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CASIMIR EFFECT: QUANTUM FLUCTUATIONS AND THERMAL FLUCTUATIONS

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BY THE COMMITTEE CONSISTING OF

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Dedicated to my respected parents-Xiaolong Li and Fengying Zhong, my virtuous wife-Sida Liu, my lovely daughter-Xiyu Li, and once confused myself.

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Norman, in epidemic

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Abstract

The Casimir effect is widely known as the force between two parallel neutral plates because of the vacuum energy. In quantum theories, the vacuum is not empty space, but very nontrivial. Any phenomenon caused by the nontrivial vacuum state of quantum fields in the presence of boundaries, nontrivial topology, varying background potentials, spacetime curvature, ect, could be referred to as a Casimir effect. At a finite temperature, Casimir effects can be significantly modified by thermal fluctuations. Despite various practical difficulties, Casimir effects have been observed in experiments. Thus, Casimir effects may be applied to practical techniques and devices, and have drawn much attention recently. This dissertation is devoted to the study of Casimir effects and their thermodynamical properties. We focus on three topics based on our research, which we will describe below.

In Chapter 1, we briefly depict the history of Casimir physics. The status of experimental investigations on Casimir physics is also briefly reviewed. We also outline possible research directions in Casimir physics, which may broaden our theoretical and technical horizons.

In Chapter 2, we show the influences of geometry and inhomogeneity on Casimir energies and stresses. Systems with high symmetries, i.e., the planar and spherical systems, are studied. We explore two media with one common surface, referred to as a two-media system. Because of the surface, there are extra divergences in Casimir energy densities and stresses at the surface besides familiar bulk divergences. We also investigate the configurations with two surfaces present, i.e., parallel configurations for planar systems and concentric configurations for spherical systems, in which finite Casimir interaction energies and Casimir forces are obtained. For planar systems, the Casimir energies and stresses are well understood in homogeneous two-media

backgrounds and parallel configurations. Some general behaviors of surface divergences in the inhomogeneous two-media background are shown. We also propose a renormalization scheme for inhomogeneous parallel configurations, which gives us finite Casimir interaction energies and forces. For spherical systems, the properties of surface divergences are largely unexplored topics, especially for inhomogeneous cases. We calculate an analytically solvable model to provide a first glimpse of surface divergences in spherical inhomogeneous two-media backgrounds. We also employ a renormalization scheme, similar to that in inhomogeneous parallel configurations, to figure out finite Casimir forces in inhomogeneous concentric configurations described by well chosen models. There is more work remaining to be done in this direction than that presented in this thesis.

In Chapter 3, we briefly demonstrate the thermal fluctuations in Casimir effects. We sketch the thermal corrections in Casimir forces. The influences of inhomogeneity on thermal Casimir forces, which may be significant in practical applications, has been investigated in a preliminary way. The Casimir interaction entropy has been intensively investigated for more than two decades, in which the negativity and consistency with Nernst's theorem are two main concerns. There are various sources for negative interaction entropy and we show two cases where the negative interaction entropies stem from the geometry. Recently, the Casimir self-entropy has attracted much attention. We pioneered this direction by considering the infinitely thin sheet and spherical shell, which are illustrated here succinctly. More realistic cases should be considered to facilitate future experiments.

In Chapter 4, we exhibit some of our thoughts about friction, which, we hope, could give us profound insights into the relations between quantum and thermal fluctuations and the irreversibility of time. As a foundation, we investigate the classical friction when a charged particle or a dipole moves in front of a dissipative conductor described with the Drude model. We also studied a two-level particle moving above a Drude conductor and two quantum oscillators in relative motion. Our investigations are still in progress.

Chapter 1

Introduction

1.1 History

Since the discovery of a nonzero attractive force between two neutral perfectly conducting plates [1], which is the famous Casimir effect, it is widely recognized that the vacuum can be highly nontrivial. The word "vacuum" means the space of no matter literally, which has been a frequently discussed topic for a long time. Plato felt it hard to accept the idea of vacuum, since he thought any realistic matter was an instantiation of its corresponding ideal pattern and the ideal form of vacuum was inconceivable to him. Aristotle considered the vacuum as logically impossible, because nothing can not be something. However, the ancient Chinese philosopher Lao Zi embraced the vacuum as the origin of the world, for example he taught "...everything in the world originates from existence, existence originates from non-existence..."

Not until Evangelista Torricelli and Blaise Pascal [2] confirmed that the "vacuum" was at the top of the mercury barometer in the 17th century, did the vacuum become an experimentally researchable object. Based on his theory of gravitation, Newton denied the existence of the vacuum by asserting that the universe was filled with ether as the mediator of gravity and light. This statement saved Kepler's hypothesis that the Moon influences the tides, which made Galileo uncomfortable.

But in 1887, the celebrated Michelson-Morley experiment ruled out the existence of the ether and showed that the speed of light is the same in different inertial frames and directions, which was regarded as one of the two clouds obscuring "the beauty and clearness of the dynamical theory" by Lord Kelvin [3] and led to Einstein's relativity. Another of Lord Kelvin's clouds, namely the inconsistency between the classical theoretical predictions on the spectrum of blackbody radiation and experiments, was also closely related to the evolution of our understanding of the vacuum. To dispel that cloud, Planck introduced a theory which heralded the upcoming quantum revolution and implied the possibility for the existence of zero-point energy (ZPE), as Einstein once stated "The existence of a zero-point energy of size $h\nu/2$ (is) probable" [4]. Unfortunately, Planck thought ZPE would had no physical consequence, while Einstein did not feel difficulty in taking the ZPE into his theory [5].

Actually if one compares the vacuum to an iceberg, the ZPE is just its tip. Nowadays, the vacuum typically refers to the ground state, which has the lowest possible energy, of a system. Regardless of the significance of the vacuum illustrated by the widely-known Dirac sea, even Heisenberg's uncertainty principle, saying that for any observables A and B the relation $\Delta A \Delta B \geq |\langle [A, B] \rangle|/2$ always holds true, suggests that fluctuations are not avoidable even in the vacuum state. Or one may roughly say the fluctuations are the main origin of the non-triviality of the vacuum state. Vacuum fluctuations are responsible for the Lamb shift [6, 7], light-light scattering in the vacuum [8], vacuum magnetic birefringence [9] and so on. The interaction with fluctuation-induced virtual particles requires any physically acceptable quantum field theory (QFT) to be renormalizable.

The modification of the vacuum also provides requisite physical mechanisms. For instance, in the Higgs mechanism, the broken symmetry of the vacuum state introduces masses to the weak interaction mediators. In 1948, Casimir [1] proposed to modify the vacuum state by inserting two parallel perfectly conducting plates so that a visible vacuum phenomenon, i.e., an attractive force on each plate, occurs. Generally, the phenomena due to the non-trivial modifications on the vacuum state caused by given boundary conditions, geometries, topologies etc., are all referred to as Casimir effects, on which I focus here[1].

Soon after Casimir's pioneering work, Lifshitz [10] and Dzyaloshinskii *et al.* [11, 12] generalized the original model to a configuration (DLP), in which two parallel dielectric media are separated by the vacuum or another homogeneous medium. Van Kampen *et al.* [13] also considered the Casimir forces between dielectrics with the zero-point energy approach. The next natural generalization is to calculate the Casimir forces in the generalized DLP configuration, in which the media are inhomogeneous. But previous endeavors [14, 15, 16, 17, 18], though valuable, do not lead to satisfactory answers. Recently, we [19] proposed a self-consistent and testable scheme to evaluate the Casimir forces in the parallel configuration comprising of inhomogeneous media.

Actually Casimir [20] suggested that the Casimir force could be the Poincaré stress compensating the repulsive electrostatic force in the classical electron model. However, in 1968 Boyer [21] demonstrated a repulsive Casimir stress in a infinitely thin perfectly conducting spherical shell, which has further been confirmed [22, 23, 24]. It rules out the Casimir stress acting as a Poincaré stress for such a simple model. A more realistic generalization of Boyer's spherical shell, i.e., a homogeneous dielectric ball immersed in a homogeneous background, was first studied by Milton [25] in 1980. Although, to the second order of the difference between the permittivities inside and outside the ball, one can remove the divergences and obtain a finite self-energy, there are unremovable divergences in higher order terms [26, 27], which obscure the interpretation on the self-energy. Brevik *et al.* [28, 29, 30, 31] introduced specific cases (diaphanous or isorefractive), in which the speed of light inside the ball equals to that outside the ball. In the diaphanous ball problems, finite well-defined self-energies are achieved. Further researches on the diaphanous spherical systems are still going on [32]. Also, people have not given up making sense of the self-energy and Casimir stress on a dielectric ball. Leonhardt *et al.* [33] claimed they had found a method to extract the finite Casimir self-stress on a dielectric ball, which cannot be regard-

[1]The natural units $\hbar = c = \varepsilon_0 = \mu_0 = k_B = G = 1$ will be utilized from now on, unless noted specifically.

ed as the summation of pair-wise Casimir-Polder interactions. Their results are inconsistent with some well-established conclusions [34, 26]; the results of [33] are argued to be erroneous because of their improper regularization and omission of the transverse contribution to the stress tensor [35]. The understanding of Casimir energies and forces in the homogeneous spherical systems are still insufficient, not to mention those effects in a inhomogeneous spherical system.

As stated above, quantum fluctuations of the vacuum state result in the Casimir effects. However, the quantum mechanism is not the only source of fluctuation. Thermal fluctuations are an important component of Casimir physics as well. The early investigations on finite-temperature corrections to the Casimir forces dated back to 1950s-60s [10, 36]. The recent controversies on proper theoretical treatments of the thermal corrections have not reach a consensus yet [37, 38, 39, 40, 41]. The consistency with thermodynamics is also involved in these controversies. Nervertheless, the pure thermodynamical aspects of the Casimir physics are quite enlightening on their own. The Casimir interaction entropy [42, 43, 44] and Casimir self-entropy [45, 46, 47, 48, 49] are the focuses in this topic, and the negativity and consistency with the third law of thermodynamics are the main concerns. The influences of the Casimir free energies, and thus entropies, on the stability of micro-structures are preliminarily studied recently [50, 51].

Furthermore, the fluctuations, both quantum-mechanical and thermodynamical, are inextricably linked to irreversible phenomena, amongst which is the quantum frictional dissipation. Although one can hardly say the quantum friction can have any practical significance [52], it is interesting to understand how the fluctuations influence or induce the frictional dissipation. There are three major methods employed in the quantum friction evaluations, i.e., the quantum statistical method [53, 54], the quantum field theory method [55, 56], and the quantum mechanical perturbation theory [52, 57]. However, there are various deviations among theoretical predictions, since no experimental trial has been done due to the smallness of the effect and possible laboratory difficulties. Experimentally testable nano-structures for quantum friction are badly needed.

Plainly, the depiction above has not covered, even a large part of, the developments of the quantum and classical fluctuation phenomena, especially dealing with Casimir physics. There are many more other interesting topics in Casimir physics, for instance, the Casimir effects in the cylindrical systems [58, 59, 60], multi-body Casimir effects [61, 62], Casimir effects out of thermal equilibrium [63, 64, 65], Casimir effects in curved spacetime [66, 67, 68], and so on. For more comprehensive reviews on Casimir physics, please see Refs. [69, 70, 71].

1.2 Experiments and applications

The Casimir forces, no matter whether they are between two perfectly conducting plates or two dielectrics, are typically very small, which obstructs the experimental measurements. There were a few experiments related to the Casimir force detection up to 1980 [72, 73, 74, 75, 76]. The pioneering work in Refs. [72, 73] only showed their results "do not contradict Casimir's theoretical prediction." The difficulty in maintaining parallelism was eliminated by measuring the force between a sphere and a plate [75, 76]. The Casimir effect was not, however, established decisively due to the lack of accuracy. The work of Sabisky *et al.* [74] is thought to be the first convincing experimental evidence of the Lifshitz theory. In 1997, Lamoreaux [77] performed a Cavendish-type experiment with a torsion balance, which confirmed the Casimir's theoretical prediction rather precisely for the first time. Soon after, Mohideen *et al.* [78, 79, 80] employed the atomic force microscope (AFM) system to perform a Casimir force measurement with 1% precision, as they claimed. Another type of apparatus, namely microelectromechanical systems (MEMSs), was utilized by Chan *et al.* [81, 82] and Decca *et al.* [83] in the Casimir force measurements. Those experimental verifications attracted attention back to Casimir physics research.

With the development of the experimental investigations of the Casimir effect, the thermal corrections to Casimir forces calculated theoretically [84] were compared with the experimental results in Ref. [77, 85]. According to Ref. [84], the dissipation and finite temperature dependances of the Casimir force predicted by the theory differ from the experimental results. Since then, there has been a controversy lasting for two decades involving which kind of permittivity model $\varepsilon(\omega)$ is more appropriate in describing a real metal at zero frequency. Many experiments [86, 87, 88] favor the nondissipative plasma model, but there are also experiments interpreted with the dissipative Drude model [89, 90]. Most recently, Mohideen *et al.* [91] performed experiments with more advanced techniques and cast one more vote in favor of the nondissipative plasma model. As we have mentioned, the dissipation of the material is also related to the negativity in the interaction entropy and the consistency with the thermodynamical laws. One can thus fairly state that the discordances between theories and experiments are still open questions to be explored. Other efforts to study the effect of dissipation in Casimir effects are just unfolding. For example, the relaxation of the free electrons in the nonequilibrium thermal Casimir effect has been considered [92].

Various other, but not all, Casimir effect experiments include the following. Researchers are trying to measure the Casimir forces in other geometries, such as, the Casimir force between two spheres [93], Casimir-Polder forces between particles and surfaces [94, 95, 96, 97], and so on. In addition, Chan *et al.* studied the Casimir forces in a microstructure on a silicon chip [98]. All those endeavors are bringing the esoteric theoretical issues of Casimir physics into our daily life as seemingly miraculous applications, though there is still a long way to go.

1.3 Future prospects

Casimir physics, caused by quantum fluctuations and significantly modified by thermal fluctuations, is a research area of broad prospects, both theoretically and experimentally.

On the theoretical side, it is still quite meaningful to further explore effects on Casimir energies and stresses due to geometry, such as the divergences and ambiguities in self-energies as stated above and those divergences due to nonsmoothness, or due to topology, such as the self-energy of a bisected sphere. Vacuum fluctuations in curved spacetime is also an intriguing topic. The energy densities and stress tensors, or energy-momentum tensor, of Casimir apparatuses in the gravitational field [99, 66] are direct generalizations of those in Minkowski space. There are also works on how to regularize and renormalize the divergences of the Casimir energy densities and stresses, which remains a problem even in flat spacetime as shown below, in curved spaces [67, 68]. The Casimir energy has long been considered as a source of the dark energy [100, 101]. The Casimir effects in the string [102, 103], superstring [104] and M-theory [105] have been intensively investigated as well.

On the experimental side, the Casimir effects in new materials, such as superconductors [106, 107], chiral media [108], topological insulators [109, 110], have been introduced. Although most of those researches are proposals and few experiments are reported, a bright future may be now within the reach of our eyesight. We anticipate contributing to the merging of Casimir physics with the rapidly developing field of new materials and the experimental techniques.

In this dissertation, we will briefly describe three topics in Casimir physics, namely, Casimir energies and stresses in systems with high symmetry, thermal corrections and Casimir entropies, and classical and quantum frictions. The narration is mainly based on our previous researches and partly of programs under study. For more detailed arguments, the reader is referred to our future publications.

Chapter 2

Casimir energies and stresses

2.1 Background

As stated above, when a system is in its ground state, the expectation values of the energy and stress of the system are called the Casimir energy and Casimir stress, which are named after Hendrik Casimir who pointed out, for the first time, the existence of a physically measurable force due to zero-point fluctuations [1]. Generally speaking, the physical phenomena caused by nontrivial Casimir energies and Casimir stresses are all known as Casimir effects, but typically Casimir effects are manifested as the forces arising from the Casimir interaction energy between rigid bodies.

Although the interaction energies are always finite and physically detectable, the Casimir energy calculations are always plagued with two types of divergences, i.e., the divergent total energies and the divergent local energy densities (and, of course, stress tensors). Total self-energies are commonly seen as divergent, even in Casimir's ideal model. So self-energies are usually less well-defined and some renormalization schemes are required to extract the finite and physically observable results from the self-energies. The most important one of the few unique and finite self-energies, is found in the perfectly conducting spherical shell with negligible thickness [21, 22, 23, 24], which excluded Casimir's proposal for the semi-classic model of electron. It could be expected that more valuable would be to extract the self-energy of a more realistic system, for instance, the self-energy in an inhomogeneous medium as shown below.

Divergences in local energy densities constantly occur at surfaces and are relatively independent of the total energies. Since gravity couples to the energy-momentum tensor locally, the Casimir energy density and stress tensor should act as the sources in Einstein's equations and have observable effects. Actually the influence of Casimir energy density and stress on gravity is basically an uncharted territory [111, 112, 113]. As an analogous and experimentally testable version of curved spacetime, inhomogeneous backgrounds and their Casimir energy densities and stresses have drawn a lot of attention. The studies in this direction are mainly focused on the spatially varing "soft" walls or boundaries, which maybe were first investigated in Ref. [114]. Efforts have been devoted to explore the properties of the Casimir energy densities and stresses [115, 116, 117, 118], and their renormalization schemes [117, 119, 120, 121] in soft wall systems. A frontier in this direction lies in computing the Casimir forces in inhomogeneous backgrounds, which is still in its initial stage.

2.2 General theory

In this chapter, we focus on the electromagnetic field, which is closer to laboratory investigations. The macroscopic Maxwell's equations, in the Euclidean space, are

$$\nabla \cdot \mathbf{D} = -\nabla \cdot \mathbf{P} + \rho,\ \nabla \times \mathbf{E} = -i\frac{\partial \mathbf{B}}{\partial \tau},\ \nabla \cdot \mathbf{B} = 0,\ \nabla \times \mathbf{H} = i\frac{\partial \mathbf{D}}{\partial \tau} + i\frac{\partial \mathbf{P}}{\partial \tau} + \mathbf{j}, \quad (2.1)$$

where $\rho, \mathbf{j}$ are the free charge density and current density involved, $\mathbf{P}$ is the external polarization source, τ is the Euclidean time, and the other parameters are defined as those in Refs. [122, 123]. In terms of the Fourier transformation, any vector $\mathbf{X}$ is expressed in the frequency space as

$$\mathbf{X}(\tau, \mathbf{r}) = \int \frac{d\zeta}{\sqrt{2\pi}} e^{i\zeta\tau} \mathbf{X}(\zeta, \mathbf{r}) = \int \frac{d\zeta}{\sqrt{2\pi}} e^{i\zeta\tau} \mathbf{X}(y), \quad (2.2)$$

in which $y = (\zeta, \mathbf{r})$ and ζ is the imaginary frequency. (Similar expressions apply for any scalars involved.) Then, with no free charge and current, the macroscopic Maxwell's equations in frequency space are

$$\nabla \cdot \mathbf{D} = -\nabla \cdot \mathbf{P},\ \nabla \times \mathbf{E} = \zeta \mathbf{B},\ \nabla \cdot \mathbf{B} = 0,\ \nabla \times \mathbf{H} = -\zeta \mathbf{D} - \zeta \mathbf{P}. \tag{2.3}$$

The action of the system S multiplied by the imaginary unit i is

$$\begin{aligned} iS &= \int d^4x_E \left[\frac{\mathbf{E}(x_E) \cdot \mathbf{D}(x_E) - \mathbf{H}(x_E) \cdot \mathbf{B}(x_E)}{2} + \mathbf{P}(x_E) \cdot \mathbf{E}(x_E) \right] \\ &= \int d^4y_1 d^4y_2 \frac{\mathbf{P}(y_1) \cdot \mathbf{\Gamma}(\bar{y}_1, y_2) \cdot \mathbf{P}(y_2)}{-2}, \end{aligned} \tag{2.4}$$

where $\bar{y} = (-\zeta, \mathbf{r})$ for any $y = (\zeta, \mathbf{r})$, and with $\mathbf{\Gamma}^{-1}(y_1, y_2)$ written in terms of the permittivity $\boldsymbol{\varepsilon}$ and permeability $\boldsymbol{\mu}$ as

$$\mathbf{\Gamma}^{-1}(y_1, y_2) = \delta(y_1 - y_2) \left[\boldsymbol{\varepsilon}(y_2) + \frac{\nabla_2 \times \boldsymbol{\mu}^{-1}(y_2) \cdot \nabla_2 \times \mathbf{1}}{\zeta_2^2} \right], \tag{2.5a}$$

and ∇_2 acts on y_2, the Green's dyadic $\mathbf{\Gamma}(y_1, y_2)$ is defined by

$$\int d^4y_2 \mathbf{\Gamma}^{-1}(y_1, y_2) \mathbf{\Gamma}(y_2, y_3) = \mathbf{1}\delta(y_1 - y_3). \tag{2.5b}$$

The relations connecting the electric and magnetic field to the electric displacement and magnetic induction in our case are $\mathbf{D}(y) = \boldsymbol{\varepsilon}(y) \cdot \mathbf{E}(y),\ \mathbf{B}(y) = \boldsymbol{\mu}(y) \cdot \mathbf{H}(y)$, where the permittivity and permeability are both localized and symmetric in the indices. By simplifying the Green's dyadic to $\mathbf{\Gamma}(\zeta, \mathbf{r}; \zeta', \mathbf{r}') = \delta(\zeta - \zeta') \mathbf{\Gamma}_\zeta(\mathbf{r}, \mathbf{r}')$, Eq. (2.5b) in a reduced form is

$$\left[\boldsymbol{\varepsilon}(\zeta, \mathbf{r}) + \frac{\nabla \times \boldsymbol{\mu}^{-1}(\zeta, \mathbf{r}) \cdot \nabla \times \mathbf{1}}{\zeta^2} \right] \cdot \mathbf{\Gamma}_\zeta(\mathbf{r}, \mathbf{r}') = \mathbf{1}\delta(\mathbf{r} - \mathbf{r}'), \tag{2.6a}$$

which leads to another useful equation

$$\left[\boldsymbol{\mu}(\zeta, \mathbf{r}) + \frac{\nabla \times \boldsymbol{\varepsilon}^{-1}(\zeta, \mathbf{r}) \cdot \nabla \times \mathbf{1}}{\zeta^2} \right] \cdot \mathbf{\Phi}_\zeta(\mathbf{r}, \mathbf{r}') = \mathbf{1}\delta(\mathbf{r} - \mathbf{r}'), \tag{2.6b}$$

where $\mathbf{\Phi}$ is expressed in terms of $\mathbf{\Gamma}$ by

$$\mathbf{\Phi}_\zeta(\mathbf{r}, \mathbf{r}') = \boldsymbol{\mu}^{-1}(\zeta, \mathbf{r}')\delta(\mathbf{r} - \mathbf{r}') - \frac{\boldsymbol{\mu}^{-1}(\zeta, \mathbf{r}) \cdot \nabla \times \mathbf{\Gamma}_\zeta(\mathbf{r}, \mathbf{r}') \times \overleftarrow{\nabla}' \cdot \boldsymbol{\mu}^{-1}(\zeta, \mathbf{r}')}{\zeta^2}. \tag{2.6c}$$

On these grounds, when no dissipation is present, the generating functional $Z = \int e^{iS}$ gives us the correlation functions

$$\langle \mathcal{T}\mathbf{E}(x)\mathbf{E}(x')\rangle = -\int \frac{d\zeta}{2\pi} e^{i\zeta(\tau-\tau')}\mathbf{\Gamma}_\zeta(\mathbf{r},\mathbf{r}'), \tag{2.7a}$$

$$\langle \mathcal{T}\mathbf{H}(x)\mathbf{H}(x')\rangle = -\int \frac{d\zeta}{2\pi} e^{i\zeta(\tau-\tau')}\left[\mathbf{\Phi}_\zeta(\mathbf{r},\mathbf{r}') - \boldsymbol{\mu}^{-1}(\zeta,\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\right], \tag{2.7b}$$

where $\mathcal{T}$ means the operators are time-ordered and $x = (\tau, \mathbf{r})$ is a spacetime point.

The energy density and momentum density transferred to the free sources, described by the free charge density ρ and free current density $\mathbf{j}$, per unit time are, respectively,

$$\mathbf{j}\cdot\mathbf{E} = -i\frac{du}{d\tau} = -i\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial\tau} - i\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial\tau} - \nabla\cdot(\mathbf{E}\times\mathbf{H}), \tag{2.8a}$$

$$\rho\mathbf{E} + \mathbf{j}\times\mathbf{B} = -i\frac{d\mathbf{p}}{d\tau} = \nabla\cdot(\mathbf{DE}+\mathbf{BH}) - D_i\nabla E_i - B_i\nabla H_i - i\frac{\partial\mathbf{D}\times\mathbf{B}}{\partial\tau}, \tag{2.8b}$$

where u and $\mathbf{p}$ are, respectively, the local energy and momentum density of the field. When there is no dissipation, the energy density u and the stress tensor $\mathbf{T}$ satisfy the relations

$$\frac{\partial u}{\partial\tau} = \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial\tau} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial\tau},\ \ \mathbf{T} = \frac{\mathbf{1}}{2}(\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H}) - \mathbf{DE} - \mathbf{BH}. \tag{2.9}$$

Therefore, the vacuum expectation values of the energy density and stress tensor of the field are

$$\overline{u}(\mathbf{r}) = -\frac{1}{2}\int\frac{d\zeta}{2\pi}\left\{\mathrm{tr}\frac{\partial[\zeta\boldsymbol{\varepsilon}(\zeta,\mathbf{r})]}{\partial\zeta}\cdot\mathbf{\Gamma}_\zeta(\mathbf{r},\mathbf{r}) + \mathrm{tr}\frac{\partial[\zeta\boldsymbol{\mu}(\zeta,\mathbf{r})]}{\partial\zeta}\cdot\mathbf{\Phi}_\zeta(\mathbf{r},\mathbf{r})\right\}, \tag{2.10a}$$

$$\begin{aligned}\overline{\mathbf{T}} \;=\; & -\int\frac{d\zeta}{2\pi}\left\{\frac{\mathbf{1}}{2}\mathrm{tr}\left[\boldsymbol{\varepsilon}(\zeta,\mathbf{r})\cdot\mathbf{\Gamma}_\zeta(\mathbf{r},\mathbf{r}) + \boldsymbol{\mu}(\zeta,\mathbf{r})\cdot\mathbf{\Phi}_\zeta(\mathbf{r},\mathbf{r})\right]\right.\\ & \left. -\boldsymbol{\varepsilon}(\zeta,\mathbf{r})\cdot\mathbf{\Gamma}_\zeta(\mathbf{r},\mathbf{r}) - \boldsymbol{\mu}(\zeta,\mathbf{r})\cdot\mathbf{\Phi}_\zeta(\mathbf{r},\mathbf{r})\right\},\end{aligned} \tag{2.10b}$$

where the obvious nonphysical δ-function terms, which are either bulk constants or only related to the structure of medium, have been omitted.

2.2.1 Planar systems

When the properties of a system varies in only one direction (usually chosen to be as the z-direction without losing any generality), we refer to this system as a *planar system*. In planar systems varying in the z-direction, the reduced Green's functions defined in Eq. (2.6) have the Fourier forms

$$(\mathbf{\Gamma}_\zeta, \mathbf{\Phi}_\zeta)(\mathbf{r}, \mathbf{r}') = \int \frac{d^2k}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{r}_\parallel - \mathbf{r}'_\parallel)} (\mathbf{g}_{\zeta,\mathbf{k}}, \mathbf{h}_{\zeta,\mathbf{k}})(z, z'). \tag{2.11}$$

For any given transverse (xy-directions) wavenumber vector $\mathbf{k}$, set $\mathbf{k}/k, k = |\mathbf{k}|$ as the unit vector along the x-axis, then it is convenient to employ the following g^E and g^H to express $\mathbf{g}_{\zeta,\mathbf{k}}$ and $\mathbf{h}_{\zeta,\mathbf{k}}$ when the medium is isotropic,

$$\left[\partial_z \frac{1}{(\mu,\varepsilon)} \partial_z - (\varepsilon, \mu)\zeta^2 - \frac{k^2}{(\mu,\varepsilon)}\right] g_{\zeta,\mathbf{k}}^{(E,H)}(z, z') = \delta(z - z'), \tag{2.12}$$

then $\mathbf{g}_{\zeta,\mathbf{k}}$ and $\mathbf{h}_{\zeta,\mathbf{k}}$, in this special frame, are

$$\mathbf{g}_{\zeta,\mathbf{k}}(z, z') = \begin{bmatrix} \frac{1}{\varepsilon\varepsilon'}\partial_z\partial_{z'} g_{\zeta,\mathbf{k}}^H + \frac{\delta(z-z')}{\varepsilon'} & & \frac{ik}{\varepsilon\varepsilon'}\partial_z g_{\zeta,\mathbf{k}}^H \\ & -\zeta^2 g_{\zeta,\mathbf{k}}^E & \\ -\frac{ik}{\varepsilon\varepsilon'}\partial_{z'} g_{\zeta,\mathbf{k}}^H & & \frac{k^2}{\varepsilon\varepsilon'} g_{\zeta,\mathbf{k}}^H + \frac{\delta(z-z')}{\varepsilon'} \end{bmatrix}, \tag{2.13a}$$

$$\mathbf{h}_{\zeta,\mathbf{k}}(z, z') = \begin{bmatrix} \frac{1}{\mu\mu'}\partial_z\partial_{z'} g_{\zeta,\mathbf{k}}^E + \frac{\delta(z-z')}{\mu'} & & \frac{ik}{\mu\mu'}\partial_z g_{\zeta,\mathbf{k}}^E \\ & -\zeta^2 g_{\zeta,\mathbf{k}}^H & \\ -\frac{ik}{\mu\mu'}\partial_{z'} g_{\zeta,\mathbf{k}}^E & & \frac{k^2}{\mu\mu'} g_{\zeta,\mathbf{k}}^E + \frac{\delta(z-z')}{\mu'} \end{bmatrix}, \tag{2.13b}$$

which, in a general coordinate system, have the forms

$$\mathbf{g}_{\zeta,\mathbf{k}} = \begin{bmatrix} \frac{k_x^2}{k^2}\frac{\partial_z\partial_{z'} g_{\zeta,\mathbf{k}}^H}{\varepsilon\varepsilon'} - \frac{k_y^2}{k^2}\zeta^2 g_{\zeta,\mathbf{k}}^E & \frac{k_x k_y}{k^2}\frac{\partial_z\partial_{z'} g_{\zeta,\mathbf{k}}^H}{\varepsilon\varepsilon'} + \frac{k_x k_y}{k^2}\zeta^2 g_{\zeta,\mathbf{k}}^E & \frac{ik_x\partial_z g_{\zeta,\mathbf{k}}^H}{\varepsilon\varepsilon'} \\ \frac{k_x k_y}{k^2}\frac{\partial_z\partial_{z'} g_{\zeta,\mathbf{k}}^H}{\varepsilon\varepsilon'} + \frac{k_x k_y}{k^2}\zeta^2 g_{\zeta,\mathbf{k}}^E & \frac{k_y^2}{k^2}\frac{\partial_z\partial_{z'} g_{\zeta,\mathbf{k}}^H}{\varepsilon\varepsilon'} - \frac{k_x^2}{k^2}\zeta^2 g_{\zeta,\mathbf{k}}^E & \frac{ik_y\partial_z g_{\zeta,\mathbf{k}}^H}{\varepsilon\varepsilon'} \\ -\frac{ik_x\partial_{z'} g_{\zeta,\mathbf{k}}^H}{\varepsilon\varepsilon'} & -\frac{ik_y\partial_{z'} g_{\zeta,\mathbf{k}}^H}{\varepsilon\varepsilon'} & \frac{k^2 g_{\zeta,\mathbf{k}}^H}{\varepsilon\varepsilon'} \end{bmatrix}, \tag{2.13c}$$

$$
\mathbf{h}_{\zeta,\mathbf{k}} = \begin{bmatrix} \frac{k_x^2}{k^2}\frac{\partial_z\partial_{z'}g^E_{\zeta,\mathbf{k}}}{\mu\mu'} - \frac{k_y^2}{k^2}\zeta^2 g^H_{\zeta,\mathbf{k}} & \frac{k_xk_y}{k^2}\frac{\partial_z\partial_{z'}g^E_{\zeta,\mathbf{k}}}{\mu\mu'} + \frac{k_xk_y}{k^2}\zeta^2 g^H_{\zeta,\mathbf{k}} & \frac{ik_x\partial_z g^E_{\zeta,\mathbf{k}}}{\mu\mu'} \\ \frac{k_xk_y}{k^2}\frac{\partial_z\partial_{z'}g^E_{\zeta,\mathbf{k}}}{\mu\mu'} + \frac{k_xk_y}{k^2}\zeta^2 g^H_{\zeta,\mathbf{k}} & \frac{k_y^2}{k^2}\frac{\partial_z\partial_{z'}g^E_{\zeta,\mathbf{k}}}{\mu\mu'} - \frac{k_x^2}{k^2}\zeta^2 g^H_{\zeta,\mathbf{k}} & \frac{ik_y\partial_z g^E_{\zeta,\mathbf{k}}}{\mu\mu'} \\ -\frac{ik_x\partial_{z'} g^E_{\zeta,\mathbf{k}}}{\mu\mu'} & -\frac{ik_y\partial_{z'} g^E_{\zeta,\mathbf{k}}}{\mu\mu'} & \frac{k^2 g^E_{\zeta,\mathbf{k}}}{\mu\mu'} \end{bmatrix}. \tag{2.13d}
$$

By separating the Casimir energy density $u(\mathbf{r})$ into the TE and TM mode contributions, then $u(\mathbf{r})$ is

$$
u(\mathbf{r}) = \int \frac{d\zeta}{2\pi}\Big[u_E(\zeta,\mathbf{r}) + u_H(\zeta,\mathbf{r})\Big], \tag{2.14a}
$$

where the TE energy density per unit frequency is

$$
u_E(\zeta,\mathbf{r}) = -\frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\left\{\frac{\partial(\zeta\mu)}{\partial\zeta}\left[\frac{\partial_z\partial_{z'}}{\mu\mu'}g^E_{\zeta,\mathbf{k}} + \frac{k^2}{\mu\mu'}g^E_{\zeta,\mathbf{k}}\right] - \frac{\partial(\zeta\varepsilon)}{\partial\zeta}\zeta^2 g^E_{\zeta,\mathbf{k}}\right\}, \tag{2.14b}
$$

and the TM contribution $u_H(\zeta,\mathbf{r})$ is obtained by making the substitutions $\varepsilon \leftrightarrow \mu$ and $E \leftrightarrow H$. Similarly, the ij-component of the stress tensor is

$$
T_{ij}(\mathbf{r}) = \int \frac{d\zeta}{2\pi}\Big[t_{E;ij}(\zeta,\mathbf{r}) + t_{H;ij}(\zeta,\mathbf{r})\Big], \tag{2.15a}
$$

where the off-diagonal terms $T_{ij}, i \neq j$ are typically zero in many cases and put aside here. The reduced diagonal components are

$$
t_{E;xx}(\zeta;\mathbf{r}) = -\int \frac{d^2k}{(2\pi)^2}\frac{1}{2\mu}\left\{\frac{k_y^2 - k_x^2}{k^2}\left[\partial_z\partial_{z'}g^E_{\zeta,\mathbf{k}} + \varepsilon\mu\zeta^2 g^E_{\zeta,\mathbf{k}}\right] + k^2 g^E_{\zeta,\mathbf{k}}\right\}, \tag{2.15b}
$$

$$
t_{E;yy}(\zeta;\mathbf{r}) = -\int \frac{d^2k}{(2\pi)^2}\frac{1}{2\mu}\left\{\frac{k_x^2 - k_y^2}{k^2}\left[\partial_z\partial_{z'}g^E_{\zeta,\mathbf{k}} + \varepsilon\mu\zeta^2 g^E_{\zeta,\mathbf{k}}\right] + k^2 g^E_{\zeta,\mathbf{k}}\right\}, \tag{2.15c}
$$

$$
t_{E;zz}(\zeta;\mathbf{r}) = -\int \frac{d^2k}{(2\pi)^2}\frac{1}{2\mu}\left[\partial_z\partial_{z'}g^E_{\zeta,\mathbf{k}} - k^2 g^E_{\zeta,\mathbf{k}} - \varepsilon\mu\zeta^2 g^E_{\zeta,\mathbf{k}}\right]. \tag{2.15d}
$$

The corresponding TM contribution is obtained by making the substitution $\varepsilon \leftrightarrow \mu$ and $E \to H$.

Define the functions $e_\pm$ and $h_\pm$ satisfying proper[1] boundary conditions

[1] The proper boundary conditions for $e_\pm$ and $h_\pm$ are usually $\lim_{z\to\pm\infty}(e,h)_\pm(z) = 0$, but not necessarily. We do see some different conditions, for example at the singularity of the

and the equations

$$\left[\partial_z \frac{1}{(\mu,\varepsilon)}\partial_z - (\varepsilon,\mu)\zeta^2 - \frac{k^2}{(\mu,\varepsilon)}\right](e_\pm, h_\pm)(\zeta,k;z) = 0, \tag{2.16}$$

then $g^E_{\zeta,\mathbf{k}}$ and its corresponding Wronskians $W^E_{\zeta,\mathbf{k}}$ are written as

$$g^E_{\zeta,\mathbf{k}}(z,z') = \frac{e_+(\zeta,k;z_>)e_-(\zeta,k;z_<)}{W^E_{\zeta,\mathbf{k}}},\ W^E_{\zeta,\mathbf{k}} = \frac{e'_+e_- - e_+e'_-}{\mu},\ e'_\pm = \frac{\partial e_\pm}{\partial z}, \tag{2.17}$$

and $g^H_{\zeta,\mathbf{k}}$ and its corresponding Wronskians $W^H_{\zeta,\mathbf{k}}$ are obtained by substituting $\varepsilon \leftrightarrow \mu$, $e \to h$. The following identities are very useful

$$\partial_z\left(\frac{e'_\sigma}{\mu}\frac{\partial}{\partial\zeta}e_\rho - e_\sigma\frac{\partial}{\partial\zeta}\frac{e'_\rho}{\mu}\right) = -\frac{1}{\mu}\frac{\partial(\varepsilon\mu\zeta^2)}{\partial\zeta}e_\sigma e_\rho + \frac{\partial\ln\mu}{\partial\zeta}\frac{\partial}{\partial z}\left(\frac{e'_\sigma e_\rho}{\mu}\right), \tag{2.18}$$

where $\sigma,\rho = \pm$. A similar identity for h can be obtained by substituting $\varepsilon \leftrightarrow \mu$, $e \to h$. So $u_E(\zeta;\mathbf{r})$ has another form

$$u_E(\zeta;\mathbf{r}) = -\partial_z\int\frac{d^2k}{(2\pi)^2}\frac{1}{2W^E_{\zeta,\mathbf{k}}}\left[\frac{e'_+e_-}{\mu} + \zeta\frac{e'_+}{\mu}\frac{\partial}{\partial\zeta}e_- - \zeta e_+\frac{\partial}{\partial\zeta}\frac{e'_-}{\mu}\right], \tag{2.19a}$$

and the reduced stress tensors are

$$t_{E;xx}(\zeta;\mathbf{r}) = \int\frac{d^2k}{(2\pi)^2}\frac{-1}{2\mu W^E_{\zeta,\mathbf{k}}}\left\{\frac{k_y^2-k_x^2}{k^2}\left[e'_+e'_- + \varepsilon\mu\zeta^2e_+e_-\right] + k^2e_+e_-\right\}, \tag{2.19b}$$

$$t_{E;yy}(\zeta;\mathbf{r}) = \int\frac{d^2k}{(2\pi)^2}\frac{-1}{2\mu W^E_{\zeta,\mathbf{k}}}\left\{\frac{k_x^2-k_y^2}{k^2}\left[e'_+e'_- + \varepsilon\mu\zeta^2e_+e_-\right] + k^2e_+e_-\right\}, \tag{2.19c}$$

$$t_{E;zz}(\zeta;\mathbf{r}) = \int\frac{d^2k}{(2\pi)^2}\frac{-1}{2\mu W^E_{\zeta,\mathbf{k}}}\left[e'_+e'_- - (k^2+\varepsilon\mu\zeta^2)e_+e_-\right]. \tag{2.19d}$$

The corresponding TM contribution is obtained by making the substitution $\varepsilon \leftrightarrow \mu$, $e \to h$ and $E \to H$, which we may refer to as EM-substitution.

When the whole space is filled with one medium, then since the responses of the medium are local, the TE Casimir energy density and stress tensors

potential. Similar argument is applied in the spherical systems discussed later.

are expressed as

$$u_E(\mathbf{r}) = -\int \frac{d\zeta d^2k}{16\pi^3}\frac{\zeta}{W^E_{\zeta,\mathbf{k}}}\left\{\frac{\partial\ln(\zeta\mu)}{\partial\zeta}\partial_z\left(\frac{e'_+e_-}{\mu}\right) - \frac{1}{\mu}\frac{\partial(\varepsilon\mu\zeta^2)}{\partial\zeta}e_+e_-\right\}, \quad (2.20a)$$

$$T_{E;xx}(\mathbf{r}) = T_{E;yy}(\mathbf{r}) = -\int \frac{d\zeta d^2k}{16\pi^3}\frac{k^2 e_+e_-}{\mu W^E_{\zeta,\mathbf{k}}}, \quad (2.20b)$$

$$T_{E;zz}(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3}\frac{\partial\ln[e_+(z_+),e_-(z)]_\mu}{\partial z_+} = \int \frac{d\zeta d^2k}{16\pi^3}\frac{\partial\ln[e_+(z),e_-(z_-)]_\mu}{-\partial z_-}, \quad (2.20c)$$

where the notation $[\cdot,\cdot]_\mu$ is defined as $\forall f,g,\ [f,g]_\mu = f'g/\mu_f - g'f/\mu_g$ and $z_+ = z_- = z$. For any given plane $z=a$, the total TE Casimir energy per unit transverse area in the $z>a$ and $z<a$ regions satisfy the expression

$$\frac{U_E^{>}(a)}{A} = -\frac{U_E^{<}(a)}{A} = \int \frac{d\zeta d^2k}{16\pi^3}\frac{1}{W^E_{\zeta,\mathbf{k}}}\left[\frac{e'_+}{\mu}\frac{\partial}{\partial\zeta}\zeta e_- - \zeta e_+\frac{\partial}{\partial\zeta}\frac{e'_-}{\mu}\right]\Bigg|_{z=a}, \quad (2.21a)$$

where A is the area of the transverse plate and

$$U_E^{\pm\infty} = \mp\int \frac{d\zeta d^2k}{16\pi^3}\frac{1}{W^E_{\zeta,\mathbf{k}}}\left[\frac{e'_+e_-}{\mu} + \zeta\frac{e'_+}{\mu}\frac{\partial}{\partial\zeta}e_- - \zeta e_+\frac{\partial}{\partial\zeta}\frac{e'_-}{\mu}\right]\Bigg|_{z=\pm\infty}, \quad (2.21b)$$

are unphysical constants, which we will always ignore in $U_E^{>}$ and $U_E^{<}$, respectively. Therefore, the total TE Casimir energy of any uniform background is zero. The TM contributions are obtained with the EM-substitution.

Consider two media (ε_1,μ_1) and (ε_2,μ_2) filling in half-spaces $z<a$ and $z>a$, respectively. Suppose $\hat{e}_{i,\pm}, i=1,2$ are solutions for Eq. (2.16) satisfying proper boundary conditions when the medium i is analytically extended to the whole space, then $e_\pm$ are solved as

$$e_+(\zeta,k;z) = \begin{cases} \hat{e}_{2,+}(z), & z>a, \\ A_e\hat{e}_{1,+}(z) + B_e\hat{e}_{1,-}(z), & z<a, \end{cases} \quad (2.22a)$$

$$e_-(\zeta,k;z) = \begin{cases} C_e \hat{e}_{2,+}(z) + D_e \hat{e}_{2,-}(z), & z > a, \\ \hat{e}_{1,-}(z), & z < a, \end{cases} \tag{2.22b}$$

and the coefficients $A_e,\ B_e,\ C_e$ and D_e are

$$A_e = \frac{[\hat{e}_{2,+}(a), \hat{e}_{1,-}(a)]_\mu}{\hat{W}_1^E}, \quad B_e = \frac{[\hat{e}_{1,+}(a), \hat{e}_{2,+}(a)]_\mu}{\hat{W}_1^E}, \tag{2.22c}$$

$$C_e = \frac{[\hat{e}_{1,-}(a), \hat{e}_{2,-}(a)]_\mu}{\hat{W}_2^E}, \quad D_e = \frac{[\hat{e}_{2,+}(a), \hat{e}_{1,-}(a)]_\mu}{\hat{W}_2^E}. \tag{2.22d}$$

The interaction induced TE Casimir energy density and stress tensors, which are the energy density and stress tensors with the corresponding bulk contributions subtracted, in the $z > a$ region are

$$\Delta u_E(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3} \frac{-\zeta C_e}{D_e \hat{W}_2^E} \left[\frac{\partial \ln(\zeta\mu_2)}{\partial\zeta} \partial_z \left(\frac{\hat{e}'_{2,+}\hat{e}_{2,+}}{\mu_2} \right) - \frac{1}{\mu_2} \frac{\partial(\varepsilon_2\mu_2\zeta^2)}{\partial\zeta} \hat{e}^2_{2,+} \right], \tag{2.23a}$$

$$\Delta T_{E;xx}(\mathbf{r}) = \Delta T_{E;yy}(\mathbf{r}) = -\int \frac{d\zeta d^2k}{(2\pi)^3} \frac{k^2 C_e}{2\mu_2 D_e \hat{W}_2^E} \hat{e}^2_{2,+}, \tag{2.23b}$$

$$\Delta T_{E;zz}(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3} \frac{-[\hat{e}_{1,-}(a), \hat{e}_{2,-}(a)]_\mu}{[\hat{e}_{2,+}(a), \hat{e}_{1,-}(a)]_\mu \hat{W}_2^E} \frac{\partial[\hat{e}_{2,+}(z), \hat{e}_{2,+}(z_-)]_\mu}{\partial z_-}, \tag{2.23c}$$

while in the $z < a$ region they are

$$\Delta u_E(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3} \frac{-\zeta B_e}{A_e \hat{W}_1^E} \left[\frac{\partial \ln(\zeta\mu_1)}{\partial\zeta} \partial_z \left(\frac{\hat{e}'_{1,-}\hat{e}_{1,-}}{\mu_1} \right) - \frac{1}{\mu_1} \frac{\partial(\varepsilon_1\mu_1\zeta^2)}{\partial\zeta} \hat{e}^2_{1,-} \right], \tag{2.24a}$$

$$\Delta T_{E;xx}(\mathbf{r}) = \Delta T_{E;yy}(\mathbf{r}) = -\int \frac{d\zeta d^2k}{(2\pi)^3} \frac{k^2 B_e}{2\mu_1 A_e \hat{W}_1^E} \hat{e}^2_{1,-}, \tag{2.24b}$$

$$\Delta T_{E;zz}(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3} \frac{-[\hat{e}_{1,+}(a), \hat{e}_{2,+}(a)]_\mu}{[\hat{e}_{2,+}(a), \hat{e}_{1,-}(a)]_\mu \hat{W}_1^E} \frac{\partial[\hat{e}_{1,-}(z), \hat{e}_{1,-}(z_-)]_\mu}{\partial z_-}. \tag{2.24c}$$

The total TE Casimir and interaction energy U_E and ΔU_E per unit transverse area satisfy the expression

$$\begin{aligned}\frac{U_E}{A} &= \frac{\Delta U_E(a) + U^<_{E,1}(a) + U^>_{E,2}(a)}{A} \\ &= \int \frac{d\zeta d^2k}{16\pi^3} - \int \frac{d\zeta d^2k}{16\pi^3}\zeta\frac{\partial \ln[\hat{e}_{2,+}(a), \hat{e}_{1,-}(a)]_\mu}{\partial\zeta},\end{aligned} \tag{2.25a}$$

while the pressure at the surface $z = a$, i.e., $P_E = T_{E;zz}(a_-) - T_{E;zz}(a_+)$, is

$$P_E = -\int \frac{d\zeta d^2k}{16\pi^3}\frac{\partial \ln[\hat{e}_{2,+}(a), \hat{e}_{1,-}(a)]_\mu}{\partial a}, \tag{2.25b}$$

which means the principle of virtual work (PVW) holds true. The corresponding TM contributions are obtained with the EM-substitution. Consider a medium (ε, μ) filling the half-space $z > a$ with a perfectly conducting plate at $z = a$, which is a special case when $(\varepsilon_1, \mu_1) \to (\infty, 1)$ and $(\varepsilon_2, \mu_2) \to (\varepsilon, \mu)$. In this special case, the total TE Casimir energy per unit transverse area and the pressure on the surface are

$$\frac{U_E}{A} = \lim_{\varepsilon_1\to\infty} \frac{\Delta U_E + U^<_{E,1} + U^>_E - U^<_{E,1}}{A}(a) = -\int \frac{d\zeta d^2k}{16\pi^3}\zeta\frac{\partial}{\partial\zeta}\ln \hat{e}_+(a), \tag{2.26a}$$

$$P_E = \lim_{\varepsilon_1\to\infty}\int \frac{d\zeta d^2k}{16\pi^3}\frac{\partial \ln[\hat{e}_+, \hat{e}_{1,-}]_\mu(a)}{-\partial a} - T_{E,1}(a_-) = \int \frac{d\zeta d^2k}{16\pi^3}\frac{\partial \ln \hat{e}_+(a)}{-\partial a}, \tag{2.26b}$$

which shows that the contributions to energy and stresses from the perfectly conductor are zero; following the same arguments, one can show that the corresponding TM contributions are

$$\frac{U_H}{A} = -\int \frac{d\zeta d^2k}{16\pi^3}\zeta\frac{\partial}{\partial\zeta}\ln\frac{\hat{h}'_+(a)}{\varepsilon(a)},\ P_H = -\int \frac{d\zeta d^2k}{16\pi^3}\frac{\partial}{\partial a}\ln\frac{\hat{h}'_+(a)}{\varepsilon(a)}. \tag{2.26c}$$

Consider the simplest physically significant planar system, i.e., three parallel isotropic media $(\varepsilon_i, \mu_i),\ i = 1, 2, 3$ filling in the regions $z < a$, $a < z < b$ and $z > b$, respectively. Suppose $\hat{e}_{i,\pm},\ i = 1, 2, 3$ are solutions satisfying proper boundary conditions for Eq. (2.16), when the medium i is analytically extended to the whole space, then $e_\pm$ are solved as

$$e_+(\zeta,k;z)=\begin{cases}\hat{e}_{3,+}(z), & z>b,\\ C_+\hat{e}_{2,+}(z)+D_+\hat{e}_{2,-}(z), & a<z<b,\\ A_+\hat{e}_{1,+}(z)+B_+\hat{e}_{1,-}(z), & z<a,\end{cases} \tag{2.27a}$$

$$e_-(\zeta,k;z)=\begin{cases}A_-\hat{e}_{3,+}(z)+B_-\hat{e}_{3,-}(z), & z>b,\\ C_-\hat{e}_{2,+}(z)+D_-\hat{e}_{2,-}(z), & a<z<b,\\ \hat{e}_{1,-}(z), & z<a,\end{cases} \tag{2.27b}$$

and the coefficients are

$$C_+=\frac{[\hat{e}_{3,+}(b),\hat{e}_{2,-}(b)]_\mu}{\hat{W}_2^E},\quad D_+=\frac{[\hat{e}_{2,+}(b),\hat{e}_{3,+}(b)]_\mu}{\hat{W}_2^E}, \tag{2.27c}$$

$$C_-=\frac{[\hat{e}_{1,-}(a),\hat{e}_{2,-}(a)]_\mu}{\hat{W}_2^E},\quad D_-=\frac{[\hat{e}_{2,+}(a),\hat{e}_{1,-}(a)]_\mu}{\hat{W}_2^E}. \tag{2.27d}$$

$$A_+=\frac{C_+[\hat{e}_{2,+}(a),\hat{e}_{1,-}(a)]_\mu+D_+[\hat{e}_{2,-}(a),\hat{e}_{1,-}(a)]_\mu}{\hat{W}_1^E}, \tag{2.27e}$$

$$B_+=\frac{C_+[\hat{e}_{1,+}(a),\hat{e}_{2,+}(a)]_\mu+D_+[\hat{e}_{1,+}(a),\hat{e}_{2,-}(a)]_\mu}{\hat{W}_1^E}, \tag{2.27f}$$

$$A_-=\frac{C_-[\hat{e}_{2,+}(b),\hat{e}_{3,-}(b)]_\mu+D_-[\hat{e}_{2,-}(b),\hat{e}_{3,-}(b)]_\mu}{\hat{W}_3^E}, \tag{2.27g}$$

$$B_-=\frac{C_-[\hat{e}_{3,+}(b),\hat{e}_{2,+}(b)]_\mu+D_-[\hat{e}_{3,+}(b),\hat{e}_{2,-}(b)]_\mu}{\hat{W}_3^E}, \tag{2.27h}$$

which means the Wronskian is

$$W^E=B_-\hat{W}_3^E=(C_+D_--D_+C_-)\hat{W}_2^E=A_+\hat{W}_1^E. \tag{2.27i}$$

The TE Casimir energy per unit transverse area is

$$\frac{U_E}{A} = -\int \frac{d\zeta d^2k}{16\pi^3}\zeta\frac{\partial}{\partial\zeta}\ln\Delta_E(a,b), \tag{2.28a}$$

and the TE pressure at $z = b$, i.e, $P_E(b) = T_{E;zz}(b_-) - T_{E;zz}(b_+)$, is

$$P_E(b) = -\int \frac{d\zeta d^2k}{16\pi^3}\frac{\partial}{\partial b}\ln\Delta_E(a,b), \tag{2.28b}$$

where the unphysical terms have been ignored and

$$\begin{aligned}\Delta_E(a,b) &= [\hat{e}_{3,+}(b), \hat{e}_{2,-}(b)]_\mu[\hat{e}_{2,+}(a), \hat{e}_{1,-}(a)]_\mu \\ &\quad -[\hat{e}_{3,+}(b), \hat{e}_{2,+}(b)]_\mu[\hat{e}_{2,-}(a), \hat{e}_{1,-}(a)]_\mu.\end{aligned} \tag{2.28c}$$

The PVW holds true as well. After subtracting the Casimir energies and pressures, demonstrated in Eq. (2.25), for the reference configuration as in Ref. [19], the interaction TE Casimir energy per unit transverse area and pressure at $z = b$ are

$$\frac{\Delta U_E}{A} = -\int \frac{d\zeta d^2k}{16\pi^3}\zeta\frac{\partial}{\partial\zeta}\ln\sigma_E(a,b),\ \ \Delta P_E(b) = -\int \frac{d\zeta d^2k}{16\pi^3}\frac{\partial}{\partial b}\ln\sigma_E(a,b), \tag{2.29a}$$

where $\sigma_E(a,b)$ is

$$\sigma_E(a,b) = 1 - \frac{[\hat{e}_{3,+}(b), \hat{e}_{2,+}(b)]_\mu[\hat{e}_{2,-}(a), \hat{e}_{1,-}(a)]_\mu}{[\hat{e}_{3,+}(b), \hat{e}_{2,-}(b)]_\mu[\hat{e}_{2,+}(a), \hat{e}_{1,-}(a)]_\mu}. \tag{2.29b}$$

The corresponding TM contributions are obtained with the EM-substitution.

2.2.2 Spherical systems

When the properties of a system varies in the radial direction, we refer to this system as a *spherical system*. In spherical systems, the reduced Green's functions defined in Eq. (2.6) can be expressed in a simple form with the vector spherical harmonics. When the permittivity and permeability of the system are isotropic, the reduced Green's functions $\mathbf{\Gamma}_\zeta(\mathbf{r},\mathbf{r}')$ and $\mathbf{\Phi}_\zeta(\mathbf{r},\mathbf{r}')$ are written simply as

$$\mathbf{\Gamma}_\zeta = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\begin{bmatrix} \frac{l(l+1)}{\varepsilon\varepsilon' rr'}g^H_{\zeta,l} + \frac{\delta(r-r')}{\varepsilon r^2} & \frac{\sqrt{l(l+1)}}{\varepsilon\varepsilon' rr'}\frac{\partial(r' g^H_{\zeta,l})}{\partial r'} & \\ \frac{\sqrt{l(l+1)}}{\varepsilon\varepsilon' rr'}\frac{\partial(r g^H_{\zeta,l})}{\partial r} & \frac{1}{\varepsilon\varepsilon' rr'}\frac{\partial^2(rr' g^H_{\zeta,l})}{\partial r\partial r'} + \frac{\delta(r-r')}{\varepsilon r^2} & \\ & & -\zeta^2 g^E_{\zeta,l} \end{bmatrix}, \tag{2.30a}$$

$$\mathbf{\Phi}_\zeta = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\begin{bmatrix} \frac{l(l+1)}{\mu\mu' rr'}g^E_{\zeta,l} + \frac{\delta(r-r')}{\mu r^2} & \frac{\sqrt{l(l+1)}}{\mu\mu' rr'}\frac{\partial(r' g^E_{\zeta,l})}{\partial r'} & \\ \frac{\sqrt{l(l+1)}}{\mu\mu' rr'}\frac{\partial(r g^E_{\zeta,l})}{\partial r} & \frac{1}{\mu\mu' rr'}\frac{\partial^2(rr' g^E_{\zeta,l})}{\partial r\partial r'} + \frac{\delta(r-r')}{\mu r^2} & \\ & & -\zeta^2 g^H_{\zeta,l} \end{bmatrix}, \tag{2.30b}$$

where the label of the matrix is given by $[1,2,3][1,2,3]^T$, which means $\mathbf{\Gamma}_\zeta$, as well as $\mathbf{\Phi}_\zeta$, has the form $\mathbf{\Gamma}_\zeta(\mathbf{r},\mathbf{r}') = \sum_{m=-l}^{l}\sum_{i,j=1}^{3} g_{i,j}(r,r')\mathbf{X}^m_{l,i}(\Omega)\mathbf{X}^m_{l,j}(\Omega')$, in which $\mathbf{X}^m_{l,i}(\Omega),\ i=1,2,3$ are defined based on the results in Appendix A.1 as

$$\mathbf{X}^m_{l,1}(\Omega) = \mathbf{Y}^m_l(\Omega),\ \mathbf{X}^m_{l,2}(\Omega) = \mathbf{\Psi}^m_l(\Omega),\ \mathbf{X}^m_{l,3}(\Omega) = \mathbf{\Phi}^m_l(\Omega), \tag{2.30c}$$

and the $g^E_{\zeta,l},\ g^H_{\zeta,l}$ are defined with the equations

$$\left[r\frac{d}{dr}\frac{1}{(\mu,\varepsilon)}\frac{d}{dr}r - \frac{l(l+1)}{(\mu,\varepsilon)} - (\varepsilon,\mu)\zeta^2 r^2\right]g^{(E,H)}_{\zeta,l}(r,r') = \delta(r-r'). \tag{2.30d}$$

The reduced TE Casimir energy density and stress tensors at $\mathbf{r}$ are thus

$$u_E(\zeta;\mathbf{r}) = \sum_{l=1}^{\infty}\frac{\nu}{4\pi}\left\{\frac{\partial(\zeta\varepsilon)}{\partial\zeta}\zeta^2 g^E_{\zeta,l} - \frac{\partial(\zeta\mu)}{\partial\zeta}\left[\frac{l(l+1)}{\mu^2 r^2}g^E_{\zeta,l} + \frac{1}{\mu^2 r^2}\frac{\partial^2(rr' g^E_{\zeta,l})}{\partial r\partial r'}\right]\right\}, \tag{2.31a}$$

$$t_{E;\theta\theta}(\zeta;\mathbf{r}) = t_{E;\varphi\varphi}(\zeta;\mathbf{r}) = -\sum_{l=1}^{\infty}\frac{\nu}{4\pi}\frac{l(l+1)}{\mu r^2}g^E_{\zeta,l}, \tag{2.31b}$$

$$t_{E;rr}(\zeta;\mathbf{r}) = -\sum_{l=1}^{\infty}\frac{\nu}{4\pi}\left[\frac{1}{\mu r^2}\frac{\partial^2(rr' g^E_{\zeta,l})}{\partial r\partial r'} - \varepsilon\zeta^2 g^E_{\zeta,l} - \frac{l(l+1)}{\mu r^2}g^E_{\zeta,l}\right], \tag{2.31c}$$

where $\nu = l + 1/2$. The TM contributions are obtained by making the substitutions $\varepsilon \leftrightarrow \mu$ and $E \rightarrow H$.

Define the functions $e_\pm$ and $h_\pm$ satisfying proper boundary conditions and the equations

$$\left[r\frac{d}{dr}\frac{1}{(\mu,\varepsilon)}\frac{d}{dr}r-\frac{l(l+1)}{(\mu,\varepsilon)}-(\varepsilon,\mu)\zeta^2r^2\right](e_\pm,h_\pm)(\zeta,l;r)=0, \tag{2.32a}$$

then $g^E_{\zeta,l}$ and $g^H_{\zeta,l}$ are written as

$$g^E_{\zeta,l}(r,r')=\frac{e_+(\zeta,l;r_>)e_-(\zeta,l;r_<)}{W^E_{\zeta,l}},\ g^H_{\zeta,l}(r,r')=\frac{h_+(\zeta,l;r_>)h_-(\zeta,l;r_<)}{W^H_{\zeta,l}}, \tag{2.32b}$$

where W_E and W_H are constants, i.e.,

$$W^E_{\zeta,l}=\frac{r^2}{\mu}(e'_+e_--e_+e'_-),\ W^H_{\zeta,l}=\frac{r^2}{\varepsilon}(h'_+h_--h_+h'_-). \tag{2.32c}$$

We further define $\mathfrak{e}(r)=re(r)$ and $\mathfrak{h}(r)=rh(r)$, which means the Eq. (2.32a) has the following forms

$$\left[\frac{d}{dr}\frac{1}{(\mu,\varepsilon)}\frac{d}{dr}-\frac{l(l+1)}{(\mu,\varepsilon)r^2}-(\varepsilon,\mu)\zeta^2\right](\mathfrak{e}_\pm,\mathfrak{h}_\pm)(\zeta,l;r)=0, \tag{2.33a}$$

which render the Wronskians as

$$W^E_{\zeta,l}=\frac{\mathfrak{e}'_+\mathfrak{e}_--\mathfrak{e}_+\mathfrak{e}'_-}{\mu},\ W^H_{\zeta,l}=\frac{\mathfrak{h}'_+\mathfrak{h}_--\mathfrak{h}_+\mathfrak{h}'_-}{\varepsilon}. \tag{2.33b}$$

The following identity is very useful

$$\begin{aligned}&\frac{\partial}{\partial r}\left[\frac{\mathfrak{e}'_+(\zeta,l;r)}{\mu(\zeta,r)}\frac{\partial}{\partial\zeta}\mathfrak{e}_-(\zeta,l;r)-\mathfrak{e}_+(\zeta,l;r)\frac{\partial}{\partial\zeta}\frac{\mathfrak{e}'_-(\zeta,l;r)]}{\mu(\zeta,r)}\right]\\ =\ &-\frac{1}{\mu}\frac{\partial(\varepsilon\mu\zeta^2)}{\partial\zeta}\mathfrak{e}_+\mathfrak{e}_-+\frac{\partial\ln\mu}{\partial\zeta}\frac{\partial}{\partial r}\left(\frac{\mathfrak{e}'_+\mathfrak{e}_-}{\mu}\right),\end{aligned} \tag{2.34}$$

and a similar identity for $\mathfrak{h}$ can be obtained by substituting $\varepsilon\leftrightarrow\mu$ and $\mathfrak{e}\to\mathfrak{h}$. So, the TE Casimir energy density and stress tensors are

$$u_E(\mathbf{r})=\frac{-1}{4\pi r^2}\sum_{l=1}^\infty\nu\int\frac{d\zeta}{2\pi}\frac{\zeta}{W^E_{\zeta,l}}\left[\frac{\partial\ln(\zeta\mu)}{\partial\zeta}\frac{\partial}{\partial r}\left(\frac{\mathfrak{e}'_+\mathfrak{e}_-}{\mu}\right)-\frac{1}{\mu}\frac{\partial(\varepsilon\mu\zeta^2)}{\partial\zeta}\mathfrak{e}_+\mathfrak{e}_-\right], \tag{2.35a}$$

$$T_{E;\theta\theta}(\mathbf{r})=T_{E;\varphi\varphi}(\mathbf{r})=\frac{-1}{4\pi r^2}\sum_{l=1}^\infty\nu\int\frac{d\zeta}{2\pi}\frac{1}{W^E_{\zeta,l}}\frac{l(l+1)}{\mu r^2}\mathfrak{e}_+\mathfrak{e}_-, \tag{2.35b}$$

$$T_{E;rr}(\mathbf{r}) = \frac{-1}{4\pi r^2}\sum_{l=1}^{\infty}\nu\int\frac{d\zeta}{2\pi}\frac{1}{W^E_{\zeta,l}}\left[\frac{\mathfrak{e}'_\pm(r)\mathfrak{e}'_\mp(r)}{\mu} - \mathfrak{e}_\pm(r)\frac{\partial}{\partial r}\frac{\mathfrak{e}'_\mp(r)}{\mu}\right], \tag{2.35c}$$

and the TE Casimir energy is

$$U_E = -\sum_{l=1}^{\infty}\nu\int\frac{d\zeta}{2\pi}\frac{1}{W^E_{\zeta,l}}\left[\frac{\mathfrak{e}'_+(r)}{\mu(r)}\frac{\partial\zeta\mathfrak{e}_-(r)}{\partial\zeta} - \zeta\mathfrak{e}_+(r)\frac{\partial}{\partial\zeta}\frac{\mathfrak{e}'_-(r)}{\mu(r)}\right]\Bigg|_{r=0}^{r=\infty}, \tag{2.35d}$$

and the TM contributions are obtained by making the EM-substitutions $\varepsilon \leftrightarrow \mu$, $\mathfrak{e} \to \mathfrak{h}$ and $E \to H$.

Consider a dielectric ball of radius a with permittivity ε_i and permeability μ_i, immersed in a medium (ε_o, μ_o). Suppose $\hat{\mathfrak{e}}_{j,\pm}$, $j = i, o$ are solutions for Eq. (2.33a) satisfying proper boundary conditions, which are usually $\lim_{r\to\infty}\hat{\mathfrak{e}}_{j,+}(r) = 0$, $\hat{\mathfrak{e}}_{j,-}(0) < \infty$, when the medium is analytically extended to the whole space, then $\hat{\mathfrak{e}}_\pm$ are solved as

$$\mathfrak{e}_+(r) = \begin{cases} \hat{\mathfrak{e}}_{o,+}(r), & r > a, \\ A_e\hat{\mathfrak{e}}_{i,+}(r) + B_e\hat{\mathfrak{e}}_{i,-}(r), & 0 < r < a, \end{cases} \tag{2.36a}$$

$$\mathfrak{e}_-(r) = \begin{cases} C_e\hat{\mathfrak{e}}_{o,+}(r) + D_e\hat{\mathfrak{e}}_{o,-}(r), & r > a, \\ \hat{\mathfrak{e}}_{i,-}(r), & 0 < r < a. \end{cases} \tag{2.36b}$$

where the coefficients and Wronskians are

$$A_e = \frac{[\hat{\mathfrak{e}}_{o,+}(a), \hat{\mathfrak{e}}_{i,-}(a)]_\mu}{\hat{W}^E_i}, \quad B_e = \frac{[\hat{\mathfrak{e}}_{i,+}(a), \hat{\mathfrak{e}}_{o,+}(a)]_\mu}{\hat{W}^E_i}, \tag{2.36c}$$

$$C_e = \frac{[\hat{\mathfrak{e}}_{i,-}(a), \hat{\mathfrak{e}}_{o,-}(a)]_\mu}{\hat{W}^E_o}, \quad D_e = \frac{[\hat{\mathfrak{e}}_{o,+}(a), \hat{\mathfrak{e}}_{i,-}(a)]_\mu}{\hat{W}^E_o}, \tag{2.36d}$$

$$W^E = D_e\hat{W}^E_o = A_e\hat{W}^E_i. \tag{2.36e}$$

Accordingly, the TE contributions to the interaction induced Casimir energy density and stress tensors, which are defined as in planar cases, in the region

$r > a$, are

$$\Delta u_E(\mathbf{r}) = \frac{-1}{4\pi r^2}\sum_{l=1}^{\infty}\nu\int\frac{d\zeta}{2\pi}\frac{\zeta}{\hat{W}_o^E}\frac{[\hat{\mathfrak{e}}_{i,-}(a),\hat{\mathfrak{e}}_{o,-}(a)]_\mu}{[\hat{\mathfrak{e}}_{o,+}(a),\hat{\mathfrak{e}}_{i,-}(a)]_\mu} \times\left[\frac{\partial\ln(\zeta\mu_o)}{\partial\zeta}\frac{\partial}{\partial r}\left(\frac{\mathfrak{e}'_{o,+}\mathfrak{e}_{o,+}}{\mu_o}\right)-\frac{1}{\mu_o}\frac{\partial(\varepsilon_o\mu_o\zeta^2)}{\partial\zeta}\mathfrak{e}_{o,+}\mathfrak{e}_{o,+}\right], \quad (2.37a)$$

$$\Delta T_{E;\theta\theta}(\mathbf{r}) = \Delta T_{E;\varphi\varphi}(\mathbf{r}) = \sum_{l=1}^{\infty}\frac{-\nu}{4\pi r^2}\int\frac{d\zeta}{2\pi}\frac{[\hat{\mathfrak{e}}_{i,-}(a),\hat{\mathfrak{e}}_{o,-}(a)]_\mu}{[\hat{\mathfrak{e}}_{o,+}(a),\hat{\mathfrak{e}}_{i,-}(a)]_\mu}\frac{l(l+1)}{\mu_o r^2}\frac{\mathfrak{e}_{o,+}^2}{\hat{W}_o^E}, \quad (2.37b)$$

$$\Delta T_{E;rr}(\mathbf{r}) = \sum_{l=1}^{\infty}\frac{-\nu}{4\pi r^2}\int\frac{d\zeta}{2\pi}\frac{1}{\hat{W}_o^E}\frac{[\hat{\mathfrak{e}}_{i,-}(a),\hat{\mathfrak{e}}_{o,-}(a)]_\mu}{[\hat{\mathfrak{e}}_{o,+}(a),\hat{\mathfrak{e}}_{i,-}(a)]_\mu}\left[\frac{\hat{\mathfrak{e}}'_{o,+}\hat{\mathfrak{e}}'_{o,+}}{\mu_o}-\hat{\mathfrak{e}}_{o,+}\frac{\partial}{\partial r}\frac{\hat{\mathfrak{e}}'_{o,+}}{\mu_o}\right], \quad (2.37c)$$

while in the region $0 < r < a$ they are

$$\Delta u_E(\mathbf{r}) = \frac{-1}{4\pi r^2}\sum_{l=1}^{\infty}\nu\int\frac{d\zeta}{2\pi}\frac{\zeta}{\hat{W}_i^E}\frac{[\hat{\mathfrak{e}}_{i,+}(a),\hat{\mathfrak{e}}_{o,+}(a)]_\mu}{[\hat{\mathfrak{e}}_{o,+}(a),\hat{\mathfrak{e}}_{i,-}(a)]_\mu} \times\left[\frac{\partial\ln(\zeta\mu_i)}{\partial\zeta}\frac{\partial}{\partial r}\left(\frac{\mathfrak{e}'_{i,-}\mathfrak{e}_{i,-}}{\mu_i}\right)-\frac{1}{\mu_i}\frac{\partial(\varepsilon_i\mu_i\zeta^2)}{\partial\zeta}\mathfrak{e}_{i,-}\mathfrak{e}_{i,-}\right], \quad (2.38a)$$

$$\Delta T_{E;\theta\theta}(\mathbf{r}) = \Delta T_{E;\varphi\varphi}(\mathbf{r}) = \sum_{l=1}^{\infty}\frac{-\nu}{4\pi r^2}\int\frac{d\zeta}{2\pi}\frac{[\hat{\mathfrak{e}}_{i,+}(a),\hat{\mathfrak{e}}_{o,+}(a)]_\mu}{[\hat{\mathfrak{e}}_{o,+}(a),\hat{\mathfrak{e}}_{i,-}(a)]_\mu}\frac{l(l+1)}{\mu_i r^2}\frac{\hat{\mathfrak{e}}_{i,-}^2}{\hat{W}_i^E}, \quad (2.38b)$$

$$\Delta T_{E;rr}(\mathbf{r}) = \sum_{l=1}^{\infty}\frac{-\nu}{4\pi r^2}\int\frac{d\zeta}{2\pi}\frac{1}{\hat{W}_i^E}\frac{[\hat{\mathfrak{e}}_{i,+}(a),\hat{\mathfrak{e}}_{o,+}(a)]_\mu}{[\hat{\mathfrak{e}}_{o,+}(a),\hat{\mathfrak{e}}_{i,-}(a)]_\mu}\left[\frac{\hat{\mathfrak{e}}'_{i,-}\hat{\mathfrak{e}}'_{i,-}}{\mu_i}-\hat{\mathfrak{e}}_{i,-}\frac{\partial}{\partial r}\frac{\hat{\mathfrak{e}}'_{i,-}}{\mu_i}\right]. \quad (2.38c)$$

The TE pressure on the surface $r = a$ and TE Casimir energy are

$$P_E(a) = \frac{-1}{4\pi a^2}\sum_{l=1}^{\infty}\nu\int\frac{d\zeta}{2\pi}\frac{\partial}{\partial a}\ln[\hat{\mathfrak{e}}_{o,+}(\zeta,l;a),\hat{\mathfrak{e}}_{i,-}(\zeta,l;a)]_\mu, \quad (2.39a)$$

$$U_E = -\sum_{l=1}^{\infty}\nu\int\frac{d\zeta}{2\pi}\zeta\frac{\partial}{\partial\zeta}\ln[\hat{\mathfrak{e}}_{o,+}(\zeta,l;a),\hat{\mathfrak{e}}_{i,-}(\zeta,l;a)]_\mu. \quad (2.39b)$$

There are two different half-space cases, i.e., I: $r < a,\ (\varepsilon_i,\mu_i) = (\varepsilon,\mu); r >$

$a,\ (\varepsilon_o,\mu_o)=(\infty,1)$ and II: $r<a,\ (\varepsilon_i,\mu_i)=(\infty,1); r>a,\ (\varepsilon_o,\mu_o)=(\varepsilon,\mu)$. For case I, the total TE and TM Casimir energy per unit transverse area and the pressure on the surface are

$$\frac{U_E}{A}=\sum_{l=1}^{\infty}\nu\int\frac{d\zeta\zeta}{-2\pi}\frac{\partial}{\partial\zeta}\ln\hat{\mathfrak{e}}_{i,-}(a),\ P_E(a)=\sum_{l=1}^{\infty}\frac{-\nu}{4\pi a^2}\int\frac{d\zeta}{2\pi}\frac{\partial}{\partial a}\ln\hat{\mathfrak{e}}_{i,-}(a),\tag{2.40a}$$

$$\frac{U_H}{A}=\sum_{l=1}^{\infty}\nu\int\frac{d\zeta\zeta}{-2\pi}\frac{\partial}{\partial\zeta}\ln\frac{\hat{\mathfrak{h}}'_{i,-}(a)}{\varepsilon(a)},\ P_H(a)=\sum_{l=1}^{\infty}\frac{-\nu}{4\pi a^2}\int\frac{d\zeta}{2\pi}\frac{\partial}{\partial a}\ln\frac{\hat{\mathfrak{h}}'_{i,-}(a)}{\varepsilon(a)}.\tag{2.40b}$$

For case II, the corresponding terms are

$$\frac{U_E}{A}=\sum_{l=1}^{\infty}\nu\int\frac{d\zeta\zeta}{-2\pi}\frac{\partial}{\partial\zeta}\ln\hat{\mathfrak{e}}_{o,+}(a),\ P_E(a)=\sum_{l=1}^{\infty}\frac{-\nu}{4\pi a^2}\int\frac{d\zeta}{2\pi}\frac{\partial}{\partial a}\ln\hat{\mathfrak{e}}_{o,+}(a),\tag{2.41a}$$

$$\frac{U_H}{A}=\sum_{l=1}^{\infty}\nu\int\frac{d\zeta\zeta}{-2\pi}\frac{\partial}{\partial\zeta}\ln\frac{\hat{\mathfrak{h}}'_{o,+}(a)}{\varepsilon(a)},\ P_H(a)=\sum_{l=1}^{\infty}\frac{-\nu}{4\pi a^2}\int\frac{d\zeta}{2\pi}\frac{\partial}{\partial a}\ln\frac{\hat{\mathfrak{h}}'_{o,+}(a)}{\varepsilon(a)}.\tag{2.41b}$$

Consider the concentric configuration, in which a ball of radius a made of the medium (ε_1,μ_1) is covered by a layer of medium (ε_2,μ_2) with the outer radius b, and the $r>b$ region space is filled with a medium (ε_3,μ_3). Suppose each of the three media is extended analytically to the whole space, then denote the solutions of Eq. (2.33a) with proper boundary conditions imposed as $\hat{\mathfrak{e}}_{i,\pm},\hat{\mathfrak{h}}_{i,\pm}$. Thus, the $\mathfrak{e}_\pm$ and their Wronskian are

$$\mathfrak{e}_+(r)=\begin{cases}\hat{\mathfrak{e}}_{3,+}(r), & r>b,\\ C_+\hat{\mathfrak{e}}_{2,+}(r)+D_+\hat{\mathfrak{e}}_{2,-}(r), & a<r<b,\\ A_+\hat{\mathfrak{e}}_{1,+}(r)+B_+\hat{\mathfrak{e}}_{1,-}(r), & r<a,\end{cases}\tag{2.42a}$$

$$\mathfrak{e}_-(r)=\begin{cases}A_-\hat{\mathfrak{e}}_{3,+}(r)+B_-\hat{\mathfrak{e}}_{3,-}(r), & r>b,\\ C_-\hat{\mathfrak{e}}_{2,+}(r)+D_-\hat{\mathfrak{e}}_{2,-}(r), & a<r<b,\\ \hat{\mathfrak{e}}_{1,-}(r), & r<a,\end{cases}\tag{2.42b}$$

where the coefficients are expressed with $\hat{\mathfrak{e}}$s and their Wronskians $\hat{W}$ as

$$C_+ = \frac{[\hat{\mathfrak{e}}_{3,+}(b), \hat{\mathfrak{e}}_{2,-}(b)]_\mu}{\hat{W}_2^E}, \ D_+ = \frac{[\hat{\mathfrak{e}}_{2,+}(b), \hat{\mathfrak{e}}_{3,+}(b)]_\mu}{\hat{W}_2^E}, \tag{2.42c}$$

$$A_+ = \frac{C_+[\hat{\mathfrak{e}}_{2,+}(a), \hat{\mathfrak{e}}_{1,-}(a)]_\mu + D_+[\hat{\mathfrak{e}}_{2,-}(a), \hat{\mathfrak{e}}_{1,-}(a)]_\mu}{\hat{W}_1^E}, \tag{2.42d}$$

$$B_+ = \frac{C_+[\hat{\mathfrak{e}}_{1,+}(a), \hat{\mathfrak{e}}_{2,+}(a)]_\mu + D_+[\hat{\mathfrak{e}}_{1,+}(a), \hat{\mathfrak{e}}_{2,-}(a)]_\mu}{\hat{W}_1^E}, \tag{2.42e}$$

$$C_- = \frac{[\hat{\mathfrak{e}}_{1,-}(a), \hat{\mathfrak{e}}_{2,-}(a)]_\mu}{\hat{W}_2^E}, \ D_- = \frac{[\hat{\mathfrak{e}}_{2,+}(a), \hat{\mathfrak{e}}_{1,-}(a)]_\mu}{\hat{W}_2^E}, \tag{2.42f}$$

$$A_- = \frac{C_-[\hat{\mathfrak{e}}_{2,+}(b), \hat{\mathfrak{e}}_{3,-}(b)]_\mu + D_-[\hat{\mathfrak{e}}_{2,-}(b), \hat{\mathfrak{e}}_{3,-}(b)]_\mu}{\hat{W}_3^E}, \tag{2.42g}$$

$$B_- = \frac{C_-[\hat{\mathfrak{e}}_{3,+}(b), \hat{\mathfrak{e}}_{2,+}(b)]_\mu + D_-[\hat{\mathfrak{e}}_{3,+}(b), \hat{\mathfrak{e}}_{2,-}(b)]_\mu}{\hat{W}_3^E}, \tag{2.42h}$$

$$W^E = B_-\hat{W}_3^E = (C_+D_- - D_+C_-)\hat{W}_2^E = A_+\hat{W}_1^E. \tag{2.42i}$$

The TE contributions to the Casimir energy are

$$U_E = -\sum_{l=1}^{\infty} \nu \int \frac{d\zeta}{2\pi} \zeta \frac{\partial}{\partial \zeta} \ln \Delta_E(a,b), \tag{2.43a}$$

where $\Delta_E(a,b)$ is

$$\begin{aligned} \Delta_E(a,b) \quad = \quad & [\hat{\mathfrak{e}}_{3,+}(b), \hat{\mathfrak{e}}_{2,-}(b)]_\mu [\hat{\mathfrak{e}}_{2,+}(a), \hat{\mathfrak{e}}_{1,-}(a)]_\mu \\ & -[\hat{\mathfrak{e}}_{3,+}(b), \hat{\mathfrak{e}}_{2,+}(b)]_\mu [\hat{\mathfrak{e}}_{2,-}(a), \hat{\mathfrak{e}}_{1,-}(a)]_\mu, \end{aligned} \tag{2.43b}$$

while the TE pressure at $r=b$, i.e, $P_E(b)=T_{E;rr}(b_-)-T_{E;rr}(b_+)$, is

$$P_E=-\frac{1}{4\pi b^2}\sum_{l=1}^{\infty}\nu\int\frac{d\zeta}{2\pi}\frac{\partial}{\partial b}\ln\Delta_E(a,b). \tag{2.43c}$$

The interaction TE Casimir energy and pressure at $r=b$, which correspond to the counterparts in Ref. [19], are obtained by replacing the $\Delta_E(a,b)$ in Eq. (2.43) with

$$\sigma_E(a,b)=1-\frac{[\hat{\mathfrak{e}}_{3,+}(b),\hat{\mathfrak{e}}_{2,+}(b)]_\mu[\hat{\mathfrak{e}}_{2,-}(a),\hat{\mathfrak{e}}_{1,-}(a)]_\mu}{[\hat{\mathfrak{e}}_{3,+}(b),\hat{\mathfrak{e}}_{2,-}(b)]_\mu[\hat{\mathfrak{e}}_{2,+}(a),\hat{\mathfrak{e}}_{1,-}(a)]_\mu}. \tag{2.44}$$

2.3 Homogeneous systems

In this section, we, for clarity, concentrate on the Casimir energies and stress tensors in two types of geometries, namely the planar and spherical geometries as before, consisting of nondissipative homogeneous media.

2.3.1 Planar systems

As described above, the Casimir effect research only developed in the planar geometry in its early days. Though most experiments testing the Casimir forces are carried out in the plate-sphere structures so that the alignment difficulty is suppressed, the proximity force approximation (PFA), which dates back to 1934 [124], based on the results of planar Casimir effects, is widely employed in the comparison between experiments and theories. Here we demonstrate some basic properties of the Casimir energy densities and stresses in three common types of planar configurations, i.e., the uniform background, two-media background and parallel configuration.

Uniform background

As is well-known, the electromagnetic field, on its own, is not sufficient to keep a self-consistent stable nontrivial background. In our analysis, we take the backgrounds as our given constraint conditions. Our studies on the background here is only limited to the simplest case, in which the permittivity and permeability of the background, denoted as ε and μ, are homogeneous and nondispersive. Then $e_+(z) = h_+(z) = e^{-\kappa z}$ and $e_-(z) = h_-(z) = e^{\kappa z}$, where $\kappa = \sqrt{k^2 + \varepsilon\mu\zeta^2}$. Regularize the physical parameters with the point-splitting regulator $\boldsymbol{\delta}$. When $\boldsymbol{\delta}$ is temporal, we have

$$u_E(\mathbf{r}) = -\int_0^\infty \frac{d\kappa\kappa^2}{8\pi^2\sqrt{\varepsilon\mu}\delta^4}\kappa\int_0^\pi d\theta\frac{\sin\theta\cos^2\theta}{e^{-i\kappa\cos\theta}} = \frac{3}{2\pi^2\sqrt{\varepsilon\mu}\delta^4}, \tag{2.45a}$$

$$T_{E;xx}(\mathbf{r}) = T_{E;yy}(\mathbf{r}) = \int_0^\infty \frac{d\kappa\kappa^2}{16\pi^2\sqrt{\varepsilon\mu}\delta^4}\kappa\int_0^\pi d\theta\frac{\sin^3\theta}{e^{-i\kappa\cos\theta}} = \frac{1}{2\pi^2\sqrt{\varepsilon\mu}\delta^4}, \tag{2.45b}$$

$$T_{E;zz}(\mathbf{r}) = -\int_0^\infty \frac{d\kappa\kappa^2}{8\pi^2\sqrt{\varepsilon\mu}\delta^4}\kappa\int_0^\pi d\theta\frac{\sin\theta}{e^{-i\kappa\cos\theta}} = \frac{1}{2\pi^2\sqrt{\varepsilon\mu}\delta^4}, \tag{2.45c}$$

where the rapidly oscillating terms are zero. When the regulator $\boldsymbol{\delta}$ is spatial, which is chosen the x-direction without losing any generality, then we have

$$u_E(\mathbf{r}) = \int_0^\infty \frac{-d\kappa\kappa^3}{16\pi^3\sqrt{\varepsilon\mu}\delta^4}\int_0^\pi d\theta\int_0^{2\pi} d\varphi\frac{\sin\theta\cos^2\theta}{e^{-i\kappa\sin\theta\cos\varphi}} = \frac{-1}{2\pi^2\sqrt{\varepsilon\mu}\delta^4}, \tag{2.46a}$$

$$T_{E;xx}(\mathbf{r}) = \int_0^\infty \frac{d\kappa\kappa^2}{16\pi^3\sqrt{\varepsilon\mu}\delta^4}\kappa\int_0^\pi d\theta\int_0^{2\pi} d\varphi\frac{\sin^3\theta\cos^2\varphi}{e^{-i\kappa\sin\theta\cos\varphi}} = \frac{-3}{2\pi^2\sqrt{\varepsilon\mu}\delta^4}, \tag{2.46b}$$

$$T_{E;yy}(\mathbf{r}) = \int_0^\infty \frac{d\kappa\kappa^2}{16\pi^3\sqrt{\varepsilon\mu}\delta^4}\kappa\int_0^\pi d\theta\int_0^{2\pi} d\varphi\frac{\sin^3\theta\sin^2\varphi}{e^{-i\kappa\sin\theta\cos\varphi}} = \frac{1}{2\pi^2\sqrt{\varepsilon\mu}\delta^4}, \tag{2.46c}$$

$$T_{E;zz}(\mathbf{r}) = -\int_0^\infty \frac{d\kappa\kappa^2}{16\pi^3\sqrt{\varepsilon\mu}\delta^4}\kappa\int_0^\pi d\theta\int_0^{2\pi} d\varphi\frac{\sin\theta}{e^{-ik\sin\theta\cos\varphi}} = \frac{1}{2\pi^2\sqrt{\varepsilon\mu}\delta^4}. \tag{2.46d}$$

We claim that the regularization scheme will affect the regularized expressions, but the relation $u_E = T_{E;xx} + T_{E;yy} + T_{E;zz}$ is always true. The corresponding TM contributions are obtained with the EM-substitution.

Two-media background

Consider two homogenous media (ε_1, μ_1) and (ε_2, μ_2) filling in half-spaces $z < a$ and $z > a$ respectively, i.e.,

$$\varepsilon(z) = \begin{cases} \varepsilon_2, & z > a, \\ \varepsilon_1, & z < a, \end{cases} \qquad \mu(z) = \begin{cases} \mu_2, & z > a, \\ \mu_1, & z < a. \end{cases} \tag{2.47}$$

Then for TE mode, $\hat{e}_{i,\pm}$, $i = 1, 2$ are solved as $\hat{e}_{i,\pm} = e^{\mp\kappa_i z}$, where $\kappa_i = \sqrt{k^2 + \varepsilon_i \mu_i \zeta^2}$. The interaction induced TE Casimir energy density and stress tensors in the $z > a$ region are

$$\Delta u_E(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3} \frac{\kappa_2\mu_1 - \kappa_1\mu_2}{\kappa_1\mu_2 + \kappa_2\mu_1} \zeta \left[\frac{\partial \ln(\zeta\mu_2)}{\partial \zeta} \kappa_2 - \frac{\partial \kappa_2}{\partial \zeta} \right] e^{-2\kappa_2(z-a)}, \tag{2.48a}$$

$$\Delta T_{E;xx}(\mathbf{r}) = \Delta T_{E;yy}(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3} \frac{k^2(\kappa_2\mu_1 - \kappa_1\mu_2)}{2\kappa_2(\kappa_1\mu_2 + \kappa_2\mu_1)} e^{-2\kappa_2(z-a)}, \tag{2.48b}$$

and $\Delta T_{E;zz}(\mathbf{r}) = 0$. The corresponding TM contributions can be obtained by making the substitution $\varepsilon \leftrightarrow \mu$. Only in the special case, in which the media are nondispersive and diaphanous, i.e., $\varepsilon_1\mu_1 = \varepsilon_2\mu_2$, we have $\Delta u(\mathbf{r}) = \Delta u_E(\mathbf{r}) + \Delta u_H(\mathbf{r}) = 0$ everywhere.

In the special case where $(\varepsilon_1, \mu_1) \to (\infty, 1)$ and $(\varepsilon_2, \mu_2) = (\varepsilon, \mu)$, the interaction values in the $z > a$ region are

$$\Delta u_E(\mathbf{r}) = -\int \frac{d\zeta d^2k}{16\pi^3} \zeta \left[\frac{\partial \ln(\zeta\mu_2)}{\partial \zeta} \kappa_2 - \frac{\partial \kappa_2}{\partial \zeta} \right] e^{-2\kappa_2(z-a)}, \tag{2.49a}$$

$$\Delta u_H(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3} \zeta \left[\frac{\partial \ln(\zeta\varepsilon_2)}{\partial \zeta} \kappa_2 - \frac{\partial \kappa_2}{\partial \zeta} \right] e^{-2\kappa_2(z-a)}, \tag{2.49b}$$

$$\Delta T_{E;xx}(\mathbf{r}) = -\Delta T_{H;xx}(\mathbf{r}) = \int \frac{d\zeta d^2k}{32\pi^3} \frac{-k^2}{\kappa_2 e^{2\kappa_2(z-a)}}, \tag{2.49c}$$

and $\Delta T_{E;yy}(\mathbf{r}) = \Delta T_{E;xx}(\mathbf{r})$, $\Delta T_{H;yy}(\mathbf{r}) = \Delta T_{H;xx}(\mathbf{r})$, $\Delta T_{E;zz}(\mathbf{r}) = \Delta T_{H;zz}(\mathbf{r}) = 0$.

When the media are nondispersive, we have

$$\Delta u_E(\mathbf{r}) = -\Delta u_H(\mathbf{r}) = \frac{-1}{16\pi^2\sqrt{\varepsilon\mu}(z-a)^4} = 2\Delta T_{E;xx}(\mathbf{r}),\ \ \Delta T_{E;zz}(\mathbf{r}) = 0. \quad (2.50)$$

Parallel configurations

Now consider the interaction in the parallel structures, i.e., the systems consisting of parallel media. In Casimir's original configuration [1], which is two perfectly conducting slab filling in half-space separated by the vacuum, the electromagnetic zero-point energy with transverse area A is

$$E_A = A\sum_{n=1}^{\infty}\int\frac{d^2k}{(2\pi)^2}\sqrt{k^2+\frac{n^2\pi^2}{4a^2}}, \quad (2.51)$$

where $2a$ is the distance between two perfectly conducting slabs, k is the amplitude of the transverse wave number $\mathbf{k}$, and the polarization of photon has been counted. With dimensional regularization, the zero-point energy per unit transverse area $\mathcal{E} = E_A/A$ is written as

$$\mathcal{E} = \sum_{n=1}^{\infty}\int\frac{d^dk}{(2\pi)^d}\int_0^{\infty}dt\frac{t^{-\frac{3}{2}}e^{-t(k^2+n^2\pi^2/4a^2)}}{\Gamma(-1/2)} = \frac{-\zeta(d+2)\Gamma(1+d/2)}{2^{d+1}\pi^{\frac{d+2}{2}}(2a)^{d+1}}, \quad (2.52)$$

which means in the special case we are calculating, $\mathcal{E}$ and hence the Casimir force per unit transverse area $\mathcal{F} = -\partial\mathcal{E}/\partial(2a)$ are obtained with $d \to 2$ as

$$\mathcal{E} = -\frac{\pi^2}{720}\frac{1}{(2a)^3}, \quad \mathcal{F} = -\frac{\pi^2}{240}\frac{1}{(2a)^4}, \quad (2.53)$$

which are just the results in Ref. [1]. Also, this problem can be solved with the Green's function method, in which $g^{(E,H)}_{\zeta,\mathbf{k}}$ in Eq. (2.12) are solved as

$$g^E_{\zeta,\mathbf{k}}(z,z') = \frac{\sinh\kappa(z_>-a)\sinh\kappa(z_<+a)}{\kappa\sinh(2\kappa a)}, \quad (2.54a)$$

$$g^H_{\zeta}(z,z') = \frac{\cosh\kappa(z_>-a)\cosh\kappa(z_<+a)}{-\kappa\sinh(2\kappa a)}, \quad (2.54b)$$

where $\kappa = \sqrt{\zeta^2 + k^2}$. So the energy density $u(\mathbf{r})$ and the zz-component of stress tensor T_{zz} at $z = a_-$ are

$$u(\mathbf{r}) = -\int \frac{d\zeta d^2k}{(2\pi)^3}\frac{\zeta^2}{\kappa}\coth 2\kappa a,\;\; T_{zz}(a_-) = -\int \frac{d\zeta d^2k}{(2\pi)^3}\kappa \coth 2\kappa a. \tag{2.55}$$

By omitting the unphysical background contributions, we get the energy per unit transverse area between the two plates $\mathcal{E}$ and the force per unit area on the plate $z = a$, denoted $\mathcal{F}_a$,

$$\mathcal{E} = -\frac{1}{48\pi^2 a^3}\int_0^\infty d\kappa\kappa^3(\coth\kappa - 1) = -\frac{\pi^2}{5760a^3}, \tag{2.56a}$$

$$\mathcal{F}_a = T_{zz}(a_-) = -\frac{1}{32\pi^2 a^4}\int_0^\infty d\kappa\kappa^3(\coth\kappa - 1) = -\frac{\pi^2}{3840a^4}, \tag{2.56b}$$

both of which are consistent with the results in Eq. (2.53).

In 1956, Lifshitz [10] generalized Casimir's original model to a more practical one, consisting of two parallel homogeneous dielectric materials separated by vacuum. Then, Dzyaloshinskii *et al.* [11, 12] used another homogeneous medium as the intervening material, i.e., the DLP model. In a DLP model, the permittivity ε and permeability μ of the system are typically

$$\varepsilon(z) = \begin{cases} \varepsilon_3, & z > b \\ \varepsilon_2, & a < z < b, \\ \varepsilon_1, & z < a, \end{cases} \qquad \mu(z) = \begin{cases} \mu_3, & z > b, \\ \mu_2, & a < z < b, \\ \mu_1, & z < a, \end{cases} \tag{2.57}$$

where $\varepsilon_i, \mu_i,\ i = 1, 2, 3$ are all homogeneous in their regions. The two-body interaction Casimir energy per unit transverse area of the TE mode is

$$\mathcal{E}_E = \frac{1}{2}\int \frac{d\zeta d^2k}{(2\pi)^3}\ln\left[1 + \frac{(\kappa_3\mu_2 - \kappa_2\mu_3)(\kappa_2\mu_1 - \kappa_1\mu_2)}{(\kappa_3\mu_2 + \kappa_2\mu_3)(\kappa_2\mu_1 + \kappa_1\mu_2)}e^{-2\kappa_2(b-a)}\right], \tag{2.58a}$$

while the pressure on the $z = b$ interface is

$$\mathcal{F}_b = T_{E;zz}(b_-) - T_{E;zz}(b_+) = -\frac{\partial}{\partial b}\mathcal{E}_E + \int \frac{d\zeta d^2k}{(2\pi)^3}\left(\frac{\kappa_2}{2} - \frac{\kappa_3}{2}\right), \tag{2.58b}$$

where $\kappa_i = \sqrt{\varepsilon_i\mu_i\zeta^2 + k^2}$ and the last term is the bulk contribution. The corresponding TM contributions can be obtained by making the substitution $\varepsilon \leftrightarrow \mu$. When $\mu_1 = \mu_2 = \mu_3 = \varepsilon_2 = 1$ and $\varepsilon_1, \varepsilon_3 \to \infty$, Eq. (2.58a) is consistent with the result in Eq. (2.53). When the media are nondispersive and diaphanous, i.e., $\varepsilon_1\mu_1 = \varepsilon_2\mu_2 = \varepsilon_3\mu_3$, the pressure on the $z = b$ surface satisfies the expression

$$\mathcal{F}_E = \mathcal{F}_H = \frac{-3}{16\pi^2\sqrt{\varepsilon_2\mu_2}(b-a)^4}\mathrm{Li}_4\left[\frac{(\mu_2-\mu_3)(\mu_2-\mu_1)}{(\mu_2+\mu_3)(\mu_1+\mu_2)}\right]. \tag{2.59}$$

2.3.2 Spherical system

As we know, except for few particular cases [21, 28], there are long-standing ambiguities in interpreting the Casimir energies and stresses of spherical configurations due to divergences, especially logarithmic ones. Arguments and works are still going on in this field. Here we briefly give some fundamental results for Casimir energy densities and stresses in two kind of spherical configurations, namely two-media backgrounds and concentric cases.

Two-media background

Consider a nondissipative, isotropic and homogeneous (NIH) ball immersed in a NIH medium. The permittivity ε and permeability μ are

$$\varepsilon(r) = \begin{cases} \varepsilon_2, & r > a, \\ \varepsilon_1, & 0 < r < a, \end{cases} \qquad \mu(r) = \begin{cases} \mu_2, & r > a, \\ \mu_1, & 0 < r < a. \end{cases} \tag{2.60}$$

Then for TE mode, $\hat{\mathfrak{e}}_{i,\pm},\ i = 1,2$ are solved as $\hat{\mathfrak{e}}_{i,+} = e_l(\kappa_i r),\ \hat{\mathfrak{e}}_{i,-} = s_l(\kappa_i r)$, where $\kappa_i = \sqrt{\varepsilon_i\mu_i\zeta^2}$. The TE contributions to the interaction induced Casimir energy density and stress tensors, in the region $r > a$, are

$$\begin{aligned} \Delta u_E(\mathbf{r}) &= \sum_{l=1}^{\infty}\frac{\nu}{4\pi r^2}\int\frac{d\zeta}{2\pi}\zeta\frac{[s_l(\kappa_1 a), s_l(\kappa_2 a)]_\mu}{[e_l(\kappa_2 a), s_l(\kappa_1 a)]_\mu} \\ &\quad \times\left[\frac{\partial\ln(\zeta\mu_2)}{\partial\zeta}\frac{\partial e_l'(\kappa_2 r)e_l(\kappa_2 r)}{\partial r} - 2\frac{\partial\kappa_2}{\partial\zeta}e_l^2(\kappa_2 r)\right], \end{aligned} \tag{2.61a}$$

$$\Delta T_{E;\theta\theta}(\mathbf{r}) = \Delta T_{E;\varphi\varphi}(\mathbf{r}) = \sum_{l=1}^{\infty} \frac{\nu l(l+1)}{4\pi r^2} \int \frac{d\zeta}{2\pi} \frac{[s_l(\kappa_1 a), s_l(\kappa_2 a)]_\mu}{[e_l(\kappa_2 a), s_l(\kappa_1 a)]_\mu} \frac{e_l^2(\kappa_2 r)}{\kappa_2 r^2}, \quad (2.61b)$$

$$\Delta T_{E;rr}(\mathbf{r}) = \sum_{l=1}^{\infty} \frac{\nu}{4\pi r^2} \int \frac{d\zeta}{2\pi} \kappa_2 \frac{[s_l(\kappa_1 a), s_l(\kappa_2 a)]_\mu}{[e_l(\kappa_2 a), s_l(\kappa_1 a)]_\mu} \left[e_l'^2(\kappa_2 r) - e_l(\kappa_2 r) e_l''(\kappa_2 r) \right], \quad (2.61c)$$

while in the region $0 < r < a$ they are

$$\begin{aligned} \Delta u_E(\mathbf{r}) &= \sum_{l=1}^{\infty} \frac{\nu}{4\pi r^2} \int \frac{d\zeta}{2\pi} \zeta \frac{[e_l(\kappa_1 a), e_l(\kappa_2 a)]_\mu}{[e_l(\kappa_2 a), s_l(\kappa_1 a)]_\mu} \\ &\quad \times \left[\frac{\partial \ln(\zeta \mu_1)}{\partial \zeta} \frac{\partial s_l'(\kappa_1 r) s_l(\kappa_1 r)}{\partial r} - 2 \frac{\partial \kappa_1}{\partial \zeta} s_l^2(\kappa_1 r) \right], \end{aligned} \quad (2.62a)$$

$$\Delta T_{E;\theta\theta}(\mathbf{r}) = \Delta T_{E;\varphi\varphi}(\mathbf{r}) = \sum_{l=1}^{\infty} \frac{\nu l(l+1)}{4\pi r^2} \int \frac{d\zeta}{2\pi} \frac{[e_l(\kappa_1 a), e_l(\kappa_2 a)]_\mu}{[e_l(\kappa_2 a), s_l(\kappa_1 a)]_\mu} \frac{s_l^2(\kappa_1 r)}{\kappa_1 r^2}, \quad (2.62b)$$

$$\Delta T_{E;rr}(\mathbf{r}) = \sum_{l=1}^{\infty} \frac{\nu}{4\pi r^2} \int \frac{d\zeta}{2\pi} \kappa_1 \frac{[e_l(\kappa_1 a), e_l(\kappa_2 a)]_\mu}{[e_l(\kappa_2 a), s_l(\kappa_1 a)]_\mu} \left[s_l'^2(\kappa_1 r) - s_l(\kappa_1 r) s_l''(\kappa_1 r) \right]. \quad (2.62c)$$

The corresponding TM contributions can be obtained by making the substitution $\varepsilon \leftrightarrow \mu$.

Consider the half-space background of type I and II as in Sec. (2.2.2), with homogeneous media (ε_1, μ_1) and (ε_2, μ_2), respectively. For type I, in the region $0 < r < a$ the parameters in Eq. (2.62) are

$$\Delta u_E(\mathbf{r}) = \sum_{l=1}^{\infty} \frac{-\nu}{4\pi r^2} \int \frac{d\zeta \zeta}{2\pi} \frac{e_l(\kappa_1 a)}{s_l(\kappa_1 a)} \left[\frac{\partial \ln(\zeta \mu_1)}{\partial \zeta} \frac{\partial s_l'(\kappa_1 r) s_l(\kappa_1 r)}{\partial r} - 2 \frac{\partial \kappa_1}{\partial \zeta} s_l^2(\kappa_1 r) \right], \quad (2.63a)$$

$$\Delta T_{E;\theta\theta}(\mathbf{r}) = \Delta T_{E;\varphi\varphi}(\mathbf{r}) = -\sum_{l=1}^{\infty} \frac{\nu l(l+1)}{4\pi r^2} \int \frac{d\zeta}{2\pi} \frac{e_l(\kappa_1 a)}{s_l(\kappa_1 a)} \frac{s_l^2(\kappa_1 r)}{\kappa_1 r^2}, \quad (2.63b)$$

$$\Delta T_{E;rr}(\mathbf{r}) = \sum_{l=1}^{\infty} \frac{-\nu}{4\pi r^2} \int \frac{d\zeta}{2\pi} \kappa_1 \frac{e_l(\kappa_1 a)}{s_l(\kappa_1 a)} \left[s_l'^2(\kappa_1 r) - s_l(\kappa_1 r) s_l''(\kappa_1 r) \right]. \quad (2.63c)$$

When ε_1 and μ_1 are nondispersive, we have

$$\Delta u_E(\mathbf{r}) = \frac{-d^2}{4\pi r^4\sqrt{\varepsilon_1\mu_1}}\sum_{l=1}^{\infty}\nu\int_0^{\infty}\frac{dxx}{\pi}\frac{e_l(x)}{s_l(x)}\left[\frac{ds_l'(xd)s_l(xd)}{d(xd)} - 2s_l^2(xd)\right], \quad (2.64)$$

where $\nu = l+1/2$ and $d = r/a$. In the region not far from the spherical center, or $d \to 0$, the nondispersive $\Delta u_E(\mathbf{r})$ is written as

$$\Delta u_E(\mathbf{r}) \approx \int_0^{\infty}\frac{dxx3d^2}{8\pi^2 r^4\sqrt{\varepsilon_1\mu_1}}\frac{e_1(x)}{s_1(x)}\left[2s_1^2(xd) - \frac{ds_1'(xd)s_1(xd)}{d(xd)}\right] = \frac{-0.11866}{a^4\sqrt{\varepsilon_1\mu_1}}. \quad (2.65)$$

Evaluate the $\Delta u_E(\mathbf{r})$ with the uniform asymptotic expansion (UAE), detailed in Appendix A.2, which means in the region $0 < r < a$, to the first order, we have

$$\Delta u_E(\mathbf{r}) = \frac{-d}{4\pi r^4\sqrt{\varepsilon_1\mu_1}}\sum_{l=1}^{\infty}\nu\int_0^{\infty}\frac{dx}{\pi}\frac{e^{2\nu\eta(zd)-2\nu\eta(z)}}{2}\left(\frac{2\nu}{\sqrt{1+z^2d^2}}+1\right), \quad (2.66a)$$

where $z = x/\nu$ and η are defined as

$$\eta(z) = \sqrt{1+z^2} + \ln\frac{z}{1+\sqrt{1+z^2}}. \quad (2.66b)$$

In the limit $a \to \infty$ and the substitutions $\nu/a \to k$, $\sum_{l=1}^{\infty}\nu/a^2 \to \int_0^{\infty} dkk$, $\Delta u_E(\mathbf{r})$ is approximated as

$$\begin{aligned}\Delta u_E(\mathbf{r}) \quad &\to \quad \frac{-1}{4\pi a^4\sqrt{\varepsilon_1\mu_1}}\sum_{l=1}^{\infty}\nu\int_0^{\infty}\frac{dx}{\pi}\frac{e^{-2\sqrt{\nu^2+x^2}\frac{a-r}{a}}}{2}\left(\frac{2\nu^2}{\sqrt{\nu^2+x^2}}+1\right)\\ &= \quad \frac{-1}{16\pi^2\sqrt{\varepsilon_1\mu_1}}\frac{1}{(a-r)^4},\end{aligned} \quad (2.66c)$$

which is consistent with the result of Eq. (2.50). For type II, in the region $r > a$ the parameters in Eq. (2.61) are

$$\Delta u_E(\mathbf{r}) = \sum_{l=1}^{\infty}\frac{-\nu}{4\pi r^2}\int\frac{d\zeta}{2\pi}\zeta\frac{s_l(\kappa_2 a)}{e_l(\kappa_2 a)}\left[\frac{\partial\ln(\zeta\mu_2)}{\partial\zeta}\frac{\partial e_l'(\kappa_2 r)e_l(\kappa_2 r)}{\partial r} - 2\frac{\partial\kappa_2}{\partial\zeta}e_l^2(\kappa_2 r)\right], \quad (2.67a)$$

$$\Delta T_{E;\theta\theta}(\mathbf{r}) = \Delta T_{E;\varphi\varphi}(\mathbf{r}) = -\sum_{l=1}^{\infty}\frac{\nu l(l+1)}{4\pi r^2}\int\frac{d\zeta}{2\pi}\frac{s_l(\kappa_2 a)}{e_l(\kappa_2 a)}\frac{e_l^2(\kappa_2 r)}{\kappa_2 r^2}, \quad (2.67b)$$

$$\Delta T_{E;rr}(\mathbf{r}) = \sum_{l=1}^{\infty} \frac{-\nu}{4\pi r^2} \int \frac{d\zeta}{2\pi} \kappa_2 \frac{s_l(\kappa_2 a)}{e_l(\kappa_2 a)} \left[e_l'^2(\kappa_2 r) - e_l(\kappa_2 r) e_l''(\kappa_2 r) \right]. \tag{2.67c}$$

Similar arguments follow. In the rest of this section, we mainly focus on the pressure at the interface between two media, which is thought to have directly measurable physical effects.

Consider a special case, where two media are separated by a infinitely thin perfectly conducting shell of radius a with (ε_1, μ_1) inside and (ε_2, μ_2) outside. When the media are not only homogeneous but also diaphanous, i.e., $\varepsilon_1\mu_1 = \varepsilon_2\mu_2$, then the pressures are

$$P_E = \sum_{l=1}^{\infty} \frac{-\nu}{4\pi a^2} \int \frac{d\zeta}{2\pi} \frac{\partial \ln e_l(\kappa a) s_l(\kappa a)}{\partial a}, \; P_H = \sum_{l=1}^{\infty} \frac{-\nu}{4\pi a^2} \int \frac{d\zeta}{2\pi} \frac{\partial \ln e_l'(\kappa a) s_l'(\kappa a)}{\partial a}, \tag{2.68}$$

which means $P = P_E + P_H$ is consistent with the principle of virtual work $P = -\partial\Delta U/\partial a$ according to Eq. (2.40) and Eq. (2.41). Those are the well known results [69, 21], which eliminate the validity of the semiclassic model for the electron proposed by Casimir [21], because of its repulsiveness.

Consider a relatively well-behaved special case, where the media are diaphanous ($\varepsilon_1\mu_1 = \varepsilon_2\mu_2$), then $\forall\zeta, \; \kappa_1 = \kappa_2 \equiv \kappa$, and the TE and TM pressures on the surface $r = a$ are written as

$$P_E(a) = \sum_{l=1}^{\infty} \frac{-\nu}{4\pi a^2} P_l(\cos\delta) \frac{\partial}{\partial a} \int \frac{d\zeta}{2\pi} e^{i\zeta\tau} \ln \left[1 + \frac{\mu_2 - \mu_1}{\mu_1} e_l(\kappa a) s_l'(\kappa a) \right], \tag{2.69a}$$

$$P_H(a) = \sum_{l=1}^{\infty} \frac{-\nu}{4\pi a^2} P_l(\cos\delta) \frac{\partial}{\partial a} \int \frac{d\zeta}{2\pi} e^{i\zeta\tau} \ln \left[1 - \frac{\mu_2 - \mu_1}{\mu_2} e_l(\kappa a) s_l'(\kappa a) \right], \tag{2.69b}$$

where the angular point-splitting regulator δ and temporal point-splitting regulator τ are included. For brevity, we further assume the dielectric ball is nondispersive and dilute, i.e., $\varepsilon_2 = \mu_2 = 1, \; \varepsilon_1 = 1 + \epsilon \to 1$. To the second order of ϵ, the pressures are

$$\begin{aligned}
P_E^{(1)}(a) = -P_H^{(1)}(a) = \sum_{l=1}^{\infty} \frac{-\epsilon\nu}{4\pi a^2} P_l(\cos\delta) \frac{\partial}{\partial a} \int \frac{d\zeta}{2\pi} e^{i\zeta\tau} e_l(|\zeta|a) s_l'(|\zeta|a) \\
= \frac{4u^6a^6 - (u^4 - 12u^2 + 48)u^2\tau^2a^4 - (3u^4 + 12u^2 - 16)\tau^4a^2 - 6(u^2 - 2)\tau^6}{-8\pi^2(4a^2 + \tau^2)^2(a^2u^2 + \tau^2)^3/\epsilon} \\
= \frac{-\epsilon}{32\pi^2a^4},\ u \to 0, \tau = 0; \quad -\frac{3\epsilon}{32\pi^2a^4} - \frac{\epsilon}{8\pi^2a^2}\frac{1}{\tau^2},\ u = 0, \tau \to 0, \qquad (2.70a)
\end{aligned}$$

where $u = \sqrt{2 - 2\cos\delta} \to \delta$, and $P_E^{(2)}(a) = P_H^{(2)}(a) - \epsilon P_E^{(1)}(a)$, in which

$$\begin{aligned}
P_E^{(2)}(a) &= \sum_{l=1}^{\infty} \frac{\epsilon^2\nu}{8\pi a^2} P_l(\cos\delta) \frac{\partial}{\partial a} \int \frac{d\zeta}{2\pi} e^{i\zeta\tau} e_l^2(|\zeta|a) s_l'^2(|\zeta|a) \\
&= -\frac{\epsilon}{2} P_E^{(1)}(a) + \frac{\epsilon^2}{8\pi^2a^4}\frac{5}{128}, \quad \delta = 0, \tau \to 0, \qquad (2.70b)
\end{aligned}$$

which are consistent with known results [32]. The total pressure $P = P_E + P_H$ at $r = a$, when evaluated with UAE to the first order, is

$$\begin{aligned}
P(a) &= \sum_{l=1}^{\infty} \frac{-\nu}{4\pi^2a^4} P_l(\cos\delta) \int_0^{\infty} dx \cos(x\tau_a) x \frac{\partial}{\partial x} \ln\left[1 - \tilde{\epsilon} e_l(x)e_l'(x)s_l(x)s_l'(x)\right] \\
&\approx \frac{3\tilde{\epsilon}}{1024\pi a^4},\ u = 0, \tau \to 0; \quad \frac{3\tilde{\epsilon}}{1024\pi a^4}\left(1 - \frac{1}{u}\right),\ u \to 0, \tau = 0, \qquad (2.71)
\end{aligned}$$

where $\tilde{\epsilon} = \epsilon^2/(\epsilon+1) \ll 1$. Except for the ambiguous divergence resulting from different regulators, we find a unique finite pressure, which starts from the second order of ϵ. It does not agree with the declaration in Ref. [33], which has been pointed out [35]. The arguments still remain [125], which makes this problem lively again.

Concentric configurations

There are not much work on the Casimir effects in concentric spherical systems done until now [126, 127, 128], as far as we know. We would like to present some basic results from our point of view here. In a homogeneous concentric configuration shown schematically in Figure 2.1(a), the permit-

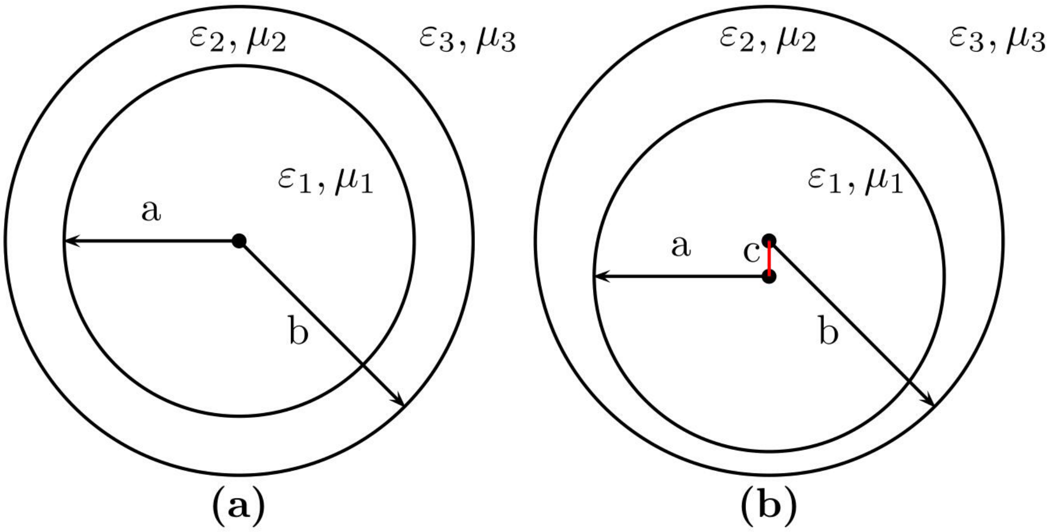

Figure 2.1: (a) The concentric configuration. (b) The eccentric configuration.

tivity ε and permeability μ of the system are typically

$$\varepsilon(r) = \begin{cases} \varepsilon_3, & r > b \\ \varepsilon_2, & a < r < b, \\ \varepsilon_1, & 0 < r < a, \end{cases} \qquad \mu(r) = \begin{cases} \mu_3, & r > b, \\ \mu_2, & a < r < b, \\ \mu_1, & 0 < r < a, \end{cases} \tag{2.72}$$

where $\varepsilon_i, \mu_i,\ i = 1, 2, 3$ are all homogeneous in their regions. The interaction induced TE pressure at $r = b$ is

$$P_E = -\sum_{l=1}^{\infty} \frac{\nu}{4\pi b^2} \int \frac{d\zeta}{2\pi} \frac{\partial}{\partial b} \ln \sigma_E(a, b), \tag{2.73}$$

where $\sigma_E(a, b)$ in this case is

$$\sigma_E(a, b) = 1 - \frac{[e_l(\kappa_3 b), e_l(\kappa_2 b)]_\mu [s_l(\kappa_2 a), s_l(\kappa_1 a)]_\mu}{[e_l(\kappa_3 b), s_l(\kappa_2 b)]_\mu [e_l(\kappa_2 a), s_l(\kappa_1 a)]_\mu}. \tag{2.74}$$

The corresponding TM contribution is obtained by making the substitutions $\varepsilon \leftrightarrow \mu$ and $E \to H$. Consider the limit case, in which $a \to \infty$ and $d = b - a$ is fixed. In the limit $a \to \infty$ and the substitutions $\nu/a \to k,\ \sum_{l=1}^{\infty} \nu/a^2 \to \int_0^\infty dkk$, P_E is

$$P_E \approx -\int \frac{d\zeta d^2k}{16\pi^3} \frac{\partial}{\partial b} \ln \left[1 + e^{-2\hat{\kappa}_2(b-a)} \frac{(\hat{\kappa}_3\mu_2 - \hat{\kappa}_2\mu_3)(\hat{\kappa}_2\mu_1 - \hat{\kappa}_1\mu_2)}{(\hat{\kappa}_3\mu_2 + \hat{\kappa}_2\mu_3)(\hat{\kappa}_2\mu_1 + \hat{\kappa}_1\mu_2)}\right], \tag{2.75}$$

where $\hat{\kappa}_i = \sqrt{k^2 + \varepsilon_i \mu_i \zeta^2},\ i = 1, 2, 3$. This result is consistent with Eq. (2.58).

For the spherical version of Casimir's original configuration, where $\varepsilon_1 = \varepsilon_3 = \infty,\ \mu_1 = \mu_3 = 1$, σ_E and σ_H are

$$\sigma_E(a,b) = 1 - \frac{e_l(\kappa_2 b) s_l(\kappa_2 a)}{s_l(\kappa_2 b) e_l(\kappa_2 a)}, \quad \sigma_H(a,b) = 1 - \frac{e_l'(\kappa_2 b) s_l'(\kappa_2 a)}{s_l'(\kappa_2 b) e_l'(\kappa_2 a)}. \tag{2.76}$$

Then further suppose ε_2, μ_2 are nondispersive for simplicity, the TE and TM pressures at the spherical shell $r = b$ are, respectively,

$$P_E = -\frac{1/\sqrt{\varepsilon_2 \mu_2}}{4\pi^2 b^2} \frac{\partial}{\partial b} \sum_{l=1}^{\infty} \nu \int_0^{\infty} dx \ln \left[1 - \frac{e_l(xb) s_l(xa)}{s_l(xb) e_l(xa)} \right], \tag{2.77a}$$

$$P_H = -\frac{1/\sqrt{\varepsilon_2 \mu_2}}{4\pi^2 b^2} \frac{\partial}{\partial b} \sum_{l=1}^{\infty} \nu \int_0^{\infty} dx \ln \left[1 - \frac{e_l'(xb) s_l'(xa)}{s_l'(xb) e_l'(xa)} \right]. \tag{2.77b}$$

In the limit $a \to 0$, the pressures are

$$P_E \to -\frac{3/\sqrt{\varepsilon_2 \mu_2}}{8\pi^2 b^2} \frac{\partial}{\partial b} \int_0^{\infty} dx \ln \left[1 - \frac{e_1(xb) s_1(xa)}{s_1(xb) e_1(xa)} \right] \approx -\frac{2.9823 a^3}{4\pi b^7 \sqrt{\varepsilon_2 \mu_2}}, \tag{2.78a}$$

$$P_H \to -\frac{3/\sqrt{\varepsilon_2 \mu_2}}{8\pi^2 b^2} \frac{\partial}{\partial b} \int_0^{\infty} dx \ln \left[1 - \frac{e_1'(xb) s_1'(xa)}{s_1'(xb) e_1'(xa)} \right] \approx -\frac{4.0491 a^3}{4\pi b^7 \sqrt{\varepsilon_2 \mu_2}}. \tag{2.78b}$$

In the limit $a \to \infty$ and $d = b - a$ fixed, they are evaluated with the uniform asymptotic expansion as

$$P_E = P_H \to \frac{-\pi^2}{480 (b-a)^4 \sqrt{\varepsilon_2 \mu_2}}, \tag{2.79}$$

which is consistent with the results in Eq. (2.56) and Eq. (2.75). It can be checked that in the limit $d = b - a \to 0$, P_E and P_H, to the leading order of UAE, satisfies Eq. (2.79), which means when the separation is small the interaction is local and the curvature effects are negligible.

Consider the generalized DLP configuration, in which the three media are nondispersive and homogeneous. Then, the general form of the TE

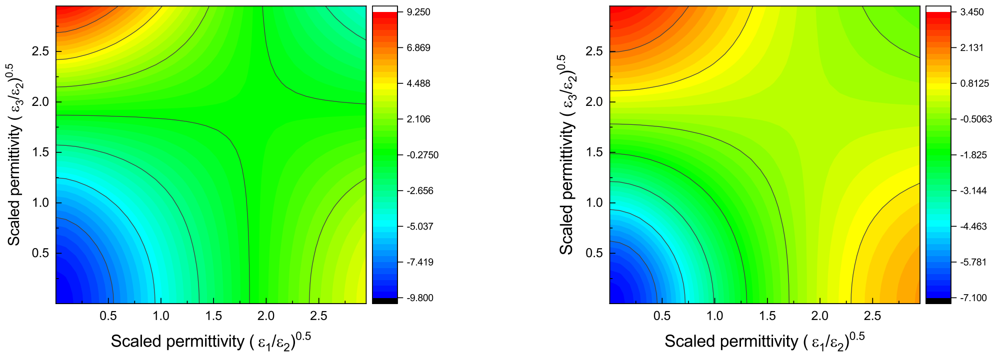

Figure 2.2: The ΔU_E (left, in unit $10^{-3}/\sqrt{\varepsilon_2}b$) and ΔU_H (right, in unit $10^{-1}/\sqrt{\varepsilon_2}b$) as functions of the scaled permittivities $\hat{\varepsilon}_1 = \sqrt{\varepsilon_1/\varepsilon_2}$ and $\hat{\varepsilon}_3 = \sqrt{\varepsilon_3/\varepsilon_2}$.

contribution to the interaction Casimir energy is

$$\Delta U_E = \sum_{l=1}^{\infty} \frac{\nu}{\sqrt{\varepsilon_2\mu_2}b} \int_0^{\infty} \frac{dx}{\pi} \ln\left\{1 - \frac{[e_l(c_{32}x), e_l(x)]_\mu [s_l(xd), s_l(c_{12}xd)]_\mu}{[e_l(c_{32}x), s_l(x)]_\mu [e_l(xd), s_l(c_{12}xd)]_\mu}\right\}, \quad (2.80)$$

where $c_{ij} = \sqrt{\varepsilon_i\mu_i}/\sqrt{\varepsilon_j\mu_j}$, $x = \kappa_2 b$ and $d = a/b \in (0,1)$. To clarify our analysis, set $\mu_1 = \mu_2 = \mu_3 = 1$, then Eq. (2.80) is

$$\Delta U_E = \sum_{l=1}^{\infty} \frac{\nu}{\sqrt{\varepsilon_2}b} \int_0^{\infty} \frac{dx}{\pi} \ln\left\{1 - \frac{[e_l(\hat{\varepsilon}_3 x), e_l(x)]_\mu [s_l(xd), s_l(\hat{\varepsilon}_1 xd)]_\mu}{[e_l(\hat{\varepsilon}_3 x), s_l(x)]_\mu [e_l(xd), s_l(\hat{\varepsilon}_1 xd)]_\mu}\right\}, \quad (2.81)$$

where $\hat{\varepsilon}_i = \sqrt{\varepsilon_i/\varepsilon_2}$ and ΔU_H can be obtained by making the substitution $\varepsilon \leftrightarrow \mu$ in the brace. For a given $d = 0.5$, the dependences of ΔU_E and ΔU_H on $(\hat{\varepsilon}_1, \hat{\varepsilon}_3)$ are shown in Figure 2.2.

For the simplest eccentric case shown schematically in Figure 2.1(b), in which a perfectly conducting ball of radius a is located in a spherical cavity of radius b inside a huge perfectly conducting bulk and the distance between the centers of the ball and cavity is c, $c + a < b$. Suppose $b - a \ll 1$, then the Casimir net force on the ball is evaluated with PFA as

$$F = -\frac{\pi^3}{90} \frac{a}{(a+c)^4} \frac{c(a+c)^4(b^2 - a^2 + c^2)}{[(b-a)^2 - c^2]^3} \approx -\frac{\pi^3}{90} \frac{(a+b)ac}{(b-a)^5}, \quad (2.82)$$

which is attractive. Obviously, when the system is concentric, i.e., $c = 0$, the net force is zero. It is more interesting to study the non-concentric Casimir force in the dielectric spherical system, where vacuum levitation of the ball may be obtained. We will proceed in this direction in the future.

2.4 Inhomogeneous media

The understanding of the properties of the Casimir energies and stresses in inhomogeneous media are actually rare and superficial. The studies in this field are mainly concentrated on the properties of divergences, the renormalization, and the inhomogeneous Casimir forces. In this section, we briefly investigate the Casimir energies and stress tensors in the presence of nondissipative, nondispersive and inhomogeneous media.

2.4.1 Planar systems

In this section, we calculate the Casimir energy densities and stress tensors in inhomogeneous two-media backgrounds. Since we have given a self-consistent renormalization scheme to get the interaction Casimir energy and forces in an inhomogeneous parallel configuration above and in Ref. [19], we will evaluate some specific inhomogeneous parallel configurations, which may lead to some further insight into the influence of inhomogeneity.

Two-media background

Consider two media (ε_1, μ_1) and (ε_2, μ_2) filling in half-spaces $z < 0$ and $z > 0$, respectively. To demonstrate the inhomogeneity effects, we assume (ε_1, μ_1) is nondispersive and homogeneous, while (ε_2, μ_2) is nondispersive but inhomogeneous. Then the interaction TE Casimir energy density and stress tensors in the $z > 0$ region are

$$\Delta u_E(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3} \frac{R_{e,+}}{\mu_2 \hat{W}_2^E} \left[\hat{e}_{2,+}'^2 + (2k^2 - \kappa_2^2)\hat{e}_{2,+}^2 \right], \tag{2.83a}$$

$$\Delta T_{E;xx}(\mathbf{r}) = \Delta T_{E;yy}(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3} \frac{k^2 R_{e,+}}{\mu_2 \hat{W}_2^E} \hat{e}_{2,+}^2, \tag{2.83b}$$

$$\Delta T_{E;zz}(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3}\frac{R_{e,+}}{\hat{W}_2^E}\frac{\partial[\hat{e}_{2,+}(z), \hat{e}_{2,+}(z_-)]_\mu}{\partial z_-}, \tag{2.83c}$$

while in the $z < a$ region they are

$$\Delta u_E(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3}\frac{R_{e,-}}{\mu_1\hat{W}_1^E}\left[\hat{e}_{1,-}'^2 + (2k^2 - \kappa_1^2)\hat{e}_{1,-}^2\right], \tag{2.84a}$$

$$\Delta T_{E;xx}(\mathbf{r}) = \Delta T_{E;yy}(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3}\frac{k^2 R_{e,-}}{\mu_1\hat{W}_1^E}\hat{e}_{1,-}^2, \tag{2.84b}$$

$$\Delta T_{E;zz}(\mathbf{r}) = \int \frac{d\zeta d^2k}{16\pi^3}\frac{R_{e,-}}{\hat{W}_1^E}\frac{\partial[\hat{e}_{1,-}(z), \hat{e}_{1,-}(z_-)]_\mu}{\partial z_-}. \tag{2.84c}$$

where $R_{e,\pm} = [\hat{e}_{2,\mp}(0), \hat{e}_{1,\mp}(0)]_\mu/[\hat{e}_{2,+}(0), \hat{e}_{1,-}(0)]_\mu$ and $\hat{e}_{i,\pm}$ are the same as in Eq. (2.27). The corresponding TM contributions are obtained by making the substitutions $\varepsilon \leftrightarrow \mu$, $e \to h$ and $E \to H$.

Consider the general behaviors of those parameters in Eq. (2.83). To this end, we further assume $(\varepsilon_1, \mu_1) = (1, 1)$ and $\varepsilon_2 = \epsilon_0 + \epsilon_1 z + \epsilon_2 z^2$, $\mu_2 = 1$ for clarity. Thus, $\hat{e}_{1,\pm} = e^{\mp\kappa z}$, $\kappa = \sqrt{k^2 + \zeta^2}$, while $\hat{e}_{2,\pm}(z) = e^{\mp\kappa_0 z} f_{e,\pm}(z)$ are determined by the equation

$$\left[\partial_z^2 \mp 2\kappa_0\partial_z - \epsilon_1\zeta^2 z - \epsilon_2\zeta^2 z^2\right] f_{e,\pm}(z) = 0, \tag{2.85}$$

where $\kappa_0 = \sqrt{k^2 + \epsilon_0\zeta^2}$. With the boundary conditions satisfied, assume the zeroth order in ϵ_1, ϵ_2 of $f_{e,\pm}$ are both 1, then to the first order we have

$$f_{e,\pm}^{(1)}(z) = -\frac{\epsilon_1\zeta^2 z}{4\kappa_0^2}(1 \pm \kappa_0 z) \mp \frac{\epsilon_2\zeta^2 z}{4\kappa_0^3}\left(1 \pm \kappa_0 z + \frac{2}{3}\kappa_0^2 z^2\right), \tag{2.86a}$$

which means, to the first order of ϵ_1, ϵ_2, $\hat{e}_{2,\pm}$ and their Wronskian satisfy

$$\hat{e}_{2,\pm}' \approx \mp e^{\mp\kappa_0 z}\left[\kappa_0 \pm \frac{\epsilon_1\zeta^2}{4\kappa_0^2}(1 \pm \kappa_0 z - \kappa_0^2 z^2) + \frac{\epsilon_2\zeta^2}{4\kappa_0^3}\left(1 \pm \kappa_0 z + \kappa_0^2 z^2 \mp \frac{2}{3}\kappa_0^3 z^3\right)\right], \tag{2.86b}$$

$$\mu_2\hat{W}_2^E \approx -2\kappa_0 - \frac{\epsilon_2\zeta^2}{2\kappa_0^3}. \tag{2.86c}$$

Then the reflection parameters $R_{e,\pm}$ in Eq. (2.83) and Eq. (2.84) are, to the first order of ϵ_1, ϵ_2,

$$R_{e,+} = \frac{\hat{e}'_{2,-}(0) - \kappa\hat{e}_{2,-}(0)}{\hat{e}'_{2,+}(0) - \kappa\hat{e}_{2,+}(0)} \approx \frac{\kappa - \kappa_0}{\kappa + \kappa_0} + \frac{\zeta^2}{2\kappa_0}\frac{\epsilon_1}{(\kappa+\kappa_0)^2} - \frac{\zeta^2\kappa}{2\kappa_0^3}\frac{\epsilon_2}{(\kappa+\kappa_0)^2}, \quad \text{(2.87a)}$$

$$R_{e,-} = \frac{\hat{e}'_{2,+}(0) + \kappa\hat{e}_{2,+}(0)}{\hat{e}'_{2,+}(0) - \kappa\hat{e}_{2,+}(0)} \approx \frac{\kappa_0 - \kappa}{\kappa + \kappa_0} + \frac{\zeta^2\kappa}{2\kappa_0^2}\frac{\epsilon_1}{(\kappa+\kappa_0)^2} + \frac{\zeta^2\kappa}{2\kappa_0^3}\frac{\epsilon_2}{(\kappa+\kappa_0)^2}. \quad \text{(2.87b)}$$

To illustrate the effects due to ϵ_1, ϵ_2, we set $\epsilon_0 = 1$. Then to the first order of ϵ_1, ϵ_2, in the $z > 0$ region we have

$$\Delta T_{E;xx}(\mathbf{r}) = \Delta T_{E;yy}(\mathbf{r}) = \frac{\Delta u_E(\mathbf{r})}{2} \approx \frac{-(\epsilon_1 - \epsilon_2 z)}{1920\pi^2 z^3}, \;\; \Delta T_{E;zz}(\mathbf{r}) \approx 0, \quad \text{(2.88a)}$$

while in the $z < 0$ region they are

$$\Delta T_{E;xx}(\mathbf{r}) = \Delta T_{E;yy}(\mathbf{r}) = \frac{\Delta u_E(\mathbf{r})}{2} \approx \frac{-(\epsilon_1 - \epsilon_2 z)}{1920\pi^2 |z|^3}, \;\; \Delta T_{E;zz}(\mathbf{r}) = 0. \quad \text{(2.88b)}$$

For TM mode, $\hat{h}_{1,\pm} = e^{\mp\kappa z}$, while $\hat{h}_{2,\pm}(z) = e^{\mp\kappa_0 z} f_{h,\pm}(z)$ are solved with

$$\left[\partial_z^2 \mp 2\kappa_0\partial_z - \frac{\epsilon_1 + 2\epsilon_2 z}{\epsilon_0 + \epsilon_1 z + \epsilon_2 z^2}(\partial_z \mp \kappa_0) - \zeta^2\epsilon_1 z - \zeta^2\epsilon_2 z^2\right] f_{h,\pm}(z) = 0. \quad \text{(2.89)}$$

With the boundary conditions satisfied, assume the zeroth order in ϵ_1, ϵ_2 of $f_{h,\pm}$ are both 1, then to the first order we have

$$f^{(1)}_{h,\pm}(z) = \frac{\epsilon_1\kappa_0 \pm \epsilon_2}{4\epsilon_0\kappa_0^3}(\kappa_0^2 + k^2)z + \left(\frac{\epsilon_2}{2\epsilon_0} - \frac{\epsilon_2 \pm \epsilon_1\kappa_0}{4\kappa_0^2}\zeta^2\right)z^2 \mp \frac{\epsilon_2\zeta^2}{6\kappa_0}z^3, \quad \text{(2.90a)}$$

which means, to the first order of ϵ_1, ϵ_2, $\hat{h}_{2,\pm}$ and their Wronskian satisfy

$$\begin{aligned} \hat{h}'_{2,\pm} \;\; \approx \;\; & e^{\mp\kappa_0 z}\Bigg\{ \mp\kappa_0 + \frac{\epsilon_1\kappa_0 \pm \epsilon_2}{4\epsilon_0\kappa_0^3}(\kappa_0^2 + k^2) + \left[\frac{\epsilon_2}{\epsilon_0} - \frac{\epsilon_2 \pm \epsilon_1\kappa_0}{4\epsilon_0\kappa_0^2}(2\kappa_0^2 + \epsilon_0\zeta^2)\right]z \\ & \mp\left[\frac{\epsilon_2(\kappa_0^2 + \epsilon_0\zeta^2)}{2\epsilon_0\kappa_0} - \frac{\epsilon_2 \pm \epsilon_1\kappa_0}{4\kappa_0}\zeta^2\right]z^2 + \frac{\epsilon_2\zeta^2}{6}z^3\Bigg\}, \end{aligned} \quad \text{(2.90b)}$$

$$\hat{W}_2^H \approx -\frac{2\kappa_0}{\epsilon_0} + \frac{\kappa_0^2 + k^2}{2\epsilon_0^2\kappa_0^3}\epsilon_2, \;\; \varepsilon_2\hat{W}_2^E \approx -2\kappa_0 - \frac{2\kappa_0 z}{\epsilon_0}\epsilon_1 + \frac{k^2 + \kappa_0^2 - 4\kappa_0^4 z^2}{2\epsilon_0\kappa_0^3}\epsilon_2. \quad \text{(2.90c)}$$

Then the reflection parameters $R_{h,\pm}$ corresponding to those in Eq. (2.83) and Eq. (2.84) are, to the first order of ϵ_1, ϵ_2,

$$R_{h,+} \approx \frac{\epsilon_0\kappa - \kappa_0}{\epsilon_0\kappa + \kappa_0} - \frac{\epsilon_1}{2\epsilon_0\kappa_0}\frac{\kappa_0^2 + k^2}{(\epsilon_0\kappa + \kappa_0)^2} + \frac{\epsilon_2\kappa}{2\kappa_0^3}\frac{\kappa_0^2 + k^2}{(\epsilon_0\kappa + \kappa_0)^2}, \tag{2.91a}$$

$$R_{h,-} \approx \frac{\kappa_0 - \epsilon_0\kappa}{\epsilon_0\kappa + \kappa_0} - \frac{\epsilon_1\kappa}{2\kappa_0^2}\frac{\kappa_0^2 + k^2}{(\epsilon_0\kappa + \kappa_0)^2} - \frac{\epsilon_2\kappa}{2\kappa_0^3}\frac{\kappa_0^2 + k^2}{(\epsilon_0\kappa + \kappa_0)^2}. \tag{2.91b}$$

When $\epsilon_0 = 1$, then to the first order of ϵ_1, ϵ_2, the parameters in the $z > 0$ region are

$$\Delta T_{H;xx}(\mathbf{r}) = \Delta T_{H;yy}(\mathbf{r}) = \frac{\Delta u_H(\mathbf{r})}{2} \approx \frac{3(\epsilon_1 - \epsilon_2 z)}{640\pi^2 z^3}, \ \Delta T_{H;zz}(\mathbf{r}) \approx 0, \tag{2.92a}$$

while in the $z < 0$ region they are

$$\Delta T_{H;xx}(\mathbf{r}) = \Delta T_{H;yy}(\mathbf{r}) = \frac{\Delta u_H(\mathbf{r})}{2} \approx \frac{3(\epsilon_1 - \epsilon_2 z)}{640\pi^2 |z|^3}, \ \Delta T_{H;zz}(\mathbf{r}) = 0. \tag{2.92b}$$

Our results here tally with those in Ref. [118].

Consider the special case where the region $z < 0$ is filled with a perfect conductor, while $\varepsilon_2 = \epsilon_0 + \epsilon_1 z + \epsilon_2 z^2$, $\mu_2 = 1$. Then $R_{e,+} = \hat{e}_{2,-}(0)/\hat{e}_{2,+}(0)$ satisfies $R_{e,+} \approx 1$, while $\Delta T_{E;xx}$ and $\Delta T_{E;zz}$, to the first order of ϵ_1, ϵ_2, are

$$\Delta T_{E;xx}(\mathbf{r}) = \Delta T_{E;yy}(\mathbf{r}) \approx -\frac{1}{32\pi^2\epsilon_0^{\frac{1}{2}} z^4} + \frac{3\epsilon_1}{320\pi^2\epsilon_0^{\frac{3}{2}} z^3} + \frac{\epsilon_2}{96\pi^2\epsilon_0^{\frac{3}{2}} z^2}, \tag{2.93a}$$

$$\Delta T_{E;zz}(\mathbf{r}) \approx -\frac{\epsilon_1 + 3\epsilon_2 z}{192\pi^2\epsilon_0^{\frac{3}{2}} z^3}, \tag{2.93b}$$

and the corresponding Δu_E is obtained with $\Delta u_E = \Delta T_{E;xx} + \Delta T_{E;yy} + \Delta T_{E;zz}$. For TM mode, $R_{h,+} = \hat{h}'_{2,-}(0)/\hat{h}'_{2,+}(0)$ satisfies

$$R_{h,+} \approx -1 - \frac{\kappa_0^2 + k^2}{2\epsilon_0\kappa_0^3}\epsilon_1, \tag{2.94}$$

while $\Delta T_{H;xx}$ and $\Delta T_{H;zz}$, to the first order of ϵ_1, ϵ_2, are

$$\Delta T_{H;xx}(\mathbf{r}) = \Delta T_{H;yy}(\mathbf{r}) \approx \frac{1}{32\pi^2\epsilon_0^{\frac{1}{2}}z^4} + \frac{3\epsilon_1}{320\pi^2\epsilon_0^{\frac{3}{2}}z^3} + \frac{\epsilon_2}{48\pi^2\epsilon_0^{\frac{3}{2}}z^2}, \tag{2.95a}$$

$$\Delta T_{H;zz}(\mathbf{r}) \approx -\frac{5(\epsilon_1 + 3\epsilon_2 z)}{192\pi^2\epsilon_0^{\frac{3}{2}}z^3}, \tag{2.95b}$$

and the corresponding Δu_E is obtained with $\Delta u_H = \Delta T_{H;xx} + \Delta T_{H;yy} + \Delta T_{H;zz}$.

Parallel configurations

The basic investigations on the inhomogeneous parallel configurations are given in our work Ref. [19]. We investigate, as a first trial, some more general behaviors of the inhomogeneous parallel configuration, in which three nondispersive media $(\varepsilon_i, \mu_i),\ i = 1, 2, 3$ fill in the regions $z < a$, $a < z < b$ and $z > b$, respectively. The Casimir pressure is determined by both the local and global properties of the media. It is interesting and essential to gain better understanding of the local and global aspects of Casimir forces, especially in the inhomogeneous cases.

Consider the generalized Casimir configuration (GCC), where the left and right media in $z < -a$ and $z > a$ satisfy $\mu_L = \mu_R = 1$, $\varepsilon_L, \varepsilon_R \to \infty$ and the permittivity and permeability of the intervening medium are

$$\varepsilon(z) = \begin{cases} \varepsilon_2, & b < z < a, \\ \varepsilon_1, & -a < z < b, \end{cases} \qquad \mu(z) = \begin{cases} \mu_2, & b < z < a, \\ \mu_1, & -a < z < b. \end{cases} \tag{2.96}$$

Then $\hat{e}_\pm$ for the intervening medium are (for all $z, x \in \mathbb{R}$, define $z_x \equiv z - x$)

$$\hat{e}_+(\zeta, k; z) = \begin{cases} t_{1,2}^E e^{-\kappa_2 z_b}, & z > b, \\ e^{-\kappa_1 z_b} - r_{1,2}^E e^{\kappa_1 z_b}, & z < b, \end{cases} \tag{2.97a}$$

$$\hat{e}_-(\zeta, k; z) = \begin{cases} e^{\kappa_2 z_b} - r_{2,1}^E e^{-\kappa_2 z_b}, & z > b, \\ t_{2,1}^E e^{\kappa_1 z_b}, & z < b, \end{cases} \tag{2.97b}$$

where $r_{i,j}^E$ and $t_{i,j}^E$ are defined as

$$r_{i,j}^E = -\frac{\kappa_i\mu_j - \kappa_j\mu_i}{\kappa_i\mu_j + \kappa_j\mu_i},\ r_{i,j}^E = -r_{j,i}^E,\ t_{i,j}^E = 1 - r_{i,j}^E = \frac{2\kappa_i\mu_j}{\kappa_i\mu_j + \kappa_j\mu_i}. \tag{2.97c}$$

The TE interaction Casimir energy is

$$\Delta U_E = \int \frac{d\zeta d^2k}{16\pi^3} \ln\left(1 - \frac{t_{1,2}^E e^{-2\kappa_2 a_b}}{1 - r_{2,1}^E e^{-2\kappa_2 a_b}} \frac{t_{2,1}^E e^{-2\kappa_1 b_{-a}}}{1 - r_{1,2}^E e^{-2\kappa_1 b_{-a}}}\right), \tag{2.98}$$

By making substitutions $E \to H,\ \varepsilon \leftrightarrow \mu$ and $\hat{e} \to \hat{h}'$, one obtains the expressions for the TM counterparts in Eq. (2.98). When $\varepsilon_1 = \varepsilon_2, \mu_1 = \mu_2$, we immediately retrieve the result of Eq. (2.56a). When $\mu_1 = 1, \varepsilon_1 \to \infty$, we find $\Delta U_E, \Delta U_H \to 0$, which means the interaction between the surfaces $z = \pm a$ is blocked by the perfectly conducting layer $-a < z < b$. For the diaphanous case $\varepsilon_1\mu_1 = \varepsilon_2\mu_2$, we have

$$\begin{aligned} \Delta U_E &= \int \frac{d\zeta d^2k}{16\pi^3 8(a-b)^3\sqrt{\varepsilon_1\mu_1}} \ln\left[1 - \frac{1 - r_{2,1}^E}{e^{\kappa} - r_{2,1}^E} \frac{1 + r_{2,1}^E}{e^{\kappa\eta} + r_{2,1}^E}\right],\ \eta = \frac{b+a}{a-b}, \\ &\approx \frac{1}{16(2a)^3\sqrt{\varepsilon_1\mu_1}}\left(\frac{r_{2,1}^{E2}}{\pi^2} - \frac{\pi^2}{90}\right),\ \eta = 1, |r_{2,1}^E| \ll 1, \end{aligned} \tag{2.99}$$

while ΔU_H can be evaluated with $r_{2,1}^H = -r_{2,1}^E$.

Now we extend our analysis to a more complicated case in the generalized Casimir configuration, where the intervening medium consists of three homogeneous slabs, whose permittivity and permeability are of the form

$$\varepsilon(z) = \begin{cases} \varepsilon_3, & c < z < a, \\ \varepsilon_2, & b < z < c, \\ \varepsilon_1, & -a < z < b, \end{cases} \qquad \mu(z) = \begin{cases} \mu_3, & c < z < a, \\ \mu_2, & b < z < c, \\ \mu_1, & -a < z < b. \end{cases} \tag{2.100}$$

Then $\hat{e}_\pm$ for the intervening medium are

$$\hat{e}_+(z) = \begin{cases} \frac{t_{1,2}^E t_{2,3}^E e^{\kappa_2 b_c}}{1 + r_{1,2}^E r_{2,3}^E e^{2\kappa_2 b_c}} e^{-\kappa_3 z_c}, & z > c, \\ \frac{t_{1,2}^E}{1 + r_{1,2}^E r_{2,3}^E e^{2\kappa_2 b_c}} e^{-\kappa_2 z_b} - \frac{t_{1,2}^E r_{2,3}^E e^{2\kappa_2 b_c}}{1 + r_{1,2}^E r_{2,3}^E e^{2\kappa_2 b_c}} e^{\kappa_2 z_b}, & b < z < c, \\ e^{-\kappa_1 z_b} - \frac{r_{1,2}^E + r_{2,3}^E e^{2\kappa_2 b_c}}{1 + r_{1,2}^E r_{2,3}^E e^{2\kappa_2 b_c}} e^{\kappa_1 z_b}, & z < b, \end{cases} \tag{2.101a}$$

$$\hat{e}_-(z) = \begin{cases} e^{\kappa_3 z_c} - \frac{r^E_{3,2}+r^E_{2,1}e^{-2\kappa_2 c_b}}{1+r^E_{3,2}r^E_{2,1}e^{-2\kappa_2 c_b}} e^{-\kappa_3 z_c}, & z > c, \\ \frac{t^E_{3,2}}{1+r^E_{3,2}r^E_{2,1}e^{-2\kappa_2 c_b}} e^{\kappa_2 z_c} - \frac{t^E_{3,2}r^E_{2,1}e^{-2\kappa_2 c_b}}{1+r^E_{3,2}r^E_{2,1}e^{-2\kappa_2 c_b}} e^{-\kappa_2 z_c}, & b < z < c, \\ \frac{t^E_{3,2}t^E_{2,1}e^{-\kappa_2 c_b}}{1+r^E_{3,2}r^E_{2,1}e^{-2\kappa_2 c_b}} e^{\kappa_1 z_b}, & z < b, \end{cases} \qquad (2.101b)$$

which means the TE interacting Casimir energy is

$$\begin{aligned} \Delta U_E &= \int \frac{d\zeta d^2k}{16\pi^3} \ln \Bigg[1 - \frac{t^E_{2,3}t^E_{2,1}e^{-2\kappa_2 c_b}}{(1-r^E_{2,3}e^{-2\kappa_2 c_b})(1-r^E_{2,1}e^{-2\kappa_2 c_b})} \\ &\quad \times \frac{t^E_{1,3}e^{-2\kappa_3 a_c}}{1-r^E_{3,1}e^{-2\kappa_3 a_c}} \frac{t^E_{3,1}e^{-2\kappa_1 b_{-a}}}{1-r^E_{1,3}e^{-2\kappa_1 b_{-a}}} \Bigg], \end{aligned} \qquad (2.102a)$$

where $r^E_{1,3},\ r^E_{3,1}$ and $t^E_{3,1} = 1 - r^E_{3,1},\ t^E_{1,3} = 1 - r^E_{1,3}$ are defined as

$$r^E_{3,1} \equiv \frac{r^E_{3,2} + r^E_{2,1}e^{-2\kappa_2 c_b}}{1 + r^E_{3,2}r^E_{2,1}e^{-2\kappa_2 c_b}},\ r^E_{1,3} \equiv \frac{r^E_{1,2} + r^E_{2,3}e^{-2\kappa_2 c_b}}{1 + r^E_{1,2}r^E_{2,3}e^{-2\kappa_2 c_b}}, \qquad (2.102b)$$

$$t^E_{3,1} = \frac{t^E_{3,2}(1 - r^E_{2,1}e^{-2\kappa_2 c_b})}{1 + r^E_{3,2}r^E_{2,1}e^{-2\kappa_2 c_b}},\ t^E_{1,3} = \frac{t^E_{1,2}(1 - r^E_{2,3}e^{-2\kappa_2 c_b})}{1 + r^E_{3,2}r^E_{2,1}e^{-2\kappa_2 c_b}}. \qquad (2.102c)$$

When $\varepsilon_2 = \varepsilon_3,\ \mu_2 = \mu_3$, Eq. (2.102a) is just Eq. (2.98).

There are attempts to explore the inhomogeneity with the step potential [129, 130, 17]. This model may also facilitate the experimental detection of the inhomogeneous Casimir forces [131]. We would like to deepen our research into this model in the future.

2.4.2 Spherical system

The current status of research into Casimir energies and stresses of inhomogeneous spherical systems is even more primitive. Only a few works [132] have been done as far as we can see. Here we will try to put forward some preliminary arguments about this topic.

Two-media background

Consider two media (ε_i, μ_i) and (ε_o, μ_o) filling in regions $0 < r < a$ and $r > a$, respectively. To demonstrate the inhomogeneous effects, we assume the media are nondispersive but inhomogeneous. Then the interaction induced TE Casimir energy density and stress tensors in the region $r > a$ are

$$\Delta u_E(\mathbf{r}) = \frac{-1}{4\pi r^2}\sum_{l=1}^{\infty}\nu\int\frac{d\zeta}{2\pi}\frac{R_{e,+}}{\hat{W}_o^E}\left[\frac{\mathfrak{e}_{o,+}'^2}{\mu_o} + \left(\frac{\nu^2-1/4}{\mu_o r^2} - \varepsilon_o\zeta^2\right)\hat{\mathfrak{e}}_{o,+}^2\right], \quad (2.103a)$$

$$\Delta T_{E;\theta\theta}(\mathbf{r}) = \Delta T_{E;\varphi\varphi}(\mathbf{r}) = \sum_{l=1}^{\infty}\frac{-\nu}{4\pi r^2}\int\frac{d\zeta}{2\pi}\frac{R_{e,+}}{\hat{W}_o^E}\frac{\nu^2-1/4}{\mu_o r^2}\mathfrak{e}_{o,+}^2, \quad (2.103b)$$

$$\Delta T_{E;rr}(\mathbf{r}) = \sum_{l=1}^{\infty}\frac{-\nu}{4\pi r^2}\int\frac{d\zeta}{2\pi}\frac{R_{e,+}}{\hat{W}_o^E}\left[\frac{\hat{\mathfrak{e}}_{o,+}'^2}{\mu_o} - \left(\frac{\nu^2-1/4}{\mu_o r^2} + \varepsilon_o\zeta^2\right)\hat{\mathfrak{e}}_{o,+}^2\right], \quad (2.103c)$$

where $R_{e,+} = [\hat{\mathfrak{e}}_{i,-}(a), \hat{\mathfrak{e}}_{o,-}(a)]_\mu/[\hat{\mathfrak{e}}_{o,+}(a), \hat{\mathfrak{e}}_{i,-}(a)]_\mu$, while in the region $0 < r < a$ those quantities are

$$\Delta u_E(\mathbf{r}) = \frac{-1}{4\pi r^2}\sum_{l=1}^{\infty}\nu\int\frac{d\zeta}{2\pi}\frac{R_{e,-}}{\hat{W}_i^E}\left[\frac{\hat{\mathfrak{e}}_{i,-}'^2}{\mu_i} + \left(\frac{\nu^2-1/4}{\mu_i r^2} - \varepsilon_i\zeta^2\right)\hat{\mathfrak{e}}_{i,-}^2\right], \quad (2.104a)$$

$$\Delta T_{E;\theta\theta}(\mathbf{r}) = \Delta T_{E;\varphi\varphi}(\mathbf{r}) = \sum_{l=1}^{\infty}\frac{-\nu}{4\pi r^2}\int\frac{d\zeta}{2\pi}\frac{R_{e,-}}{\hat{W}_i^E}\frac{\nu^2-1/4}{\mu_i r^2}\hat{\mathfrak{e}}_{i,-}^2, \quad (2.104b)$$

$$\Delta T_{E;rr}(\mathbf{r}) = \sum_{l=1}^{\infty}\frac{-\nu}{4\pi r^2}\int\frac{d\zeta}{2\pi}\frac{R_{e,-}}{\hat{W}_i^E}\left[\frac{\hat{\mathfrak{e}}_{i,-}'^2}{\mu_i} - \left(\frac{\nu^2-1/4}{\mu_i r^2} + \varepsilon_i\zeta^2\right)\hat{\mathfrak{e}}_{i,-}^2\right]. \quad (2.104c)$$

where $R_{e,-} = [\hat{\mathfrak{e}}_{i,+}(a), \hat{\mathfrak{e}}_{o,+}(a)]_\mu/[\hat{\mathfrak{e}}_{o,+}(a), \hat{\mathfrak{e}}_{i,-}(a)]_\mu$. The corresponding TM contribution is obtained by making the substitution $\varepsilon \leftrightarrow \mu$, $\mathfrak{e} \to \mathfrak{h}$ and $E \to H$.

Consider a special radial inhomogeneity, i.e., the permittivity $\varepsilon(r) = \lambda/r^2$ and permeability $\mu = 1$, to naively illustrate its influences on the vacuum energy density and stress tensors. For the case I, in which $\varepsilon_o \to \infty, \mu_o = 1$ and $\varepsilon_i = \lambda/r^2, \mu_i = 1$, then the TE interaction stress tensors are $\mathfrak{e}_\pm(\zeta, l; r) = r^{\frac{1}{2}\mp\sqrt{\nu^2+\lambda\zeta^2}}$, $\mathfrak{h}_\pm(\zeta, l; r) = r^{-\frac{1}{2}\mp\sqrt{\nu^2+\lambda\zeta^2}}$

$$\Delta T_{E;\theta\theta}(\mathbf{r}) = \Delta T_{E;\varphi\varphi}(\mathbf{r}) = \sum_{l=1}^{\infty}\frac{-\nu(\nu^2-1/4)}{8\pi^2 r^3\sqrt{\lambda}}\int_0^\infty d\zeta\frac{e^{-2\ln\frac{a}{r}\sqrt{\nu^2+\zeta^2}}}{\sqrt{\nu^2+\zeta^2}}, \quad (2.105a)$$

$$\Delta T_{E;rr}(\mathbf{r}) = \sum_{l=1}^{\infty} \frac{-\nu}{8\pi^2 r^3 \sqrt{\lambda}} \int_0^{\infty} d\zeta \frac{e^{-2\ln\frac{a}{r}\sqrt{\nu^2+\zeta^2}}}{\sqrt{\nu^2+\zeta^2}} \left(\frac{1}{2} + \sqrt{\nu^2+\zeta^2}\right), \quad (2.105b)$$

which means in the vicinity of the surface ($r \approx a$) they are

$$\Delta T_{E;\theta\theta}(\mathbf{r}) = \Delta T_{E;\varphi\varphi}(\mathbf{r}) \approx \frac{-1}{32\pi^2 \sqrt{\lambda}(a-r)^2} \left[\frac{a}{(a-r)^2} - \frac{1}{4a}\right], \quad (2.105c)$$

$$\Delta T_{E;rr}(\mathbf{r}) \approx \frac{-1}{32\pi^2 \sqrt{\lambda}(a-r)^2} \left(\frac{1}{2a} + \frac{1}{a-r}\right). \quad (2.105d)$$

For the TM mode, the corresponding terms are

$$\begin{aligned} &\Delta T_{H;\theta\theta}(\mathbf{r}) = \Delta T_{H;\varphi\varphi}(\mathbf{r}) \\ =\ &\sum_{l=1}^{\infty} \frac{-\nu(\nu^2 - 1/4)}{8\pi^2 r^3 \sqrt{\lambda}} \int_0^{\infty} d\zeta \frac{e^{-2\ln\frac{a}{r}\sqrt{\nu^2+\zeta^2}}}{\sqrt{\nu^2+\zeta^2}} \frac{1+2\sqrt{\nu^2+\zeta^2}}{1-2\sqrt{\nu^2+\zeta^2}}, \end{aligned} \quad (2.106a)$$

$$\Delta T_{H;rr}(\mathbf{r}) = \sum_{l=1}^{\infty} \frac{-\nu}{8\pi^2 r^3 \sqrt{\lambda}} \int_0^{\infty} d\zeta \frac{e^{-2\ln\frac{a}{r}\sqrt{\nu^2+\zeta^2}}}{\sqrt{\nu^2+\zeta^2}} \left(\frac{1}{2} + \sqrt{\nu^2+\zeta^2}\right) = \Delta T_{E;rr}(\mathbf{r}), \quad (2.106b)$$

which means in the vicinity of the surface ($r \approx a$) they are

$$\Delta T_{H;\theta\theta}(\mathbf{r}) = \Delta T_{H;\varphi\varphi}(\mathbf{r}) \approx \frac{-1}{32\pi^2 \sqrt{\lambda}(a-r)^2} \left[\frac{1}{4a} - \frac{a}{(a-r)^2}\right], \quad (2.106c)$$

and $\Delta T_{H;rr}(\mathbf{r}) = \Delta T_{E;rr}(\mathbf{r})$. For the case II, in which $\varepsilon_i \to \infty, \mu_i = 1$ and $\varepsilon_o = \lambda/r^2, \mu_o = 1$, then the TE stress tensors are expressed as

$$\Delta T_{E;\theta\theta}(\mathbf{r}) = \Delta T_{E;\varphi\varphi}(\mathbf{r}) = \frac{-1}{8\pi^2 r^3 \sqrt{\lambda}} \sum_{l=1}^{\infty} \nu \left(\nu^2 - \frac{1}{4}\right) K_0\left(2\nu \ln\frac{r}{a}\right), \quad (2.107a)$$

$$\Delta T_{E;rr}(\mathbf{r}) = \frac{-1}{8\pi^2 r^3 \sqrt{\lambda}} \left[\frac{1}{2}\sum_{l=1}^{\infty} \nu K_0\left(2\nu \ln\frac{r}{a}\right) - \sum_{l=1}^{\infty} \nu^2 K_1\left(2\nu \ln\frac{r}{a}\right)\right], \quad (2.107b)$$

which means in the vicinity of the surface ($r \approx a$) they are

$$\Delta T_{E;\theta\theta}(\mathbf{r}) = \Delta T_{E;\varphi\varphi}(\mathbf{r}) \approx \frac{-1}{32\pi^2 \sqrt{\lambda}(r-a)^2} \left[\frac{a}{(r-a)^2} - \frac{1}{4a}\right], \quad (2.107c)$$

$$\Delta T_{E;rr}(\mathbf{r}) \approx \frac{-1}{32\pi^2\sqrt{\lambda}(r-a)^2}\left(\frac{1}{2a} - \frac{1}{r-a}\right). \tag{2.107d}$$

For the TM mode, the corresponding terms are

$$\begin{aligned} &\Delta T_{H;\theta\theta}(\mathbf{r}) = \Delta T_{H;\varphi\varphi}(\mathbf{r}) \\ =& \sum_{l=1}^{\infty} \frac{\nu(\nu^2-1/4)}{8\pi^2 r^3\sqrt{\lambda}} \int_0^\infty d\zeta \frac{e^{-2\ln\frac{r}{a}\sqrt{\nu^2+\zeta^2}}}{\sqrt{\nu^2+\zeta^2}}\left[1 - \frac{2}{1+2\sqrt{\nu^2+\zeta^2}}\right], \end{aligned} \tag{2.108a}$$

$$\Delta T_{H;rr}(\mathbf{r}) = \sum_{l=1}^{\infty} \frac{-\nu}{8\pi^2 r^3\sqrt{\lambda}} \int_0^\infty d\zeta \frac{1-2\sqrt{\nu^2+\zeta^2}}{2e^{2\ln\frac{r}{a}\sqrt{\nu^2+\zeta^2}}\sqrt{\nu^2+\zeta^2}} = \Delta T_{E;rr}(\mathbf{r}), \tag{2.108b}$$

which means in the vicinity of the surface ($r \approx a$) they are

$$\begin{aligned} \Delta T_{H;\theta\theta}(\mathbf{r}) \;=\; & \Delta T_{H;\varphi\varphi}(\mathbf{r}) \approx \frac{-1}{32\pi^2\sqrt{\lambda}(r-a)^2}\left[-\frac{a}{(r-a)^2} - \frac{4}{3(r-a)} - \frac{1}{4a}\right. \\ & \left. -\frac{r-a}{12a^2} - \frac{(r-a)^2}{90a^3}\left(8 + 15\gamma_E + 15\ln\frac{r-a}{a}\right)\right], \end{aligned} \tag{2.108c}$$

and $\Delta T_{H;rr}(\mathbf{r}) = \Delta T_{E;rr}(\mathbf{r})$. In the limit $a \to \infty$, the results above are consistent with those in Eq. (2.50). We recognize that except for the curvature-dependent parts in $\Delta T_{H;rr}(\mathbf{r})$ and $\Delta T_{E;rr}(\mathbf{r})$, the pressures on the inner and outer sides of a infinitely thin perfectly conducting spherical shell, when it is immersed in a medium with the permittivity $\varepsilon(r) = \lambda/r^2$ and permeability $\mu = 1$, are both attractive. This phenomenon may facilitate the experimental detection on the self-stress in spherical systems.

Concentric configurations

The pressure on interfaces of a concentric system can be obtained by using the results in Eq. (2.43) and Eq. (2.44). As a preliminarily illustration of inhomogeneous Casimir forces, we just discuss a simple and analytically solvable system, i.e., the spherical version of generalized Casimir configuration (SGCC) with the permittivity and permeability of the intervening medium being $\varepsilon(r) = \lambda/r^2, \mu = 1$ and with inner and outer spherical regions being perfectly conductors. The TE Casimir pressure at $r = b$ is

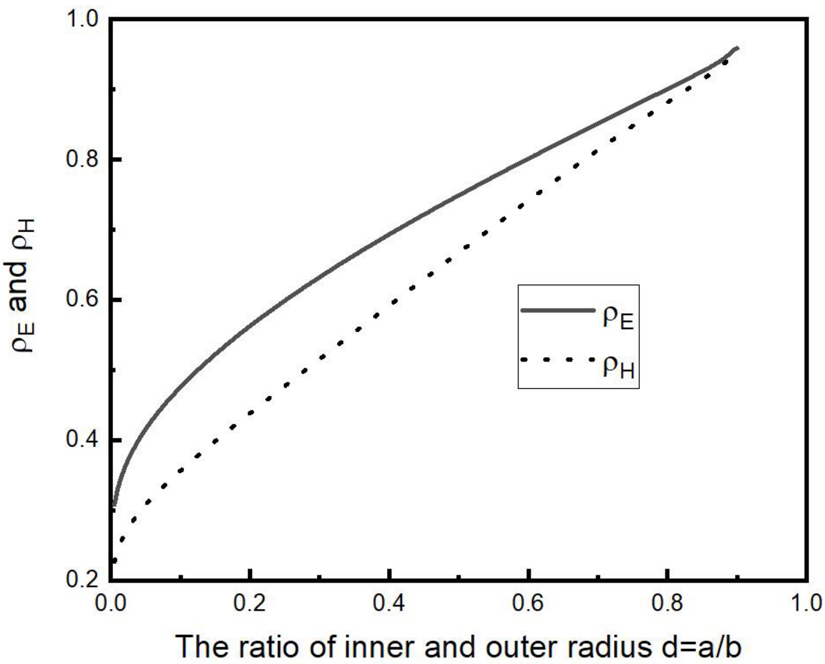

Figure 2.3: The $d = a/b$ dependences of ratio $\rho_E = U_E/U_E^{(0)}$ and $\rho_H = U_H/U_H^{(0)}$, with $\sqrt{\lambda} = b = 1$.

$$P_E = -\frac{1}{4\pi b^2}\frac{\partial}{\partial b}\sum_{l=1}^{\infty}\nu\int\frac{d\zeta}{2\pi}\ln\sigma_E(a,b),\ \ \sigma_E(a,b) = 1 - \frac{\hat{\mathfrak{e}}_+(b)\hat{\mathfrak{e}}_-(a)}{\hat{\mathfrak{e}}_-(b)\hat{\mathfrak{e}}_+(a)}, \tag{2.109a}$$

while the corresponding TM pressure is

$$P_H = -\frac{1}{4\pi b^2}\frac{\partial}{\partial b}\sum_{l=1}^{\infty}\nu\int\frac{d\zeta}{2\pi}\ln\sigma_H(a,b),\ \ \sigma_H(a,b) = 1 - \frac{\hat{\mathfrak{h}}'_+(b)\hat{\mathfrak{h}}'_-(a)}{\hat{\mathfrak{h}}'_-(b)\hat{\mathfrak{h}}'_+(a)}. \tag{2.109b}$$

In our situation, $\hat{\mathfrak{e}}_+(r)$ and $\hat{\mathfrak{h}}_+(r)$ are solved as

$$\hat{\mathfrak{e}}_\pm(\zeta,l;r) = r^{\frac{1}{2}\mp\sqrt{\nu^2+\lambda\zeta^2}},\ \ \hat{\mathfrak{h}}_\pm(\zeta,l;r) = r^{-\frac{1}{2}\mp\sqrt{\nu^2+\lambda\zeta^2}}, \tag{2.110}$$

which means P_E and P_H are

$$P_E = P_H = -\frac{1}{2\pi^2 b^3\sqrt{\lambda}}\sum_{l=1}^{\infty}\nu^3\int_0^\infty d\zeta\frac{\sqrt{1+\zeta^2}d^{2\nu\sqrt{1+\zeta^2}}}{1-d^{2\nu\sqrt{1+\zeta^2}}}, \tag{2.111}$$

where $d = a/b \in (0,1)$. In the limit $d \to 0$, we have

$$\begin{aligned} P_E = P_H &\approx -\frac{9}{16\pi^2 b^3\sqrt{\lambda}}\left[3K_0(-3\ln d) - \frac{K_{-1}(-3\ln d)}{\ln d}\right] \\ &\to -\frac{9}{16\pi^2 b^3\sqrt{\lambda}}\frac{2695\sqrt{\pi/6}}{2359296}\frac{d^3}{|\ln d|^{\frac{13}{2}}} \approx 0,\ d \to 0. \end{aligned} \tag{2.112}$$

In the limit $d \to 1$, we have

$$\begin{aligned} P_E = P_H &\approx -\frac{1}{2\pi^2 b^3\sqrt{\lambda}}\sum_{l=1}^{\infty}\nu\int_0^\infty d\zeta\frac{\sqrt{\nu^2+\zeta^2}e^{-2(1-d)\sqrt{\nu^2+\zeta^2}}}{1-e^{-2(1-d)\sqrt{\nu^2+\zeta^2}}} \\ &\to -\frac{b/\sqrt{\lambda}}{2\pi^2(b-a)^4}\int_0^\infty d\kappa\kappa^3\frac{e^{-2\kappa}}{1-e^{-2\kappa}}, \end{aligned} \tag{2.113}$$

which is consistent with the results in Eq. (2.56). It agrees with the PFA argument and the intuition, of course. To show the general effects of inhomogeneity, we briefly compare the interaction energy of the inverse square SGCC

$$U_E = U_H = \frac{1}{\pi}\sum_{l=1}^{\infty}\frac{\nu^2}{\sqrt{\lambda}}\int_0^\infty d\zeta\ln\left[1 - d^{2\nu\sqrt{1+\zeta^2}}\right], \tag{2.114a}$$

with the TE and TM interaction energies of the vacuum SGCC, in which the TE interaction energy is

$$U_E^{(0)} = \frac{1}{\pi}\sum_{l=1}^{\infty}\frac{\nu}{b}\int_0^\infty d\zeta\ln\left[1 - \frac{e_l(\zeta)s_l(\zeta d)}{s_l(\zeta)e_l(\zeta d)}\right], \tag{2.114b}$$

and the TM contribution $U_H^{(0)}$ is obtained by making the substitution $X \to X'$, $X = e, s$. The results are demonstrated in Figure 2.3, which does not violate Eq. (2.113), though further numerical calculations are needed for this case and more complicated cases. Definitely, much more work should be done in this novel field.

2.5 Summary

In this chapter, we briefly outlined research on Casimir energies and stresses, which may be the earliest objects studied in the Casimir physics. For planar systems composed of homogeneous media, even with dispersion included, there are widely accepted regularization and renormalization schemes to obtain physically measurable results, lots of which have been verified experimentally. However, for spherical systems, there is no well-recognized method to get renormalized Casimir energies and stresses, except for a few special cases. We show the basic results for homogeneous planar and spherical systems. We also study the elementary concentric systems and show that they approach their planar counterparts when curvature effects can be ignored. When media of a system are inhomogeneous, problems are much more complicated. We derive our previous results about divergent properties of the Casimir energy densities and stress tensors in inhomogeneous two-media backgrounds. The first step toward achieving further insight into the inhomogeneous parallel configurations is given. We also demonstrate our first systematic considerations about the inhomogeneous spherical systems. More thorough investigations are in progress.

Of course, studies on the Casimir energies and stresses are not limited to the planar and spherical geometries [133, 134, 135], the inhomogeneity can be in the directions other than the "longitudinal" direction, and even the anisotropy could be included [136, 137, 138, 139]. Moreover, it is worth while to consider the microscopic structures of materials explicitly when investigating the Casimir energies and stresses. All those factors merit diversified possible applications.

Chapter 3

Thermal Casimir effects

3.1 Background

As stated above, the thermal corrections to the Casimir forces are important and various experiments have resulted in controversies about the properties of the media in the low-frequency domain at finite temperature. As an issue related to those debates, the negativity of the Casimir interaction entropies and its consistency with the third law of thermodynamics have been investigated in detail. Recently, we explored the Casimir self-entropies in some ideal models. There exist many more unsolved problems about the Casimir self-entropies than those solved. Originally, the Casimir self-entropies were once thought to be only a theoretical subject, anticipated to compensate the negative interaction entropies and render the total entropy positive. However, the even more fascinating aspect of the Casimir self-entropy is its implications for realistic phenomena [51]. Since the ground state may be nontrivial because of constraints, it is not surprising that the corresponding thermodynamic response of the system, characterized by the Casimir entropy, is highly nontrivial.

Until now, studies on the thermal Casimir effects are executed for systems in thermodynamic equilibrium. Some efforts have been put into the explorations on Casimir effects in nonequilibrium systems [63, 96, 64, 65], but more needed in this nascent field.

3.2 General theory

As we haven mentioned, the action of the electromagnetic system in the Euclidean space satisfies

$$iS = \int d^4x_E \frac{\mathbf{E}\cdot\mathbf{D} - \mathbf{H}\cdot\mathbf{B}}{2} = \frac{1}{2}\int d^4y_1 d^4y_2 \mathbf{E}^*(y_1)\cdot\mathbf{\Gamma}^{-1}(y_1, y_2)\cdot\mathbf{E}(y_2), \quad (3.1)$$

where at any imaginary frequency-space point $y = (\zeta, \mathbf{r})$ the permittivity $\boldsymbol{\varepsilon}$ and permeability $\boldsymbol{\mu}$ are defined with $\mathbf{D} = \boldsymbol{\varepsilon}\cdot\mathbf{E},\ \mathbf{B} = \boldsymbol{\mu}\cdot\mathbf{H}$, and the definition of the operator $\mathbf{\Gamma}^{-1}(y_1, y_2)$ and $\mathbf{\Gamma}(y_1, y_2)$ are given in Eq. (2.5). Therefore, the corresponding generating functional, or quantum partition function, Z is expressed as

$$\begin{aligned} Z &= \int D\mathbf{E}^*(y)D\mathbf{E}(y)\exp\left[\frac{1}{2}\int d^4y_1 d^4y_2 \mathbf{E}^*(y_1)\cdot\mathbf{\Gamma}^{-1}(y_1, y_2)\cdot\mathbf{E}(y_2)\right] \\ &= C_\infty \exp\left[\frac{\delta(0)}{2}\int d\zeta \mathrm{Tr}\ln\mathbf{\Gamma}_\zeta(\mathbf{r}, \mathbf{r}')\right], \end{aligned} \quad (3.2)$$

where C_∞ is an physically irrelevant constant normalization coefficient and $\mathbf{\Gamma}_\zeta(\mathbf{r}, \mathbf{r}')$ is defined in Eq. (2.6a). We also know that $\beta_0 = \int dt = 2\pi\delta(0)$ and the partition functional Z at zero temperature can be expressed with the energy U as $Z \propto e^{-\beta_0 E}$, which means U, in a static situation, is

$$U = -\frac{1}{2}\int \frac{d\zeta}{2\pi}\mathrm{Tr}\ln\mathbf{\Gamma}_\zeta(\mathbf{r}, \mathbf{r}'). \quad (3.3)$$

On the other hand, the nonzero temperature partition function is obtained, by taking the periodic condition into account, as

$$\begin{aligned} Z &= \int D\mathbf{E}_n^* D\mathbf{E}_n \exp\left[-\frac{1}{2}\sum_{n=-\infty}^{\infty}\int d^3x \mathbf{E}_n^*\cdot\mathbf{\Gamma}_n^{-1}\cdot\mathbf{E}_n\right] \\ &= C_\beta \exp\left(\frac{1}{2}\sum_{n=-\infty}^{\infty}\mathrm{Tr}\ln\mathbf{\Gamma}_n\right), \end{aligned} \quad (3.4)$$

where $\beta = 1/T$ is the inverse of the temperature T and, by defining the Matsubara frequency $\zeta_n = 2\pi nT$, we have $\mathbf{\Gamma}_n(\mathbf{r}, \mathbf{r}') = \mathbf{\Gamma}_{\zeta_n}(\mathbf{r}, \mathbf{r}')$. So the free energy expressed with the partition function is

$$F = -T\ln Z = -\frac{T}{2}\sum_{n=-\infty}^{\infty}\mathrm{Tr}\ln\mathbf{\Gamma}_n \Rightarrow T\to 0, T\sum_{n=-\infty}^{n}\to\int\frac{d\zeta}{2\pi}, F\to U, \quad (3.5)$$

which is consistent with the law of thermodynamics and the definition of Helmholtz free energy $F = U - TS$, i.e., when the temperature is zero, the free energy F is just the energy of the system U as in Eq. (3.3).

From here in this section, we use the integral representation instead of the summation in Eq. (3.5) as the default setting for simplicity. Suppose $\mathbf{\Gamma}_\zeta^{-1}(\mathbf{r},\mathbf{r}')$ could be separated into two parts, i.e., $\mathbf{\Gamma}_\zeta^{-1}(\mathbf{r},\mathbf{r}') = \mathbf{\Gamma}_{0;\zeta}^{-1}(\mathbf{r},\mathbf{r}') + \mathbf{V}_\zeta(\mathbf{r},\mathbf{r}')$, which means $\mathbf{\Gamma}_\zeta = (\mathbf{1}+\mathbf{\Gamma}_{0;\zeta}\cdot\mathbf{V}_\zeta)^{-1}\cdot\mathbf{\Gamma}_{0;\zeta}$ and the extra free energy introduced by the potential $\mathbf{V}$ is

$$\begin{aligned}\Delta F &= -\frac{1}{2}\int\frac{d\zeta}{2\pi}\mathrm{Tr}\ln\mathbf{\Gamma}_\zeta(\mathbf{r},\mathbf{r}') + \frac{1}{2}\int\frac{d\zeta}{2\pi}\mathrm{Tr}\ln\mathbf{\Gamma}_{0;\zeta}(\mathbf{r},\mathbf{r}') \\ &= \frac{1}{2}\int\frac{d\zeta}{2\pi}\mathrm{Tr}\ln(\mathbf{1}+\mathbf{\Gamma}_{0;\zeta}\cdot\mathbf{V}_\zeta). \end{aligned} \quad (3.6)$$

When separating $\mathbf{V}_\zeta$ into two parts as $\mathbf{V}_\zeta = \mathbf{V}_{1;\zeta}+\mathbf{V}_{2;\zeta}$, then ΔF is

$$\Delta F = \frac{1}{2}\int\frac{d\zeta}{2\pi}\mathrm{Tr}\ln(\mathbf{1}+\mathbf{\Gamma}_{0;\zeta}\cdot\mathbf{V}_\zeta) = \Delta F_1 + \Delta F_2 + F_{12}, \quad (3.7)$$

where, by defining $\mathbf{\Gamma}_{i;\zeta} = (\mathbf{1}+\mathbf{\Gamma}_{0;\zeta}\cdot\mathbf{V}_{i;\zeta})^{-1}\cdot\mathbf{\Gamma}_{0;\zeta}$, the self-free energies ΔF_i and interaction free energy F_{12} are

$$\Delta F_i = \frac{1}{2}\int\frac{d\zeta}{2\pi}\mathrm{Tr}\ln(\mathbf{1}+\mathbf{\Gamma}_{0;\zeta}\cdot\mathbf{V}_{i;\zeta}), \quad i=1,2, \quad (3.8a)$$

$$F_{12} = \frac{1}{2}\int\frac{d\zeta}{2\pi}\mathrm{Tr}\ln(\mathbf{1}-\mathbf{\Gamma}_{1;\zeta}\cdot\mathbf{V}_{1;\zeta}\cdot\mathbf{\Gamma}_{2;\zeta}\cdot\mathbf{V}_{2;\zeta}). \quad (3.8b)$$

By introducing the scattering matrix $\mathbf{T}_{i;\zeta} = \mathbf{V}_{i;\zeta}\cdot(\mathbf{1}+\mathbf{\Gamma}_{0;\zeta}\cdot\mathbf{V}_{i;\zeta})^{-1}$, F_{12} can be written in terms of the famous TGTG formula [140]

$$F_{12} = \frac{1}{2}\int\frac{d\zeta}{2\pi}\mathrm{Tr}\ln(\mathbf{1}-\mathbf{\Gamma}_{0;\zeta}\cdot\mathbf{T}_{1;\zeta}\cdot\mathbf{\Gamma}_{0;\zeta}\cdot\mathbf{T}_{2;\zeta}), \quad (3.9)$$

where $\mathbf{T}_{i;\zeta}$ is usually expanded as follows to facilitate the calculation process

$$\mathbf{T}_{i;\zeta} = \mathbf{V}_{i;\zeta} - \mathbf{V}_{i;\zeta} \cdot \mathbf{\Gamma}_{0;\zeta} \cdot \mathbf{V}_{i;\zeta} + \mathbf{V}_{i;\zeta} \cdot \mathbf{\Gamma}_{0;\zeta} \cdot \mathbf{V}_{i;\zeta} \cdot \mathbf{\Gamma}_{0;\zeta} \cdot \mathbf{V}_{i;\zeta} - \cdots . \tag{3.10}$$

The pure thermodynamical quantity is the entropy, which is derived from the relation $S = -\partial F/\partial T$. One can obtain other thermodynamical quantities from the Helmholtz free energy.

3.3 Thermal Casimir forces

Here the thermal corrections to Casimir forces are roughly sketched. Except for the planar geometry, the thermal Casimir forces in spherical configurations are also considered. The thermal corrections in inhomogeneous systems are roughly demonstrated.

3.3.1 Parallel configurations

For the simplicity, consider the temperature correction to the Casimir force in the original Casimir configuration, which means the internal energy per unit area and pressure at finite temperature $T \neq 0$ are

$$\frac{\Delta U}{A} = \frac{T}{\pi}\sum_{n=1}^{\infty} \zeta_n^2 \ln(1 - e^{-4\zeta_n a}),\ \mathcal{F}_a = \frac{-1}{32\pi}\frac{T}{a^3}\left(\zeta(3) + \sum_{n=1}^{\infty}\int_{4\zeta_n a}^{\infty}\frac{dx x^2}{e^x - 1}\right). \tag{3.11}$$

In the low-temperature (low-T) limit $T \to 0$, we have

$$\frac{\Delta U}{A} \approx \frac{-\pi^2}{5760a^3} + \frac{\zeta(3)}{\pi}T^3 - \frac{2\pi^2}{15}T^4 a + \frac{16\pi^3}{45}T^5 a^2, \tag{3.12a}$$

$$\begin{aligned} \mathcal{F}_a &= \frac{-1}{256\pi^2 a^4}\int_0^{\infty} dn \int_n^{\infty}\frac{dx x^2}{e^x - 1} + \frac{-1}{256\pi^2 a^4}\sum_{k=1}^{\infty}\frac{(8\pi T a)^{2k} B_{2k}}{(2k)!} f^{(2k-2)}(0) \\ &= \frac{-\pi^2}{3840a^4} - \frac{\pi^2}{45}T^4, \quad f(n) = \frac{n^2}{e^n - 1}, \end{aligned} \tag{3.12b}$$

which agrees well with the results in Eq. (2.53) and the definition of Helmholtz free energy. In the high-temperature (high-T) limit $T \to \infty$, they are

$$\frac{\Delta U}{A} \approx -4\pi T^3 e^{-8\pi T a},\ \mathcal{F}_a \approx -\frac{\zeta(3)}{32\pi a^3}T - \frac{2\pi}{a}T^3 e^{-8\pi a T}. \tag{3.13}$$

Consider the finite temperature correction to the Casimir force of the configuration defined in Eq. (2.57), which means for the nondispersive and diaphanous case, in the low-T limit $T \to 0$ we have

$$\mathcal{F}_E^{T\to 0} = \mathcal{F}_H^{T\to 0} \to \frac{-3}{16\pi^2\sqrt{\varepsilon_2\mu_2}(b-a)^4}\mathrm{Li}_4\left[\frac{(\mu_2-\mu_3)(\mu_2-\mu_1)}{(\mu_2+\mu_3)(\mu_1+\mu_2)}\right], \tag{3.14a}$$

while in the high-T limit $T \to \infty$

$$\mathcal{F}_E^{T\to \infty} = \mathcal{F}_H^{T\to \infty} \approx \frac{-T}{8\pi(b-a)^3}\mathrm{Li}_3\left[\frac{(\mu_2-\mu_3)(\mu_2-\mu_1)}{(\mu_2+\mu_3)(\mu_1+\mu_2)}\right], \tag{3.14b}$$

in which the exponential decaying corrections are ignored.

Consider one of the analytically solvable inhomogeneous case, where the medium with the permittivity $\varepsilon(z) = \lambda/(c-z)^2$ and permeability $\mu = 1$ is sandwiched between two perfect conductors. The interfaces are at $z = a, z = b$ and $a < b < c$. The TE and TM contributions to the pressure on the interface $z = b$ at zero-temperature (zero-T) are

$$\mathcal{F}_E(b) = -\int \frac{d\zeta d^2k}{16\pi^3}\frac{\partial}{\partial b}\ln\sigma_E(a,b),\ \sigma_E(a,b) = 1 - \frac{\hat{e}_+(b)\hat{e}_-(a)}{\hat{e}_-(b)\hat{e}_+(a)}, \tag{3.15a}$$

$$\mathcal{F}_H(b) = -\int \frac{d\zeta d^2k}{16\pi^3}\frac{\partial}{\partial b}\ln\sigma_H(a,b),\ \sigma_H(a,b) = 1 - \frac{\hat{h}'_+(b)\hat{h}'_-(a)}{\hat{h}'_-(b)\hat{h}'_+(a)}, \tag{3.15b}$$

where $\hat{e}_\pm(z)$ and $\hat{h}'_\pm(z)$ are ($\nu = \sqrt{\lambda\zeta^2 + 1/4}$)

$$\hat{e}_+(z) = \sqrt{c-z}I_\nu[k(c-z)],\ \hat{e}_-(z) = \sqrt{c-z}K_\nu[k(c-z)], \tag{3.15c}$$

$$\hat{h}'_+(z) = \frac{I_\nu[k(c-z)] - 2k(c-z)I'_\nu[k(c-z)]}{2\sqrt{c-z}^3}, \tag{3.15d}$$

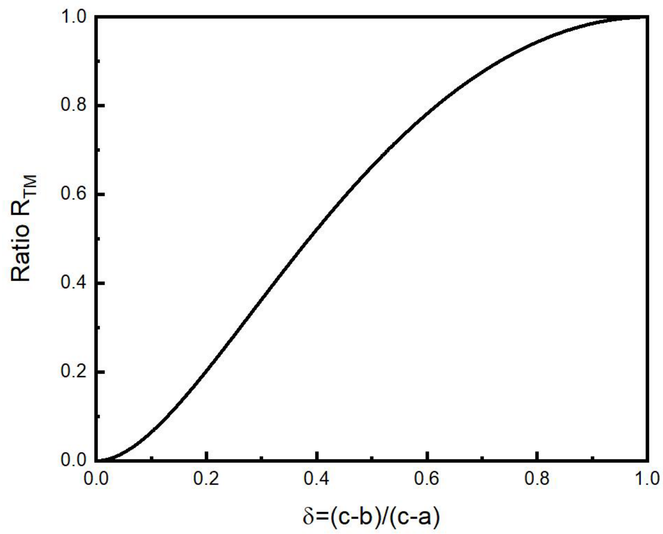

Figure 3.1: The ration between $\mathcal{F}_H^{T\to\infty}(b)$ in Eq. (3.17b) and the its inhomogeneous counterpart in Eq. (3.13), i.e., $R_{\rm TM} = \mathcal{F}_H^{T\to\infty}(b)/[-\zeta(3)T/8\pi(b-a)^3]$, as a function of $\delta = (c-b)/(c-a)$.

$$\hat{h}'_-(z) = \frac{K_\nu[k(c-z)] - 2k(c-z)K'_\nu[k(c-z)]}{2\sqrt{c-z}^3}. \tag{3.15e}$$

For the nonzero temperature $T \neq 0$, the $\mathcal{F}_E(b)$ and $\mathcal{F}_H(b)$ are ($\nu_n^2 = \lambda\zeta_n^2 + 1/4$)

$$\mathcal{F}_E(b) = \frac{T}{4\pi(c-a)^3}\frac{\partial}{\partial\delta}\sum_{n=-\infty}^{\infty}\int_0^\infty dkk\ln\left[1 - \frac{I_{\nu_n}(k\delta)K_{\nu_n}(k)}{I_{\nu_n}(k)K_{\nu_n}(k\delta)}\right], \tag{3.16a}$$

$$\begin{aligned}\mathcal{F}_H(b) &= \frac{T}{4\pi(c-a)^3}\frac{\partial}{\partial\delta}\sum_{n=-\infty}^{\infty}\int_0^\infty dkk\ln\left\{1 - \frac{K_{\nu_n}(k) - 2kK'_{\nu_n}(k)}{I_{\nu_n}(k) - 2kI'_{\nu_n}(k)}\right.\\ &\quad\left.\times\frac{I_{\nu_n}(k\delta) - 2k\delta I'_{\nu_n}(k\delta)}{K_{\nu_n}(k\delta) - 2k\delta K'_{\nu_n}(k\delta)}\right\},\end{aligned} \tag{3.16b}$$

where $\delta = (c-b)/(c-a) \in (0,1)$. In the high-T limit, they are evaluated with the uniform asymptotic expansion as

$$\mathcal{F}_E^{T\to\infty}(b) \to \frac{-T}{4\pi(c-a)^3}\int_0^\infty dkk^2\left\{\coth\left[k(1-\delta)\right] - 1\right\} = \frac{-\zeta(3)T}{8\pi(b-a)^3}, \tag{3.17a}$$

$$\mathcal{F}_H^{T\to\infty}(b) \to \frac{T\delta^2}{2\pi(b-a)^3}\int_0^\infty \frac{dkk^4(1-\delta+k\delta)^{-2}e^{-2k}}{\frac{1-\delta-k}{1-\delta+k} - \frac{(1-\delta)^2-k^2\delta^2}{(1-\delta+k\delta)^2}e^{-2k}} \approx -\frac{\zeta(3)T\delta^2}{8\pi(b-a)^3},\ \delta\to 1. \tag{3.17b}$$

The results of Eq. (3.17) are evidently consistent with that in Eq. (3.13). Figure 3.1 signifies the nontrivial significance of inhomogeneity. More attention should be paid to the influences of the inhomogeneity on the thermal corrections.

3.3.2 Concentric configurations

For the finite-temperature situation, consider the TE and TM contributions to the pressure on the interface $r = b$ in the case of two concentric perfectly conducting spheres separated by the vacuum studied in Eq. (2.114b), i.e.,

$$\mathcal{F}_{(E,H)} = -\frac{T}{4\pi b^2}\frac{\partial}{\partial b}\sum_{l=1}^{\infty}\nu\sum_{n=-\infty}^{\infty}\ln\sigma_{(E,H),n}(a,b), \tag{3.18a}$$

where $\sigma_E(a,b)$ and $\sigma_H(a,b)$ are

$$\sigma_{E,n}(a,b) = 1 - \frac{e_l(|\zeta_n|b)s_l(|\zeta_n|a)}{s_l(|\zeta_n|b)e_l(|\zeta_n|a)},\ \ \sigma_{H,n}(a,b) = 1 - \frac{e_l'(|\zeta_n|b)s_l'(|\zeta_n|a)}{s_l'(|\zeta_n|b)e_l'(|\zeta_n|a)}. \tag{3.18b}$$

In the low-T limit, the temperature-dependent corrections to the zero-T results are trivial, namely $\Delta\mathcal{F}_E, \Delta\mathcal{F}_H \approx 0$, and do not depend on the temperature polynomially. In the high-T limit, then $\mathcal{F}_E$ and $\mathcal{F}_H$ are ($d = a/b \in (0,1)$)

$$\mathcal{F}_E \approx \mathcal{F}_H \approx -\frac{T}{4\pi b^2}\frac{\partial}{\partial b}\sum_{l=1}^{\infty}\nu\ln(1-d^{2\nu}) \sim -\frac{\zeta(3)T}{8\pi(b-a)^3},\ \ d\to 1, \tag{3.19}$$

which is consistent with the results in Eq. (3.13).

Consider the pressure on the interface $r = b$ of the inhomogeneous SGCC, as described by Eq. (2.109). Then at finite temperature, the TE and TM contributions to the pressure are

$$\mathcal{F}_E = \mathcal{F}_H = -\frac{T}{4\pi b^2}\frac{\partial}{\partial b}\sum_{l=1}^{\infty}\nu\sum_{n=-\infty}^{\infty}\ln\left[1 - d^{2\sqrt{\nu^2+\lambda\zeta_n^2}}\right], \tag{3.20}$$

which means in the high-T limit $T \to \infty$, they are approximated as

$$\mathcal{F}_E = \mathcal{F}_H \approx \frac{Ta}{4\pi b^4}\frac{\partial}{\partial d}\sum_{l=1}^{\infty}\nu\ln(1-d^{2\nu}) + \frac{\sqrt{\lambda}T^2 d^{4\pi T\sqrt{\lambda}}}{2b^3\ln^2 d}\left(1 - 4\pi T\sqrt{\lambda}\ln d\right). \tag{3.21}$$

Therefore, the inhomogeneity in this case does not affect the high-T pressure significantly, since $d \in (0,1)$ and the last term of Eq. (3.21) decays exponentially as $T \to \infty$.

3.4 Casimir self-entropies

The body is usually modeled as a potential, which means in the case of a dielectric medium, the potential should be $\mathbf{V}_\zeta(\mathbf{r},\mathbf{r}') = [\boldsymbol{\varepsilon}(\zeta,\mathbf{r}) - \mathbf{1}]\delta(\mathbf{r}-\mathbf{r}') = \boldsymbol{\chi}(\zeta,\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')$. So the single-body induced free energy is

$$F = \frac{T}{2}\sum_{n=-\infty}^{\infty} \mathrm{Tr}\ln(\mathbf{1}+\mathbf{V}\cdot\boldsymbol{\Gamma}_{n,b}) = \frac{T}{2}\sum_{n=-\infty}^{\infty} \mathrm{Tr}\ln\left[\mathbf{1}+\boldsymbol{\chi}(\zeta_n,\mathbf{r})\cdot\boldsymbol{\Gamma}_{\zeta_n,b}(\mathbf{r},\mathbf{r})\right], \quad (3.22)$$

based on which we, in this section, evaluate the Casimir self-entropies in two nontrivial configurations, i.e., the planar thin sheet and the spherical shell [45, 46].

3.4.1 Thin sheet

Suppose there is an infinitely thin planar dielectric sheet located at $z = 0$ in the vacuum and its potential $\boldsymbol{\chi}(\zeta)\delta(z)$ is homogeneous, then the free energy induced by this sheet, expressed with g in Eq. (2.13), is

$$F = \frac{T}{2}\sum_{n=-\infty}^{\infty}\int \frac{d^2k}{(2\pi)^2}\mathbf{tr}\ln\left[1+\boldsymbol{\chi}(\zeta_n)\cdot\mathbf{g}_{\zeta_n,\mathbf{k};b}(z,z)\right]. \quad (3.23)$$

Further assume [141] that $\boldsymbol{\chi}(\zeta) = \mathrm{diag}[\chi(\zeta),\chi(\zeta),0]$, then the TE and TM contributions to the free energy per unit area F_E and F_H is

$$F_E = \frac{T}{4\pi}\sum_{n=-\infty}^{\infty} e^{i\zeta_n\tau}\int_0^\infty dk k J_0(k\delta)\ln\left[1+\chi(\zeta_n)\frac{\zeta_n^2}{2\kappa_n}\right], \quad (3.24a)$$

$$F_H = \frac{T}{4\pi}\sum_{n=-\infty}^{\infty} e^{i\zeta_n\tau}\int_0^\infty dk k J_0(k\delta)\ln\left[1+\chi(\zeta_n)\frac{\kappa_n}{2}\right], \quad (3.24b)$$

where $\kappa_n = \sqrt{k^2+\zeta_n^2}$, τ and δ are regulators which are set to zero at the end of the calculation.

Consider the plasma model, in which $\chi(\zeta) = 2\lambda_0/\zeta^2$, then F_E is

$$\begin{aligned} F_E &= \frac{\lambda_0^2 T}{4\pi\delta_\lambda}\int_0^\infty dk\frac{J_1(k)}{k+\delta_\lambda} + \frac{\lambda_0^2 T}{2\pi\delta_\lambda}\frac{e^{-\rho\delta_\lambda}}{1-e^{-\rho\delta_\lambda}} - \frac{\lambda_0^2 T}{4\pi}\sum_{n=1}^\infty K_0(n\rho\delta_\lambda) \\ &+ \frac{\lambda_0^2 T}{2\pi\delta_\lambda}\sum_{m=2}^\infty \frac{(-\delta_\lambda)^m}{2^{\frac{m+1}{2}}\Gamma(\frac{m+3}{2})}\sum_{n=1}^\infty \frac{K_{\frac{1-m}{2}}(n\rho\delta_\lambda)}{(n\rho\delta_\lambda)^{\frac{m-1}{2}}}, \end{aligned} \tag{3.25a}$$

where $\rho = 2\pi T/\lambda_0$, $\delta_\lambda = \lambda_0\delta$, and τ is set to zero. To keep the nontrivial terms with $\delta_\lambda \to 0$, the first three terms of F_E in Eq. (3.25a)

$$F_{E,1} = \frac{\lambda_0^3}{4\pi^2\delta_\lambda^2} - \frac{\lambda_0^3}{16\pi\delta_\lambda} - \frac{\lambda_0^3}{32\pi^2}\rho + \frac{\lambda_0^3}{48\pi^2}\rho^2 - \frac{\lambda_0^3}{8\pi^2}\rho\frac{\ln\rho}{2} + \frac{\lambda_0^3}{8\pi^2}\rho\frac{\ln(2\pi)}{2}, \tag{3.25b}$$

while the last term is

$$\begin{aligned} F_{E,2} &= -\frac{\lambda_0^3}{4\pi^2}\frac{\ln(\rho\delta_\lambda)}{3} + \frac{\lambda_0^3}{4\pi^2}\left[\frac{1}{4} - \frac{\gamma_E}{3} + \frac{\rho}{8} - \frac{\rho^2}{24} - \frac{\ln(2\pi)}{4}\rho + \frac{\zeta(3)}{8\pi^2}\rho^3\right] \\ &+ \frac{\lambda_0^3}{4\pi^2}\left[\frac{\rho^3}{2}\zeta^{(1,0)}\Big(-2, 1+\frac{1}{\rho}\Big) - \rho^2\zeta^{(1,0)}\Big(-1, 1+\frac{1}{\rho}\Big)\right], \end{aligned} \tag{3.25c}$$

where $\zeta(x,y)$ is Hurwitz zeta function. Then the total TE free energy $F_E = F_{E,1} + F_{E,2}$ is

$$\begin{aligned} F_E &= F_E^{(0)} - \frac{\lambda_0^3}{12\pi^2}\ln\rho - \frac{\lambda_0^3}{16\pi^2}\rho\ln\rho + \frac{\lambda_0^3}{96\pi^2}\rho^2 + \frac{\lambda_0^3}{32\pi^4}\zeta(3)\rho^3 \\ &+ \frac{\lambda_0^3}{4\pi^2}\left[\frac{\rho^3}{2}\zeta^{(1,0)}\Big(-2, 1+\frac{1}{\rho}\Big) - \rho^2\zeta^{(1,0)}\Big(-1, 1+\frac{1}{\rho}\Big)\right], \end{aligned} \tag{3.25d}$$

where the temperature-independent part is

$$F_E^{(0)} = \frac{\lambda_0^3}{4\pi^2\delta_\lambda^2} - \frac{\lambda_0^3}{16\pi\delta_\lambda} - \frac{\lambda_0^3}{4\pi^2}\frac{\ln\delta_\lambda}{3} + \frac{\lambda_0^3}{4\pi^2}\left(\frac{1}{4} - \frac{\gamma_E}{3}\right). \tag{3.25e}$$

The TM contribution to the free energy F_H is

$$\begin{aligned} F_H &= \frac{\lambda_0^3\rho}{8\pi^2}\sum_{n=-\infty}^\infty e^{in\rho\tau_\lambda}\int_0^\infty dkkJ_0(k\delta_\lambda)\ln\left[1+\frac{n^2\rho^2}{\sqrt{k^2+n^2\rho^2}}\right] \\ &+ \frac{\lambda_0^3\rho}{8\pi^2}\sum_{n=-\infty}^\infty e^{in\rho\tau_\lambda}\lim_{x\to 0}\frac{d}{dx}\int_0^\infty dkkJ_0(k\delta_\lambda)(k^2+n^2\rho^2)^{\frac{x}{2}}, \end{aligned} \tag{3.26a}$$

where both δ and τ are used, and $\tau_\lambda = \lambda_0 \tau$. To keep the nontrivial terms with $\delta_\lambda, \tau_\lambda \to 0$, the second term of F_H in Eq. (3.26a) is

$$F_{H,2} = -\frac{\lambda_0^3 \rho}{8\pi^2 \delta_\lambda^2} \sum_{n=-\infty}^{\infty} |n|\rho\delta_\lambda K_1(|n|\rho\delta_\lambda) = -\frac{\lambda_0^3}{8\pi\delta_\lambda^3} - \frac{\lambda_0^3\rho^3}{4\pi^2}\frac{\zeta(3)}{8\pi^2}, \qquad (3.26b)$$

while the first term is

$$\begin{aligned} F_{H,1} &= \frac{\lambda_0^3}{2\pi^2}\sum_{m=2}^{\infty}\frac{(-2)^m\Gamma(\frac{m+4}{2})\Gamma(\frac{2m+3}{2})}{\delta_\lambda^{m+4}\Gamma(\frac{m+3}{2})} - \frac{9\lambda_0^3}{16\pi\delta_\lambda^5} + \frac{\lambda_0^3}{2\pi^2\delta_\lambda^4} + \frac{2\lambda_0^3}{225\pi^2} \\ &+ \frac{\lambda_0^3}{8\pi^2}\left[\frac{2}{15}\ln\rho + \frac{\rho}{8} + \frac{\rho^2}{18} - \frac{\zeta(3)}{4\pi^2}\rho^3 - \frac{3\zeta(5)}{2\pi^4}\rho^5 \right. \qquad (3.26c) \\ &\left. + 2\rho^2\psi^{(-2)}(\rho^{-1}) - 10\rho^3\psi^{(-3)}(\rho^{-1}) + 24\rho^4\psi^{(-4)}(\rho^{-1}) - 24\rho^5\psi^{(-5)}(\rho^{-1})\right], \end{aligned}$$

where $\psi^{(n)}(x)$ is the polygamma function. Then the total TM free energy $F_H = F_{H,1} + F_{H,2}$ is

$$\begin{aligned} F_H &= F_H^{(0)} + \frac{\lambda_0^3}{8\pi^2}\left[\frac{2}{15}\ln\rho + \frac{\rho}{8} + \frac{\rho^2}{18} - \frac{\zeta(3)}{2\pi^2}\rho^3 - \frac{3\zeta(5)}{2\pi^4}\rho^5 + 2\rho^2\psi^{(-2)}(\rho^{-1}) \right. \\ &\left. - 10\rho^3\psi^{(-3)}(\rho^{-1}) + 24\rho^4\psi^{(-4)}(\rho^{-1}) - 24\rho^5\psi^{(-5)}(\rho^{-1})\right], \qquad (3.26d) \end{aligned}$$

where the temperature-independent term is

$$F_H^{(0)} = \frac{4}{225} - \frac{\pi}{4\delta_\lambda^3} + \frac{1}{\delta_\lambda^4} - \frac{9\pi}{8\delta_\lambda^5} + \frac{\lambda_0^3}{2\pi^2}\left[\sum_{m=2}^{\infty}\frac{(-2)^m\Gamma(\frac{m+4}{2})\Gamma(\frac{2m+3}{2})}{\delta_\lambda^{m+4}\Gamma(\frac{m+3}{2})}\right]. \qquad (3.26e)$$

The corresponding entropies in terms of their reduced forms $s_X = 4\pi S_X/\lambda_0^2$ are expressed as

$$\begin{aligned} s_E &= -\frac{\rho}{6} - \frac{3\zeta(3)}{4\pi^2}\rho^2 + \frac{1}{2} + \frac{1}{2}\ln\rho + \frac{2}{3\rho} - 3\rho^2\zeta^{(1,0)}(-2, 1+\rho^{-1}) \qquad (3.27a) \\ &+ 4\rho\zeta^{(1,0)}(-1, 1+\rho^{-1}) + \rho\zeta^{(1,1)}(-2, 1+\rho^{-1}) - 2\zeta^{(1,1)}(-1, 1+\rho^{-1}), \end{aligned}$$

$$\begin{aligned} s_H &= -\frac{2}{15\rho} - \frac{1}{8} - \frac{1}{9}\rho + \frac{3\zeta(3)}{2\pi^2}\rho^2 + \frac{15\zeta(5)}{2\pi^4}\rho^4 + 2\ln\Gamma(\rho^{-1}) - 14\rho\psi^{(-2)}(\rho^{-1}) \\ &+ 54\rho^2\psi^{(-3)}(\rho^{-1}) - 120\rho^3\psi^{(-4)}(\rho^{-1}) + 120\rho^4\psi^{(-5)}(\rho^{-1}). \qquad (3.27b) \end{aligned}$$

In the low-temperature or strong-coupling limit, i.e., $\rho \to 0$, s_E and s_H behave as

$$s_E = -\frac{3\zeta(3)}{4\pi^2}\rho^2 + \frac{\rho^3}{45} - \frac{\rho^5}{315} + \frac{\rho^7}{525} + o(\rho^9), \tag{3.28a}$$

$$s_H = \frac{3\zeta(3)}{4\pi^2}\rho^2 + \frac{\rho^3}{15} + \frac{15\zeta(5)}{4\pi^2}\rho^4 + \frac{\rho^5}{63} - \frac{\rho^7}{225} + o(\rho^9), \tag{3.28b}$$

while in the high temperature or weak coupling limit, i.e., $\rho \to \infty$, they are of the forms

$$s_E = -\frac{\rho}{3} + \frac{3 - 2\ln 2\pi}{4} + \frac{\ln \rho}{2} + \frac{2}{3\rho} + o(\rho^{-2}), \tag{3.29a}$$

$$s_H = \frac{15\zeta(5)}{2\pi^2}\rho^4 + \frac{3\zeta(3)}{2\pi^2}\rho^2 - \frac{\rho}{9} + \frac{1}{8} - \frac{2}{15\rho} + o(\rho^{-2}). \tag{3.29b}$$

The third law of thermodynamics is satisfied for both TE and TM mode according to Eq. (3.28), in that the entropy vanishes as temperature approaching zero. Although Eq. (3.27a) shows that the s_E is negative for any ρ, the contribution from the TM mode is always positive, whose absolute value is larger than that of s_E. So the total entropy $s = s_E + s_H$ of the single sheet described by the plasma model is always positive, which is just as expected. When the Drude model, $\lambda(\zeta_n) = 2\lambda_0/(\zeta_n^2 + \gamma\zeta_n)$, is used, only the $n = 0$ term of F_E is not present, which results in a divergent contribution to the total entropy for small damping factor $\gamma \to 0$, which may imply some deficiency of the Drude model.

3.4.2 Spherical shell

Suppose there is an infinitely thin spherical shell in the vacuum with its center located at $\mathbf{r} = \mathbf{0}$ and radius a, and its potential is $\boldsymbol{\chi}(\zeta, \mathbf{r}) = \lambda(\mathbf{1} - \hat{\mathbf{r}}\hat{\mathbf{r}})\delta(r - a)$, then the free energy $F = F_E + F_H$ induced by this shell could be expressed with g in Eq. (2.30) in terms of the TE and TM contributions F_E and F_H, in which F_E and F_H are

$$\begin{aligned}
F_E &= \frac{T}{2}\sum_{n=-\infty}^{\infty} e^{i\zeta_n\tau}\sum_{l=1}^{\infty}(2l+1)P_l(\cos\delta)\ln\left[1-\lambda\zeta_n^2a^2g^E_{\zeta_n,l}(a,a)\right]\\
&= \frac{T}{2}\sum_{n=-\infty}^{\infty} e^{i\zeta_n\tau}\sum_{l=1}^{\infty}(2l+1)P_l(\cos\delta)\ln\left[1+\lambda|\zeta_n|e_l(|\zeta_n|a)s_l(|\zeta_n|a)\right], \quad (3.30a)
\end{aligned}$$

$$\begin{aligned}
F_H &= \frac{T}{2}\sum_{n=-\infty}^{\infty} e^{i\zeta_n\tau}\sum_{l=1}^{\infty}(2l+1)P_l(\cos\delta)\ln\left[1+\lambda\frac{\partial^2 rr' g^H_{\zeta_n,l}(r,r')}{\partial r\partial r'}\right]_{r=r'=a}\\
&= \frac{T}{2}\sum_{n=-\infty}^{\infty} e^{i\zeta_n\tau}\sum_{l=1}^{\infty}(2l+1)P_l(\cos\delta)\ln\left[1-\lambda|\zeta_n|e_l'(|\zeta_n|a)s_l'(|\zeta_n|a)\right]. \quad (3.30b)
\end{aligned}$$

Consider the regularized plasma model, in which $\chi(\zeta) = \lambda_0/(\zeta^2 a + \mu^2 a)$ and the regulator μ satisfies $\mu \to 0$, then F_E and F_H are expressed as

$$F_E = \frac{T}{2}\sum_{n=-\infty}^{\infty} e^{in\alpha\tau_a}\sum_{l=1}^{\infty}(2l+1)P_l(\cos\delta)\ln\left[1+\lambda_0\frac{\alpha|n|e_l(\alpha|n|)s_l(\alpha|n|)}{\alpha^2n^2+\mu_a^2}\right], \quad (3.31a)$$

$$F_H = \frac{T}{2}\sum_{n=-\infty}^{\infty} e^{in\alpha\tau_a}\sum_{l=1}^{\infty}(2l+1)P_l(\cos\delta)\ln\left[1-\lambda_0\frac{\alpha|n|e_l'(\alpha|n|)s_l'(\alpha|n|)}{\alpha^2n^2+\mu_a^2}\right], \quad (3.31b)$$

where $\alpha = 2\pi aT$, $\tau_a = \tau/a$ and $\mu_a = \mu a$. In the weak coupling limit $\lambda_0 \to 0$, the free energies are

$$\begin{aligned}
F_E^{\lambda_0\to 0} &= \lambda_0\frac{T}{2}\sum_{n=-\infty}^{\infty} e^{in\alpha\tau_a}\sum_{l=1}^{\infty}(2l+1)P_l(\cos\delta)\frac{e_l(\alpha|n|)s_l(\alpha|n|)}{\alpha|n|}\\
&= \frac{\lambda_0}{2\pi a}\left[\frac{1}{u^2}+\frac{\ln(\tau_a/2)}{2}+\frac{\alpha^2}{12}-\frac{1}{2}\ln\frac{\sinh(\alpha)}{\alpha}\right], \quad (3.32a)
\end{aligned}$$

$$\begin{aligned}
F_H^{\lambda_0\to 0} &= \frac{\lambda_0}{2\pi a}\left[\frac{4+u^2}{-u^3}\frac{\pi^2}{24\alpha}+\frac{1}{u^2}+\frac{\ln(\tau_a/2u^2)}{2}-\left(1+\frac{1}{2u}+\frac{4+u^2}{4u^3\mu_a^2}\right)\frac{\alpha}{2}\right]\\
&\quad+\frac{\lambda_0}{2\pi a}\left[\frac{\alpha^2}{36}+\frac{1}{2}\ln\frac{\sinh(\alpha)}{\alpha}\right], \quad (3.32b)
\end{aligned}$$

where $u = \sqrt{2-2\cos\delta}$. The first line of Eq. (3.32b) is not consistent with the third law of thermodynamics, which strongly suggest that it should be ignored. To justify this, we can rewrite Eq. (3.32) as

$$F_E^{\lambda_0\to 0} = \Lambda_{(1)}^{\mathrm{TE}} - \frac{\lambda_0\alpha}{\pi a}\sum_{l=1}^{\infty}(2l+1)\int_0^{\infty} dn \frac{\mathrm{Im} w_l(0,0;in\alpha)}{e^{2\pi n}-1}, \tag{3.33a}$$

$$F_H^{\lambda_0\to 0} = \Lambda_{(1)}^{\mathrm{TM}} + \frac{\lambda_0\alpha}{\pi a}\sum_{l=1}^{\infty}(2l+1)\int_0^{\infty} dn \frac{\mathrm{Im} v_l(0,0;in\alpha)}{e^{2\pi n}-1}. \tag{3.33b}$$

where w_l and v_l are defined as

$$w_l(\tau,\mu;x) = \cos(\tau x)\frac{x e_l(x) s_l(x)}{x^2+\mu^2},\ v_l(\tau,\mu;x) = \cos(\tau x)\frac{x e_l'(x) s_l'(x)}{x^2+\mu^2}, \tag{3.33c}$$

and the Abel-Plana formula is used, the temperature-independent terms are written as

$$\Lambda_{(1)}^{(\mathrm{TE},\mathrm{TM})} = \pm\frac{\lambda_0}{2\pi a}\sum_{l=1}^{\infty}(2l+1)P_l(\cos\delta)\int_0^{\infty} dx (w_l, v_l)(\tau_a,\mu_a;x). \tag{3.33d}$$

Then the temperature-dependent parts of $F_X^{\lambda_0\to 0}$, i.e., $\Delta F_X^{\lambda_0\to 0} = F_X^{\lambda_0\to 0} - \Lambda_{(1)}^{\mathrm{TX}}$, are evaluated as

$$\Delta F_E^{\lambda_0\to 0} = \frac{\lambda_0}{4\pi a}\left(\frac{\alpha^2}{6} - \ln\frac{\sinh\alpha}{\alpha}\right),\ \Delta F_H^{\lambda_0\to 0} = \frac{\lambda_0}{4\pi a}\left(\frac{\alpha^2}{18} + \ln\frac{\sinh(\alpha)}{\alpha}\right), \tag{3.34}$$

which means the corresponding entropies in the weak-coupling limit are obtained with $S = -\partial F/\partial T$ as

$$S_E^{\lambda_0\to 0} = -\frac{\lambda_0}{2}\left(\frac{\alpha}{3} + \frac{1}{\alpha} - \coth\alpha\right),\ S_H^{\lambda_0\to 0} = -\frac{\lambda_0}{2}\left(\frac{\alpha}{9} - \frac{1}{\alpha} + \coth\alpha\right), \tag{3.35}$$

which are consistent with the third law of thermodynamics and negative.

In the strong-coupling limit $\lambda_0 \to \infty$, the free energies are $F_E^{\lambda_0\to\infty} = F_{E,n>0}^{\lambda_0\to\infty}$ and $F_H^{\lambda_0\to\infty} = F_{H,n=0}^{\lambda_0\to\infty} + F_{H,n>0}^{\lambda_0\to\infty}$, in which

$$F_E^{\lambda_0\to\infty} = \frac{\alpha}{\pi a}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\nu P_l(\cos\delta)\cos(\alpha_n\tau_a)\ln\left[\lambda_0\frac{\alpha_n e_l(\alpha_n) s_l(\alpha_n)}{\alpha_n^2+\mu_a^2}\right], \tag{3.36a}$$

$$F_{H,n>0}^{\lambda_0\to\infty} = \frac{\alpha}{\pi a}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\nu P_l(\cos\delta)\cos(\alpha_n\tau_a)\ln\left[-\lambda_0\frac{\alpha_n e_l'(\alpha_n)s_l'(\alpha_n)}{\alpha_n^2+\mu_a^2}\right], \tag{3.36b}$$

where $\nu = l + 1/2$, $\alpha_n = |n|\alpha$ and we have used the fact that the $n = 0$ term of F_E and F_H can be written as

$$F_{E,n=0} = 0,\ \ F_{H,n=0} = \frac{\alpha}{2\pi a}\sum_{l=1}^{\infty}\nu P_l(\cos\delta)\ln\left[1+\frac{\lambda_0}{\mu_a^2}\left(\nu-\frac{1}{4\nu}\right)\right]. \tag{3.36c}$$

It is obvious that $F_{H,n=0}^{\lambda\to\infty}$ is

$$\begin{aligned} F_{H,n=0}^{\lambda\to\infty} &= -\frac{T}{2}\ln\frac{\lambda_0}{\mu_a^2}+T\sum_{l=1}^{\infty}\nu\ln\nu P_l(\cos\delta)-T\frac{1}{4}F\left(\frac{\pi}{2},\cos\frac{\delta}{2}\right)\\ &\quad -\frac{T}{12}\left(6+7\ln 2-3\gamma_E-36\ln G\right), \end{aligned} \tag{3.37}$$

where G is the Glaisher constant and $F(a,x)$ is the elliptic integral of the first kind. The following term

$$F_{X,c}^{\lambda_0\to\infty} = \frac{\alpha}{2\pi a}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\nu P_l(\cos\delta)\cos(\alpha_n\tau_a)\ln\left[\lambda_0\frac{\alpha_n}{2(\alpha_n^2+\mu_a^2)}\right], \tag{3.38a}$$

is common in $F_{E,n>0}^{\lambda_0\to\infty}$ and $F_{H,n>0}^{\lambda_0\to\infty}$ and is evaluated as

$$F_{X,c}^{\lambda_0\to\infty} = -\frac{\alpha}{4\pi a}\ln\frac{\alpha}{\pi\lambda_0}, \tag{3.38b}$$

but when using analytic regulation method, it is

$$F_{X,c}^{\lambda_0\to\infty} = \frac{\alpha}{2\pi a}\sum_{l=1}^{\infty}\nu\sum_{n=1}^{\infty}\ln\frac{\lambda_0}{2\alpha_n} = -\frac{11\alpha}{48\pi a}\ln\frac{\alpha}{\pi\lambda_0}. \tag{3.38c}$$

The sensitivity of the coefficient to the regularization method suggests the $F_{X,c}^{\lambda_0\to\infty}$ term is unphysical and should be ignored. Denote the rest of $F_{X,n>0}^{\lambda_0\to\infty}$ as $\Delta F_{X,n>0}^{\lambda_0\to\infty} = F_{X,n>0}^{\lambda_0\to\infty} - F_{X,c}^{\lambda_0\to\infty}$, then $\Delta F_{n>0}^{\lambda_0\to\infty} = \Delta F_{E,n>0}^{\lambda_0\to\infty} + \Delta F_{H,n>0}^{\lambda_0\to\infty}$ is

$$\Delta F_{n>0}^{\lambda_0\to\infty} = \frac{\alpha}{\pi a}\sum_{l=1}^{\infty}\nu P_l(\cos\delta)\sum_{n=1}^{\infty}\cos(\alpha_n\tau_a)\ln\left[-4e_l(\alpha_n)s_l(\alpha_n)e_l'(\alpha_n)s_l'(\alpha_n)\right]. \tag{3.39}$$

When evaluated with the uniform asymptotic expansion (UAE), the leading term of $\Delta F_{n>0}^{\lambda_0\to\infty}$ is

$$\begin{aligned}\Delta F_{n>0,0}^{\lambda_0\to\infty} &= \frac{\alpha}{2\pi a}\sum_{l=1}^{\infty}\frac{P_l(\cos\delta)}{4\nu} - \frac{3}{64a}\sum_{l=1}^{\infty}P_l(\cos\delta) \\ &\quad -\frac{1}{32a}\left[\frac{2-e^y(y^2-2y+4)-e^{2y}(y^2+2y-2)}{2(e^y-1)^3} - 4\frac{\ln(1-e^{-y})}{y}\right] \\ &\quad +\frac{1}{64a}\left(3-3y\frac{\partial}{\partial y}+y^2\frac{\partial^2}{\partial y^2}\right)\int_0^\infty \frac{dx}{e^{\pi x}-1}\frac{\sin(xy)}{\cos(xy)-\cosh(y)} \\ &\to -\frac{T}{4}\left[\ln(aT)+0.71351\right],\ T\to\infty, \end{aligned} \tag{3.40}$$

where $y=\pi/\alpha$. This means the leading behavior of the self-entropy in the strong-coupling high-T limit is $S\sim 0.25\ln T$.

In the low-temperature limit $T\to 0$, the free energies are

$$\begin{aligned}F_E^{T\to 0} &= \frac{\alpha}{\pi a}\sum_{l=1}^{\infty}\nu P_l(\cos\delta)\sum_{n=1}^{\infty}\cos(\alpha_n\tau_a)\ln\left[1+\lambda_0\frac{\alpha_n e_l(\alpha_n)s_l(\alpha_n)}{\alpha_n^2+\mu_a^2}\right] \\ &\approx \frac{\Lambda_E^{T\to 0}(\delta,\tau_a,\mu_a)}{\pi a} + \frac{(\pi a)^3}{15}T^4\frac{\lambda_0}{\lambda_0+3}, \end{aligned} \tag{3.41a}$$

$$\begin{aligned}F_H^{T\to 0} &= \frac{\alpha}{\pi a}\sum_{l=1}^{\infty}\nu P_l(\cos\delta)\sum_{n=1}^{\infty}\cos(\alpha_n\tau_a)\ln\left[1-\lambda_0\frac{\alpha_n e_l'(\alpha_n)s_l'(\alpha_n)}{\alpha_n^2+\mu_a^2}\right] \\ &\approx \frac{\Lambda_H^{T\to 0}(\delta,\tau_a,\mu_a)}{\pi a} - \frac{2}{15}(\pi a)^3T^4, \quad aT\ll\lambda_0, \end{aligned} \tag{3.41b}$$

which are consistent with the results in Ref. [46].

For the general case, the free energies F_E and F_H are evaluated with the Abel-Plana formula as

$$F_E = \frac{\Lambda_E(\delta,\tau_a,\mu_a)}{\pi a} - \frac{1}{\pi a}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty dn\frac{\arg[1+\lambda_0 f_E(l,in)]}{e^{\frac{2\pi}{\alpha}n}-1}, \tag{3.42a}$$

$$F_H = \frac{\Lambda_H(\delta,\tau_a,\mu_a)}{\pi a} - \frac{1}{\pi a}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty dn\frac{\arg[-n^2-\lambda_0 f_H(l,in)]}{e^{\frac{2\pi}{\alpha}n}-1}, \tag{3.42b}$$

where $f_E(l,x)=e_l(x)s_l(x)/x$ and $f_H(l,x)=xe_l'(x)s_l'(x)$. Numerically calculate the self-entropies S_E and S_H derived from the temperature-dependent parts

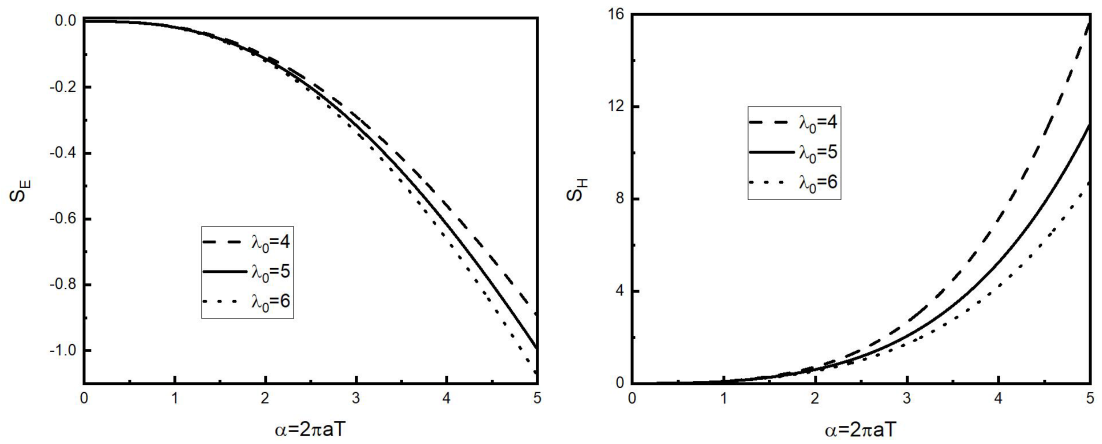

Figure 3.2: The TE and TM self-entropies S_E and S_H as functions of $\alpha = 2\pi aT$ evaluated numerically, based on the results of Eq. (3.42).

ΔF_E and ΔF_H with the properties given in Eq. (A.6), as shown in Figure 3.2. According to our numerical result, the total self-entropy of the spherical shell is consistent with the third law of thermodynamics and always positive, while the TE contribution is always negative. These results are completely similar to those in the thin sheet case, but disagree with some results, for instance, in the weak-coupling limit shown in Eq. (3.35) the TM self-entropy is not positive. Further investigations are indispensable in our future work.

3.5 Casimir interaction entropies

The Casimir interaction entropies were originally investigated as a part of the arguments about the proper low-frequency model for medium. Some researchers claimed the Drude model leads to results violating the Nernst's theorem [142, 143, 144], while others did not agree with them [38, 145, 89, 90]. On the other hand, the Casimir interaction entropies are interesting on their own, since there exist parameter intervals allowing for negative interaction entropies. For a given system, this negativity clearly signifies the abnormally altered structure of quantum levels of that system, which is commonly thought to be related to the repulsive Casimir forces. Dissipation may result in the negative Casimir interaction entropy [146, 147, 142], and the geometry is also a source [148, 149]. The joint effects of geometry and dissipation were also investigated [150, 43].

3.5.1 Interacting particles

As a brief illustration, we consider the electrically polarizable particles for clarity.[1] Then the interaction free energy is

$$F_{12} = \frac{1}{2}\int \frac{d\zeta}{2\pi} \mathrm{Tr}\ln(\mathbf{1} - \mathbf{\Gamma}_{v;\zeta}\cdot\mathbf{T}^{E}_{1;\zeta}\cdot\mathbf{\Gamma}_{v;\zeta}\cdot\mathbf{T}^{E}_{2;\zeta}), \tag{3.43}$$

where $\mathbf{T}^{E}_{i;\zeta} = \mathbf{V}^{E}_{i;\zeta}\cdot(\mathbf{1}+\mathbf{\Gamma}_{v;\zeta}\cdot\mathbf{V}^{E}_{i;\zeta})^{-1},\ i=1,2$ are the scattering matrices and $\mathbf{\Gamma}_{v;\zeta}$ is the Green's diadic of the vacuum. $\mathbf{V}^{E}_{i;\zeta},\ i=1,2$ are susceptibilities of the two particles, which are denoted as $\mathbf{V}^{E}_{i;\zeta}(\mathbf{r},\mathbf{r}') = \boldsymbol{\alpha}_i\delta(\mathbf{r}-\mathbf{R}_i)\delta(\mathbf{r}-\mathbf{r}')$, where $\mathbf{R}_i$ is the position of particle i and $\boldsymbol{\alpha}_i$ may be dispersive. In this case, the scattering matrices are written as

$$\mathbf{T}^{E}_{i;\zeta}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{R}_i)\boldsymbol{\alpha}_i\cdot\Big[\mathbf{1}+\mathbf{\Gamma}_{v;\zeta}(\mathbf{R}_i,\mathbf{R}_i)\cdot\boldsymbol{\alpha}_i\Big]^{-1}\delta(\mathbf{r}'-\mathbf{R}_i), \tag{3.44}$$

which results in the expression for the interaction free energy of the two particles as

$$\begin{aligned} F_{12} &= \frac{1}{2}\int\frac{d\zeta}{2\pi}\mathrm{tr}\ln\Big\{\mathbf{1}-\mathbf{\Gamma}_{v;\zeta}(\mathbf{R}_2,\mathbf{R}_1)\cdot\boldsymbol{\alpha}_1\cdot\Big[\mathbf{1}+\mathbf{\Gamma}_{v;\zeta}(\mathbf{R}_1,\mathbf{R}_1)\cdot\boldsymbol{\alpha}_1\Big]^{-1} \\ &\quad \cdot\mathbf{\Gamma}_{v;\zeta}(\mathbf{R}_1,\mathbf{R}_2)\cdot\boldsymbol{\alpha}_2\cdot\Big[\mathbf{1}+\mathbf{\Gamma}_{v;\zeta}(\mathbf{R}_2,\mathbf{R}_2)\cdot\boldsymbol{\alpha}_2\Big]^{-1}\Big\}. \end{aligned} \tag{3.45}$$

Usually the limit $|\boldsymbol{\alpha}_i| \ll 1$ holds true, so we can keep the leading order to write the interaction free energy as

$$F_{12} \approx -\frac{1}{2}\int\frac{d\zeta}{2\pi}\mathrm{tr}\Big[\mathbf{\Gamma}_{v;\zeta}(\mathbf{R}_2,\mathbf{R}_1)\cdot\boldsymbol{\alpha}_1\cdot\mathbf{\Gamma}_{v;\zeta}(\mathbf{R}_1,\mathbf{R}_2)\cdot\boldsymbol{\alpha}_2\Big]. \tag{3.46}$$

[1] Consider two particles in the vacuum which are both electrically and magnetically polarizable. Then by defining $\mathbf{\Gamma}_{i;\zeta} = (\mathbf{1}+\mathbf{\Gamma}^{H}_{i;\zeta}\cdot\mathbf{V}^{E}_{i;\zeta})^{-1}\cdot\mathbf{\Gamma}^{H}_{i;\zeta},\ i=1,2$, the interaction free energy F_{12} is

$$F_{12} = \frac{1}{2}\int\frac{d\zeta}{2\pi}\mathrm{Tr}\ln(\mathbf{1}-\mathbf{\Gamma}^{H}_{1;\zeta}\cdot\mathbf{T}^{E}_{1;\zeta}\cdot\mathbf{\Gamma}^{H}_{2;\zeta}\cdot\mathbf{T}^{E}_{2;\zeta}),$$

where $\mathbf{T}^{E}_{i;\zeta} = \mathbf{V}^{E}_{i;\zeta}\cdot(\mathbf{1}+\mathbf{\Gamma}^{H}_{i;\zeta}\cdot\mathbf{V}^{E}_{i;\zeta})^{-1},\ i=1,2$ and

$$\begin{aligned} \mathbf{\Gamma}^{H}_{i;\zeta}(\mathbf{r},\mathbf{r}') &= \mathbf{1}\delta(\mathbf{r}-\mathbf{r}') - \frac{\nabla\times\mathbf{\Phi}^{E}_{i,\zeta}(\mathbf{r},\mathbf{r}')\times\overleftarrow{\nabla}'}{\zeta^2},\ \mathbf{\Phi}^{E}_{i;\zeta} = (\mathbf{1}+\mathbf{\Gamma}_{v;\zeta}\cdot\mathbf{V}^{H}_{i;\zeta})^{-1}\cdot\mathbf{\Gamma}_{v;\zeta} \\ &= \mathbf{\Gamma}_{v;\zeta}+\mathbf{\Phi}_{v;\zeta}\cdot\mathbf{V}^{H}_{i;\zeta}\cdot(1+\mathbf{\Gamma}_{v;\zeta}\cdot\mathbf{V}^{H}_{i;\zeta})^{-1}\cdot\mathbf{\Phi}_{v;\zeta},\ \mathbf{\Phi}_{v;\zeta} = -\frac{\nabla\times\mathbf{\Gamma}_{v;\zeta}}{\zeta}. \end{aligned}$$

We assume $\mathbf{R}_i = (0, 0, z_i)$ without losing any generality. With a given finite temperature $T > 0$, when $\boldsymbol{\alpha}_i = \hat{\mathbf{z}}\hat{\mathbf{z}}\alpha_i$, the interaction free energy F_{12} is

$$F_{12}^{\parallel} = -\frac{\alpha_1\alpha_2 z_T}{64\pi^3|z_1 - z_2|^7}\left[4 + 8\sum_{n=1}^{\infty}(1 + z_T n)^2 e^{-2z_T n}\right], \tag{3.47a}$$

where $z_T = 2\pi|z_1 - z_2|T$; when $\boldsymbol{\alpha}_i = \hat{\mathbf{x}}\hat{\mathbf{x}}\alpha_i$, F_{12} is

$$F_{12}^{\perp} = -\frac{\alpha_1\alpha_2 z_T}{64\pi^3|z_1 - z_2|^7}\left[1 + 2\sum_{n=1}^{\infty}(1 + z_T n + z_T^2 n^2)^2 e^{-2z_T n}\right]; \tag{3.47b}$$

when $\boldsymbol{\alpha}_1 = \hat{\mathbf{x}}\alpha_1$ and $\boldsymbol{\alpha}_2 = \hat{\mathbf{z}}\alpha_2$, then $F_{12} = 0$. In the zero-temperature limit $T \to 0$, F_{12}s in Eq. (3.47a) and Eq. (3.47b) are, respectively,

$$F_{12}^{\parallel} \to E_{12}^{\parallel} = -\frac{5\alpha_1\alpha_2}{32\pi^3|z_1 - z_2|^7}, \; F_{12}^{\perp} \to E_{12}^{\perp} = -\frac{13\alpha_1\alpha_2}{128\pi^3|z_1 - z_2|^7}, \tag{3.48}$$

while the high-T limits of $F_{12}^{\parallel}$ and $F_{12}^{\perp}$ are obvious. For the isotropic particle with $\boldsymbol{\alpha}_i = \alpha_i\mathbf{1}$, the interaction free energy is $F_{12} = F_{12}^{\parallel} + 2F_{12}^{\perp}$, which means its low-T limit is just the famous result in Ref. [151], i.e.,

$$F_{12} \to E_{12} = -\frac{23\alpha_1\alpha_2}{64\pi^3|z_1 - z_2|^7}. \tag{3.49}$$

When one of the particle is anisotropic, say $\boldsymbol{\alpha}_1 = \mathbf{1}\alpha, \; \boldsymbol{\alpha}_2 = \alpha(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}}) + \beta\hat{\mathbf{z}}\hat{\mathbf{z}}$, then the Casimir interaction entropy is $S_{12} = 2S_{12}^{\perp} + \beta S_{12}^{\parallel}/\alpha$, where $S_{12}^{\parallel}$ and $S_{12}^{\perp}$ are derived according to Eq. (3.47) except for $\alpha_1 = \alpha_2 = \alpha$. In the unit of $\alpha^2/32\pi^2|z_1 - z_2|^6$, the reduced Casimir interaction entropies, in the low-T limit, are

$$s_{12}^{\parallel} \sim \frac{8}{45}z_T^3, \; s_{12}^{\perp} \sim -\frac{4}{45}z_T^3 \Rightarrow s_{12} \sim \frac{\beta - \alpha}{\alpha}\frac{8}{45}z_T^3. \tag{3.50}$$

Therefore, although those interaction entropies are always positive with high enough temperature, there is a region, where the total interaction entropy is negative, if $\beta < \alpha$. For more information, please see our Ref. [44].

3.5.2 Concentric spherical shells

Consider two concentric spherical shells with radii $a_i, i = 1, 2,\ a_1 < a_2$, permeabilities $\mu_1 = \mu_2 = 1$ and susceptibilities

$$\mathbf{V}_{i;\zeta}(\mathbf{r}, \mathbf{r}') = \delta(r - a_i)\frac{\delta(r - r')}{r^2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left[\lambda_i \mathbf{\Psi}_l^m(\Omega)\mathbf{\Psi}_l^{m*}(\Omega') + \rho_i \mathbf{\Phi}_l^m(\Omega)\mathbf{\Phi}_l^{m*}(\Omega')\right]. \tag{3.51}$$

The interaction free energy of the first order scattering $F_{12}^{(1)}$ can be written in terms of the sum of TE and TM contributions as $F_{12}^{(1)} = F_{12}^{(1),\mathrm{TE}} + F_{12}^{(1),\mathrm{TM}}$, in which $F_{12}^{(1),\mathrm{TE}}$ and $F_{12}^{(1),\mathrm{TM}}$ are

$$F_{12}^{(1),\mathrm{TE}} = -\rho_1\rho_2\sum_{l=1}^{\infty}\nu T\sum_{n=-\infty}^{\infty}\zeta_n^2 e_l^2(|\zeta_n|a_2)s_l^2(|\zeta_n|a_1), \tag{3.52a}$$

$$F_{12}^{(1),\mathrm{TM}} = -\lambda_1\lambda_2\sum_{l=1}^{\infty}\nu T\sum_{n=-\infty}^{\infty}\zeta_n^2 e_l'^2(|\zeta_n|a_2)s_l'^2(|\zeta_n|a_1). \tag{3.52b}$$

According to the result of Eq. (A.13c), the TE contribution is evaluated as

$$\begin{aligned} F_{12}^{(1),\mathrm{TE}} &= -\rho_1\rho_2 T\sum_{n=1}^{\infty}\zeta_n^2\left\{\frac{\zeta_n^2 a_1 a_2}{2}\left[\mathrm{Ei}[-2|\zeta_n|(a_2+a_1)] - \mathrm{Ei}[-2|\zeta_n|(a_2-a_1)]\right]\right. \\ &\left. -e^{-2|\zeta_n|a_2}\sinh^2(|\zeta_n|a_1)\right\}, \end{aligned} \tag{3.53a}$$

which means in the low-T and high-T limits, $F_{12}^{(1),\mathrm{TE}}$ has the forms

$$\Delta F_{12,T\to 0}^{(1),\mathrm{TE}} \approx -\frac{16\pi^7}{135}\rho_1\rho_2 a_2 a_1^4 T^8 \Rightarrow S_{12,T\to 0}^{(1),\mathrm{TE}} \to \frac{128\pi^7}{135}\rho_1\rho_2 a_2 a_1^4 T^7, \tag{3.53b}$$

$$\Delta F_{12,T\to\infty}^{(1),\mathrm{TE}} \approx -\frac{2\pi^3\rho_1\rho_2 a_1 a_2}{a_2 - a_1}T^4 e^{-4\pi(a_2-a_1)T}, \tag{3.53c}$$

where the temperature-independent parts have been ignored. In principle, the TM contribution can also be calculated analytically according to the result in Eq. (A.13b). However, the complexity is unacceptable. In the low-

T limit, $F_{12}^{(1),\mathrm{TM}}$ satisfies

$$\begin{aligned}\Delta F_{12,T\to 0}^{(1),\mathrm{TM}} &\approx \frac{16\pi^3 a_1^2\lambda_1\lambda_2}{135 a_2}T^4\left(1-\varphi_2 T^2+\varphi_4 T^4\right)\\ &\Rightarrow S_{12,T\to 0}^{(1),\mathrm{TM}} \to -\frac{32\pi^3 a_1^2\lambda_1\lambda_2}{135 a_2}T^3\left(2-3\varphi_2 T^2+4\varphi_4 T^4\right),\end{aligned} \tag{3.54a}$$

in which coefficients φ_2 and φ_4, satisfying $9\varphi_2^2 < 32\varphi_4^2$, are

$$\varphi_2 = \frac{2\pi^2}{7}\frac{13a_1^2+70a_2^2}{15},\ \ \varphi_4 = \pi^4\frac{164a_1^4+2429a_1^2a_2^2+4760a_2^4}{1225}, \tag{3.54b}$$

while in high-T limit, it is

$$\begin{aligned}\Delta F_{12,T\to\infty}^{(1),\mathrm{TM}} &\approx -\frac{\lambda_1\lambda_2}{4a_2^2}T\left[\frac{1}{4}\frac{3d-1}{(1-d)^2}+\frac{\arctan(\sqrt{d})}{4\sqrt{d}}+\frac{2d}{3}{}_4F_3(1.5,2,2,2;1,1,2.5;d)\right.\\ &\left.+\frac{d}{3}{}_5F_4(1.5,2,2,2,2;1,1,1,2.5;d)\right],\ d=\frac{a_1^2}{a_2^2},\\ &\to -\frac{\lambda_1\lambda_2}{3a_2^2}Td,\ d\to 0;\quad -\frac{\lambda_1\lambda_2}{8a_2^2}Td\frac{d^2+4d+1}{(1-d)^4},\ d\to 1,\end{aligned} \tag{3.54c}$$

where ${}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;x)$ denotes a generalized hypergeometric function. The total interaction entropy, in the low-T region, is

$$\begin{aligned}S_{12,T\to 0}^{(1)} &= S_{12,T\to 0}^{(1),\mathrm{TE}}+S_{12,T\to 0}^{(1),\mathrm{TM}}\\ &\to -\frac{32\pi^3 a_1^2\lambda_1\lambda_2}{135a_2}T^3\left(2-3\varphi_2T^2+\left(4\varphi_4-\frac{4\rho_1\rho_2}{\lambda_1\lambda_2}\pi^4a_2^2a_1^2\right)T^4\right],\end{aligned} \tag{3.55}$$

which means the total interaction entropy $S_{12}^{(1)}$ is consistent with the third law of thermodynamics and it is possible that there is a temperature range in which $S_{12}^{(1)}$ is negative, for instance, if $\rho_1\rho_2\to 0$, $S_{12,T\to 0}^{(1)}$ can be negative in the whole low-T region.

Therefore, we see another example where the negative entropy of purely geometric origin occurs. Looking closely into the details of the quantum state distribution of the concentric configuration may unveil more properties of the origin of negative interaction entropy. Also, since the negative interaction entropy phenomenon is believed to be related the repellency of Casimir force, the concentric configuration here could be a proper point of penetration into the geometry-facilitated Casimir levitation, which is typically caused by the properties of medium.

3.6 Summary

In this chapter, we demonstrate our researches on the Casimir interaction entropies and self-entropies briefly. Although those entropies have been studied for more than two decades, our understanding is still not profound enough, especially for the self-entropy. As we have shown, the Casimir self-entropy is only well-defined for some extremely special cases. Generally, it is not clear how to interpret the Casimir self-entropy because of the divergences, even logarithmic ones, depending on the temperature. Nevertheless, we see illuminating phenomena in our self-entropy investigations, such as the vanishing self-entropy of the thin sheet in the strong-coupling limit and the negative TE and TM self-entropies of the thin spherical shell. Over all, our knowledge about the Casimir self-entropy is pretty superficial and we are just getting started. Evidently, any experimentally testable self-entropy effects, such as the negative specific heat and the modified melting thickness of a hailstone, will be of great help.

The Casimir interaction entropy is much easier to be detected. However, experimental results diverge. Since the negative Casimir interaction entropy almost always means the negative interaction specific heat, a properly designed experiment may find the negative Casimir interaction entropy in the laboratory. Given that more detailed knowledge about the dissipation could throw much light on the origin of the negative interaction entropy, evaluating the dissipation of the system explicitly, for example with the approach pointed out in Refs. [152, 153], is valuable.

Chapter 4

Classical and quantum friction

4.1 Background

Friction is a well-known concept, a force which resists the relative motion of bodies. The irreversible dissipation of energy is a distinct characteristic of friction, which is also closely related to the time-reversibility of the system involved. Usually friction is seen between bodies in contact, but quantum fluctuations, perhaps modified by thermal fluctuations, predicts the probability of a non-contact frictional force, called Casimir friction.

Casimir friction has been studied for more than four decades [154, 155, 53, 52], and we saw a renaissance of this topic since about 2010, in which Ref. [156], claiming no Casimir friction exists, may have inflamed passions. Most researchers think Casimir friction is real, for instance Pendry [157] derives a nonzero friction by considering the interaction of surface plasmons in two parallel dielectric plates mediated by the vacuum fluctuation of the electromagnetic field. It is widely believed that the dissipation of the media and the thermalization of the dynamical system are sources of the Casimir friction, which makes sense since both of these effects are irreversible. In their series of papers on the quantum oscillators in relative motion [158, 159, 160, 161, 162, 163, 164], Høye and Brevik show that there is dissipation of energy, and thus frictional force, in a thermal dynamical system. Barton also has his own series papers on oscillator systems [165, 166], the results of which have some discrepancies with those of Høye *et al.* There are studies with the dissipation included as well. The main difficulty lies in the proper management of the dissipation. The quantization of the macroscopic Maxwell's equations with the media satisfying the Kramers-Kronig relations has been given in Refs. [152, 153], and similar methods have been utilized in some investigations on the Casimir friction [167, 168, 169, 170].

As is known, there are discrepancies among studies on the Casimir friction. For more details, please see Ref. [171]. In this chapter, based on our research, we demonstrate both classical and quantum friction. The systems explored are simple yet illustrative, in order to clarify our arguments.

4.2 Classical friction

Suppose a particle with the charge q is moving in a medium [172] and its charge density and current density are, respectively, $\rho(t,\mathbf{r}) = q\delta[\mathbf{r}-\mathbf{R}(t)]$ and $\mathbf{j}(t,\mathbf{r}) = q\dot{\mathbf{R}}(t)\delta[\mathbf{r}-\mathbf{R}(t)]$, where $\mathbf{R}(t)$ is the trajectory of the particle. So $\mathbf{E}(t,\mathbf{r})$ is expressed as

$$\mathbf{E}(\omega,\mathbf{r}) = \int \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega t}\mathbf{E}(\omega,\mathbf{r}),\ \mathbf{E}(\omega,\mathbf{r}) = \frac{1}{i\omega}\int d\mathbf{r}'\mathbf{\Gamma}_\omega(\mathbf{r},\mathbf{r}')\cdot\mathbf{j}(\omega,\mathbf{r}'), \tag{4.1a}$$

$$\left[\boldsymbol{\varepsilon} - \frac{\nabla\times\boldsymbol{\mu}^{-1}\cdot\nabla\times\mathbf{1}}{\omega^2}\right]\cdot\mathbf{\Gamma}_\omega(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \tag{4.1b}$$

which means the energy loss rate of the particle W can be written as

$$W = -\int d\mathbf{r}\ \mathbf{j}(t,\mathbf{r})\cdot\mathbf{E}(t,\mathbf{r}) = -q^2\int\frac{d\omega dt'}{2\pi i\omega}e^{i\omega(t'-t)}\dot{\mathbf{R}}(t)\cdot\mathbf{\Gamma}_\omega[\mathbf{R}(t),\mathbf{R}(t')]\cdot\dot{\mathbf{R}}(t'). \tag{4.2}$$

Let the background be vacuum for $z>0$ and an isotropic and homogeneous medium (ε,μ) for $z<0$, then the propagator can be written as

$$\mathbf{\Gamma}_\omega(\mathbf{r},\mathbf{r}') = \int\frac{d^2k}{(2\pi)^2}e^{i\mathbf{k}\cdot(\mathbf{r}_\parallel-\mathbf{r}'_\parallel)}\mathbf{g}_{\omega,\mathbf{k}}(z,z'), \tag{4.3a}$$

where $\mathbf{g}_{\omega,\mathbf{k}}$ is the Minkowskian version of Eq. (2.13c), i.e.,

$$\mathbf{g}_{\omega,\mathbf{k}} = \begin{bmatrix} \frac{k_x^2}{k^2}\frac{\partial_z\partial_{z'}g^H_{\zeta,\mathbf{k}}}{\varepsilon\varepsilon'} + \frac{k_y^2}{k^2}\omega^2 g^E_{\zeta,\mathbf{k}} & \frac{k_xk_y}{k^2}\frac{\partial_z\partial_{z'}g^H_{\zeta,\mathbf{k}}}{\varepsilon\varepsilon'} - \frac{k_xk_y}{k^2}\omega^2 g^E_{\zeta,\mathbf{k}} & \frac{ik_x\partial_z g^H_{\zeta,\mathbf{k}}}{\varepsilon\varepsilon'} \\ \frac{k_xk_y}{k^2}\frac{\partial_z\partial_{z'}g^H_{\zeta,\mathbf{k}}}{\varepsilon\varepsilon'} - \frac{k_xk_y}{k^2}\omega^2 g^E_{\zeta,\mathbf{k}} & \frac{k_y^2}{k^2}\frac{\partial_z\partial_{z'}g^H_{\zeta,\mathbf{k}}}{\varepsilon\varepsilon'} + \frac{k_x^2}{k^2}\omega^2 g^E_{\zeta,\mathbf{k}} & \frac{ik_y\partial_z g^H_{\zeta,\mathbf{k}}}{\varepsilon\varepsilon'} \\ -\frac{ik_x\partial_{z'} g^H_{\zeta,\mathbf{k}}}{\varepsilon\varepsilon'} & -\frac{ik_y\partial_{z'} g^H_{\zeta,\mathbf{k}}}{\varepsilon\varepsilon'} & \frac{k^2 g^H_{\zeta,\mathbf{k}}}{\varepsilon\varepsilon'} \end{bmatrix}, \tag{4.3b}$$

and g^E, g^H satisfy the equations

$$\left[\partial_z\frac{1}{(\mu,\varepsilon)}\partial_z + (\varepsilon,\mu)\omega^2 - \frac{k^2}{(\mu,\varepsilon)}\right]g^{(E,H)}_{\zeta,\mathbf{k}}(z,z') = \delta(z-z'). \tag{4.3c}$$

By defining $\tilde{\kappa} = \sqrt{k^2 - \varepsilon\mu\omega^2}$, $\kappa = \sqrt{k^2-\omega^2}$ and the functions $e_\pm$ as

$$e_+(z) = \begin{cases} e^{-\kappa z}, & z > 0, \\ \frac{-\kappa-\tilde{\kappa}/\mu}{-2\tilde{\kappa}/\mu}e^{-\tilde{\kappa}z} + \frac{\kappa-\tilde{\kappa}/\mu}{-2\tilde{\kappa}/\mu}e^{\tilde{\kappa}z}, & z < 0, \end{cases} \tag{4.4a}$$

$$e_-(z) = \begin{cases} \frac{\tilde{\kappa}/\mu-\kappa}{-2\kappa}e^{-\kappa z} + \frac{-\tilde{\kappa}/\mu-\kappa}{-2\kappa}e^{\kappa z}, & z > 0, \\ e^{\tilde{\kappa}z}, & z < 0, \end{cases} \tag{4.4b}$$

g^E is expressed in the region $z, z' > 0$ as

$$g^E(z,z') = \frac{\tilde{\kappa}/\mu - \kappa}{\tilde{\kappa}/\mu + \kappa}\frac{e^{-\kappa(z+z')}}{2\kappa} - \frac{e^{-\kappa|z-z'|}}{2\kappa}, \tag{4.4c}$$

while in the region $z, z' < 0$ it is

$$g^E(z,z') = \frac{\kappa - \tilde{\kappa}/\mu}{\kappa + \tilde{\kappa}/\mu}\frac{e^{\tilde{\kappa}(z+z')}}{2\tilde{\kappa}/\mu} - \frac{e^{-\tilde{\kappa}|z-z'|}}{2\tilde{\kappa}/\mu}, \tag{4.4d}$$

where the second terms on the right sides are the bulk terms when each medium filling in the whole space. By making the substitution $\varepsilon \leftrightarrow \mu$, we obtain the corresponding g^Hs.

Firstly, set the particle moving with a constant velocity. Assume that the particle is not in the dielectric and its position at time t is $\mathbf{R}(t) = vt\hat{\mathbf{x}} + a\hat{\mathbf{z}}$, $a > 0$, then W is

$$W = -q^2v^2\int\frac{d\omega dt'}{2\pi}\frac{1}{i\omega}e^{-i\omega(t-t')}\int\frac{d^2k}{(2\pi)^2}e^{ik_xv(t-t')}g_{\omega,\mathbf{k};xx}(a,a). \tag{4.5}$$

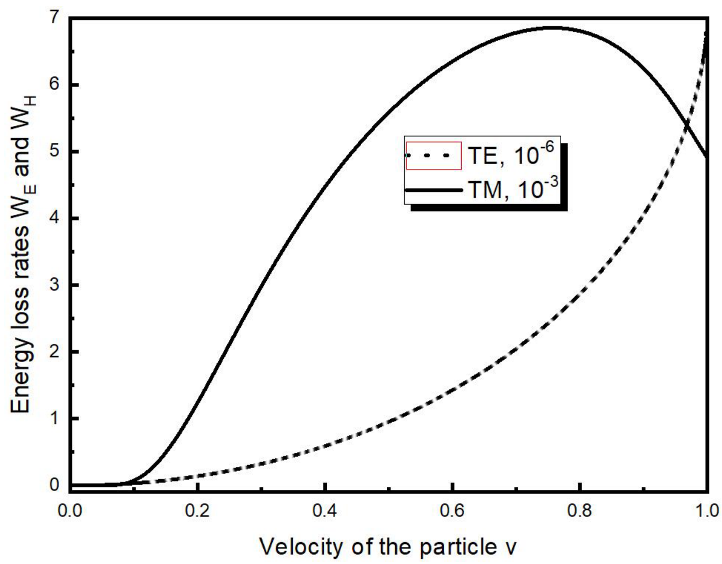

Figure 4.1: The energy loss rates as functions of the velocity of particle outside the medium, with the distance between particle and surface being $a = 10\text{nm}$, $\hbar\omega_p = 9.0\text{eV}$ and $\hbar\nu = 0.035\text{eV}$ (in our unit convention $a = 1$, $\omega_p = 0.45$ and $\nu = 0.00175$).

It is obvious that $W = 0$ always holds true when the dielectric is nondissipative. For the conductor described with the Drude model $\varepsilon = 1 - \omega_p^2/(\omega^2 + i\nu\omega), \mu = 1$, then the TE and TM contributions to W are

$$\begin{aligned} W_E &= \frac{q^2\omega_p^2}{4i\pi^2}v^3\gamma^2 \int d^2k \frac{k_x k_y^2 e^{-k\omega_a}}{\gamma^2 k_x^2 + k_y^2} \frac{1}{\sqrt{k^2 + \frac{k_x v\gamma}{k_x v\gamma + i\nu_\omega}} + k} \\ &\approx \frac{q^2}{8\pi^2}v^4 \int d^2k \frac{\omega_p^2 k_x^2 k_y^2}{4k^5\nu} e^{-2ka} = \frac{\omega_p^2 q^2}{256\pi\nu a}v^4, \;\; v \to 0. \end{aligned} \tag{4.6a}$$

$$\begin{aligned} W_H &= \frac{q^2\omega_p^2}{4i\pi^2}v\gamma^2 \int d^2k \frac{k_x k^2 e^{-k\omega_a}}{\gamma^2 k_x^2 + k_y^2} \frac{(1 - \frac{1}{k_x^2 v^2\gamma^2 + i\nu_\omega k_x v\gamma})}{\sqrt{k^2 + \frac{k_x v\gamma}{k_x v\gamma + i\nu_\omega}} + (1 - \frac{1}{k_x^2 v^2\gamma^2 + i\nu_\omega k_x v\gamma})k} \\ &\approx \frac{\nu q^2}{\omega_p^2}v^2\gamma^2 \int \frac{d^2k}{(2\pi)^2} e^{-2ak}\frac{k_x^2}{k} = \frac{\nu q^2}{16\pi\omega_p^2}\frac{v^2}{a^3}, \;\; v \to 0, \end{aligned} \tag{4.6b}$$

where $\gamma = 1/\sqrt{1-v^2}$, $\omega_a = 2\omega_p a$ and $\nu_\omega = \nu/\omega_p$. In the high-velocity limit $v \to 1$, W_E and W_H behave as

$$W_E \to \frac{q^2\omega_p^2}{4i\pi^2} \int d^2k \frac{k_x k_y^2 e^{-|k_y|\omega_a}}{k^2} \frac{1}{\sqrt{k_y^2 + \frac{k_x}{k_x + i\nu_\omega}} + |k_y|}, \tag{4.7a}$$

$$W_H \to \frac{q^2\omega_p^2}{4i\pi^2}\int d^2k \frac{k_x k_y^2 e^{-|k_y|\omega_a}}{k^2}\frac{(1-\frac{1}{k_x^2+i\nu_\omega k_x})}{\sqrt{k_y^2+\frac{k_x}{k_x+i\nu_\omega}}+(1-\frac{1}{k_x^2+i\nu_\omega k_x})|k_y|}. \quad (4.7b)$$

There are arguments claiming that in the weak-damping limit $\nu \to 0$ the friction approaches a constant, which is definitely a novel phenomenon and should be interpreted properly.[1]

When the particle is in the Drude conductor, whose permittivity and permeability are $\varepsilon = 1 - \omega_p^2/(\omega^2 + i\nu\omega), \mu = 1$ as above, and its position at time t is $\mathbf{R}(t) = vt\hat{\mathbf{x}} + a\hat{\mathbf{z}},\ a < 0$, then the TE and TM contributions to W,

[1] In the polar coordinate, W_E and W_H are written as

$$W_E = \frac{q^2 v}{4\pi^2 a^2}\int_0^\infty dkke^{-k}\int_0^{v\gamma} dx \frac{x\sqrt{1-\frac{x^2}{v^2\gamma^2}}}{1+x^2}\mathrm{Im}\left[1+\sqrt{1+\frac{\omega_a^2}{k^2}\left(1+i\frac{\nu_a}{kx}\right)^{-1}}\right]^{-1},$$

$$\begin{aligned} W_H &= \frac{q^2\gamma}{4\pi^2 a^2}\int_0^\infty dkke^{-k}\int_0^{v\gamma}\frac{dxx}{x^2+1}\frac{1}{\sqrt{v^2\gamma^2-x^2}} \\ &\quad\times \mathrm{Im}\left[1+\left(1-\frac{\omega_a^2}{k^2x^2+i\nu_a kx}\right)^{-1}\sqrt{1+\frac{\omega_a^2}{k^2}\left(1+i\frac{\nu_a}{kx}\right)^{-1}}\right]^{-1}, \end{aligned}$$

where $\nu_a = 2\nu a$ and $x = v\gamma\cos\theta$. $W_E^{\nu=0} = 0$ is always true since the ν_ω dependence of W_E is analytic, while for W_H we have

$$\begin{aligned} &\lim_{\nu_\omega\to 0}\mathrm{Im}\left[1+\left(1-\frac{\omega_a^2}{k^2x^2+i\nu_a kx}\right)^{-1}\sqrt{1+\frac{\omega_a^2}{k^2}\left(1+i\frac{\nu_a}{kx}\right)^{-1}}\right]^{-1} \\ &= \frac{\pi\sqrt{1+\omega_a^2/k^2}\,\omega_a^2/k^2}{(1+\sqrt{1+\omega_a^2/k^2})^2}\delta\left[x^2-\frac{\omega_a^2/k^2}{1+\sqrt{1+\omega_a^2/k^2}}\right], \end{aligned}$$

which means

$$\begin{aligned} W_H &= \frac{q^2\gamma}{8\pi^2 a^2}\int_{\frac{\omega_a}{\sqrt{\gamma^4-1}}}^\infty \frac{dkke^{-k}}{\frac{\omega_a^2/k^2}{1+\sqrt{1+\omega_a^2/k^2}}+1}\frac{1}{\sqrt{v^2\gamma^2-\frac{\omega_a^2/k^2}{1+\sqrt{1+\omega_a^2/k^2}}}}\frac{\pi\sqrt{1+\omega_a^2/k^2}\,\omega_a^2/k^2}{(1+\sqrt{1+\omega_a^2/k^2})^2} \\ &= \frac{q^2\gamma}{8\pi a^2}\int_{\frac{\omega_a}{\sqrt{\gamma^4-1}}}^\infty dkke^{-k}\frac{1}{\sqrt{\gamma^2-\sqrt{1+\omega_a^2/k^2}}}\frac{(\sqrt{1+\omega_a^2/k^2}-1)^2}{\omega_a^2/k^2} \\ &= \frac{q^2}{8\pi a^2\omega_a^2}\int_0^\infty dkk^3e^{-k}(\sqrt{1+\omega_a^2/k^2}-1)^2 \neq 0,\ \gamma\to\infty, \end{aligned}$$

which is consistent with the results in Eq. (4.7b) and Figure 4.1.

with the bulk contribution ignored, are

$$\begin{aligned}\Delta W_E &= \frac{q^2 v^3 \gamma^2 \omega_p^2}{8i\pi^2}\int \frac{d^2k k_y^2}{k_x^2\gamma^2+k_y^2}\frac{k_x e^{\sqrt{k^2+\frac{k_x v\gamma}{k_x v\gamma+i\nu_\omega}}\omega_a}}{\sqrt{k^2+\frac{k_x v\gamma}{k_x v\gamma+i\nu_\omega}}}\frac{\sqrt{k^2+\frac{k_x v\gamma}{k_x v\gamma+i\nu_\omega}}-k}{\sqrt{k^2+\frac{k_x v\gamma}{k_x v\gamma+i\nu_\omega}}+k}\\ &\to -\frac{q^2 v^2\omega_p^2}{8\pi^2}\int \frac{d^2k}{k_x v}\frac{k_x^3 k_y^2 v^3}{4k^5\nu_\omega}e^{k\omega_a} = -\frac{q^2 v^4\omega_p^2}{128\pi\nu_\omega|\omega_a|}, \quad v\to 0, \end{aligned} \tag{4.9a}$$

$$\begin{aligned}\Delta W_H &= \frac{q^2 v\gamma^2\omega_p^2}{8i\pi^2}\int d^2k\frac{k_x e^{\sqrt{k^2+\frac{k_x v\gamma}{k_x v\gamma+i\nu_\omega}}\omega_a}}{k_x^2\gamma^2+k_y^2}\frac{\sqrt{k^2+\frac{k_x v\gamma}{k_x v\gamma+i\nu_\omega}}}{(1-\frac{1}{k_x^2v^2\gamma^2+i\nu_\omega k_x v\gamma})}\\ &\quad\times\frac{\sqrt{k^2+\frac{k_x v\gamma}{k_x v\gamma+i\nu_\omega}}-k(1-\frac{1}{k_x^2v^2\gamma^2+i\nu_\omega k_x v\gamma})}{\sqrt{k^2+\frac{k_x v\gamma}{k_x v\gamma+i\nu_\omega}}+k(1-\frac{1}{k_x^2v^2\gamma^2+i\nu_\omega k_x v\gamma})}\\ &\to \frac{q^2v^2\omega_p^2}{8\pi^2}\int\frac{d^2k}{k_x v}\frac{k_x v k_x^2\nu_\omega}{k}e^{k\omega_a} = \frac{\nu_\omega q^2 v^2\omega_p^2}{4\pi|\omega_a|^3}, \quad v\to 0. \end{aligned} \tag{4.9b}$$

The results of Eq. (4.9) are plotted in Figure 4.2. In the limit $v \to 1, \gamma \to \infty$, the limiting values of ΔW_E and ΔW_H are

$$\Delta W_E \to \frac{q^2\omega_p^2}{8i\pi^2}\int\frac{d^2k}{k_x}\frac{k_y^2}{k^2}\frac{k_x^2 e^{\sqrt{k_y^2+\frac{k_x}{k_x+i\nu_\omega}}\omega_a}}{\sqrt{k_y^2+\frac{k_x}{k_x+i\nu_\omega}}}\frac{\sqrt{k_y^2+\frac{k_x}{k_x+i\nu_\omega}}-|k_y|}{\sqrt{k_y^2+\frac{k_x}{k_x+i\nu_\omega}}+|k_y|}, \tag{4.10a}$$

$$\begin{aligned}\Delta W_H &\to \frac{q^2\omega_p^2}{8i\pi^2}\int\frac{d^2k}{k_x}\frac{k_x^2 e^{\sqrt{k_y^2+\frac{k_x}{k_x+i\nu_\omega}}\omega_a}}{k^2}\frac{\sqrt{k_y^2+\frac{k_x}{k_x+i\nu_\omega}}}{(1-\frac{1}{k_x^2+i\nu_\omega k_x})}\\ &\quad\times\frac{\sqrt{k_y^2+\frac{k_x}{k_x+i\nu_\omega}}-|k_y|(1-\frac{1}{k_x^2+i\nu_\omega k_x})}{\sqrt{k_y^2+\frac{k_x}{k_x+i\nu_\omega}}+|k_y|(1-\frac{1}{k_x^2+i\nu_\omega k_x})}. \end{aligned} \tag{4.10b}$$

As shown in Figure 4.2, the total energy loss rate turns from positive to negative as the velocity of the particle increases, which implies an accelerating force. But it is only the interaction contribution. It is easy to check that the bulk contributions to W are divergent, which leads to some ambiguity. Suppose the whole space is filled with the conductor $\varepsilon = 1-\omega_p^2/(\omega^2+i\nu\omega), \mu = 1$, then the energy loss rates are

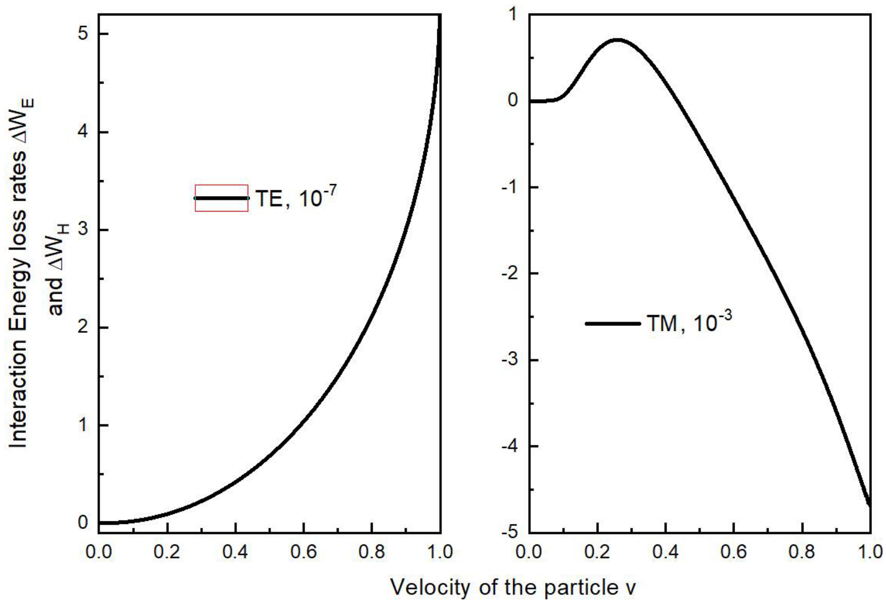

Figure 4.2: The interaction energy loss rates as functions of the velocity of particle inside the medium, with the distance between particle and surface being $a = 10\text{nm}$, $\hbar\omega_p = 9.0\text{eV}$ and $\hbar\nu = 0.035\text{eV}$ (in our unit convention $a = 1$, $\omega_p = 0.45$ and $\nu = 0.00175$).

$$\begin{aligned} W_E &= \frac{\omega_p^2 q^2 v^3 \gamma^2}{8i\pi^2} \int d^2k \frac{k_x k_y^2}{k_x^2\gamma^2 + k_y^2} \frac{e^{-\omega_p \delta \sqrt{k^2 + \frac{k_x v\gamma}{k_x v\gamma + i\nu_\omega}}}}{\sqrt{k^2 + \frac{k_x v\gamma}{k_x v\gamma + i\nu_\omega}}} \\ &\approx \frac{\omega_p^2 q^2 v^4}{128\pi\nu_\omega}\left[3 - 4\gamma_E - 4\ln\left(\frac{\omega_p v}{4\nu_\omega}\delta\right)\right], \ v \to 0, \end{aligned} \tag{4.11a}$$

$$\begin{aligned} W_H &= -\frac{\omega_p^2 q^2 v \gamma^2}{8i\pi^2} \int d^2k \frac{k_x \sqrt{k^2 + \frac{k_x v\gamma}{k_x v\gamma + i\nu_\omega}}}{k_x^2\gamma^2 + k_y^2} \frac{e^{-\omega_p \delta \sqrt{k^2 + \frac{k_x v\gamma}{k_x v\gamma + i\nu_\omega}}}}{1 - \frac{1}{k_x^2 v^2 \gamma^2 + i\nu_\omega k_x v\gamma}} \\ &\approx \frac{\nu_\omega q^2 v^2}{4\pi\omega_p \delta^3}, \ v \to 0, \end{aligned} \tag{4.11b}$$

where $\delta > 0$ is a point-splitting regulator in the z-direction. Obviously, it is not sufficient to interpret the nontrivial bulk contributions with the point-splitting regularization.

Consider a neutral particle with a dipole $\mathbf{d}$. According to the Maxwell's equations, the electric field can be expressed as

$$\hat{\mathbf{E}}(t, \mathbf{r}) = -\int dt' \int d\mathbf{r}' \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \mathbf{\Gamma}_\omega(\mathbf{r}, \mathbf{r}') \cdot \mathbf{P}(t', \mathbf{r}'), \tag{4.12}$$

where $\mathbf{P}$ is the polarization source due to the particle. In our case, $\mathbf{P}(t, \mathbf{r}) =$

$\mathbf{d}(t)\delta[\mathbf{r}-\mathbf{R}(t)]$ where $\mathbf{R}(t)$ is the trajectory of the particle, and the force acting on the particle is

$$\mathbf{F}(t) = -\int dt' \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \mathbf{tr} \nabla_{\mathbf{R}(t)} \mathbf{\Gamma}_\omega[\mathbf{R}(t), \mathbf{R}(t')] \cdot \mathbf{d}(t')\mathbf{d}(t), \qquad (4.13)$$

which means when $\hat{\mathbf{d}} = d\hat{\mathbf{z}}$ and the particle is fixed above a dielectric half-space with the permittivity $\varepsilon =$ constant and permeability $\mu = 1$, the force acting on the particle is[2]

$$\mathbf{F}(t) = \int \frac{dt' d\omega}{2\pi} e^{-i\omega(t-t')} \hat{\mathbf{z}} \partial_z \int \frac{d^2 k k^2 g^H_{\zeta,\mathbf{k}}(z,z')}{-(2\pi)^2/d^2}\Bigg|_{z=z'=a} = -\frac{\varepsilon-1}{\varepsilon+1}\frac{3d^2\hat{\mathbf{z}}}{2\pi(2a)^4}, \quad (4.14)$$

in which the nonphysical divergent self-interacting term has been ignored.

Now suppose the particle is moving in the x-direction with a constant velocity, i.e., $\mathbf{R}(t) = vt\hat{\mathbf{x}} + a\hat{\mathbf{z}},\ a > 0$, then the force parallel to the motion caused by the dielectric slab is

$$\begin{aligned}\Delta F_x(t) &= d^2\omega_p^4\gamma^2 \int \frac{d^2k}{(2\pi)^2} \frac{ik_x(k_x^2\gamma^2 + k_y^2)(1 - \frac{1}{k_x^2 v^2\gamma^2 + i\nu_\omega k_x v\gamma})}{\sqrt{k^2 + \frac{k_x v\gamma}{k_x v\gamma + i\nu_\omega}} + (1 - \frac{1}{k_x^2v^2\gamma^2 + i\nu_\omega k_x v\gamma})k} e^{-\omega_a k} \\ &\approx -\frac{3d^2\nu}{16\pi\omega_p^2 a^5} v,\ v \to 0; \qquad (4.15a)\end{aligned}$$

when the dipole is not transverse but longitudinal, namely $\hat{\mathbf{d}} = d\hat{\mathbf{x}}$, then

$$\begin{aligned}\Delta F_x(t) &= d^2\omega_p^4\gamma^2 \int \frac{d^2k}{(2\pi)^2} \frac{ik_x^3\gamma^2}{k_x^2\gamma^2 + k_y^2} \Bigg[k_y^2 v^2 \frac{1}{\sqrt{k^2 + \frac{k_x v\gamma}{k_x v\gamma + i\nu_\omega}} + k} \\ &\quad + k^2 \frac{(1 - \frac{1}{k_x^2v^2\gamma^2 + i\nu_\omega k_x v\gamma})}{\sqrt{k^2 + \frac{k_x v\gamma}{k_x v\gamma + i\nu_\omega}} + (1 - \frac{1}{k_x^2v^2\gamma^2 + i\nu_\omega k_x v\gamma})k}\Bigg] e^{-\omega_a k} \\ &\approx -\frac{9d^2\nu}{64\pi\omega_p^2 a^5} v,\ v \to 0; \qquad (4.15b)\end{aligned}$$

when the dipole is parallel to the dielectric but not longitudinal, namely

[2] Consider the same situation except for $\varepsilon \to \infty$, then the electrostatic potential U is

$$U = -\frac{q^2}{4\pi(2a-2r)} - \frac{q^2}{4\pi(2a+2r)} + \frac{2q^2}{4\pi(2a)} \approx -\frac{d^2}{2\pi(2a)^3},\ d = 2qr,$$

which is consistent with the result in Eq. (4.14), since $\mathbf{F} = -\hat{\mathbf{z}}\partial U/\partial(2a)$.

$\hat{\mathbf{d}} = d\hat{\mathbf{y}}$, then

$$\begin{aligned}\Delta F_x(t) &= d^2\omega_p^4\gamma^2 \int \frac{d^2k}{(2\pi)^2}\frac{ik_x}{k_x^2\gamma^2+k_y^2}\Bigg[k_x^4\gamma^4v^2\frac{1}{\sqrt{k^2+\frac{k_xv\gamma}{k_xv\gamma+i\nu_\omega}}+k} \\ &\quad +k_y^2k^2\frac{(1-\frac{1}{k_x^2v^2\gamma^2+i\nu_\omega k_xv\gamma})}{\sqrt{k^2+\frac{k_xv\gamma}{k_xv\gamma+i\nu_\omega}}+(1-\frac{1}{k_x^2v^2\gamma^2+i\nu_\omega k_xv\gamma})k}\Bigg]e^{-\omega_a k} \\ &\approx -\frac{3d^2\nu}{64\pi\omega_p^2a^5}v,\ \ v\to 0. \end{aligned} \tag{4.15c}$$

Classical friction in various situations is under investigation, such as magnetic dipoles and Vavilov-Čerenkov radiation. Besides, the time-dependence of the dipole and thermal fluctuations may introduce interesting properties when the dipole is moving [173, 174].

4.3 Quantum friction

Consider a neutral polarizable particle, modeled as a two-level system, with the Hamiltonian and dipole operators

$$\hat{H}_0 = \Delta\hat{s}_z + \frac{\omega_e+\omega_g}{2}, \quad \hat{\mathbf{d}} = \mathbf{d}(\hat{s}_+ + \hat{s}_-), \tag{4.16}$$

where $\Delta = \omega_e - \omega_g$, ω_g and ω_e are eigenenergies of the ground and excited states of the particle. Suppose the particle is moving above a medium located in $z < 0$ with a constant velocity $\mathbf{v} = v\hat{\mathbf{x}}$ according to the trajectory $\mathbf{R}(t) = vt\hat{\mathbf{x}} + a\hat{\mathbf{z}},\ t > 0,\ a > 0$, then $\hat{\mathbf{E}}(\omega, \mathbf{r})$ is

$$\begin{aligned}\hat{\mathbf{E}}(t,\mathbf{r}) &= -\int_0^\infty \frac{d\omega}{\sqrt{2\pi}}\int d\mathbf{r}'\left[e^{-i\omega t}\boldsymbol{\Gamma}_\omega(\mathbf{r},\mathbf{r}')\cdot\hat{\mathbf{P}}(\omega,\mathbf{r}')+h.c.\right] \qquad (4.17a) \\ &= -\int_0^\infty \frac{d\omega}{2\pi}\int_{-\infty}^\infty dt'\left\{e^{-i\omega(t-t')}\theta(t-t')\boldsymbol{\Gamma}_\omega[\mathbf{r},\mathbf{R}(t')]\cdot\hat{\mathbf{d}}(t')+h.c.\right\},\end{aligned}$$

which means the interaction Hamiltonian, in the Heisenberg picture, is

$$\begin{aligned}\hat{H}_i(t) &= \int_0^\infty d\omega\int_{-\infty}^\infty \frac{dt'}{2\pi}\Big\{e^{-i\omega(t-t')}\mathbf{tr}\boldsymbol{\Gamma}_\omega[\mathbf{R}(t),\mathbf{R}(t')]\cdot\hat{\boldsymbol{\alpha}}(t,t') \\ &\quad +e^{i\omega(t-t')}\mathbf{tr}\boldsymbol{\Gamma}_\omega^*[\mathbf{R}(t),\mathbf{R}(t')]\cdot\hat{\boldsymbol{\alpha}}(t,t')\Big\},\end{aligned} \tag{4.17b}$$

where $\hat{\boldsymbol{\alpha}}(t,t') = \hat{\boldsymbol{\alpha}}(t-t')$ and its Fourier component $\hat{\boldsymbol{\alpha}}(\omega)$ is defined as

$$\hat{\boldsymbol{\alpha}}(t,t') = \frac{\hat{\mathbf{d}}(t)\hat{\mathbf{d}}(t') + \hat{\mathbf{d}}(t')\hat{\mathbf{d}}(t)}{2}\theta(t-t'),\ \hat{\boldsymbol{\alpha}}(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i\omega t}\hat{\boldsymbol{\alpha}}(t), \quad (4.17c)$$

and the causality condition is included as the step function $\theta(t-t')$. The dipole-dipole correlation function is $\boldsymbol{\alpha}(t,t') = \langle\hat{\boldsymbol{\alpha}}(t,t')\rangle = \mathrm{tr}\big[\rho(0)\hat{\boldsymbol{\alpha}}(t,t')\big]$, then when $\mathbf{v}=0$, the interaction energy is

$$\begin{aligned} E_i &= \int_0^{\infty} d\omega \mathrm{tr}\Big[\boldsymbol{\Gamma}_\omega(a\hat{\mathbf{z}}, a\hat{\mathbf{z}}) \cdot \boldsymbol{\alpha}(\omega) + \boldsymbol{\Gamma}^*_\omega(a\hat{\mathbf{z}}, a\hat{\mathbf{z}}) \cdot \boldsymbol{\alpha}(-\omega)\Big] \quad (4.17d) \\ &= 2\mathrm{Re}\int_0^{\infty} d\omega \mathrm{tr}\Big[\boldsymbol{\Gamma}_\omega(a\hat{\mathbf{z}}, a\hat{\mathbf{z}}) \cdot \boldsymbol{\alpha}(\omega)\Big] = \int_{-\infty}^{\infty} d\omega \mathrm{tr}\Big[\boldsymbol{\Gamma}_\omega(a\hat{\mathbf{z}}, a\hat{\mathbf{z}}) \cdot \boldsymbol{\alpha}(\omega)\Big], \end{aligned}$$

and the force on the particle is $\mathbf{F}(t) = -\nabla_{\mathbf{R}(t)}\langle\hat{H}_i(t)\rangle$, which, in our case, has the form

$$F_x = 2\int_0^{\infty} d\omega \int \frac{d^2k}{(2\pi)^2} k_x \mathrm{Imtr}\Big[\mathbf{g}_{\omega,\mathbf{k}}(a,a) \cdot \boldsymbol{\alpha}(k_x v - \omega)\Big]. \quad (4.17e)$$

The equation of motion for $\hat{s}_+$ is $\dot{\hat{s}}_+ = i\Delta\hat{s}_+ - i[\hat{s}_+, \hat{H}_i]$, which is solved as

$$\begin{aligned} \hat{s}_+(t) &= \hat{s}_+(0)e^{i\Delta t} - i\int_0^t dt' e^{i\Delta(t-t')}[\hat{s}_+(t'), \hat{H}_i(t')] \\ &\approx \hat{s}_+(0)e^{i\Delta t} - ie^{i\Delta t}\int_0^t dt'[\hat{s}_+(0), \hat{H}_i(t')], \end{aligned} \quad (4.18)$$

in which the approximation is made to the first order. Then the leading terms of $\hat{d}(t)$ and $\boldsymbol{\alpha}(t,t')$ are $\hat{\mathbf{d}}^{(0)}(t) = \mathbf{d}[\hat{s}_+(0)e^{i\Delta t} + \hat{s}_-(0)e^{-i\Delta t}]$ and $\hat{\boldsymbol{\alpha}}^{(0)}(t,t') = \mathbf{dd}\theta(t-t')\cos\Delta(t-t') = \boldsymbol{\alpha}^{(0)}(t,t')$ and its ω-transform is

$$\boldsymbol{\alpha}^{(0)}(\omega) = \frac{\mathbf{dd}}{4}\Big[\delta(\omega+\Delta) + \delta(\omega-\Delta)\Big] + \frac{\mathbf{dd}}{4\pi}P\frac{2i\omega}{\omega^2-\Delta^2}. \quad (4.19)$$

To the first order we have

$$\hat{\mathbf{d}}^{(1)}(t) = \int_0^t dt' \frac{[\hat{\mathbf{d}}^{(0)}(t), \hat{H}_i(t')]}{i},\ \hat{\boldsymbol{\alpha}}^{(1)}(t,t') = \int_0^t dt'' \frac{[\hat{\boldsymbol{\alpha}}^{(0)}(t,t'), \hat{H}_i(t'')]}{i} = 0, \quad (4.20)$$

which can be repeatedly checked that $\forall n \geq 1,\ \hat{\boldsymbol{\alpha}}^{(n)} = 0$, meaning that $\boldsymbol{\alpha}(t,t') = \boldsymbol{\alpha}^{(0)}(t,t')$. By using the oddness of the integral over k_x, the frictional force on the particle is

$$\begin{aligned}F_x(t) &= \int_{\frac{\Delta}{v}}^{\infty} dk_x \int_{-\infty}^{\infty} \frac{dk_y}{(2\pi)^2} k_x \mathbf{tr}\Big[\mathrm{Im}\mathbf{g}_{k_x v-\Delta,\mathbf{k}}(a,a)\cdot\mathbf{dd}\Big] \\ &+ \int_{-\frac{\Delta}{v}}^{\infty} dk_x \int_{-\infty}^{\infty} \frac{dk_y}{(2\pi)^2} k_x \mathbf{tr}\Big[\mathrm{Im}\mathbf{g}_{k_x v+\Delta,\mathbf{k}}(a,a)\cdot\mathbf{dd}\Big], \qquad (4.21)\end{aligned}$$

which is zero when the velocity is zero. Assume the $\Delta = 0$ and $\mathbf{d} = d\hat{\mathbf{z}}$ for simplicity, then $F_x(t)$ is

$$\begin{aligned}F_x(t) &= -d^2\omega_p^4\gamma^2 \int \frac{dk_y}{(2\pi)^2}\int_{-\infty}^{\infty} dk_x (k_x^2\gamma^2 + k_y^2)\mathrm{Im}\frac{k_x e^{-\omega_a k}}{\sqrt{k^2 + \frac{k_x v\gamma}{k_x v\gamma + i\nu_\omega}} + k} \\ &\to -d^2\omega_p^4\gamma\frac{\nu_\omega(\gamma^2+1)}{8\pi v}\int_0^{\infty}\frac{dk k^3 e^{-\omega_a k}}{\sqrt{k^2+1}(\sqrt{k^2+1}+k)^2},\ \nu_\omega \ll v\gamma. \qquad (4.22)\end{aligned}$$

Obviously, if the substrate is nondissipative, i.e., $\nu = 0$, then $F_x(t) = 0$, since the imaginary part of $\mathbf{g}_{\omega,\mathbf{k}}$ vanishes.

To further explore the relation between irreversibility and friction, consider another well-known model, in which two neutral polarizable particles are in relative motion. Previous papers [158, 165] on this model usually ignored retardation in the interaction between two oscillators. Here the retardation has been introduced. The Hamiltonian and dipole operators are

$$\hat{H}_0 = \sum_{i=1,2}\frac{\hat{\mathbf{p}}_i^2}{2m_i} + \frac{1}{2}m_i(\boldsymbol{\omega}_i\cdot\hat{\mathbf{r}}_i)^2 = \sum_{i=1,2}\sum_{b=x,y,z}\omega_{i,b}\left(a_{i,b}^{\dagger}a_{i,b}+\frac{1}{2}\right),\ \hat{\mathbf{d}} = q_i\hat{\mathbf{r}}_i, \qquad (4.23a)$$

where $\boldsymbol{\omega}_i = \mathrm{diag}(\omega_{i,x},\omega_{i,y},\omega_{i,z})$, and the operators $a_{i,c}$ are defined as

$$\hat{r}_{i,b} = \frac{a_{i,b}+a_{i,b}^{\dagger}}{\sqrt{2m_i\omega_{i,b}}},\ \hat{p}_{i,b} = \sqrt{\frac{m_i\omega_{i,b}}{2}}\frac{a_{i,b}-a_{i,b}^{\dagger}}{i}, \qquad (4.23b)$$

which give us the commutation relations $[a_{i,b}, a_{j,c}] = 0,\ [a_{i,b}, a_{j,c}^{\dagger}] = \delta_{ij}\delta_{bc}$. Suppose the particle 1 is located at $\mathbf{R}_1(t) = \mathbf{0}$ and the particle 2 is moving with the trajectory $\mathbf{R}_2(t)$. For clarity, assume the two particles only have the freedom to move in the z-direction, then the interaction Hamiltonian, with the retardation included, is

$$\hat{H}_i(t) = \int_{-\infty}^{\infty} dt' \psi[t-t', \mathbf{R}_2(t)] \frac{\hat{r}_2(t)\hat{r}_1(t') + \hat{r}_1(t')\hat{r}_2(t)}{2}, \tag{4.24a}$$

where the index z is ignored and the coupling coefficient is

$$\begin{aligned} \psi[t-t', \mathbf{R}_2(t)] &= q_1 q_2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left\{ e^{-i\omega(t-t')} \Gamma_{\omega;zz}[\mathbf{R}_2(t), \mathbf{0}] \theta(t-t') \right\} \\ &= q_1 q_2 \left[\frac{\delta(t-|\mathbf{R}_2(t)|-t')}{2\pi|\mathbf{R}_2(t)|^3} + \frac{\delta'(t-|\mathbf{R}_2(t)|-t')}{2\pi|\mathbf{R}_2(t)|^2} \right], \ \mathbf{R}_2(t) = R_2(t)\hat{\mathbf{z}}, \end{aligned} \tag{4.24b}$$

where the retarded Green's functions is used. Let the interaction start at the initial time $t = 0$, the equations of motion for the two particles are

$$\dot{a}_j(t) = -i\omega_j a_j(t) - i[a_j(t), \hat{H}_i(t)], \ j = 1, 2, \tag{4.25a}$$

which are formally satisfy the relations

$$\hat{r}_j(t) = \hat{r}_{j;0}(t) - i \int_0^t dt' [\hat{r}_{j;t'}(t), \hat{H}_i(t')], \ \hat{r}_{j;t'}(t) = \frac{a_j(t')e^{-i\omega_j(t-t')} + h.c.}{\sqrt{2m_j\omega_j}}. \tag{4.25b}$$

The direct interaction energy $\hat{H}_i^{(0)}(t)$, which is to the first order of ψ, is

$$\hat{H}_i^{(0)}(t) = \int_0^t dt' \psi[t-t', \mathbf{R}_2(t)] \hat{r}_{2;0}(t)\hat{r}_{1;0}(t'), \tag{4.26}$$

which means the averge of $\hat{H}_i^{(0)}(t)$ is proportional to $\langle \hat{r}_{2;0}(t)\hat{r}_{1;0}(t') \rangle$. When the two particles are initially disentangled, $\hat{H}_i^{(0)}$ just corresponds to the classical result found in Eq. (4.13). When the particles are entangled and assume the initial state described by the density matrix $\rho(0) = |t=0\rangle \langle t=0|$, $|t=0\rangle = (|0_1 1_2\rangle + |1_1 0_2\rangle)/\sqrt{2}$, then $\langle \hat{r}_{2;0}(t)\hat{r}_{1;0}(t') \rangle$ is nonzero, i.e.,

$$\langle \hat{r}_{2;0}(t)\hat{r}_{1;0}(t') \rangle = \frac{\cos(\omega_2 t - \omega_1 t')}{\sqrt{4m_1 m_2 \omega_1 \omega_2}} \neq 0. \tag{4.27}$$

For simplicity, let $m_1 = m_2 = m$, $\omega_1 = \omega_2 = \omega_o$, then when particle 2 is moving with a constant velocity in the z-direction and the trajectory $\mathbf{R}_2(t) =$

$(vt+a)\hat{\mathbf{z}},\ a>0$, the force on particle 2 parallel to $\mathbf{R}_2$ is

$$F_z^{(0)}(t) = \frac{q_1q_2}{2m\omega_o}\frac{(3-\omega_o^2z^2)\cos(\omega_oz)+3\omega_oz\sin(\omega_oz)}{2\pi z^4}\Bigg|_{z=a+vt},\ t\geq\frac{a}{1-v}. \quad (4.28)$$

The magnitude of $F_z^{(0)}(t)$ decays in an oscillatorily way as particle 2 moves away from particle 1. The average of $F_z^{(0)}$ satisfies the expression

$$\overline{F_z^{(0)}}T = -\frac{q_1q_2}{4\pi m\omega_ovz_i^3}\left[\cos(\omega_oz_i)+\omega_oz_i\sin(\omega_oz_i)\right],\ z_i=\frac{a}{1-v}, \quad (4.29)$$

which is just the change of the interaction energy with a factor of v^{-1}. So it depends on the initial position whether the averge force is attractive or not. Also it is clear that the expectation value of each dipole moment is always zero, i.e., $\langle\hat{d}_{i;0}(t)\rangle = q_i\mathrm{tr}[\rho(0)\hat{r}_{i;0}(t)] = 0$, so the nonzero $F_z^{(0)}(t)$ in Eq. (4.29) is purely a quantum effect. Although the quantum entanglement can facilitate the transfer of energy, it is unlikely to be a source of energy dissipation, since the dissipation typically means time-irreversible aspects of a process.

Of course, much more work, which may be fruitful, could be done. For example, by including heat reservoirs, it is possible to track the path of energy dissipation. For further discussions on this topic, please see our future papers.

4.4 Summary

In this chapter, we briefly depicted dissipative frictional forces in both classical and quantum systems, based on our recent studies on this topic. In the framework of classical electrodynamics, we investigate the friction acting on a charged particle moving parallel to an imperfect conducting slab described by the Drude model. In nonrelativistic and ultrarelativistic regimes, the properties of friction due to the TE and TM modes are quite different. The velocity dependence of the friction is non-monotonic. The friction may be nonzero in the low-resistivity limit when the particle is moving, even close to the speed of light. The properties of friction due to a time-independent dipole moving with a constant velocity are also studied. But it is much more complicated if the dipole evolving with time. Even when the dipole moves inside a homogeneous medium, the radiation may also result in a force in the opposite direction of motion, although there are divergences which plague physical interpretation.

Also, we introduce two models to catch a glimpse of the properties of Casimir friction or quantum friction. When a two-level particle is moving in front of a Drude conducting slab, the particle feels a frictional force due to its interaction with the slab. If the slab is nondissipative, the friction is zero. That is, the dissipation of the conductor leads to the friction. To further explore quantum friction, we study two quantum oscillators in relative motion with the retardation included. If the oscillators have a quantum entanglement, we find a longitudinal force which is not a dissipative force. Our arguments are limited to the leading order and multi-scattering corrections are also nontrivial, though usually too small for any precise experimental detection.

We will, of course, keep working on the classical and quantum friction topics, which, we anticipate, can enrich our knowledge about the relations between quantum friction and irriversibility of time.

Chapter 5

Conclusions and perspectives

5.1 Conclusions

In Chapter 2, we study the Casimir energies, stresses, and forces in some planar and spherical systems. For homogeneous cases, we see bulk divergences of Casimir stresses in a uniform background and we also see divergences at the surface between two dielectrics. We reproduce the expression for the Casimir-Lifshitz force in a DLP model, which has already been justified experimentally [74, 175, 176]. We briefly investigate Casimir stresses and forces in spherical systems and show they are consistent with corresponding planar cases but much more nontrivial due to the curvature. For inhomogeneous cases, based on our work [118], we find divergences depending on discontinuity properties of two media at their surface. Bulk divergences and special cases are also given in Ref. [118]. In Chapter 2, we take a first step to further understand our renormalization scheme, which is introduced to calculate inhomogeneous Casimir forces in planar systems [19], by considering interaction between step homogeneous media. We try to generalize our scheme to concentric spherical cases with some specific examples. More general arguments should be carried out in the future.

In Chapter 3, we study the influence of thermal fluctuations on Casimir effects. First we derive some well-known thermal corrections Casimir forces for some homogeneous cases. The inhomogeneity of media has significant effects on thermal Casimir forces, which is illustrated with some specific examples here, for planar and spherical systems. Based on our pioneering studies [45, 46], we demonstrate Casimir self-entropies of an infinitely thin plasma sheet and an infinitely thin plasma spherical shell. For the thin sheet, its Casimir self-entropies have analytic forms. Both TE and TM Casimir entropies are consistent with the third law of thermodynamics. The TE contribution is always negative, the TM contribution is always positive, and the total Casimir self-entropy is always positive. These results are overall satisfactory, but when the plasma model for this thin sheet is replaced by the Drude model, we see divergences in the Casimir self-entropy [45]. For the thin spherical shell, no analytic forms are found for TE and TM Casimir self-entropies, but consistent limiting arguments are given. The general properties of TE and TM Casimir self-entropies are evaluated numerically, which contradict some limiting results. More investigations are needed. Also we show negative Casimir interaction entropies due to geometry.

In Chapter 4, we calculate classical electromagnetic frictions acting on a charged particle moving with a constant velocity above a Drude conductor [172]. We see a maximum in the velocity dependence of the TM frictional force. Even when the dissipation of conductor disappears, the TM frictional force remains. We also briefly investigate the frictions due to quantum fluctuations in atom-plate and oscillator-oscillator systems, where the retardation of electromagnetic fields has been included. The frictional properties depend strongly on the details of quantum evolution process. Lots of effort should be put into the researches on this topic.

5.2 Perspectives

Definitely, it is not our intension to trap ourselves in theories that are purely abstract, since we recognize that we are always in a "practice-theory" loop when trying to understand the world. Actually for Casimir physics, we see a clear tendency towards practical applications. For example, the autosuspension has been implemented for a nanoplate [131] in a system similar to that described in Sec. (2.4.1), which demonstrates the possibility of Casimir forces keeping micro-structures nontouching and the whole system robust. For another example, Munday *et al.* measured the Casimir torque with a liquid in front of a birefringent substrate [139], which implies the Casimir torque may be used as an actuating scheme for nanomechanics. We expect our research could act as a guidance for experiments or even applications.

As shown in previous chapters, there are many problems in the topics mentioned, which could and should be investigated. Casimir stress tensors and forces in inhomogeneous system with other geometries, for instance the spherical geometry in Chapter 2, or even topologies, are worthy of research. It is interesting to discover properties apparently different from planar cases. Though the divergences of Casimir stresses at the surface of inhomogeneous media seem preposterous, it is believed those divergences should be finite within the atomic scale, which should be precisely studied. It is also an interesting proposal to test the potential influence of Casimir stresses on the surface structure deformable media [119, 120], but the electrostrictive contributions should be included [177]. Inhomogeneous Casimir forces have a good latent capacity to be applied to micromechanical systems, which implies the significance of experimentally accessible systems. How Casimir stresses couple to gravity attracts much attention, but studies mainly focus on simple cases of scalar fields [113, 68]. The influences of electromagnetic Casimir stress tensors on gravity, especially with inhomogeneous media, are largely unknown.

The Casimir entropy, especially the Casimir self-entropy, is a relatively novel research object, and plenty of unsolved questions are waiting for us to put forward and answer. Regularization methods and their consequences should be understood. For example, in our investigations on the Casimir self-entropy of a thin spherical shell, we see divergences inconsistent with the third law of thermodynamics, which are omitted *ad hoc*. Renormalization schemes in Casimir self-entropy calculations should be introduced, since we expect the self-specific heat $C = T\partial S/\partial T$ to be finite. Factors, such as geometry, topology, dimension of spacetime *et al.*, may also affect the properties of Casimir self-entropy. As a counterpart, the dimensional dependences of Casimir interaction entropy are given in Ref. [126]. We plan to figure out some experimental implication due to the Casimir self-entropy [51].

We would like to make our own contributions to old yet active topics, namely classical and quantum frictions. Studies on classical frictional self-forces of electric and magnetic dipoles due to dipole radiation and Vavilov-Čerenkov radiation, and their frictions when dipoles are moving in front an imperfectly conducting surface, are in progress. The frictions in various systems, such as oscillator-oscillator, briefly depicted in Chapter 4, atom-atom, atom-dielectrics and so on, should be considered. We will study thermal corrections to quantum and classical frictions, which we think may facilitate experiments [171]. Experimental proposals [178] are also welcome.

Appendix A

Some mathematical tools

In this appendix, we outline several mathematics utilized in our research. As an appendix of a thesis in physics, we do not pursue the mathematical completeness. For more details, please refer to professional math materials.

A.1 Vector Spherical Harmonics

The vector spherical harmonics (VSH) are defined as $\mathbf{Y}_l^m = Y_l^m \hat{\mathbf{r}}$,

$$\begin{aligned} \mathbf{\Psi}_l^m &= \frac{r\nabla Y_l^m}{\sqrt{l(l+1)}} \\ &= \frac{1}{\sqrt{l(l+1)}}\Big[\hat{\boldsymbol{\theta}}\Big(m\frac{\cos\theta}{\sin\theta}Y_l^m + \sqrt{(l+\frac{1}{2})^2-(m+\frac{1}{2})^2}e^{-i\varphi}Y_l^{m+1}\Big) + \hat{\boldsymbol{\varphi}}\frac{im}{\sin\theta}Y_l^m\Big], \end{aligned} \tag{A.1a}$$

$$\begin{aligned} \mathbf{\Phi}_l^m &= \frac{\mathbf{r}\times\nabla Y_l^m}{\sqrt{l(l+1)}} \\ &= \frac{1}{\sqrt{l(l+1)}}\Big[\hat{\boldsymbol{\varphi}}\Big(m\frac{\cos\theta}{\sin\theta}Y_l^m + \sqrt{(l+\frac{1}{2})^2-(m+\frac{1}{2})^2}e^{-i\varphi}Y_l^{m+1}\Big) - \hat{\boldsymbol{\theta}}\frac{im}{\sin\theta}Y_l^m\Big], \end{aligned} \tag{A.1b}$$

where $\mathbf{\Psi}_0^0 = \mathbf{\Phi}_0^0 = \mathbf{0}$. With VSH, any vector field $\mathbf{E}$ can be written as

$$\mathbf{E} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Big[E_{l,m;Y}\mathbf{Y}_l^m + E_{l,m;\Psi}\mathbf{\Psi}_l^m + E_{l,m;\Phi}\mathbf{\Phi}_l^m\Big], \tag{A.2}$$

where any $E_{l,m;Y}, E_{l,m;\Psi}, E_{l,m;\Phi}$ only depends on r. The basic properties of VSH are listed as follows:

1) $\mathbf{Y}_l^{-m} = (-1)^m\mathbf{Y}_l^{m*}$, $\mathbf{\Psi}_l^{-m} = (-1)^m\mathbf{\Psi}_l^{m*}$, $\mathbf{\Phi}_l^{-m} = (-1)^m\mathbf{\Phi}_l^{m*}$

2) The orthogonal relations are $\mathbf{Y}_l^m \cdot \mathbf{\Psi}_l^m = \mathbf{Y}_l^m \cdot \mathbf{\Phi}_l^m = \mathbf{\Psi}_l^m \cdot \mathbf{\Phi}_l^m = 0$ and

$$\int \mathbf{Y}_{l'}^{m'*}\cdot\mathbf{Y}_l^m d\Omega = \int \mathbf{\Psi}_{l'}^{m'*}\cdot\mathbf{\Psi}_l^m d\Omega = \int \mathbf{\Phi}_{l'}^{m'*}\cdot\mathbf{\Phi}_l^m d\Omega = \delta_{ll'}\delta_{mm'},$$

$$\int \mathbf{Y}_{l'}^{m'*}\cdot\mathbf{\Phi}_l^m d\Omega = \int \mathbf{Y}_{l'}^{m'*}\cdot\mathbf{\Psi}_l^m d\Omega = \int \mathbf{\Phi}_{l'}^{m'*}\cdot\mathbf{\Psi}_l^m d\Omega = 0,$$

3) The divergence and curl of any field $\mathbf{E}$ are

$$\nabla\cdot\mathbf{E} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Big[\frac{1}{r^2}\frac{d(r^2E_{l,m;Y})}{dr} - \frac{\sqrt{l(l+1)}}{r}E_{l,m;\Psi}\Big]Y_l^m,$$

$$
\begin{aligned}
\nabla \times \mathbf{E} \quad = \quad & \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ -\frac{\sqrt{l(l+1)}}{r} E_{l,m;\Phi} \mathbf{Y}_l^m - \frac{1}{r} \frac{d(rE_{l,m;\Phi})}{dr} \mathbf{\Psi}_l^m \right. \\
& \left. + \left[\frac{1}{r} \frac{d(rE_{l,m;\Psi})}{dr} - \frac{\sqrt{l(l+1)}}{r} E_{l,m;Y} \right] \mathbf{\Phi}_l^m \right\}.
\end{aligned}
$$

When $\Omega = \Omega'$, the VHS are also satisfies

$$
\sum_{m=-l}^{l} \mathbf{Y}_l^m(\Omega) \mathbf{Y}_l^{m*}(\Omega') = \hat{\mathbf{r}}\hat{\mathbf{r}} \sum_{m=-l}^{l} Y_l^m(\Omega) Y_l^{m*}(\Omega') = \frac{2l+1}{4\pi} P_l(\cos\theta_x) \hat{\mathbf{r}}\hat{\mathbf{r}} = \frac{2l+1}{4\pi} \hat{\mathbf{r}}\hat{\mathbf{r}}, \quad \text{(A.3a)}
$$

$$
\sum_{m=-l}^{l} \mathbf{\Psi}_l^m(\Omega) \mathbf{\Psi}_l^{m*}(\Omega') = \frac{2l+1}{4\pi} \frac{r^2}{l(l+1)} \nabla\nabla' P_l(\cos\theta_x) = \frac{2l+1}{8\pi} (\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}}), \quad \text{(A.3b)}
$$

$$
\sum_{m=-l}^{l} \mathbf{\Phi}_l^m(\Omega) \mathbf{\Phi}_l^{m*}(\Omega') = \frac{2l+1}{4\pi} \frac{r^2}{l(l+1)} (\hat{\mathbf{r}} \times \nabla)(\hat{\mathbf{r}} \times \nabla') P_l(\cos\theta_x) = \frac{2l+1}{8\pi} (\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}}), \quad \text{(A.3c)}
$$

$$
\sum_{m=-l}^{l} \mathbf{Y}_l^m(\Omega) \mathbf{\Psi}_l^{m*}(\Omega') = \frac{\hat{\mathbf{r}} r}{\sqrt{l(l+1)}} \nabla' \sum_{m=-l}^{l} Y_l^m(\Omega) Y_l^{m*}(\Omega') = \mathbf{0}, \quad \text{(A.3d)}
$$

where θ_x is the angle between the directions of Ω and Ω', i.e. (θ, φ) and (θ', φ'), so $\cos\theta_x = \sin\theta\cos\varphi\sin\theta'\cos\varphi' + \sin\theta\sin\varphi\sin\theta'\sin\varphi' + \cos\theta\cos\theta'$.

A.2 Uniform Asymptotic Expansion

For large order $\nu \to \infty$, the modified Bessel functions can be uniformly expanded as

$$
I_\nu(x) = \frac{e^{\nu\eta}}{\sqrt{2\pi\nu}(1+z^2)^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{1}{\nu^k} U_k(p), \quad \text{(A.4a)}
$$

$$
K_\nu(x) = \sqrt{\frac{\pi}{2\nu}} \frac{e^{-\nu\eta}}{(1+z^2)^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\nu^k} U_k(p), \quad \text{(A.4b)}
$$

$$
I'_\nu(x) = \frac{e^{\nu\eta}(1+z^2)^{\frac{1}{4}}}{\sqrt{2\pi\nu} z} \sum_{k=0}^{\infty} \frac{1}{\nu^k} V_k(p), \quad \text{(A.4c)}
$$

$$
K'_\nu(x) = -\sqrt{\frac{\pi}{2\nu}} \frac{e^{-\nu\eta}(1+z^2)^{\frac{1}{4}}}{z} \sum_{k=0}^{\infty} \frac{(-1)^k}{\nu^k} V_k(p), \quad \text{(A.4d)}
$$

where z, p, η are defined as

$$
z = \frac{x}{\nu}, \ p = (1+z^2)^{-\frac{1}{2}}, \ \eta = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}}, \quad \text{(A.4e)}
$$

and the polynomials used are $U_0(p) = V_0(p) = 1$ and

$$U_1(p) = \frac{-5p^3 + 3p}{24}, \quad V_1(p) = \frac{7p^3 - 9p}{24}; \tag{A.4f}$$

$$U_2(p) = \frac{385p^6 - 462p^4 + 81p^2}{1152}, \; V_2(p) = \frac{-455p^6 + 594p^4 - 135p^2}{1152}. \tag{A.4g}$$

Since the modified spherical Bessel functions we usually use are defined as $s_l(x) = xi_l(x) = \sqrt{\pi x/2} I_\nu(x)$ and $e_l(x) = xk_l(x) = \sqrt{2x/\pi} K_\nu(x)$ where $\nu = l + 1/2$, they can be expanded uniformly as

$$s_l(x) = \frac{\sqrt{z} e^{\nu\eta}}{2(1+z^2)^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{1}{\nu^k} U_k(p), \; e_l(x) = \frac{\sqrt{z} e^{-\nu\eta}}{(1+z^2)^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\nu^k} U_k(p), \tag{A.5a}$$

$$s_l'(x) = \frac{e^{\nu\eta}(1+z^2)^{\frac{1}{4}}}{2\sqrt{z}} \sum_{k=0}^{\infty} \frac{1}{\nu^k} \left[\frac{pU_k(p)}{2\nu} + V_k(p) \right], \tag{A.5b}$$

$$e_l'(x) = \frac{e^{-\nu\eta}(1+z^2)^{\frac{1}{4}}}{\sqrt{z}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\nu^k} \left[\frac{pU_k(p)}{2\nu} - V_k(p) \right]. \tag{A.5c}$$

Also, for the functions $f_E(l, x) = e_l(x)s_l(x)/x$ and $f_H(l, x) = xe_l'(x)s_l'(x)$ defined with e and s, we have the following useful properties

$$f_E(l, ix) = K_\nu(ix)I_\nu(ix) = -\frac{\pi}{2} J_\nu(x)Y_\nu(x) - i\frac{\pi}{2} J_\nu^2(x), \tag{A.6a}$$

$$f_H(l, ix) = -\frac{\pi}{2} \mathcal{J}_\nu(x)\mathcal{Y}_\nu(x) - i\frac{\pi}{2} \mathcal{J}_\nu^2(x), \tag{A.6b}$$

where we have defined the functions

$$\mathcal{J}_\nu(x) = \left(\nu - \frac{1}{2}\right) J_\nu(x) - xJ_{\nu-1}(x), \; \mathcal{Y}_\nu(x) = \left(\nu - \frac{1}{2}\right) Y_\nu(x) - xY_{\nu-1}(x). \tag{A.6c}$$

A.3 Summation Formulas

Consider the summation $\sum_{n=0}^{\infty} f(n)$, in which $f(x)$ has no singularity in $\{z|\mathrm{Re}z \geq 0\}$ and satisfies $\lim_{|z|\to\infty} f(z) = o(|z|^{-1-\delta(z)}), \; \delta(z) > 0$. Then S is

$$\begin{aligned} \sum_{n=0}^{\infty} f(n) &= f(0) + \lim_{\epsilon\to 0_+} \left[\int_{i\infty}^{i\epsilon} dx \frac{f(x)}{e^{2\pi i x} - 1} + \int_{-i\epsilon}^{-i\infty} dx \frac{f(x)}{e^{2\pi i x} - 1} - \frac{1}{2} f(0) \right] \\ &= \int_0^\infty dx f(x) + \frac{1}{2} f(0) + \lim_{\epsilon\to 0_+} \left[i \int_\epsilon^\infty dx \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1} \right]. \end{aligned} \tag{A.7}$$

Since the relation $f^*(ix) = f(-ix) \Leftrightarrow \mathrm{Re}f(ix) = \mathrm{Re}f(-ix), \; \mathrm{Im}f(ix) = -\mathrm{Im}f(-ix)$ is usually satisfied, we can safely say

$$\sum_{n=0}^{\infty} f(n) = \int_0^\infty dx f(x) + \frac{1}{2} f(0) + i \int_0^\infty dx \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1}, \tag{A.8}$$

which is just the *Abel-Plana formula*.

Consider the summation $\sum_{a=m}^{n} f(a)$. For any integer n we have

$$\int_n^{n+1} dx f(x) = \int_n^{n+1} f(x) d\tilde{B}_1(x) = \frac{f(n+1)+f(n)}{2} - \int_n^{n+1} f'(x)\tilde{B}_1(x)dx, \tag{A.9}$$

where $\tilde{B}_n(x)$ is the periodic Bernoulli function. So in our case we have

$$\sum_{a=m}^{n} f(a) = \int_m^n f(x)dx + \frac{f(n)+f(m)}{2} + \int_m^n f'(x)\tilde{B}_1(x)dx, \tag{A.10}$$

which, by employing $\tilde{B}'_{n+1}(x) = (n+1)\tilde{B}_n(x)$, leads us to the *Euler-Maclaurin formula* [179]

$$\begin{aligned}\sum_{a=m}^{n} f(a) &= \int_m^n f(x)dx + \frac{f(n)+f(m)}{2} + \sum_{k=2}^{p-1} \frac{B_k}{k!}\left[f^{(k-1)}(n) - f^{(k-1)}(m)\right] \\ &\quad + R_p, \ R_p = \frac{(-1)^p}{p!}\int_m^n f^{(p)}(x)\tilde{B}_p(x)dx,\end{aligned} \tag{A.11}$$

where we assume $f(x)$ is integrable (usually Riemannian) in $[m,n]$ up to pth order of derivative, B_n is the Bernoulli number, and R_p is referred to as the remaining term. If $f(x)$ has no singularity in $[0,\infty)$ and the conditions $\forall n \in \mathbf{Z}_+ \cup \{0\}$, $\lim_{x\to\infty} f^{(n)}(x) = 0$ and $\lim_{p\to\infty} R_p = 0$, we have

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} dn f(n) + \frac{f(0)}{2} - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0). \tag{A.12}$$

Some summations involving special functions are also useful in our research. For example, those involving the modified spherical Bessel functions as follows [180]

$$\sum_{l=1}^{\infty}(2l+1)P_l(q)e_l(x)s_l(y) = \frac{xye^{-\rho}}{\rho} - e^{-x}\sinh(y), \ \rho = \sqrt{x^2+y^2-2qxy} \tag{A.13a}$$

$$\begin{aligned}\sum_{l=1}^{\infty}(2l+1)P_l(q)e'_l(x)s'_l(y) &= \left[\rho - 3 + \frac{1}{\rho} + \frac{2(x^2+y^2)}{\rho^2}\left(1+\frac{1}{\rho}\right)\right. \\ &\quad \left. - \frac{(x^2-y^2)^2}{\rho^3}\left(1+\frac{3}{\rho}+\frac{3}{\rho^2}\right)\right]\frac{e^{-\rho}}{4} + e^{-x}\cosh(y),\end{aligned} \tag{A.13b}$$

which gives the relation

$$\begin{aligned}\sum_{l=1}^{\infty} \nu e_l^2(x)s_l^2(y) &= \frac{1}{4}\int_{-1}^{1} dq \left[\frac{xye^{-\rho}}{\rho} - e^{-x}\sinh(y)\right]^2 \\ &= \frac{xy}{4}\Big[\mathrm{Ei}[-2(x+y)] - \mathrm{Ei}[-2(x-y)]\Big] - \frac{1}{2}e^{-2x}\sinh^2(y),\end{aligned} \tag{A.13c}$$

where $\mathrm{Ei}(x)$ is the exponential integral.

List of Publications

Kimball A. Milton, Yang Li, Pushpa Kalauni, Prachi Parashar, Romain Guérout, Gert-Ludwig Ingold, Astrid Lambrecht, Serge Reynaud. Negative entropies in Casimir and Casimir-Polder interactions. *Fortschritte der Phys.*, **65**: 1600047, 2017.

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