \overrightarrow{\nabla} & -\frac{\partial}{\partial t}\end{pmatrix}*\begin{pmatrix}-\frac{\partial}{\partial t} & \overrightarrow{\nabla}\\ -\overrightarrow{\nabla} & \frac{\partial}{\partial t}\end{pmatrix}*\begin{pmatrix}\text{ï¿œ}_{-} & -\overrightarrow{\text{ï¿œ}_{+}}\\ \overrightarrow{\text{ï¿œ}_{-}} & \text{ï¿œ}_{+}\end{pmatrix}\\ &=\begin{pmatrix}\nabla^{2}-\frac{\partial^{2}}{\partial t^{2}} & 0\\ 0 & \nabla^{2}-\frac{\partial^{2}}{\partial t^{2}}\end{pmatrix}*\begin{pmatrix}(\varphi-\phi) & -\left(\overrightarrow{A}+\overrightarrow{B}\right)\\ \left(\overrightarrow{A}-\overrightarrow{B}\right) & (\varphi+\phi)\end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} (\Box\varphi - \Box\phi) & -\left(\Box\overrightarrow{A} + \Box\overrightarrow{B}\right) \\ \left(\Box\overrightarrow{A} - \Box\overrightarrow{B}\right) & (\Box\varphi + \Box\phi) \end{pmatrix}$$

$$\Longrightarrow - \begin{pmatrix} \jmath_{-} & -\vec{\jmath}_{+} \\ \vec{\jmath}_{-} & \jmath_{+} \end{pmatrix} = - \begin{pmatrix} (\varrho - \rho) & -\left(\overrightarrow{j} + \overrightarrow{k}\right) \\ \left(\overrightarrow{j} - \overrightarrow{k}\right) & (\varrho + \rho) \end{pmatrix}. \qquad (3.38)$$

This can further be reduced to following form of wave equations

$$\Box\phi = -\rho, \qquad \Box\varphi = -\varrho,$$
$$\Box \mathrm{A}_{\mu} = -j_{\mu}, \qquad \Box B_{\mu} = -k_{\mu}. \qquad (3.39)$$

Using the definitions (3.24) , we get the following expanded form of equation (3.39) as

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = \rho,$$
$$\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = \varrho,$$
$$\frac{\partial^2 A_{\mu}}{\partial t^2} - \nabla^2 A_{\mu} = j_{\mu},$$
$$\frac{\partial^2 B_{\mu}}{\partial t^2} - \nabla^2 B_{\mu} = k_{\mu}; \qquad (3.40)$$

which are the potential wave equations for generalized fields of dyons.

3.4.2 Generalized Split-Octonion Current Wave Equations

The electromagnetic wave equation is a second-order partial differential equation that describes the propagation of electromagnetic waves through a medium or in a vacuum. Thus, in the case of split-octonion current wave equations, we operate $\overline{\boxdot}$ given by equation (3.23) to split octonion current $\mathbb{J}$ (3.34) in term of 2×2 Zorn's vector matrix as [44]

$$\overline{\boxdot}\,\mathbb{J}=\begin{pmatrix}-\frac{\partial}{\partial t} & \overrightarrow{\nabla}\\ -\overrightarrow{\nabla} & \frac{\partial}{\partial t}\end{pmatrix}*\begin{pmatrix}(\varrho-\rho) & -\left(\overrightarrow{j}+\overrightarrow{k}\right)\\ \left(\overrightarrow{j}-\overrightarrow{k}\right) & (\varrho+\rho)\end{pmatrix}$$
$$=\begin{pmatrix}-\frac{\partial\varrho}{\partial t}+\frac{\partial\rho}{\partial t}+\overrightarrow{\nabla}\cdot\overrightarrow{j}-\overrightarrow{\nabla}\cdot\overrightarrow{k} & \begin{array}{c}\frac{\partial\overrightarrow{j}}{\partial t}+\frac{\partial\overrightarrow{k}}{\partial t}+\overrightarrow{\nabla}\varrho+\overrightarrow{\nabla}\rho\\ -\overrightarrow{\nabla}\times\vec{j}+\overrightarrow{\nabla}\times\overrightarrow{k}\end{array}\\ \begin{array}{c}-\overrightarrow{\nabla}\varrho+\overrightarrow{\nabla}\rho+\frac{\partial\overrightarrow{j}}{\partial t}-\frac{\partial\overrightarrow{k}}{\partial t}\\ +\overrightarrow{\nabla}\times\vec{j}+\overrightarrow{\nabla}\times\overrightarrow{k}\end{array} & \overrightarrow{\nabla}\cdot\overrightarrow{j}+\overrightarrow{\nabla}\cdot\overrightarrow{k}+\frac{\partial\varrho}{\partial t}+\frac{\partial\rho}{\partial t}\end{pmatrix};\tag{3.41}$$

which can be compactified as

$$\overline{\boxdot}\,\mathbb{J}=\mathbb{S},\tag{3.42}$$

where

$$\mathbb{S}=(\Im_m-\Im_e)\,u_0^*+(\Im_m+\Im_e)\,u_0+(\overrightarrow{r}-\overrightarrow{s})\,u_i^*+(\overrightarrow{r}+\overrightarrow{s})\,u_i$$
$$=\begin{pmatrix}(\Im_m-\Im_e) & -(\overrightarrow{r}+\overrightarrow{s})\\ (\overrightarrow{r}-\overrightarrow{s}) & (\Im_m+\Im_e)\end{pmatrix}\longmapsto\begin{pmatrix}\Im_- & \overrightarrow{S_+}\\ \overrightarrow{S_-} & \Im_+\end{pmatrix}.\tag{3.43}$$

Here $\Im_-\ \rightarrow (\Im_m-\Im_e)\,,\ \overrightarrow{S_-}\rightarrow(\overrightarrow{r}-\overrightarrow{s})\,,\ \Im_+\rightarrow(\Im_m+\Im_e)\,,\ \overrightarrow{S_+}\rightarrow$ $(\overrightarrow{r}+\overrightarrow{s})$. So, equation (3.42) leads to following four equations

$$\Im_m=\overrightarrow{\nabla}\cdot\overrightarrow{k}+\frac{\partial\varrho}{\partial t};\qquad \Im_e=\overrightarrow{\nabla}\cdot\overrightarrow{j}+\frac{\partial\rho}{\partial t};$$
$$\overrightarrow{r}=-\overrightarrow{\nabla}\rho-\frac{\partial\overrightarrow{j}}{\partial t}-\overrightarrow{\nabla}\times\overrightarrow{k};\qquad \overrightarrow{s}=-\overrightarrow{\nabla}\varrho-\frac{\partial\overrightarrow{k}}{\partial t}+\overrightarrow{\nabla}\times\overrightarrow{j};\tag{3.44}$$

which are the split octonion current wave equations for the components of generalized fields of dyons. In equation (3.44) $\Im_m$ and $\Im_e$ are vanishing due to Lorentz gauge conditions applied for the cases of electric and magnetic charges.

3.4.3 Generalized Split-Octonion Field Equations (Continuity Equation)

A continuity equation in physics is an equation that describes the transport of a conserved quantity. Since mass, energy, momentum, electric charge and other natural quantities are conserved under their respective appropriate conditions, a variety of physical phenomena may be described using continuity equations. A continuity equation is a special case of the more general transport equation. In the case of split-octonion field equations, we operate $\overline{\boxdot}$ both sides to the equation (3.33) as

$$\overline{\boxdot}\left(\boxdot\mathbb{F}\right)=\left(\overline{\boxdot}\boxdot\right)\mathbb{F}=-\,\overline{\boxdot}\mathbb{J}\longrightarrow-\mathbb{S}, \tag{3.45}$$

where $\mathbb{F}$, $\mathbb{J}$, and $\mathbb{S}$ are defined in equations (3.31),(3.34) and (3.43). So, in terms of 2×2 Zorn's vector matrix, the left hand side of equation (3.45) can be written as

$$\begin{aligned}\left(\overline{\boxdot}\boxdot\right)\mathbb{F}&=\begin{pmatrix}\Box & 0\\ 0 & \Box\end{pmatrix}*\begin{pmatrix}0 & -\left(\overrightarrow{H}+\overrightarrow{E}\right)\\ \overrightarrow{H}-\overrightarrow{E} & 0\end{pmatrix}\\ &=\begin{pmatrix}0 & -\left(\Box\overrightarrow{H}+\Box\overrightarrow{E}\right)\\ \left(\Box\overrightarrow{H}-\Box\overrightarrow{E}\right) & 0\end{pmatrix};\end{aligned} \tag{3.46}$$

whereas the right hand side of equation (3.45) is expressed as

$$\boxdot\mathbb{J}=\begin{pmatrix} -\frac{\partial\varrho}{\partial t}+\frac{\partial\rho}{\partial t}+\overrightarrow{\nabla}\cdot\overrightarrow{j}-\overrightarrow{\nabla}\cdot\overrightarrow{k} & \frac{\partial\overrightarrow{j}}{\partial t}+\frac{\partial\overrightarrow{k}}{\partial t}+\overrightarrow{\nabla}\varrho+\overrightarrow{\nabla}\rho \\ & -\overrightarrow{\nabla}\times\overrightarrow{j}+\overrightarrow{\nabla}\times\overrightarrow{k} \\ -\overrightarrow{\nabla}\varrho+\overrightarrow{\nabla}\rho+\frac{\partial\overrightarrow{j}}{\partial t}-\frac{\partial\overrightarrow{k}}{\partial t} & \\ +\overrightarrow{\nabla}\times\overrightarrow{j}+\overrightarrow{\nabla}\times\overrightarrow{k} & \overrightarrow{\nabla}\cdot\overrightarrow{j}+\overrightarrow{\nabla}\cdot\overrightarrow{k}+\frac{\partial\varrho}{\partial t}+\frac{\partial\rho}{\partial t} \end{pmatrix}. \tag{3.47}$$

So equation (3.45) thus leads to following differential equations

$$\begin{aligned} \Im_m &\longrightarrow \overrightarrow{\nabla}\cdot\overrightarrow{k}+\frac{\partial\varrho}{\partial t}\Rightarrow 0, \\ \Im_e &\longrightarrow \overrightarrow{\nabla}\cdot\overrightarrow{j}+\frac{\partial\rho}{\partial t}\Rightarrow 0; \end{aligned} \tag{3.48}$$

which are the "*Continuity equations*" of generalized fields of dyons in split octonion formulation for moving dyons. Accordingly, from equation (3.44), we get

$$\begin{aligned} \overrightarrow{r} &\longrightarrow -\overrightarrow{\nabla}\rho-\frac{\partial\overrightarrow{j}}{\partial t}-\overrightarrow{\nabla}\times\overrightarrow{k}=\square\overrightarrow{E}; \\ \overrightarrow{s} &\longrightarrow -\overrightarrow{\nabla}\varrho-\frac{\partial\overrightarrow{k}}{\partial t}+\overrightarrow{\nabla}\times\overrightarrow{j}=\square\overrightarrow{H}; \end{aligned} \tag{3.49}$$

which may be reduced to

$$\begin{aligned} \nabla^2\overrightarrow{H}-\frac{\partial^2\overrightarrow{H}}{\partial t^2} &= \overrightarrow{\nabla}\varrho+\frac{\partial\overrightarrow{k}}{\partial t}-\overrightarrow{\nabla}\times\overrightarrow{j}\longrightarrow\overrightarrow{s} \\ \nabla^2\overrightarrow{E}-\frac{\partial^2\overrightarrow{E}}{\partial t^2} &= \overrightarrow{\nabla}\rho+\frac{\partial\overrightarrow{j}}{\partial t}+\overrightarrow{\nabla}\times\overrightarrow{k}\longrightarrow\overrightarrow{r} \end{aligned} \tag{3.50}$$

These are generalized wave equations of dyons in split octonions.

3.5 Energy-Momentum Conservation in Split Octonion Electrodynamics

The laws of energy and momentum conservation [45] are probably the most frequently quoted laws in physics. The law of conservation of energy is a law of physics, i.e. the total amount of energy in an isolated system remains constant over time. The total energy is said to be conserved over time. For an isolated system, this law means that energy can change its location within the system, and that it can change form within the system, for instance chemical energy can become kinetic energy, but that energy can be neither created nor destroyed. And the other hand the law of conservation of momentum is a fundamental law of nature, and it states that if no external force acts on a closed system of objects, the momentum of the closed system remains constant. Conservation of momentum is a mathematical consequence of the homogeneity (shift symmetry) of space (position in space is the canonical conjugate quantity to momentum). That is, conservation of momentum is equivalent to the fact that the physical laws do not depend on position.

In the case of split octonion electrodynamics, we have used the equation (3.33). Operating $\overline{\mathbb{F}}$ on both sides of equation (3.33) as

$$\overline{\mathbb{F}}\,(\boxdot\mathbb{F}) = -\overline{\mathbb{F}}\,\mathbb{J}, \tag{3.51}$$

where the left hand side of equation (3.51) , i.e. $\overline{\mathbb{F}}\,(\boxdot\mathbb{F})$ may now be expressed in terms of 2×2 Zorn's vector matrix as

$$\overline{\mathbb{F}}\left(\boxdot\mathbb{F}\right)=\begin{pmatrix}0 & \overrightarrow{H}-\overrightarrow{E}\\ -\overrightarrow{H}-\overrightarrow{E} & 0\end{pmatrix}*\begin{pmatrix}-\overrightarrow{\nabla}.\overrightarrow{H}+\overrightarrow{\nabla}.\overrightarrow{E} & -\frac{\partial\overrightarrow{H}}{\partial t}-\frac{\partial\overrightarrow{E}}{\partial t}\\ & +\overrightarrow{\nabla}\times\overrightarrow{H}-\overrightarrow{\nabla}\times\overrightarrow{E}\\ -\frac{\partial\overrightarrow{H}}{\partial t}+\frac{\partial\overrightarrow{E}}{\partial t} & \\ -\overrightarrow{\nabla}\times\overrightarrow{H}-\overrightarrow{\nabla}\times\overrightarrow{E} & -\overrightarrow{\nabla}.\overrightarrow{H}-\overrightarrow{\nabla}.\overrightarrow{E}\end{pmatrix}$$

$$=\begin{pmatrix}B-A & -\left(C+D\right)\\ C-D & B+A\end{pmatrix}. \tag{3.52}$$

Here $A,B,C,D,$ the reduced forms of the matrix multiplication are described as

$$\begin{aligned}A=&\overrightarrow{H}.\frac{\partial\overrightarrow{H}}{\partial t}+\overrightarrow{E}.\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)-\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right);\\ B=&\overrightarrow{H}.\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{E}.\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right);\\ C=&\overrightarrow{H}\times\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{E}\times\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)-\overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)\\ &+\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right);\\ D=&\overrightarrow{E}\times\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{H}\times\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)-\overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)\\ &+\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right);\end{aligned} \tag{3.53}$$

whereas the right hand side of the equation (3.51) may also be expressed as

$$-\overline{\mathbb{F}}\mathbb{J}=-\begin{pmatrix}0 & \left(\overrightarrow{H}-\overrightarrow{E}\right)\\ -\left(\overrightarrow{H}+\overrightarrow{E}\right) & 0\end{pmatrix}*\begin{pmatrix}\left(\varrho-\rho\right) & -\left(\overrightarrow{j}+\overrightarrow{k}\right)\\ \left(\overrightarrow{j}-\overrightarrow{k}\right) & \left(\varrho+\rho\right)\end{pmatrix}$$

$$=\begin{pmatrix}B'-A' & -\left(C'+D'\right)\\ C'-D' & B'+A'\end{pmatrix}, \tag{3.54}$$

where A',B',C',D', again the reduced forms, are described as

$$
\begin{aligned}
A' &= -\overrightarrow{H}.\overrightarrow{k} - \overrightarrow{E}.\overrightarrow{j};\\
B' &= -\overrightarrow{H}.\overrightarrow{j} - \overrightarrow{E}.\overrightarrow{k};\\
C' &= -\varrho\overrightarrow{H} - \rho\overrightarrow{E} + \left(\overrightarrow{H}\times\overrightarrow{j}\right) - \left(\overrightarrow{E}\times\overrightarrow{k}\right);\\
D' &= -\rho\overrightarrow{H} + \varrho\overrightarrow{E} - \left(\overrightarrow{H}\times\overrightarrow{k}\right) + \left(\overrightarrow{E}\times\overrightarrow{j}\right).
\end{aligned} \tag{3.55}
$$

The above analysis shows that the left hand and right sides of equations (3.51) resemble to one another if the coefficients A, B, C, D and A', B', C', D' coincide to each other (i.e. $A \cong A'$, $B \cong B'$, $C \cong C'$, $D \cong D'$). Let us discuss the various consequences of the above analysis in following subsections.

3.5.1 Conservation of Energy of the Octonion Electrodynamics

In order to discuss the conservation of energy of the octonion electrodynamics, let us use equations (3.53) and (3.55) for A and A'. So, we get

$$
\overrightarrow{H}.\frac{\partial\overrightarrow{H}}{\partial t} + \overrightarrow{E}.\frac{\partial\overrightarrow{E}}{\partial t} + \overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right) - \overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right) = -\overrightarrow{H}.\overrightarrow{k} - \overrightarrow{E}.\overrightarrow{j}, \tag{3.56}
$$

which reduces to

$$
\frac{1}{2}\frac{\partial H^2}{\partial t} + \frac{1}{2}\frac{\partial E^2}{\partial t} + \overrightarrow{\nabla}.\left(\overrightarrow{E}\times\overrightarrow{H}\right) = -\overrightarrow{H}.\overrightarrow{k} - \overrightarrow{E}.\overrightarrow{j}. \tag{3.57}
$$

This expression (3.57) may be visualized as the *"work-energy theorem"* or "*Poynting Theorem*" [45] to the case generalized octonion electrodynamics.

Poynting's theorem is analogous to the work-energy theorem of classical mechanics reproducing the continuity equation, so that it relates the energy stored in generalized electromagnetic field to the work done on a charge distribution , through energy flux. It should be noted that the Poynting theorem is not valid in electrostatics and magnetostatics, since electric and magnetic fields change with time when electromagnetic energy flows. Equation (3.57) may then be reduced to

$$\frac{1}{2}\frac{\partial}{\partial t}\left(E^{2}+H^{2}\right)+\overrightarrow{\nabla}.\left(\overrightarrow{E}\times\overrightarrow{H}\right)+\left(\overrightarrow{H}.\overrightarrow{k}+\overrightarrow{E}.\overrightarrow{j}\right)=0, \tag{3.58}$$

where the energy due to electric field is given by

$$W_{e}=\frac{1}{2}\int E^{2}d\tau, \tag{3.59}$$

whereas the energy due to magnetic field is discussed as

$$W_{m}=\frac{1}{2}\int H^{2}d\tau, \tag{3.60}$$

So, the total energy stored in generalized electromagnetic fields of dyons is

$$W_{em}=\frac{1}{2}\int\left(E^{2}+H^{2}\right)d\tau. \tag{3.61}$$

As such, the energy density i.e. the energy per unit time, per unit area, transported by the fields is called the *Poynting vector* ($\overrightarrow{S}$) given by

$$\overrightarrow{S}=\left(\overrightarrow{E}\times\overrightarrow{H}\right), \tag{3.62}$$

which represents the directional energy flux density (the rate of energy transfer per unit area, in W/m^{2}) of an electromagnetic field. Thus, the *Poynting*

Theorem also may be generalized as the conservation of energy i.e.

$$\frac{dW}{dt}=-\frac{\partial W_{em}}{\partial t}-\overrightarrow{\nabla}.\overrightarrow{S}-\left(\overrightarrow{H}.\overrightarrow{k}+\overrightarrow{E}.\overrightarrow{j}\right). \tag{3.63}$$

Similarly equating the coefficients B and B' of the equations (3.53) and (3.55), we get

$$\overrightarrow{H}.\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{E}.\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)=-\overrightarrow{H}.\overrightarrow{j}-\overrightarrow{E}.\overrightarrow{k}, \tag{3.64}$$

which is reduced to the following generalized Dirac-Maxwell's Equations (GDM) of dyons from octonion electrodynamics, i.e.

$$\begin{aligned}\frac{\partial\overrightarrow{H}}{\partial t}+(\overrightarrow{\nabla}\times\overrightarrow{E})=-\overrightarrow{k}\ \longmapsto(\overrightarrow{\nabla}\times\overrightarrow{E})=-\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{k};\\ \frac{\partial\overrightarrow{E}}{\partial t}-(\overrightarrow{\nabla}\times\overrightarrow{H})=-\overrightarrow{j}\ \longmapsto\overrightarrow{\nabla}\times\overrightarrow{H})=\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{j}.\end{aligned} \tag{3.65}$$

3.5.2 Conservation of Momentum for Octonion Electrodynamics

The conservation of momentum is a fundamental concept of physics along with the conservation of energy and the conservation of mass. In order to understand the conservation of momentum for octonion electrodynamics, let us equate C and C' of the equations (3.53) and (3.55). So, we get

$$\begin{aligned}\overrightarrow{H}\times\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{E}\times\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)-\overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)\\ +\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)\\ =-\varrho\overrightarrow{H}-\rho\overrightarrow{E}+\left(\overrightarrow{H}\times\overrightarrow{j}\right)-\left(\overrightarrow{E}\times\overrightarrow{k}\right).\end{aligned} \tag{3.66}$$

Now using the following identities

$$\begin{aligned}\overrightarrow{H}\times\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{E}\times\frac{\partial\overrightarrow{H}}{\partial t}=&-\frac{\partial}{\partial t}\left(\overrightarrow{E}\times\overrightarrow{H}\right)\Longrightarrow-\frac{\partial\overrightarrow{S}}{\partial t};\\ \overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)=&\frac{1}{2}\nabla\left(H^{2}\right)-\left(\overrightarrow{H}.\overrightarrow{\nabla}\right)\overrightarrow{H};\\ \overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)=&\frac{1}{2}\nabla\left(E^{2}\right)-\left(\overrightarrow{E}.\overrightarrow{\nabla}\right)\overrightarrow{E};\end{aligned}\tag{3.67}$$

we get the following reduced form of equation (3.66), i.e.

$$\begin{aligned}\frac{\partial\overrightarrow{S}}{\partial t}+\frac{1}{2}\nabla\left(E^{2}+H^{2}\right)-\left(\overrightarrow{H}.\overrightarrow{\nabla}\right)\overrightarrow{H}-\left(\overrightarrow{E}.\overrightarrow{\nabla}\right)\overrightarrow{E}-\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)-\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)\\ =\varrho\overrightarrow{H}+\rho\overrightarrow{E}-\left(\overrightarrow{H}\times\overrightarrow{j}\right)+\left(\overrightarrow{E}\times\overrightarrow{k}\right),\end{aligned}\tag{3.68}$$

which gives rise the connection between electromagnetic energy and the force due to presence of electric and magnetic energy of dyons in the following manner

$$\begin{aligned}\overrightarrow{\boldsymbol{F}}=&-\frac{\partial\overrightarrow{S}}{\partial t}-\frac{1}{2}\nabla\left(E^{2}+H^{2}\right)+\left(\overrightarrow{H}.\overrightarrow{\nabla}\right)\overrightarrow{H}+\left(\overrightarrow{E}.\overrightarrow{\nabla}\right)\overrightarrow{E}+\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)\\ &+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)+\varrho\overrightarrow{H}+\rho\overrightarrow{E}-\left(\overrightarrow{H}\times\overrightarrow{j}\right)+\left(\overrightarrow{E}\times\overrightarrow{k}\right).\end{aligned}\tag{3.69}$$

3.5.3 Maxwell Stress Tensor

The Maxwell Stress Tensor [45] is a second rank tensor used in classical electromagnetism to represent the interaction between electromagnetic forces and mechanical momentum. So, in order to get the solution for the electromagnetic force discussed by equation (3.69), we start with the following expression of Maxwell Stress Tensor T_{ij} for generalized electromagnetic field as [45]

$$T_{ij} = \left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \left(H_i H_j - \frac{1}{2}\delta_{ij}H^2\right), \tag{3.70}$$

where the indices i and j refer to the coordinates x, y and z. So the stress tensor (3.70) has a total of nine components. Thus, we may also write (3.70) in terms of its following components

$$\begin{aligned} T_{xx} =& \frac{1}{2}\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2}\left(H_x^2 - H_y^2 - H_z^2\right), \\ T_{xy} =& E_x E_y + H_x H_y, \end{aligned} \tag{3.71}$$

and so on. Maxwell Stress Tensor is usually denoted by a double arrow $\overleftrightarrow{T}$ to carry out two indices (i & j) where one of the indices represent vector. So, the divergence of $\overleftrightarrow{T}$ is associated with its jth component as [45]

$$\begin{aligned} \left(\nabla.\overleftrightarrow{T}\right)_j =& \left[\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)E_j + \left(\overrightarrow{E}.\overrightarrow{\nabla}\right)E_j - \frac{1}{2}\nabla_j E^2\right] \\ &+ \left[\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)H_j + \left(\overrightarrow{H}.\overrightarrow{\nabla}\right)H_j - \frac{1}{2}\nabla_j H^2\right]. \end{aligned} \tag{3.72}$$

Hence the total octonionic representation of electromagnetic field given by (3.69) may be written as

$$\overrightarrow{\boldsymbol{F}} = \left(\nabla.\overleftrightarrow{T}\right) - \frac{\partial\overrightarrow{S}}{\partial t} + \overrightarrow{f}, \tag{3.73}$$

where $\overrightarrow{f}$ is given by

$$\begin{aligned}\overrightarrow{f} =& \varrho\overrightarrow{H} + \rho\overrightarrow{E} - \left(\overrightarrow{H} \times \overrightarrow{j}\right) + \left(\overrightarrow{E} \times \overrightarrow{k}\right) \\ =& \left(\rho\overrightarrow{E} + \overrightarrow{j} \times \overrightarrow{H}\right) + \left(\varrho\overrightarrow{H} - \overrightarrow{k} \times \overrightarrow{E}\right). \end{aligned} \tag{3.74}$$

Thus, equation (3.74) may be identified as an expression of *"generalized electromagnetic force"* of dyons. Let us use here the Newton's second law

$$\overrightarrow{\boldsymbol{F}} = \frac{\partial\overrightarrow{P_{mech}}}{\partial t} \qquad \text{and} \qquad \overrightarrow{f} = \frac{\partial\overrightarrow{P_{dyons}}}{\partial t}, \tag{3.75}$$

where $\overrightarrow{P_{mech}}$ and $\overrightarrow{P_{dyons}}$ are the mechanical and dyonic momentum respectively. So, the expression (3.75) describes the conservation of momentum in the following manner, i.e.

$$\begin{aligned}\frac{\partial\overrightarrow{P_{mech}}}{\partial t} =& \left(\nabla.\overleftrightarrow{T}\right) - \frac{\partial\overrightarrow{S}}{\partial t} + \frac{\partial\overrightarrow{P_{dyons}}}{\partial t} \\ =& \left(\nabla.\overleftrightarrow{T}\right) - \frac{\partial\left(\overrightarrow{S} - \overrightarrow{P_{dyons}}\right)}{\partial t}, \end{aligned} \tag{3.76}$$

which can further be reduced in the following form of *continuity equation as*

$$\frac{\partial}{\partial t}\left(\overrightarrow{P_{mech}} + \overrightarrow{P_{Gem}}\right) = \nabla.\overleftrightarrow{T}\ ; \tag{3.77}$$

where $\overrightarrow{P_{Gem}} \longmapsto \left(\overrightarrow{S} - \overrightarrow{P_{dyons}}\right)$ is the total generalized electromagnetic momentum. Comparing the coefficients D and D' of equations (3.53) and (3.55) respectively, we get

$$\begin{aligned}&\overrightarrow{E} \times \frac{\partial\overrightarrow{E}}{\partial t} - \overrightarrow{H} \times \frac{\partial\overrightarrow{H}}{\partial t} - \overrightarrow{H} \times \left(\overrightarrow{\nabla} \times \overrightarrow{E}\right) - \overrightarrow{E} \times \left(\overrightarrow{\nabla} \times \overrightarrow{H}\right) \\ &\qquad + \overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right) + \overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right) \\ &\qquad = -\rho\overrightarrow{H} + \varrho\overrightarrow{E} - \left(\overrightarrow{H} \times \overrightarrow{k}\right) + \left(\overrightarrow{E} \times \overrightarrow{j}\right), \end{aligned} \tag{3.78}$$

which can further be reduced to the complete set of Generalized Dirac Maxwell equations (2.30) of dyons.

3.6 Split-Octonionic realization for Lorentz invariants

Now instead of operating equation (3.33) by $\mathbb{F}$ from the left, let us operate it both sides from the left by $\mathbb{F}$. Accordingly, we get

$$\mathbb{F}\left(\boxdot\mathbb{F}\right)= \; -\mathbb{F}\left(\mathbb{J}\right). \tag{3.79}$$

The left hand side of the equation (3.79) may be written as

$$\begin{aligned}\mathbb{F}\left(\boxdot\mathbb{F}\right)= & \begin{pmatrix}0 & -\left(\overrightarrow{H}+\overrightarrow{E}\right)\\ \overrightarrow{H}-\overrightarrow{E} & 0\end{pmatrix}*\begin{pmatrix}\overrightarrow{\nabla}.\overrightarrow{H}-\overrightarrow{\nabla}.\overrightarrow{E} & \begin{array}{c}\frac{\partial\overrightarrow{H}}{\partial t}+\frac{\partial\overrightarrow{E}}{\partial t}\\ -\overrightarrow{\nabla}\times\overrightarrow{H}+\overrightarrow{\nabla}\times\overrightarrow{E}\end{array}\\ \begin{array}{c}\frac{\partial\overrightarrow{H}}{\partial t}-\frac{\partial\overrightarrow{E}}{\partial t}\\ +\overrightarrow{\nabla}\times\overrightarrow{H}+\overrightarrow{\nabla}\times\overrightarrow{E}\end{array} & \overrightarrow{\nabla}.\overrightarrow{H}+\overrightarrow{\nabla}.\overrightarrow{E}\end{pmatrix}\\ = & \begin{pmatrix}\beta-\alpha & -\left(\gamma+\zeta\right)\\ \gamma-\zeta & \beta+\alpha\end{pmatrix};\end{aligned} \tag{3.80}$$

where the coefficients of the equation (3.80) expressed as

$$\begin{aligned}\alpha= & \overrightarrow{H}.\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{E}.\frac{\partial\overrightarrow{H}}{\partial t}+\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right);\\ \beta= & \overrightarrow{H}.\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{E}.\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)-\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right);\\ \gamma= & \overrightarrow{H}\times\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{E}\times\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)\\ & -\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right);\end{aligned}$$

$$\zeta = -\overrightarrow{E}\times\frac{\partial\overrightarrow{E}}{\partial t} - \overrightarrow{H}\times\frac{\partial\overrightarrow{H}}{\partial t} - \overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right) + \overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)$$
$$-\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right) + \overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right); \quad (3.81)$$

Similarly, the right hand side of the equation (3.79) expressed as

$$-\mathbb{F}(\mathbb{J}) = -\begin{pmatrix} 0 & -\left(\overrightarrow{H}+\overrightarrow{E}\right) \\ \overrightarrow{H}-\overrightarrow{E} & 0 \end{pmatrix} * \begin{pmatrix} (\varrho-\rho) & -\left(\overrightarrow{j}+\overrightarrow{k}\right) \\ \left(\overrightarrow{j}-\overrightarrow{k}\right) & (\varrho+\rho) \end{pmatrix}$$
$$= \begin{pmatrix} \beta'-\alpha' & -(\gamma'+\zeta') \\ \gamma'-\zeta' & \beta'+\alpha' \end{pmatrix}, \quad (3.82)$$

where

$$\alpha' = \overrightarrow{H}.\overrightarrow{k} - \overrightarrow{E}.\overrightarrow{j};$$
$$\beta' = \overrightarrow{H}.\overrightarrow{j} - \overrightarrow{E}.\overrightarrow{k};$$
$$\gamma' = -\varrho\overrightarrow{H} - \rho\overrightarrow{E} + \left(\overrightarrow{H}\times\overrightarrow{j}\right) + \left(\overrightarrow{E}\times\overrightarrow{k}\right);$$
$$\zeta' = -\rho\overrightarrow{H} - \varrho\overrightarrow{E} - \left(\overrightarrow{H}\times\overrightarrow{k}\right) - \left(\overrightarrow{E}\times\overrightarrow{j}\right). \quad (3.83)$$

Using equations (3.81) and (3.83) for α and α', we get

$$\overrightarrow{H}.\frac{\partial\overrightarrow{E}}{\partial t} - \overrightarrow{E}.\frac{\partial\overrightarrow{H}}{\partial t} + \overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right) + \overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right) = \overrightarrow{H}.\overrightarrow{k} - \overrightarrow{E}.\overrightarrow{j}; \quad (3.84)$$

which is further reduced to

$$\frac{\partial}{\partial t}\left(\vec{E}^2 - \vec{H}^2\right) = \overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right) + \overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right) - \overrightarrow{H}.\overrightarrow{k} + \overrightarrow{E}.\overrightarrow{j}. \quad (3.85)$$

This expression leads to the relation for first *Lorentz invariant* [46] of $\left(\vec{E}^2 - \vec{H}^2\right)$. Similarly equating coefficients β and β' from equations (3.81) and (3.83), we get

$$\overrightarrow{H}.\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{E}.\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)-\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)=\overrightarrow{H}.\overrightarrow{j}-\overrightarrow{E}.\overrightarrow{k};\quad(3.86)$$

which is reduced to

$$\frac{\partial}{\partial t}\left(\overrightarrow{E}.\overrightarrow{H}\right)=\overrightarrow{H}.\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}.\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)+\overrightarrow{H}.\overrightarrow{j}-\overrightarrow{E}.\overrightarrow{k}.\quad(3.87)$$

This expression leads to the relation for second *Lorentz invariant* [46] of $\left(\overrightarrow{E}.\overrightarrow{H}\right)$. Accordingly equating the coefficients γ and γ' from equations (3.81) and (3.83), we get

$$\begin{aligned}\overrightarrow{H}\times\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{E}\times\frac{\partial\overrightarrow{H}}{\partial t}-\overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)+\overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)\\-\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)\\=-\varrho\overrightarrow{H}-\rho\overrightarrow{E}+\left(\overrightarrow{H}\times\overrightarrow{j}\right)+\left(\overrightarrow{E}\times\overrightarrow{k}\right),\quad(3.88)\end{aligned}$$

which is further reduced to

$$\begin{aligned}\frac{1}{2}\overrightarrow{\nabla}\left(\vec{E}^{2}-\vec{H}^{2}\right)=\left(\overrightarrow{E}.\overrightarrow{\nabla}\right)\overrightarrow{E}-\left(\overrightarrow{H}.\overrightarrow{\nabla}\right)\overrightarrow{H}+\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)-\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)\\-\overrightarrow{H}\times\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{E}\times\frac{\partial\overrightarrow{H}}{\partial t}-\varrho\overrightarrow{H}-\rho\overrightarrow{E}+\left(\overrightarrow{H}\times\overrightarrow{j}\right)+\left(\overrightarrow{E}\times\overrightarrow{k}\right).\\(3.89)\end{aligned}$$

This expression leads to the relation for the gradient of first *Lorentz invariant* [46] i.e. $\overrightarrow{\nabla}\left(\vec{E}^{2}-\vec{H}^{2}\right)$. Consequently equating the coefficients ζ and ζ' from equations (3.81) and (3.83), we get,

$$
\begin{aligned}
-\overrightarrow{E}\times\frac{\partial\overrightarrow{E}}{\partial t}-\overrightarrow{H}\times\frac{\partial\overrightarrow{H}}{\partial t} & -\overrightarrow{H}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right)+\overrightarrow{E}\times\left(\overrightarrow{\nabla}\times\overrightarrow{H}\right)\\
 & -\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)+\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)\\
 & =-\rho\overrightarrow{H}-\varrho\overrightarrow{E}-\left(\overrightarrow{H}\times\overrightarrow{k}\right)-\left(\overrightarrow{E}\times\overrightarrow{j}\right), \qquad (3.90)
\end{aligned}
$$

which is further simplified to

$$
\begin{aligned}
\overrightarrow{\nabla}\left(\overrightarrow{E}\cdot\overrightarrow{H}\right)= & \left(\overrightarrow{E}.\overrightarrow{\nabla}\right)\overrightarrow{H}-\left(\overrightarrow{H}.\overrightarrow{\nabla}\right)\overrightarrow{E}+\overrightarrow{H}\left(\overrightarrow{\nabla}.\overrightarrow{E}\right)\\
 & -\overrightarrow{E}\left(\overrightarrow{\nabla}.\overrightarrow{H}\right)+\overrightarrow{E}\times\frac{\partial\overrightarrow{E}}{\partial t}+\overrightarrow{H}\times\frac{\partial\overrightarrow{H}}{\partial t}\\
 & -\rho\overrightarrow{H}-\varrho\overrightarrow{E}-\left(\overrightarrow{H}\times\overrightarrow{k}\right)-\left(\overrightarrow{E}\times\overrightarrow{j}\right). \qquad (3.91)
\end{aligned}
$$

These expression leads to the relation for the gradient of second *Lorentz invariant* [46] i.e. $\overrightarrow{\nabla}\left(\overrightarrow{E}\cdot\overrightarrow{H}\right)$.

3.7 Discussion and Conclusion

The lack of associativity in octonion formulation of dyons forbids their group theoretical study in terms of abelian and non-Abelian gauge structure. However, split octonion basis of octonions presented in section-3.2, gives rise to their isomorphic matrix representation associated with 2×2 Zorn's vector matrices. As such, any four dimensional relativistic four vector may be reproduced in terms of split octonion as its bi-valued representation of Zorn's vector matrices by taking scalar component along principle diagonal and vector component as off-diagonal elements. Split basis of octonions is related to Pauli-spin matrices by equation (3.14). Split octonion conjugate is defined by equation (3.18), while the norm of split octonion is given by equation (3.19). Equation (3.21) is the representation of Euclidean four-dimensional space-time vector in terms of split octonion or Zorn's vector realization. Equation (3.22) and (3.23) are the split octonion equivalent of four differential operator and its conjugate respectively giving rise to invariant D' Alembertian operator (3.24).

In section-3.3 , we have discussed the various quantum equation of dyons in split octonion formulation. Equation (3.25) represent the split octonion form of generalized four - potential of dyons and equation (3.26) describes the field equation of dyons in split octonion formalism. The Lorentz gauge condition for the dynamics of electric and magnetic charges of dyon is expressed by equation (3.27) and the equation (3.28) is the reduced form of field equation (3.26). The split octonion equivalent of generalized electromagnetic fields of dyons in terms of 2×2 Zorn's vector matrix realization is given by equation (3.29). Equation (3.30) may be visualized the generalized electric and magnetic fields associated with dyons. The split octonion representation of generalized electromagnetic field vector $\mathbb{F}$ of dyons is defined in equation (3.31) in terms of 2×2 Zorn's vector matrix realization. Equation (3.32) represents the wave equation of dyons in split octonion realization, while equation (3.33) is the reduced form of equation (3.32). The split octonion equivalent of generalized four - current of dyons $\mathbb{J}$ in terms of 2×2 Zorn's vector matrix realization has been given by equation (3.34). Equation (3.35) and (3.36) are the split octonion equivalents of generalized Dirac-Maxwell's equations of dyons.

In section-3.4, we have described the case of generalized electromagnetic fields of dyons (particles carrying simultaneous existence of electric and magnetic charge) for which we may define generalized octonion valued potential wave equations, generalized octonion current wave equations and generalized octonion fields wave equations in eight dimensional formalism by combining two four-dimensional spaces. Equation (3.37) is the split octonion form of potential wave equation, whereas equation (3.38) is the expended form of equation (3.37) in terms of 2×2 Zorn's vector matrix. Equation (3.39) is the reduced compect form of potential wave equation (3.37) for generalized fields of dyons, while the equation (3.40) is usual and known form of equation (3.39). It is is shown that the electromagnetic wave equation is a second-order partial differential equation that describes the propagation of electromagnetic waves through a medium or in a vacuum. Accordingly, equation (3.41) has been derived as the split octonion current wave equation in terms of 2×2 Zorn's vector matrix realization. The compect form of equation (3.41) is derived by equation (3.42). Equation (3.44) are the split octonion current wave equation for the components of generalized fields of dyons after applying the Lorentz gauge condition for the case of electric and magnetic charges.

A continuity equation in physics is an equation that describes the transport of a conserved quantity. Since mass, energy, momentum, electric charge and other natural quantities are conserved under their respective appropriate conditions, a variety of physical phenomena may be described using continuity equations. Thus, the continuity equation is regarded as a special case of the more general transport equation. Equation (3.45) thus describes the split octonion field equation in compact form, whereas equation (3.46) and (3.47) are the field equation of dyons in terms of 2×2 Zorn's vector matrix. Equation (3.48) thus represents the continuity equation of generalized fields of dyons in split octonion formulation. The continuity equation says that if charge is moving out of a differential volume (i.e. divergence of current density is positive) then the amount of charge within that volume is going to decrease, so the rate of charge density is negative. Accordingly, equation (3.49) and (3.50) are investigated as the generalized wave equation of dyons in split octonion.

In section-3.5, we have briefly discussed the energy momentum conservation for the case of split octonion electrodynamics. Operating the generalized conjugate electromagnetic field of dyons $\overline{\mathbb{F}}$ on both sides of equation (3.33), we have derived the equation (3.51), whose left hand side part is expressed in terms of 2×2 Zorn's vector matrix realization and has been expended as equation (3.52) with its components are given in (3.53). Consequently, the right hand side part of equation (3.51) has been expended in terms of 2×2 Zorn's vector matrix realization ant thus expressed by equation (3.54) whose components are described in equation (3.55). The above analysis shows that the left and right hand sides of equations (3.51) resemble to one another only if the coefficients A, B, C, D and A', B', C', D' coincide to each other. As such, the equivalence of A and A' gives rise to equations (3.56) and (3.57). Hence, the expression (3.57) may be visualized as the *"work-energy theorem"* or "*Poynting Theorem*" to the case of generalized octonion electrodynamics. Poynting's theorem is analogous to the work-energy theorem of classical mechanics reproducing the continuity equation, so that it relates the energy stored in generalized electromagnetic field to the work done on a charge distribution , through energy flux. It should be noted that the Poynting theorem is not valid in electrostatics and magnetostatics, since electric and magnetic fields change with time when electromagnetic energy flows. Accordingly, the equation (3.58) is obtained for *"work-energy theorem"* or "*Poynting Theorem*". Equation (3.59) defines the energy due to electric field while equation (3.60) reproduced the energy due to magnetic field of dyons. As such, the total energy stored in generalized electromagnetic fields of dyons is expressed by equation (3.61). Hence, the energy density i.e. the energy per unit time per unit area transported by the field is called *"Poynting vector"* has been investigated by equation (3.62). It represents the directional energy flux density (the rate of energy transfer per unit area, in W/m^2) of an electromagnetic field. Furthermore, the Poynting theorem has been generalized as the conservation of energy by equation (3.63). On the other hand, equivalence of B and B' of equation (3.53) and (3.55) provides equation (3.64) which has been investigated to the generalized Dirac-Maxwell's (GDM) equations of dyons given by equation (3.65). In order to develop the conservation of momentum and generalized electrodynamics, we have equated C and C' of equation (3.53) and accordingly obtained equation (3.66) which on using the identities (3.67) provides equation (3.59). It is emphasized that the equation (3.69) gives rise the connection between electromagnetic energy and the force due to presence of electric and magnetic energy of dyons. The Maxwell stress tensor is a second rank tensor which provides the force in classical electromagnetism to represent the interaction between electromagnetic forces and mechanical momentum. The Maxwell stress tensor for generalized electromagnetic field has been developed in terms of total nine components expressed by equation (3.71). The divergence of Maxwell stress tensor has been obtained as equation (3.72). Equation (3.74) gives rise the expression of *"generalized electromagnetic force"* of dyons. Equation (3.76) and (3.77) respectively describe the conservation of momentum and continuity equation. Like wise comparison of D and D' of equations (3.53) and (3.55) also gives the equation (3.78) which can be further be reduced to generalized Dirac Maxwell's (GDM) equations. In section-3.6, we have discussed the split octonion realization for Lorentz invariants of generalized electromagnetic fields of dyons. Here we have operated equation (3.33) by $\mathbb{F}$ from the left instead of right and accordingly derived equation (3.79). Following the previous method of equating the coefficients we have derived equations (3.80) - (3.91). Accordingly, we have obtained the first Lorentz invariant of $\left(\vec{E}^2 - \vec{H}^2\right)$ in terms of equation (3.85), second Lorentz invariant of $\left(\overrightarrow{E}.\overrightarrow{H}\right)$ in terms of equation (3.87). Hence, the gradient of first Lorentz invariant i.e. $\overrightarrow{\nabla}\left(\vec{E}^2 - \vec{H}^2\right)$ has been derived by equation (3.89) while the gradient of second Lorentz invariant i.e. $\overrightarrow{\nabla}\left(\overrightarrow{E}.\overrightarrow{H}\right)$ has been investigated by equation (3.91). It should be noted that the theory of classical electrodynamics has been generalized consistently to the case of dyons by means of split octonions and it is shown that fore going analysis is compact, simpler and manifestly covariant.

Bibliography

[1] P. A. M. Dirac, **"Quantized Singularities in the Electromagnetic Field"**, Proc. Roy. Soc London, **A133** (1931), 60.

[2] J. Schwinger, **"Magnetic Charge and Quantum Field Theory"**, Phys. Rev., **144** (1966), 1087.

[3] D. Zwanziger, **"Quantum Field Theory of Particles with Both Electric and Magnetic Charges"**, Phys. Rev., **176** (1968), 1489.

[4] C. N. Yang and T. T. Wu, **"Dirac monopole without strings: Monopole Harmonics",** Nucl. Phys., **B107** (1976), 365.

[5] F. Rohrlich, **"Classical Theory of Magnetic Monopoles",** Phys. Rev., **150** (1966), 1104.

[6] P. Goddard and D. Olive, **"Magnetic monopoles in gauge field theories"**, Rep. Prog. Phys., **41** (1978), 1357.

[7] N. Craigie, **"Theory and Detection of magnetic Monopoles in Gauge Theories"**, World Scientific, Singapur, (1986).

[8] G. 't Hooft, **"Magnetic monopole in unifield gauge theories"**, Nucl. Phys., **B79** (1974), 276.

[9] A. M. Polyakov, **"Particle spectrum in quantum field theory"**, JETP Lett., **20** (1974), 194.

[10] B. Julia and A. Zee, **"Poles with both magnetic and electric charges in non-Abelian gauge theory"**, Phys. Rev., **D11** (1975), 2227.

[11] E. Witten, **"Dyons of charge $e\vartheta/2\pi$",** Phys. Lett., **B86** (1979), 283.

[12] P. S. Bisht, O. P. S. Negi and B. S. Rajput, **"Null-tetrad formulation of dyons"**, Nuovo Cimento, **104A** (1991), 337.

[13] P. S. Bisht, O. P. S. Negi and B. S. Rajput, **"Quaternion gauge theory of dyonic fields"**, Prog. Theo. Phys., **85** (1991), 157.

[14] J. C. Maxwell, **"A dynamical theory of the electromagnetic field",** Philosophical Transactions of the Royal Society of London, **155** (1865), 492.

[15] O. Heaviside, **"Electromagnetic Theory"**, The Electrician Printing and Publishing Co., London, (1894).

[16] A. Einstein, **"Importance of Static Gravitational Fields Theory"**, Phys. Z., **14** (1913), 1261.

[17] H. Thirring, **"On the Effect of Rotating Distant Masses in Einstein's Theory of Gravitation"**, Zeitschrift fi¿œr Physik, **19** (1918), 204.

[18] J. Lense and H. Thirring, **"On the Influence of the Proper Rotation of Central Bodies on the Motions of Planets and Moons According to Einstein's Theory of Gravitation"**, Phys. Z., **19** (1918), 156.

[19] M. L. Ruggiero and A. Tartaglia, **"Gravito magnetic effects",** arXiv:gr-qc/ 027065 v2, (2002).

[20] C. Castro, **"On the noncommutative and nonassociative geometry of octonionic space time, modified dispersion relations and grand unification",** J. Math. Phys. **48** (2007), 73517 .

[21] L. E. Dickson, **"On Quaternions and Their Generalization and the History of the Eight Square Theorem",** Ann. Math., **20** (1919), 155.

[22] W. R. Hamilton, **"Elements of quaternions**", Chelsea Publications Co., New York, (1969).

[23] P. G. Tait, **"An elementary Treatise on Quaternions**", Oxford Univ. Press, New York, (1875).

[24] D. Finklestein, J. M. Jauch, S. Schiminovich and D. Speiser, **"Principle of general quaternion covariance"**, J. Math. Phys., **4** (1963), 788.

[25] S. L. Adler, **"Quaternion Quantum Mechanics and Quantum Fields**", Oxford Univ. Press, New York, (1995).

[26] B. S. Rajput, S. R. Kumar and O. P. S. Negi, **"Quaternionic formulation for dyons",** Lett. Nuovo Cimento, **34** (1982), 180 .

[27] V. Majernik, **"Quaternionic formulation of the classical fields"**, Adv. Cliff. Alg., **9** (1999), 119.

[28] V. V. Kravchenkov, **"Applied Quaternion Analysis**", Helderman Verlag, Germany (2003).

[29] Shalini Bisht, P. S. Bisht and O. P. S. Negi, **"Revisiting quaternion formulation and electromagnetism",** Nuovo Cimento, **B113** (1998), 1449.

[30] K. Morita, **"Octonions, Quarks and QCD",** Prog. Theor. Phys., **65** (1981), 787 .

[31] R. P. Graves, "**Life of Sir William Rowan Hamilton**", 3 volumes, Arno Press, New York, (1975).

[32] J. C. Baez, **"The Octonions"**, Bull. Amer. Math. Soc., **39** (2001), 145.

[33] S. Catto, **"Exceptional Projective Geometries and Internal Symmetries"**, hep-th/0302079 v1 (2003).

[34] R. Foot and G. C. Joshi, **"Space-time symmetries of super string and Jordan algebras",** Int. J. Theor. Phys., **28** (1989), 1449 .

[35] J. Lukierski and F. Toppan, **"Generalized space-time supersymmetries, division algebras and octonionic M-theory"**, Phys. Lett., **B539** (2002), 266.

[36] J. Schray, **"Octonions and Supersymmetry"**, Ph.D. thesis, Department of Physics, Oregon State University, Corvallis, (1994).

[37] K. Imaeda, "**Quaternionic formulation of tachyons, superluminal transformations and a complex space-time**", Lett. Nuovo Cimento, **50** (1979), 271.

[38] R. Penny, **"Octonions and the Dirac equation"**, Amer. J. Phys., **36** (1968), 871.

[39] A. Gamba, **"Peculiarities of the Eight Dimensional Space"**, J. Math. Phys., **8** (1967), 775.

[40] M. Gogberashvili, **"Octonionic electrodynamics"**, J. Phys. A: Math.Gen., **39** (2006), 7099.

[41] P. S. Bisht and O. P. S. Negi, **"Quaternion-Octonion Analyticity for Abelian and non-Abelian gauge theories of Dyons"**, Int. J. Theor. Phys., **47** (2008), 1497.

[42] P. S. Bisht, B. Pandey and O. P. S. Negi, **"Octonion wave Equation and Generalized fields of Dyons"**, Proc. National Symp. Mathematical Sciences, Nagpur Ed. By T. M. Karade and G. S. Khadekar, **12** (2001), 137.

[43] B. C. Chanyal, P. S. Bisht and O. P. S. Negi, **"Generalized Octonion Electrodynamics"**, Int. J. Theor. Phys., **49** (2010), 137.

[44] B. C. Chanyal, P. S. Bisht and O. P. S. Negi, **"Generalized Split-Octonion Electrodynamics"**, Int. J. Theor. Phys., **50** (2011), 1919.

[45] D. J. Griffiths, "**Introduction to Electrodynamics**", (Book) Pearson Prentice Hall, Third Edition (2006).

[46] V. L. Mironov and S. V. Mironov, "**Octonic representation of electromagnetic field equations**", J. Math. Phys., **50** (2009), 12901.

CHAPTER 4

Octonion Electrodynamics in chiral medium

A part of this chapter is communicated to the publication in the Int. J. Theor. Phys.

ABSTRACT

Keeping in view the consequences of the present theory of dyons in isotropic medium, we have undertaken the study of the octonion analysis of time dependent Generalized Dirac - Maxwell's equations of dyons in chiral medium. Consequently, the octonionic forms of potential, field and current equation are developed in simple and compact manners in the case of homogeneous (isotropic) medium and it is emphasized that the corresponding quantum equations derived in terms of octonions are invariant under Lorentz and duality transformations. Accordingly, the generalized electrodynamics in chiral medium has been developed in terms of compact and simpler forms of octonion representations in presence of electric and magnetic charges of dyons.

Chapter 4

Octonion Electrodynamics in chiral medium

4.1 Introduction

There has been a revival in the formulation of natural laws so that there exists [1] four division algebras consisting the algebra of real numbers, complex numbers, quaternions and octonions. Octonion analysis has been widely discussed by Baez [2]. It has also played an important role in the context of various physical problems of higher dimensional supersymmetry, supergravity and super strings etc. Few interest in the subject of monopoles and dyons was enhanced by the work of t' Hooft [3] and polyakov [4] and its extension by Julia and Zee [5]. The work of the Schwinger [6] was the first exception to the argument against the existence of monopoles. At the same time so many paradoxes were related to the theory of pure Abelian monopoles, as Dirac's veto [7, 8], wrong spin-statistics connection [9] and many others [10, 11]. Several problems were soon resolved by the invention of dyons [12, 13, 14] particles carrying simultaneous existence of electric and magnetic charge. Fresh interest in this subject was enhanced by the idea given by t' Hooft [15] and Polyakov [16] showing that monopoles are the intrinsic parts of grand unified theories. The Dirac monopoles is an elementary particle but the t' Hooft - Polyakov monopoles [15, 16] is complicated extended object having a definite mass and finite size inside of which massive fields plays a role in providing a smooth structure and outside it they vanish rapidly leaving the field configuration identical to abelian Dirac monopole. Julia and Zee [17] have extended the idea of t' Hooft [15] and polyakov [16] to construct the classical solutions for non-Abelian dyons. Consequently, Prasad and Sommerfield [18, 19] have derive the analytic stable solutions for the non-Abelian monopoles and dyons of finite mass by keeping the symmetry of vacuum broken but letting the self-interaction of Higgs field approaching zero. Such solutions, satisfying the Bogomonlys condition [20] are described as Bogomolnys-Prasad-Sommerfield (BPS) monopoles. On the other hand, quaternions were invented by Hamilton [21] to extend the theory of complex numbers to three dimensions. The quaternion formalism has been rediscovered at regular intervals and the Maxwell's differential equations are rewritten as one quaternion equations [22, 23]. Finklestein et al [24] developed quaternionic quantum mechanics and Adler [25] described the theory of the algebraic structure of quantum chromo dynamics for strong interactions. Naturally, although quaternions from noncommutative but associative algebra in four dimensions, octonions [2] passes eight components and their algebra is both non commutative and non associative. Sedenions also carry noncommutative and nonassociative algebra in sixteen dimensions in the same way as complex octonions [26]. Quaternions and octonions are defined with different dimensions and structures, be it real, complex, split, dual, and hyperbolic. These algebras are successfully used in many fields and represent the problems found in physics. In physics Clifford algebra [27] features highly in various studies, also quaternions and octonions. Kravchenko and co-authers [28, 29] discussed the Maxwell's equations in homogeneous media and accordingly developed [30] the quaternionic reformulation of the time-dependent Maxwell's equations along with the classical solution of a moving source i.e. electron. Kravchenko et. al [31] have also demonstrated the electromagnetic fields in chiral media and their quaternionic form in a simple and consistent manner. The potential importance of monopole and the results of Witten [32] that monopoles are necessarily dyon, Bisht et. al [33] have constructed a self-consistent co-variant theory of generalized electromagnetic fields associated with dyons each carrying the generalized charge as complex quantity with its real and imaginary part as electric and magnetic constituents. Recently, the generalized Dirac Maxwell's equations in homogeneous (isotropic) medium [34, 35] and their quaternionic forms have been discussed in a unique and consistent way. The solution of time independent generalized Dirac Maxwell's (GDM) equations in presence of electric and magnetic sources have been discussed [35, 36] in chiral media and inhomogeneous media.

In this chapter, we have discussed the octonion electrodynamics in homogeneous (isotropic) and chiral medium. In section (4.2), we have discussed the definition of chiral medium. The chiral media are isotropic birefringent substances that responses to either electric or magnetic excitation with both electric and magnetic polarizations. In section (4.3), we have obtained the generalized electromagnetic fields equations of dyons in isotropic medium. Thus, we have derived the generalized Dirac-Maxwell's equations and other various quantum equations in the homogeneous (isotropic) medium. It has been shown that the field equations of dyons remain invariant under the duality transformations in isotropic homogeneous medium and the equation of motion reproduces the rotationally symmetric gauge invariant angular momentum of dyons. Keeping in view the consequences of the present theory of dyons in isotropic medium, we have also undertaken the study of the octonion analysis of time dependent Maxwell's equations in chiral medium for dyons in presence of electric and magnetic charges (sources) are obtained in unique, simpler and consistent manner. Thus, in section (4.4), we have discussed the generalized octonion Maxwell's equations in the case of isotropic medium. Accordingly, the octonionic forms of potential, field and current equation are developed in simple and compact manners in the case of homogeneous (isotropic) medium and it is emphasized that the corresponding quantum equations derived in terms of octonions are invariant under Lorentz and duality transformations. Thus, we have discussed the generalized octonion electrodynamics in chiral medium (section 4.5). It provides the field equations, wave equations and other quantum equations of dyons in the case of chiral medium by means of octonionic eight dimensional representations. As such, we have described the chiral parameter and pairing constant in terms of octonionic representation of Drude-Born-Fedorov constitutive relations. Hence, we have derived the generalized theory of Dirac-Maxwell's equations in presence of electric and magnetic charges of dyons in the case of chiral media in simple, compact and consistent manner.

4.2 Chiral Media

Chiral media [37, 38] are isotropic birefringent substances that respond to either electric or magnetic excitation with both electric and magnetic polarizations. The understanding of the properties of such media, the differences from ordinary dielectrics, and their possible applications require detailed mathematical modeling. The mathematical modeling of chiral media is done [37, 38] through the modification of the constitutive relations for normal dielectrics. For a normal dielectric material the electric displacement $\vec{D}$ depends solely on the electric field $\vec{E}$, and the magnetic field $\vec{B}$ depends on the magnetic induction $\vec{H}$, while in a chiral medium, $\vec{D}$ and $\vec{B}$ depend on a combination of $\vec{E}$ and $\vec{H}$ [39, 40]. In many cases of interest these constitutive laws describe non-local relations containing $\vec{E}$ and $\vec{H}$. This is a common model for time-dispersive chiral media. Also these constitutive laws may be either linear or nonlinear relations of the fields corresponding to the modeling of linear or nonlinear chiral media respectively. Let us discuss the octonion reformulation of generalized fields of dyons in normal (isotropic / homogeneous) and chiral medium.

4.3 Generalized Electromagnetic Fields of Dyons in Isotropic Medium

In order to write the various quantum equations of dyons in isotropic medium, we start with the definition of homogeneous (isotropic) medium [41] in the generalized electromagnetic fields as,

$$\vec{D} = \epsilon \vec{\mathcal{E}} \quad (\epsilon = \epsilon_0 \epsilon_r), \tag{4.1}$$

and

$$\vec{\mathcal{B}} = \mu \vec{H} \quad (\mu = \mu_0 \mu_r); \tag{4.2}$$

where $\vec{D}$ and $\vec{\mathcal{B}}$ are respectively the electric and magnetic induction vectors while $\vec{\mathcal{E}}$ and $\vec{H}$ are generalized electromagnetic fields, described in chapter-2. Here ϵ_0 is the free space permittivity, μ_0 is the permeability of free space, ϵ_r and μ_r are defined respectively as relative permittivity and permeability associated with electric and magnetic fields. So the following identity be used

[34] as

$$\mu\epsilon = \frac{1}{v^2}, \tag{4.3}$$

which leads to

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\mu_r \epsilon_r}}, \tag{4.4}$$

where

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}, \tag{4.5}$$

is the velocity of light in free space (vacuum) and v is considered as the speed of electromagnetic wave in homogeneous (isotropic) medium. On using the equations (4.1) and (4.2), the Maxwell's equations (2.30), given in chapter-2 take the following differential form,

$$\begin{aligned}
\vec{\nabla} \cdot \vec{\mathcal{E}} &= \frac{\rho}{\epsilon}; \\
\vec{\nabla} \cdot \vec{\mathcal{B}} &= \mu \varrho; \\
\vec{\nabla} \times \vec{\mathcal{E}} &= -\frac{\partial \vec{\mathcal{B}}}{\partial t} - \frac{\tilde{\mathrm{k}}}{\epsilon}; \\
\vec{\nabla} \times \vec{\mathcal{B}} &= \frac{1}{v^2} \frac{\partial \vec{\mathcal{E}}}{\partial t} + \mu \vec{\mathrm{j}}.
\end{aligned} \tag{4.6}$$

Differential equations (4.6) are thus referred as the generalized field equations of dyons in homogeneous (isotropic) medium and the corresponding electric and magnetic fields are accordingly referred as the generalized electromagnetic fields of dyons in isotropic medium. These electric and magnetic fields of dyons in homogeneous (isotropic) medium may then be written in terms of two potentials [34] as

$$\overrightarrow{\mathcal{E}} = - \frac{\partial \overrightarrow{A}}{\partial t} - \overrightarrow{\nabla}\phi - \overrightarrow{\nabla} \times \overrightarrow{B}; \tag{4.7}$$

$$\overrightarrow{\mathcal{B}} = -\frac{1}{v^2}\frac{\partial \overrightarrow{B}}{\partial t} - \overrightarrow{\nabla}\varphi + \overrightarrow{\nabla} \times \overrightarrow{A}. \tag{4.8}$$

Here in the equations (4.7) and (4.8) $\{A^{\mu}\} = \left\{\phi, v\vec{A}\right\}$ and $\{B^{\mu}\} = \left\{v\varphi, \vec{B}\right\}$ are the potentials respectively associated with electric and magnetic charges, consisting of two vector potentials ($\vec{A}$, $\vec{B}$) and two scalar potentials (ϕ, φ) due to the presence of electric and magnetic charges.

So, the electric and magnetic fields are symmetrically invariant under the following transformations as

$$\begin{aligned}
\overrightarrow{\mathcal{E}} &\to v\overrightarrow{\mathcal{B}};\\
\overrightarrow{\mathcal{B}} &\to -\frac{\overrightarrow{\mathcal{E}}}{v};\\
\phi &\to v\varphi;\\
\varphi &\to -\frac{\phi}{v};\\
\vec{\mathrm{j}} &\to v\vec{k};\\
\vec{k} &\to -\frac{\vec{\mathrm{j}}}{v};\\
\rho &\to \frac{\varrho}{v};\\
\varrho &\to -v\rho.
\end{aligned} \tag{4.9}$$

Maxwell's equations (4.6) are thus invariant under the generalized continuous linear transformations [34];

$$\overrightarrow{\mathcal{E}} = \overrightarrow{\mathcal{E}}\cos\theta + \overrightarrow{\mathcal{B}}v\sin\theta; \tag{4.10}$$

and

$$\overrightarrow{\mathcal{B}}v = -\overrightarrow{\mathcal{E}}\sin\theta + \overrightarrow{\mathcal{B}}v\cos\theta. \tag{4.11}$$

which reduces to equation $\overrightarrow{\mathcal{E}} \to v\overrightarrow{\mathcal{B}}$ and $\overrightarrow{\mathcal{B}} \to -\frac{\overrightarrow{\mathcal{E}}}{v}$ for $\theta = \frac{\pi}{2}$ and thus are recalled as duality transformations. Consequently, these two equations (4.10) and (4.11) are also expressed as duality transformations between electric and magnetic constituents of dyons. As such, the generalized Dirac-Maxwell's (GDM) equations given by equations (4.6) automatically be considered as manifestly covariant and dual invariant field equations of dyons moving in isotropic (homogeneous) medium. Defining the complex vector field $\vec{\psi}$ as

$$\vec{\psi} = \overrightarrow{\mathcal{E}} - iv\overrightarrow{\mathcal{B}}, \tag{4.12}$$

and using the equations (4.7) and (4.8), we get the following relations between generalized field $\vec{\psi}$ and the components of four-potential as

$$\vec{\psi} = -\frac{\partial\vec{V}}{\partial t} - \vec{\nabla}\Phi - iv\left(\vec{\nabla}\times\vec{V}\right), \tag{4.13}$$

where $\{V_\mu\}$ is the generalized four-potential of dyons in homogeneous medium with its constituents as,

$$V_\mu = \left\{\Phi, \vec{V}\right\}; \tag{4.14}$$

where

$$\Phi = \phi - iv\varphi. \tag{4.15}$$

As such we may write the Maxwell's field (GDM) equations in terms of generalized field $\vec{\psi}$ (4.13) as follows,

$$\begin{aligned}\vec{\nabla}\cdot\vec{\psi} =&\frac{\boldsymbol{\rho}}{\epsilon};\\ \vec{\nabla}\times\vec{\psi} =&-iv\left(\mu\vec{J}+\frac{1}{v^{2}}\frac{\partial\vec{\psi}}{\partial t}\right).\end{aligned} \tag{4.16}$$

Here $\boldsymbol{\rho}$ and $\vec{J}$ are the generalized charge and current source densities of dyons in homogeneous medium described as

$$\begin{aligned}\boldsymbol{\rho} =&\left(\rho-i\frac{\varrho}{v}\right);\\ \vec{J} =&\left(\vec{j}-iv\vec{k}\right).\end{aligned} \tag{4.17}$$

So, the new parameter $\vec{S}$ be expressed in the following form in terms of source densities [34] as

$$\vec{S}=\Box\vec{\psi}=-\mu\frac{\partial\vec{J}}{\partial t}-\frac{1}{\epsilon}\vec{\nabla}\boldsymbol{\rho}-iv\mu\left(\vec{\nabla}\times\vec{J}\right) \tag{4.18}$$

where $\Box$ is the D' Alembertian operator. Thus, the Generalized Dirac-Maxwell's (GDM) equations are expressed as the wave equations in terms of generalized four potential of dyon [34] as

$$\begin{aligned}\Box\Phi =&v\mu\boldsymbol{\rho};\\ \Box\vec{V} =&\mu\vec{J}.\end{aligned} \tag{4.19}$$

4.4 Generalized Octonion Maxwell's Equations for Isotropic Medium

In order to write the quantum equations of dyons in isotropic media in terms of compact notations of octonions, let us start with the four dimensional representation of differential operator [42, 43] expressed in terms of octonion units as

$$\boxdot = e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z} - \frac{i}{v}e_7\frac{\partial}{\partial t}, \tag{4.20}$$

whereas the consequent octonion conjugate differential operator is defined as

$$\overline{\boxdot} = -e_1\frac{\partial}{\partial x} - e_2\frac{\partial}{\partial y} - e_3\frac{\partial}{\partial z} + \frac{i}{v}e_7\frac{\partial}{\partial t}. \tag{4.21}$$

So, the octonionic potential $\mathbb{V}$ (2.91) may now be expressed in isotropic medium as

$$\begin{aligned}\mathbb{V} =& e_1(A_x + ie_7\frac{B_x}{v}) + e_2(A_y + ie_7\frac{B_y}{v}) + e_3(A_z + ie_7\frac{B_z}{v}) + (\varphi + ie_7\frac{\phi}{v}) \\ =& e_1\mathrm{V_x} + e_2\mathrm{V_y} + e_3\mathrm{V_z} + ie_7\emptyset \end{aligned} \tag{4.22}$$

where $(\emptyset, \mathrm{V_x}, \mathrm{V_y}, \mathrm{V_z}) = (\emptyset, \overrightarrow{\mathrm{V}}) = \{\mathrm{V}_\mu\}$ are the components of generalized four potential $\{\mathrm{V}_\mu\}$ associated with generalized charge $(q = e - i\,g)$ (where e and g are respectively known as electric and magnetic charges) of dyons.

Now operating $\overline{\boxdot}$ of the equation (4.21) to octonion potential $\mathbb{V}$ (4.22), we get

$$\begin{aligned}\boxdot\mathbb{V} =& e_1(-\frac{\partial\varphi}{\partial x}+\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}-\frac{1}{v^2}\frac{\partial B_x}{\partial t})\\ &+e_2(-\frac{\partial\varphi}{\partial y}+\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial zx}-\frac{1}{v^2}\frac{\partial B_y}{\partial t})\\ &+e_3(-\frac{\partial\varphi}{\partial z}+\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}-\frac{1}{v^2}\frac{\partial B_z}{\partial t})\\ &-ie_4\frac{1}{v}(-\frac{\partial\phi}{\partial x}-\frac{\partial B_z}{\partial y}+\frac{\partial B_y}{\partial z}-\frac{\partial A_x}{\partial t})\\ &-ie_5\frac{1}{v}(-\frac{\partial\phi}{\partial y}-\frac{\partial B_x}{\partial z}+\frac{\partial B_z}{\partial x}-\frac{\partial A_y}{\partial t})\\ &-ie_6\frac{1}{v}(-\frac{\partial\phi}{\partial z}-\frac{\partial B_y}{\partial x}+\frac{\partial B_x}{\partial y}-\frac{\partial A_z}{\partial t}).\end{aligned} \tag{4.23}$$

which on the application of following Lorentz Gauge condition ,

$$\begin{aligned}\overrightarrow{\nabla}\cdot\overrightarrow{A}+\frac{1}{v^2}\frac{\partial\phi}{\partial t} =& 0;\\ \overrightarrow{\nabla}\cdot\overrightarrow{B}+\frac{1}{v^2}\frac{\partial\varphi}{\partial t} =& 0;\end{aligned} \tag{4.24}$$

reduces to the following octonion form as

$$\boxdot\mathbb{V} = \mathbb{F}; \tag{4.25}$$

Here the generalized electromagnetic field $\mathbb{F}$ is discussed in chapter-2. The components of the generalized electromagnetic field $\mathbb{F}$ may then be written as

$$\begin{aligned}F_0 =& F_7 = 0;\\ F_1 =& (-\frac{\partial\varphi}{\partial x}+\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}-\frac{1}{v^2}\frac{\partial B_x}{\partial t});\\ F_2 =& (-\frac{\partial\varphi}{\partial y}+\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial zx}-\frac{1}{v^2}\frac{\partial B_y}{\partial t});\end{aligned}$$

$$\begin{aligned}
F_3 =&(-\frac{\partial\varphi}{\partial z}+\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}-\frac{1}{v^2}\frac{\partial B_z}{\partial t});\\
F_4 =&-\frac{i}{v}(-\frac{\partial\phi}{\partial x}-\frac{\partial B_z}{\partial y}+\frac{\partial B_y}{\partial z}-\frac{\partial A_x}{\partial t});\\
F_5 =&-\frac{i}{v}(-\frac{\partial\phi}{\partial y}-\frac{\partial B_x}{\partial z}+\frac{\partial B_z}{\partial x}-\frac{\partial A_y}{\partial t});\\
F_6 =&-\frac{i}{v}(-\frac{\partial\phi}{\partial z}-\frac{\partial B_y}{\partial x}+\frac{\partial B_x}{\partial y}-\frac{\partial A_z}{\partial t});
\end{aligned} \tag{4.26}$$

which gives rise the connection with generalized electromagnetic fields as

$$\begin{aligned}
F_1 \longmapsto&\mathcal{B}_x, \qquad F_4 \longmapsto -i\frac{\mathcal{E}_x}{v};\\
F_2 \longmapsto&\mathcal{B}_y, \qquad F_5 \longmapsto -i\frac{\mathcal{E}_y}{v};\\
F_3 \longmapsto&\mathcal{B}_z, \qquad F_6 \longmapsto -i\frac{\mathcal{E}_z}{v}.
\end{aligned} \tag{4.27}$$

Thus, the generalized electromagnetic field of dyons $\mathbb{F}$ (2.98) may now be expressed in the following octonionic form in isotropic medium i.e.

$$\begin{aligned}
\mathbb{F} =&e_1(\mathcal{B}_x+ie_7\frac{\mathcal{E}_x}{v})+e_2(\mathcal{B}_y+ie_7\frac{\mathcal{E}_y}{v})+e_3(\mathcal{B}_z+ie_7\frac{\mathcal{E}_z}{v})\\
=&e_1\Psi_x+e_2\Psi_y+e_3\Psi_z
\end{aligned} \tag{4.28}$$

where $\overrightarrow{\Psi} = \overrightarrow{\mathcal{B}} + i\, e_7 \frac{\overrightarrow{\mathcal{E}}}{v}$ is the generalized vector field of dyons. Now applying the differential operator $\boxdot$ to the equation (4.20) , we get

$$\begin{aligned}
\boxdot\,\mathbb{F} =&-e_0(\overrightarrow{\nabla}.\overrightarrow{\mathcal{B}})+e_1[(\overrightarrow{\nabla}\times\overrightarrow{\mathcal{B}})_x-\frac{1}{v^2}\frac{\partial\mathcal{E}_x}{\partial t}]\\
&+e_2[(\overrightarrow{\nabla}\times\overrightarrow{\mathcal{B}})_y-\frac{1}{v^2}\frac{\partial\mathcal{E}_y}{\partial t}]+e_3[(\overrightarrow{\nabla}\times\overrightarrow{\mathcal{B}})_z-\frac{1}{v^2}\frac{\partial\mathcal{E}_z}{\partial t}]\\
&-ie_4\frac{1}{v}[(\overrightarrow{\nabla}\times\overrightarrow{\mathcal{E}})_x-\frac{\partial\mathcal{B}_x}{\partial t}]-ie_5\frac{1}{v}[(\overrightarrow{\nabla}\times\overrightarrow{\mathcal{E}})_y-\frac{\partial\mathcal{B}_y}{\partial t}]\\
&-ie_6\frac{1}{v}[(\overrightarrow{\nabla}\times\overrightarrow{\mathcal{E}})_z-\frac{\partial\mathcal{B}_z}{\partial t}]+ie_7\frac{1}{v}(\overrightarrow{\nabla}.\overrightarrow{\mathcal{E}}).
\end{aligned} \tag{4.29}$$

So, the wave equation in isotropic medium of generalized octonion may be written as

$$\boxdot \mathbb{F} = \mathbb{J}; \tag{4.30}$$

where $\mathbb{J}$, the generalized octonion current discussed in chapter-2, may be written in the following manner in isotropic (chiral) medium, i.e.

$$\mathbb{J} = \mu(\mathrm{e_0}\varrho + \mathrm{e_1 j_x} + \mathrm{e_2 j_y} + \mathrm{e_3 j_z}) - \frac{1}{\epsilon} \cdot \frac{\mathrm{i}}{\mathrm{v}}(\mathrm{e_4 k_x} + \mathrm{e_5 k_y} + \mathrm{e_6 k_z} + \mathrm{e_7}\rho), \tag{4.31}$$

where $(\rho, \overrightarrow{j}) = \{\mathrm{j}_\mu\}$, $(\varrho, \overrightarrow{j}) = \{\mathrm{k}_\mu\}$ and $(\mathrm{J_0}, \overrightarrow{J}) = \{\mathrm{J}_\mu\}$ are respectively the four currents associated with electric charge, magnetic monopole and generalized fields of dyons. Thus, the equation (4.30) lead to following differential equations

$$\begin{aligned}
(\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{B}}) &= \mu\varrho; \\
(\overrightarrow{\nabla} \cdot \overrightarrow{\mathcal{E}}) &= \frac{\rho}{\epsilon}; \\
(\overrightarrow{\nabla} \times \overrightarrow{\mathcal{B}})_x &= \frac{1}{v^2}\frac{\partial E_x}{\partial t} + \mathrm{j_x}; \\
(\overrightarrow{\nabla} \times \overrightarrow{E})_x &= -\frac{\partial \mathcal{B}_x}{\partial t} - \frac{\mathrm{k_x}}{\epsilon}; \\
(\overrightarrow{\nabla} \times \overrightarrow{\mathcal{B}})_y &= \frac{1}{v^2}\frac{\partial E_y}{\partial t} + \mathrm{j_y}; \\
(\overrightarrow{\nabla} \times \overrightarrow{E})_y &= -\frac{\partial \mathcal{B}_y}{\partial t} - \frac{\mathrm{k_y}}{\epsilon}; \\
(\overrightarrow{\nabla} \times \overrightarrow{\mathcal{B}})_z &= \frac{1}{v^2}\frac{\partial E_z}{\partial t} + \mathrm{j_z}; \\
(\overrightarrow{\nabla} \times \overrightarrow{E})_z &= -\frac{\partial \mathcal{B}_z}{\partial t} - \frac{\mathrm{k_z}}{\epsilon};
\end{aligned} \tag{4.32}$$

which may further be generalized as

$$\begin{aligned}
\overrightarrow{\nabla}\cdot\overrightarrow{\mathcal{B}} =&\mu\varrho;\\
\overrightarrow{\nabla}\cdot\overrightarrow{\mathcal{E}} =&\frac{\rho}{\epsilon};\\
\overrightarrow{\nabla}\times\overrightarrow{\mathcal{B}} =&\frac{1}{v^2}\frac{\partial\vec{\mathcal{E}}}{\partial t}+\mu\vec{\mathrm{j}};\\
\overrightarrow{\nabla}\times\overrightarrow{\mathcal{E}} =&-\frac{\partial\overrightarrow{\mathcal{B}}}{\partial t}-\frac{\tilde{\mathrm{k}}}{\epsilon};
\end{aligned} \tag{4.33}$$

These differential equations are analogous to the Generalized Maxwell's Equations (GDM) of dyons in isotropic medium given by equation (4.6).

4.5 Generalized Octonion Electrodynamic in Chiral Medium

In chiral medium [36-38], the electric and magnetic fields are paired with each other. As such the constitutive relations $\vec{D}=\epsilon\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}=\mu\vec{H}$ will be written in paired form as

$$\begin{aligned}
\vec{D} =&\epsilon\vec{\mathcal{E}}+\epsilon'\vec{H} \qquad \Longrightarrow \qquad \vec{D}=\epsilon\left(\vec{\mathcal{E}}+\beta\left(\nabla\times\vec{\mathcal{E}}\right)\right)\\
\vec{\mathcal{B}} =&\mu\vec{H}+\mu'\vec{\mathcal{E}} \qquad \Longrightarrow \qquad \vec{\mathcal{B}}=\mu\left(\vec{H}+\beta\left(\nabla\times\vec{H}\right)\right)
\end{aligned} \tag{4.34}$$

where β is chiral parameter and ϵ', μ' are pairing constant, and these relation are known as Drude-Born-Fedorov constitutive relations [37, 38], which are expressed as

$$\begin{aligned}
\vec{\mathcal{B}}=&\left(\vec{\mathcal{B}}_x,\vec{\mathcal{B}}_y,\vec{\mathcal{B}}_z\right),\\
\vec{D}=&\left(\vec{D}_x,\vec{D}_y,\vec{D}_z\right);
\end{aligned} \tag{4.35}$$

where $\vec{\mathcal{B}}$, in the equation (4.35) is describes the octonion basis vectors as

$$\begin{aligned}\vec{\mathcal{B}}_x =&\mu H_x e_1 - e_7\mu\beta\left(\nabla\times H\right)_x\cdot e_4;\\ \vec{\mathcal{B}}_y =&\mu H_y e_2 - e_7\mu\beta\left(\nabla\times H\right)_y\cdot e_5;\\ \vec{\mathcal{B}}_z =&\mu H_z e_3 - e_7\mu\beta\left(\nabla\times H\right)_z\cdot e_6.\end{aligned}\tag{4.36}$$

Thus, equation (4.36) may now be written as

$$\vec{\mathcal{B}} =\mu\vec{H}e_j - e_7\mu\beta\left(\nabla\times H\right)\cdot e_{j+3};\qquad (j=1,2,3)\,.\tag{4.37}$$

Which represents the generalized magnetic field in chiral medium in terms of octonion. Similarly, the electric induction vector $\vec{D}$ form the equation (4.35) expressed in octonionic form as

$$\begin{aligned}\vec{D}_x =&\epsilon\mathcal{E}_x e_1 - e_7\epsilon\beta\left(\nabla\times\mathcal{E}\right)_x\cdot e_4;\\ \vec{D}_y =&\epsilon\mathcal{E}_y e_2 - e_7\epsilon\beta\left(\nabla\times\mathcal{E}\right)_y\cdot e_5;\\ \vec{D}_z =&\epsilon\mathcal{E}_z e_3 - e_7\epsilon\beta\left(\nabla\times\mathcal{E}\right)_z\cdot e_6;\end{aligned}\tag{4.38}$$

which may also be expressed as

$$\vec{D} =\epsilon\vec{\mathcal{E}}e_j - e_7\epsilon\beta\left(\nabla\times\mathcal{E}\right)\cdot e_{j+3};\qquad (j=1,2,3)\,.\tag{4.39}$$

So, the generalized electromagnetic field of dyons $\vec{\mathcal{F}}$ of octonion in chiral medium may then be expressed as

$$\vec{\mathcal{F}} =\vec{\mathcal{B}} + ie_7\vec{D}.\tag{4.40}$$

Substituting $\vec{\mathcal{B}}$ and $\vec{D}$ from the equations (4.37) and (4.39) into equation (4.40), we get

$$\vec{\mathcal{F}} = \left(\mu\vec{H}e_j - e_7\mu\beta\left(\nabla\times H\right)\cdot e_{j+3}\right) + ie_7\left(\epsilon\vec{\mathcal{E}}e_j - e_7\epsilon\beta\left(\nabla\times\mathcal{E}\right)\cdot e_{j+3}\right). \tag{4.41}$$

which may further be elaborated as

$$\begin{aligned}\vec{\mathcal{F}} = &\left(\mu H_x + ie_7\epsilon\mathcal{E}_x\right)\cdot e_1\\ &+\left(\mu H_y + ie_7\epsilon\mathcal{E}_y\right)\cdot e_2\\ &+\left(\mu H_z + ie_7\epsilon\mathcal{E}_z\right)\cdot e_3\\ &-e_7\beta\left(\mu(\nabla\times H)_x\right.\\ &\left.+ie_7\epsilon\left(\nabla\times\mathcal{E}\right)_x\right)\cdot e_4\\ &-e_7\beta\left(\mu(\nabla\times H)_y\right.\\ &\left.+ie_7\epsilon\left(\nabla\times\mathcal{E}\right)_y\right)\cdot e_5\\ &-e_7\beta\left(\mu(\nabla\times H)_z\right.\\ &\left.+ie_7\epsilon\left(\nabla\times\mathcal{E}\right)_z\right)\cdot e_6.\end{aligned} \tag{4.42}$$

We may also write the equation (4.42) in the following manner

$$\begin{aligned}\vec{\mathcal{F}} = &\left\{\mu H_x + \mu\beta\left(\nabla\times H\right)_x\right\}\cdot e_1 + \left\{\mu H_y + \mu\beta\left(\nabla\times H\right)_y\right\}\cdot e_2\\ &+\left\{\mu H_z + \mu\beta\left(\nabla\times H\right)_z\right\}\cdot e_3 + \left\{\epsilon\mathcal{E}_x + \epsilon\beta\left(\nabla\times\mathcal{E}\right)_x\right\}\cdot e_4\\ &+\left\{\epsilon\mathcal{E}_y + \epsilon\beta\left(\nabla\times\mathcal{E}\right)_y\right\}\cdot e_5 + \left\{\epsilon\mathcal{E}_z + \epsilon\beta\left(\nabla\times\mathcal{E}\right)_z\right\}\cdot e_6.\end{aligned} \tag{4.43}$$

Now operating $\boxdot$ from equation (4.20) to the chiral octonion field $\vec{\mathcal{F}}$ of the equation (4.43) , we get

$$
\begin{aligned}
&\boxdot\mathcal{F} = \\
&-\{\mu\frac{\partial}{\partial x}H_x+\mu\beta\frac{\partial}{\partial x}\left(\nabla\times H\right)_x+\mu\frac{\partial}{\partial y}H_y+\mu\beta\frac{\partial}{\partial y}\left(\nabla\times H\right)_y+\mu\frac{\partial}{\partial z}H_z \\
&\quad+\mu\beta\frac{\partial}{\partial z}\left(\nabla\times H\right)_z\}\cdot e_0 \\
&+\{\mu\frac{\partial}{\partial y}H_z+\mu\beta\frac{\partial}{\partial y}\left(\nabla\times H\right)_z-\mu\frac{\partial}{\partial z}H_y-\mu\beta\frac{\partial}{\partial z}\left(\nabla\times H\right)_y-\epsilon\frac{\partial}{\partial t}\mathcal{E}_x \\
&\quad-\epsilon\beta\frac{\partial}{\partial t}\left(\nabla\times\mathcal{E}\right)_x\}\cdot e_1 \\
&+\{\mu\frac{\partial}{\partial z}H_x+\mu\beta\frac{\partial}{\partial z}\left(\nabla\times H\right)_x-\mu\frac{\partial}{\partial x}H_z-\mu\beta\frac{\partial}{\partial x}\left(\nabla\times H\right)_z-\epsilon\frac{\partial}{\partial t}\mathcal{E}_y \\
&\quad-\epsilon\beta\frac{\partial}{\partial t}\left(\nabla\times\mathcal{E}\right)_y\}\cdot e_2 \\
&+\{\mu\frac{\partial}{\partial x}H_y+\mu\beta\frac{\partial}{\partial x}\left(\nabla\times H\right)_y-\mu\frac{\partial}{\partial y}H_x-\mu\beta\frac{\partial}{\partial y}\left(\nabla\times H\right)_x-\epsilon\frac{\partial}{\partial t}\mathcal{E}_z \\
&\quad-\epsilon\beta\frac{\partial}{\partial t}\left(\nabla\times\mathcal{E}\right)_z\}\cdot e_3 \\
&+i\{\epsilon\frac{\partial}{\partial z}\mathcal{E}_y+\epsilon\beta\frac{\partial}{\partial z}\left(\nabla\times\mathcal{E}\right)_y-\epsilon\frac{\partial}{\partial y}\mathcal{E}_z-\epsilon\beta\frac{\partial}{\partial y}\left(\nabla\times\mathcal{E}\right)_z-\mu\frac{\partial}{\partial t}H_x \\
&\quad-\mu\beta\frac{\partial}{\partial t}\left(\nabla\times H\right)_x\}\cdot e_4 \\
&+i\{\epsilon\frac{\partial}{\partial x}\mathcal{E}_z+\epsilon\beta\frac{\partial}{\partial x}\left(\nabla\times\mathcal{E}\right)_z-\epsilon\frac{\partial}{\partial z}\mathcal{E}_x-\epsilon\beta\frac{\partial}{\partial z}\left(\nabla\times\mathcal{E}\right)_x-\mu\frac{\partial}{\partial t}H_y \\
&\quad-\mu\beta\frac{\partial}{\partial t}\left(\nabla\times H\right)_y\}\cdot e_5 \\
&+i\{\epsilon\frac{\partial}{\partial y}\mathcal{E}_x+\epsilon\beta\frac{\partial}{\partial y}\left(\nabla\times\mathcal{E}\right)_x-\epsilon\frac{\partial}{\partial x}\mathcal{E}_y-\epsilon\beta\frac{\partial}{\partial x}\left(\nabla\times\mathcal{E}\right)_y-\mu\frac{\partial}{\partial t}H_z \\
&\quad-\mu\beta\frac{\partial}{\partial t}\left(\nabla\times H\right)_z\}\cdot e_6 \\
&+i\{\epsilon\frac{\partial}{\partial x}\mathcal{E}_x+\epsilon\beta\left(\nabla\times\mathcal{E}\right)_x+\epsilon\frac{\partial}{\partial y}\mathcal{E}_y+\epsilon\beta\left(\nabla\times\mathcal{E}\right)_y+\epsilon\frac{\partial}{\partial z}\mathcal{E}_z+\epsilon\beta\left(\nabla\times\mathcal{E}\right)_z\}\cdot e_7
\end{aligned}
\tag{4.44}
$$

which may further be reduced to

$$\begin{aligned}\boxdot\mathcal{F} = & -\{\mu(\overrightarrow{\nabla}\cdot\overrightarrow{H})+\mu\beta\overrightarrow{\nabla}\cdot(\overrightarrow{\nabla}\times\overrightarrow{H})\}.e_1\\ & +\{\mu(\overrightarrow{\nabla}\times\overrightarrow{H})-\epsilon\frac{\partial}{\partial t}\vec{\mathcal{E}}+\mu\beta\overrightarrow{\nabla}\cdot(\overrightarrow{\nabla}\times\overrightarrow{H})-\epsilon\beta\frac{\partial}{\partial t}\left(\overrightarrow{\nabla}\times\vec{\mathcal{E}}\right)\\ & -\mu\beta\overrightarrow{\nabla}\cdot(\overrightarrow{\nabla}\times\overrightarrow{H})\}\cdot e_j\\ & +i\{-\epsilon\left(\overrightarrow{\nabla}\times\vec{\mathcal{E}}\right)+\epsilon\beta\overrightarrow{\nabla}\cdot\left(\overrightarrow{\nabla}\times\vec{\mathcal{E}}\right)-\mu\frac{\partial}{\partial t}\vec{H}-\mu\beta\frac{\partial}{\partial t}(\overrightarrow{\nabla}\times\overrightarrow{H})\\ & -\epsilon\beta\overrightarrow{\nabla}\cdot\left(\overrightarrow{\nabla}\times\vec{\mathcal{E}}\right)\}\cdot e_{j+3}\\ & +i\{\epsilon\left(\overrightarrow{\nabla}\cdot\vec{\mathcal{E}}\right)+\epsilon\beta\overrightarrow{\nabla}\cdot\left(\overrightarrow{\nabla}\times\vec{\mathcal{E}}\right)\}\cdot e_7\end{aligned}\tag{4.45}$$

It leads to the octonion GDM wave equation in Chiral medium of dyons as

$$\boxdot\mathcal{F}=\mathcal{J};\tag{4.46}$$

where octonion current source density $\mathcal{J}$ of dyons be expressed as

$$\begin{aligned}\mathcal{J} = & e_0J_0+e_1J_1+e_2J_2+e_3J_3+e_4J_4+e_5J_5+e_6J_6+e_7J_7\\ = & -e_0\varrho+e_j\vec{j}-ie_{j+3}\vec{k}+ie_7\rho.\end{aligned}\tag{4.47}$$

Here, we may obtain the following relations from equation (4.46) as

$$\begin{aligned}\mu(\overrightarrow{\nabla}\cdot\overrightarrow{H}) = & \varrho;\\ \mu(\overrightarrow{\nabla}\times\overrightarrow{H}) = & \epsilon\frac{\partial}{\partial t}\vec{\mathcal{E}}+\epsilon\beta\frac{\partial}{\partial t}\left(\overrightarrow{\nabla}\times\vec{\mathcal{E}}\right)+\vec{j};\\ \epsilon\left(\overrightarrow{\nabla}\times\vec{\mathcal{E}}\right) = & -\mu\frac{\partial}{\partial t}\vec{H}-\mu\beta\frac{\partial}{\partial t}(\overrightarrow{\nabla}\times\overrightarrow{H})-\vec{k};\\ \epsilon\left(\overrightarrow{\nabla}\cdot\vec{\mathcal{E}}\right) = & \rho.\end{aligned}\tag{4.48}$$

These are octonionic GDM equations of dyons in Chiral medium. Using the following relations for generalized electromagnetic field as

$$\begin{aligned}
\frac{\partial}{\partial t}\vec{D} =& \epsilon\frac{\partial}{\partial t}\left(\vec{\mathcal{E}}+\beta\left(\vec{\nabla}\times\vec{\mathcal{E}}\right)\right)=\epsilon\frac{\partial}{\partial t}\vec{\mathcal{E}}+\epsilon\beta\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{\mathcal{E}}\right)\\
\frac{\partial}{\partial t}\vec{\mathcal{B}} =& \mu\frac{\partial}{\partial t}\left(\vec{H}+\beta(\vec{\nabla}\times\vec{H})\right)=\mu\frac{\partial}{\partial t}\vec{H}+\mu\beta\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})\\
\vec{\nabla}\cdot\vec{D} =& \epsilon\vec{\nabla}\cdot\left(\vec{\mathcal{E}}+\beta\left(\vec{\nabla}\times\vec{\mathcal{E}}\right)\right)=\epsilon\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right)\\
\vec{\nabla}\cdot\vec{\mathcal{B}} =& \mu\vec{\nabla}\cdot\left(\vec{H}+\beta(\vec{\nabla}\times\vec{H})\right)=\mu(\vec{\nabla}\cdot\vec{H})
\end{aligned} \tag{4.49}$$

we get

$$\begin{aligned}
\mu(\vec{\nabla}\cdot\vec{H}) =& \varrho;\\
\mu(\vec{\nabla}\times\vec{H}) =& \frac{\partial}{\partial t}\vec{D}+\vec{j};\\
\epsilon\left(\vec{\nabla}\times\vec{\mathcal{E}}\right) =& -\frac{\partial}{\partial t}\vec{\mathcal{B}}-\vec{k};\\
\epsilon\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right) =& \rho;
\end{aligned} \tag{4.50}$$

which is the alternative form of generalized Dirac-Maxwell's equations of dyon in Chiral medium in terms of generalized octonion electrodynamics.

4.6 Discussion and Conclusion

The generalized Dirac Maxwell's equations associated with dyons in isotropic (chiral) medium are obtained consistently in section (4.3). Equations (4.1) and (4.2) describe the relationship between the electric field and electric induction as well as the magnetic field and magnetic induction with the introduction of isotropic medium. These relations describe the rich variety of physical phenomenon representing the properties and response of the medium in order to find out the application of generalized electromagnetic field of dyons. The relation between the velocity of electromagnetic wave and medium parameter is established in equation (4.3) which are further explained in equation (4.4) and (4.5). So, the generalized field equations of dyons in homogeneous (isotropic) medium are obtained in equation (4.6) where the electric and magnetic fields are recalled as the generalized electromagnetic fields of dyons in isotropic medium. As such, the equations (4.7) and (4.8) are expressions for electric and magnetic fields of dyons in homogeneous (isotropic) medium with their relationship to the corresponding potential. It is shown that the generalized electric and magnetic field (4.7) and (4.8) are symmetric and dual invariant. Equations (4.9) describe duality relations (transformations) between electric and magnetic constituents of dyons which are responsible in order to check the duality invariance for field equations and other quantum equation of dyons. So, it is emphasized that the generalized Dirac-Maxwell's equations (4.6) are invariant under the generalized continuous linear transformations given by equations (4.10) and (4.11) which are reduced to the duality transformations for $\theta = \frac{\pi}{2}$ and thus recalls as duality transformations. Here, $\overrightarrow{\mathcal{E}} \to v\overrightarrow{\mathcal{B}}$ and $\overrightarrow{\mathcal{B}} \to -\frac{\overrightarrow{\mathcal{E}}}{v}$ are described as the duality transformations between electric and magnetic constituents of dyons. Equations (4.12) represents the generalized field vector in terms electric and magnetic fields of dyons and may then be related with the components of generalized four potential of dyons by equations (4.13-4.15). As such, the four different generalized Dirac-Maxwell's equation (4.6) are reduced to two complex linear differential equation (4.16) in which the generalized charge and current source densities of dyons in homogeneous medium is described by equation (4.17). Accordingly, we have established the relationship between the components of generalized four current and a new parameter $\vec{S}$ given by equation (4.18). Hence, the GDM equations are expressed in terms of components of generalized potential and current by equation (4.19).

In the section (4.4), we have discussed the octonionic formulation of generalized field equations of dyons for isotropic medium. The octonion differential operator is given by equation (4.20) and its conjugate has been defined by equation (4.21) for the case of homogeneous isotropic medium. Accordingly, equation (4.22) represents the octonionic representation of the generalized four potential. Octonion wave equation (4.23) provides the connection between the conjugate differential operator and the electric and magnetic four potentials. so, the Lorentz Gauge condition are discussed by equation (4.24). The octonion wave equation (4.24) has been reduced to its compact form by equation (4.25) which is further been expanded by equation (4.26) in terms of the connection between the components of generalized field and potentials. Equation (4.28) thus establishes the relation of generalized field vector of dyons in isotropic medium, which has further been expended into equation (4.29). As such, we have established the homogeneous octonion wave equation given by (4.30), which is octonionic form of generalized Dirac Maxwell's equation in compact, simpler and consistent manner. Accordingly, we have obtained the usual differential form of generalized Dirac Maxwell's equation in terms of equation (4.32) and (4.33) for generalized field of dyons in homogeneous isotropic medium.

In section (4.5), we have under taken the study of generalized electromagnetic fields in chiral media form the definition of the generalized Dirac Maxwell's equation of dyons. We have defined the connection between the generalized electromagnetic fields and induction vector in terms of chiral parameters from Drude-Born-Fedorov constitutive equation. Here it should be noted that the electric and magnetic fields are paired with each other given by the equations(4.34). It is shown that in the absence of chiral parameter the electric displacement and magnetic induction vectors are reduced to equations (4.1) and (4.2) in homogeneous (isotropic) medium. Chirality is described as the asymmetry in the molecular structure where a molecule is assumed to be a chiral if it can not be superimposed onto its mirror image. We have written the components of chirality dependent electric displacement vector and magnetic induction vector in terms of Cartesian coordinate for an isotropic medium given in equation (4.35), whereas the octonion representations of the parameters $\overrightarrow{B}$ and $\vec{\mathcal{D}}$ are provided by the equations (4.36) - (4.39). The octonion reformulation of the generalized electromagnetic field of dyons in chiral media has been investigated by equations (4.40) and (4.41) whereas its components associated with electric and magnetic vectors in the presence of chiral parameter are discussed by equation (4.42) and (4.43). Accordingly, the octonions wave equation has been solved to describe the components of generalized electromagnetic fields equations (4.44) and (4.45). Thus, the octonionic wave equation (4.46) is regarded as the octonionic representation of generalized Dirac Maxwell's (GDM) equation of dyons in chiral medium. It is compact, simpler, manifestly covariant and consistent as well. It reduces to the octonionic wave equation for generalized fields of dyons in isotropic medium in the absence of chiral parameter. It can further reproduces the field equation of moving charged particle like electron (monopole) in the absence of monopole (electron) in vacuum if we consider neither chiral nor isotropic medium parameters. Accordingly, the time derivative and divergence of field equations in chiral medium are obtained equations (4.49) and (4.50) which is also supports the above conclusion.

Bibliography

[1] L. E. Dickson, **"On quaternions and their generalization and the history of the eight square theorem"**, Ann. Math., **20** (1919), 155 .

[2] J. C. Baez, **"The Octonions"**, Bull. Am. Math. Soc., **39** (2001), 145.

[3] G. t' Hooft, **"Magnetic monopoles in unified gauge theories"**, Nucl. Phys., **B79** (1974), 276.

[4] A. M. Polyakov, **"Particle spectrum in quantum field theory",** JEPT Letter, **20** (1974), 194.

[5] B. Julia and A. Zee, **"Poles with both magnetic and electric charges in non-Abelian gauge theory",** Phys. Rev., **D11** (1975), 2227.

[6] J. Schwinger, **"Magnetic Charge and Quantum Field Theory"**, Phys. Rev., **144** (1966), 1087.

[7] P. A. M. Dirac, "**Quantized Singularities in the Electromagnetic Field**", Proc. R. Soc. London Sec., **A60** (1931), 133.

[8] M. N. Saha, "**Note on Dirac's theory of magnetic poles**", Phys. Rev., **75** (1949), 1968.

[9] A. S. Goldhaber, **"Connection of Spin and Statistics for Charge-Monopole Composites"**, Phy. Rev. Lett., **36** (1976), 1122.

[10] D. Zwanziger, **"Quantum Field Theory of Particles with Both Electric and Magnetic Charges"**, Phys. Rev., **176** (1968), 1489.

[11] K. Seo, **"Mechanism of baryon number violation around a monopole in an $SU(5)$ grand unified model"**, Phys. Lett., **B126** (1983), 201.

[12] L. D. Faddev, "**Les Houches lecture, session 20**", (BOOK) North Holland, (1976).

[13] S. Coleman, "**Ettore Majorana**", Lectures dilivered at the Int. school of sub nuclear Phys. (1975).

[14] N. Cabibbo and E. Ferrari, **"Quantum electrodynamics with Dirac monopoles",** Nuovo Cim., **23** (1962, 1147.

[15] G. 't Hooft, **"Magnetic monopoles in unified gauge theories"**, Nucl. Phys., **B79** (1974), 276.

[16] A. M. Polyakov, **"Particle spectrum in the quantum field theory",** JETP. Lett., **20** (1974), 194.

[17] B. Julia and A. Zee, "**Poles with both magnetic and electric charges in non-Abelian gauge theory**", Phys. Rev. **D11** (1975), 2227.

[18] M. K. Prasad, **"Yang-Mills-Higgs monopole solutions of arbitrary topological charge",** Comm. Math. Phys., **80** (1981), 137.

[19] M. Prasad and C. Sommerfield, "**An exact classical solution for the 't Hooft monopole and the Julia–Zee dyon**", Phys. Rev. Lett., **35** (1975), 760.

[20] E. P. Bogomol'nyi, **"The stability of classical solutions",** Sov. J. Nucl. Phys., **24** (1976), 449.

[21] W. R. Hamilton, **"Elements of quaternions"**, Vol. I, II and III (Chelsea, New York), (1899).

[22] A. Singh, **"On the Quaternion Form of Electromagnetlc-Current Equations",** Nuovo Cimento Lett., **31** (1981), 145.

[23] B. S. Rajput, S. R. Kumar and O. P. S. Negi, **"Quaternionic formulation for dyons"**, Nuovo Cimento Lett., **36** (1983), 75.

[24] D. Finkelstein, J. M. Jauch, S. Schiminovish and D. Speises, **"Principle of General Quaterniona Covariance",** J. Math. Phys., **4** (1963), 788.

[25] S. L. Adler, "**Quaternionic quantum field theory**", Commun. Math. Phys., **104** (1986), 611.

[26] S. Okubo, "**Introduction to octonion and Non-Associative Algebras in Physics**", Cambridge University Press, Cambridge, UK, (1995).

[27] B. Jancewwicz, "**Multivectors and Clifford Algebra in Electrodynamics**", World Scientific, Singapore, (1989).

[28] V. V. Kravchenko, **"Applied Quaternionic Analysis**", Heldermann Verlage, Germany, **28** (2003).

[29] V. V. Kravchenko, "**A new approach for describing electromagnetic wave propagation in inhomogeneous media**", Zeitschrift, Fur Analysis and Anwenduhgen, **21** (2002), 21.

[30] K. V. Khmelnytskaya, V. V. Kravchenko, and V. S. Rabinovich, **"Quaternionic Fundamental Solutions for Electromagnetic Scattering Problems and Application"**, Zeitschrift fi¿œr Analysis und ihre Anwendungen, **22** (2003), 147.

[31] S. M. Grudsky, K. V. Khmelnytskaya and V. V. Kravchenko, "**On a quaternionic Maxwell equation for the time-dependent electromagnetic field in a chiral medium**", J. Phys. A. Math. Gen., **37** (2004), 4641.

[32] E. Witten, **"Dyons of charge e$\vartheta/2\pi$"**, Phys. Lett. **86B** (1979), 283.

[33] P. S. Bisht , O. P. S. Negi and B. S. Rajput, **"Quaternion gauge theory of dyonic fields"**, Prog. Theor. Phys., **85** (1991), 151.

[34] J. Singh, P. S. Bisht and O. P. S. Negi, **"Generalized electromagnetic fields of dyons in isotropic medium"**, Commn. in Phys., **17** (2007), 83.

[35] J. Singh, P. S. Bisht and O. P. S. Negi, **"Generalized electromagnetic fields in chiral medium"**, J. Phys. A. Math. and Gen., (2007).

[36] M. Tanisli and M. E. Kansu, **"Octonionic Maxwell's equations for bi-isotropic media"**, J. Math. Phys., **52** (2011), 053511.

[37] A. Lakhtakia and Beltrami, **"Fields in Chiral Media",** World Scientific, Singapore (1994).

[38] I. V. Lindell, A. H. Sihvola, S. A. Tretyakov and A. J. Viitanen, **"Electromagnetic wave in chiral and bi-isotropic media"**, Artech House Publisher, Boston, MA (1994).

[39] A. Lakhtakia, **"Beltrami Fields in Chiral Media"**, World Scientific, London, (1994).

[40] V. K. Mel'nikov, **"Exact solutions of the Korteweg-de Vries equation with a self-consistent source"**, Phys. Lett., **A128** (1988), 488.

[41] J. A. Starton, **“Electromagnetic Theory”**, Mc Graw Hill Company, New York, (1941).

[42] B. C. Chanyal, P. S. Bisht and O. P. S. Negi, **“Generalized Octonion Electrodynamics”,** Int. J. Theor. Phys., **49** (2010), 1333.

[43] B. C. Chanyal, P. S. Bisht and O. P. S. Negi, **“Generalized Split - Octonion Electrodynamics”,** Int. J. Theor. Phys., **50** (2011), 1919.

CHAPTER 5

Octonion Gauge Formulation And Quantum Chromodynamics

A part of this chapter has been published in Int. J. Thoer. Phys., 51, (2012) 3410

ABSTRACT

An attempt to describe the octonionic reformulation of Abelian and non-Abelian gauge theory of dyons has been made to discuss the $U(1)_e \times U(1)_m$ ***abelian gauge theory,*** $U(1) \times SU(2)$ ***electroweak gauge theory and also the*** $SU(2)_e \times SU(2)_m$ ***non-Abelian gauge theories of in term of 2×2 Zorn vector matrix realization of split octonions. It is shown that*** $SU(2)_e$ ***characterizes the usual theory of the Yang Mill's field (isospin or weak interactions) due to presence of electric charge while the gauge group*** $SU(2)_m$ ***predicts the existence of t-Hooft-Polyakov monopole in non-Abelian Gauge theory. Accordingly, we have established the relations between octonion basis elements and Gell-Mann*** λ ***matrices of*** $SU(3)$ ***symmetry on comparing the multiplication tables of these two. Consequently, the quantum chromodynamics (QCD) has been reformulated and it is shown that the theory of strong interactions could be explained better in terms of non-associative octonion algebra.***

Chapter 5

Octonion Gauge Formulation And Quantum Chromodynamic

5.1 Introduction

In spite of the symmetry, conservation laws and gauge fields describe elementary particle in terms of their field quanta and interactions. Nevertheless, the role of number system (hyper complex numbers) has been an important factor in understanding the various theories of physics from macroscopic to microscopic level. In fact , there has been a revival in the formulation of natural laws in terms of numbers. So, according to celebrated Hurwitz theorem there exists [1] four division algebra consisting of $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions) [2, 3] and $\mathbb{O}$ (octonions) [4-6]. All these four algebra's are alternative with totally anti symmetric associators. Real number explains will the classical Newtonian mechanics, complex number plays an important role for the explanation beyond the framework of quantum theory and relativity. The division algebras are by no means new to physics; in most theories, both classical mechanics and quantum mechanics are described already in terms of the real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$. Real and complex numbers are limited only up to two dimensions, quaternions are extended to four dimensions (one real and three imaginaries) while octonions represent eight dimensions (one real and seven imaginaries). Real and complex numbers are commutative and associative. Furthermore, the group $SU(2)$ is everywhere in physics [7], and its connection to the quaternions $\mathbb{H}$ is well known. Quaternions are having relations with Pauli matrices which explain non Abelian gauge theory. Quaternions were very first example of hyper complex numbers having the significant impacts on Mathematics and Physics. Because of their beautiful and unique properties, quaternions attracted many to study the laws of nature over the field of these numbers. Quaternions naturally unify [8, 9] electromagnetism and weak force, producing the electroweak $SU(2)\times U(1)$ sector of standard model. Quaternion are associative but not commutative while its next generalization to octonions is neither commutative nor associative. Rather, the laws of alternatively and distributivity are obeyed by octonions. Quaternions and octonions are extensively used in the various branches of physics and mathematics. The octonion analysis has also played an important role in the context of various physical problems [10-17] higher dimensional supersymmetry, super gravity and super strings etc while the quaternions have an important role to unify [7-9] electromagnetism and weak forces to represent the electroweak $SU(2)\times U(1)$ sector of standard model. Octonion is a last member of the normed division algebra, which its sequence consists of $\mathbb{R}\longmapsto\mathbb{C}\longmapsto\ \mathbb{H}\ \longmapsto\mathcal{O}$. Octonions are used for unification program of strong interaction with successful gauge theory of fundamental interaction i.e. octonions naturally unify [13-15] strong, electromagnetism and weak force, producing $SU(3)_c\times SU(2)_w\times U(1)_Y$. A theoretical description of the leptons and quark structure of hadrons has been proposed by Gunaydin and Gursey [18] in the context of octonionic quantum mechanics and considered the possibility of constructing an octonionic Hilbert space. They identified a natural decomposition of space with the representation of leptons and quarks. Likewise, the octonions are extensively studied [18, 19] for the description of color quarks and played an important role for unification programme of fundamental interactions in terms of successful gauge theories. Furthermore, the quaternionic formulation of Yang–Mill's field equations and octonion reformulation of quantum chromo dynamics (QCD) has also been developed [20] by taking magnetic monopoles [21-23] and dyons (particles carrying electric and magnetic charges) [24-27] into account. It is shown that the three quaternion units explain the structure of Yang-Mill's field while the seven octonion units provide the consistent structure of $SU(3)_c$ gauge symmetry of quantum chromo dynamics.

The remarkable success of division algebra prompted us for this chapter to reformulate the abelian and non-Abelian gauge theory of dyons with the application of split octonions and their Zorn vector matrix realization. In section (5.2) we have discussed the U(1)$\times$U(1) abelian gauge theory of dyons from the invariance principles of Lagrangian formulation in order to obtain the dyonic field equations. It has been shown that this formalism provides better understanding to explain the duality conjunctive for the justification of existence of monopoles and dyons. In section (5.3), we have discussed the U(1)$\times$U(1) octonion gauge formulation in terms of 2$\times$2 Zorn vector matrix realization of split octonion in compact and consistent manner. As such, we have developed the octonion covariant derivative for U(1)$\times$U(1) gauge theory of dyons in terms of 2$\times$2 Zorn matrix realization of split octonions. It is shown that the commutation relation between the octonion covariant derivative leads to two types of gauge field strength of generalized electromagnetic fields of dyons responsible for the simultaneous existence of electric charge and magnetic monopole. It is also shown that the generalized Dirac Maxwell's equations of dyons leads to two types of two photon in terms of two four currents associated with electric charge and magnetic monopole. In section (5.4), we have discussed the octonion gauge fields as the combination of two quaternion gauge fields. The covariant derivative, abelian and non-Abelian gauge structure and the gauge current equation are described in split octonion formulation of gauge theory. As such, we have investigated the U(1)×SU(2) octonionic gauge formulation in simpler and compact manner. Our U(1)×SU(2) theory of weak interaction describes two fold symmetry of electroweak interactions. The first fold describes the gauge boson of standard electroweak theory while the second one has be investigated to describe the structure of alternative electroweak interaction to the presence of magnetic monopole. In section (5.5), we have extended U(1)×SU(2) to the non-Abelian $SU(2)_e \times SU(2)_m$ gauge formulation in terms of 2×2 Zorn vector matrix of split octonions. Accordingly, the octonion gauge theory has been reconnected to the 't Hooft Polyakov magnetic monopole theory (section 5.6) in order to satisfy the existence of magnetic monopole in Grand Unified Theories (GUTs). In section (5.7), we have discussed the SU(3) generators (Gell-Mann matrices) and their multiplication properties and accordingly the resemblance between the octonion basis elements and the SU(3) generators are discussed in section (5.8), where a proper mapping between two has been investigated. Accordingly, the SU(3) symmetry as been developed interms of non-associativity of octonion basis elements which does not effect the invariance of SU(2) spin (i-spin) multiplets. Further more, it is concluded that the algebra of strong interactions correspond to SU(3) automorphism of octonion algebra and supports earlier results of Gï¿œnaydin [19]. In section (5.9), we have discussed the relationship of octonions and the parameters of quantum chromodynamics (QCD). Consequently, the exact SU(3) symmetry of colors has been investigated in terms of octonion algebra in order to describe quantum chromodynamics (QCD). Hence, we have reformulated the theory of strong interaction (i.e. the quantum chromodynamics (QCD)) based on colors $SU(3)_c$ whose generators satisfy the non-associative algebra of octonions. It is shown that in this theory the gluonic field strength tensor of QCD behaves like to the electromagnetic field strength tensor of QED. More over, the SU(3) gauge theory of strong interactions and the invariant Lagrangian formulation has been suitably handled in terms of non-associativity of octonion in section (5.10), where gauge transformations are octonionic, and the octonion affinity describes the Yang-Mill's field. It is concluded that octonionic colored quarks are dyons where the generalized field of dyons are discussed as the two fold gauge symmetries of SU(3) non-Abelian gauge group associated respectively with electric and magnetic charges.

5.2 $\mathbf{U(1)} \times \mathbf{U(1)}$ Gauge Formulation of Dyons

Let us introduce the following two four component spinor [27] as,

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \tag{5.1}$$

where Ψ_1 and Ψ_2 are four component spinors. Negi-Dehnen [27] identified Ψ_1 as the Dirac spinor for a electric charge (like electron) while the other spinor Ψ_2 has been identified as the Dirac iso-spinors acting on the magnetic monopole. Thus the Ψ may be visualized as the bi-spinor for dyons in terms of its electric and magnetic constituents. Each spinor Ψ_1 and Ψ_2 satisfy the free particle Dirac equation [27]

$$\begin{aligned}\mathcal{L}_0 =&\overline{\Psi}(i\gamma^{\mu}\partial_{\mu}+m)\,\hat{1}\Psi\\ =&\left(\overline{\Psi}_1,\overline{\Psi}_2\right)\begin{bmatrix}(i\gamma^{\mu}\partial_{\mu}+m) & 0\\ 0 & (i\gamma^{\mu}\partial_{\mu}+m)\end{bmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\\ =&\sum_{a=1}^{a=2}\overline{\Psi}_a(i\gamma^{\mu}\partial_{\mu}+m)\Psi_a\,\hat{1}\end{aligned} \tag{5.2}$$

where $\hat{1}$ is 2×2 unit matrix. So, the Unitary transformations taking part for the invariance of free particle Dirac equation for bi-spinor are the global $U=U^{(e)}(1)\times U^{(m)}(1)$ two component spinors Ψ_1 and Ψ_2 . In this case Ψ_1 acts on unitary gauge group $U^{(e)}(1)$ whereas the iso-spinor Ψ_2 acts on the other unitary group $U^{(m)}(1)$ with the symbols (e) and (m) are used for the electric and magnetic charges. Thus equation (5.2) is invariant under global gauge transformation

$$U=U^{(e)}\times U^{(m)}=\exp\left(i\Lambda_j\tau_a^{j\,b}\right) \tag{5.3}$$

where

$$\tau_a^{j\,b}=\tau_a^{1\,b}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\ \ and\ \ \tau_a^{j\,b}=\tau^2=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \tag{5.4}$$

and

$$\left[\tau_a^{j\,b},\tau_a^{k\,b}\right]=\varepsilon_l^{jk}\tau_a^{j\,b}=0 \tag{5.5}$$

because we have $j,\,k,\,l=1,2$. Accordingly the spinor transforms [27] as

$$\begin{aligned}\Psi_1\longmapsto\Psi_1'\longmapsto\left[U^{(e)}\right]\Psi_1&=\exp\{i\Lambda_1\}\Psi_1\\ \Psi_2\longmapsto\Psi_2'\longmapsto\left[U^{(m)}\right]\Psi_2&=\exp\{i\Lambda_2\}\Psi_2\\ \overline{\Psi}_1\longmapsto\overline{\Psi}_1'\longmapsto\overline{\Psi}_1\left[U^{(e)}\right]^{-1}&=\overline{\Psi}_1\exp\{i\Lambda_1\}\\ \overline{\Psi}_2\longmapsto\overline{\Psi}_2'\longmapsto\overline{\Psi}_2\left[U^{(m)}\right]^{-1}&=\overline{\Psi}_2\exp\{i\Lambda_2\}\\ \Psi\longmapsto\Psi'\longmapsto U\Psi&=\begin{pmatrix}\exp\{i\Lambda_1\}&0\\0&\exp\{i\Lambda_2\}\end{pmatrix}\begin{pmatrix}\Psi_1\\\Psi_2\end{pmatrix}\\ \overline{\Psi}\longmapsto\overline{\Psi}'\longmapsto\overline{\Psi}U^{-1}&=\left(\overline{\Psi}_1,\overline{\Psi}_2\right)\begin{pmatrix}\exp\{i\Lambda_1\}&0\\0&\exp\{i\Lambda_2\}\end{pmatrix}\end{aligned}\tag{5.6}$$

In equations (5.3) and (5.5) $\Lambda_j(\forall\, j=1,2)$ are independent of space and time for global gauge transformations. If we extend this symmetry to invariance under local gauge transformations where $\Lambda_j\Longrightarrow\Lambda_j(x)$ $(\forall\, j=1,2)$ in equations (5.3) and (5.5), the Lagrangian (5.2) transforms as

$$\overline{\Psi}(i\gamma^\mu\partial_\mu+m)\,\hat{1}\Psi\longmapsto\overline{\Psi}(i\gamma^\mu D_\mu+m)\,\hat{1}\Psi\tag{5.7}$$

where partial derivative ∂_μ has been replaced by the covariant derivative D_μ [27] as

$$D_\mu=D^b_{\mu a}=\partial_\mu\delta^b_a+\beta_{\mu j}\tau^{j\,b}_a\tag{5.8}$$

and $\tau^{j\,b}_a$ are given by equation (5.4) along with

$$\beta_{\mu j}\longmapsto\beta_{\mu 1}=\,A_\mu\tag{5.9}$$

$$\beta_{\mu j}\longmapsto\beta_{\mu 2}=\,C_\mu.\tag{5.10}$$

These are the gauge potentials respectively associated with the dynamics of electric and magnetic charges with the following gauge transformations [27]

$$\beta_{\mu1} = A_{\mu} \longmapsto A'_{\mu} \longmapsto \left[U^{(e)}\right] A_{\mu} \left[U^{(e)}\right]^{-1} + \frac{1}{e}\left[U^{(e)}\right] \partial_{\mu} \left[U^{(e)}\right]^{-1} \quad (5.11)$$

$$\beta_{\mu2} = C_{\mu} \longmapsto C'_{\mu} \longmapsto \left[U^{(m)}\right] C_{\mu} \left[U^{(m)}\right]^{-1} + \frac{1}{g}\left[U^{(m)}\right] \partial_{\mu} \left[U^{(m)}\right]^{-1} \quad (5.12)$$

where

$$\left[U^{(e)}\right] \Longrightarrow \exp\{i\Lambda_1(x)\} \quad (5.13)$$

$$\left[U^{(m)}\right] \Longrightarrow \exp\{i\Lambda_2(x)\}. \quad (5.14)$$

As such, we may write [27] the co-variant derivative D_{μ} (5.8) as

$$D_{\mu}\Psi = \begin{bmatrix} \partial_{\mu} - ieA_{\mu} & 0 \\ 0 & \partial_{\mu} - igC_{\mu} \end{bmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \quad (5.15)$$

which transforms as

$$D_{\mu}\Psi \longmapsto D'_{\mu}\Psi' \longmapsto \begin{pmatrix} \exp\{i\Lambda_1(x)\} & 0 \\ 0 & \exp\{i\Lambda_2(x)\} \end{pmatrix} \begin{pmatrix} (\partial_{\mu} - ieA_{\mu})\Psi_1 \\ (\partial_{\mu} - igC_{\mu})\Psi_2 \end{pmatrix}$$

$$= U\left(D_{\mu}\Psi\right). \quad (5.16)$$

Hence, we get

$$[D_{\mu},\, D_{\nu}]\Psi(x) = \begin{bmatrix} -ieF_{\mu\nu} & 0 \\ 0 & -ig\mathcal{F}_{\mu\nu} \end{bmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \quad (5.17)$$

and it leads to the Jacobi identity

$$[D_{\mu},[D_{\nu},\, D_{\lambda}]] + [D_{\nu},[D_{\lambda},\, D_{\mu}]] + [D_{\lambda},[D_{\mu},\, D_{\nu}]] = 0 \quad (5.18)$$

along with the Bianchi identities

$$D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu} = 0 \tag{5.19}$$

$$D_\mu \mathcal{F}_{\nu\lambda} + D_\nu \mathcal{F}_{\lambda\mu} + D_\lambda \mathcal{F}_{\mu\nu} = 0 \tag{5.20}$$

As such, the total Lagrangian for generalized fields of dyons is described [27] as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \overline{\Psi}(i\gamma^\mu\partial_\mu + m)\Psi - A_\mu j^\mu - C_\mu k^\mu \tag{5.21}$$

where

$$j^\mu = e\,\overline{\Psi}_1\gamma^\mu\Psi_1 \tag{5.22}$$

and

$$k^\mu = g\,\overline{\Psi}_2\gamma^\mu\Psi_2 \tag{5.23}$$

are the the four currents associated respectively with electric and magnetic charges on dyons. These four- currents obtained from the Dirac spinor Ψ_1 and the Dirac iso-spinor Ψ_2 satisfy the following conserved relations

$$\begin{aligned} \partial_\mu j^\mu &= j^\mu{}_{,\mu} = 0 \\ \partial_\mu k^\mu &= k^\mu{}_{,\mu} = 0. \end{aligned} \tag{5.24}$$

5.3 U(1) × U(1) Octonion Gauge Formulation

Let us write [32] the split octonion valued space time vector $\mathbb{Z}^{\mu}$ $(\mu=0,1,2,3)$ in terms of the 4×4 (space time vector-valued) Zorn matrix $\mathbb{Z}^{\mu}_{ab}$ as

$$\begin{aligned}\mathbb{Z}^{\mu}=&x_{0}^{\mu}u_{0}^{*}+y_{0}^{\mu}u_{0}+x_{j}^{\mu}u_{j}^{*}+y_{j}^{\mu}u_{j}\\=&\begin{pmatrix}x_{0}^{\mu}e_{0} & -x_{j}^{\mu}e_{j}\\ y_{j}^{\mu}e_{j} & y_{0}^{\mu}e_{0}\end{pmatrix},\qquad(\forall\, j=1,2,3)\end{aligned}\tag{5.25}$$

where $(\mu=0,1,2,3)$ represent the internal four dimensional space with $(\mu=0)$ representing $U(1)$ abelian gauge structure while $\mu=j$ $(\forall\, j=1,2,3)$ may be used for non-Abelian gauge structure. Here x_{0}^{μ}, x_{j}^{μ}, y_{0}^{μ}, y_{j}^{μ} are real valued variables for abelian and non-Abelian gauge fields. When the space time metric is $\eta_{\mu\nu}\hat{1}_{4\times4}$, the bi linear term

$$\begin{aligned}\frac{1}{4}Trace[\eta_{\mu\nu}\mathbb{Z}^{\mu}.\mathbb{Z}^{\nu}]=&\frac{1}{4}\eta_{\mu\nu}[x_{0}^{\mu}x_{0}^{\nu}+y_{0}^{\mu}y_{0}^{\nu}+x_{j}^{\mu}x_{j}^{\nu}+y_{j}^{\mu}y_{j}^{\nu}]\,Trace[\hat{1}_{2\times2}]\\=&\frac{1}{2}\eta_{\mu\nu}[x_{0}^{\mu}x_{0}^{\nu}+y_{0}^{\mu}y_{0}^{\nu}+x_{j}^{\mu}x_{j}^{\nu}+y_{j}^{\mu}y_{j}^{\nu}]\end{aligned}\tag{5.26}$$

describe the inner product. The octonion conjugation is accordingly defined as

$$\overline{\mathbb{Z}}^{\mu}=x_{0}^{\mu}u_{0}+y_{0}^{\mu}u_{0}^{*}-x_{j}^{\mu}u_{j}^{*}-y_{j}^{\mu}u_{j}=\begin{pmatrix}y_{0}^{\mu}e_{0} & x_{j}^{\mu}e_{j}\\ -y_{j}^{\mu}e_{j} & x_{0}^{\mu}e_{0}\end{pmatrix},\tag{5.27}$$

while the Hermitian conjugation is described [32] as

$$\begin{aligned}(\mathbb{Z}^{\mu})^{\dagger}=&(x_{0}^{\mu})^{*}u_{0}+(y_{0}^{\mu})^{*}u_{0}^{*}-(x_{j}^{\mu})^{*}u_{j}^{*}-(y_{j}^{\mu})^{*}u_{j}\\=&\begin{pmatrix}(y_{0}^{\mu})^{*}e_{0} & (x_{j}^{\mu})^{*}e_{j}\\ -(y_{j}^{\mu})^{*}e_{j} & (x_{0}^{\mu})^{*}e_{0}\end{pmatrix}.\end{aligned}\tag{5.28}$$

So, we may write $x_j^\mu = y_j^\mu = (x_j^\mu)^* = (y_j^\mu)^* = 0$ for Abelian gauge fields. Thus, the split octonion differential operator is written as,

$$\boxdot = u_0^* \partial_\mu + u_0 \partial_\mu \longmapsto \partial_\mu e_0 \hat{1}, \tag{5.29}$$

where $u_0 = \frac{1}{2}(e_0 + ie_7)$, $u_0^* = \frac{1}{2}(e_0 - ie_7)$ are split octonion basis given by equation (3.1). Thus, the equation (5.29) in term of 2×2 Zorn matrix may be written as

$$\boxdot = \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix} \approx \partial_\mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \approx \partial_\mu \hat{1}_{2\times2}. \tag{5.30}$$

Hence, the covariant derivative for $U(1)\times U(1)$ gauge theory of dyons defined by (5.8) and (5.15) may be written as split octonion valued in terms of 2×2 Zorn vector matrix realization as

$$D_\mu = \begin{pmatrix} \partial_\mu + A_\mu & 0 \\ 0 & \partial_\mu + B_\mu \end{pmatrix}; \tag{5.31}$$

which yields

$$D_\mu D_\nu = \begin{pmatrix} \partial_\mu \partial_\nu + \partial_\mu A_\nu + A_\mu \partial_\nu + A_\mu A_\nu & 0 \\ 0 & \partial_\mu \partial_\nu + \partial_\mu B_\nu + B_\mu \partial_\nu + B_\mu B_\nu \end{pmatrix}, \tag{5.32}$$

and

$$D_\nu D_\mu = \begin{pmatrix} \partial_\nu \partial_\mu + \partial_\nu A_\mu + A_\nu \partial_\mu + A_\nu A_\mu & 0 \\ 0 & \partial_\nu \partial_\mu + \partial_\nu B_\mu + B_\nu \partial_\mu + B_\nu B_\mu \end{pmatrix}. \tag{5.33}$$

On subtraction, i.e. $[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu$, these equations reduce to

$$[D_\mu, D_\nu] = \begin{pmatrix} \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu & 0 \\ 0 & \partial_\mu B_\nu - \partial_\nu B_\mu + B_\mu B_\nu - B_\nu B_\mu \end{pmatrix}, \tag{5.34}$$

which reproduces

$$[D_\mu, D_\nu] = \begin{pmatrix} F_{\mu\nu} & 0 \\ 0 & \mathcal{F}_{\mu\nu} \end{pmatrix} \longmapsto \mathbb{F}_{\mu\nu}; \tag{5.35}$$

where

$$\begin{aligned} F_{\mu\nu} =& \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu \longmapsto E_{\mu\nu}; \\ \mathcal{F}_{\mu\nu} =& \partial_\mu B_\nu - \partial_\nu B_\mu + B_\mu B_\nu - B_\nu B_\mu \longmapsto H_{\mu\nu}; \end{aligned} \tag{5.36}$$

Here $E_{\mu\nu}$ and $H_{\mu\nu}$ represent the octonionic forms of generalized field tensors of electromagnetic fields of dyons. Now operating D_μ given by the equation (5.31) to the generalized four electromagnetic fields $\mathbb{F}_{\mu\nu}$ (5.35), we get

$$\begin{aligned} D_\mu \mathbb{F}_{\mu\nu} =& \begin{pmatrix} \partial_\mu + A_\mu & 0 \\ 0 & \partial_\mu + B_\mu \end{pmatrix} * \begin{pmatrix} E_{\mu\nu} & 0 \\ 0 & H_{\mu\nu} \end{pmatrix} \\ =& \begin{pmatrix} \partial_\mu E_{\mu\nu} & 0 \\ 0 & \partial_\mu H_{\mu\nu} \end{pmatrix} \longmapsto \begin{pmatrix} j_\nu & 0 \\ 0 & k_\nu \end{pmatrix} \Longrightarrow \mathbb{J}_\nu; \end{aligned} \tag{5.37}$$

where

$$j_\nu = \partial_\mu E_{\mu\nu}; \qquad k_\nu = \partial_\mu H_{\mu\nu}; \tag{5.38}$$

are respectively the four currents associated with electric charge and magnetic monopole (i.e. the constituents of dyons) in the case of $U(1)\times U(1)$ octonion gauge formalism. Thus, we have obtained the justification of $U(1)\times U(1)$ gauge theory of dyons in terms of split octonions and their correspondence with 2×2 Zorn vector matrix realization. Here we may infer that the $U(1)\times U(1)$ gauge theory is described well by split octonion formulation where the spinor and iso spinor take part together.

5.4 $\mathbf{U(1)\times SU(2)}$ Octonion Gauge Formulation

So, by virtue of split octonion formulation we may extend $U(1)$ gauge theory to the $U(1)\times SU(2)$ gauge theory. Accordingly, we write an octonion as the combination of two gauge fields expanded in terms of quaternions i.e.

$$\begin{aligned} A_{\mu} &\mapsto A_{\mu}^{0}+A_{\mu}^{a}e_{a}, \\ B_{\mu} &\mapsto B_{\mu}^{0}+B_{\mu}^{a}e_{a}, \qquad (\forall a=1,2,3.) \end{aligned} \tag{5.39}$$

So, the co-variant derivative in case of $U(1)\times SU(2)$ octonion gauge field in the split octonion form may be expressed as

$$D_{\mu}=\begin{pmatrix} \partial_{\mu}+A_{\mu}^{0}+A_{\mu}^{a}e_{a} & 0 \\ 0 & \partial_{\mu}+B_{\mu}^{0}+B_{\mu}^{a}e_{a} \end{pmatrix}, \tag{5.40}$$

where the components of electric A_{μ}^{0} and magnetic B_{μ}^{0} are the four potentials of dyons in case of $U(1)$ while A_{μ}^{a} and B_{μ}^{a} describe of the $SU(2)$ gauge field theory. Similarly

$$D_{\nu}=\begin{pmatrix} \partial_{\nu}+A_{\nu}^{0}+A_{\nu}^{a}e_{a} & 0 \\ 0 & \partial_{\nu}+B_{\nu}^{0}+B_{\nu}^{a}e_{a} \end{pmatrix}, \tag{5.41}$$

and

$$[D_{\mu},D_{\nu}]=\begin{pmatrix} G_{\mu\nu}^{0}+G_{\mu\nu}^{a}e_{a} & 0 \\ 0 & \mathsf{G}_{\mu\nu}^{0}+\mathsf{G}_{\mu\nu}^{\mathsf{a}}e_{a} \end{pmatrix}\longmapsto\ \mathbb{G}_{\mu\nu}; \tag{5.42}$$

which is $U(1)\times SU(2)$ octonion gauge field strength for dyons in 2×2 Zorn matrix realization. where

$$\begin{aligned} G_{\mu\nu}^{0}= & \partial_{\mu}A_{\nu}^{0}-\partial_{\nu}A_{\mu}^{0}+\left[A_{\mu}^{0},A_{\nu}^{0}\right]\longmapsto\ E_{\mu\nu}^{0}, \\ G_{\mu\nu}^{a}= & \partial_{\mu}A_{\nu}^{a}-\partial_{\nu}A_{\mu}^{a}+e_{a}\left[A_{\mu}^{a},A_{\nu}^{a}\right]\longmapsto\ E_{\mu\nu}^{a}, \end{aligned} \tag{5.43}$$

which are respectively abelian and non-Abelian $U(1)_{e}\times SU(2)_{e}$ gauge structures in presence of electric charge. Similarly

$$\begin{aligned} \mathsf{G}_{\mu\nu}^{0}= & \partial_{\mu}B_{\nu}^{0}-\partial_{\nu}B_{\mu}^{0}+\left[B_{\mu}^{0},B_{\nu}^{0}\right]\longmapsto\ H_{\mu\nu}^{0}, \\ \mathsf{G}_{\mu\nu}^{\mathsf{a}}= & \partial_{\mu}B_{\nu}^{a}-\partial_{\nu}B_{\mu}^{a}+e_{a}\left[B_{\mu}^{a},B_{\nu}^{a}\right]\longmapsto\ H_{\mu\nu}^{a}, \end{aligned} \tag{5.44}$$

is $U(1)_{m}\times SU(2)_{m}$ gauge structure associated with the presence of magnetic monopole. Accordingly, we get

$$\begin{aligned} D_{\mu}\mathbb{G}_{\mu\nu}= & \begin{pmatrix} \partial_{\mu}+A_{\mu}^{0}+A_{\mu}^{a}e_{a} & 0 \\ 0 & \partial_{\mu}+B_{\mu}^{0}+B_{\mu}^{a}e_{a} \end{pmatrix}*\begin{pmatrix} G_{\mu\nu}^{0}+G_{\mu\nu}^{a}e_{a} & 0 \\ 0 & \mathsf{G}_{\mu\nu}^{0}+\mathsf{G}_{\mu\nu}^{\mathsf{a}}e_{a} \end{pmatrix} \\ = & \begin{pmatrix} \partial_{\mu}G_{\mu\nu}^{0}+\partial_{\mu}G_{\mu\nu}^{a}e_{a} & 0 \\ 0 & \partial_{\mu}\mathsf{G}_{\mu\nu}^{0}+\partial_{\mu}\mathsf{G}_{\mu\nu}^{\mathsf{a}}e_{a} \end{pmatrix}, \end{aligned} \tag{5.45}$$

which may further be reduced in terms of compect notation of split octonion formulation i.e.

$$D_\mu \mathbb{G}_{\mu\nu} = \mathbb{J}_\nu. \tag{5.46}$$

Here $\mathbb{J}_\nu$ is $U(1) \times SU(2)$ form of octonion gauge current for dyons which may be expressed in term of 2×2 Zorn matrix as

$$\mathbb{J}_\nu = \begin{pmatrix} j_\nu^0 + j_\nu^a e_a & 0 \\ 0 & k_\nu^0 + k_\nu^a e_a \end{pmatrix}, \tag{5.47}$$

where

$$\begin{aligned} j_\nu^0 &= \partial_\mu G_{\mu\nu}^0; \\ j_\nu^a &= \partial_\mu G_{\mu\nu}^a; \\ k_\nu^0 &= \partial_\mu G_{\mu\nu}^0; \\ k_\nu^a &= \partial_\mu G_{\mu\nu}^a. \end{aligned} \tag{5.48}$$

Here j_ν^0 and j_ν^a are generalized octonion current for $U(1)_e \times SU(2)_e$ (electric case) and k_ν^0 and k_ν^a for $U(1)_m \times SU(2)_m$ (magnetic case). The analogous continuity equation then changes to be

$$D_\mu \mathbb{J}_\mu = \begin{pmatrix} \partial_\mu j_\mu^0 + \partial_\mu j_\mu^a e_a & 0 \\ 0 & \partial_\mu k_\nu^0 + \partial_\mu k_\nu^a e_a \end{pmatrix} = 0. \tag{5.49}$$

5.5 Non-Abelian $\mathbf{SU(2)_e \times SU(2)_m}$ Gauge Formulation

In order to describe $SU(2)_e \times SU(2)_m$ gauge formulation, let us write the covariant derivative D_μ as

$$D_\mu = \partial_\mu + \mathbb{V}_\mu; \tag{5.50}$$

where $\mathbb{V}_\mu$ is the octonion form of generalized four potential expressed as

$$\mathbb{V}_\mu = e_0 \left(A_\mu^\tau e_\tau\right) + ie_7 \left(B_\mu^\tau e_\tau\right). \tag{5.51}$$

Here $\tau \mapsto 1, 2, 3$ denotes $SU(2)$ generator. Thus, the covariant derivative (5.50) may be expressed as

$$\begin{aligned} D_\mu =& \partial_\mu + e_0 \left(A_\mu^\tau e_\tau\right) + ie_7 \left(B_\mu^\tau e_\tau\right) \\ =& u_0^* \left(\partial_\mu + A_\mu^\tau e_\tau + B_\mu^\tau e_\tau\right) + u_0 \left(\partial_\mu + A_\mu^\tau e_\tau - B_\mu^\tau e_\tau\right). \end{aligned} \tag{5.52}$$

The split octonion equivalent of equation (5.52) in term of 2×2 Zorn's vector matrix realization may be expressed as

$$D_\mu = \begin{pmatrix} \partial_\mu + \left(A_\mu^\tau + B_\mu^\tau\right) e_\tau & 0 \\ 0 & \partial_\mu + \left(A_\mu^\tau - B_\mu^\tau\right) e_\tau \end{pmatrix}. \tag{5.53}$$

Similarly

$$D_\nu = \begin{pmatrix} \partial_\nu + \left(A_\nu^\tau + B_\nu^\tau\right) e_\tau & 0 \\ 0 & \partial_\nu + \left(A_\nu^\tau - B_\nu^\tau\right) e_\tau \end{pmatrix}; \tag{5.54}$$

which gives rise to

$$[D_\mu, D_\nu] = \begin{pmatrix} (G^\tau_{\mu\nu} + \mathsf{G}^\tau_{\mu\nu})e_\tau & 0 \\ 0 & (G^\tau_{\mu\nu} - \mathsf{G}^\tau_{\mu\nu})e_\tau \end{pmatrix} \longmapsto \mathbb{G}^\tau_{\mu\nu}; \tag{5.55}$$

where

$$\begin{aligned} G^\tau_{\mu\nu} =& \partial_\mu A^\tau_\nu - \partial_\nu A^\tau_\mu + e_\tau \left[A^\tau_\mu, A^\tau_\nu\right] \longmapsto E^\tau_{\mu\nu}; \\ \mathsf{G}^\tau_{\mu\nu} =& \partial_\mu B^\tau_\nu - \partial_\nu B^\tau_\mu + e_\tau \left[B^\tau_\mu, B^\tau_\nu\right] \longmapsto H^\tau_{\mu\nu}; \end{aligned} \tag{5.56}$$

respectively represent $SU(2)$ non-Abelian gauge structure associated with electric charge and magnetic monopole.
So, we may write

$$D_\mu \mathbb{G}^\tau{}_{\mu\nu} = \begin{pmatrix} \left(\partial_\mu G^\tau_{\mu\nu} + \partial_\mu \mathsf{G}^\tau_{\mu\nu}\right) e_\tau & 0 \\ 0 & \left(\partial_\mu G^\tau_{\mu\nu} - \partial_\mu \mathsf{G}^\tau_{\mu\nu}\right) e_\tau \end{pmatrix}, \tag{5.57}$$

which may further be reduced to the following compact notation of an octonion formulation as

$$D_\mu \mathbb{G}^\tau{}_{\mu\nu} = \mathbb{J}^\tau_\nu; \tag{5.58}$$

where $\mathbb{J}^\tau_\nu$ $(\forall\, \tau = 1, 2, 3)$, the octonion gauge current in terms of 2×2 Zorn's matrix realization of $SU(2)_e \times SU(2)_m$, may be expressed as

$$\mathbb{J}^\tau_\nu = \begin{pmatrix} (j^\tau_\nu + k^\tau_\nu)\, e_\tau & 0 \\ 0 & (j^\tau_\nu - k^\tau_\nu)\, e_\tau \end{pmatrix}, \tag{5.59}$$

from which we may write following field equations for non-Abelian gauge fields of dyons

$$\begin{aligned} j_{\nu}^{\tau} &= \partial_{\mu} G_{\mu\nu}^{\tau}; \\ k_{\nu}^{\tau} &= \partial_{\mu} \mathsf{G}_{\mu\nu}^{\tau}; \end{aligned} \tag{5.60}$$

Here j_{ν}^{τ} and k_{ν}^{τ} are octonion non-Abelian currents respectively used for electric charge and magnetic monopole for the case of $SU(2)_e \times SU(2)_m$ gauge field theory. Accordingly the continuity equation generalizes as

$$D_{\nu} \mathbb{J}_{\nu}^{\tau} = \begin{pmatrix} (\partial_{\nu} j_{\nu}^{\tau} + \partial_{\nu} k_{\nu}^{\tau})\, e_{\tau} & 0 \\ 0 & (\partial_{\nu} j_{\nu}^{\tau} - \partial_{\nu} k_{\nu}^{\tau})\, e_{\tau} \end{pmatrix} = 0. \tag{5.61}$$

5.6 Condition of 't Hooft Polyakov Monopole

From the fore going analysis, we may easily obtain the case of 't Hooft Polyakov [22, 23] theory of magnetic monopoles. Let us write the complex conjugate of differential operator (5.53) as

$$D_{\mu}^{*} = \partial_{\mu} + e_0 \left(A_{\mu}^{\tau} e_{\tau} \right) - i e_7 \left(B_{\mu}^{\tau} e_{\tau} \right), \tag{5.62}$$

which may further be reduced to 2×2 vector matrix realization of split octonion as

$$D_{\mu}^{*} = \begin{pmatrix} \partial_{\mu} + \left(A_{\mu}^{\tau} - B_{\mu}^{\tau} \right) e_{\tau} & 0 \\ 0 & \partial_{\mu} + \left(A_{\mu}^{\tau} + B_{\mu}^{\tau} \right) e_{\tau} \end{pmatrix}. \tag{5.63}$$

Equation (5.53) and (5.62) gives rise to

$$\begin{aligned}\mathscr{D}_\mu =&\frac{1}{2}\left(D_\mu + D_\mu^*\right)\\ =&\begin{pmatrix} \partial_\mu + A_\mu^\tau e_\tau & 0 \\ 0 & \partial_\mu + A_\mu^\tau e_\tau \end{pmatrix}\\ =&\left(\partial_\mu + A_\mu^\tau e_\tau\right)\hat{1},\end{aligned} \tag{5.64}$$

which describes the covariant derivative of Yang-Mill's field of $SU(2)$ gauge theory.

We may now use the covariant derivative (5.62) in order to discuss Lagrangian used in the Georgi-Glashow model for the description of the $SO(3) \sim SU(2)$ Yang Mills field theory coupled to a Higgs field [22] as

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\mathscr{D}_\mu\vec{\phi}\right)\cdot\left(\mathscr{D}^\mu\vec{\phi}\right) - V(\phi), \tag{5.65}$$

where

$$F_{\mu\nu} =\partial_\mu C_\nu - \partial_\nu C_\mu + e f_{abc} C_b^\mu C_c^\nu, \tag{5.66}$$

and

$$C_\mu =A_\mu^\tau e_\tau; \quad F_{\mu\nu} = F_{\mu\nu}^\tau e_\tau; \tag{5.67}$$

with

$$F_{\mu\nu}^\tau =\partial_\mu A_\nu^\tau - \partial_\nu A_\mu^\tau + e_\tau\left[A_\mu^\tau, A_\nu^\tau\right]. \tag{5.68}$$

Here C_μ is gauge potential of $SO(3)$, and $\vec{\phi}$ is Higgs field in three dimension of $SO(3)$. Thus, the gauge covariant derivative may be written as

$$\mathscr{D}_\mu\vec{\phi} = (\partial_\mu + eC_\mu)\,\vec{\phi}, \tag{5.69}$$

and Higgs potential $V(\phi) \mapsto \frac{1}{4}\lambda\,(\phi^2 - a^2)$. As such, we may develop accordingly the theory of extended (Soliton like) monopole given by 't - Hooft and Polyakov. The equation of motion and other mathematical formulation has already been given in chapter-2 section (2.3.3).

5.7 SU(3) Generators (Gell-Mann Matrices)

The Gell-Mann λ matrices are the representations of the infinitesimal generators of the special unitary group called $SU(3)$. This group has eight dimension, and therefore it has some set with eight linearly independent generators, which can be written as G_A, with A taking values from 1,2,3,..........,8. They obey the following commutation relations as

$$[G_A\,, G_B] = iF^{ABC}G_C; \tag{5.70}$$

where F^{ABC} is the structure constants, and it is completely antisymmetric. i.e. $F^{123} = +1$; $F^{147} = F^{165} = F^{246} = F^{257} = F^{354} = F^{367} = \frac{1}{2}$ and $F^{458} = F^{678} = \frac{\sqrt{3}}{2}$.

Independent generators G_A $(\forall\, A = 1, 2, 3, ..., 8)$ of $SU(3)$ symmetry group are related with the 3×3 Gell-Mann matrices as

$$G_A = \frac{\lambda_A}{2}; \tag{5.71}$$

where λ_A ($\forall\, A = 1, 2, 3, ..., 8$) are defined as

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$
$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \; \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix}; \qquad (5.72)$$

which satisfy the following properties

$$\begin{aligned} (\lambda_A)^\dagger &= \lambda_A; \\ Tr(\lambda_A) &= 0; \\ Tr(\lambda_A \lambda_B) &= 2\delta_{AB}; \\ [\lambda_A, \lambda_B] &= 2iF^{ABC}\lambda_C. \end{aligned} \qquad (5.73)$$

where $A, B, C = 1, 2, 3, ..., 8$. As such, we may summarize the multiplication rules for the generators (in terms of λ matrices) of $SU(3)$ symmetry in the following table [35] as;

$\cdot$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
λ_1	Λ_1	$i\lambda_3$	$-i\lambda_2$	$\frac{i}{2}\lambda_7$	$-\frac{i}{2}\lambda_6$	$\frac{i}{2}\lambda_5$	$-\frac{i}{2}\lambda_4$	$\frac{1}{\sqrt{3}}\lambda_1$
λ_2	$-i\lambda_3$	Λ_2	$i\lambda_1$	$\frac{i}{2}\lambda_6$	$\frac{i}{2}\lambda_7$	$-\frac{i}{2}\lambda_4$	$-\frac{i}{2}\lambda_5$	$\frac{1}{\sqrt{3}}\lambda_2$
λ_3	$i\lambda_2$	$-i\lambda_1$	Λ_3	$-\frac{i}{2}\lambda_5$	$\frac{i}{2}\lambda_4$	$\frac{i}{2}\lambda_7$	$-\frac{i}{2}\lambda_6$	$\frac{1}{\sqrt{3}}\lambda_3$
λ_4	$-\frac{i}{2}\lambda_7$	$-\frac{i}{2}\lambda_6$	$\frac{i}{2}\lambda_5$	Λ_4	$-\frac{i}{2}\lambda_3$	$\frac{i}{2}\lambda_2$	$\frac{i}{2}\lambda_1$	$-\frac{\sqrt{3}}{2}i\lambda_5$
λ_5	$\frac{i}{2}\lambda_6$	$-\frac{i}{2}\lambda_7$	$-\frac{i}{2}\lambda_4$	$\frac{i}{2}\lambda_3$	Λ_5	$-\frac{i}{2}\lambda_1$	$\frac{i}{2}\lambda_2$	$\frac{\sqrt{3}}{2}i\lambda_4$
λ_6	$-\frac{i}{2}\lambda_5$	$\frac{i}{2}\lambda_4$	$-\frac{i}{2}\lambda_7$	$-\frac{i}{2}\lambda_2$	$\frac{i}{2}\lambda_1$	Λ_6	$\frac{i}{2}\lambda_3$	$-\frac{\sqrt{3}}{2}i\lambda_7$
λ_7	$\frac{i}{2}\lambda_4$	$\frac{i}{2}\lambda_5$	$\frac{i}{2}\lambda_6$	$-\frac{i}{2}\lambda_1$	$-\frac{i}{2}\lambda_2$	$-\frac{i}{2}\lambda_3$	Λ_7	$\frac{\sqrt{3}}{2}i\lambda_6$
λ_8	$-\frac{1}{\sqrt{3}}\lambda_1$	$-\frac{1}{\sqrt{3}}\lambda_2$	$-\frac{1}{\sqrt{3}}\lambda_3$	$\frac{\sqrt{3}}{2}i\lambda_5$	$-\frac{\sqrt{3}}{2}i\lambda_4$	$\frac{\sqrt{3}}{2}i\lambda_7$	$-\frac{\sqrt{3}}{2}i\lambda_6$	Λ_8

Table 5.1: Multiplication table for Gell-Mann λ matrices of $SU(3)$ symmetry

Here the value of $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6, \Lambda_7, \Lambda_8$ are described [35] in terms of 3×3 matrices as

$$\Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Longrightarrow (\lambda_1)^2 = (\lambda_2)^2 = (\lambda_3)^2 = \Lambda_{123};$$

$$\Lambda_4 = \Lambda_5 = \Lambda_{45} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Longrightarrow (\lambda_4)^2 = (\lambda_5)^2 = \Lambda_{45};$$

$$\Lambda_6 = \Lambda_7 = \Lambda_{67} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Longrightarrow (\lambda_6)^2 = (\lambda_7)^2 = \Lambda_{67};$$

$$\Lambda_8 = \frac{4}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Longrightarrow (\lambda_8)^2 = \frac{4}{3}I. \tag{5.74}$$

where $\hat{1}$ is 3×3 unit matrix.

5.8 Relation between Octonions basis and SU(3) Generators

Comparing Table-2.1 and Table-5.1, we may observe a resemblance between the octonions and the Gell-Mann λ matrices of $SU(3)$ symmetry on using simultaneously the relations (2.14) and (5.74) in the following table [35] as

Octonions basis	**$SU(3)$ generators**
$e_1 \longmapsto$	$i\lambda_1$
$e_2 \longmapsto$	$i\lambda_2$
$e_3 \longmapsto$	$i\lambda_3$
$e_4 \longmapsto$	$\frac{i}{2}\lambda_4$
$e_5 \longmapsto$	$\frac{i}{2}\lambda_5$
$e_6 \longmapsto$	$-\frac{i}{2}\lambda_6$
$e_7 \longmapsto$	$-\frac{i}{2}\lambda_7$
$e_0 \longmapsto$	$\frac{\sqrt{3}}{2}\lambda_8$

Table 5.2: Relation between Octonion basis and SU(3) generators

As such, we have the freedom to establish [20, 35] a connection between the octonion basis elements e_A and 3×3 Gell-Mann λ matrices of $SU(3)$ in the following manner i.e.

$$\begin{aligned}
e_1 =& i\lambda_1,\ e_2 = i\lambda_2,\ e_3 = i\lambda_3 \ \Longrightarrow e_A \Longleftrightarrow i\lambda_A;\ (\forall\, A = 1, 2, 3.)\\
e_4 =& \frac{i}{2}\lambda_4,\ e_5 = \frac{i}{2}\lambda_5, \Longrightarrow e_A \Longleftrightarrow \frac{i}{2}\lambda_A;\ (\forall\, A = 4, 5);\\
e_6 =& -\frac{i}{2}\lambda_6,\ e_7 = -\frac{i}{2}\lambda_7 \ \Longrightarrow e_A \Longleftrightarrow -\frac{i}{2}\lambda_A;\ (\forall\, A = 6, 7);\\
e_0 =& \frac{\sqrt{3}}{2}\lambda_8.
\end{aligned} \tag{5.75}$$

Theses results are similar to the those derived earlier by Gunaydin Gursey [18] for octonion units and λ matrices of $SU(3)$ symmetry. Equation (5.75) satisfies the Cayley algebra followed by the octonion multiplication rule $e_A \cdot e_B = -\delta_{AB} + f_{ABC}e_C$. So, we may establish the following relations among structure constants of octonions and $SU(3)$ symmetry [35] as

$$\begin{aligned}
F^{ABC} =& f^{ABC};\ (\forall\, ABC = 123)\\
F^{ABC} =& \frac{1}{2}f^{ABC};\ (\forall\, ABC = 147, 246, 257, 435, 516, 673)\\
F^{ABC} =& \frac{\sqrt{3}}{2}f^{ABC}.\ (\forall\, ABC = 458, 678)
\end{aligned} \tag{5.76}$$

Hence, we get

$$\begin{aligned}
[e_A, e_B] =& i\,[\lambda_A, \lambda_B] \qquad (e_c = i\lambda_c);\ (\forall\, ABC = 123)\\
[e_A, e_B] =& \frac{i}{2}\,[\lambda_A, \lambda_B] \qquad (e_c = \frac{i}{2}\lambda_c);\ (\forall\, ABC = 147, 246, 257, 435, 516, 673)\\
[e_A, e_B] =& \frac{\sqrt{3}}{2}i\,[\lambda_A, \lambda_B] \qquad (e_c = \frac{\sqrt{3}}{2}i\lambda_c).\ (\forall\, ABC = 458, 678)
\end{aligned} \tag{5.77}$$

which are the commutation relations among octonions basis elements and Gell-Mann λ matrices of $SU(3)$ symmetry the so called Eight fold way. The benefit to write the octonions in terms of Gell-Mann λ matrices of $SU(3)$

symmetry may be described [35] as

- Non-associativity of octonions does not effect the invariance of the symmetry group $SU(2)$ spin (or isospin) multiplets for the given values of structure constants f^{ABC}.
- It is better to describe the $SU(3)$ symmetry in terms of compact notations of octonions. Accordingly, the theory of strong interactions could be described better in terms of non associative Cayley algebra.
- The eight Gell-Mann λ matrix could be designated in terms of hyper charge which may have the direct link with the scalar octonion unit e_0.
- It may be concluded that the algebra of strong interactions corresponds to the $SU(3)$ automorphisms of the octonion algebra which is in support of the results obtained earlier by Gï¿œnaydin [19].

5.9 Octonions and QCD

The color group $SU(3)$ corresponds to the local symmetry whose gauging gives rise to Quantum Chromodynamics (QCD). There are two different types of $SU(3)$ symmetry. The first one is the symmetry that acts on the different colors of quarks. This symmetry is an exact gauge symmetry mediated by the gluons. Other $SU(3)$ symmetry is a flavor symmetry which rotates different flavors of quarks to each other, or flavor $SU(3)$. Flavor $SU(3)$ is an approximate symmetry of the vacuum of QCD, and is not a fundamental symmetry at all. It is an accidental consequence of the small mass of the three lightest quarks. Here, we are interested in exact $SU(3)$

symmetry of colors in terms of octonion algebra, so as to describe quantum chromodynamics (QCD). For this, let us substitute the values of octonion units e_A in terms of λ_A so that we may express an octonion x [35] as

$$\begin{aligned} x =& x_0\left(\frac{\sqrt{3}}{2}\lambda_8\right) + x_1\left(i\lambda_1\right) + x_2\left(i\lambda_2\right) + x_3\left(i\lambda_3\right) + x_4\left(\frac{i}{2}\lambda_4\right) + x_5\left(\frac{i}{2}\lambda_5\right) \\ &+ x_6\left(-\frac{i}{2}\lambda_6\right) + x_7\left(-\frac{i}{2}\lambda_7\right) \end{aligned} \tag{5.78}$$

which can further be reduced as

$$x = x_0\mathcal{O}_0 + x_1\mathcal{O}_1 + x_2\mathcal{O}_2 + x_3\mathcal{O}_3 + x_4\mathcal{O}_4 + x_5\mathcal{O}_5 + x_6\mathcal{O}_6 + x_7\mathcal{O}_7. \tag{5.79}$$

where $\mathcal{O}_0 \rightarrow \frac{\sqrt{3}}{2}\lambda_8, \mathcal{O}_1 \rightarrow i\lambda_1, \mathcal{O}_2 \rightarrow i\lambda_2, \mathcal{O}_3 \rightarrow i\lambda_3, \mathcal{O}_4 \rightarrow \frac{i}{2}\lambda_4, \mathcal{O}_5 \rightarrow \frac{i}{2}\lambda_5, \mathcal{O}_6 \rightarrow -\frac{i}{2}\lambda_6, \mathcal{O}_7 \rightarrow -\frac{i}{2}\lambda_7$.
Thus the octonion conjugate be written as

$$\overline{x} = x_0\mathcal{O}_0 - x_1\mathcal{O}_1 - x_2\mathcal{O}_2 - x_3\mathcal{O}_3 - x_4\mathcal{O}_4 - x_5\mathcal{O}_5 - x_6\mathcal{O}_6 - x_7\mathcal{O}_7. \tag{5.80}$$

Here the new octonion units $\mathcal{O}_A$ associated with the $SU(3)$ symmetry satisfy the octonion algebra [35], i.e.

$$\mathcal{O}_A \cdot \mathcal{O}_B = -\delta_{AB} + f_{ABC}\mathcal{O}_C. \tag{5.81}$$

As such, we may reformulate the theory of strong interactions, the quantum Chromodynamics (QCD) based on colour $SU(3)_C$ whose generators satisfy the non-associative algebra of octonions. Let us consider the triplet $(u,\ d,\ s)$ (i.e. the up, down, and strange) flavors of quarks as the three objects of the group namely the $SU(3)$ group of flavor symmetry. Each quark has been described in terms of three colors namely the red, blue and green. The dynamics of the quarks and gluons are controlled by the quantum Chromodynamics Lagrangian. The gauge invariant QCD Lagrangian [35] is described as

$$\begin{aligned}\mathcal{L} =&\overline{\psi}_j(i\gamma^{\mu}(D_{\mu})_{jk} - m\delta_{jk})\psi_k - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a \\ =&\overline{\psi}_j(i\gamma^{\mu}\partial_{\mu} - m)\psi_j - gG^a_{\mu}\overline{\psi}_j\gamma^{\mu}T^a_{jk}\psi_k - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a,\end{aligned} \tag{5.82}$$

where $j, k\,(= 1, 2, 3)$ are labeled for three quark fields associated with three colors (namely the red, blue and green) so that we have

$$\psi_j = \begin{pmatrix}\psi_R \\ \psi_B \\ \psi_G\end{pmatrix}; \qquad \bar{\psi}_j = \left(\bar{\psi}_R, \bar{\psi}_B, \bar{\psi}_G\right). \tag{5.83}$$

which is a dynamical function of space-time, in the fundamental representation of the $SU(3)$ gauge group, indexed by $(j,\ k\ =\ 1, 2, 3)$. In equation (5.82), G^a_{μ} is the octet of gluon fields which is also a dynamical function of space-time in the adjoint representation of the SU(3) gauge group, indexed by $a,\ b, ... = 1, 2, ..., 8$; the γ^{μ} are the Dirac matrices connecting the spinor representation to the vector representation of the Lorentz group; and T^a_{jk} are the generators connecting the fundamental, anti-fundamental and adjoint representations of the $SU(3)$ gauge group. In our case, the octonion units connecting to Gell-Mann λ matrices provide one such representation for the generators of $SU(3)$ gauge group. In equation (5.82), the symbol $G^a_{\mu\nu}$ represents the gauge invariant gluon field strength tensor, analogous to the electromagnetic field strength tensor $F_{\mu\nu}$ in electrodynamics. It is described by

$$G^a_{\mu\nu} =\partial_{\mu}G^a_{\nu} - \partial_{\nu}G^a_{\mu} - g\,f^{abc}G^b_{\mu}G^c_{\mu} \quad (\forall\, a, b, c = 1, 2, 3, ..., 8) \tag{5.84}$$

where f^{abc} are the structure constants of $SU(3)$ groups as described above in terms of octonions. In equation (5.82), the constants m and g control the quark mass and coupling constants of the theory, subject to renormalization in the full quantum theory. Here we may introduce a local phase transformation in color space. Under $SU(3)$ symmetry the spinor ψ transforms as

$$\psi \longrightarrow \psi^{'} = U\psi = \exp\{i\lambda.\alpha(x)\}\psi; \qquad (\lambda = 1, 2, ..., 8) \tag{5.85}$$

where

$$\lambda.\alpha(x) = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3 + \lambda_4\alpha_4 + \lambda_5\alpha_5 + \lambda_6\alpha_6 + \lambda_7\alpha_7 + \lambda_8\alpha_8 \tag{5.86}$$

which on using Table-5.2, may directly be written in the in the following form in terms of octonions [35] i.e.

$$\begin{aligned}\lambda.\alpha(x) = & -ie_1\alpha_1 - ie_2\alpha_2 - ie_3\alpha_3 - 2ie_4\alpha_4 \\ & - 2ie_5\alpha_5 - 2ie_6\alpha_6 - 2ie_7\alpha_7 + \frac{2}{\sqrt{3}}e_0\alpha_8\end{aligned} \tag{5.87}$$

As such, the the quantum Chromodynamics (QCD) may be reformulated in terms of octonions and non-associative algebra in order to explain its interesting consequences like

- Quarks confinement
- Color blindness of nature
- Asymptotic freedom
- Calculation for the masses of mesons and baryons etc.

5.10 Split-Octonions $SU(3)$ Gauge Theory

The automorphism group of the Octonion algebra is the 14-dimensional exceptional G_2 group that admits a $SU(3)$ subgroup leaving invariant the idempotents u_0 and u_0^* described by chapter-3 equation (3.1). This $SU(3)_C$ was identified as the color group acting on the quark and anti-quark triplets [19,32]. As such, the automorphism group $SU(3)$ of the quantum mechanical Hilbert space should be considered as an exact symmetry and can not be identified as the symmetry of broken unitary spin gauge group. It is like the $SU(3)_C$ color gauge group of quantum chromo-dynamics (QCD). Therefore, in order to describe the $SU(3)$ gauge theory suitably handled with split octonions, let us start with the split octonion equivalent of any four vector A_μ and its conjugate in terms of 2×2 Zorn matrix realization (3.21) which is given in chapter-3. The Octonion covariant derivative or $\mathcal{O}$- derivative of an octonion K is defined [32-34] as

$$K_{\|\mu} = K_{,\mu} + [\Im_\mu, K], \tag{5.88}$$

where $\Im_\mu$ is the Octonion affinity. It is the object that makes $K_{\|\mu}$ transform like an octonion under $\mathcal{O}$ transformations i.e.

$$\begin{aligned} K' &= UKU^{-1} \\ K'_{\|\mu} &= UK_{\|\mu}U^{-1} \\ \Im'_\mu &= U\Im_\mu U^{-1} - \frac{\partial U}{\partial x^\mu}U^{-1}, \end{aligned} \tag{5.89}$$

where the $U(x)$ are octonions which define local (Octonion) unitary transformations and are isomorphic to the rotation group $O(3)$. Thus, equation

(5.89) describes $SU(2)$ nature of octonion $\mathcal{O}$ transformations resulting to the octonion affinity (gauge potential) $\Im_{\mu}$ of Yang-Mill's type field and is expressed [34] as,

$$\Im_{\mu}=-L_{\mu j}u_{j}^{*}-K_{\mu j}u_{j}=\begin{bmatrix}0_{2} & L_{\mu j}e_{j}\\ -K_{\mu j}e_{j} & 0_{2}\end{bmatrix}\quad(\forall\, j=1,2,3)\tag{5.90}$$

where the quaternion units $e_j=-i\sigma_j$ are suitably handled [31] with Pauli spin matrices σ_j. Now, we have the freedom to extend $SU(2)$ gauge theory to the case of $SU(3)$ Yang Mills gauge theory of colored quarks by replacing the Pauli spin matrices to Gellmann λ matrices. So, from equation (5.90), it is clear that octonion covariant derivative (5.90) is subjected by two real (or one complex) gauge potential transformations. Hence, G_{μ}^{a} the octet of gluon fields describing Lagrangian (5.82) is either a complex gauge field or comprises the order pair of two real gauge fields. So, we may write [35] the covariant derivative D_{μ} for $SU(3)$ Lagrangian (5.82) as

$$D_{\mu}=\partial_{\mu}+\mathbb{V}_{\mu};\tag{5.91}$$

where $\mathbb{V}_{\mu}$ is the octonion form of generalized four potential [35] described as

$$\mathbb{V}_{\mu}=e_{0}\left(A_{\mu}^{\alpha}e_{\alpha}\right)+ie_{7}\left(B_{\mu}^{\alpha}g_{\alpha}\right).\qquad(\forall\,\mu=0,1,2,3;\;\alpha=1,2,......,8.)\tag{5.92}$$

The beauty of the equation (5.92) reinforces the $SU(3)$ symmetry of colored quarks with two gauge potentials as the consequence of automorphism group of split octonions in terms of 2×2 Zorn vector matrix realization. Here the two gauge potentials A_{μ}^{α} and $B_{\mu}^{\alpha}(\forall\,\mu=0,1,2,3;\;\alpha=1,2,......,8.)$ may be identified as the gauge potentials for two chromo charges supposed to be responsible for the existence of electric and magnetic chromo-charges. It may, therefore, be concluded that octonion colored quarks are dyons i.e. the particles which carry the simultaneous existence of electric and magnetic charges [24-27]. Substituting the value of $SU(3)_C$ octonion gauge potential $\mathbb{V}_\mu$ (5.92) in to the equation (5.91), we may write the covariant derivative D_μ [35] as

$$\begin{aligned} D_\mu =& \partial_\mu + e_0\left(A_\mu^\alpha e_\alpha\right) + ie_7\left(B_\mu^\alpha g_\alpha\right) \\ =& u_0^*\left(\partial_\mu + A_\mu^\alpha e_\alpha + B_\mu^\alpha g_\alpha\right) + u_0\left(\partial_\mu + A_\mu^\alpha e_\alpha - B_\mu^\alpha g_\alpha\right), \end{aligned} \tag{5.93}$$

which may be written in term of 2×2 Zorn matrix realization [34, 35] as

$$D_\mu = \begin{pmatrix} \partial_\mu + \left(e_\alpha A_\mu^\alpha + g_\alpha B_\mu^\alpha\right) & 0 \\ 0 & \partial_\mu + \left(e_\alpha A_\mu^\alpha - g_\alpha B_\mu^\alpha\right) \end{pmatrix}. \tag{5.94}$$

It yields to

$$[D_\mu, D_\nu] = \begin{pmatrix} \begin{matrix} \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + e_\alpha\left[A_\mu^\alpha,\, A_\nu^\alpha\right] \\ +\partial_\mu B_\nu^\alpha - \partial_\nu B_\mu^\alpha + g_\alpha\left[B_\mu^\alpha,\, B_\nu^\alpha\right] \end{matrix} & 0 \\ 0 & \begin{matrix} \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + e_\alpha\left[A_\mu^\alpha,\, A_\nu^\alpha\right] \\ -\partial_\mu B_\nu^\alpha + \partial_\nu B_\mu^\alpha - g_\alpha\left[B_\mu^\alpha,\, B_\nu^\alpha\right] \end{matrix} \end{pmatrix}. \tag{5.95}$$

Equation (5.95) may be reduced as

$$[D_\mu, D_\nu] = \begin{pmatrix} G_{\mu\nu}^\alpha e_\alpha + \mathsf{G}_{\mu\nu}^\alpha g_\alpha & 0 \\ 0 & G_{\mu\nu}^\alpha e_\alpha - \mathsf{G}_{\mu\nu}^\alpha g_\alpha \end{pmatrix} \longmapsto \mathbb{G}_{\mu\nu}^\alpha; \tag{5.96}$$

where

$$G_{\mu\nu}^{\alpha} = \partial_{\mu}A_{\nu}^{\alpha} - \partial_{\nu}A_{\mu}^{\alpha} + e_{\alpha}\left[A_{\mu}^{\alpha},\, A_{\nu}^{\alpha}\right] \longmapsto E_{\mu\nu}^{\alpha}; \tag{5.97}$$

$$\mathsf{G}_{\mu\nu}^{\alpha} = \partial_{\mu}B_{\nu}^{\alpha} - \partial_{\nu}B_{\mu}^{\alpha} + g_{\alpha}\left[B_{\mu}^{\alpha},\, B_{\nu}^{\alpha}\right] \longmapsto H_{\mu\nu}^{\alpha}; \tag{5.98}$$

are two $SU(3)$ non-Abelian gauge field strengths in term of electric ($E_{\mu\nu}^{\alpha}$) and magnetic field ($H_{\mu\nu}^{\alpha}$) field strengths obtained respectively from electric (A_{μ}^{α}) and magnetic (B_{μ}^{α}) gauge potentials, i.e.

$$\mathbb{G}_{\mu\nu}^{\alpha} = \begin{pmatrix} E_{\mu\nu}^{\alpha}e_{\alpha} + H_{\mu\nu}^{\alpha}g_{\alpha} & 0 \\ 0 & E_{\mu\nu}^{\alpha}e_{\alpha} - H_{\mu\nu}^{\alpha}g_{\alpha} \end{pmatrix}. \tag{5.99}$$

Operating the covariant derivative D_{μ} (5.94) to the the generalized field strength $\mathbb{G}_{\mu\nu}^{\alpha}$ of dyons (5.99), we get [35]

$$D_{\mu}\mathbb{G}^{\alpha}{}_{\mu\nu} = \begin{pmatrix} \partial_{\mu}G_{\mu\nu}^{\alpha}e_{\alpha} + \partial_{\mu}\mathsf{G}_{\mu\nu}^{\alpha}g_{\alpha} & 0 \\ 0 & \partial_{\mu}G_{\mu\nu}^{\alpha}e_{\alpha} - \partial_{\mu}\mathsf{G}_{\mu\nu}^{\alpha}g_{\alpha} \end{pmatrix} \Longrightarrow \mathbb{J}_{\nu}^{\alpha}; \tag{5.100}$$

where $\mathbb{J}_{\nu}^{\alpha}$ is the generalized octonion gauge current in term of 2×2 Zorn realization of split octonion $SU(3)$ gauge theory. It also comprises the electric and magnetic four currents of dyons as

$$\mathbb{J}_{\nu}^{\alpha} = \begin{pmatrix} J_{\nu}^{\alpha}e_{\alpha} + K_{\nu}^{\alpha}g_{\alpha} & 0 \\ 0 & J_{\nu}^{\alpha}e_{\alpha} - K_{\nu}^{\alpha}g_{\alpha} \end{pmatrix}. \tag{5.101}$$

Here $J_{\nu}^{\alpha} = \partial_{\mu}G_{\mu\nu}^{\alpha}$ and $K_{\nu}^{\alpha} = \partial_{\mu}\mathsf{G}_{\mu\nu}^{\alpha}$ are the four currents respectively associated with the presence of electric and magnetic charges. So, it is concluded that split octonion $SU(3)$ gauge theory of colored quarks describes dyons which are the particles carrying the simultaneous existence of electric and magnetic monopoles.

5.11 Discussion and Conclusion

In the this chapter, we have applied the method of split octonion (i.e. Zorn's vector matrix realization) to reformulate the gauge field equations of dyons as the direct extension of split octonion formulation of electrodynamics. Since, for the case of split-octonion we have the freedom to write the components of Zorn's vector matrices either real or complex. It should be noted that the dyons are considered as the particles carrying generalized fields and charge as complex quantity whose real part is identified to discuss the usual electrodynamics while the imaginary part is concerned with the existence of monopole and their fields. The lack of associativity in octonion formulation of dyons developed in previous chapter forbids their group theoretical study in terms of abelian and non-Abelian group structures. The advantages of split octonion formalism are discussed here in terms of compact and simpler notations of split octonion valued potential, field and currents of dyons free from the non associativity. So, in the section (5.2), we have started with the U(1)×U(1) gauge theory of dyons in terms of the Lagrangian equation, global gauge transformation, spinor transformation, covariant derivative, field equations and other quantum equations in compact, simpler and consistent manner.

In section (5.3), we have investigated the U(1)×U(1) octonion gauge formulation in terms of 2×2 Zorn vector matrix realization of split octonion. Here, the split octonion valued space time vector has been defined by equation (5.25) in terms of 4×4 matrix representation. Equation (5.26) expressed as the space time metric has been obtained as bi linear in order to described the inner product. The octonion conjugate of space time vector has been obtained in equation (5.27), while the split octonion Hermitian conjugate has been established by equation (5.28) followed by the split octonion differential operator in equation (5.29). The octonion differential operator (5.29) has been investigate in terms of 2×2 Zorn vector matrix realization given by equation (5.30). Hence, we have developed the octonion covariant derivative for U(1)×U(1) gauge theory of dyons in terms of 2×2 Zorn vector matrix realization. It is shown that the commutation relation between the octonion covariant derivative leads to two types of gauge field strength of generalized electromagnetic fields of dyons. The two gauge field strengths are present due to the existence of electric and magnetic charges in dyons. So, these are named as the electromagnetic field tensors associated respectively with electric and magnetic charges. Equation (5.37) has been investigated as the octonionic representation of generalized Dirac Maxwell's equations of dyons. It leads to two photons whose currents are defined as j_μ (associated with electric charge) and the k_μ (associated with magnetic charge). Therefore, the two photons group are associated with two U(1) groups in terms of two four-currents given by equation (5.38) of octonion gauge theory. So, it is the octonion gauge theory which provides simultaneously the existence of U(1)×U(1) group of generalized electromagnetic fields of dyons. Thus, it is clear that the octonion gauge theory leads two types of gauge fields which are supposed to be the existence of magnetic monopole (dyon).

Accordingly, we have constructed the octonion gauge fields as the combination of two quaternion gauge fields by equation (5.39). The covariant derivative of U(1)×SU(2) octonion gauge field in terms of 2×2 Zorn's vector matrix realization has been constructed in equations (5.40) and (5.41). It is shown that the commutation relation of the octonion covariant derivative leads to U(1)×SU(2) generalized fields vector given by equation (5.42) in terms of 2×2 Zorn vector realization of split octonions. Thus, equation (5.42) leads to the abelian and non-Abelian gauge structures in the presence of electric charge and magnetic monopole. Equations (5.43) provides the expressions for U(1)×SU(2) gauge field strengths associated with electric charge. Thus in U(1)×SU(2) gauge theory of electroweak interaction. On the other hand, equation (5.44) provides the U(1)×SU(2) gauge theory due to the presence of magnetic monopole. Equation (5.44) has been designated as the $U(1)_m \times SU(2)_m$ gauge theory while the subscript m denotes the magnetic monopole. It behaves like dual electroweak theory where the massive gauge particles are taking part due to the presence of magnetic monopole. An current has been obtain in equation (5.46) which shows that the U(1)×SU(2) gauge theory of dyons leads to the conservation of U(1)×SU(2) current. As such, we have two fold symmetry whose first fold exhibit due to presence of the electric charge while the second field gives rise the presence of magnetic monopole. Hence, we get the two U(1) and two SU(2) currents given by equation (5.48) and the analogous continuity equation has been investigated by equation (5.49).

In section (5.5), we have extended U(1)×SU(2) to the split octonionic non-Abelian $SU(2)_e \times SU(2)_m$ gauge formulation in terms of 2×2 Zorn vector matrix. Accordingly, in order to described $SU(2)_e \times SU(2)_m$ gauge formulation, we have obtained the covariant derivative in equation (5.50) which are further explained in equations (5.52) and (5.54), while the octonionic representation of generalized four potential has been expressed in equation (5.51). As such, we have obtained the two fold SU(2) non-Abelian gauge structure in equation (5.55) whose electric and magnetic components are expressed by equation (5.56). So, equation (5.57) represents the non-Abelian extension of GDM equation which has been written by equation (5.58) in simple, compact and consistent notation. The octonion gauge current in terms of 2×2 Zorn's matrix realization of $SU(2)_e \times SU(2)_m$ has been obtained in equation (5.59) while the electric and magnetic currents are investigated by equation (5.60). Analogous continuity equation has been obtained by equation (5.61). So, we can correlated our theory to the 't Hooft Polyakov theory of magnetic monopoles. Defining the complex conjugate of covariant differential operator by equation (5.62), we have established the covariant derivative of octonion gauge theory in terms 2×2 Zorn vector matrix realization given equation (5.63). As such, the covariant derivative has been obtained by equation (5.64) as the covariant of Yang-Mill's field SU(2) gauge theory. Hence, it is emphasized that there is the inter relationship between the octonion gauge theory and the 't-Hooft-Polyakov monopole theory. So, we may describe the Lagrangian formulation in terms of equations (5.65) - (5.69). Thus, we have correlated the Lagrangian formulation of 't-Hooft-Polyakov theory to our octonion gauge theory and reestablished the covariant derivative by equation (5.69). It is concluded that our theory established the existence of soliton like magnetic monopole provided by 't-Hooft Polyakov theory.

In section (5.7), we have discussed the Gell-Mann λ- matrices which represents the infinitesimal generators of the special unitary group SU(3). The properties of eight linearly independent generator has been defined by equations (5.70) and (5.71). The generators of SU(3) symmetry group are defined in terms of 3×3 Gell-Mann λ- matrices equation (5.72) with their properties by equation (5.73). As such, we may summarize the multiplication properties of SU(3) generators in terms of multiplication table (5.1) whose parameters are defined by equation (5.74). In section (5.8), we have established the relation between the octonions basis elements and the SU(3) generators. In table (5.2), we have shown the isomorphism between the octonion basis elements and the Gell-Mann λmatrices of SU(3) group. As such, we has been established the a proper mapping between the octonion basis elements e_A and the generators of SU(3) symmetry in equation (5.75). Accordingly, the structure constants of octonions and SU(3) generators are related by equation (5.76) while the commutation relations among octonion basis elements and SU(3) generators are related by equation (5.77). The benefit to write the SU(3) generators in terms of octonion basis elements are

- Non-associativity of octonions does not effect the invariance of the symmetry group $SU(2)$ spin (or isospin) multiplets for the given values of structure constants f^{ABC}.

- It is better to describe the $SU(3)$ symmetry in terms of compact notations of octonions. Accordingly, the theory of strong interactions could be described better in terms of non associative Cayley algebra.

- The eighth Gell-Mann λ matrix could be designated in terms of hyper charge which may have the direct link with the scalar octonion unit e_0.

- It may be concluded that the algebra of strong interactions corresponds to the $SU(3)$ automorphisms of the octonion algebra which is in support of the results obtained earlier by Gï¿œnaydin [19].

In section (5.9), we have discussed the relationship of octonions and the parameters of Quantum Chromodynamics (QCD). The color group $SU(3)$ corresponds to the local symmetry whose gauging gives rise to Quantum Chromodynamics (QCD). As such, the values of octonion basis elements in terms of Gell-Mann λ- matrices has been established in equations (5.78) and (5.79). Here, we are concerned with the exact $SU(3)$ symmetry of colors in terms of octonion algebra as to describe quantum chromodynamics (QCD). The octonions conjugate of equation (5.79) has been obtained by equation (5.80), which is further be expressed by equation (5.81). As such, we have re-formulated the theory of strong interactions, the quantum Chromodynamics (QCD), based on colour $SU(3)_C$ whose generators satisfy the non-associative algebra of octonions. The dynamics of the quarks and gluons are controlled by the quantum chromodynamics Lagrangian, which is given by equation (5.82). In our case, the octonion units connecting to Gell-Mann λ matrices provide one such representation for the generators of $SU(3)$ gauge group. So, in equation (5.82), the symbol $G^a_{\mu\nu}$ represents the gauge invariant gluon field strength tensor, analogous to the electromagnetic field strength tensor $F_{\mu\nu}$ in electrodynamics. Equation (5.83) represents the dynamic function of space time in the SU(3) gauge group. The gluonic field strength tensor which is analogous to the electromagnetic field strength tensor has been obtained by equation (5.84). The local phase transformation in color space has been expressed by equation (5.85) while the equation (5.86) and (5.87) represent the octonionic multiplication with the λ- matrices. As such, the the quantum Chromodynamics (QCD) may be reformulated in terms of octonions and non-associative algebra in order to explain its interesting consequences like

- Quarks confinement
- Color blindness of nature
- Asymptotic freedom
- Calculation for the masses of mesons and baryons etc.

The automorphism group of the Octonion algebra is the 14-dimensional exceptional G_2 group that admits a $SU(3)$ subgroup leaving invariant the idempotents u_0 and u_0^*. This $SU(3)_C$ was identified as the color group acting on the quark and anti-quark triplets. As such, the automorphism group $SU(3)$ of the quantum mechanical Hilbert space should be considered as an exact symmetry and can not be identified as the symmetry of broken unitary spin gauge group. It is like the $SU(3)_C$ color gauge group of quantum chromodynamics (QCD). Therefore, we have investigated the $SU(3)$ gauge theory suitably handled with split octonions in section (5.10). So, the octonion covariant derivative of an octonion representation has been written in equation (5.88), while the octonionic transformations are represented by equation

(5.89). Equation (5.89) describes $SU(2)$ nature of octonion $\mathcal{O}$ transformations resulting to the octonion affinity (gauge potential) $\Im_{\mu}$ of Yang-Mill's type field given by equation (5.90). Accordingly, the covariant derivative for SU(3) Lagrangian has been established in equation (5.91). Equation (5.92) defines the octonion form of generalized four potential in the case of SU(3) gauge theory. It is emphasized that the beauty of the equation (5.92) reinforces the $SU(3)$ symmetry of colored quarks with two gauge potentials as the consequence of automorphism group of split octonions in terms of 2 $\times$ 2 Zorn vector matrix realization. From the foregoing analysis one can draw conclusion that octonion colored quarks behave as dyons (i.e. the particles which carry the simultaneous existence of electric and magnetic charges). Accordingly, the covariant derivative in terms of $SU(3)_{c}$ octonion gauge potential has been written in equation (5.93)-(5.95). As such, the generalized field strengths of dyons has been developed in equation (5.96), as the combination of two SU(3) non-Abelian gauge field strengths respectively known as electric and magnetic field strength given equations (5.97)-(5.99). We have established the generalized octonion gauge current equation (5.100) in terms of 2×2 Zorn realization of split octonion $SU(3)$ gauge theory. It also comprises the electric and magnetic four currents of dyons in equation (5.101). So, it is concluded that split octonion $SU(3)$ gauge theory of colored quarks describes dyons saying that the quarks are dyons.

Bibliography

[1] L. E. Dickson, **“On Quaternions and Their Generalization and the History of the Eight Square Theorem”**, Ann. Math., **20** (1919), 155.

[2] W. R. Hamilton, **“Elements of Quaternions”**, Chelsea Publications Co., New York, (1969).

[3] P. G. Tait, **"An elementary Treatise on Quaternions",** Oxford Univ. Press (1875).

[4] A. Cayley, **“An Jacobi’s elliptic functions, in reply to the Rev. B. Bornwin; and on quaternion”**, Phil. Mag., **26** (1845), 208.

[5] R. P. Graves, **“Life of Sir William Rowan Hamilton”,** 3 volumes, Arno Press, New York, (1975).

[6] J. C. Baez, **“The Octonions”**, Bull. Amer. Math. Soc., **39** (2001), 145.

[7] K. Morita, **“Quaternionic Variational Formalism for Poincarï¿œ Gauge Theory and Supergravity”**, Prog. Theor. Phys., **73** (1985), 999.

[8] K. Morita, **“Gauge Theories over Quaternions and Weinberg-Salam Theory”,** Prog. Theor. Phys., **65** (1981), 2071.

[9] K. Morita, **"Quaternionic Weinberg-Salam Theory",** Prog. Theor. Phys., **67** (1982), 1860.

[10] S. Catto, **" Exceptional Projective Geometries and Internal Symmetries"**, eprint: arXiv: hep-th/0302079 (2003).

[11] S. Catto, **"Symmetries in Science:from the rotation group to quantum algebras"**, Ed. B. Gruber, Plenum Press, **6** (1993), 129.

[12] R. Foot and G. C. Joshi, **"Space-time symmetries of superstring and Jordan Algebras"**, Int. J. Theor. Phys., **28** (1989), 1449.

[13] R. Foot and G. C. Joshi, **"String theories and the Jordan algebras"**, Phys. Lett., **B199** (1987), 203.

[14] T. Kugo and P. Townsend, **"Supersymmetry and the Division Algebras"**, Nucl Phys., **B221** (1983), 357.

[15] C. A. Manogue and A. Sudbery, **"General solutions of covariant superstring equations of motion"**, Phys. Rev., **D40** (1989), 4073.

[16] J. Schray, **"Octonions and Supersymmetry",** Ph. D. thesis, Department of Physics, Oregon State University, (1994) (un published).

[17] K. S. Abdel-Khalek Mostafa, **"Ring Division Algebra, Self Duality and Supersymmetry"**, eprint: arXiv: hep-th/0002155 (2000).

[18] M. Gï¿œnaydin and F. Gï¿œrsey, **"Quark structure and octonions"**, J. Math. Phys., **14** (1973), 1651.

[19] M. Gï¿œnaydin, **"Octonionic Hilbert spaces, the Poincarï¿œ group and SU(3)"**, J. Math. Phys., **17** (1976), 1875.

[20] Pushpa, P. S. Bisht, Tianjun Li and O. P. S. Negi, **"Quaternion Octonion Reformulation of Quantum Chromodynamics"**, Int. J. Theor. Phys., **50** (2011), 594.

[21] P. A. M. Dirac, **"Quantised Singularities in the Electromagnetic Field"**, Proc. Royal Society, **A133** (1931), 60.

[22] G. 't Hooft, **"Magnetic monopoles in unified gauge theories"**, Nucl. Phys., **B79** (1974), 276.

[23] A. M. Polyakov, **"Particle spectrum in quantum field theory"**, JETP Lett., **20** (1974), 194.

[24] J. Schwinger, **"Dyons Versus Quarks"**, Science, **166** (1969), 690.

[25] D. Zwanzinger, **"Quantum Field Theory of Particles with Both Electric and Magnetic Charges"**, Phys. Rev., **176** (1968), 1489.

[26] B. Julia and A. Zee , **"Poles with both magnetic and electric charges in non-Abelian gauge theory"**, Phys. Rev., **D11** (1975), 2227.

[27] O. P. S. Negi and H. Dehnen , **"Gauge Formulation for Two potential Theory of Dyons"**, Int. J. Theor. Phys., **50** (2011), 2446.

[28] B. C. Chanyal, P. S. Bisht and O. P. S. Negi , **"Generalized Octonion Electrodynamics"**, Int. J. Theor. Phys., **49** (2010), 1333.

[29] B. C. Chanyal, P. S. Bisht and O. P. S. Negi, **"Generalized Split-Octonion Electrodynamics"**, Int. J. Theor. Phys., **50** (2011), 1919.

[30] P. S. Bisht, B. Pandey and O. P. S. Negi, **"Interpretations of octonion wave equations"**, FIZIKA (Zagreb), **B17** (2008), 405.

[31] Pushpa, P. S. Bisht and O. P. S. Negi , **"Spontaneous symmetry breaking in presence of electric and magnetic charges"**, Int. J. Theor. Phys., **50** (2011), 1927.

[32] C. Castro, **"On the noncommutative and nonassociative geometry of octonionic space time, modified dispersion relations and grand unification"**, J. Math. Phys., **48** (2007), 073517.

[33] S. Marques and C. G. Oliveira, **"An extension of quaternionic metrics to octonions"**, J. Math. Phys., **26** (1983), 3131.

[34] S. Dangwal, P S Bisht and O P S Negi, **"Octonionic Gauge Formulation for Dyonic Fields"**, eprint: arXiv: hep-th/0608061 (2006).

[35] B. C. Chanyal, P. S. Bisht, Tianjun Li and O. P. S. Negi, **"Octonion Quantum Chromodynamics",** Int. J. Theor. Phys., **51** (2012), 3410.

CHAPTER 6

Role of Octonions in Physics Beyond Standard Model (BSM)

A part of this chapter is communicated to the publication in the Int. J. Theor. Phys.

ABSTRACT

In this chapter, we have made an attempt to discuss the role of octonions in physics beyond standard model. Thus, we have discussed the role of octonions in grand unified theories (GUTs) gauge group of which is describe is $SU(3) \times SU(2) \times U(1)$ ***followed by the role of octonions in supersymmetry. Further more, we have analyzed the role of octonions in gravity and dark matter where, we have described the octonion space as the combination of two quaternionic spaces namely gravitational G-space and electromagnetic EM-space. It is shown that octonionic hot dark matter contains the photon and graviton (i.e. massless particles) while the octonionic cold dark matter is associated with the*** $W^{\pm}, Z^{0}$ ***(massive) bosons. At last, we have described the role of octonion consistently in superstring theory (i.e. a theory of everything to describe the unification of all four types of forces namely gravitational, electromagnetic, weak and strong).***

Chapter 6

Role of Octonions in Physics Beyond Standard Model (BSM)

6.1 Introduction:- The Standard Model

The Standard Model (SM) [1-5] of particle physics summarizes all [6-11] we know about the fundamental forces of electromagnetism, as well as the weak and strong interactions [12] (without gravity). The Standard Model consists of elementary particles grouped into two classes: bosons [12] (particles that transmit forces) and fermions [12] (particles that make up matter). The bosons have particle spin that is either 0, 1 or 2. The fermions have spin 1/2. On the other hand, particle physics strives to identify the building blocks of matter and describe the interactions that bind them: the set of instructions needed to create a universe. Our most succinct and (we believe) accurate set of instructions is encapsulated in a quantum field theory [1,3,4] called the Standard Model, which describes a universe [13] made up of six types of quarks and six types of leptons, bound together by three fundamental forces: strong, weak, and electromagnetic. The standard model is a relativistic quantum field theory [1-3] that incorporates the basic principles of quantum mechanics and special relativity. Like quantum electrodynamics (QED) the standard model is a gauge theory [14]. However, with the non-Abelian gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ instead of the simple Abelian $U(1)_{em}$ gauge group of QED. The gauge bosons are the photons mediating the electromagnetic interactions, the W- and Z-bosons mediating the weak interactions [12], as well as the gluons mediating the strong interactions [2,3,12]. Gauge theories can exist in several phases: in the Coulomb phase with massless gauge bosons (like in QED), in the Higgs-phase with spontaneously broken gauge symmetry [14] and with massive gauge bosons (e.g. the W- and Z-bosons), and in the confinement phase, in which the gauge bosons do not appear in the spectrum (like the gluons in quantum chromodynamics (QCD)). On the other hand, The Standard Model was formulated in the 1970s and tentatively established by experiments in the early 1980s. Nearly three decades of exacting experiments have tested and verified the theory in meticulous detail, confirming all of its predictions. Thus, the Standard Model of particle physics is the most successful theory of nature in history, but increasingly there are signs that it must be extended by adding new particles that play roles in high-energy reactions.

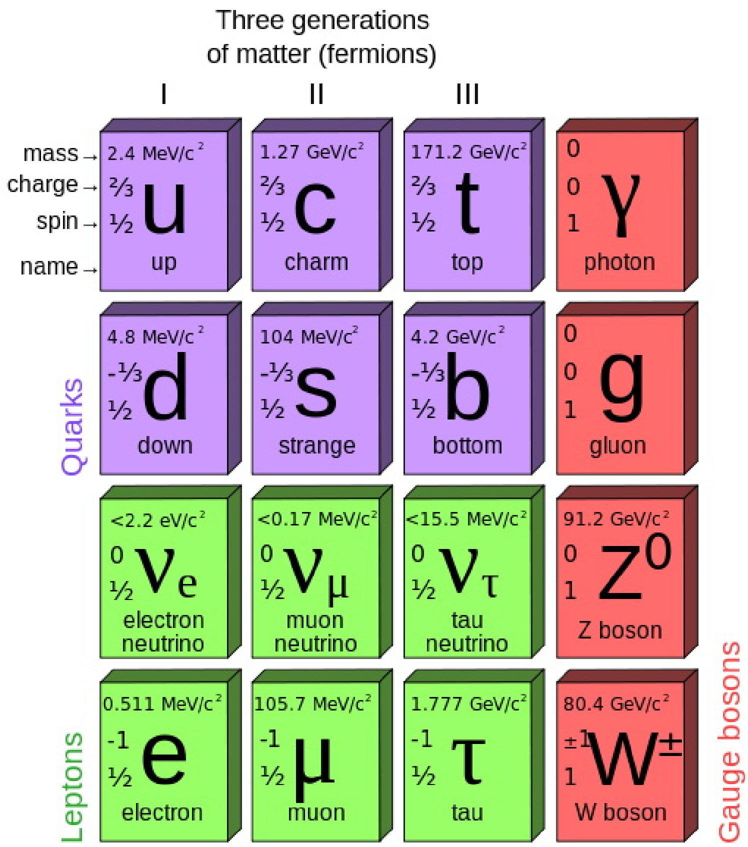

Figure 6.1: The SM particle content

The main ingredients of the SM are shown in Fig. 6.1. The particles involved are characterized by their spin, their mass, and the quantum numbers (charges). In the Standard Model Fig. 6.1, includes 12-elementary particles of spin known as fermions and 4-gauge bosons. According to the spin-statistics theorem [12], fermions respect the Pauli exclusion principle. Each fermion has a corresponding antiparticle. The fermions of the Standard Model are classified according to how they interact (or equivalently, by what charges they carry). There are six quarks (up, down, charm, strange, top, bottom), and six leptons (electron, electron neutrino, muon, muon neutrino, tau, tau neutrino). The gauge bosons are defined as force carriers that mediate the strong, weak, and electromagnetic fundamental interactions. All the Standard Model particles given in Fig. 6.1 have been detected by experiment already [12,15], except the Higgs boson which remained the mystery and challenge to the physicists. Thanks god, on July 4, 2012, the Compact Muon Solenoid (CMS) and the Argonne Tandem Linear Accelerator System (ATLAS) experimental teams at the Large Hadron Collider (LHC) independently announced that they each confirmed the formal discovery of a Higgs boson of mass between 125 and 127 GeV/c^2 [16]. So, the Higgs model is completely tested and verified by the experiment.

6.1.1 Problems with the Standard Model

Despite being the most successful theory of particle physics to date, the Standard Model is not perfect [17,18]. The deficiencies of the Standard Model on the bans of experimental observations which are not yet explain, are described as

- **Gravity**: The standard model does not provide an explanation of gravity [19]. Moreover it is incompatible with the most successful theory of gravity to date, general relativity.
- **Dark matter and dark energy:** Cosmological observations tell us that the standard model is able to explain only about 4% of the energy present in the universe. Of the missing 96%, about 24% should be dark matter [20], i.e. matter that behaves just like the other matter we know, but which interacts only weakly with the standard model fields. The rest should be dark energy, a constant energy density for the vacuum.

Attempts to explain the dark energy in terms of vacuum energy of the standard model lead to a mismatch of 120 orders of magnitude.

- **Neutrino masses:** According to the standard model the neutrinos are massless particles [21]. However, neutrino oscillation experiments have shown that neutrinos do have mass. Mass terms for the neutrinos can be added to the standard model by hand, but these lead to new theoretical problems [21]. (For example, the mass terms need to be extraordinarily small).

- **Matter/antimatter asymmetry:** The universe is made out of mostly matter. However, the standard model predicts that matter and antimatter [22] should have been created in (almost) equal amounts, which would have annihilated each other as the universe cooled.

On the other hand the standard model is incomplete with respect to theoretical problems associated with

- **Hierarchy problem** – the standard model introduces particle masses through a process known as spontaneous symmetry breaking caused by the Higgs field. Within the standard model, the mass of the Higgs gets some very large quantum corrections due to the presence of virtual particles (mostly virtual top quarks) [23]. These corrections are much larger than the actual mass of the Higgs. This means that the bare mass parameter of the Higgs in the standard model must be fine tuned in such a way that almost completely cancels the quantum corrections. This level of fine tuning is deemed unnatural by many theorists.

- **Strong CP problem** – theoretically it can be argued that the standard model should contain a term that breaks CP symmetry [24] —relating matter to antimatter— in the strong interaction sector. Experimentally, however, no such violation has been found, implying that the coefficient of this term is very close to zero. This fine tuning is also considered unnatural.

- **Number of parameters** – the standard model depends on 19 numerical parameters. Their values are known from experiment, but the origin of the values is unknown. Some theorists have tried to find relations between different parameters, for example, between the masses of particles in different generations.

6.1.2 Physics Beyond the Standard Model

Physics beyond the Standard Model [25,26] refers to the theoretical developments needed to explain the deficiencies of the Standard Model, such as the origin of mass, the strong CP problem, neutrino oscillations, matter–antimatter asymmetry, and the nature of dark matter and dark energy [22,26]. Another problem lies within the mathematical framework of the Standard Model itself. The Standard Model is inconsistent with that of general relativity to the point that one or both theories break down in their descriptions under certain conditions (for example within known space-time singularities like the Big Bang and black hole event horizons). Theories that lie beyond the Standard Model include various extensions of the standard model are given in following subsections.

6.1.2.1 Grand Unified theories

A Grand Unified Theory (GUT) [12,26,27], is a model in particle physics in which at high energy, the three gauge interactions of the Standard Model which define the electromagnetic, weak, and strong interactions, are merged into one single interaction characterized by one larger gauge symmetry and thus one unified coupling constant. In contrast, the experimentally verified Standard Model of particle physics is based on three independent interactions, symmetries and coupling constants.

The standard model has three gauge symmetries [12]; the colour $SU(3)$, the weak isospin $SU(2)$, and the hypercharge $U(1)$ symmetry, corresponding to the three fundamental forces. Due to renormalization the coupling constants of each of these symmetries vary with the energy at which they are measured. Around 10^{16} GeV these couplings become approximately equal. This has led to speculation that above this energy the three gauge symmetries of the standard model are unified in one single gauge symmetry with a simple group gauge group, and just one coupling constant. Below this energy the symmetry is spontaneously broken to the standard model symmetries [28]. Unifying gravity with the other three interactions would provide a theory of everything (TOE), rather than a GUT. Nevertheless, GUTs are often seen as an intermediate step towards a TOE. The new particles predicted by models of grand unification cannot be observed directly at particle colliders because their masses are expected to be of the order of the so-called GUT scale, which is predicted to be just a few orders of magnitude below the Planck scale and thus far beyond the reach of currently foreseen collision experiments. Instead, effects of grand unification might be detected through indirect observations such as proton decay, electric dipole moments of elementary particles, or the properties of neutrinos [29]. Some grand unified theories predict the existence of magnetic monopoles. Popular choices for the unifying group are the special unitary group in five dimensions SU(5) and the special orthogonal group in ten dimensions SO(10) [30].

- **SU(5)**: $SU(5)$ [31] is the simplest GUT. The smallest simple Lie group which contains the standard model, and upon which the first Grand Unified Theory was based, is $SU(5) \supset SU(3) \times SU(2) \times U(1)$ [12]. Such group symmetries allow the reinterpretation of several known particles as different states of a single particle field. However, it is not obvious that the simplest possible choices for the extended "Grand Unified" symmetry [29] should yield the correct inventory of elementary particles. The fact that all matter particles fit nicely into three copies of the smallest group representations of $SU(5)$ and immediately carry the correct observed charges, is one of the first and most important reasons why people believe that a Grand Unified Theory might actually be realized in nature. The two smallest irreducible representations [12,29] of $SU(5)$ are 5 and 10. In the standard assignment, the 5 contains the charge conjugates of the right-handed down-type quark color triplet and a left-handed lepton isospin doublet, while the 10 contains the six up-type quark components, the left-handed down-type quark color triplet, and the right-handed electron. This scheme has to be replicated for each of the three known generations of matter. It is notable that the theory is anomaly free with this matter content.

- **SO(10)**: The next simple Lie group which contains the standard model is $SO(10) \supset SU(5) \supset SU(3) \times SU(2) \times U(1)$ [32]. The unification of matter is even more complete, since the irreducible spinor representation 16 contains both the and 10 of $SU(5)$ and a right-handed neutrino, and thus the complete particle content of one generation of the extended standard model with neutrino masses [12]. This is already the largest simple group which achieves the unification of matter in a scheme involving only the already known matter particles (apart from the Higgs sector). Since different standard model fermions are grouped together in larger representations, GUTs specifically predict relations [28-32] among the fermion masses, such as between the electron and the down quark, the muon and the strange quark, and the tau lepton and the bottom quark for SU(5) and SO(10).

6.1.2.2 Supersymmetry

Supersymmetry (SUSY) [2,3,28,33] is a theory beyond the Standard Model which introduces a new symmetry between fermions and bosons. In this theory, each known particle of the Standard Model is associated with a yet to discover supersymmetric particle. Each fermion is associated to a new boson, or "sfermion" (selectron, sneutrino, etc.) [33], and each boson is associated with a new fermion, or "bosino" (gluino, higgsino, etc.), all other quantum numbers being identical. No supersymmetric particle has been discovered so far: supersymmetry must hence be a broken symmetry to allow the superparticles to be more massive than their Standard Model partner.

This theory has many attractive features, such as:

- It can solve the so-called hierarchy problem which is related to the extreme fine-tuning needed in the Standard Model to stabilize the Higgs mass at the electroweak scale.

- It allows a unification of the forces at high energy.
- It can provide a good candidate to explain the cold non-baryonic dark matter found in the universe.

If supersymmetry exists close to the TeV energy scale, it allows for a solution of the hierarchy problem of the Standard Model [28,29,33], i.e., the fact that the Higgs boson mass is subject to quantum corrections which barring extremely fine-tuned cancellations among independent contributions would make it so large as to undermine the internal consistency of the theory. In supersymmetric theories [33], on the other hand, the contributions to the quantum corrections coming from Standard Model are naturally canceled by the contributions of the corresponding superpartners. Other attractive features of TeV-scale supersymmetry are the fact that it allows for the high-energy unification of the weak interactions, the strong interactions and electromagnetism, and the fact that it provides a candidate for dark matter and a natural mechanism for electroweak symmetry breaking.

6.1.2.3 Neutrinos

In the standard model neutrinos have exactly zero mass [21]. This is a consequence of the standard model containing only left-handed neutrinos. With no suitable right-handed partner it is impossible to add a renormalizible mass term to the standard model [12]. Measurements however indicated that neutrinos spontaneously change flavor, which implies that neutrinos have a mass. These measurements only give the relative masses of the different flavors. The best constraint on the absolute mass of the neutrinos comes from precision measurements of tritium decay, providing an upper limit 2 eV, which makes them at least five orders of magnitude lighter than the other particles in the standard model [28]. This necessitates an extension of the standard model, which not only needs to explain how neutrinos get their mass, but also why the mass is so tiny [34].

6.1.2.4 String theory

Extensions, revisions, replacements, and reorganizations of the Standard Model exist in attempt to correct for these and other issues. String theory [35] is one such reinvention, and many theoretical physicists think that such theories are the next theoretical step toward a true Theory of Everything. Theories of quantum gravity such as loop quantum gravity and others are thought by some to be promising candidates to the mathematical unification of quantum field theory and general relativity, requiring less drastic changes to existing theories [36]. However recent work places stringent limits on the putative effects of quantum gravity on the speed of light, and disfavors some current models of quantum gravity [37]. Among the numerous variants of String Theory, M-theory, whose mathematical existence was first proposed at a String Conference in 1995, is believed by many to be a proper "ToE" candidate, notably by physicists Brian Greene and Stephen Hawking. Though a full mathematical description is not yet known, solutions to the theory exist for specific cases [38]. Recent works have also proposed alternate string models, some of which lack the various harder-to-test features of M-theory (e.g. the existence of Calabi–Yau manifolds, many extra dimensions, etc.) including works done by Lisa Randall et. al [39,40].

So, keeping in view the above facts in mind, in this chapter we have made an attempt to discuss the role of octonions in physics beyond standard model.

In section (6.2), we have discussed the role of octonions in grand unified theory (GUT) gauge group of which is describes $SU(3) \times SU(2) \times U(1)$. Here we have extended $SU(2) \times U(1)$ (electroweak) gauge theory to the $SU(3) \times SU(2) \times U(1)$ gauge theory in terms of 2×2 Zorn vector matrix realization of split octonions. Thus, we have established the covariant derivative, gauge field strength and field equation for the case of grand unified theory in terms of 2×2 Zorn vector matrix realization of split octonion. As such, the octonionic formulation regardless a generalization of GUTs for the mixing of gauge current used for U(1), SU(2) and SU(3) sectors associated respectively with the electromagnetic, weak and strong interactions in presence of dyons. In section (6.3), we have undertake the study of role of octonions in supersymmetry and features of octonions realization of supersymmetry. Accordingly, we have discussed the supersymmetry algebra and their properties in terms of 2×2 split octonionic valued matrices in simple, compact and consistent manner. In section (6.4), we have analyzed the role of octonions in gravity and dark matter. Here, we have described the octonion space as the combination of two quaternionic spaces namely gravitational G-space and electromagnetic EM-space. As such, the octonionic differential operator, octonionic valued potential, octonionic field equation and other quantum equations have been reformulated in gravitational - electromagnetic space of octonion representation in simpler and consistent way. Consequently, we have discussed the radius vector, velocity representation and generalized charge and generalized mass of the particle in terms of octonion representations. It is shown that the gravitational - electromagnetic fields has been divided in terms of four type of sub-fields namely G-G, EM-G, EM-EM and G-EM subfields. Further more in subsection (6.4.1), we have reformulated the theory dark matter in terms of octonion variables. It is emphasized that the dark matter neither emits nor absorbs light or electromagnetic radiation at any significant level. Instead, its existence and properties have been analyzed from its gravitational effects on visible matter, radiation and large scale structure of the universe. Here the dark matter (nonbaryonic) has been investigated in terms of octonion hot-dark matter and octonion cold-matter. As such, we have derived the various quantum equations for octonionic hot dark matter and cold dark matter. It is shown that octonionic hot dark matter contains the photon and graviton (i.e. massless particles) while the octonionic cold dark matter is associated with the $W^{\pm}, Z^{0}$ (massive) bosons. At last in section (6.5), we have discussed the role of octonion in superstring theory (i.e. a theory of everything to describe the unification of all four types of forces namely gravitational, electromagnetic, weak and strong). The octonionic differential operator, octonionic valued potential wave equation, octonionic field equation and other various quantum equations has been discussed the framework of superstring theory in simpler, compact and consistent manner. Consequently, the generalized Dirac-Maxwell's equations are studied with the preview of superstring theory by means of octonions.

6.2 Role of Octonions in GUTs

Let us start with the local $SU(3) \times SU(2) \times U(1)$ gauge symmetry which is an internal symmetry that Standard Model. The smallest simple Lie group which contains the standard model, and upon which the first Grand Unified Theory was based, is $SU(5) \supset SU(3) \times SU(2) \times U(1)$. Here we may extend $SU(2) \times U(1)$ gauge theory given by chapter-5 to the $SU(3) \times SU(2) \times U(1)$ gauge theory in terms of split octonion formulation [41,42]. We may described the $SU(3)\times SU(2)\times U(1)$ gauge field as

$$\begin{aligned}A_{\mu} \mapsto & A_{\mu}^{0}+A_{\mu}^{a}e_{a}+A_{\mu}^{\alpha}e_{\alpha},\\ B_{\mu} \mapsto & B_{\mu}^{0}+B_{\mu}^{a}e_{a}+B_{\mu}^{\alpha}e_{\alpha}, \qquad (\forall\mu=0,1,2,3;\; a=1,2,3;\; \alpha=1,2,......,8.)\end{aligned} \tag{6.1}$$

where the components of electric A_{μ}^{0} and magnetic B_{μ}^{0} are the four potentials of dyons in case of $U(1)$ while respectively the A_{μ}^{a},B_{μ}^{a} and A_{μ}^{α},B_{μ}^{α} describe of the $SU(2)$ and $SU(3)$ gauge field theory. So, the covariant derivative in the case of $SU(3)\times SU(2)\times U(1)$ octonion gauge field in the split octonion form (2×2 Zorn's matrix) may be expressed as

$$D_{\mu}=\left(\begin{array}{cc}\partial_{\mu}+A_{\mu}^{0}+A_{\mu}^{a}e_{a}+A_{\mu}^{\alpha}e_{\alpha} & 0\\ 0 & \partial_{\mu}+B_{\mu}^{0}+B_{\mu}^{a}e_{a}+B_{\mu}^{\alpha}e_{\alpha}\end{array}\right). \tag{6.2}$$

Similarly

$$D_{\nu}=\left(\begin{array}{cc}\partial_{\nu}+A_{\nu}^{0}+A_{\nu}^{a}e_{a}+A_{\nu}^{\alpha}e_{\alpha} & 0\\ 0 & \partial_{\nu}+B_{\nu}^{0}+B_{\nu}^{a}e_{a}+B_{\nu}^{\alpha}e_{\alpha}\end{array}\right), \tag{6.3}$$

On subtraction, i.e. $[D_{\mu},D_{\nu}]=D_{\mu}D_{\nu}-D_{\nu}D_{\mu}$, these equations reduce to

$$[D_{\mu},D_{\nu}]=\left(\begin{array}{cc}G_{\mu\nu}^{0}+G_{\mu\nu}^{a}e_{a}+G_{\mu\nu}^{\alpha}e_{\alpha} & 0\\ 0 & \mathsf{G}_{\mu\nu}^{0}+\mathsf{G}_{\mu\nu}^{\mathsf{a}}e_{a}+\mathsf{G}_{\mu\nu}^{\alpha}e_{\alpha}\end{array}\right)\longmapsto\ \mathbb{G}_{\mu\nu}^{\alpha}; \tag{6.4}$$

which is $SU(3)\times SU(2)\times U(1)$ octonion gauge field strength for dyons in 2×2 Zorn matrix realization.

In equation (6.4),

$$\begin{aligned}G_{\mu\nu}^{0} &=\partial_{\mu}A_{\nu}^{0}-\partial_{\nu}A_{\mu}^{0}+\left[A_{\mu}^{0},A_{\nu}^{0}\right]\longmapsto E_{\mu\nu}^{0}, \quad (U(1)_{e}\, gauge)\\ G_{\mu\nu}^{a} &=\partial_{\mu}A_{\nu}^{a}-\partial_{\nu}A_{\mu}^{a}+e_{a}\left[A_{\mu}^{a},A_{\nu}^{a}\right]\longmapsto E_{\mu\nu}^{a}, \quad (SU(2)_{e}\, gauge)\\ G_{\mu\nu}^{\alpha} &=\partial_{\mu}A_{\nu}^{\alpha}-\partial_{\nu}A_{\mu}^{\alpha}+e_{a}\left[A_{\mu}^{\alpha},A_{\nu}^{\alpha}\right]\longmapsto E_{\mu\nu}^{\alpha}, \quad (SU(3)_{e}\, gauge)\end{aligned} \tag{6.5}$$

are the constituents of $U(1)_{e}\times SU(2)_{e}\times SU(3)_{e}$ gauge structures in presence of electric charge.
Similarly

$$\begin{aligned}\mathsf{G}_{\mu\nu}^{0} &=\partial_{\mu}B_{\nu}^{0}-\partial_{\nu}B_{\mu}^{0}+\left[B_{\mu}^{0},B_{\nu}^{0}\right]\longmapsto H_{\mu\nu}^{0}, \quad (U(1)_{m}\, gauge)\\ \mathsf{G}_{\mu\nu}^{\mathsf{a}} &=\partial_{\mu}B_{\nu}^{a}-\partial_{\nu}B_{\mu}^{a}+e_{a}\left[B_{\mu}^{a},B_{\nu}^{a}\right]\longmapsto H_{\mu\nu}^{a}, \quad (SU(2)_{m}\, gauge)\\ \mathsf{G}_{\mu\nu}^{\alpha} &=\partial_{\mu}B_{\nu}^{\alpha}-\partial_{\nu}B_{\mu}^{\alpha}+e_{a}\left[B_{\mu}^{\alpha},B_{\nu}^{\alpha}\right]\longmapsto H_{\mu\nu}^{\alpha}, \quad (SU(3)_{m}\, gauge)\end{aligned} \tag{6.6}$$

are the constituents of $U(1)_{m}\times SU(2)_{m}\times SU(3)_{m}$ gauge structure in presence of magnetic monopole. Now operating D_{μ} given by the equation (6.2) to the octonion gauge field strength $\mathbb{G}_{\mu\nu}^{\alpha}$ (6.4), we get

$$\begin{aligned}D_{\mu}\mathbb{G}_{\mu\nu}^{\alpha} &= \begin{pmatrix}\partial_{\mu}+A_{\mu}^{0}+A_{\mu}^{a}e_{a}+A_{\mu}^{\alpha}e_{\alpha} & 0\\ 0 & \partial_{\mu}+B_{\mu}^{0}+B_{\mu}^{a}e_{a}+B_{\mu}^{\alpha}e_{\alpha}\end{pmatrix}\\ &\quad * \begin{pmatrix}G_{\mu\nu}^{0}+G_{\mu\nu}^{a}e_{a}+G_{\mu\nu}^{\alpha}e_{\alpha} & 0\\ 0 & \mathsf{G}_{\mu\nu}^{0}+\mathsf{G}_{\mu\nu}^{\mathsf{a}}e_{a}+\mathsf{G}_{\mu\nu}^{\alpha}e_{\alpha}\end{pmatrix}\\ &= \begin{pmatrix}\partial_{\mu}G_{\mu\nu}^{0}+\partial_{\mu}G_{\mu\nu}^{a}e_{a}+\partial_{\mu}G_{\mu\nu}^{\alpha}e_{\alpha} & 0\\ 0 & \partial_{\mu}\mathsf{G}_{\mu\nu}^{0}+\partial_{\mu}\mathsf{G}_{\mu\nu}^{\mathsf{a}}e_{a}+\partial_{\mu}\mathsf{G}_{\mu\nu}^{\alpha}e_{\alpha}\end{pmatrix},\end{aligned} \tag{6.7}$$

which may further be reduced in terms of compect notation of split octonion formulation i.e.

$$D_\mu \mathbb{G}^{\alpha}_{\mu\nu} = \mathbb{J}^{\alpha}_{\nu}. \tag{6.8}$$

Here $\mathbb{J}^{\alpha}_{\nu}$ is $U(1) \times SU(2) \times SU(3)$ form of octonion gauge current for dyons which may be expressed in term of 2×2 Zorn matrix as

$$\mathbb{J}^{\alpha}_{\nu} = \begin{pmatrix} j^0_\nu + j^a_\nu e_a + j^\alpha_\nu e_\alpha & 0 \\ 0 & k^0_\nu + k^a_\nu e_a + k^\alpha_\nu e_\alpha \end{pmatrix}, \tag{6.9}$$

from which we may get following field equations of dyons

$$\begin{aligned} j^0_\nu &= \partial_\mu G^0_{\mu\nu}; \quad (\forall\, \mu, \nu = 0, 1, 2, 3) \\ j^a_\nu &= \partial_\mu G^a_{\mu\nu}; \quad (\forall\, a = 1, 2, 3) \\ j^\alpha_\nu &= \partial_\mu G^\alpha_{\mu\nu}; \quad (\forall\, \alpha = 1, 2, 3, ..., 8) \\ k^0_\nu &= \partial_\mu G^0_{\mu\nu}; \quad (\forall\, \mu, \nu = 0, 1, 2, 3) \\ k^a_\nu &= \partial_\mu G^a_{\mu\nu}; \quad (\forall\, a = 1, 2, 3) \\ k^\alpha_\nu &= \partial_\mu G^\alpha_{\mu\nu}. \quad (\forall\, \alpha = 1, 2, 3, ..., 8) \end{aligned} \tag{6.10}$$

were j^0_ν is the $U(1)$ current for electric charge, j^a_ν is the $SU(2)$week current associated with electric charge and j^α_ν is the current associated with $SU(3)_c$ used for chromo electric charge. On the other hand k^0_ν is $U(1)$ the counterpart of the four current, k^a_ν is the $SU(2)$ weak current while the k^α_ν is $SU(3)_c$ gluonic current due to the presence of magnetic monopole. As such, the octonionic formulation regardless a generalization of GUTs for the mixing of gauge currents used for $U(1), SU(2)$ and $SU(3)_c$ sectors associated respectively with the electromagnetic, weak and strong interactions in presence of dyons showing the duality invariance as well. Consequently, the continuity equation is generalized as

$$D_\mu \mathbb{J}_\mu^\alpha = \begin{pmatrix} \partial_\mu j_\mu^0 + \partial_\mu j_\mu^a e_a + \partial_\mu j_\mu^\alpha e_\alpha & 0 \\ 0 & \partial_\mu k_\nu^0 + \partial_\mu k_\nu^a e_a + \partial_\mu k_\nu^\alpha e_\alpha \end{pmatrix} = 0. \quad (6.11)$$

6.3 Role of Octonions in SUSY

We start with the following features of octonion realization of supersymmetry [43-45]:

- Supercharges are realized as multiplication by octonion units.
- Super space is an octonionic space spanned by these octonion units.
- Rotation group acts as an algebraic automorphism of this super space.
- Lorentz transformations acting an octonion units belong to the gauge group that leaves octonionic norm.

For a theory to be supersymmetric, it is necessary that its particle content form a representation of the supersymmetry algebra. Using the gamma matrices representation, we may describe the representation of the supersymmetry algebra in $d = 3, 4, 6, 10$. Thus, the supersymmetry algebra may be written [44] as

$$\{Q_a, Q_b\} = 2(\Gamma)_{ab}\partial_\mu = -2(\Gamma)_{ab}P_\mu \quad (6.12)$$

where Q_a, Q_b are the supersymmetry generators and transform as spin-half operators under the angular momentum algebra (P_μ). The supersymmetry generator commute with momentum operator. The Γ- represent gamma matrices representation [43,44] as

$$\Gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a & 0 \end{pmatrix} \tag{6.13}$$

The split octonion realizations are recovered by setting $\sigma_a, \tilde{\sigma}_a$as matrices with octonion-valued entries, instead of being real matrices. The one-to-one correspondence exists of 1D extended supersymmetry [43,44] and Weyl type real - valued Clifford algebras [46] is obtained by expressing the supersymmetry generator Q_a satisfying the supersymmetry algebra [44],

$$\{Q_a, Q_b\} = \eta_{ab} H, \tag{6.14}$$

for the generalized pseudo-Euclidean metric η_{ab} of (p, q) signature. So,

$$Q_a = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sigma_a \\ \tilde{\sigma}_a H & 0 \end{pmatrix} \tag{6.15}$$

where H is the Hamiltonian. Thus, the octonionic realization of an one dimensional supersymmetry is given by following 2×2 split octonionic valued matrices [44] as

$$Q_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ H & 0 \end{pmatrix}; \quad Q_a = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & e_a \\ e_a H & 0 \end{pmatrix}. \tag{6.16}$$

Here Q_0 and $Q_a(a = 1, 2, 3......, 7)$ are eight supersymmetry generators. So, we may written as

$$\begin{aligned} Q_0 =& \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e_0 \\ e_0 H & 0 \end{pmatrix}; \quad Q_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e_1 \\ e_1 H & 0 \end{pmatrix}; \\ Q_2 =& \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e_2 \\ e_2 H & 0 \end{pmatrix}; \quad Q_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e_3 \\ e_3 H & 0 \end{pmatrix}; \\ Q_4 =& \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e_4 \\ e_4 H & 0 \end{pmatrix}; \quad Q_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e_5 \\ e_5 H & 0 \end{pmatrix}; \\ Q_6 =& \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e_6 \\ e_6 H & 0 \end{pmatrix}; \quad Q_7 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e_7 \\ e_7 H & 0 \end{pmatrix}. \end{aligned} \tag{6.17}$$

Thus, the above supersymmetry can also be expressed in an octonionic representation. It corresponds to the simplest (for dimensional D=1) case of a class of higher - dimensional generalized octonionic super symmetries. So, we may introduce the octonionic supercharge $\mathcal{Q}$ [44] as

$$\begin{aligned} \mathcal{Q} =& e_0Q_0 + e_1Q_1 + e_2Q_2 + e_3Q_3 + e_4Q_4 + e_5Q_5 + e_6Q_6 + e_7Q_7 \\ =& Q_0 + \sum_{a=1}^{7} e_a Q_a. \end{aligned} \tag{6.18}$$

The complex conjugate of equation (6.18) may be expressed as

$$\begin{aligned} \overline{\mathcal{Q}} =& e_0Q_0 - e_1Q_1 - e_2Q_2 - e_3Q_3 - e_4Q_4 - e_5Q_5 - e_6Q_6 - e_7Q_7 \\ =& Q_0 - \sum_{a=1}^{7} e_a Q_a. \end{aligned} \tag{6.19}$$

Consequently, the octonionic $N = 8$ can be written [43,44] as

$$\{\mathcal{Q}, \mathcal{Q}\} = \{\overline{\mathcal{Q}}, \overline{\mathcal{Q}}\} = 2H; \qquad \{\mathcal{Q}, \overline{\mathcal{Q}}\} = 0. \tag{6.20}$$

The octonionic $N = 8$ is an in-equivalent realization of the one dimensional $N = 8$ supersymmetry with respect to standard $N = 8$, obtained by replacing the seven imaginary octonion basis $e_a (\forall a = 0, 1, 2, ..., 7)$.

6.4 Role of octonion in Gravity and Dark Matter

Let us identify the octonion space (eight dimensional) as the combination of two quaternionic spaces namely associated with the gravitational interaction (G-space) and electromagnetic interaction (EM-space) [47,48]. So, we may write the octonionic (gravitational-electromagnetic) space as

$$\mathcal{O} = (\mathcal{O}_{g-space}\,,\ \mathcal{O}_{em-space}) \Longrightarrow ((e_0, e_1, e_2, e_3)\,,\ (e_4, e_5, e_6, e_7))\,, \tag{6.21}$$

where $(\mathcal{O}_{g-space})$ is octonionic gravitational space consists e_0, e_1, e_2, e_3 octonion basis and $(\mathcal{O}_{em-space})$ is octonionic electromagnetic space consists e_4, e_5, e_6, e_7. So

$$\mathcal{O} = (e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7) = (\mathcal{O}_g + \mathcal{O}_{em}). \tag{6.22}$$

Any physical quantity $X \in \mathcal{O}$ may be written as

$$\begin{aligned} X = X_g + X_{em} =& (X_{g_0} e_0 + X_{g_1} e_1 + X_{g_2} e_2 + X_{g_3} e_3) \\ & + (X_{em_0} e_4 + X_{em_1} e_5 + X_{em_2} e_6 + X_{em_3} e_7) \\ =& \sum_{j=0}^{3} X_{g_j} e_j + e_7 \sum_{j=0}^{3} X_{em_j} e_j. \end{aligned} \tag{6.23}$$

Accordingly, the octonion differential operator $\boxdot$ also may be written as the combination of the two quaternionic space (G-space & EM-space) [48] in the terms of eight dimensional space as

$$\begin{aligned}\boxdot = \boxdot_g + \boxdot_{em} =&(\partial_{g_0}e_0 + \partial_{g_1}e_1 + \partial_{g_2}e_2 + \partial_{g_3}e_3)\\ &+(\partial_{em_0}e_4 + \partial_{em_1}e_5 + \partial_{em_2}e_6 + \partial_{em_3}e_7)\\ &=\sum_{j=0}^{3}\partial_{g_j}e_j + e_7\sum_{j=0}^{3}\partial_{em_j}e_j.\end{aligned} \tag{6.24}$$

Thus, the octonion conjugate of equation (6.24) may then be written as

$$\begin{aligned}\overline{\boxdot} = \overline{\boxdot}_g + \overline{\boxdot}_{em} =&(\partial_{g_0}e_0 - \partial_{g_1}e_1 - \partial_{g_2}e_2 - \partial_{g_3}e_3)\\ &+(-\partial_{em_0}e_4 - \partial_{em_1}e_5 - \partial_{em_2}e_6 - \partial_{em_3}e_7)\\ &=\partial_{g_0}e_0 - \sum_{j=1}^{3}\partial_{g_j}e_j - e_7\sum_{j=0}^{3}\partial_{em_j}e_j.\end{aligned} \tag{6.25}$$

Accordingly, the octonion valued potential, in eight dimensional formalism may also be written as the combinations of two four dimensional quaternionic spaces (i.e. G-space and EM-space) as

$$\begin{aligned}\mathbb{V} = (V_g\,,\ V_{em}) &= ((V_0, V_1, V_2, V_3)\,,\ (V_4, V_5, V_6, V_7))\\ &= ((V_{g_0}, V_{g_1}, V_{g_2}, V_{g_3})\,,\ (V_{em_0}, V_{em_1}, V_{em_2}, V_{em_3}))\,,\end{aligned} \tag{6.26}$$

which can further be reduced to

$$\begin{aligned}\mathbb{V} =&(V_{g_0}e_0 + V_{g_1}e_1 + V_{g_2}e_2 + V_{g_3}e_3) + (V_{em_0}e_4 + V_{em_1}e_5 + V_{em_2}e_6 + V_{em_3}e_7)\\ =&\sum_{j=0}^{3}V_{g_j}e_j + e_7\sum_{j=0}^{3}V_{em_j}e_j.\end{aligned} \tag{6.27}$$

As such, we may obtain the octonion potential wave equation for gravitational-electromagnetic space by operating $\overline{\boxdot}$ given by equation (6.25) to octonion potential $\mathbb{V}$ (6.27) in the following manner,

$$\begin{aligned}
\overline{\boxdot}\,\mathbb{V} =& e_0\{(\partial_{g_0}V_{g_0}+\partial_{g_1}V_{g_1}+\partial_{g_2}V_{g_2}+\partial_{g_3}V_{g_3}) \\
&+(\partial_{em_0}V_{em_0}+\partial_{em_1}V_{em_1}+\partial_{em_2}V_{em_2}+\partial_{em_3}V_{em_3})\} \\
&+e_1\{(\partial_{g_0}V_{g_1}-\partial_{g_1}V_{g_0}-\partial_{g_2}V_{g_3}+\partial_{g_3}V_{g_2}) \\
&+(-\partial_{em_0}V_{em_3}+\partial_{em_1}V_{em_2}-\partial_{em_2}V_{em_1}+\partial_{em_3}V_{em_0})\} \\
&+e_2\{(\partial_{g_0}V_{g_2}-\partial_{g_2}V_{g_0}+\partial_{g_1}V_{g_3}-\partial_{g_3}V_{g_1}) \\
&+(-\partial_{em_0}V_{em_2}-\partial_{em_1}V_{em_3}+\partial_{em_2}V_{em_0}+\partial_{em_3}V_{em_1})\} \\
&+e_3\{(\partial_{g_0}V_{g_3}-\partial_{g_3}V_{g_0}-\partial_{g_1}V_{g_2}+\partial_{g_2}V_{g_1}) \\
&+(-\partial_{em_1}V_{em_0}+\partial_{em_0}V_{em_1}-\partial_{em_2}V_{em_3}+\partial_{em_3}V_{em_2})\} \\
&+e_4\{(\partial_{g_0}V_{em_0}+\partial_{g_1}V_{em_3}+\partial_{g_2}V_{em_2}-\partial_{g_3}V_{em_1}) \\
&+(-\partial_{em_0}V_{g_0}+\partial_{em_1}V_{g_3}-\partial_{em_2}V_{g_2}-\partial_{em_3}V_{g_1})\} \\
&+e_5\{(\partial_{g_0}V_{em_1}-\partial_{g_1}V_{em_2}+\partial_{g_2}V_{em_3}+\partial_{g_3}V_{em_0}) \\
&+(-\partial_{em_1}V_{g_0}-\partial_{em_0}V_{g_3}+\partial_{em_2}V_{g_1}-\partial_{em_3}V_{g_2})\} \\
&+e_6\{(\partial_{g_0}V_{em_2}+\partial_{g_1}V_{em_1}-\partial_{g_2}V_{em_0}+\partial_{g_3}V_{em_3}) \\
&+(-\partial_{em_2}V_{g_0}+\partial_{em_0}V_{g_2}-\partial_{em_1}V_{g_1}-\partial_{em_3}V_{g_3})\} \\
&+e_7\{(\partial_{g_0}V_{em_3}-\partial_{g_1}V_{em_0}-\partial_{g_2}V_{em_1}-\partial_{g_3}V_{em_2}) \\
&+(-\partial_{em_3}V_{g_0}+\partial_{em_0}V_{g_1}+\partial_{em_1}V_{g_2}+\partial_{em_2}V_{g_3})\}. \qquad (6.28)
\end{aligned}$$

which can further be reduced to

$$\overline{\boxdot}\,\mathbb{V} \;=\; \mathbb{F}=\left(\left(F_0,F_1,F_2,F_3\right),\,\left(F_4,F_5,F_6,F_7\right)\right), \qquad (6.29)$$

where $\mathbb{F}(F_0,F_1,F_2,F_3,F_4,F_5,F_6,F_7)$ is also an octonion reproduces the field strength of generalized gravitational-electromagnetic fields of dyons. Thus, we may be express $\mathbb{F}$ as

$$\begin{aligned}\mathbb{F}=F_g+F_{em}= & \left(\left(F_{g_0},F_{g_1},F_{g_2},F_{g_3}\right),\left(F_{em_0},F_{em_1},F_{em_2},F_{em_3}\right)\right)\\ = & (F_{g_0}e_0+F_{g_1}e_1+F_{g_2}e_2+F_{g_3}e_3)\\ & +(F_{em_0}e_4+F_{em_1}e_5+F_{em_2}e_6+F_{em_3}e_7),\end{aligned} \tag{6.30}$$

where the component of $\mathbb{F}\left(F_{g_0},F_{g_1},F_{g_2},F_{g_3},F_{em_0},F_{em_1},F_{em_2},F_{em_3}\right)$ are expressed as

$$\begin{aligned}
F_{g_0}= & \{(\partial_{g_0}V_{g_0}+\partial_{g_1}V_{g_1}+\partial_{g_2}V_{g_2}+\partial_{g_3}V_{g_3})\\
& +e_7(-\partial_{em_3}V_{g_0}+\partial_{em_0}V_{g_1}+\partial_{em_1}V_{g_2}+\partial_{em_2}V_{g_3})\}\\
F_{g_1}= & \{(\partial_{g_0}V_{g_1}-\partial_{g_1}V_{g_0}-\partial_{g_2}V_{g_3}+\partial_{g_3}V_{g_2})\\
& +e_7(-\partial_{em_0}V_{g_0}+\partial_{em_1}V_{g_3}-\partial_{em_2}V_{g_2}-\partial_{em_3}V_{g_1})\}\\
F_{g_2}= & \{(\partial_{g_0}V_{g_2}-\partial_{g_2}V_{g_0}+\partial_{g_1}V_{g_3}-\partial_{g_3}V_{g_1})\\
& +e_7(-\partial_{em_1}V_{g_0}-\partial_{em_0}V_{g_3}+\partial_{em_2}V_{g_1}-\partial_{em_3}V_{g_2})\}\\
F_{g_3}= & \{(\partial_{g_0}V_{g_3}-\partial_{g_3}V_{g_0}-\partial_{g_1}V_{g_2}+\partial_{g_2}V_{g_1})\\
& +e_7(-\partial_{em_2}V_{g_0}+\partial_{em_0}V_{g_2}-\partial_{em_1}V_{g_1}-\partial_{em_3}V_{g_3})\}\\
F_{em_0}= & \{(\partial_{g_0}V_{em_0}+\partial_{g_1}V_{em_3}+\partial_{g_2}V_{em_2}-\partial_{g_3}V_{em_1})\\
& +e_7(-\partial_{em_0}V_{em_3}+\partial_{em_1}V_{em_2}-\partial_{em_2}V_{em_1}+\partial_{em_3}V_{em_0})\}\\
F_{em_1}= & \{(\partial_{g_0}V_{em_1}-\partial_{g_1}V_{em_2}+\partial_{g_2}V_{em_3}+\partial_{g_3}V_{em_0})\\
& +e_7(-\partial_{em_0}V_{em_2}-\partial_{em_1}V_{em_3}+\partial_{em_2}V_{em_0}+\partial_{em_3}V_{em_1})\}\\
F_{em_2}= & \{(\partial_{g_0}V_{em_2}+\partial_{g_1}V_{em_1}-\partial_{g_2}V_{em_0}+\partial_{g_3}V_{em_3})\\
& +e_7(-\partial_{em_1}V_{em_0}+\partial_{em_0}V_{em_1}-\partial_{em_2}V_{em_3}+\partial_{em_3}V_{em_2})\}\\
F_{em_3}= & \{(\partial_{g_0}V_{em_3}-\partial_{g_1}V_{em_0}-\partial_{g_2}V_{em_1}-\partial_{g_3}V_{em_2})\\
& +e_7(\partial_{em_0}V_{em_0}+\partial_{em_1}V_{em_1}+\partial_{em_2}V_{em_2}+\partial_{em_3}V_{em_3})\}
\end{aligned} \tag{6.31}$$

using the Lorentz Gauge conditions in the equation (6.31), i.e. $F_{g_0}=F_{em_3}=0$. Thus, equation (6.30) may be written as

$$\mathbb{F} = F_g + F_{em} = (F_{g_1}e_1 + F_{g_2}e_2 + F_{g_3}e_3) + (F_{em_0}e_4 + F_{em_1}e_5 + F_{em_2}e_6). \tag{6.32}$$

Here the first term ($F_g = F_{g_1}, F_{g_2}, F_{g_3}$) is defined as the field strength of the gravitational interaction in G-space while the second term ($F_{em} = F_{em_0}, F_{em_1}, F_{em_2}$) is associated with the field strength of the electromagnetic interaction in EM-space. Hence, we may obtain the octonionic field equation in gravitational-electromagnetic space on applying the differential operator (6.24) to equation (6.32) as

$$\begin{aligned}
\boxdot\,\mathbb{F} = &- e_0\{(\partial_{g_1}F_{g_1} + \partial_{g_2}F_{g_2} + \partial_{g_3}F_{g_3}) + (\partial_{em_0}F_{em_0} + \partial_{em_1}F_{em_1} + \partial_{em_2}F_{em_2})\}\\
&+e_1\{(\partial_{g_0}F_{g_1} + \partial_{g_2}F_{g_3} + \partial_{g_3}F_{g_2}) + (-\partial_{em_1}F_{em_2} + \partial_{em_2}F_{em_1} - \partial_{em_3}F_{em_0})\}\\
&+e_2\{(\partial_{g_0}F_{g_2} - \partial_{g_1}F_{g_3} + \partial_{g_3}F_{g_1}) + (-\partial_{em_2}F_{em_0} + \partial_{em_0}F_{em_2} - \partial_{em_3}F_{em_1})\}\\
&+e_3\{(\partial_{g_0}F_{g_3} + \partial_{g_1}F_{g_2} - \partial_{g_2}F_{g_1}) + (-\partial_{em_0}F_{em_1} + \partial_{em_1}F_{em_0} - \partial_{em_3}F_{em_2})\}\\
&+e_4\{(\partial_{g_0}F_{em_0} - \partial_{g_2}F_{em_2} + \partial_{g_3}F_{em_1}) + (-\partial_{em_1}F_{g_3} + \partial_{em_2}F_{g_2} + \partial_{em_3}F_{g_1})\}\\
&+e_5\{(\partial_{g_0}F_{em_1} + \partial_{g_1}F_{em_2} - \partial_{g_3}F_{em_0}) + (-\partial_{em_2}F_{g_1} + \partial_{em_0}F_{g_3} + \partial_{em_3}F_{g_2})\}\\
&+e_6\{(\partial_{g_0}F_{em_2} - \partial_{g_1}F_{em_1} + \partial_{g_2}F_{em_0}) + (-\partial_{em_0}F_{g_2} + \partial_{em_1}F_{g_1} + \partial_{em_3}F_{g_3})\}\\
&+e_7\{(\partial_{g_1}F_{em_0} + \partial_{g_2}F_{em_1} + \partial_{g_3}F_{em_2}) + (-\partial_{em_0}F_{g_1} - \partial_{em_1}F_{g_2} - \partial_{em_2}F_{g_3})\}.
\end{aligned} \tag{6.33}$$

which is further reduced to the compact notation in terms of an octonionic gravitational-electromagnetic space as

$$\boxdot\,\mathbb{F} \;=\; \mathbb{J} = ((J_0, J_1, J_2, J_3)\,,\,(J_4, J_5, J_6, J_7))\,, \tag{6.34}$$

where $\mathbb{J}(J_0, J_1, J_2, J_3, J_4, J_5, J_6, J_7)$ is also an octonion reproduces the field current source of dyons. So, it may be expressed as

$$\begin{aligned}
\mathbb{J} &= (J_0, J_1, J_2, J_3, J_4, J_5, J_6, J_7) \\
&= (J_{(g-g)_0} + J_{(em-em)_0})e_0 + (J_{(g-g)_1} + J_{(em-em)_1})e_1 \\
&+ (J_{(g-g)_2} + J_{(em-em)_2})e_2 + (J_{(g-g)_3} + J_{(em-em)_3})e_3 \\
&+ \{J_{(em-g)_0} + J_{(g-em)_0})e_4 + (J_{(em-g)_1} + J_{(g-em)_1})e_5 \\
&+ (J_{(em-g)_2} + J_{(g-em)_2})e_6 + (J_{(em-g)_3} + J_{(g-em)_3})e_7.
\end{aligned} \tag{6.35}$$

Here $J_{(g-g)}, J_{(em-em)}, J_{(em-g)}, J_{(g-em)}$ are defined for the octonionic current source respectively for gravitational-gravitational, electromagnetic-electromagnetic, electromagnetic-gravitational, gravitational-electromagnetic interaction [47,48]. As such, the components of octonionic current source $\mathbb{J}$ are described as

$$\begin{aligned}
J_{(g-g)_0} &= (\partial_{g_1} F_{g_1} + \partial_{g_2} F_{g_2} + \partial_{g_3} F_{g_3}), \quad J_{(em-em)_0} = (\partial_{em_0} F_{em_0} + \partial_{em_1} F_{em_1} + \partial_{em_2} F_{em_2}); \\
J_{(g-g)_1} &= (\partial_{g_0} F_{g_1} + \partial_{g_2} F_{g_3} + \partial_{g_3} F_{g_2}), \quad J_{(em-em)_1} = (-\partial_{em_1} F_{em_2} + \partial_{em_2} F_{em_1} - \partial_{em_3} F_{em_0}); \\
J_{(g-g)_2} &= (\partial_{g_0} F_{g_2} - \partial_{g_1} F_{g_3} + \partial_{g_3} F_{g_1}), \quad J_{(em-em)_2} = (-\partial_{em_2} F_{em_0} + \partial_{em_0} F_{em_2} - \partial_{em_3} F_{em_1}); \\
J_{(g-g)_3} &= (\partial_{g_0} F_{g_3} + \partial_{g_1} F_{g_2} - \partial_{g_2} F_{g_1}), \quad J_{(em-em)_3} = (-\partial_{em_0} F_{em_1} + \partial_{em_1} F_{em_0} - \partial_{em_3} F_{em_2}); \\
J_{(em-g)_0} &= (\partial_{g_0} F_{em_0} - \partial_{g_2} F_{em_2} + \partial_{g_3} F_{em_1}), \quad J_{(g-em)_0} = (-\partial_{em_1} F_{g_3} + \partial_{em_2} F_{g_2} + \partial_{em_3} F_{g_1}); \\
J_{(em-g)_1} &= (\partial_{g_0} F_{em_1} + \partial_{g_1} F_{em_2} - \partial_{g_3} F_{em_0}), \quad J_{(g-em)_1} = (-\partial_{em_2} F_{g_1} + \partial_{em_0} F_{g_3} + \partial_{em_3} F_{g_2}); \\
J_{(em-g)_2} &= (\partial_{g_0} F_{em_2} - \partial_{g_1} F_{em_1} + \partial_{g_2} F_{em_0}), \quad J_{(g-em)_2} = (-\partial_{em_0} F_{g_2} + \partial_{em_1} F_{g_1} + \partial_{em_3} F_{g_3}); \\
J_{(em-g)_3} &= (\partial_{g_1} F_{em_0} + \partial_{g_2} F_{em_1} + \partial_{g_3} F_{em_2}), \quad J_{(g-em)_3} = (-\partial_{em_0} F_{g_1} - \partial_{em_1} F_{g_2} - \partial_{em_2} F_{g_3});
\end{aligned} \tag{6.36}$$

which are analogous to the generalized Dirac-Maxwell's (GDM) equations in presence of gravitational-gravitational (G-G), electromagnetic-electromagnetic (EM-EM), electromagnetic-gravitational (EM-G), gravitational-electromagnetic (G-EM) interaction.

Consequently, the octonionic radius vector ($\mathbb{R} = R_0, R_1, R_2, R_3, R_4, R_5, R_6, R_7$)

is defined the combination of two quaternionic space in the following manner,

$$\begin{aligned}\mathbb{R} =&(R_0, R_1, R_2, R_3)\ ,\ (R_4, R_5, R_6, R_7)\\ =&(R_0e_0 + R_1e_1 + R_2e_2 + R_3e_3) + (R_4e_4 + R_5e_5 + R_6e_6 + R_7e_7). \quad (6.37)\end{aligned}$$

which yields the velocity ($\boldsymbol{v}$) in the octonionic (gravitational-electromagnetic) representation as

$$\begin{aligned}\boldsymbol{v} =&\frac{\partial\mathbb{R}}{\partial t} = \frac{\partial}{\partial t}\{(R_0e_0 + R_1e_1 + R_2e_2 + R_3e_3) + (R_4e_4 + R_5e_5 + R_6e_6 + R_7e_7)\}\\ =&(v_0e_0 + v_1e_1 + v_2e_2 + v_3e_3) + (v_4e_4 + v_5e_5 + v_6e_6 + v_7e_7), \quad (6.38)\end{aligned}$$

we may also described the charge and mass [48] of the particle in octonionic space as

$$\begin{aligned}
J_{(g-g)_0} &\cong Q_{(g-g)_0}v_0, & J_{(em-em)_0} &\cong Q_{(em-em)_0}v_0;\\
J_{(g-g)_1} &\cong Q_{(g-g)_1}v_1, & J_{(em-em)_1} &\cong Q_{(em-em)_1}v_1;\\
J_{(g-g)_2} &\cong Q_{(g-g)_2}v_2, & J_{(em-em)_2} &\cong Q_{(em-em)_2}v_2;\\
J_{(g-g)_3} &\cong Q_{(g-g)_3}v_3, & J_{(em-em)_3} &\cong Q_{(em-em)_3}v_3;\\
J_{(em-g)_0} &\cong Q_{(em-g)_0}v_4, & J_{(g-em)_0} &\cong Q_{(g-em)_0}v_4;\\
J_{(em-g)_1} &\cong Q_{(em-g)_1}v_5, & J_{(g-em)_1} &\cong Q_{(g-em)_1}v_5;\\
J_{(em-g)_2} &\cong Q_{(em-g)_2}v_6, & J_{(g-em)_2} &\cong Q_{(g-em)_2}v_6;\\
J_{(em-g)_3} &\cong Q_{(em-g)_3}v_7, & J_{(g-em)_3} &\cong Q_{(g-em)_3}v_7; \quad (6.39)
\end{aligned}$$

where $Q_{(g-g)}, Q_{(em-g)}, Q_{(g-em)}$ are respectively denoted the "Mass" of the gravitational - gravitational (G-G), electromagnetic - gravitational (EM-G), gravitational - electromagnetic (G-EM) interactions while $Q_{(em-em)}$ represent the "Charge" of the electromagnetic - electromagnetic (EM-EM) interaction.

So, the $Q_{(g-g)}$,$Q_{(em-g)}$, $Q_{(g-em)}$ and $Q_{(em-em)}$ respectively describe the "Generalized mass" and "Generalized charge" [48]. From the equations (6.35), (6.36) and (6.39), we may obtain the four-type of subfields in the octonionic electromagnetic-gravitational fields [47,48] as

- Gravitational-Gravitational (G-G) subfield.
- Electromagnetic-Gravitational (EM-G) subfield.
- Electromagnetic-Electromagnetic (EM-EM) subfield.
- Gravitational-Electromagnetic (G-EM) subfield.

Thus, from above four subfields, we have describe the dark matter in the following subsections.

6.4.1 The Dark Matter

The *Dark Matter* [20,22] is a type of matter hypothesized to account for a large part of the total mass in the universe. Dark matter cannot be seen directly with telescopes which is neither emits nor absorbs light or other electromagnetic radiation at any significant level. Instead, its existence and properties are inferred from its gravitational effects on visible matter, radiation and the large scale structure of the universe. The majority of dark matter in the universe cannot be baryons, and thus does not form atoms. It also cannot interact with ordinary matter as electromagnetic forces, i.e. the dark matter particles do not carry any electric charge. The nonbaryonic dark matter may include the photon, graviton, intermediate bosons and neutrinos, or supersymmetric particles. Unlike baryonic matter, nonbaryonic dark matter does not contribute to the formulation of the elements in the universe as its presence is revealed only via its gravitational attraction. Thus, the nonbaryonic dark matter [20-22] is evident through its gravitational effect only. There are two type of nonbaryonic dark matter respectively defined as hot dark matter and cold dark matter. Here, we have made an attempt to express the nonbaryonic dark matter in terms of octonion representation in the following manner.

- **Octonions Hot Dark Matter (OHDM):** Octonions hot dark matter assumed to compose of particles that have zero or near-zero mass. The special theory of relativity requires that massless particles move at the speed of light while near-zero mass particles move at nearly the speed of light. Thus, the octonionic hot dark matter may be associated with the gravitational-gravitational (G-G) and electromagnetic-electromagnetic (EM-EM) subfields. Thus, the octonionic hot dark matter (OHDM) includes the photon and graviton. As such, we may write the quantum equation for octonionic hot dark matter in terms of potential, field and current equations. So,the potential wave equations from (6.23) and (6.24), may be written in the quaternionic (G-G) space as

$$\begin{aligned}\boxdot_g X_g =&(\partial_{g_0}e_0+\partial_{g_1}e_1+\partial_{g_2}e_2+\partial_{g_3}e_3)\cdot(X_{g_0}e_0+X_{g_1}e_1+X_{g_2}e_2+X_{g_3}e_3)\\ =&V_{(g-g)_0}e_0+V_{(g-g)_1}e_1+V_{(g-g)_2}e_2+V_{(g-g)_3}e_3,\quad\text{(for G-G space)}\end{aligned}\tag{6.40}$$

which may further be written for EM-EM sector as

$$\begin{aligned}\boxdot_{em} X_{em} =&(\partial_{em_0} e_4 + \partial_{em_1} e_5 + \partial_{em_2} e_6 + \partial_{em_3} e_7) \\ &\cdot (X_{em_0} e_4 + X_{em_1} e_5 + X_{em_2} e_6 + X_{em_3} e_7) \\ =&V_{(em-em)_0} e_0 + V_{(em-em)_1} e_1 + V_{(em-em)_2} e_2 + V_{(em-em)_3} e_3, \\ &(\text{for EM-EM space}) \end{aligned} \tag{6.41}$$

Thus, equations (6.25) and (6.27) reduces to

$$\begin{aligned}\overline{\boxdot}_g V_g =&(\partial_{g_0} e_0 - \partial_{g_1} e_1 - \partial_{g_2} e_2 - \partial_{g_3} e_3) \cdot (V_{g_0} e_0 + V_{g_1} e_1 + V_{g_2} e_2 + V_{g_3} e_3) \\ =&F_{(g-g)_0} e_0 + F_{(g-g)_1} e_1 + F_{(g-g)_2} e_2 + F_{(g-g)_3} e_3, \quad (\text{for G-G space}) \end{aligned} \tag{6.42}$$

and

$$\begin{aligned}\overline{\boxdot}_{em} V_{em} =&(-\partial_{em_0} e_4 - \partial_{em_1} e_5 - \partial_{em_2} e_6 - \partial_{em_3} e_7) \\ &\cdot (V_{em_0} e_4 + V_{em_1} e_5 + V_{em_2} e_6 + V_{em_3} e_7) \\ =&F_{(em-em)_0} e_0 + F_{(em-em)_1} e_1 + F_{(em-em)_2} e_2 + F_{(em-em)_3} e_3, \\ &(\text{for EM-EM space}) \end{aligned} \tag{6.43}$$

Accordingly, the field source equations from (6.23) and (6.32), are respectively described as

$$\begin{aligned}\boxdot_g F_g =&(\partial_{g_0} e_0 + \partial_{g_1} e_1 + \partial_{g_2} e_2 + \partial_{g_3} e_3) \cdot (F_{g_0} e_0 + F_{g_1} e_1 + F_{g_2} e_2 + F_{g_3} e_3) \\ =&J_{(g-g)_0} e_0 + J_{(g-g)_1} e_1 + J_{(g-g)_2} e_2 + J_{(g-g)_3} e_3, \quad (\text{for G-G space}) \end{aligned} \tag{6.44}$$

and

$$\begin{aligned}\boxdot_{em}F_{em} =&(\partial_{em_0}e_4+\partial_{em_1}e_5+\partial_{em_2}e_6+\partial_{em_3}e_7)\\ &\cdot(F_{em_0}e_4+F_{em_1}e_5+F_{em_2}e_6+F_{em_3}e_7)\\ =&J_{(em-em)_0}e_0+J_{(em-em)_1}e_1+J_{(em-em)_2}e_2+J_{(em-em)_3}e_3.\\ &(\text{for EM-EM space})\end{aligned} \tag{6.45}$$

These two equations (6.44), (6.45) describe the generalized Dirac-Maxwell's equations of dyons in terms of octonionic hot dark matter comparizing gravitational-gravitational (G-G) and electromagnetic-electromagnetic (EM-EM) interactions. Hence, we may conclude that the quantum equations for octonionic hot dark matter (i.e. photon and graviton) are expressed in the terms of quaternionic representations of octonions.

- **Octonions Cold Dark Matter (OCDM):** Like wise, the octonions cold dark matter may be described as the composition of the massive objects moving at sub-relativistic velocities. So, the difference between the octonions cold dark matter (OCDM) and the octonions hot dark matter (OHDM) is significant in the formulation of structure, because the velocities of octonions hot dark matter cause it to wipe out structure on small scales. Thus, the octonions cold dark matter is associated with the electromagnetic-gravitational (EM-G) and gravitational-electromagnetic (G-EM) subfields. Hence, the octonions cold dark matter (OCDM) is assumed to include intermediate particles (i.e. $W^{\pm}, Z^{o}$ particles). So, we may write the quantum equations for octonions cold dark matter in terms of potential, field and current equations. The potential wave equations from (6.23) and (6.24) may then be written respectively as

$$\begin{aligned}\boxdot_{em} X_g =&(\partial_{em_0} e_4 + \partial_{em_1} e_5 + \partial_{em_2} e_6 + \partial_{em_3} e_7)\\ &\cdot (X_{g_0} e_0 + X_{g_1} e_1 + X_{g_2} e_2 + X_{g_3} e_3)\\ =&V_{(em-g)_0} e_4 + V_{(em-g)_1} e_5 + V_{(em-g)_2} e_6 + V_{(em-g)_3} e_7,\\ &(\text{for EM-G space})\end{aligned} \tag{6.46}$$

and

$$\begin{aligned}\boxdot_{g} X_{em} =&(\partial_{g_0} e_0 + \partial_{g_1} e_1 + \partial_{g_2} e_2 + \partial_{g_3} e_3)\\ &\cdot (X_{em_0} e_4 + X_{em_1} e_5 + X_{em_2} e_6 + X_{em_3} e_7)\\ =&V_{(g-em)_0} e_0 + V_{(g-em)_1} e_1 + V_{(g-em)_2} e_2 + V_{(g-em)_3} e_3\\ &+V_{(g-em)_4} e_4 + V_{(g-em)_5} e_5 + V_{(g-em)_6} e_6 + V_{(g-em)_7} e_7.\\ &(\text{for G-EM space})\end{aligned} \tag{6.47}$$

Accordingly, the field equations from (6.25) and (6.27) are respectively described as

$$\begin{aligned}\overline{\boxdot}_{em} V_g =&(\partial_{em_0} e_4 - \partial_{em_1} e_5 - \partial_{em_2} e_6 - \partial_{em_3} e_7)\\ &\cdot (V_{g_0} e_0 + V_{g_1} e_1 + V_{g_2} e_2 + V_{g_3} e_3)\\ =&F_{(em-g)_0} e_4 + F_{(em-g)_1} e_5 + F_{(em-g)_2} e_6 + F_{(em-g)_3} e_7,\\ &(\text{for EM-G space})\end{aligned} \tag{6.48}$$

and

$$\begin{aligned}\overline{\boxdot}_{g} V_{em} =&(\partial_{g_0} e_0 - \partial_{g_1} e_1 - \partial_{g_2} e_2 - \partial_{g_3} e_3) \cdot (V_{em_0} e_4 + V_{em_1} e_5 + V_{em_2} e_6 + V_{em_3} e_7)\\ =&F_{(g-em)_0} e_0 + F_{(g-em)_1} e_1 + F_{(g-em)_2} e_2 + F_{(g-em)_3} e_3\\ &+F_{(g-em)_4} e_4 + F_{(g-em)_5} e_5 + F_{(g-em)_6} e_6 + F_{(g-em)_7} e_7.\\ &(\text{for G-EM space})\end{aligned} \tag{6.49}$$

On the other hand the field source equations (6.23) and (6.32) are expressed as

$$\begin{aligned}\boxdot_{em}F_g =&(\partial_{em_0}e_4+\partial_{em_1}e_5+\partial_{em_2}e_6+\partial_{em_3}e_7)\\&\cdot(F_{g_0}e_0+F_{g_1}e_1+F_{g_2}e_2+F_{g_3}e_3)\\=&J_{(em-g)_0}e_4+J_{(em-g)_1}e_5+J_{(em-g)_2}e_6+J_{(em-g)_3}e_7,\\&\text{(for EM-G space)}\end{aligned}\quad (6.50)$$

and

$$\begin{aligned}\boxdot_g F_{em} =&(\partial_{g_0}e_4+\partial_{g_1}e_5+\partial_{g_2}e_6+\partial_{g_3}e_7)\cdot(F_{g_0}e_0+F_{g_1}e_1+F_{g_2}e_2+F_{g_3}e_3)\\=&J_{(g-em)_0}e_0+J_{(g-em)_1}e_1+J_{(g-em)_2}e_2+J_{(g-em)_3}e_3\\&+J_{(g-em)_4}e_4+J_{(g-em)_5}e_5+J_{(g-em)_6}e_6+J_{(g-em)_7}e_7.\\&\text{(for G-EM space)}\end{aligned}\quad (6.51)$$

These equation on simplification, describe the generalized Dirac-Maxwell's equations of dyons for octonionic cold dark matter in the presence of electromagnetic-gravitational (EM-G) and gravitational-electromagnetic (G-EM) interactions. So, the quantum equations for octonionic cold dark matter (i.e. $W^{\pm}, Z^{o}$ particles) may easily be expressed in the terms of simpler and compact notation of octonions representations.

6.5 Role of octonion in Superstring Theory

The octonionic representation of the super-string (SS) theory may be consider as the combination of four complex ($\mathbb{C}$) spaces namely associated with the gravitational (G-space), electromagnetic (EM-space), weak (W-space) and strong (S-space) interactions [48,49], i.e. unification of the four fundamental forces. So, we may write the octonionic superstring space as

$$\begin{aligned}\mathcal{O}_{SS} =& \mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}\\ &= (\mathcal{O}_g\,,\,\mathcal{O}_{em}\,,\,\mathcal{O}_w\,,\,\mathcal{O}_s) \Longrightarrow ((e_0,e_1)\,,\,(e_2,e_3)\,,\,(e_4,e_5)\,,\,(e_6,e_7))\,,\end{aligned} \tag{6.52}$$

where $\mathcal{O}_g$, $\mathcal{O}_{em}$, $\mathcal{O}_w$, $\mathcal{O}_s$ are respectively known as gravitational, electromagnetic, weak and strong spaces in superstring theory related with the octonionic basis $(e_0,e_1),(e_2,e_3),(e_4,e_5),(e_6,e_7)$. Thus, the octonionic physical quantity $\mathbb{X}\in\mathcal{O}_{SS}$ is expressed as

$$\begin{aligned}\mathbb{X} =& X_g + X_{em} + X_w + X_s\\ =& (X_{g_0}e_0 + X_{g_1}e_1) + (X_{em_0}e_2 + X_{em_1}e_3) + (X_{w_0}e_4 + X_{w_1}e_5) + (X_{s_0}e_6 + X_{s_1}e_7).\end{aligned} \tag{6.53}$$

The octonionic differential operator in case of superstring theory (i.e. unification of four differential operator) may be written as

$$\begin{aligned}\boxplus_{SS} =& \boxdot_g + \boxdot_{em} + \boxdot_w + \boxdot_s\\ =& (\partial_{g_0}e_0 + \partial_{g_1}e_1) + (\partial_{em_0}e_2 + \partial_{em_1}e_3) + (\partial_{w_0}e_4 + \partial_{w_1}e_5) + (\partial_{s_0}e_6 + \partial_{s_1}e_7).\end{aligned} \tag{6.54}$$

Octonionic conjugate of equation (6.54) is described as

$$\overline{\boxplus}_{SS} = (\partial_{g_0}e_0 - \partial_{g_1}e_1) - (\partial_{em_0}e_2 + \partial_{em_1}e_3) - (\partial_{w_0}e_4 + \partial_{w_1}e_5) - (\partial_{s_0}e_6 + \partial_{s_1}e_7). \tag{6.55}$$

Thus, the octonionic superstring valued potential with the combination of four potentials may be expressed as

$$\begin{aligned}\mathbb{V}_{SS} = (V_g\,,\, V_{em}\,,\, V_w\,,\, V_s) &= ((V_0, V_1)\,,\,(V_2, V_3)\,,\,(V_4, V_5)\,,\,(V_6, V_7)) \\ &= ((V_{g_0}, V_{g_1})\,,\,(V_{em_0}, V_{em_1})\,,\,(V_{w_0}, V_{w_1})\,,\,(V_{s_0}, V_{s_1}))\,,\end{aligned} \tag{6.56}$$

which is further simplified to

$$\mathbb{V}_{SS} = (V_{g_0}e_0 + V_{g_1}e_1) + (V_{em_0}e_2 + V_{em_1}e_3) + (V_{w_0}e_4 + V_{w_1}e_5) + (V_{s_0}e_6 + V_{s_1}e_7). \tag{6.57}$$

In order to obtain the octonionic potential wave equations of the superstring space, let us operate $\overline{\boxplus}_{SS}$ given by equation (6.55) to octonion superstring valued potential $\mathbb{V}_{SS}$ of equation (6.57) and we get

$$\begin{aligned}&\overline{\boxplus}_{SS}\,\mathbb{V}_{SS} = \\ &e_0\{\partial_{g_0}V_{g_0} + \partial_{g_1}V_{g_1} + \partial_{em_0}V_{em_0} + \partial_{em_1}V_{em_1} + \partial_{w_0}V_{w_0} + \partial_{w_1}V_{w_1} + \partial_{s_0}V_{s_0} + \partial_{s_1}V_{s_1}\} \\ &+e_1\{\partial_{g_0}V_{g_1} - \partial_{g_1}V_{g_0} - \partial_{em_0}V_{em_1} + \partial_{em_1}V_{em_0} - \partial_{w_0}V_{s_1} + \partial_{w_1}V_{s_0} - \partial_{s_0}V_{w_1} + \partial_{s_1}V_{w_0}\} \\ &+e_2\{\partial_{g_0}V_{em_0} + \partial_{g_1}V_{em_1} - \partial_{em_0}V_{g_0} - \partial_{em_1}V_{g_1} - \partial_{w_0}V_{s_0} - \partial_{w_1}V_{s_1} + \partial_{s_0}V_{w_0} + \partial_{s_1}V_{w_1}\} \\ &+e_3\{\partial_{g_0}V_{em_1} - \partial_{g_1}V_{em_0} + \partial_{em_0}V_{g_1} - \partial_{em_1}V_{g_0} + \partial_{w_0}V_{w_1} - \partial_{w_1}V_{w_0} - \partial_{s_0}V_{s_1} + \partial_{s_1}V_{s_0}\} \\ &+e_4\{\partial_{g_0}V_{w_0} + \partial_{g_1}V_{s_1} + \partial_{em_0}V_{s_0} - \partial_{em_1}V_{w_1} - \partial_{w_0}V_{g_0} + \partial_{w_1}V_{em_1} - \partial_{s_0}V_{em_0} - \partial_{s_1}V_{g_1}\} \\ &+e_5\{\partial_{g_0}V_{w_1} - \partial_{g_1}V_{s_0} + \partial_{em_0}V_{s_1} + \partial_{em_1}V_{w_0} - \partial_{w_0}V_{em_1} - \partial_{w_1}V_{g_0} + \partial_{s_0}V_{g_1} - \partial_{s_1}V_{em_0}\} \\ &+e_6\{\partial_{g_0}V_{s_0} + \partial_{g_1}V_{w_1} - \partial_{em_0}V_{w_0} + \partial_{em_1}V_{s_1} + \partial_{w_0}V_{em_0} - \partial_{w_1}V_{g_1} - \partial_{s_0}V_{g_0} - \partial_{s_1}V_{em_1}\} \\ &+e_7\{\partial_{g_0}V_{s_1} - \partial_{g_1}V_{w_0} - \partial_{em_0}V_{w_1} - \partial_{em_1}V_{s_0} + \partial_{w_0}V_{g_1} + \partial_{w_1}V_{em_0} + \partial_{s_0}V_{em_1} - \partial_{s_1}V_{g_0}\},\end{aligned} \tag{6.58}$$

which provides the following octonionic analogous of superstring theory as

$$\overline{\boxplus}_{SS}\,\mathbb{V}_{SS} \;=\; \mathbb{F}_{SS} = ((F_0, F_1),\,(F_2, F_3),\,(F_4, F_5),\,(F_6, F_7))\,; \tag{6.59}$$

where $\mathbb{F}_{SS}(F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7)$ is an octonion which reproduces the superstring field strengths. So, equation (6.59)may further be expressed as

$$\begin{aligned}\mathbb{F}_{SS} = F_g + F_{em} + F_w + F_s = &((F_{g_0}, F_{g_1}), (F_{em_0}, F_{em_1}), (F_{w_0}, F_{w_1}), (F_{s_0}, F_{s_1}))\\ =&(F_{g_0}e_0 + F_{g_1}e_1) + (F_{em_0}e_2 + F_{em_1}e_3)\\ &+ (F_{w_0}e_4 + F_{w_1}e_5) + (F_{s_0}e_6 + F_{s_1}e_7),\end{aligned} \tag{6.60}$$

where the first term ($F_g = F_{g_0}, F_{g_1}$) is defined as the gravitational field strength in G-space, the second term ($F_{em} = F_{em_0}, F_{em_1}$) is described as the electromagnetic field strength in EM-space, the third term ($F_w = F_{w_0}, F_{w_1}$) provides the weak interaction field strength in W-space and the forth term ($F_s = F_{s_0}, F_{s_1}$) is responsible for the strong field strength in S-space. Thus, the component of $\mathbb{F}_{SS}\{(F_{g_0}, F_{g_1}), (F_{em_0}, F_{em_1}), (F_{w_0}, F_{w_1}), (F_{s_0}, F_{s_1})\}$ are expressed as the following octonionic representation

$$\begin{aligned}
F_{g_0}e_0 &= \{\partial_{g_0}V_{g_0}e_0 - \partial_{g_1}V_{g_0}e_1 - \partial_{em_0}V_{g_0}e_2 - \partial_{em_1}V_{g_0}e_3\\
&-\partial_{w_0}V_{g_0}e_4 - \partial_{w_1}V_{g_0}e_5 - \partial_{s_0}V_{g_0}e_6 - \partial_{s_1}V_{g_0}e_7\}\\
F_{g_1}e_1 &= \{\partial_{g_1}V_{g_1}e_0 + \partial_{g_0}V_{g_1}e_1 - \partial_{em_1}V_{g_1}e_2 + \partial_{em_0}V_{g_1}e_3\\
&-\partial_{s_1}V_{g_1}e_4 + \partial_{s_0}V_{g_1}e_5 - \partial_{w_1}V_{g_1}e_6 + \partial_{w_0}V_{g_1}e_7\}\\
F_{em_0}e_2 &= \{\partial_{em_0}V_{em_0}e_0 + \partial_{em_1}V_{em_0}e_1 + \partial_{g_0}V_{em_0}e_2 - \partial_{g_1}V_{em_0}e_3\\
&-\partial_{s_0}V_{em_0}e_4 - \partial_{s_1}V_{em_0}e_5 + \partial_{w_0}V_{em_0}e_6 + \partial_{w_1}V_{em_0}e_7\}\\
F_{em_1}e_3 &= \{\partial_{em_1}V_{em_1}e_0 - \partial_{em_0}V_{em_1}e_1 + \partial_{g_1}V_{em_1}e_2 + \partial_{g_0}V_{em_1}e_3\\
&+\partial_{w_1}V_{em_1}e_4 - \partial_{w_0}V_{em_1}e_5 - \partial_{s_1}V_{em_1}e_6 + \partial_{s_0}V_{em_1}e_7\}\\
F_{w_0}e_4 &= \{\partial_{w_0}V_{w_0}e_0 + \partial_{s_1}V_{w_0}e_1 + \partial_{s_0}V_{w_0}e_2 - \partial_{w_1}V_{w_0}e_3\\
&+\partial_{g_0}V_{w_0}e_4 + \partial_{em_1}V_{w_0}e_5 - \partial_{em_0}V_{w_0}e_6 - \partial_{g_1}V_{w_0}e_7\}\\
F_{w_1}e_5 &= \{\partial_{w_1}V_{w_1}e_0 - \partial_{s_0}V_{w_1}e_1 + \partial_{s_1}V_{w_1}e_2 + \partial_{w_0}V_{w_1}e_3\\
&-\partial_{em_1}V_{w_1}e_4 + \partial_{g_0}V_{w_1}e_5 + \partial_{g_1}V_{w_1}e_6 - \partial_{em_0}V_{w_1}e_7\}\\
F_{s_0}e_6 &= \{\partial_{s_0}V_{s_0}e_0 + \partial_{w_1}V_{s_0}e_1 - \partial_{w_0}V_{s_0}e_2 + \partial_{s_1}V_{s_0}e_3\\
&+\partial_{em_0}V_{s_0}e_4 - \partial_{g_1}V_{s_0}e_5 + \partial_{g_0}V_{s_0}e_6 - \partial_{em_1}V_{s_0}e_7\}\\
F_{s_1}e_7 &= \{\partial_{s_1}V_{s_1}e_0 - \partial_{w_0}V_{s_1}e_1 - \partial_{w_1}V_{s_1}e_2 - \partial_{s_0}V_{s_1}e_3\\
&+\partial_{g_1}V_{s_1}e_4 + \partial_{em_0}V_{s_1}e_5 + \partial_{em_1}V_{s_1}e_6 + \partial_{g_0}V_{s_1}e_7\}.
\end{aligned} \tag{6.61}$$

So, in order to obtain the octonionic superstring field equations, we apply the differential operator (6.54) to equation (6.60) as

$$\begin{aligned}
&\boxplus_{SS}\ \mathbb{F}_{SS} = \\
&-e_0\{\partial_{g_0}F_{g_0}+\partial_{g_1}F_{g_1}+\partial_{em_0}F_{em_0}+\partial_{em_1}F_{em_1}+\partial_{w_0}F_{w_0}+\partial_{w_1}F_{w_1}+\partial_{s_0}F_{s_0}+\partial_{s_1}F_{s_1}\} \\
&+e_1\{\partial_{g_0}F_{g_1}+\partial_{g_1}F_{g_0}+\partial_{em_0}F_{em_1}-\partial_{em_1}F_{em_0}+\partial_{w_0}F_{s_1}-\partial_{w_1}F_{s_0}+\partial_{s_0}F_{w_1}-\partial_{s_1}F_{w_0}\} \\
&+e_2\{\partial_{g_0}F_{em_0}-\partial_{g_1}F_{em_1}+\partial_{em_0}F_{g_0}+\partial_{em_1}F_{g_1}+\partial_{w_0}F_{s_0}+\partial_{w_1}F_{s_1}-\partial_{s_0}F_{w_0}-\partial_{s_1}F_{w_1}\} \\
&+e_3\{\partial_{g_0}F_{em_1}+\partial_{g_1}F_{em_0}-\partial_{em_0}F_{g_1}+\partial_{em_1}F_{g_0}-\partial_{w_0}F_{w_1}+\partial_{w_1}F_{w_0}+\partial_{s_0}F_{s_1}-\partial_{s_1}F_{s_0}\} \\
&+e_4\{\partial_{g_0}F_{w_0}-\partial_{g_1}F_{s_1}-\partial_{em_0}F_{s_0}+\partial_{em_1}F_{w_1}+\partial_{w_0}F_{g_0}-\partial_{w_1}F_{em_1}+\partial_{s_0}F_{em_0}+\partial_{s_1}F_{g_1}\} \\
&+e_5\{\partial_{g_0}F_{w_1}+\partial_{g_1}F_{s_0}-\partial_{em_0}F_{s_1}-\partial_{em_1}F_{w_0}+\partial_{w_0}F_{em_1}+\partial_{w_1}F_{g_0}-\partial_{s_0}F_{g_1}+\partial_{s_1}F_{em_0}\} \\
&+e_6\{\partial_{g_0}F_{s_0}-\partial_{g_1}F_{w_1}+\partial_{em_0}F_{w_0}-\partial_{em_1}F_{s_1}-\partial_{w_0}F_{em_0}+\partial_{w_1}F_{g_1}+\partial_{s_0}F_{g_0}+\partial_{s_1}F_{em_1}\} \\
&+e_7\{\partial_{g_0}F_{s_1}+\partial_{g_1}F_{w_0}+\partial_{em_0}F_{w_1}+\partial_{em_1}F_{s_0}-\partial_{w_0}F_{g_1}-\partial_{w_1}F_{em_0}-\partial_{s_0}F_{em_1}+\partial_{s_1}F_{g_0}\},
\end{aligned} \tag{6.62}$$

which can further be reduced and be written in following compact notation in terms of an octonionic superstring representation as

$$\boxplus_{SS}\,\mathbb{F}_{SS} \;=\; \mathbb{J}_{SS}=\left(\left(J_0,J_1\right),\left(J_2,J_3\right),\left(J_4,J_5\right),\left(J_6,J_7\right)\right). \tag{6.63}$$

Here $\mathbb{J}_{SS}(J_0,J_1,J_2,J_3,J_4,J_5,J_6,J_7)$ is an octonionic superstring current source, which may be expressed in the following matrix form,

$$\begin{pmatrix}
J_{(g-g)} & J_{(g-g)} & J_{(g-em)} & J_{(g-em)} & J_{(g-w-s)} & J_{(g-w-s)} & J_{(g-s-w)} & J_{(g-s-w)} \\
J_{(em-em)} & J_{(em-em)} & J_{(em-g)} & J_{(em-g)} & J_{(em-s-w)} & J_{(em-s-w)} & J_{(em-w-s)} & J_{(em-w-s)} \\
J_{(w-w)} & J_{(w-s)} & J_{(w-s)} & J_{(w-w)} & J_{(w-g-em)} & J_{(w-em-g)} & J_{(w-em-g)} & J_{(w-g-em)} \\
J_{(s-s)} & J_{(s-w)} & J_{(s-s)} & J_{(s-w)} & J_{(s-em-g)} & J_{(s-g-em)} & J_{(s-g-em)} & J_{(s-em-g)}
\end{pmatrix}
\begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{pmatrix},
\Longrightarrow \mathbb{J}_{SS}. \tag{6.64}$$

Here $J_{(g-g)}, J_{(em-em)}, J_{(em-g)}, J_{(g-em)}$ are defined as the octonionic representation of gravitational - gravitational, electromagnetic - electromagnetic, electromagnetic - gravitational, gravitational - electromagnetic current source and $J_{(w-w)}, J_{(s-s)}, J_{(s-w)}, J_{(w-s)}$ sectors of the octonionic current source respectively associated the weak-weak, strong-strong, strong-weak, weak strong interactions. Consequently, the remaining terms of equation (6.64) are also denoted as the parts of octonionic current sources for the combination of different three interactions. Thus, the present formulation may be provides the superstring theory in the terms of octonionic representations.

6.6 Discussion and Conclusion

In section (6.1), we have seen that the standard model is incomplete and needs modification in order to explain the problems like the origin of mass, the strong CP problem, supersymmetry, neutrino oscillations, matter–antimatter asymmetry, the nature of dark matter and dark energy and the unification of gravity with strong and electroweak interactions. Accordingly, in section (6.2), we have described the role of octonions in grand unified theories of the gauge group is $SU(3) \times SU(2) \times U(1)$. The smallest simple Lie group which contains the standard model, and upon which the first Grand Unified Theory was based, is $SU(5) \supset SU(3) \times SU(2) \times U(1)$. Here, we have extended $SU(2) \times U(1)$ gauge theory to the $SU(3) \times SU(2) \times U(1)$ gauge theory in terms of split octonion formulation. Thus, equation (6.1) describes the two types of $SU(3) \times SU(2) \times U(1)$ gauge theories for particles carrying simultaneously electric and magnetic charges namely dyons. Accordingly, we have established the covariant derivative in equation (6.2) and (6.3) in the case of $SU(3)\times SU(2)\times U(1)$ octonion gauge field in the terms of split octonion 2×2 Zorn's vector matrix realization. The commutation relation of two covariant derivative are obtained by equation (6.4), which defines the $SU(3)\times SU(2)\times U(1)$ octonion gauge field strength for dyons in 2×2 Zorn matrix realization. Equations (6.5) and (6.6) represents the gauge field structures corresponding to U(1), SU(2) and SU(3) gauge groups exhibit the symmetry of gauge field strength corresponding to U(1), SU(2) and SU(3) gauge groups in presence of electric charge and magnetic monopole. Hence, we have obtained the field equation (6.8) for the grand unified theory i.e. $SU(3)\times SU(2)\times U(1)$ gauge for the fields associated with dyons. It is rather the octonion gauge current for dyons in $U(1)\times SU(2)\times SU(3)$ gauge symmetry given by equation (6.9) in terms of 2×2 Zorn matrix realization. Equation (6.10) represents the various gauge currents of GUTs namely the U(1) gauge current, SU(2) gauge current and SU(3) gauge current in presence of electric (magnetic) and magnetic (electric) charges. Here the j_{ν}^{0} describes $U(1)$ gauge current for electric charge, j_{ν}^{a} is the $SU(2)$ week current associated with electric charge provides $W^{\pm},Z^{0}$ bosons and j_{ν}^{α} is the current associated with $SU(3)_{c}$ used for chromo electric charge. Consequently, k_{ν}^{0} is $U(1)$ magnetic the counterpart of the U(1) gauge current, k_{ν}^{a} is the $SU(2)$ weak current while the k_{ν}^{α} is $SU(3)_{c}$ gluonic current due to the presence of magnetic monopole. As such, the octonionic formulation regardless a generalization of GUTs for the mixing of gauge currents used for $U(1),SU(2)$ and $SU(3)_{c}$ sectors associated respectively with the electromagnetic, weak and strong interactions in presence of dyons showing the duality invariance as well in terms of the continuity equation obtained in equation (6.11).

In section (6.3), we have discussed the following features of octonion realization of supersymmetry:

- Supercharges are realized as multiplication by octonion units.
- Super space is an octonionic space spanned by these octonion units.
- Rotation group acts as an algebraic automorphism of this super space.
- Lorentz transformations acting an octonion units belong to the gauge group that leaves octonionic norm.

For a theory to be supersymmetric, it is necessary that its particle content form a representation of the supersymmetry algebra (6.12), where the supersymmetry generator commute with momentum operator. Equation (6.13) describes the explicit representation of gamma matrices representation. The split octonion realizations have been recovered by setting $\sigma_a, \tilde{\sigma}_a$as matrices with octonion-valued entries, instead of being real matrices. Thus, the supersymmetry algebra is defined by equation (6.14), while the supersymmetry generator Q_a has been expressed by equation (6.15)-(6.16) in terms of 2×2 split octonions values matrices. Accordingly, the eight supersymmetry generators has been investigated in equation (6.17). The octonions supercharge is defined by equation (6.18), while the complex conjugate of octonionic supercharge is expressed by equation (6.19) in terms of eight dimensional representation. So, the supersymmetry has been investigated by equation (6.20) in terms of octonions.

In section (6.4), we have established the role of octonions in gravity and dark matter. Thus, we have defined the octonion space (eight dimensional) as the combination of two quaternionic spaces namely associated with the gravitational interaction (G-space) and electromagnetic interaction (EM-space). Equation (6.21) and (6.22) define the gravitational-electromagnetic space in terms of octonionic eight dimension space. Any physical quantity has been obtained by equation (6.23) in octonionic gravitational-electromagnetic space. Accordingly, the octonionic differential operator has been written in equation (6.24) as the combination of gravitational space (G-space) and electromagnetic space (EM-space). Equation (6.25) represents the octonion conjugate of equation (6.24). So, the octonion valued potential in eight dimensional space has been established in equations (6.26) and (6.27) as the combination of two four dimensional quaternionic space (i.e. G-space & EM-space). The octonionic field equation has been investigated in equation (6.28) which has been compactified by equation (6.29), whose components are described by equation (6.30). Accordingly, the field strength of octonionic generalized gravitational-electromagnetic fields of dyons has been investigated in equation (6.30). Equation (6.31) represents the components of gravitational-electromagnetic fields which are expressed in terms of octonionic representation given by equation (6.31). Thus, we have established the octonionic field strength due to gravitational interaction (G-space) and electromagnetic interaction (EM-space) given by equation (6.32). The components of octonionic wave equation has been expressed in equation (6.33), which has been compactified to equation (6.34). Equation (6.34) thus describes the compactified form of the octonion wave equation representing the octonionic gravitational-electromagnetic vector space. It shows the existence of generalized Dirac-Maxwell's (GDM) equations in terms of generalized gravitational-electromagnetic fields of dyons. Thus, the octonion current source has been obtained by equation (6.35) while their components are expressed by equation (6.36) in terms of gravitational gravitational, electromagnetic - electromagnetic, electromagnetic - gravitational, gravitational - electromagnetic interactions. So, equation (6.36) describes as the analogue of the generalized Dirac-Maxwell's (GDM) equations in presence of G-G, EM-EM, EM-G, G-EM interactions. Equation (6.37) represents the octonionic radius vector as the combination of two quaternionic spaces. The velocity representation of octonionic eight dimension has been obtained in equation (6.38). Accordingly, we have established the generalized charge and generalized mass of the particle given by the equation (6.39) in terms of octonion representation. So, we have been obtained the four-type of sub-fields in the octonionic Electromagnetic - Gravitational fields i.e. Gravitational - Gravitational (G-G) sub-field, Electromagnetic - Gravitational (EM-G) sub-field, Electromagnetic - Electromagnetic (EM-EM) subfield, Gravitational - Electromagnetic (G-EM) subfield.

In subsection (6.4.1), we have discussed the dark matter in terms of octonionic representation. The *'Dark Matter'* has been considered as a type of matter hypothesized to account for a large part of the total mass in the universe. Dark matter cannot be seen directly with telescopes which is neither emits nor absorbs light or other electromagnetic radiation at any significant level. Instead, its existence and properties are inferred from its gravitational effects on visible matter, radiation and the large scale structure of the universe. The majority of dark matter in the universe cannot be baryons, and thus does not form atoms. It also cannot interact with ordinary matter as electromagnetic forces, i.e. the dark matter particles do not carry any electric charge. The nonbaryonic dark matter may include the photon, graviton, intermediate bosons and neutrinos, or supersymmetric particles. Unlike baryonic matter, nonbaryonic dark matter does not contribute to the formulation of the elements in the universe as its presence is revealed only via its gravitational attraction. Thus, the nonbaryonic dark matter is evident through its gravitational effect only. Octonions hot dark matter is composed of particles that have zero or near-zero mass. Thus, the octonionic hot dark matter (OHDM) includes the photon and graviton. As such, we have established the various quantum equation for octonionic hot dark matter in terms of potential, field and current equations given by equations (6.40)-(6.45). It is concluded that the quantum equations for octonionic hot dark matter (i.e. photon and graviton) are expressed in the terms of quaternionic representations of octonions. Accordingly, the octonions cold dark matter has been described as the composition of the massive objects moving at sub-relativistic velocities. Hence, the octonions cold dark matter (OCDM) includes the intermediate particles (i.e. $W^{\pm}, Z^{o}$ particles). So, we have established the quantum equations for octonions cold dark matter in terms of potential, field and current equations given by equations (6.46)-(6.51).

In section (6.7), we have described the role of octonion in superstring theory. As such, the octonionic representation of the super-string (SS) theory has been established as the combination of four complex ($\mathbb{C}$) spaces namely associated with the gravitational (G-space), electromagnetic (EM-space), weak (W-space) and strong (S-space) interactions, i.e. unification of the four fundamental forces. Thus, from equation (6.52) it is clear that the octonionic superstring spaces can be defined as the combination of gravitational, electromagnetic, weak and strong spaces and a physical quantity can be written in octonionic superstring space by equation (6.53). The octonionic differential operator in the case of superstring theory has been obtained by equation (6.54). The octonion conjugate of equation (6.54) is described by equation (6.55). Thus, the octonionic valued superstring potential is investigation in equation (6.56) and (6.57) as the combination of four potentials. In order to obtain the octonion valued potential wave equation for superstring theory, we have operated the covariant derivative to octonionic superstring potential by equation (6.58) and thus obtained the octonion wave equation (6.59) for superstring theory in simple and compact octonion notation. The components of octonionic valued superstring potential with octonionic superstring conjugate differential operator has been expressed in equation (6.58). The compactified form of equation (6.58) has been defined by equation (6.59). Equation (6.60) represents the octonion superstring field strength, which is the combination of four interactions namely gravitational, electromagnetic, weak and strong interactions. It is shown that the octonion wave equation for superstring theory contains the various fields namely $F_g, F_{em}, F_w \,\&\, F_s$, where the first term ($F_g = F_{g0}, F_{g_1}$) defines the gravitational field strength in G-space, the second term ($F_{em} = F_{em_0}, F_{em_1}$) describes the electromagnetic field strength in EM-space, the third term ($F_w = F_{w_0}, F_{w_1}$) provides the weak interaction field strength in W-space and the forth term ($F_s = F_{s_0}, F_{s_1}$) is responsible for the strong field strength in S-space. Further more, the various components of octonionic superstring field are described in equation (6.61). In order to obtain the octonionic superstring field equation which has been obtained in equation (6.62) which has been obtained as octonionic wave equation (6.63) in compactified notation. Accordingly, the generalized Dirac-Maxwell's is visualized in superstring theory by means of octonions and the generalized current has been discussed in terms of matrix by equation (6.64). The generalized current described the various terms where $J_{(g-g)}, J_{(em-em)}, J_{(em-g)}, J_{(g-em)}$ defines as the octonionic representation of gravitational - gravitational, electromagnetic - electromagnetic, electromagnetic - gravitational, gravitational - electromagnetic current source and $J_{(w-w)}, J_{(s-s)}, J_{(s-w)}, J_{(w-s)}$ sectors of the octonionic current source respectively associated the weak-weak, strong-strong, strong-weak, weak strong interactions. Consequently, the remaining terms of equation (6.64) denotes as the parts of octonionic current sources for the combination of different three interactions. Thus, the present formulation has been provides the superstring theory in the terms of octonionic representations.

Bibliography

[1] M. E. Peskin and D. V. Schroeder, "**An introduction to quantum field theory**", Book, HarperCollins, (1995).

[2] T. P. Cheng and L. F. Li, "**Gauge theory of elementary particle physics**", Book, Oxford University Press, (1982).

[3] S. Weinberg, "**The quantum theory of fields**", Book, Cambridge University Press, **2** (1996).

[4] A. Zee, "**Quantum Field Theory in a Nutshell**", Book, Princeton University Press, Second Edition (2010).

[5] S. F. Novaes, "**Standard Model: An Introduction**", arXiv: hep-ph/0001283, (2000).

[6] S. L. Glashow, "**Partial-symmetries of weak interactions**", Nuclear Physics **22** (1961), 579.

[7] S. Weinberg, "**A Model of Leptons**", Physical Review Letters, **19** (1967), 1264.

[8] A. Salam, "**Elementary Particle Physics: Relativistic Groups and Analyticity**", Eighth Nobel Symposium, Stockholm: Almquvist and Wiksell, (1968) 367.

[9] F. Englert and R. Brout, "**Broken Symmetry and the Mass of Gauge Vector Mesons**", Physical Review Letters **13** (1964), 321.

[10] P. W. Higgs, "**Broken Symmetries and the Masses of Gauge Bosons**", Physical Review Letters, **13** (1964), 508.

[11] G. S. Guralnik, C. R. Hagen and T. W. B. Kibble, "**Global Conservation Laws and Massless Particles**", Physical Review Letters, **13** (1964), 585.

[12] D. J. Griffiths, “**Introduction to Elementary Particles**”, John Wiley & Sons (1987).

[13] R. Harnik, G. D. Kribs, and G. Perez, "**A universe without weak interactions**", Physical Review D, **74** (2006), 035006.

[14] C. Becchi, **“Introduction to Gauge Theories”**, arXiv: hep-ph/9705211, (1997).

[15] K. Nakamura, "**Review of Particle Physics**", Journal of Physics G: Nuclear and Particle Physics, **37** (2010), 075021.

[16] A. Boyle, "**Milestone in Higgs quest: Scientists find new particle**", MSNBC, Retrieved July 5, (2012)

[17] C. Arzt, M. B. Einhorn and J. Wudka, “**Patterns of deviation from the standardmodel**”, Nuclear Physics B, **433** (1995), 41.

[18] A. G. Cohen, D. B. Kaplan and A. E. Nelson, “**The more Minimal Supersymmetric StandardModel**”, Physics Letters B, **388** (1996), 588.

[19] V. A. Kosteleckï¿œ, “**Gravity, Lorentz violation, and the standard model**”, Phys. Rev. D, **69** (2004), 105009.

[20] G. Bertone, D. Hooper and J. Silk, “**Particle darkmatter: evidence, candidates and constraints**”, Physics Reports, Vol. **405** (2005), 279.

[21] G. Altarelli and F. Feruglio, “**Neutrino masses and mixings: a theoretical perspective**”, Physics Reports, **320** (1999), 295.

[22] M. Dine, “**Origin of the matter-antimatter asymmetry**”, Rev. Mod. Phys., **76** (2003), 1.

[23] H. C. Cheng and I. Low, “**TeV symmetry and the little hierarchy problem**”, JHEP, **09** (2003), 051.

[24] H. Y. Cheng, “**The strong CP problem revisited**”, Physics Reports, **158** (1988), 1.

[25] Lykken, "**Beyond the Standard Model**", arXiv: hep-ph/1005.1676 (2010).

[26] J. W. F. Valle, **“Physics Beyond the Standard Model”**, arXiv: hep-ph/9603307, (1996).

[27] S. Raby, **“Grand Unified Theories”**, arXiv: hep-ph/0608183, (2006).

[28] M. E. Peskin and D. V. Schroeder, “**An introduction to quantum field theory**”, Addison-Wesley, (1995) 786.

[29] G. Ross, “**Grand Unified Theories**”, Book, Westview Press. (1984).

[30] Buchmï¿œller, "**Neutrinos, Grand Unification and Leptogenesis**", arXiv: hep-ph/0204288v2,(2002).

[31] L. Ibï¿œï¿œez, **“Locally supersymmetric $SU(5)$ grand unification”**, Physics Letters B, **118** (1982), 73.

[32] L. J. Hall, R. Rattazzi and U. Sarid, **“Top quark mass in supersymmetric $SO(10)$ unification”**, Phys. Rev. D, **50** (1994), 7048.

[33] H. P. Nilles, “**Supersymmetry, supergravity and particle physics**”, Physics Reports, **110** (1984), 1.

[34] Wells, "**Lectures on Higgs Boson Physics in the Standard Model and Beyond**", arXiv: hep-ph/0909.4541, (2009).

[35] E. Witten, "**Stringtheory dynamics in various dimensions**", Nuclear Physics B, **443** (1995), 85.

[36] L. Smolin and R. Sundrum, "**Three Roads to Quantum Gravity**", Basic Books, ISBN 0-465-07835-4 (2001).

[37] A. A. Abdo, "**A limit on the variation of the speed of light arising from quantum gravity effects**", Nature **462** (7271).,331, arXiv:0908.1832 (Fermi GBM/LAT Collaborations) (2009).

[38] J. Maldacena, A. Strominger and E. Witten, "**Black hole entropy in M-Theory**", Journal of High Energy Physics, **12** (1997), 002.

[39] L. Randall and R. Sundrum, "**Large Mass Hierarchy from a Small Extra Dimension**", Physical Review Letters, **83** (1999), 3370.

[40] L. Randall and R. Sundrum, "**An Alternative to Compactification**", Physical Review Letters, **83** (1999), 4690.

[41] B. C. Chanyal, P. S. Bisht and O. P. S. Negi, **"Generalized Split-Octonion Electrodynamics"**, Int. J. Theor. Phys., **50** (2011), 1919.

[42] B. C. Chanyal, P. S. Bisht, Tianjun Li and O. P. S. Negi, **"Octonion Quantum Chromodynamics",** Int. J. Theor. Phys., **51** (2012), 3410.

[43] F. Toppan, "**Quaternionic and Octonionic Spinors**", arXiv: hep-th/0503210 v1, (2005).

[44] F. Toppan, "**On the Octonionic M-algebra and superconformal M-algebra**", arXiv: hep-th/0307070 v1, (2003).

[45] R. Foot and G. C. Joshi, "**A Natural Framework for the Minimal Supersymmetric Gauge Theories**", Letters in Mathematical Physics, **15** (1988), 237.

[46] A. Pashnev and F. Toppan, "**On the Classification of N-extended Supersymmetric Quantum Mechanical Systems**", J. Math. Phys. **42** (2001), 5257.

[47] Z. Weng, "**Octonionic Quantum Interplays of Dark Matter and Ordinary Matter**", arXiv: physics.gen-ph/0702019v4, (2008).

[48] Z. Weng, "**Octonionic electromagnetic and gravitational interactions and dark matter**", arXiv: physics.class-ph/0612102v8, (2009).

[49] Z. Weng, "**Octonionic Strong and Weak Interactions and Their Quantum Equations**", arXiv: physics/0702054, (2007).

LIST OF PUBLICATION

1. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi , **"Generalized Octonion Electrodynamics"** International Journal of Theoretical Physics, **49** (2010), 1333-1343.

2. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, **"Generalized Split-Octonion Electrodynamics"** International Journal of Theoretical Physics, **50** (2011), 1919-1926.

3. **B. C. Chanyal**, P. S. Bisht, Tianjun Li and O. P. S. Negi, **"Octonion Quantum Chromodynamics",** International Journal of Theoretical Physics, **51** (2012), 3410.

4. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi , **"Generalized Octonion Electrodynamics"** Abstract book of **"Gravitational Theories And Astronomy, CGTA",** Bhadrawati, District - Chandrapur-M.S. (India) during December 28-30 (2009), pp 50.

5. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, **"Octonion Analysis of Dyons"** Abstract book of **"Role of Science and Technology in the Development of Uttarakhand Stats"** Department of Physics, Govt. P. G. College, Bageshwar, Uttarakhand (India) during October 25-26, (2010) pp 28, (Communicated).

6. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, **"Generalized Split-Octonion Electrodynamics"** Abstract book of 5^{th} **UCOST**, Doon University, Dehradun, Uttarakhand (India) during November 10-12 (2010) pp 164.

7. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, "**Role of octonion in non - Abelian Gauge Theory**" Abstract book of **"National Seminar on Recent Trends in Micro and Macro Physics (NSRTMMP-2011)"** Department of Physics, Govt. P. G. College, Gopeshwar, Chamoli, Uttarakhand (India) during October 12-13, (2011) pp 34 (Communicated).

8. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, **"Role of Split-Octonions in Gauge Theory"** Abstract book of **"Recent Trends in Material Science and Nano Structure (RTMSNS-12)"** Department of Physics, Govt. P. G. College, Rudrapur (U.S. Nagar), Uttarakhand (India) during January 3-4, (2012) pp 48, (Communicated).

9. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, **"Split Octonion Electrodynamics and Energy-Momentum Conservation Laws for Dyons"** (Communicated) 2012.

10. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, **"Octonion Electrodynamics in chiral medium",** (Communicated) 2012.

11. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, **"Octonionic non-Abelian Gauge Theory",** e-print, hepp- viXra: 1210.0116, (2012) (To be published)

12. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, **"Octonion Dark Matter",** (Communicated) 2012.

CONFERENCES PARTICIPATED

1. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi , **"Generalized Octonion Electrodynamics"** International Level Conference on **"Gravitational Theories And Astronomy, CGTA"** December 28-30 (2009), Nilkanthrao Shinde Science & Arts College, Bhadrawati, District - Chandrapur (M.S.) India.

2. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, **"Octonion Analysis of Dyons"** State Level Conference on **"Role of Science and Technology in the Development of Uttarakhand Stats"** October 25-26 (2010), Department of Physics, Govt. P. G. College, Bageshwar, Uttarakhand (India).

3. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, **"Generalized Split-Octonion Electrodynamics"** State Level Conference 5^{th} **UCOST**, November 10-12 (2010) at Doon University, Dehradun, Uttarakhand (India).

4. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, "**Role of octonion in non - Abelian Gauge Theory**" National Level Conference on **"National Seminar on Recent Trends in Micro and Macro Physics (NSRTMMP-2011)"** October 12-13 (2011) at Department of Physics, Govt. P. G. College, Gopeshwar, Chamoli, Uttarakhand (India).

5. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, **"Octonions and non - Abelian Gauge Theory"** State Level Conference 6^{th} **UCOST**, November 14-16 (2011) at Kumaun University S. S. J. Campus Almora, Uttarakhand (India).

6. **B. C. Chanyal**, P. S. Bisht and O. P. S. Negi, **"Role of Split-Octonions in Gauge Theory"** National Level Conference on **"Recent Trends in Material Science and Nano Structure (RTMSNS-12)"** January 3-4 (2012) at Department of Physics, Govt. P. G. College, Rudrapur (U.S. Nagar), Uttarakhand (India).

........⋆⋆⋆........