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## PAPER

# Changes of orbital electron correlations due to the coupling between quantum dots and Majorana zero modes

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#### Abstract

We present a detailed analysis about the changes of the orbital electron-correlation effects in one quantum-dot circuit, by considering finite couplings between the quantum dots and Majorana zero modes (MZMs). It is found that the dot-MZM couplings complicate the orbital-Kondo effect, because the orbital correlation occurs between the localized states in the quantum dots and the continuum hybridized states induced by the indirect metal-MZM couplings. When two of such correlation exist in pair, they have an opportunity to induce a long-range RKKY correlation, which is related to the MZMs. Further investigation shows that this RKKY interaction leads to the anomalous fractional Josephson effect. Our work can be helpful in clarifying the influence of MZM on the orbital electron correlation effects.

## 1. Introduction

Topological superconductor (TS), a kind of fermionic symmetric protected topological material, has attracted extensive interest in the field of mesoscopic physics because Majorana zero modes (MZMs) appear at the ends of the one-dimensional TS which are of potential application in the fault-tolerant topological quantum computation [1–3]. Owning to the presence of MZM, various important transport behaviors of the TS have been observed, such as the fractional Josephson effect [4–7], the resonant Andreev reflection [8–11], and the special interplay between the local and crossed Andreev reflections in the two-terminal structures [12–14]. Moreover, MZM takes nontrivial effect to the transport behaviors of the non-topological systems, e.g., the quantum-dot (QD) systems, via coupling to the electronic bound state. When MZM is laterally connected with one closed QD circuit, the conductance magnitude is suppressed by one half [15, 16]. If it couples to two QD sub-circuits serially, nonlocal phenomenon can be achieved with its application in the nonlocal quantum entanglement [17, 18].

In addition to its influence on the conventional quantum transport, MZM plays the nontrivial role in modifying the electron correlation in QD systems [19–25]. In the structure that MZM couples to one Kondo QD (KQD), the Kondo physics shows new results. Chirla et al have observed that aside from the Kondo scale  $T_K$ , a new energy scale  $T^* \ll T_K$  emerges, which controls the low-energy physics of the system. At low temperatures, the ac conductance is suppressed for frequencies below  $T^*$ , whereas at high temperatures, the regular logarithmic dependence in the differential conductance is affected [19]. Other groups report that except the Kondo fixed point, this system flows to a new fixed point controlled by the MZM-induced coupling, which is characterized by the correlations between KQD and the fermion parity of the TS and metal [20]. Besides, the interplay between the Kondo effect and Andreev reflection has been found to modify the conductance to a great extent [21]. When the QD couples to the MZM and the s-wave superconductor simultaneously, the interplay between the two superconducting mechanisms adjusts the subgap Kondo effect in a special way [22]. For a two-terminal circuit with the MZM–KQD coupling, the unitary-limit value of the linear conductance will be reduced by exactly a factor  $\frac{3}{4}$  in the weak-coupling

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regime, and in the strong-coupling regime, the spin-split Kondo resonance occurs due to the MZM-induced Zeeman splitting [23]. On the other hand, it has been shown that if a Coulomb-blockaded topological superconductor hosting MZMs couples to the metal leads, the non-Kondo many-body physics will take place [26, 27].

With respect to the Kondo effect, it usually originates from the antiferromagnetic correlation between the localized quantum state and the conduction electron states [28]. As a matter of fact, it can also be achieved by changing the spin degree of freedom to be any other two-valued quantum number. Such a kind of Kondo effect is accordingly named as the orbital-Kondo effect. The development of the mesoscopic physics promotes the realization of it. According to the previous works, the simplest realization of the orbital-Kondo phenomenon is based on the coupling between two discrete levels to the leads, e.g., two single-level QDs [29, 30]. Compared with the spin-Kondo physics, the orbital-Kondo effect possesses the advantages that the structural parameters, e.g., the discrete levels and their coupling to the leads, can be independently controlled or tuned by external fields, which are more helpful in understanding the Kondo physics [31]. Due to this reason, the orbital-Kondo effect is an important concern in the mesoscopic physics, and lots of results have been reported [32, 33]. For instance, it can be tuned geometrically by the external magnetic flux in an Aharonov–Bohm interferometer [34]. And one flux-dependent Kondo temperature comes into being in this system. Also, the orbital-Kondo effect can be observed in a spinless system of two single-level QDs coupled to electron reservoirs. It shows that the splitting caused by level renormalization can be compensated by external gate voltages, and then the full Kondo anomaly takes place [35]. More recently, some researchers have investigated the orbital-Kondo effect by placing two antidots between the edges of an integer  $\nu = 1$  or fractional  $\nu = \frac{1}{3}$  quantum Hall bar [36]. It has been found that the inter-antidot tunneling can destabilize the Kondo fixed point for the  $\nu = \frac{1}{3}$  fractional Hall state, producing an effective interedge tunneling.

The influences of the MZM on the spin-Kondo effect motivate us to think about its effect on the orbital electron-correlation physics. In this work, we present an analysis about the orbital electron correlation in one QD circuit, by considering the presence of QD-MZM couplings (see figure 1). We expect that the MZMs can introduce new physics to the orbital electron correlation effects. The calculation results show that the QD–MZM couplings cause the orbital-Kondo correlation to occur between the localized state in the QD and the continuum hybridized states caused by the indirect metal–MZM couplings. Such a result motivates us to consider the case where two orbital correlation mechanisms co-exist in one circuit. And then, we find a kind of long-range RKKY correlation, which can also be adjusted by the superconducting phase difference between the two pairs of MZMs.

## 2. Theoretical model

The structure that we consider is illustrated in figure 1. Two normal metallic leads couple to two pairs of MZMs via QDs, respectively. Between each pair of QDs, there exists the strong interdot Coulomb interaction. Some groups have reported that the interdot Coulomb interaction can be observed experimentally [37]. It is admitted that the orbital-RKKY interaction has an opportunity to come into being in such a system, in the presence of appropriate structural parameters. The Hamiltonian for this system is written as

$$H = H_0 + H_{\rm M} + H_{\rm T}.\tag{1}$$

The first term in equation (1) is the Hamiltonian for the two normal metallic leads and the two QDs. It takes the form as

$$H_0 = \sum_{lk} \varepsilon_{lk} c_{lk}^{\dagger} c_{lk} + \sum_{l\alpha} \varepsilon_{l\alpha} d_{l\alpha}^{\dagger} d_{l\alpha} + \sum_{\alpha} U_{\alpha} n_{1\alpha} n_{2\alpha}.$$
(2)

 $c_{lk}^{\dagger}(c_{lk})$  is an operator to create (annihilate) an electron of the continuous state  $|k\rangle$  in lead-l (l = 1, 2), and  $\varepsilon_{lk}$  is the corresponding energy level.  $d_{l\alpha}^{\dagger}(d_{l\alpha})$  is the creation (annihilation) operator in QD- $l\alpha$  with level  $\varepsilon_{l\alpha}$  ( $\alpha = L, R$ ).  $U_{\alpha}$  denotes the interdot Coulomb interaction with  $n_{l\alpha} = d_{l\alpha}^{\dagger} d_{l\alpha}$ .  $H_{\rm M}$  denotes the Hamiltonian of the Majorana bound states (MBSs) at the ends of the one-dimensional TS- $\alpha$ . Since we are only interested in the leading physics induced by the MBSs, we would like to write out their low-energy effective forms [38, 39]

$$H_{\rm M} = i \sum_{\alpha} \xi_{\alpha M} \gamma_{1\alpha} \gamma_{2\alpha},\tag{3}$$

where  $\gamma_{l\alpha}$  is the self-Hermitian operator for the  $l\alpha$ -th MBS with  $\gamma_{l\alpha} = \gamma_{l\alpha}^{\dagger}$ . In the case of  $\xi_{\alpha M} = 0$ , the MZMs are achieved. Next  $H_T$ , the last term of H, describes the QD-lead and QD-MZM couplings. It reads

$$H_{\rm T} = \sum_{lk,\alpha} V_{lk,\alpha} c_{lk}^{\dagger} d_{l\alpha} + \sum_{l\alpha} W_{l\alpha} \gamma_{l\alpha} d_{l\alpha} + \text{h.c..}$$
(4)

 $V_{lk,\alpha}$  (l = 1, 2 and  $\alpha = L, R$ ) is the coupling coefficient between the QD- $l\alpha$  and lead-l.  $W_{l\alpha}$  reflects the coupling between MBS- $l\alpha$  and QD- $l\alpha$ .

We would like to say that the MBSs are achieved in the situation of strong magnetic field [40–42]. When the magnetic field covers the QDs, spin polarization takes place, which causes one spin state to exist in each QD in the low-energy region. This exactly suppresses the intradot Coulomb repulsion, and also, the spin index in the system can be neglected. Therefore, the Hamiltonian of the whole system is reasonable and feasible. In addition, it is easy to understand that the strong interdot Coulomb interaction between QD-1 $\alpha$ and QD-2 $\alpha$  can induce the occurrence of orbital-Kondo effect when the structural parameters are set below the Kondo temperature. Otherwise, the correlations on the two sides of this system have an opportunity to exhibit the orbital-RKKY correlation. Our purpose is to clarify the properties of orbital-Kondo and orbital-RKKY physics modulated by the presence of QD–MZM couplings.

#### 3. Results analysis and discussion

#### 3.1. Orbital-Kondo effect

We first aim to investigate the properties of orbital-Kondo effect influenced by the QD–MZM coupling. To do so, we suppose that the metallic leads only couple to one pair of MZMs via two QDs. In such a case, the system Hamiltonian is simplified as

$$H = \sum_{lk} \varepsilon_k c_{lk}^{\dagger} c_{lk} + \sum_l \varepsilon d_l^{\dagger} d_l + U n_1 n_2 + \sum_{lk} V_{lk} c_{lk}^{\dagger} d_l + \sum_l W_l \gamma_l d_l + \text{h.c.}.$$
(5)

In equation (5), we take  $\varepsilon_l = \varepsilon$  to maintain the symmetry of the system.

The Kondo physics is generally discussed by transforming the Anderson-type Hamiltonian into its s–d exchange form. And then, we employ the path-integral approach to perform this transformation. From the starting point of the above Anderson model, the partition function within the path-integral framework is written as  $Z = \int \mathcal{D}[c_l^{\dagger}(k, \tau), c_l(k, \tau), d_l^{\dagger}(\tau), d_l(\tau), \gamma_l(\tau)] e^{-S}$  [43], where the action is given by

$$S = \int d\tau \left\{ \sum_{lk} c_l^{\dagger}(k,\tau) (\partial_{\tau} + \varepsilon_k) c_l(k,\tau) + \sum_l d_l^{\dagger}(\tau) (\partial_{\tau} + \varepsilon) d_l(\tau) + U n_1(\tau) n_2(\tau) \right. \\ \left. + \sum_{lk} \left[ V_{lk} c_l^{\dagger}(k,\tau) d_l(\tau) + V_{lk}^* d_l^{\dagger}(\tau) c_l(k,\tau) \right] + \sum_l \left[ W_l \gamma_l(\tau) d_l(\tau) + W_l^* d_l^{\dagger}(\tau) \gamma_l(\tau) \right] \right\}.$$
(6)

Following the complicated derivation shown in appendix A, we are able to find the leading physics picture in our system. It should be noticed that under the condition of non-symmetric coupling, an anisotropic s–d exchange model similar to the result from Affleck [45] can be obtained, which is used to discuss the Kondo problem in such a condition. On the other hand, for the symmetric-coupling case, a simple s–d exchange interaction appears, i.e.,

$$H_{\rm sd} = \sum_{k,k'} \frac{J_1}{2} \psi_k^{\dagger} \boldsymbol{\sigma} \psi_{k'} \cdot \boldsymbol{S} + \sum_k \frac{J_2}{2} (\psi_k^{\dagger} \boldsymbol{\sigma} \gamma \cdot \boldsymbol{S} + \text{h.c.}) + \frac{J_3}{2} \underbrace{\gamma^{\dagger} \boldsymbol{\sigma} \gamma \cdot \boldsymbol{S}}_{=i\gamma_1\gamma_2 S^{\rm x}},$$
(7)

where 
$$J_1 = V_k V_{k'} \left( \frac{1}{\varepsilon_k - \varepsilon} + \frac{1}{\varepsilon_{k'} - \varepsilon} + \frac{1}{\varepsilon + U - \varepsilon_k} + \frac{1}{\varepsilon + U - \varepsilon_{k'}} \right)$$
,  $J_2 = V_k W \left( \frac{1}{\varepsilon_k - \varepsilon} + \frac{1}{\varepsilon + U - \varepsilon_k} - \frac{1}{\varepsilon} + \frac{1}{\varepsilon + U} \right)$ , and  $J_3 = W^2 \left( \frac{1}{\varepsilon + U} - \frac{1}{\varepsilon} \right)$ . We here have taken  $|W_l| = W$  for presenting the Kondo physics with this s-d exchange Hamiltonian.

In equation (7), we find that the first term describes the normal orbital-Kondo effect, with the Kondo temperature expressed as  $T_K^{(0)} \sim D e^{-\frac{1}{2\rho l_1}}$  (*D* and  $\rho$  denote the bandwidth and density of state of the leads, respectively). For the second term, it also exhibits an orbital-Kondo effect which is driven by the pseudo-spin correlation between the localized states in the QDs and the continuum hybridized states induced by indirect metal-MZM couplings. After a series of calculation, its Kondo temperature is expressed as  $T_K^{(1)} \sim D e^{-\frac{|\varepsilon|}{2\rho l_2}}$ . Therefore, different structural parameters cause this system to exhibit two Kondo

correlations, respectively. However, the third term can not be viewed as the pseudo-spin correlation. Note, additionally, that the electron near the Fermi surface contributes dominantly to the Kondo correlation, so  $\varepsilon_k$  in the formula of  $J_j$  can be approximated as  $\varepsilon_F$  for calculation.

With respect to  $W_l$ , we know that many works differentiate them by considering  $W_1 = |W_1|$  and  $W_2 = i|W_2|$ , respectively [46, 47]. This seems to be inconsistent with our discussion. However, note that the *i* factor can be absorbed into the Majorana operator, i.e.,  $\gamma_2 \rightarrow i\gamma_2 = e^{i\pi/2}\gamma_2$ . And then, the corresponding coupling term can be rewritten as  $|W_2|(e^{i\pi/2}\gamma_2d_2 + e^{-i\pi/2}d_2^{\dagger}\gamma_2)$  [48], which is feasible in our discussion. Under the condition of  $|W_l| = W$ , the Kondo Hamiltonian in equation (7) is obtained. Also, we are sure that the MZMs can be realized in two independent nanowires, so the phase difference between the coupling coefficients can be avoided directly.

#### 3.2. Orbital-RKKY effect

With the discussion of the orbital-Kondo effect, we next investigate the orbital-RKKY physics modified by the QD–MZM couplings. We start from the s–d exchange Hamiltonian shown in equation (7) and assume that each correlation term is doubled for achieving the RKKY effect. For convenience, the sub-indexes of the same kind of parameters are ignored by supposing their identical magnitudes.

To begin with, in our considered orbital-RKKY model, the partition function in the frequency-momentum space is directly written as

$$Z = \int \mathcal{D}\psi_k^{\dagger}(\omega_n) \mathcal{D}\psi_k(\omega_n) \exp(-\mathcal{F}), \qquad (8)$$

where  $\mathcal{F} = \sum_{mn;kp} \psi_k^{\dagger}(\omega_n) \left[ (i\omega_n - \varepsilon_k) \delta_{kp} \delta_{mn} + \sum_{\alpha} \frac{J_{\alpha,kp}^{(1)}}{2} \boldsymbol{\sigma} \cdot \boldsymbol{S}_{\alpha} \right] \psi_p(\omega_m) + \sum_{n;k\alpha} \frac{J_{\alpha,k}^{(2)}}{2} \psi_k^{\dagger}(\omega_n) (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})$ 

 $S_{\alpha}\gamma_{\alpha}(\omega_n)\} + \sum_{m;p\alpha'} \frac{J_{\alpha'p}^{(2)}}{2} \gamma_{\alpha'}^{\dagger}(\omega_m)(\boldsymbol{\sigma} \cdot \boldsymbol{S}_{\alpha'})\psi_p(\omega_m)\}.$  After integrating out the conduction-electron field, we get the new expression of the partition function, i.e.,

$$Z = \det\left[(i\omega_n - \varepsilon_k)\delta_{kp}\delta_{mn} + \sum_{\alpha} \frac{J^{(1)}_{\alpha,kp}}{2}\boldsymbol{\sigma} \cdot \boldsymbol{S}_{\alpha}\delta_{mn}\right] \exp\left\{\sum_{mn;kp;\alpha\alpha'} \frac{J^{(2)}_{\alpha,k}J^{(2)}_{\alpha',p}}{4}\gamma^{\dagger}_{\alpha}(\omega_n)(\boldsymbol{\sigma} \cdot \boldsymbol{S}_{\alpha})G(k,\omega_n)\right.$$
$$\left. \left. \left. \left(\boldsymbol{\sigma} \cdot \boldsymbol{S}_{\alpha'}\right)\gamma_{\alpha'}(\omega_m)\delta_{mn}\right\}\right\},\tag{9}$$

with  $G(k, \omega_n) = \left[ (i\omega_n - \varepsilon_k) \delta_{kp} + \sum_{\alpha''} \frac{J_{\alpha'',kp}^{(1)}}{2} \boldsymbol{\sigma} \cdot \boldsymbol{S}_{\alpha''} \right]^{-1}$ . The relevant parameters are given as  $J_{L,kp}^{(1)} = J_1 e^{i(k-p)\cdot r_L}|_{r_L=0}$  and  $J_{R,kp}^{(1)} = J_1 e^{i(k-p)\cdot r_R}|_{r_R=\ell}$ ;  $J_{L,k}^{(2)} = J_2 e^{ik\cdot r_L}|_{r_L=0}$  and  $J_{R,p}^{(2)} = J_2 e^{ip\cdot r_R}|_{r_R=\ell}$ . We would like to point out that by evaluating the logarithm of the partition function, the RKKY

We would like to point out that by evaluating the logarithm of the partition function, the RKKY correlation can be observed. The detailed processes are shown in appendix B. After derivation, one finds that in addition to the normal RKKY term, i.e.,  $H_{R0} \simeq -J_1^2 \chi(\ell) S_L \cdot S_R$ , two new correlations come into play, i.e.,

$$H_{R2} = \frac{\beta J_2^2}{4} \sum_{kn} Q_{nk}^{(2)} [(\boldsymbol{S}_L \cdot \boldsymbol{S}_R)(\gamma_L^{\dagger} \gamma_R + \gamma_R^{\dagger} \gamma_L) + i(\boldsymbol{S}_L \times \boldsymbol{S}_R) \cdot (\gamma_L^{\dagger} \boldsymbol{\sigma} \gamma_R - \gamma_R^{\dagger} \boldsymbol{\sigma} \gamma_L)],$$
(10)

where  $Q_{nk}^{(2)} = \frac{1}{\beta} \frac{e^{ik \cdot \ell}}{i\omega_n - \varepsilon_k}$ . The results in equation (10) clearly show that due to the presence of the MZM–QD couplings, the RKKY correlation mechanism of our system undergoes new and interesting changes, which are more complicated than the orbital-Kondo correlation. The first change can be viewed as the direct transformation of the RKKY interaction induced by the MZMs, whereas the second describes a new kind of MZM-assisted correlation.

We would like to say that the two terms in equation (10) are caused by the coexistence of two orbital correlations between the localized states in the QDs and the MZM-induced continuum hybridized states (see the second term in equation (7)). It is not difficult to understand that when these two correlations interact, the RKKY interaction is also allowed to take place, meanwhile, the MZMs on the two sides get the effective couplings. In fact, the co-existence of the MZMs induces the anisotropy of this system, so new correlation mechanism arises inevitably, as described by the second term in equation (10). Although it is similar to the Dzyaloshinskii–Moriya interaction in the spin–orbit coupled conduction electrons [49, 50], alternative physics mechanism is involved here, due to the special role of the MZMs. Therefore, the lead-MZM couplings make important contribution to the modulation of the orbital RKKY correlation. Next, one can estimate the real-space oscillation of the coefficient in equation (10) by performing the frequency-momentum summation calculation, i.e.,  $\frac{1}{\beta} \sum_{kn} \frac{e^{ikr} e^{i\omega_n 0^+}}{i\omega_n - \varepsilon_k} \frac{T \to 0}{2\pi^2 r} \int_0^{k_F} dkk \sin(kr)$ . And then, the corresponding nonlocal susceptibility can be given as  $\chi_1(\ell) = \frac{\rho}{2\pi^2 r^2} \left[\sin(k_F r) - k_F r \cos(k_F r)\right]|_{r=\ell}$ . Comparison with the normal RKKY interaction, it is certainly a long-range parameter which decays along the distance in the  $\ell^{-2}$  form. Besides this analysis, we anticipate that the second new correlation is usually very weak, since the leading correlation in this system, i.e., the RKKY effect, tends to arrange the spin in the ferromagnetic or anti-ferromagnetic ways.

In view of equation (10), the MZMs indeed make key contributions to the occurrence of new RKKY correlations. It should be noticed that the MZM pairs exist in the topological superconductors, and they are also allowed to carry the superconducting phases, i.e.,  $\gamma_{L(R)} = [\gamma_{1L(R)}e^{i\theta_{L(R)}/2}, i\gamma_{2L(R)}e^{i\theta_{L(R)}/2}]^T$ . Thus, the new correlation terms depend on the superconducting phase difference between the two pairs of MZMs. This exactly means that they introduce intricate physics picture to the fractional Josephson effect. In other words, the new RKKY correlation is tunable by considering the Josephson phase difference. We would like to discuss this point as follows. For the first term in equation (10), it can be rewritten as  $\tau_{R2}^{(1)} = (S_L \cdot S_R)[2i(\gamma_{1L}\gamma_{1R} + \gamma_{2L}\gamma_{2R})] \sin \frac{\theta}{2}$  with  $\theta = \theta_R - \theta_L$ . If we define  $f_L = \frac{1}{\sqrt{2}}(\gamma_{1L} + i\gamma_{2L})$  and

 $f_R = \frac{1}{\sqrt{2}}(\gamma_{1R} - i\gamma_{2R}), \tau_{R2}^{(1)}$  will be transformed into  $\tau_{R2}^{(1)} \propto 2(\mathbf{S}_L \cdot \mathbf{S}_R) \sin \frac{\theta}{2} f_L f_R + \text{h.c.}$  It does give rise to the anomalous fractional Josephson effect. The second term in equation (10) can be deduced in the same way, and then it reads

$$\tau_{R2}^{(2)} = 2\left[i(\mathbf{S}_L \times \mathbf{S}_R)_y + (\mathbf{S}_L \times \mathbf{S}_R)_z\right] \cos\frac{\theta}{2} f_L^{\dagger} f_R - 2i(\mathbf{S}_L \times \mathbf{S}_R)_x \sin\frac{\theta}{2} f_L f_R + \text{h.c.}.$$
(11)

Considering the parity conservation, the two parts in this equation will govern the fractional Josephson effect independently.

Up to now, with the help of the above discussions, we have known the roles of the MZMs in modifying the RKKY interaction. Just due to the appearance of the new correlations terms, the fractional Josephson effect can be induced in a complicated way. An explicit result is that at the limit of small superconducting phase difference, finite super current can be driven. This is helpful for understanding the long-range RKKY interaction induced by the MZMs. Therefore, we ascertain that the long-range RKKY interaction can be observed via measuring the modification of the fractional Josephson effect.

## 4. Summary

To summarize, we have performed investigations about the orbital-RKKY effect in a two-lead mesoscopic circuit where each metallic lead couples to two single-level QDs, by considering the presence of finite QD–MZM couplings. Firstly, in the case of each lead coupling to one QD, the orbital s–d exchange Hamiltonian is obtained. It has been found that due to the presence of MZMs, its form becomes complicated, because one new orbital correlation occurs between the localized state in the QD and the continuum hybridized states induced by indirect metal–MZM couplings. As a result, different orbital Kondo physics can be induced by tuning the structural parameters. Next, when the above s–d exchange correlation is paired, the new orbital correlation leads to the appearance of long-range RKKY correlation, which is tightly related to the MZMs. And then, this RKKY correlation enables to interact with the

anomalous fractional Josephson effect, in the case of finite superconducting phase difference between the MZM pairs. Thus, our considered system contains rich physics picture, which connects the correlation mechanism with the Josephson effect. Also surely, our results suggest that the MZM–QD couplings play special and important roles in modulating the orbital-RKKY effect. Therefore, this work provides useful information for clarifying the special influence of MZM on the electron correlation effects.

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## Appendix A

The strong-interaction term in equation (6) can be divided into charge and 'spin' parts in the sense of orbit where  $Un_1n_2 = \frac{U}{4}n - US_z^2$  with charge density  $n = n_1 + n_2$  and the orbital–spin density in the projective *z*-direction with  $S_z = \frac{1}{2}(n_1 - n_2)$ . And then, by performing the Hubbard–Stratonovich transformation, we obtain the new form of the partition function, i.e.,

obtain the new form of the partition function, i.e.,  $Z = \int \mathcal{D}[\phi, \Delta] \int \mathcal{D}[\psi^{\dagger}, \psi, \chi^{\dagger}, \chi, \gamma^{\dagger}, \gamma] e^{-S[\psi^{\dagger}, \psi, \chi^{\dagger}, \chi, \gamma^{\dagger}, \gamma, \phi, \Delta]}$ in which  $\psi^{\dagger} = [c_{1k}^{\dagger}, c_{2k}^{\dagger}], \chi^{\dagger} = [d_{1}^{\dagger}, d_{2}^{\dagger}],$   $\gamma^{\dagger} = \gamma^{T} = [\gamma_{1}, \gamma_{2}]$  with the auxiliary fields  $\phi$  and  $\Delta$ . Within the fermion representation for spin  $S_{z} = \frac{1}{2}\chi^{\dagger}\sigma^{z}\chi$ , the action reads

$$S = \int d\tau \left\{ \sum_{k} \psi^{\dagger} (\partial_{\tau} + \varepsilon_{k}) \psi + \chi^{\dagger} (\partial_{\tau} + \varepsilon) \chi + \frac{\phi^{2} + \Delta^{2}}{U} - i\phi \chi^{\dagger} \chi - \Delta \chi^{\dagger} \sigma^{z} \chi \right. \\ \left. + \left( \sum_{k} \psi^{\dagger} \mathcal{V}_{k} + \gamma^{\dagger} \mathcal{W} \right) \chi + \chi^{\dagger} \left( \sum_{k} \mathcal{V}_{k}^{\dagger} \psi + \mathcal{W}^{\dagger} \gamma \right) \right\},$$
(A1)

where  $\mathcal{V}_k = \frac{1}{2}[(V_{1k} + V_{2k})\mathbb{I} + (V_{1k} - V_{2k})\sigma_z]$  and  $\mathcal{W} = \frac{1}{2}[(W_1 + W_2)\mathbb{I} + (W_1 - W_2)\sigma_z]$ .

The following derivation about *S* can be performed with the help of the effective mean-field theory. We then employ the saddle-point method by separating the auxiliary fields into static mean part and the fluctuation part, i.e.,  $\phi(\tau) = \phi_0 + \delta\phi(\tau)$  and  $\Delta(\tau) = \Delta_0 + \delta\Delta(\tau)$ . Note that the effective action should keep invariant after the rotation of the quantization axis which is a *SU*(2)-gauge-invariant transformation with  $\mathcal{U}^{\dagger}\sigma^{z}\mathcal{U} \rightarrow \mathbf{n} \cdot \boldsymbol{\sigma}$  [ $\mathcal{U} \in \mathbb{SU}(2)$ ]. Accordingly, in the gauge-invariant action, the pseudo-fermion fields undergo a unitary transformation  $\tilde{\chi} = \mathcal{U}^{\dagger}$ , which does not modify the integral measure because the Jacobian determinant between the Grassmann variables is 1. This is manifested as the fact that  $\mathcal{D}[\tilde{\chi}^{\dagger}, \tilde{\chi}] = \mathcal{D}[\chi^{\dagger}, \chi] \left\| \frac{\partial(\tilde{\chi}^{\dagger}, \tilde{\chi})}{\partial(\chi^{\dagger}, \chi)} \right\| = \mathcal{D}[\chi^{\dagger}, \chi] \det[\mathcal{U}^{\dagger}\mathcal{U}] = \mathcal{D}[\chi^{\dagger}, \chi]$ . And then, it is understandable that such a gauge invariant theory should include all possible gauge trajectories in the path integral, i.e.,  $Z = \int \mathcal{D}[\mathbf{n}(\tau)] e^{-S[\mathbf{n}]}$ . After integrating out  $\tilde{\chi}$  and  $\tilde{\chi}^{\dagger}$ , we get the effective action

$$S_{\text{eff}} = \int d\tau \left\{ \sum_{k} \psi^{\dagger} (\partial_{\tau} + \varepsilon_{k}) \psi + \frac{\phi^{2} + \Delta^{2}}{U} + \left( \sum_{k} \psi^{\dagger} \mathcal{V}_{k} + \gamma^{\dagger} \mathcal{W} \right) \mathcal{U}^{\dagger} G_{d} \mathcal{U} \left( \sum_{k'} \mathcal{V}_{k'}^{*} \psi + \mathcal{W}^{*} \gamma \right) \right\} - \text{Tr} \ln \left[ -G_{d}^{-1} \right].$$
(A2)

Here, we introduce an effective Green function  $G_d = [g_d^{-1} - \Sigma]^{-1}$  for the QDs, in which the free Green function is defined by  $g_d = -(\partial_\tau + \varepsilon - i\phi_0 - \Delta_0\sigma^z)^{-1}$  with the self-energy correction expressed as  $\Sigma = \mathcal{U}\partial_\tau \mathcal{U}^{\dagger} + i\delta\phi(\tau) + \delta\Delta(\tau)$ . In the Kondo regime,  $\varepsilon < 0$  and U > 0, whereas  $|V_k|/|\varepsilon| \ll 1$  and  $|V_k|/U \ll 1$  ( $V_{lk} = V_k$  is assumed for the orbital-Kondo effect). This means that the fluctuations around the static charge configuration can be forbidden. Thus, we set  $|\delta\phi|, |\delta\Delta| \to 0$ , which accordingly yields  $\Sigma \approx \mathcal{U}\partial_\tau \mathcal{U}^{\dagger}$ . After this discussion, the saddle-point conditions can be clarified from equations  $\frac{\delta}{\delta\phi_0} \ln Z = 0$  and  $\frac{\delta}{\delta\Delta_0} \ln Z = 0$  with the help of the action in equation (13). They are

$$\frac{2}{U}\phi_0 - i\langle \chi^{\dagger}\chi \rangle = 0, \tag{A3}$$

$$\frac{2}{U}\Delta_0 - \langle \chi^{\dagger}\sigma^z \chi \rangle = 0. \tag{A4}$$

In equation (A3) we should restrict to the single-occupation configuration  $\langle \chi^{\dagger}\chi \rangle = \langle n \rangle = 1$ , while equation (A4) indicates the z-projective component of orbital spin  $\langle \chi^{\dagger}\sigma^{z}\chi \rangle = 2|S_{z}| = 1$ . Therefore, we obtain the mean-field values, i.e.,  $\phi_{0} = iU/2$  and  $\Delta_{0} = U/2$ . If we present the calculation with the effective action in equation (13), the corrected QD Green function should be encountered, i.e., Tr  $\ln[-G_{d}^{-1}] =$  $\ln \det[-G_{d}^{-1}] = \sum_{\omega_{l}} \ln[(-i\omega_{l} + \varepsilon - i\phi_{0})^{2} - \Delta_{0}^{2}]$ . This implies that the corrections to  $\phi_{0}$  and  $\Delta_{0}$ , however, are exponentially small in 1/T, which can be ignored properly on the saddle-point level [44].

Thereafter we derive the coupling term as

$$\mathcal{U}^{\dagger}G_{d}(\tau)\mathcal{U} \xrightarrow{\omega \to 0} \mathcal{U}^{\dagger}[(-\varepsilon + i\phi_{0})\mathbb{I} + \Delta_{0}\sigma^{z}]^{-1}\mathcal{U} = \mathcal{U}^{\dagger}[(-\varepsilon - U/2)\mathbb{I} + (U/2)\sigma^{z}]^{-1}\mathcal{U}$$
$$= \mathcal{U}^{\dagger}\left[\frac{U\sigma^{z}/2 + (\varepsilon + U/2)}{|\varepsilon|(\varepsilon + U)}\right]\mathcal{U} = (\frac{1}{\varepsilon + U} - \frac{1}{\varepsilon})\boldsymbol{n} \cdot \frac{\boldsymbol{\sigma}}{2} - \frac{\varepsilon + U/2}{\varepsilon(\varepsilon + U)}.$$
(A5)

 $\boldsymbol{n}(\tau)$ , the spin direction vector, is orientated on the Bloch sphere  $\mathbb{S}^2 \cong \mathbb{SU}(2)/\mathbb{U}(1)$ . It is actually the spin vector unit  $\boldsymbol{n} = z^{\dagger}\boldsymbol{\sigma}z$  represented in the coherent-state spinor  $|\boldsymbol{n}(\theta(\tau),\varphi(\tau))\rangle = z = (z_1,z_2)^T = (e^{-i\varphi(\tau)} \cos \frac{\theta(\tau)}{2}, \sin \frac{\theta(\tau)}{2})^T$ , which is usually used in the path-integral formalism. Here, an additional emergent Berry-phase term originates from the self-energy correction which contributes to the action, i.e.,  $-\operatorname{Tr}[g_d \Sigma] \approx \int d\tau [\mathcal{U}^{\dagger} \partial_{\tau} \mathcal{U}]_{1,1} = \frac{i}{2} \int d\tau (1 - \cos \varphi(\tau))\dot{\theta}(\tau)$  [44]. We also prove that it is indeed an equivalent expression of geometric phase in the spin coherent state spinor form  $\int d\tau z^{\dagger} \dot{z}$ . Instead, it can be fully replaced by quantum spin-1/2 operator  $\boldsymbol{S} = S\boldsymbol{n}$  in the operator form.

## Appendix B

The normal RKKY term is involved in the first term of  $\ln Z$  in equation (9), and it is solved in the following way:

$$H_{R0} = \ln \det \left[ \delta_{kp} \delta_{mn} + (i\omega_n - \varepsilon_k)^{-1} \sum_{\alpha} \frac{J_{\alpha,kp}^{(1)}}{2} (\boldsymbol{\sigma} \cdot \boldsymbol{S}_{\alpha}) \right]$$
  
$$= \operatorname{Tr} \left[ (i\omega_n - \varepsilon_k)^{-1} \sum_{\alpha} \frac{J_{\alpha,kp}^{(1)}}{2} (\boldsymbol{\sigma} \cdot \boldsymbol{S}_{\alpha}) \right] - \frac{1}{2} \operatorname{Tr} \left\{ \left[ (i\omega_n - \varepsilon_k)^{-1} \sum_{\alpha} \frac{J_{\alpha,kp}^{(1)}}{2} (\boldsymbol{\sigma} \cdot \boldsymbol{S}_{\alpha}) \right]^2 \right\}$$
  
$$= \sum_{n;k,p} \frac{J_1}{2} \frac{\operatorname{Tr} \left[ \boldsymbol{\sigma} \cdot \left( \boldsymbol{S}_L + e^{i(k-p) \cdot \ell} \boldsymbol{S}_R \right) \right]}{i\omega_n - \varepsilon_k} - \frac{1}{2} \sum_{m,n;k,p} \frac{J_1^2}{4} \frac{\operatorname{Tr} \left\{ \left[ \boldsymbol{\sigma} \cdot \left( \boldsymbol{S}_L + \tilde{\boldsymbol{S}}_R \right) \right] \left[ \boldsymbol{\sigma} \cdot \left( \boldsymbol{S}_L + \tilde{\boldsymbol{S}}_R \right) \right] \right\}}{(i\omega_n - \varepsilon_p)}$$
  
$$= -\frac{1}{2} \sum_{m,n;k,p} \frac{J_1^2}{4} \frac{(\boldsymbol{S}_L + \tilde{\boldsymbol{S}}_R) \cdot (\boldsymbol{S}_L + \tilde{\boldsymbol{S}}_R)}{(i\omega_n - \varepsilon_k)(i\omega_m - \varepsilon_p)} \simeq \Delta E \underbrace{S(S+1)}_{=3/4} - \frac{\beta J_1^2}{4} \sum_{m,n;k,p} Q_{mn,kp}^{(1)} \boldsymbol{S}_L \cdot \boldsymbol{S}_R. \tag{B1}$$

Here,  $Q_{mn,kp}^{(1)} = \frac{1}{\beta} \frac{e^{iq\cdot\ell}}{\varepsilon_k - \varepsilon_{k+q} + i\nu_l} \left[ \frac{1}{i\omega_n - \varepsilon_k} - \frac{1}{i\omega_{n+l} - \varepsilon_{k+q}} \right]$  and  $\tilde{S_R} = e^{i(k-p)\cdot\ell} S_R$ . The momentum-frequency summation of  $Q_{mn,kp}$  gives out exactly the susceptibility of the conduction electron which is formally expressed as

$$\chi_{c}(q)\delta^{\alpha\alpha'} = -\frac{1}{\beta}\sum_{k} \operatorname{Tr}\left[\sigma^{\alpha}G_{c}(k+q)\sigma^{\alpha'}G_{c}(k)\right] = -\frac{1}{\beta}\sum_{k,\omega_{n}}G_{c}(k+q,i\omega_{n}+i\nu_{b})G_{c}(k,i\omega_{n})$$
$$= \int_{|k|< k_{F}}\frac{\mathrm{d}^{3}k}{(2\pi)^{3}}\frac{f(k)-f(k+q)}{\varepsilon_{k+q}-\varepsilon_{k}-i\nu_{b}}.$$
(B2)

It is known that at the low temperature, the bosonic Matsubara frequency component  $\nu_b$  can be ignored. We now arrive at the classical RKKY form interaction  $-J_1^2 \chi(\ell) S_L \cdot S_R$  with the non-local susceptibility  $\chi(\ell) = \int_{|k| < k_F} \frac{d^3q}{(2\pi)^3} \chi_c(q) e^{iq \cdot \ell}$ . For the normal metal, such an integral can be evaluated as the Lindhard function, i.e.,  $\chi(r)|_{r=\ell} = \frac{\rho}{2\pi^2 r^3} \left[ \cos(2k_F r) - \frac{\sin(2k_F r)}{2k_F r} \right]|_{r=\ell}$ , which reproduces the well-known Friedel oscillation. This result also proves the feasibility of our deduction method.

The rest correlation terms in equation (9) can be analyzed by means of the perturbative technique. To do so, it is necessary to expand  $G(k, \omega_n)$  into the Dyson series, i.e.,

 $G(k,\omega_n) = G_0(k) + G_0(k) \left[ \sum_{\alpha} \frac{J_{\alpha,kp}^{(1)}}{2} (\boldsymbol{\sigma} \cdot \boldsymbol{S}_{\alpha}) \right] G_0(p) + \cdots \text{ in which } G_0(k) = (i\omega_n - \varepsilon_k)^{-1}.$  According to the order of energy scale, we only need to consider the zero-order term of the series to describe the new RKKY correlation mechanism. As a result, two new parts arise. For the first one, it is given as

$$H_{R1} = \frac{J_2^2}{4} \sum_{\alpha, kpmn} \frac{\delta_{kp} \delta_{mn}}{i\omega_n - \varepsilon_k} \gamma_\alpha^{\dagger} (\boldsymbol{\sigma} \cdot \boldsymbol{S}_\alpha)^2 \gamma_\alpha$$
$$= \frac{3J_2^2}{16} \sum_{\alpha, kn} \frac{\gamma_\alpha^{\dagger} \gamma_\alpha}{i\omega_n - \varepsilon_k} = \frac{3J_2^2}{4} \sum_{k, n} \frac{1}{i\omega_n - \varepsilon_k}.$$
(B3)

It is evident that such a result does not induce any correlation but only modifies the vacuum energy which originates from the correction on the Fermi-liquid vacuum energy in the metal. It can be revealed with the trivial momentum summation above  $\sum_{k,\omega_n} \frac{\delta_{n,0}}{i\omega_n - \varepsilon_k} \simeq -\rho \int_{D'}^{D} \frac{d\varepsilon_k}{\varepsilon_k} = -\rho \ln \frac{D}{D'}$  where  $\varepsilon_F \in (D', D)$ . Therefore, it can be omitted by re-scaling the measurement energy level. Alternatively, the second part, i.e., the new correlation term, reads

$$H_{R2} = \frac{J_2^2}{4} \sum_{\alpha kn} \frac{e^{-ip \cdot \ell} \delta_{kp}}{i\omega_n - \varepsilon_k} \gamma_{\alpha}^{\dagger} (\boldsymbol{\sigma} \cdot \boldsymbol{S}_{\alpha}) (\boldsymbol{\sigma} \cdot \boldsymbol{S}_{\bar{\alpha}}) \gamma_{\bar{\alpha}}$$
  
$$= \frac{\beta J_2^2}{4} \sum_{kn} Q_{nk}^{(2)} [(\boldsymbol{S}_L \cdot \boldsymbol{S}_R) (\gamma_L^{\dagger} \gamma_R + \gamma_R^{\dagger} \gamma_L) + i (\boldsymbol{S}_L \times \boldsymbol{S}_R) \cdot (\gamma_L^{\dagger} \boldsymbol{\sigma} \gamma_R - \gamma_R^{\dagger} \boldsymbol{\sigma} \gamma_L)], \qquad (B4)$$

where  $Q_{nk}^{(2)} = \frac{1}{\beta} \frac{e^{ik \cdot \ell}}{i\omega_n - \varepsilon_k}$ .

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