UNITARITY AND PRODUCTION AMPLITUDES (APPLICATION TO πN HIGHER RESONANCES)

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I would like to discuss a model of higher resonances in the πN scattering that has been worked out extensively in the past year, and for which still further works are being done. In this model which I am about to discuss, we consider the three-particle intermediate states $N + 2\pi$ in which the two pions are strongly correlated in the T = J = 1 state as an unstable particle we now call the ρ meson [1, 2]. As this new channel becomes energetically open, the ρN channel becomes coupled to the πN channel. In fact the πN system can make a virtual transition to the ρN channel even below the ρN threshold. In our model, we assume a strong coupling between the πN and ρN channels, eventually developing a resonance either above or below the ρN threshold. This model is very similar to, but significantly different in one respect from, the model suggested by Dalitz and Miller for the Y_1^*. The analogy of the Dalitz-Miller model in our problem would be to assume a strong force between the ρ and N particles to sustain a bound state which would be stable, were it not for the "weak" coupling between the ρN and πN channels. In our model, however, it is precisely the coupling of the two channels that is responsible for the resonance. There is very little, if any, difference between these two models phenomenologically, but at a deeper dynamical level, there is a difference in outlook.

The reason that the model we are considering is capable of accounting for the higher resonances is seen as follows. Let L_J be the "orbital" angular momentum and the total angular momentum of the πN system and I_J the same for the ρN system. Note that there is no change in the intrinsic parity in the reaction $N + \pi \rightarrow N + \rho$. Therefore

πN L_J	ρN I_J	
$S_{1/2}$	$S_{1/2}$	$D_{1/2}$
$P_{1/2}$	$P_{1/2}$	
$P_{3/2}$	$P_{3/2}$	$F_{3/2}$
$D_{3/2}$	$S_{3/2}$	$D_{3/2}$
$D_{5/2}$	$D_{5/2}$	$G_{5/2}$
$F_{5/2}$	$P_{5/2}$	$F_{5/2}$

Now as the ρN channel becomes energetically open, the I = 0 ($L_J = D_{3/2}$, $S_{1/2}$) state will be excited first, then the I = 1 ($F_{5/2}$, $P_{3/2}$, $P_{1/2}$) state, and so on. Since the statistical weight is proportional to $(2J + 1)$, we expect that the first resonance is predominantly in the state $L_J = D_{3/2}$ and the second one in the $L_J = F_{5/2}$, $P_{3/2}$, $P_{1/2}$ states, which of course agree with the experimental findings. The isotopic dependance is determined by the specific "primary" interaction we choose, and we will discuss it when the appropriate moment comes.

Our formal approach will be based on the unitarity and analyticity of the relevant amplitudes. Actually the analytic properties of the production and three-particle-to-three-particle scattering amplitudes are only scantly known, and we shall proceed by assuming a particular diagram as giving the main contribution to the left hand cuts of the production amplitude. On the unitarity relations of the coupled processes, I shall rely heavily on the recent work of Ball, Frazer and Nauenberg.

While for the scattering amplitude the unitarity relation

$$T - T^* = 2\pi i T \rho T^* \tag{1}$$

gives directly the discontinuity across the right hand cut, this is not the case in general for the production and 3-particle-to-3-particle amplitudes. Let us denote by subscript 1 the πN channel, and by 2 the ρN channel. Then various elements of the T-matrix are defined as

$$\begin{aligned}
T_{11}(s_+) &= \sqrt{2k_0'}\sqrt{p_0'/m}\langle \pi(k') N(p')^{out} | J_N^\dagger(0) | \pi(k) \rangle \sqrt{2k_0}\, u(p), \\
T_{21}(s_+, \sigma_+) &= \sqrt{4k_{10}'k_{20}'}\sqrt{q_0'/m}\langle \pi(k_1')\pi(k_2') N(q')^{out} | J_N^\dagger(0) | \pi(k) \rangle \sqrt{2k_0}\, u(p), \qquad (2) \\
&\quad \cdots\cdots \\
T_{22}(s_+, \sigma_+', \sigma_+) &= \sqrt{4k_{10}'k_{20}'}\sqrt{q_0''/m}\langle \pi(k_1')\pi(k_2') N(q')^{out} | J_N^\dagger(0) | \pi(k_1)\pi(k_2)^{in} \rangle \\
&\quad \times \sqrt{4k_{10}k_{20}}\, u(q).
\end{aligned}$$

In the above definitions of the T-matrix elements, we have deliberately contracted the nucleon rather than the pion operators so as to keep the two pions in the bra or ket together. To specify the kinematics of the reactions $\pi + N \longleftrightarrow \pi + N$, $\pi + N \longleftrightarrow \pi + \pi + N$, and $\pi + \pi + N \longleftrightarrow \pi + \pi + N$ we need to specify 2, 5 and 8 variables, respectively. We specify the total energy of the system in the centre of mass which is common in three processes:

$$\begin{aligned}
s &= -(p+k)^2 = -(p'+k')^2 \\
&= -(q + k_1 + k_2)^2 = -(q' + k_1' + k_2')^2.
\end{aligned} \tag{3}$$

For three-particle states, we will denote by σ the total energy square of the two pion system in its own c.m.:

$$\begin{aligned}
\sigma &= -(k_1 + k_2)^2 \\
\sigma' &= -(k_1' + k_2')^2.
\end{aligned} \tag{4}$$

The other remaining variables are appropriately chosen angles.

Suppressing the angular variables, we deduce the discontinuity of the T_{ij} across the real s-axis, $s \geqslant (m+\mu)^2$ as

$$
\begin{aligned}
1/2i\,[T_{11}(s_+) - T_{11}(s_-)] &= \Sigma_2\, T_{11}(s_+)\rho_1(s)\, T_{11}(s_-) \\
&\quad + \Sigma_{3,\sigma}\, T_{12}(s_+,\sigma_+)\rho_2(s,\sigma)\, T_{21}(s_-,\sigma_-), \\
1/2i\,[T_{12}(s_+,\sigma) - T_{12}(s_-,\sigma)] &= \Sigma_2\, T_{11}(s_+)\rho_1(s)\, T_{12}(s_-) \\
&\quad + \Sigma_{3,\sigma'}\, T_{12}(s_+,\sigma'_+)\rho_2(s,\sigma)\, T^c_{22}(s_-,\sigma'_-,\sigma), \qquad (5) \\
1/2i\,[T^c_{22}(s_+,\sigma',\sigma) - T^c_{22}(s_-,\sigma',\sigma)] &= \Sigma_2\, T_{21}(s_+,\sigma')\rho_1(s)\, T_{12}(s_-,\sigma) \\
&\quad + \Sigma_{3,\sigma''} T^c_{22}(s_+,\sigma',\sigma'')\rho_2(s,\sigma'')\, T^c_{22}(s_-,\sigma'',\sigma)
\end{aligned}
$$

where Σ_2 is the two-particle phase space integral for fixed s:

$$\Sigma_2 \propto \int_{-1}^{1} d\cos\theta \int^{2\pi} d\varphi$$

and $\rho_1(s)$ is the phase space factor of the πN system:

$$\rho_1(s) \equiv \rho_1(s;m,\mu) \propto \frac{P}{\sqrt{s}}\,\theta(s-(m+\mu)^2) = \frac{[s-(m+\mu)^2]^{1/2}[s-(m-\mu)^2]^{1/2}}{2s}\,\theta(s-(m+\mu)^2).$$

$\Sigma_{3,\sigma}$ is the three-particle phase space integral including the integration over the continuous mass variable σ, and $\rho_2(s,\sigma)$ is the phase space factor for the ρN system:

$$\rho_2(s,\sigma) \propto \rho_1(s;m,\sqrt{\sigma})\,\rho_1(\sigma;\mu,\mu)\,\theta(\sqrt{s}-m-\sqrt{\sigma})\,\theta(\sigma-4\mu^2)\,\theta(s-(m+\mu)^2).$$

Eqs. (5) are derivable (at least heuristically!) using the L.S.Z. formalism. It must be emphasized that Eqs. (5) are not the unitarity relations, albeit they are intimately related to the latter. Before exhibiting the connection between those two, we wish to note the topological structure of the T^c_{22} in Eqs. (5). From the definition of T^c_{22} in Eq. (2), we see that

$$S_{22} = \frac{q_0}{m}\,\delta(\vec{q}-\vec{q}')\,S_{\pi\pi} + (2\pi)^4 i\,\delta(q+k_1+k_2-q'-k_1'-k_2')\ \underset{\uparrow\,\text{factors}}{\ldots}\ T^c_{22}$$

so that T^c_{22} is defined as

$$T^c_{22} = T_{22} - \frac{q_0}{m}\delta(\vec{q}-\vec{q}')\,T_{\pi\pi}\ .$$

That is to say, the "connected" amplitude excludes the disconnected group in which the nucleon is non-interacting (Fig. 1).

To get rid of the complicating angular dependence in Eqs. (5), it is convenient to decompose the amplitudes into partial waves. The decomposition of the elastic amplitude is well-known, so I shall not elaborate upon this.

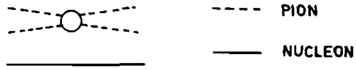

Fig. 1

To decompose the production amplitude T_{12} we first go to the 2π centre-of-mass system and choose the $\tilde{z}$-axis along the direction of the q (see Fig. 2).

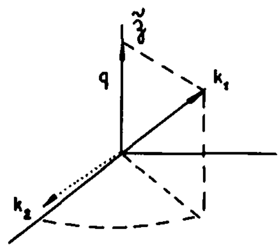

Fig. 2

In this frame, we project out a particular angular momentum state ℓ of the 2π-system with the quantization axis along the $\tilde{z}$-axis. Let $\ell_{\tilde{z}}$ be the projection of ℓ onto the $\tilde{z}$-axis. Now we Lorentz-transform the system to the total centre of mass. Since the Lorentz transformation is along the $\tilde{z}$-axis with the velocity

$$v = |\vec{q}| \Big/ \sqrt{|\vec{q}|^2 + \sigma}$$

the projection $\ell_{\tilde{z}}$ remains invariant and $\ell_{\tilde{z}}$ acquires the role of the helicity of the 2π system. As the two-pion system is now equivalent to a particle of mass $\sqrt{\sigma}$, spin ℓ, helicity $\ell_{\tilde{z}}$ as far as kinematics is concerned, the decomposition into definite (J,π) states follows in the standard manner of Jacob and Wick. The three-particles amplitude T_{22} may be decomposed in a similar manner. If we assume, as we shall do, that only one particular ℓ dominates, and that the mass-distribution in $\sqrt{\sigma}$ is sharply peaked near $\sigma = \sigma_r$, the picture of an unstable particle can be naturally incorporated into our scheme. From now on we shall retain only the $\ell = 1$ amplitudes in our considerations.

Once the amplitudes T_{ij} are decomposed into partial waves, the angular integrations in Eqs. (5) can be performed trivially. For given quantum numbers J, π, we obtain

$$T_{11}^{J\pi}(s_+) - T_{11}^{J\pi}(s_-) = 2i\, T_{11}^{J\pi}(s_+)\, \rho_1(s)\, T_{11}^{J\pi}(s_-) + 2i \sum_a \int_{4\mu^2}^{(\sqrt{s}\, m)^2} d\sigma\, T_{12}^{J\pi}(s_+, \sigma_+)\, \rho_2(s,\sigma)\, T_{21}^{J\pi}(s_-, \sigma_-), \qquad (6)$$

where, for $\ell = 1$, $\alpha = 1$, 2 or 3 is the polarization index corresponding to two transverse, one longitudinal polarizations.

At this juncture let us clarify the connection between the discontinuity Eq. (6) and the unitarity relations. For T_{12}, for example, the unitarity relation asserts that (suppressing $J, \pi, \sum_a, \alpha$ hereafter)

$$T_{12}(s_+, \sigma_+) - T_{12}^*(s_-, \sigma_-) = 2i\, T_{11}^*(s)\, \rho_1(s)\, T_{12}(s,\sigma) + 2i\int d\sigma'\, T_{12}^*(s,\sigma')\rho_2(s,\sigma')\, T_{22}(s,\sigma',\sigma). \quad (7)$$

We assume $T_{ij}^*(s, x, --) = T_{ij}(s^*, x^*, ---)$, i. e., T_{ij} are real analytic functions in the energy variables. We are unable to give a rigorous, general proof for this, but this is true in perturbation theory as far as can be ascertained. Then we may write Eq. (7) as

$$[T_{12}(s_+, \sigma_+) - T_{12}(s_-, \sigma_+)] + [T_{12}(s_-, \sigma_+) - T_{12}(s_-, \sigma_-)]$$

$$= 2i\, T_{11}(s - i\epsilon)\, \rho_1(s)\, T_{11}(s + i\epsilon) + 2i\int d\sigma'\, T_{12}(s_-, \sigma'_-)\rho_2(s,\sigma)\, T_{22}(s_+, \sigma'_+, \sigma_+). \quad (8)$$

Now we note that

$$T_{12}(s, \sigma_+) - T_{12}(s, \sigma_-) = 2i\, e^{i\delta(\sigma)} \sin\delta(\sigma)\, T_{12}(s_-, \sigma_-)$$

$$= 2i\int d\sigma'(T_{22} - T_{22}^c)\, \rho_2(s, \sigma')\, T_{12}(s_-, \sigma_-). \quad (9)$$

Therefore, Eqs. (8) and (9) imply the second (not written out) line of Eq. (6). What it means may become clearer if you consider a particular diagram (see Fig. 3).

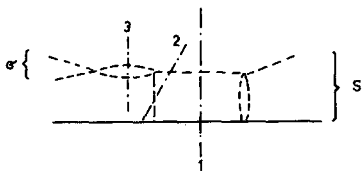

Fig. 3

When we consider the imaginary part, there are contributions from partitions 2 and 3 of the diagram as well as that from the partition 1. In computing the "absorptive" part, however, the contributions from 2 and 3 should <u>not</u> be included (since σ is fixed). It may be further remarked that the partitions 2 and 3 give rise to terms of the form

$$\int d\sigma'\, T_{12}(s_-, \sigma_-)\, \rho_2(s, \sigma')\, T_{22}^D(s_+, \sigma'_+, \sigma)$$

where T_{22}^D is the excluded disconnected part

$$T_{22}^D = T_{22} - T_{22}^c.$$

So much for the unitarity aspect of the problem, let us now look at the dynamics. We would like to take into account the longest range "force" that contributes to the process $\pi + N \longleftrightarrow \rho + N$. One-pion exchange is possible between the nucleon and the meson (see Fig. 4).

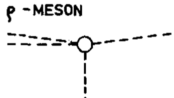

Fig. 4

In fact, the reason why we have singled out the $N + \rho$ intermediate state, but neglected other states such as $N + \omega$ (η) is, that this is the only state consisting of one nucleon and one unstable meson that can be reached from the $N + \pi$ state through one-pion exchange. The matrix element for Fig. 4 is

$$\bar{u}(q)\gamma_5 u(p)\, ig\, \frac{1}{t-\mu^2} F_{\pi\pi}(\sigma, \cdots) \tag{10}$$

where $t = -(p-q)^2$, and $F_{\pi\pi}$ is

$$F_{\pi\pi}(\sigma, \cdots) \propto 3\sqrt{\frac{\sigma}{\sigma-4\mu^2}}\, e^{i\delta(\sigma)} \sin\delta(\sigma)\, P_1(\hat{k}_1 \cdot \hat{k})\Big|_{2\pi.\,c.\,m.}$$

$$\equiv f_{\pi\pi}(\sigma)\, 3\, P_1(\hat{k}_1 \cdot \hat{k}) .$$

When we make a partial wave projection of (10) we get two branch lines, one extending from $s = 0$ to $-\infty$, the other one between the two branch points $s^{\pm}$, given by

$$s^{\pm}(\sigma) = m^2 + \sigma/2 \pm \frac{\sqrt{\sigma}}{2\mu}[(4m^2-\mu^2)(4\mu^2-\sigma)]^{1/2} .$$

If we give a small positive imaginary part to $\sigma : \sigma \to \sigma + i\eta,\ \eta > 0$, and increase σ from a certain small value, the branch points $s^{\pm}(\sigma)$ move as indicated in Fig. 5.

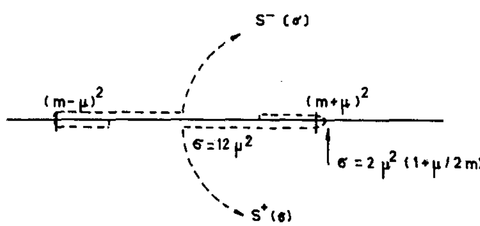

Fig. 5

That is, the projection of the one-pion exchange amplitude develops an anomalous singularity at $\sigma = 2\mu^2(1+\mu/2m)$, a complex singularity at $\sigma = 4\mu^2$. If, instead, we give a small imaginary part to σ, the loci of the singularities in Fig. 5 are reflected about the real axis.

The amplitudes T_{ij} contain some kinematical cuts. We define the new set of amplitudes M_{ij} by

$$M_{ij} = \frac{1}{\sqrt{g_i}}\, T_{ij}\, \frac{1}{\sqrt{g_j}}$$

where the g_i are factors proportional to $p_i^{2L_i}$, p_i being the magnitude of the channel momentum, L_i the lowest channel orbital angular momentum ($L_i = L$ for $i = 1$, $L_i = I$ for $i = 2$, where L and I are defined as before). On defining new quantities $\rho_1^L(s)$, $\rho_2^I(s,\sigma)$ by

$$\rho_1^L(s) = g_1\rho_1(s) \sim [p^{2L+1} \text{as} \quad s \to (m+\mu)^2] ,$$

$$\rho_2^I(s,\sigma) = g_2\rho_2(s,\sigma) \sim [\, q(s,\sigma)^{2I+1} \text{as} \quad s \to (m+2\mu)^2] ,$$

we can write the discontinuity conditions, Eq. (6), in the form

$$M_{11}^{J\pi}(s_+) - M_{11}^{J\pi}(s_-) = 2i\, M_{11}^{J\pi}(s_-)\rho_1^L\, M_{11}(s_+)$$
$$+ 2i\, \Sigma \int d\sigma\, M_{12}^{J\pi}(s_-,\sigma_-)\, \rho_2^I(s,\sigma)\, M_{21}^{J\pi}(s_+,\sigma_+). \qquad (11)$$

The factors ρ_1^L and ρ_2^I express the combined effect of the variation of the available phase space and the centrifugal barrier. As a consequence, the dominant energy dependence of the amplitudes near thresholds is removed from M_{ij}: the M_{ij} are approximately constant near the thresholds.

We can now write down the dispersion equations for the M_{ij}. The "input" amplitude $B_{21}(s,\sigma)$ has a spectral representation of the form

$$B_{21}(s,\sigma) = \frac{f_{\pi\pi}(\sigma)}{\pi}\left[\int_{s+}^{s^-} \frac{ds'}{s'-s}\, \alpha(s',\sigma) + \int_{-\infty}^{0} \frac{ds'}{s'-s}\, \beta(s',\sigma)\right] .$$

If we fix the value of σ below $2\mu^2(1+\mu/2m)$, we can write down the dispersion equation for $M_{21}(s,\sigma)$:

$$M_{21}(s,\sigma) = B_{21}(s,\sigma) + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{ds'}{s'-s}\, M_{21}(s'_+)\,\rho_1(s')\,M_{11}(s'_-)$$

$$+ \frac{1}{\pi}\int_{(m+2\mu)^2}^{\infty} \frac{ds'}{s'-s} \cdots \quad . \qquad (12)$$

Let us now consider the analytic continuation of Eq. (11) in the mass σ. As we have seen, M_{21} will develop an anomalous singularity, and we must deform the contour in the second term of (12) to avoid the intruding cut of $M_{21}(s'_+,\sigma)$.

The amplitude $M_{21}(s'_+,\sigma)$ in the integrand must be continued to the second sheet through the two-particle unitarity cut:

$$M_{21}(s_+,\sigma) = M_{21}^{II}(s_-,\sigma)$$
$$= M_{21}(s_-,\sigma)[1 + 2i\rho_1(s)\, M_{11}(s_-,\sigma)]^{-1}$$
$$\rho_1^{II}(s-i\epsilon) = \rho_1(s+i\epsilon) = -\,\rho_1(s-i\epsilon) .$$

Therefore the continuation of Eq. (11) in σ now gives

$$M_{21}(s,\sigma) = B_{21}(s,\sigma) + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{ds'}{s'-s} \cdots$$

$$+ \frac{1}{\pi}\int_{s^+(\sigma)}^{(m+\mu)^2} \frac{ds'}{s'-s}\, \text{disc}\, M_{21}^{II}(s')\,\rho_1(s')\,M_{11}^{II}(s'). \qquad (13)$$

One can show that $M_{21}(s,\sigma)$ given by Eq. (13) does not satisfy the discontinuity condition in σ, namely

$$M_{21}(s,\sigma_+) - M_{21}(s,\sigma_-) = 2i f_{\pi\pi}(\sigma_+)\,\rho(\sigma)\,M_{21}(s,\sigma_-);\ \rho(\sigma) = \sqrt{\frac{\sigma - 4\mu^2}{\sigma}}. \qquad (14)$$

Eq. (14) means that

$$\operatorname*{disc}_{\sigma}\left[\frac{M_{21}(s,\sigma+i\epsilon)}{f_{\pi\pi}(\sigma+i\epsilon)}\right] = \sigma,$$

but because of the unsymmetrical (in the s-plane) complex anomalous singularity, Eq. (13) gives

$$\operatorname*{disc}_{\sigma}\left[\frac{M_{21}(s,\sigma+i\epsilon)}{f_{\pi\pi}(\sigma+i\epsilon)}\right] = \frac{1}{\pi}\int_{s^+}^{(m+\mu)^2}\frac{ds'}{s'-s}\left[\operatorname{disc}\frac{M_{21}^{II}(s',\sigma_+)}{f_{\pi\pi}(\sigma_+)}\right]\rho_1^{II}(s')\,M_{11}(s')$$
$$+\frac{1}{\pi}\int_{(m+\mu)^2}^{s^-}\frac{ds'}{s'-s}\left[\operatorname{disc}\frac{M_{21}(s',\sigma_-)}{f_{\pi\pi}(\sigma_-)}\right]\rho_1(s')\,M_{11}^{II}(s')$$
$$\neq 0.$$

Ball, Frazer and Nauenberg noted that the diagram shown in Fig. 6 has a cut from s^+ to s^-, corresponding to the nuclear line indicated by arrow on the mass shell.

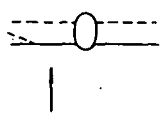

Fig. 6

The contribution from the branch cut from s^+ to s^- of this diagram can be shown to be

$$\frac{1}{\pi\rho(\sigma)}\left[\int_{s^+}^{(m+\mu)^2}\frac{ds'}{s'-s}\,\alpha(s',\sigma)\,\rho_1^{II}(s')\,M_{11}^{II}(s') + \int_{(m+\mu)^2}^{s^-}\frac{ds'}{s'-s}\,\alpha(s',\sigma)\,\rho_1(s')\,M_{11}(s')\right].$$

When we add this term to Eq. (13) and manipulate a little bit, we obtain

$$\frac{M_{21}(s,\sigma)}{f_{\pi\pi}(\sigma)} = \frac{B_{21}(s,\sigma)}{f_{\pi\pi}(\sigma)} + \frac{1}{\pi\rho(\sigma)f_{\pi\pi}^{\kappa}(\sigma)}\int_{s^+(\sigma)}^{(m+\mu)^2}\frac{ds'}{s'-s'}\,\alpha(s',\sigma)\,\rho_1^{II}(s')\,M_{11}(s')$$
$$+\frac{1}{\pi\rho(\sigma)f_{\pi\pi}^{\sigma}}\int_{(m+\mu)^2}^{s^-(\sigma)}\frac{ds'}{s'-s}\,\alpha(s',\sigma)\rho_1(s')\,M_{11}(s') + \text{unitary contributions}. \qquad (15)$$

By virtue of the relations

$$\alpha^*(s,\sigma) = -\alpha(s^*,\sigma^*),$$

$$\frac{1}{f_{\pi\pi}(\sigma)} - \frac{1}{f^*_{\pi\pi}(\sigma)} = -2i\rho(\sigma),$$

the amplitude $M_{21}(s,\sigma)$ satisfies Eq. (15).

The above consideration may be understood better in terms of diagrams. Consider, for example, a Cutkosky diagram of Fig. 7.

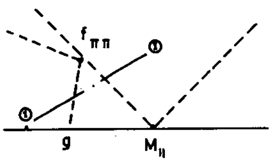

Fig. 7

The discontinuity in σ corresponds to the partition of the diagram along line 1, and this must be written in the form (see Eq. 14)

$$2i f_{\pi\pi}(\sigma)\,\rho(\sigma)\,M_{12}(s,\sigma_-).$$

Therefore, M_{21} must include the diagram shown in Fig. 8.

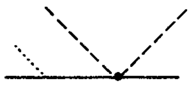

Fig. 8

In this model both the elastic amplitude M_{11} and the 3 particle amplitude M_{22} are driven by the inelastic process, which in turn is generated by the one-pion exchange mechanism and the unitarity. Schematically the whole coupled processes can be shown as in Fig. 9 .

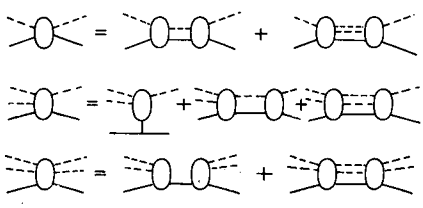

Fig. 9

The non-linear set of equations (11) and (15) can be solved by the matrix N/D method. Of course a slight modification is necessary to accommodate the complex anomalous singularities. An important result is that, while individual elements D_{ij} do have complex singularities, the determinant of D does not.

Moreover we observe that

$$S = \rho^{1/2} D(s - i\epsilon) D^{-1}(s + i\epsilon) \rho^{-1/2}$$

so that

$$\ln \det D(s_-) - \ln \det D(s_+) = \ln \det S = \sum_{\alpha \text{ eigenchannels}} \delta_\alpha(s)$$

and from this and the fact that det D has only the normal unitarity cut, the multichannel Levinson theorem follows :

$$\sum_\alpha \left\{ \delta_\alpha(m + \mu) - \delta_\alpha(\infty) \right\} = \pi (n_{\text{bound}} - n_{\text{c.D.D}}) \; .$$

Instead of demonstrating the N/D solution, let us be content with the semiphenomenological K-matrix solution, to glean the nature of the more elaborate solution. Let us adopt the matrix notation and write

$$M = K + iK \cdot \rho^{J\pi} - M \tag{16}$$

where the rows and columns of the matrices are labelled by the discrete channel indeces i and the continuous σ

$$(M) = \begin{pmatrix} M_{11} \cdots & M_{2\sigma,1} \cdots \\ \vdots & \\ M_{1,2\sigma'} & M_{2\sigma,2\sigma'} \\ \vdots & \end{pmatrix} \text{ etc.}$$

and the matrix ρ is diagonal

$$\left(\rho^{J\pi}\right) = \begin{pmatrix} \rho_1^L & 0 & 0 \\ 0 & \rho_2^I(s,\sigma) & \\ 0 & & \end{pmatrix} \; .$$

We write

$$m_{11} = M_{11}$$
$$m_{12}(s,\sigma) = M_{12}(s,\sigma) f_{\pi\pi}^{-1}(\sigma)$$
$$m_{22}(s,\sigma',\sigma) = f_{\pi\pi}^{-1}(\sigma') M_{22}(s,\sigma',\sigma) f_{\pi\pi}^{-1}(\sigma) \; .$$

Now we make an essential approximation that $f_{\pi\pi}(\sigma)$ is sharply peaked at $\sigma = \sigma_r$, and, in the spirit of the steepest descent approximation,

$$m_{12}(s,\sigma) \simeq m_{12}(s,\sigma_r) \equiv m_{12}(s), \text{ etc.}$$

Likewise we define κ_{ij} by removing the sharp dependance of κ_{ij} on the mass distribution σ. On defining the 3 particle factor ρ_2^{-I} corrected for the final interaction

$$\bar{\rho}_2^I(s) = \int_{4\mu^2}^{(\sqrt{s}-m)^2} d\sigma \left| f_{\pi\pi}(\sigma)\right|^2 \rho_2(s,\sigma) .$$

We obtain a reduced 2 X 2 matrix relation between m and κ :

$$m_{11}(s) = \kappa_{11}(s) + i\,\kappa_{11}(s)\,\rho_1^L(s)\,m_{11}(s) + i\,\kappa_{12}\,\bar{\rho}_2^I m_{21}, \tag{17}$$

In order that the M_{ij} satisfy the relation, Eq. (12), it is necessary and sufficient that κ be real symmetric; the M_{ij} can be written as

$$m = (1 - i\bar{\rho}\cdot\kappa)^{-1}\kappa \quad ,$$

the condition for resonance is

$$\mathrm{Re}\,[\det\,(1 - i\bar{\rho}\cdot\kappa)] = 0$$

or

$$0 = \mathrm{Re}\,[1 + \rho_1^L\,\bar{\rho}_2^I\,(\kappa_{12}\kappa_{21} - \kappa_{11}\kappa_{22}) - i\rho_1^L\,\kappa_{11} - i\bar{\rho}_2^I\kappa_{22}] \ . \tag{18}$$

Now in the range $(m+\mu)^2 \leq s \leq (m+\sqrt{\sigma_r})^2 \rho_1$ is purely real, while $\bar{\rho}_2^I$ is predominantly imaginary, so that Eq. (18) reduces approximately to

$$| + |\bar{\rho}_2^I|\,\kappa_{22} \simeq 0 . \tag{18}$$

A necessary condition for resonance below the ρ N threshold is then

$$\kappa_{22} < 0.$$

A detailed calculation shows that κ_{22} is in fact negative both in the $D_{3/2}$, and $F_{5/2}$ states (T = 1/2) for the interaction considered (one-pion exchange), and the ratio of the κ_{22}, in the T = 1/2 state to that in the T = 3/2 is $\simeq$ 4 :1(see Fig.10). Since $|\bar{\rho}_2^I| \simeq |\bar{\rho}|^{2I+1}$, we see that, assuming $\kappa_{22}^{I=0} \simeq \kappa_{22}^{I=1}$, the $D_{3/2}$ resonance lies below the $F_{5/2}$ resonance in the T = 1/2 channel.

A crude calculation made by Cook and myself may be summarized in the following graph (Fig. 11).

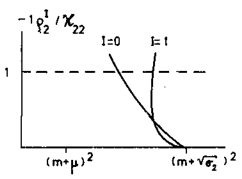

Fig. 10

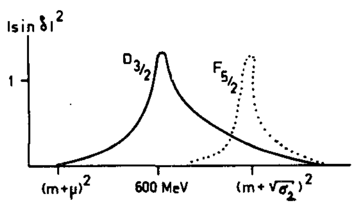

Fig. 11

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