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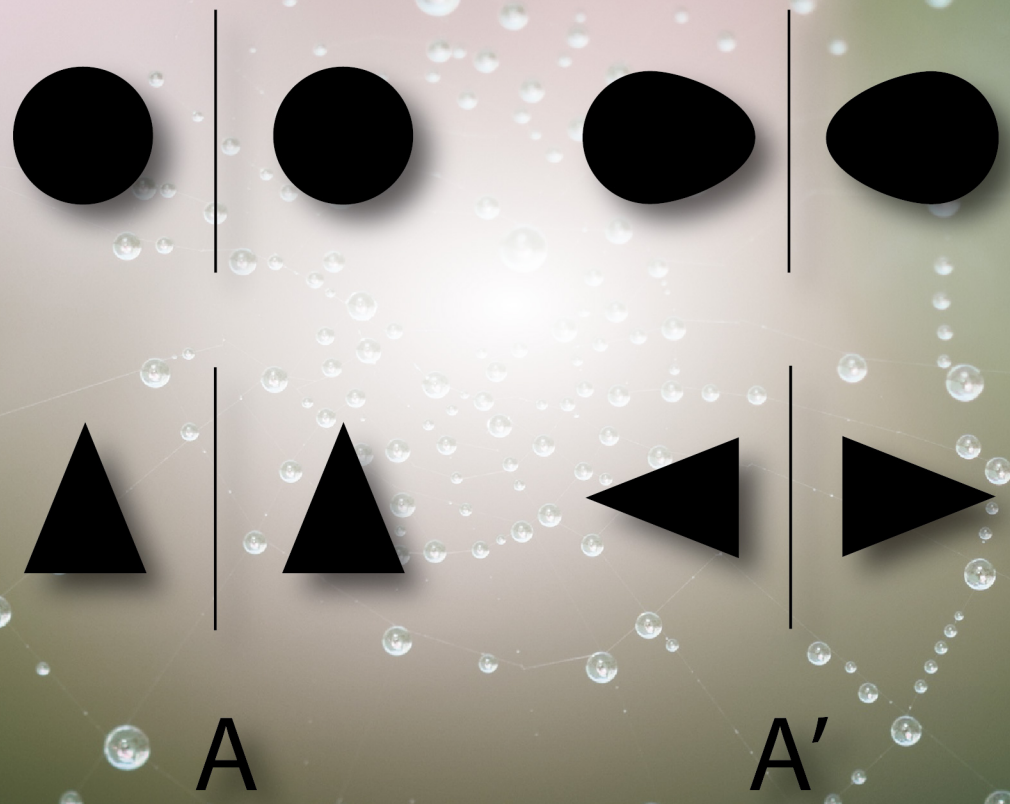

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Schrödinger Equations with Logarithmic Self-Interactions: From Antilinear $\mathcal{PT}$-Symmetry to the Nonlinear Coupling of Channels

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Abstract: Schrödinger equations with non-Hermitian, but $\mathcal{PT}$-symmetric quantum potentials $V(x)$ found, recently, a new field of applicability in classical optics. The potential acquired there a new physical role of an "anomalous" refraction index. This turned attention to the nonlinear Schrödinger equations in which the interaction term becomes state-dependent, $V(x) \to W(\psi(x), x)$. Here, the state-dependence in $W(\psi(x), x)$ is assumed logarithmic, and some of the necessary mathematical assumptions, as well as some of the potential phenomenological consequences of this choice are described. Firstly, an elementary single-channel version of the nonlinear logarithmic model is outlined in which the complex self-interaction $W(\psi(x), x)$ is regularized via a deformation of the real line of x into a self-consistently constructed complex contour C. The new role played by $\mathcal{PT}$-symmetry is revealed. Secondly, the regularization is sought for a multiplet of equations, coupled via the same nonlinear self-interaction coupling of channels. The resulting mathematical structures are shown to extend the existing range of physics covered by the logarithmic Schrödinger equations.

Keywords: PT symmetry; nonlinear Schrödinger equations; logarithmic nonlinearities; coupled-channel systems; regularizations

1. Introduction

During the early days of quantum theory, the typical model-building recipe started from the description of dynamics in classical mechanics. Subsequently, the model was "quantized", i.e., basically, the scalar quantities were replaced by operators, etc. [1]. At present, the most successful model-building strategies are almost entirely opposite: the mathematically much more complicated quantum models are considered primary, while the theoretical verification of their acceptability is based on the analysis of their classical limit (cf., e.g., [2]).

It is not too surprising that the inversion of the recipe proves productive. *Pars pro toto*, let us mention the recent success of $\mathcal{PT}$-symmetric quantum models in which one defines a quantum system first [3]. Thus, in the pioneering letter [4], the authors initiated the study of *bona fide* quantum Hamiltonian operators $H = -d^2/dx^2 + \mathrm{i}x^3$ and $H = -d^2/dx^2 + x^2(\mathrm{i}x)^\delta$, $\delta > 0$, with the analysis of the related classical limits only performed several years later [5].

One of the main physical messages delivered by the latter studies (see also the recent reviews [6,7]) may be seen in an encouragement of the building of quantum models with anomalous properties ranging from the unusual forms of supersymmetry [8–11] up to the mathematically-consistent incorporation of the relativistic kinematics in quantum mechanics [12]. In our present paper, we intend to point out and demonstrate that one of the truly promising next steps might be also seen in a move beyond the linear $\mathcal{PT}$-symmetric quantum mechanics.

2. Effective Hamiltonians

2.1. The Concept of Open Quantum Systems

The general linear quantum evolution equation:

$$\mathrm{i}\frac{d}{dt}|\Psi(t)\rangle = H(t)\,|\Psi(t)\rangle \tag{1}$$

(called, usually, the "time-dependent Schrödinger equation") prescribes the evolution of a ket-vector $|\Psi(t)\rangle \in \mathcal{H}^{(T)}$. Thus, pure states of a quantum system $\mathcal{S}$ are represented in a convenient "textbook" Hilbert space $\mathcal{H}^{(T)}$ [1]. During any measurement (performed, presumably, at a suitable positive time $t = t_{fin} > 0$), our knowledge of the solution $|\Psi(t_{fin})\rangle$ of Equation (1) enables us to consider any (self-adjoint) operator $Q(t)$ of an observable quantity and to predict the probability distribution of its measured values in terms of matrix elements $\langle\Psi(t_{fin})|Q(t_{fin})|\Psi(t_{fin})\rangle$. In the applications of such a theory, one has to keep in mind the conventional tacit assumptions, e.g.,

{A.1} about the pure-state nature of the initial conditions. This must be guaranteed by the preparation of the system at $t = t_{ini} = 0$;

{A.2} about the unitarity of the evolution requiring, due to the Stone theorem [13], the self-adjointness of the Hamiltonian in $\mathcal{H}^{(T)}$, $H(t) = H^{\dagger}(t)$;

{A.3} about the absence of any "external" forces and/or interactions that would couple $\mathcal{S}$ (called "open quantum system") to a dynamically uncontrolled "environment" $\mathcal{S}'$ [14,15].

The validity of all of these assumptions must be carefully checked in applications. In fact, it is rather easy to weaken Assumption {A.1} and to move to quantum statistics working with mixed states. Recently, it has been also shown that it is not so prohibitively complicated to weaken also the self-adjointness Assumption {A.2} while still staying inside the linear quantum theory of textbooks (cf., e.g., the set of review chapters in the recent book [7]). In contrast, the numerous theoretical and mathematical (pars pro toto, let us mention [2]), as well as the more straightforward and pragmatic (cf. [16–18]) analyses of the role of Assumption {A.3} seem to lead to the necessity of a reconsideration of the very foundations and consistent formulations of the linear quantum theory itself.

We believe that it is necessary to pay more attention to the simultaneous weakening of the two apparently separate traditional postulates {A.2} and {A.3} concerning the self-adjointness and linearity of the quantum observable quantities, respectively. In this sense, our present study may be perceived as originating from the recent brief account of the recent history of quantum theory [19] in which the possible respective weakening of Postulates {A.2} and {A.3} was still presented as strictly independent.

This independence was emphasized by the notation. Thus, in the former setting of Postulate {A.2}, the self-adjoint nature of the Hamiltonian in the conventional Schrödinger Equation (1) was emphasized, whenever needed, by the replacement of the ambiguous symbol $H(t)$ by its lower-case version $h(t)$ (cf., e.g., Equation (1.5.1) in [19]). Then, a transition was considered to certain generalized, non-self-adjoint Hamiltonian-resembling operators $G(t)$ controlling the evolution of wave kets (for the rigorous definitions and for further details, the readers may consult also review [20]).

2.2. The Feshbach's Concept of Model Space

In the alternative, more conventional context of the weakening of Postulate {A.3}, let us recall Feshbach's projection-operator method [21]. In it, one partitions the identity operator of the textbook Hilbert space $\mathcal{H}^{(T)}$ into a pair of projectors, $I = R + Q$. According to the traditional philosophy, one interprets projectors R and Q as corresponding to the "almost decoupled" quantum systems $\mathcal{S}$ and $\mathcal{S}'$, respectively. This enables us to partition the Hamiltonian eigenvalue problem in the complete space $\mathcal{H}^{(T)}$,

$$(R+Q)\,H\,(R+Q)\,|\Psi_n\rangle = E_n\,(R+Q)\,|\Psi_n\rangle\,.$$

In the subsequent step, one replaces this eigenvalue problem by its strictly equivalent nonlinear alternative defined in the smaller, R-projected and $\mathcal{S}$-representing open-system subspace alias model space (cf. Equation (1.1.1) in [19]),

$$H_{eff}(E_n)\,|\Phi_n\rangle = E_n\,|\Phi_n\rangle\,, \quad |\Phi_n\rangle = R\,|\Psi_n\rangle\,. \tag{2}$$

Naturally, the price to pay for the full preservation of the physical contents of the theory is that the effective model-space Hamiltonian:

$$H_{eff}(E) = R\,H\,R + R\,H\,Q\,\left[E\,I - Q\,H\,Q\right]^{-1}\,Q\,H\,R \tag{3}$$

becomes state-dependent. This is a nonlinearity feature that leads, also, to the necessity of a re-evaluation of the unitarity {A.2} of the evolution (cf. also [22] for a few related interesting technical details).

3. Antilinear Versus Nonlinear Interactions

3.1. Quantum $\mathcal{PT}$-Symmetric Schrödinger Equations

In 2015, Jorge Cham [23] classified the "parity-time alias $\mathcal{PT}$ symmetry in optics" as the research subject, which belongs among the top ten physics discoveries of the decade. During the same period of time, the subject also (re-)attracted the attention of mathematicians [7]. Still, one should not forget about the decisive role played, especially during the early stages of these developments, by quantum physicists. For the reasons as summarized, e.g., in the reviews [3,6,19], the current quick growth of interest in this brand new branch of classical physics and experimental optics would not probably emerge without its deep theoretical motivation provided by the numerous quantum $\mathcal{PT}$ symmetric models.

In the context of classical physics the most interesting and specific role has been played by the models in which one dealt with certain nonlinear parts of the interactions. These efforts may be found preceded by one of the oldest published papers on the quantum field version of $\mathcal{PT}$ symmetry [24]. In this paper the authors found and proved, constructively, that the expectation value of the field does not vanish in a broad class of the D-dimensional scalar quantum field cases. In the language of mathematics, their interaction Lagrangian was chosen nonlinear and non-Hermitian, but $\mathcal{PT}$ symmetric,

$$\mathcal{L}_{int} \sim \phi^2(\mathrm{i}\phi)^\delta = \phi^2 + \delta\phi^2\ln(\mathrm{i}\phi) + \delta^2\phi^2[\ln(\mathrm{i}\phi)]^2 + \ldots\,. \tag{4}$$

In this formula, the (presumably, small) parameter δ played the role of a measure of the nonlinearity of the interaction. Due to the immediate emergence of certain mathematical difficulties and/or serious open questions [7], the attention has been, subsequently, limited to the linear $\mathcal{PT}$ symmetric quantum models, which remained compatible with the textbook quantum mechanics (see, e.g., the early papers [4,25] or the thorough review [6]).

3.2. The Turn of Attention to Classical Optics

Around the year 2006, many Schrödinger equations describing the quantum-mechanical models were revealed to be mathematically-equivalent to Maxwell equations in media [26,27], see also more recent works [28,29]. This opened a way towards a natural, phenomenologically-motivated introduction of nonlinear interaction terms and, in particular, towards the nonlinear Schrödinger equations.

In our present paper, we decided to start the more specific considerations from one of the simplest special forms of the latter family of equations, viz., from the nonlinear logarithmic Schrödinger equations (LSE),

$$i\partial_t\psi(\vec{x},t) = (-\Delta + V_{LSE})\,\psi(x,t)\,, \quad V_{LSE} = -b\,\ln|\psi(x,t)|^2 \tag{5}$$

containing the d-dimensional Laplacian Δ and a constant real-valued parameter b.

The motivation of such a choice was three-fold. First of all, we took into consideration that the models with logarithmic nonlinearities are exceptional in admitting the existence of spatial solitons and bullets [30,31]. This makes certain nonlinear media well described by these models, especially when the singularity of the logarithm is assumed suitably regularized. In some cases, even the closed-form solitonic solutions can be obtained [31].

Secondly, we felt inspired by the possibility of transition from a single-channel model (5) to its various possible coupled-channel generalizations. In more detail, this motivation will be explained in Section 3.3 below. Our third source of interest in the applicability of the nonlinear and non-Hermitian, but $\mathcal{PT}$-symmetric models preceded the above-cited note [23], which emphasized the role of $\mathcal{PT}$-symmetry in classical optics.

The discovery of this role originated in fact in quantum mechanics [4,26]. In the language of mathematics, the paradox and challenge lied in the reality of the spectra generated by the manifestly non-Hermitian quantum Hamiltonians. What proved worth the generalization was the possibility of a move from the real line of the "coordinate" $x \in \mathbb{R}$ in Equation (5) to its more general, complex-contour alternative $x \in \mathcal{C} \subset \mathbb{C}$ (cf. Section 4 below).

In between the two (viz., real-x and complex-x) extremes, one can find the models in which the coordinates x are discrete (cf., e.g., [32], with further references). This might make the contact with experiments in optics even closer. In a broader context of applications of $\mathcal{PT}$-symmetry to the other areas in mathematics and physics, the readers are recommended to consult the recent review paper [33].

3.3. The Turn of Attention to the Coupling of Channels

The phenomenological success of the linear Schrödinger equations of the single-channel, as well as multiple-channel forms (for example, in quantum chemistry) was based on the computer-supported success of various sophisticated constructive and variational methods. Among them, we felt particularly addressed by the efficiency of the so-called exp S alias coupled-cluster method (CCM) [34]. The search for the wave function $\psi(\vec{x}, t)$ has been found facilitated there, plus in multiple non-chemical applications [35–37], via a pre-multiplication ansatz:

$$\psi = \hat{\mathcal{A}}\, \psi_0. \tag{6}$$

A trivial (plus, usually, time-independent) Slater-determinant reference function ψ_0 is multiplied here by an ad hoc operator written in an exponential-function form, $\hat{\mathcal{A}} = \exp \hat{S}$. In the language of numerical mathematics, the latter ansatz is called a "preconditioning" of the wave function [38]). The essence of the CCM construction of a specific physical state ψ may be then seen in the successful re-construction of the "operator logarithm" $\hat{S}$. This is achieved in the form of its infinite-series expansion, usually in a suitable basis of the standard creation and annihilation operators.

4. Nonlinear Schrödinger Equations on Complex-Plane Contours of the "Coordinate"

The appeal of the single-channel nonlinear Schrödinger equations:

$$i\partial_t \psi(x,t) = -\Delta\, \psi(x,t) + g\, F[\psi(x,t)]\, \psi(x,t) \tag{7}$$

is not only mathematical, but also phenomenological (as we already mentioned, these equations find applications ranging from classical optics [39,40] to the advanced studies of complicated quantum systems [41]). In all of these contexts, the choice of the nonlinearity is desirable, but in practice, mostly restricted to its local-probability-density-like quadratic special case:

$$F_{(quadratic)}[\psi(x,t)] = \varrho(x,t) = \psi^*(x,t)\psi(x,t)\,.$$

The study of the more general and non-polynomial versions of the theory is more difficult and, hence, perceivably less popular. As we already indicated, a special role is played by the logarithmic choice of:

$$F_{(logarithmic)}[\psi(x,t)] = \ln\,[\psi^*(x,t)\psi(x,t)]\ .$$

From the purely formal point of view, the conventional admissibility of the complex values of the wave functions $\psi(x,t)$ may cause serious technical problems in the latter case because the negativity of $\varrho(x,t)$ would imply that one has to treat $F_{(logarithmic)}[\psi(x,t)]$ as a multi-sheeted analytic function with essential singularity in the origin.

4.1. Linear Equations on the Complex Contours

One of the possible ways of circumventing the latter technical obstacle has been recently outlined, in a broader framework of $\mathcal{PT}$-symmetric quantum theory, by the authors of [42]. These authors started from the deep knowledge of the underlying formalism (cf., e.g., its detailed description in [3]). First of all, they emphasized that once the self-interaction in the (linear or nonlinear) Schrödinger equation acquires the form of a multi-sheeted analytic function, one has to keep in mind that the integration contours of x need not be kept real. Without any change in the measurable predictions, they may be, inside the domains of the analyticity of $\psi(x)$, deformed.

Once we admit that x may move along a complex line or contour (say, C), it becomes very natural to introduce the concept of a nonlocal quantum probability density $\sigma(z) = \psi^*(-z,t)\psi(z,t)$, which would be defined in the whole complex plane of the "coordinate" z (let us note that the latter variable z need not necessarily lie on the "admissible" contour C; that is why we changed the symbol). In this context, it is now easy to imagine that once we admit that the choice of the contour C is at our disposal, we may control its choice, and we may guarantee that the conventional positivity of the local probability density $\varrho(x,t) = \psi^*(x,t)\psi(x,t)$ (for which we had $x \in \mathbb{R}$) may find its complex-plane nonlocal (and, for the sake of simplicity, stationary) generalization $\sigma(z) = \psi^*(-z,t)\psi(z,t)$.

The authors of [42] developed this idea in an impressive study of some of its purely technical aspects. Upon their hypothetical nonlocal probability density $\sigma(z) = \psi^*(-z,t)\psi(z,t)$, they imposed the strict direction-dependent reality requirement:

$$\text{Im}\ \sigma(z)dz = 0 \tag{8}$$

and complemented it by the second, direction-dependent "strict positivity" requirement:

$$\text{Re}\ \sigma(z)dz > 0\,. \tag{9}$$

This study proved truly inspiring. In the context of nonlinear Schrodinger equations living on the real line, such a possibility, without the use of the complex direction-determining constraints (8) and (9), was considered by Ablowitz and Musslimani in [43,44].

4.2. Nonlinear Equations on the Complex Contours

The authors of [42] postulated that $\psi^*(-z,t) = \psi(z,t)$. This requirement may be called "unbroken $\mathcal{PT}$-symmetry", implying that in the linear quantum models:

- the latter two equations may be solved to determine the "correct" contour C,
- in a technically highly nontrivial manner, one can also guarantee that for one of the resulting contours, the "candidate for the anomalous quantum probability density" $\sigma(z)$ can be globally normalized to one, $\int_C \sigma(z)dz = 1$.

In the context of our present paper, we imagined that the goal achieved in [42] may prove also useful in the nonlinear-equation setting. We may claim that the demonstration of the existence of the contours C and of the practical feasibility of their reconstruction from the pair of constraints (8) and (9)

may find its immediate application also in the development of the new class of the nonlinear Schrödinger Equation (7) in which one would set:

$$F_{(logarithmic\ \mathcal{PT}-symmetric)}[\psi(x,t)] = \ln\ [\psi^*(-x,t)\psi(x,t)]\ . \tag{10}$$

In other words, interactions $-\log\sigma(z)$ will possess the nice properties of having the self-interaction term real and, hence, easily interpreted in the spirit of standard textbooks. It is important that the construction circumvents the necessity of the analysis of the influence of the essential singularity upon the very definition of the (nonlinear) Hamiltonian operator [45].

It is worth adding that the "optimal" complex contour C depends on the wave functions. This means that it varies with time in general. For this reason, even the authors of the pioneering paper [42] had to accept the adiabatic approximation, set, for simplicity, $\psi(z,t) = \exp(iE_nt)\psi_n(z)$, and treat the contour C as time independent, having postponed "consideration of time-dependent contours to a future paper". Still, their results, incomplete as they are at present, proved important and close to our present interests, especially in emphasizing that although the "mathematics needed to analyze these contours is subtle and involves the use of asymptotics beyond all orders", one should point out that " the existence of such contours is the essential element in establishing the correspondence between complex quantum and classical mechanics ... in the high-quantum-number limit" [42]. We may just add here that the same mathematics also lies in the grounds of the possible extension of these results to the logarithmically nonlinear complex quantum/classical mechanics.

5. Effective Nonlinearities

Initially, our considerations were restricted to the standard linear quantum theory in which we found several sources of motivation (cf. Section 6 below). Then, we turned attention to the potentially useful phenomenological (cf. Section 7) and mathematical (cf. Section 8) aspects of various nonlinear extensions of the conventional linear models.

We came to the conclusion that at least some of the nonlinear Schrödinger equations deserve a deeper study and, perhaps, an extension to matrix models. The preliminary illustration of such an expectation has been found via the study of several special cases and analytical solutions; cf. Section 9 and the discussion in Section 10.

5.1. Coupled Cluster Wave-Function Ansatz

There exist several reasons for the (a priori, not quite expected) practical success of the CCM approach in which one replaces the construction of the Hilbert-space vector ψ by the construction of the Hilbert-space operator (i.e., of the $M \times M$ matrix $\hat{A}$ with $M = \infty$). Among them, let us mention here just the fast a posteriori convergence of the results obtained, within the framework of quantum chemistry, via truncations (i.e., via finite-dimensional approximations using $M < \infty$).

We feel impressed and inspired by the practical success of applications of the CCM idea. Its assumptions of the stationarity of the system and of the constancy of ψ_0 lead to the reduction of Equation (13) to the time-independent eigenvalue problem. It can be rewritten in the specific, user-friendly CCM linear equation:

$$\left(\hat{H} - E\right)\ \exp\hat{S} = 0\,. \tag{11}$$

The equation is to be satisfied by the eligible preconditioning operators $\hat{S} = \ln\hat{A}$ (cf., e.g., review papers [46] and/or [47] for more details). In this context, the most natural matrix, channel-coupling realistic upgrade of Equation (5) would be the following coupled set of M^2 logarithmic Schrödinger equations:

$$i\partial_t\mathcal{A} + \Delta\mathcal{A} - V_{(NL)}\mathcal{A} = 0\,, \quad V_{(NL)} = -b\ln(\mathcal{A}^\dagger\mathcal{A})\,. \tag{12}$$

They may be expected to contain the manifestly $\mathcal{A}$-dependent interaction term, generating the time-evolution of the M^2-plet of the time-dependent matrix elements of $\mathcal{A} = \mathcal{A}(\vec{x},t)$. We intend to pay

attention to a few consequences of the scalar-to-matrix generalizations of the nonlinear interactions in their specific logarithmic-nonlinearity sampling.

5.2. Broader Context in Physics

One of the key motivations of the efforts aimed at a suitable nonlinear reformulation of quantum theory may be seen in the lack of any consistent formal bridge between linear theory and general relativity [48]. Typically, the role of the environment $\mathcal{S}'$ would be played there by the self-gravity of the particles themselves. Thus, in the language of the Newton–Schrödinger equation, the Hamiltonian in Equation (1) (or, at least, its "effective potential" part V_{eff}) would vary with the wave function in a way controlled, say, by an independent (say, Poisson) equation [2,31].

For the sake of the feasibility of calculations, the structure of V_{eff} is often postulated in advance. One of the best known examples is provided by [16,17] in which the toy models are considered in the form of the nonlinear logarithmic Schrödinger Equation (5) with the wave-function solutions $\psi \in L^2(\Re^d)$ studied in an interval of time $t \in (t_0, t_1)$. This equation, along with its relativistic analogue, finds multiple applications in the physics of quantum fields and particles [49–55], extensions of quantum mechanics [16,56], optics and transport or diffusion phenomena [57–60], nuclear physics [61, 62], the theory of dissipative systems and quantum information [63–68], the theory of superfluidity [69–72] and the effective models of the physical vacuum and classical and quantum gravity [73–76], where one can utilize the well-known fluid/gravity analogy between inviscid fluids and pseudo-Riemannian manifolds [77–81]. The relativistic analogue of Equation (5) is obtained by replacing the derivative part with the d'Alembert operator and is not considered here.

Naturally, in the general nonlinear context, the physical meaning of the solutions ψ, as well as the proper interpretation of their evolution depend on the phenomenological background of the application one has in mind. The strict coincidence of the physical predictions between the standard linear quantum model and its effective nonlinear alternative will decrease with the growth of the uncertainty, i.e., of the size of the space eliminated, say, by Feshbach's partitioning in Equation (3). The simplification may enhance the errors. Still, the merits of Equation (5) and, in particular, of the logarithmic form of the nonlinearity are shared by many concrete physical applications. The close limiting-transition correspondence between the nonlinear and linear models may be preserved, in principle at least.

6. Roots in Linear Theory

The evolution of quantum systems is commonly described in the Schrödinger picture [1], by the Schrödinger equation:

$$i\partial_t \psi(\vec{x}, t) = \hat{H}\psi(\vec{x}, t)\,, \quad \hat{H} = -\Delta + V(\vec{x}), \tag{13}$$

which is linear. Besides, in this equation the choice of a form of the potential $V(\vec{x})$ is usually dictated by a combination of pragmatic and theoretical considerations. In this section, we give some arguments for how the effective nonlinearities can arise in quantum theory despite the linearity of the "global" Schrödinger equations themselves.

6.1. Pilot-Wave Approach

For the purposes of illustration, one can select, as a starting point, the context of (linear) quantum mechanics. In its de-Broglie and Bohm formulation [82] (often called "pilot-wave" formulation [83]), one uses the Madelung representation of a conventional wave function, in which the latter factorizes into the product:

$$\psi(\vec{x}, t) = R(\vec{x}, t)\, \exp[iS(\vec{x}, t)] \tag{14}$$

where both of the auxiliary functions $R(\vec{x}, t)$ and $S(\vec{x}, t)$ are real-valued (see also [84,85] for more commentaries). Such an assumption becomes particularly relevant in the subsequent step of

a nonlinearization of the theory in which one inserts ansatz (14) in Equation (5). One reveals that the nonlinear interaction term:

$$V_{LSE} \sim \ln|\psi(\vec{x},t)|^2 = 2\,\ln R(\vec{x},t) \tag{15}$$

becomes perceivably simplified and entirely independent of the phase $S(\vec{x},t)$, so one feels encouraged to analyze the consequences.

The scalar, single-component nature of the real functions $\psi(\vec{x},t)$, $R(\vec{x},t)$ and $S(\vec{x},t)$ may be perceived as an artificial restriction. Indeed, the simplifying role of the polar decomposition (14) could have been easily extended to matrices. This is one of the guiding ideas of our present work. Indeed, as long as there exists a very natural polar decomposition for matrices, we decided to pay attention to the replacement of the complex scalar quantity $\psi(\vec{x},t)$ in Equation (5) by an (say, invertible) M by M complex matrix $\mathcal{A}(\vec{x},t)$.

6.2. Solvable and Partially-Solvable Models

The relevance of both of the latter aspects of the choice of $V(x)$ may be exemplified by the tremendous popularity of the exactly solvable (ES) harmonic oscillator $V_{(HO)}(\vec{x}) \sim |\vec{x}|^2$, which combines the non-numerical, analytic (or algebraic) nature of Equation (13) with the appeal of the equidistant, purely vibrational spectrum of the low-lying bound-state energies $E_{(HO)}$ and of the localized closed-form wave functions:

$$\psi_{(HO)}(\vec{x},t) \sim \exp(-iE_{(HO)}t)\,\exp\left(-|\vec{x}|^2/2 + \mathcal{O}(\ln|\vec{x}|)\right), \tag{16}$$

which may be recalled an explicit illustration of the polar decomposition (14).

The other choices of the phenomenologically useful interactions may emphasize either the formal friendliness of $V(\vec{x}) \neq V_{(HO)}(\vec{x})$ (e.g., the closed-form tractability of the ES Schrödinger equations [86–88]) or its capability of providing an insight into the variability of the dynamics (thus, one may add perturbations to $V_{(HO)}(\vec{x})$), etc.

Furthermore, coexistence between mathematics and phenomenology has been also achieved via the so-called quasi-exactly solvable (QES) models [89–91]. In this setting, one starts from a qualified guess of a few suitable input wave-function candidates. The idea proves particularly productive when the logarithmic corrections in (16) are replaced by an explicit ansatz. In the case of an ℓ-th partial wave and for an n-th radial excitation, for example, the formula [92,93]:

$$\psi^{(QES)}_{n,\ell}(r,t) \sim \exp(-iE_n t)\,\exp(-r^2/2)\,\sum_{k=0}^{N} a_k^{(n)}\, r^{k+\ell+1}, \tag{17}$$

where $r = |\vec{x}|$, can be recalled as providing one of the analytically-solvable illustrations (the readers may consult, e.g., [94–98] for multiple extensions of such an approach to quantum model building).

In both of the ES and QES studies, an interest in the formal questions prevails (see, e.g., the most recent review [99]). We intend to pursue here a slightly different implementation of the ansatz idea. The emphasis will be shifted from the detailed study of correction terms $\mathcal{O}(\ln|\vec{x}|)$ in Equation (16) to the generic large$-|\vec{x}|$ behavior of the time-independent modulus of the asymptotically harmonic wave functions as prescribed by Formula (16),

$$\psi^*_{(HO)}(\vec{x},t)\psi_{(HO)}(\vec{x},t) \sim \exp(-|\vec{x}|^2 + \mathcal{O}(\ln|\vec{x}|))\,. \tag{18}$$

Such an asymptotic estimate of wave functions may be perceived as related to a certain equivalence-class property of potentials,

$$V_{(initial)}(\vec{x}) = |\vec{x}|^2 + \mathcal{O}(\ln|\vec{x}|) \sim -\ln[\psi^*(\vec{x},t)\psi(\vec{x},t)] = V_{(generalized)}(\vec{x})\,. \tag{19}$$

Virtually the same relationship could have been also deduced from our QES toy model (17):

$$V_{(generalized)}(r) = -b\,\ln[\psi^*_{(QES)}(r,t)\psi_{(QES)}(r,t)] \sim r^2 + \mathcal{O}(\ln r) = V_{(initial)}(r), \tag{20}$$

if one imposes $b = 2$.

Notice that the correction term $\mathcal{O}(\ln|\vec{x}|)$ is only negligible in the asymptotic domain. Our illustrative example (17) demonstrates that the non-exponential, power-law components of the wave function become dominant near the origin. In this domain of coordinates, Equation (18) becomes replaced by an alternative estimate,

$$\psi^*(\vec{x},t)\psi(\vec{x},t) \sim \mathcal{O}(|\vec{x}|^{const}) \tag{21}$$

so that, in full analogy with the standard $M = 1$ nonlinear logarithmic Schrödinger Equation (5), one encounters one of the natural limitations of the parallels with linear models.

In light of all of these formulae, it is not too surprising that the special, logarithmically-nonlinear Schrödinger Equation (5) was proposed as an eligible generalization of its linear predecessor in quantum theory [16].

7. Roots in Phenomenology

An important formal support for the turn of attention to the matrix nonlinear wave-equation model (12) may be found again in the non-commutative matrix version:

$$\mathcal{A}(\vec{x},t) = U(\vec{x},t)P(\vec{x},t) \tag{22}$$

of the polar decomposition formula in which the auxiliary $M \times M$ matrix $U(\vec{x},t)$ is required to be unitary, while the second $M \times M$ matrix factor $P(\vec{x},t)$ is demanded to be positive definite and Hermitian, $P = P^\dagger > 0$. Thus, for any given (and, say, invertible) $M \times M$ matrix $\mathcal{A}$, a full formal analogy between fundamental Formulae (14) and (22) is established. The existence and uniqueness of the latter formula form a mathematical core of our present message.

Naturally, at the nontrivial matrix dimensions $M > 1$, the attempted (and, say, logarithmic) nonlinearization of the theory encounters several challenges on the conceptual level. Nevertheless, the underlying mathematical idea of factorization (22) preserves the closest analogy with its conventional scalar $M = 1$ special case (14).

In this sense, also the choice of the matrix interaction term in its logarithmic form:

$$V_{(NL)} = -b\ln(\mathcal{A}^\dagger\mathcal{A}) = -2\,b\,\ln P(\vec{x},t) \tag{23}$$

seems optimal. It is worth emphasizing that even the first nontrivial case with $M = 2$ and with:

$$\mathcal{A} = \begin{pmatrix} \psi_{11}(x,t) & \psi_{12}(x,t) \\ \psi_{21}(x,t) & \psi_{22}(x,t) \end{pmatrix} \tag{24}$$

may be perceived as a dynamical evolution model of four (in general, of M^2) complex-valued functions $\psi_{ab}(x,t)$.

As we already emphasized, our interest in the matrix model of the evolution (12), (23) is mainly motivated, among all of its eligible physical backgrounds, by the conventional (linear) quantum mechanics. In such a framework, one often employs the point-particle interpretation of the single-channel wave functions ψ_j and moves to the coupled-channel scenario in which the dynamics is still linear and controlled by a suitable matrix potential [100].

Apparently, a nonlinear generalization of the coupled-channel evolution equation might seem natural and straightforward, but one has to avoid several technical, as well as conceptual obstacles. Firstly, although the experimental tests as performed for conserved systems in atomic physics

excluded any quantitatively predictive implementation of the "effective nonlinearity" hypothesis of the logarithmic type, the reality-mimicking situation appeared perceivably more encouraging in nuclear-physics phenomenology where the spatial separation of the individual fermions may be expected to be reduced [61,62], as well as in a theory of superfluidity, where many-body interactions become strongly nonlinear with an increase of density [69–72]. The appealing possibility of making quantum theory slightly nonlinear survived as a challenging theoretical option.

In this spirit, keeping the mathematical structure of equations sufficiently close to the quantum-lattice-like picture of interacting harmonic oscillators, the resulting effective, nonlinear forms of quantum mechanics could also shorten its distance from the formalism of relativistic quantum field theory [101], in principle at least.

In the opposite direction, let us finally contemplate the transition from scalar quantities ψ to matrices $\mathcal{A}$ in a way aimed at a mathematical parallelism between the non-scalar $M \times M$ nonlinear theory and its linear, quantum-mechanical predecessors in which there emerges the direct concept of the coupling of channels. What comes immediately in mind are (linear) Schrödinger equations in which the single wave function $\psi(\vec{x}, t)$ (describing, say, the motion of a single particle or quasiparticle) is replaced by a K-plet $\vec{\psi}(\vec{x}, t) = \psi_1(\vec{x}, t), \psi_2(\vec{x}, t), \ldots, \psi_K(\vec{x}, t)$ of independent wave functions describing, in the language of physics, separate "channels". In this sense, the replacement of ψ by $\mathcal{A}$ can be perceived as just one of the special choices.

The resulting picture may be given a slightly modified interpretation in which the individual single-channel wave functions become treated as infinite, N-dimensional bra-vectors with $N = \infty$. Once the coordinates may be discretized, one obtains an N-dimensional column-vector $\psi(x_i, t)$, i = 1, 2, ..., N representing the single-channel wave function. A K-plet of these functions might be then perceived as a non-square, $N \times K$ matrix $\mathcal{A}$.

Although we do not intend to follow the latter, alternative line of consideration in what follows, the study of the latter possibility should still be kept in mind (we are skipping this idea here mainly because we would have to work with non-square matrices, losing the polar decomposition (22) and having to replace the usual Laplacian Δ by its discrete alternative). Still, having matrices $\mathcal{A} = \mathcal{A}(\vec{x}, t)$ defined as the K-plets of wave-function components $\psi_j(\vec{x}, t)$ with $j = 1, 2, \ldots, K$, the underlying quantum theory could be also made nonlinear along very similar lines.

8. Constructions Strategies

Let us now formulate some general remarks about the scenario in which matrix $\mathcal{A}$ has rank M.

The transition from scalars ψ to matrices $\mathcal{A}$ opens several technical problems related to the generic non-commutativity between $\mathcal{A}$ and its Hermitian conjugate $\mathcal{A}^\dagger$. One of the most natural technical tools of simplification of the general $M \times M$ problem may be seen in various reparametrizations of these matrices.

As we already emphasized, the most promising sample of such a reparametrization is provided by the polar decomposition (22). Unfortunately, in spite of its feasibility at the sufficiently small matrix dimensions M, such a reparametrization becomes complicated if we encounter some special forms of $\mathcal{A}$ at larger M. One sample of such a sparse-matrix assumption and scenario will be provided below.

One can expect that our choice of the forms of the (not necessarily Hermitian) matrix $\mathcal{A}$ will be dictated by its algebraic (i.e., Lie-algebraic) characteristics. In such a setting, one of the promising (and, at the same time, not necessarily perturbative) technical ingredients may be seen in the use of the Baker–Campbell–Hausdorff identity [102]:

$$\ln(\exp A \exp B) = A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [B, [B, A]]) -$$

$$- \frac{1}{4!}[B, [A, [A, B]]] - \frac{1}{6!}([A, [A, [A, [A, B]]]] + [B, [B, [B, [B, A]]]]) + \ldots . \quad (25)$$

Indeed, once we reparametrize $\mathcal{A} = \exp B$ and $\mathcal{A}^\dagger = \exp A$, we may try to require, tentatively, that starting from a certain level of nesting, the nested commutators between matrices A and B would vanish. Then, the series (25) will be truncated.

Let us now describe one of the preliminary possibilities of a consistent bridging of the gap. We shall restrict our attention just to the most elementary possibility in which one perceives the quantum (and, perhaps, free or just weakly self-interacting) field as a set of harmonic oscillators. In such a perception, one could very naturally assign nonlinear generalizations (5) to every one of these oscillators. Next, we may mimic the field dynamics by a suitable weak (i.e., say, nearest-neighbor) mutual interaction between these oscillators.

On this background, we analyze the possibility in which the coupling between the individual oscillators (i.e., between the initially decoupled linear Schrödinger Equation (13)) gets mediated by a tentative, model-forming replacement of a vector ψ by a (say, very weakly non-diagonal) matrix $\mathcal{A}$.

Once one accepts the effective theory interpretation of the single nonlinear Equation (5) and once one re-writes it in the equivalent form (12) for a single scalar function $\mathcal{A} = \psi(\vec{x}, t) = \psi_1(\vec{x}, t)$, one gets immediately interested in a transition to an M-plet of such equations, which are only coupled by the logarithmic interaction term. For such a purpose it is sufficient to re-interpret symbol $\mathcal{A}$, say, as a weighted diagonal matrix,

$$\mathcal{A} = \begin{bmatrix} g_1\,\psi_1(\vec{x},t) & 0 & \dots & 0 \\ 0 & g_2\,\psi_2(\vec{x},t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & g_M\,\psi_M(\vec{x},t) \end{bmatrix}. \tag{26}$$

One of the most natural tentative physical interpretations of such a generalization should proceed strictly along the lines as discussed in Section 9 below.

In between the two extremes of a diagonal and of a fully-general matrix form of $\mathcal{A}$, there exists a natural intermediate structure in which one stays as close to the most elementary diagonal extreme as possible, but in which one recalls the nearest-neighbor coupling idea as used most often in linear Schrödinger equations. This leads to the ansatz:

$$\mathcal{A} = \begin{bmatrix} g_1\,\psi_1(\vec{x},t) & h_1\,\chi_1(\vec{x},t) & 0 & \dots & 0 \\ f_2\,\lambda_2(\vec{x},t) & g_2\,\psi_2(\vec{x},t) & h_2\,\chi_2(\vec{x},t) & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & f_{M-1}\,\lambda_{M-1}(\vec{x},t) & g_{M-1}\,\psi_{M-1}(\vec{x},t) & h_{M-1}\,\chi_{M-1}(\vec{x},t) \\ 0 & \dots & 0 & f_M\,\lambda_M(\vec{x},t) & g_M\,\psi_M(\vec{x},t) \end{bmatrix}. \tag{27}$$

In its full generality, such an ansatz looks fairly complicated. At the same time, the flexibility of an interplay between the linear and nonlinear channel-coupling mechanisms seems phenomenologically highly promising.

The way out of the dilemma leads via a return to the $M = 2$ models of preceding sections. At the same time, the growth of M still keeps the matrix (27) tractable as factorized into the product of an upper-triangular and lower-triangular matrix. Thus, whenever the coupling constants h_j and f_k remain sufficiently small, this opens a way towards the tractability of similar systems at arbitrarily large dimensions M.

9. Analytical Solutions

In this section, we pay attention to the possible simulations of the interaction between a system and its environment, which would be based on the logarithmic type of the interaction's nonlinearity. Let us, therefore, start from the choice of the first nontrivial case and consider solutions of the matrix LSE of the form (12) with $\mathcal{A}$ being a 2×2 matrix (24). For the sake of simplicity, let us also limit our attention to the case of the single spatial dimension replacing vector $\vec{x}$ by scalar x. Then, Equation (12) becomes:

$$i\partial_t \mathcal{A} + \partial_{xx}\mathcal{A} + b\ln(\mathcal{A}^\dagger \mathcal{A})\mathcal{A} = 0, \tag{28}$$

and we also impose that:

$$\int_{x_0}^{x_1} \mathrm{tr}(\mathcal{A}^\dagger \mathcal{A})dx = \mathcal{N}, \tag{29}$$

assuming that the system is confined to the interval of $x \in [x_0, x_1]$. The latter formula comes from the normalization condition: $\int d\vec{x}\,\mathrm{tr}(\mathcal{A}^\dagger \mathcal{A}) \equiv \langle \mathcal{A}|\mathcal{A}\rangle = \mathcal{N}$, where integration is taken over a spatial volume taken by the system, $\mathcal{N}$ being a constant usually interpreted as a number of particles inside such a volume, while tr here is the standard matrix-trace operation.

Furthermore, below, we consider the cases when we managed to obtain exact analytical solutions of Equation (28).

9.1. Diagonal Case

We shall start our analysis from the simple, yet nontrivial case of the two coupled LSEs, written in the form of (12), (24), $\mathcal{A}$ being a diagonal matrix:

$$\mathcal{A} = \begin{pmatrix} \psi_1(x,t) & 0 \\ 0 & \psi_2(x,t) \end{pmatrix}, \tag{30}$$

where $\psi_a(x,t)$ are complex-valued functions. In this model, the auxiliary cross-coupling terms are neglected. Considering this case enables us to contemplate, more easily, a pair of uncoupled logarithmic Schrödinger equations in a way that would emphasize also certain parallelism with the concept of the coupled channels in the linear theory setting. Moreover, in the language of phenomenology, any pair of similar individual equations, linear or nonlinear, may be perceived as mimicking a fully-separated evolution of an isolated system $\mathcal{S}_1$ (of our immediate interest) and of its remote and irrelevant, "switched-off" environment $\mathcal{S}_2$.

In the more realistic situations, one can only rarely neglect the possible cross-interaction between subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ completely. Still, the most immediate reward of the study of such a mutual interaction usually comes when one assumes that the resulting "perturbation" of the relevant subsystem $\mathcal{S}_1$ remains weak.

The ground-state solution of Equation (28) can be found exactly. In a rest frame, it has the form of the gausson, i.e., by the Gaussian packet modulated by the de Broglie plane wave,

$$\psi_a(x,t) = C_a \exp\left(-\frac{1}{2}b\,x^2 + \nu_a x - iE_a t\right), \tag{31}$$

where $a = 1, 2$ and where:

$$E_a = b(1 - \ln C_a^2) - \nu_a^2 \tag{32}$$

is the energy of a wave in the a-th channel. Quantities C_a and ν_a are integration constants related, together with b, to the mean and variance of the Gaussian packet.

If one imposes also the normalization condition (29), then one obtains an additional constraint for the integration constants:

$$F_1(x_0) - F_1(x_1) + F_2(x_0) - F_2(x_1) = 2\sqrt{\frac{b}{\pi}}\mathcal{N}, \tag{33}$$

where we denoted $F_a(x) = C_a^2 \exp\left(\frac{\nu_a^2}{b}\right) \operatorname{erf}\left(\frac{\nu_a - bx}{\sqrt{b}}\right)$. For instance, when x_1 and x_0 are set to plus and minus infinity, then this constraint takes a simple form: $\sum_{a=1}^{2} C_a^2 \exp\left(\frac{\nu_a^2}{b}\right) = \sqrt{\frac{b}{\pi}}\mathcal{N}$.

Note that due to the Galilean symmetry of LSE, from the solution (31), one can always obtain gausson solutions whose center of mass propagates with velocity v_a, independently for each channel. For instance, one can check that the following function:

$$\psi_a(x,t) = C_a \exp\left[-\frac{1}{2}(x - v_a t)\left(b(x - v_a t) - 2\nu_a - i v_a\right)\right], \tag{34}$$

is also a solution of Equations (28) and (31), provided:

$$(v_a/2)^2 + \nu_a^2 = b(1 - \ln C_a^2), \tag{35}$$

v_a being another real-valued integration constant.

Finally, one can check that the solution (30) naturally satisfies the commutativity condition:

$$\left[\mathcal{A}^\dagger, \mathcal{A}\right] = 0, \tag{36}$$

which eliminates the question of ordering inside the logarithmic term. This condition can be also considered as a requirement that one works with two observables described by $\mathcal{A}^\dagger$ and $\mathcal{A}$ and having common eigenvectors, i.e., being measurable simultaneously.

9.2. Off-Diagonal Case

Contrary to the previous case, let us assume that it is the channel-coupling terms that are dominating now. Therefore, one can assume $\mathcal{A}$ to be an off-diagonal matrix:

$$\mathcal{A} = \begin{pmatrix} 0 & \bar{\psi}_1(x,t) \\ \bar{\psi}_2(x,t) & 0 \end{pmatrix}, \tag{37}$$

where $\bar{\psi}_a(x,t)$ are complex-valued functions.

Still, the ground-state solution of Equation (28) can be found exactly. In a rest frame, it also has the form of the Gaussian packet modulated by the de Broglie plane wave,

$$\bar{\psi}_a(x,t) = C_a \exp\left(-\frac{1}{2} b\, x^2 + \nu_a x - i\bar{E}_a t\right), \tag{38}$$

where $a = 1, 2$, and:

$$\bar{E}_a = \nu_a^2 - b(1 - \ln C_a^2) \tag{39}$$

is the energy of a wave, while C_a and ν_a are the integration constants related, together with b, to the mean and variance of the Gaussian packet. If one imposes also the normalization condition (29), then one obtains an additional constraint for the integration constants, which is identical to Equation (33).

Comparing the solution (38) with its diagonal analogue (31), one can see an interesting feature: two solutions can be transformed from one another by a simple time inversion:

$$\psi_a(x,t) = \bar{\psi}_a(x,-t), \;\; E_a = -\bar{E}_a, \tag{40}$$

which can indicate that the diagonal and off-diagonal terms of the matrix $\mathcal{A}$ (24) describe the processes happening in opposite directions of time or, alternatively, having opposite signs of their energy eigenvalues.

Finally, one can also check that, unlike the diagonal case, the commutator $\left[\mathcal{A}^{\dagger}, \mathcal{A}\right]$ does not vanish:

$$\left[\mathcal{A}^{\dagger}, \mathcal{A}\right] = \mathrm{e}^{-b\,x^2}\left(C_2^2\mathrm{e}^{2\nu_2 x} - C_1^2\mathrm{e}^{2\nu_1 x}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{41}$$

unless $C_1^2 = C_2^2$ and $\nu_1 = \nu_2$; some features of the case with a vanishing commutator were discussed after Equation (36) above.

10. Discussion

Based on the motivations coming from different areas of quantum mechanics, such as the coupled-channel systems, the logarithmically nonlinear quantum wave equation of the matrix type was introduced. In the realm of the effective nonlinear quantum theories (i.e., say, for open quantum systems or in the theory of superfluidity or Bose–Einstein condensates [69]), the matrix logarithmic model might be expected to keep resemblance with its "single-channel" predecessor, sharing with it also some of the more advanced applications. Beyond that realm, the model is capable of playing several other roles ranging from one of the specific nonlinear generalizations of the classical wave equation up to some of their non-classical, perturbative or non-perturbative extensions.

Besides, the matrix model could offer new inspiration to the current experimental studies in quantum optics (cf. [57]) or to the methodical considerations in information theory (cf. [63–68]). One of the particularly encouraging recent mathematical results obtained by Babin and Figotin [103] indicates that in the semi-classical range, the combination of the logarithmic nonlinearities with the coupled-equation structures may prove unexpectedly productive. In the quantum limit of their formalism, remarkably enough, the energy levels of the hydrogen atom get close to their quantum textbook values. At the same time, a contact between the classical and quantum phenomenology is achieved via an innovative physical reinterpretation of wave functions $\mathcal{A}$ treated as charges; not possessing the usual quantum probabilistic interpretation, but degenerating, correctly, to the trajectories of classical point-particles in appropriate limits.

Some special cases and analytical solutions, for the case of 2×2 matrices, seem to have a general feature of possessing a Gaussian wave packet shape modulated by the de Broglie plane waves. The diagonal and off-diagonal components of the matrix turn out to be describing the waves with opposite signs of energy or, alternatively, moving in opposite directions of time. Other types of matrix solutions can be the subject of future studies.

11. Conclusions

With the motivation originating in several different areas of physics, a new, logarithmically nonlinear form of wave equations was proposed and studied. A key motivation of our generalizations of the logarithmic nonlinearity to complex functions and/or to matrices was found in quantum physics. In both of the particular contexts of the quantum mechanics of $\mathcal{PT}$-symmetric and/or coupled-channel systems, the respective complex and/or scalar matrix generalizations appeared surprisingly natural. Exact analytic solutions of our equation were also obtained. For some of the most elementary special cases, the solutions coincide with the matrix versions of the gaussons (i.e., of the Gaussian packets modulated by de Broglie plane waves). The individual components of our toy model solution matrices describe the waves that move in opposite directions so that the model intertwines the incoming and outgoing waves via an innovative version of their nonlinear self-interaction.

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References

1. Messiah, A. *Quantum Mechanics*; North-Holland Publishing Company: Amsterdam, The Netherlands, 1961; Volume I.
2. Brody, D.C.; Houghston, L.F. Geometric Quantum Mechanics. *J. Geom. Phys.* **2001**, *38*, 19–53.
3. Bender, C.M. Making sense of non-Hermitian Hamiltonians. *Rep. Prog. Phys.* **2007**, *70*, 947.
4. Bender, C.M.; Boettcher, S. Real spectra in non-Hermitian Hamiltonians having PT symmetry. *Phys. Rev. Lett.* **1998**, *80*, 5243.
5. Mostafazadeh, A. Metric operator in pseudo-Hermitian quantum mechanics and the imaginary cubic potential. *J. Phys. A Math. Gen.* **2006**, *39*, 10171.
6. Mostafazadeh, A. Pseudo-Hermitian representation of quantum mechanics. *Int. J. Geom. Methods Mod. Phys.* **2010**, *7*, 1191.
7. *Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects*; Bagarello, F., Gazeau, J.-P., Szafraniec, F. H., Znojil, M., Eds.; John Wiley & Sons, Inc.: Hoboken, NJ, USA, 2015.
8. Andrianov, A.A.; Ioffe, M.V. Cannata, F.; Dedonder, J.-P. SUSY quantum mechanics with complex superpotentials and real energy spectra. *Int. J. Mod. Phys. A* **1999**, *14*, 2675–2688.
9. Znojil, M.; Cannata, F.; Bagchi, B.; Roychoudhury, R. Supersymmetry without hermiticity within PT symmetric quantum mechanics. *Phys. Lett. B* **2000**, *483*, 284–289.
10. Znojil, M. Non-Hermitian SUSY and singular PT-symmetrized oscillators. *J. Phys. A Math. Gen.* **2002**, *35*, 2341–2352.
11. Znojil, M.; Jakubsky, V. Supersymmetric quantum mechanics living on topologically nontrivial Riemann surfaces. *Pramana J. Phys.* **2009**, *73*, 397–404.
12. Mostafazadeh, A. Hilbert space structures on the solution space of Klein-Gordon type evolution equations. *Class. Quantum Gravit.* **2003**, *20*, 155–171.
13. Stone, M.H. On one-parameter unitary groups in Hilbert space. *Ann. Math.* **1932**, *33*, 643–648.
14. Moiseyev, N. *Non-Hermitian Quantum Mechanics*; Cambridge University Press: Cambridge, UK, 2011.
15. Eleuch, H.; Rotter, I. Resonances in open quantum systems. *Phys. Rev. A* **2017**, *95*, 022117.
16. Bialynicki-Birula, I.; Mycielski, J. Nonlinear Wave Mechanics. *Ann. Phys.* **1976**, *100*, 62–93.
17. Bialynicki-Birula, I.; Mycielski, J. Uncertainty relations for information entropy in wave mechanics. *Commun. Math. Phys.* **1975**, *44*, 129–132.
18. Mielnik, B. Empty Bottle: The Revenge of Schrodinger's Cat? *J. Phys.Conf. Ser.* **2017**, *839*, 012006.
19. Znojil, M. Non-selfadjoint operators in quantum physics: Ideas, people and trends. In *Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects*; John Wiley & Sons, Inc.: Hoboken, NJ, USA, 2015; pp. 7–58.
20. Znojil, M. Three-Hilbert-space formulation of Quantum Mechanics. *Symmetry Integrability Geomi. Methods Appl.* **2009**, *5*, doi:10.3842/SIGMA.2009.001.
21. Feshbach, H. Unified theory of nuclear reactions. *Ann. Phys.* **1958**, *5*, 357–390.
22. Znojil, M. Linear representation of energy-dependent Hamiltonians (quant-ph/0403223). *Phys. Lett. A* **2004**, *326*, 70–76.
23. Cham, J. Top 10 physics discoveries of the last 10 years. *Nat. Phys.* **2015**, *11*, 799. doi:10.1038/nphys3500.
24. Bender, C.M.; Milton, K.A. Nonperturbative Calculation of Symmetry Breaking in Quantum Field Theory. *Phys. Rev. D* **1997**, *55*, R3255.
25. Buslaev, V.; Grechi, V. Equivalence of unstable anharmonic oscillators and double wells. *J. Phys. A Math. Gen.* **1993**, *26*, 5541–5549.
26. Musslimani, Z.H.; Makris, K.G.; El-Ganainy, R.; Christodoulides, D.N. Optical Solitons in PT Periodic Potentials. *Phys. Rev. Lett.* **2008**, *100*, 030402.

27. Makris, M.G.; El-Ganainy, R.; Christodoulides, D.N.; Musslimani, Z.H. Beam dynamics in PT symmetric optical lattices. *Phys. Rev. Lett.* **2008**, *100*, 103904.
28. Zloshchastiev, K.G. Quantum-statistical approach to electromagnetic wave propagation and dissipation inside dielectric media and nanophotonic and plasmonic waveguides. *Phys. Rev. B* **2016**, *94*, 115136.
29. Zloshchastiev, K.G. Sustainability of Environment-Assisted Energy Transfer in Quantum Photobiological Complexes. *Annalen der Physik* **2017**, *529*, 1600185, doi; 10.1002/andp.201600185.
30. Snyder, A.W.; Mitchell, D.J. Mighty morphing spatial solitons and bullets. *Opt. Lett.* **1997**, *22*, 16–18.
31. Christodoulides, N.; Coskun, T.H.; Mitchell, M.; Segev, M. Multimode incoherent spatial solitons in logarithmically saturable nonlinear media. Phys. *Rev. Lett.* **1998**, *80*, 2310.
32. Znojil, M. Bound states emerging from below the continuum in a solvable PT-symmetric discrete Schroedinger equation. *Phys. Rev. A* **2017**, *96*, 012127. doi:10.1103/PhysRevA.96.012127.
33. Konotop, V.V.; Yang, J.; Zezyulin, D.A. Nonlinear waves in PT-symmetric systems. *Rev. Mod. Phys.* **2016**, *88*, 035002.
34. Čížek, J. On the Correlation Problem in Atomic and Molecular Systems. Calculation of Wavefunction Components in Ursell-Type Expansion Using Quantum-Field Theoretical Methods. *J. Chem. Phys.* **1966**, *45*, 4256.
35. Kümmel, H.G. A biography of the coupled cluster method. In *Recent Progress in Many-Body Theories*; Brandes, R.F., Brandes, T. Gernoth, K.A. Walet, N.R., Xian, Y., Eds.; World Scientific Publishing: Singapore, 2002; pp. 334–348.
36. McClain, J.; Lischner, J.; Watson, T.; Matthews, D.A.; Ronca, E.; Louie, S.G.; Chan, G.K.L. Spectral functions of the uniform electron gas via coupled-cluster theory and comparison to theGWand related approximations. *Phys. Rev. B* **2016**, *93*, 235139.
37. Hagen, G.; Hjorth-Jensen, M.; Jansen, G.R.; Papenbrock, T. Emergent properties of nuclei fromab initiocoupled-cluster calculations. *Phys. Scr.* **2016**, *91*, 063006.
38. Acton, F.S. *Numerical Methods That Work*; Harper & Row: New York, NY, USA, 1970.
39. Rüter, C.E.; Makris, K.; El-Ganainy, R.; Christodoulides, D.N.; Segev, M.; Kip, D. Observation of parity-time symmetry in optics. *Nat. Phys.* **2010**, *6*, 192.
40. Znojil, M. Parity-Time Symmetry and the Toy Models of Gain-Loss Dynamics near the Real Kato's Exceptional Points. *Symmetry* **2016**, *8*, 52.
41. Cartarius, H.; Dast, D.; Haag, D.; Wunner, G.; Eichler, R.; Main, J. Stationary and dynamical solutions of the Gross-Pitaevskii equation for a Bose-Einstein condensate in a PT symmetric double well. *Acta Polytech.* **2013**, *53*, 259–267.
42. Bender, C.M.; Hook, D.W.; Meisinger, P.N.; Wang, Q. Probability Density in the Complex Plane. *Ann. Phys.* **2010**, *325*, 2332–2362.
43. Ablowitz, M.J.; Musslimani, Z.H. Integrable nonlocal nonlinear Schrodinger equation. *Phys. Rev. Lett.* **2013**, *110*, 064105.
44. Ablowitz, M.J.; Musslimani, Z.H. Integrable nonlocal nonlinear equations. *Stud. Appl. Math.* **2017**, *139*, 7–59.
45. Bagchi, B.; Modak, S.; Panigrahi, P.K.; Ruzicka, F.; Znojil, M. Exploring branched Hamiltonians for a class of nonlinear systems. *Mod. Phys. Lett. A* **2015**, *30*, 1550213, doi:10.1142/S0217732315502132.
46. Bishop, R.F. An overview of coupled cluster theory and its applications in physics. *Theor. Chim. Acta* **1991**, *80*, 95–148.
47. Bartlett, R.J.; Musial, M. Coupled-cluster theory in quantum chemistry. *Rev. Mod. Phys.* **2007**, *79*, 291.
48. Mielnik, B. Generalized quantum mechanics. *Commun. Math. Phys.* **1974**, *37*, 221–256.
49. Rosen, G. Dilatation covariance and exact solutions in local relativistic field theories. *Phys. Rev.* **1969**, *183*, 1186.
50. Rosen, G. Particle-like solutions to nonlinear complex scalar field theories with positive-definite energy densities. *J. Math. Phys.* **1968**, *9*, 996.
51. Enqvist, K.; Mazumdar, A. Cosmological consequences of MSSM flat directions. *Phys. Rept.* **2003**, *380*, 99.
52. Dzhunushaliev, V.; Zloshchastiev, K.G. Singularity-free model of electric charge in physical vacuum: Non-zero spatial extent and mass generation. *Central Eur. J. Phys.* **2013**, *11*, 325–335.
53. Gulamov, I.E.; Nugaev, E.Y.; Smolyakov, M.N. Theory of U(1) gauged Q-balls revisited. *Phys. Rev. D* **2014**, *89*, 085006.

54. Gulamov, I.E.; Nugaev, E.Y.; Panin, A.G.; Smolyakov, M.N. Theory of U(1) gauged Q-balls revisited. *Phys. Rev. D* **2015**, *92*, 045011.
55. Dzhunushaliev, V.; Makhmudov, A.; Zloshchastiev, K.G. Singularity-free model of electrically charged fermionic particles and gauged Q-balls. *Phys. Rev. D* **2016**, *94*, 096012.
56. Bialynicki-Birula, I.; Mycielski, J. Gaussons: Solitons Of The Logarithmic Schrodinger Equation. *Phys. Scripta* **1979**, *20*, 539.
57. Buljan, H.; Šiber, A.; Soljačić, M.; Schwartz, T.; Segev M.; Christodoulides, D.N. Incoherent white light solitons in logarithmically saturable noninstantaneous nonlinear media. *Phys. Rev. E* **2003**, *68*, 036607.
58. De Martino, S.; Falanga, M.; Godano, C.; Lauro, G. Logarithmic Schrödinger-like equation as a model for magma transport. *Europhys. Lett.* **2003**, *63*, 472–475.
59. De Martino, S.; Lauro, G. Soliton like solutions for a capillary fluid. In Proceedings of the 12th conference on waves and stability in continuous media, Villasimius, Italy, 1–7 June 2003; Monaco, R; Pennisi, S; Rionero, S; Ruggeri, T., Eds.; World Scientific Publishing, Singapore, 2004, pp. 148–152.
60. Hansson, T.; Anderson, D.; Lisak, M. Propagation of partially coherent solitons in saturable logarithmic media: A comparative analysis. *Phys. Rev. A* **2009**, *80*, 033819.
61. Hefter, E.F. Application of the nonlinear Schrodinger equation with a logarithmic inhomogeneous term to nuclear physics. *Phys. Rev. A* **1985**, *32*, 1201.
62. Kartavenko, V.G.; Gridnev, K.A.; Greiner, W. Nonlinear Effects in Nuclear Cluster Problem. *Int. J. Mod. Phys. E* **1998**, *7*, 287.
63. Yasue, K. Quantum Mechanics Of Nonconservative Systems. *Ann. Phys.* **1978**, *114*, 479.
64. Lemos, N.A. Dissipative forces and the algebra of operators in stochastic quantum mechanics. *Phys. Lett. A* **1980**, *78*, 239.
65. Brasher, J.D. Nonlinear-wave mechanics, information-theory, and thermodynamics. *Int. J. Theor. Phys.* **1991**, *30*, 979–984.
66. Schuch, D. Nonunitary connection between explicitly time-dependent and nonlinear approaches for the description of dissipative quantum systems. *Phys. Rev. A* **1997**, *55*, 935.
67. Davidson, M.P. Comments on the nonlinear Schrodinger equation. *Nuov. Cim. B* **2001**, *116*, 1291.
68. Lopez, J.L. Nonlinear Ginzburg-Landau-type approach to quantum dissipation. *Phys. Rev. E* **2004**, *69*, 026110.
69. Avdeenkov, A.V.; Zloshchastiev, K.G. Quantum Bose liquids with logarithmic nonlinearity: Self-sustainability and emergence of spatial extent. *J. Phys. B At. Mol. Opt. Phys.* **2011**, *44*, 195303.
70. Zloshchastiev, K.G. Volume element structure and roton-maxon-phonon excitations in superfluid helium beyond the Gross-Pitaevskii approximation. *Eur. Phys. J. B* **2012**, *85*, 273.
71. Bouharia, B. Stability of logarithmic Bose—Einstein condensate in harmonic trap. *Mod. Phys. Lett. B* **2015**, *29*, 1450260.
72. Zloshchastiev, K.G. Stability and Metastability of Trapless Bose-Einstein Condensates and Quantum Liquids. *Z. Naturforsch. A* **2017**, *72*, 677–687.
73. Zloshchastiev, K.G. Logarithmic nonlinearity in theories of quantum gravity: Origin of time and observational consequences. *Gravit. Cosmol.* **2010**, *16*, 288–297.
74. Zloshchastiev, K.G. Spontaneous symmetry breaking and mass generation as built-in phenomena in logarithmic nonlinear quantum theory. *Acta Phys. Pol. B* **2011**, *42*, 261–292.
75. Zloshchastiev, K.G. Vacuum Cherenkov effect in logarithmic nonlinear quantum theory. *Phys. Lett. A* **2011**, *375*, 2305–2308.
76. Scott, T.C.; Zhang, X.; Mann, R.B.; Fee, G.J. Canonical reduction for dilatonic gravity in 3+1 dimensions. *Phys. Rev. D* **2016**, *93*, 084017.
77. Unruh, W.G. Experimental black-hole evaporation? *Phys. Rev. Lett.* **1981**, *46*, 1351.
78. Jacobson, T.A.; Volovik, G.E. Event horizons and ergoregions in 3 He. *Phys. Rev. D* **1998**, *58*, 064021.
79. Visser, M. Acoustic black holes: Horizons, ergospheres and Hawking radiation. *Class. Quantum Gravit.* **1998**, *15*, 1767.
80. Zloshchastiev, K.G. Acoustic phase lenses in superfluid helium as models of composite spacetimes in general relativity: Classical and quantum features. *Acta Phys. Pol. B* **1999**, *30*, 897–905.
81. Volovik, G.E. *The Universe in a Helium Droplet*; International Series of Monographs on Physics; Oxford University Press: Oxford, UK, 2003; Volume 117, pp. 1–507.

82. Bohm, D. A suggested interpretation of the quantum theory in terms of "hidden" variables. *Phys. Rev.* **1952**, *35*, 166–193.
83. Styer, D.F.; Balkin, M.S.; Becker, K.M.; Burns, M.R.; Dudley, C.E.; Forth, S.T.; Rinkoski, M.T. Nine formulations of quantum mechanics. *Am. J. Phys.* **2002**, *70*, 288–297.
84. Philippidis, C.; Dewdney, C.; Hiley, B.J. Quantum interference and the quantum potential. *Nuovo Cim. B* **1979**, *52*, 15–28.
85. Stapp, H.P. The Undivided Universe: An ontological interpretation of Quantum Theory. *Am. J. Phys.* **1994**, *62*, 958–960.
86. Znojil, M. Comment on the Green function for the anharmonic-oscillators. *Phys. Rev. D* **1982**, *26*, 3750–3751.
87. Lévai, G. A search for shape-invariant solvable potentials. *J. Phys. A Math. Gen.* **1989**, *22*, 689.
88. Lévai, G.; Znojil, M. Systematic search for PT-symmetric potentials with real energy spectra. *J. Phys. A Math. Gen.* **2000**, *33*, 7165–7180.
89. Ushveridze, A.G. *Quasi-Exactly Solvable Models in Quantum Mechanics*; IOP: Bristol, UK, 1994.
90. Znojil, M. Harmonic oscillator well with a screened Coulombic core is quasiexactly solvable. *J. Phys. A Math. Gen.* **1999**, *32*, 4563–4570.
91. Znojil, M. Non-Hermitian matrix description of the PT-symmetric anharmonic oscillators. *J. Phys. A Math. Gen.* **1999**, *32*, 7419–7428.
92. Flessas, G.P. On the Schrodinger equation for the $x^2 + \lambda x^2/(1 + gx^2)$ interaction. *Phys. Lett. A* **1981**, *83*, 121–122.
93. Znojil, M. Potential $r^2 + \lambda r^2/(1 + gr^2)$ and the analytic continued fractions. *J. Phys. A Math. Gen.* **1983**, *16*, 293–301.
94. Roy, P.; Roychoudhury, R. New exact solutions of the non-polynomial oscillator $V(x) = x^2 + \lambda x^2(1 + gx^2)$ and supersymmetry. *Phys. Lett. A* **1987**, *122*, 275–279.
95. Fernández, F.M. Convergent power-series solutions to the Schrodinger equation with the potential $x^2 + \lambda x^2/(1 + gx^2)$. *Phys. Lett. A* **1991**, *160* 116–118.
96. Znojil, M. Quasi-exact states in the Lanczos recurrent picture. *Phys. Lett. A* **1991**, *161*, 191–196.
97. Znojil, M.; Leach, P.G.L. On the elementary Schrodinger bound states and their multiplets. *J. Math. Phys.* **1992**, *33*, 2785–2794.
98. Znojil, M. Tridiagonal PT-symmetric N-by-N Hamiltonians and a fine-tuning of their observability domains in the strongly non-Hermitian regime. *J. Phys. A Math. Theor.* **2007**, *43*, 13131–13148.
99. Turbiner, A. One-dimensional quasi-exactly solvable Schrödinger equations. *Phys. Rep.* **2016**, *642*, 1–71.
100. Newton, R.G. *Scattering Theory of Waves and Particles*; Springer: Berlin, Germany, 2013.
101. Greiner, W. *Relativistic Quantum Mechanics*; Springer: Berlin, Germany, 2000; p. 339.
102. Dynkin, E.B. On the representation of the series log $(e^x e^y)$ for noncommutative x, y by commutators. *Math. Sbornik* **1949**, *25*, 155–162.
103. Babin, A.; Figotin, A. Some mathematical problems in a neoclassical theory of electric charges. *Discret. Contin. Dyn. Syst.* **2010**, *A27*, 1283–1326.