Aspects of Dynamical Locality and Locally Covariant Canonical Quantization

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Abstract

In this thesis we consider a number of different aspects of dynamical locality, an axiom on locally covariant theories proposed by Fewster and Verch [25] that is closely related to the question of whether a theory describes the same physics in all spacetimes.

After some introductory material, in Chapters 3 and 4 we examine dynamical locality for the nonminimally coupled scalar field and its enlarged algebra of observables. We show that dynamical locality holds at all masses, including non-zero masses, for the nonminimally coupled scalar field theory. We also demonstrate that dynamical locality holds in the massive minimally coupled and massive conformally coupled cases for the enlarged algebra of observables, and fails to hold in the massless minimally coupled case.

In Chapter 5, we discuss a number of categorical structures that can be used in the construction of classical theories that may be quantized using canonical anticommutation relations (CAR), and their subsequent quantization. We prove a number of results pertaining to dynamical locality of classical theories and their CAR-quantized counterparts.

In Chapters 6 and 7, we give a simplified version of the locally covariant classical and quantum Dirac theories, using the machinery developed in Chapter 5. We also formulate for the first time versions of these theories that are entirely independent of the choice of a global reference frame for the spacetime, and depend only on an equivalence class of these frames. We demonstrate that both the simplified frame-dependent theories and the frame-independent theories are dynamically local.

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Author's declaration

This thesis represents the results of original research by the author, except where acknowledgement has been made to other sources.

Most of the material presented in Chapters 3 and 4, and Appendices B and C, has previously been published by the author [24].

The work in Chapters 6 and 7 concerning the simplified construction of the frame-dependent Dirac theories has been undertaken in collaboration with Dr Chris Fewster. To the extent that material herein represents the results of joint research, the author's contribution has been significant.

The author has not submitted this work previously for examination for any degree at the University of York or any other institution.

Chapter 1

Introduction

The area of quantum field theory in curved spacetime attempts to explore the problem of quantum gravity by treating the spacetime metric as fixed and unaffected by the presence of quantum fields. This is in contrast to reality, in the sense that the presence of matter affects the metric in a way that is predictable by general relativity in large-scale, low energy situations. We should not, therefore, expect QFT in curved spacetime to give accurate predictions in all cases: the situations in which it is liable to fail include (but are not limited to) regions with extremely high curvature or energy density, or at extremely small length scales.

However, QFT in curved spacetime is useful because it is able to predict many phenomena that cannot be explained in traditional quantum field theory on Minkowski space. Examples include the observation by Hawking [34] (and later in a more rigorous fashion, by Dimock and Kay [21]) that collapsing black holes radiate as a black body with a certain temperature dependent on the surface gravity; the demonstration by Kay and Wald [42, 41] that related thermal properties arising from geometrical properties of the spacetime may be observed in a wider range of spacetimes, including the maximally extended Schwarzschild solution; and the Unruh effect [28, 16, 60].

It is equally interesting to see what aspects of QFT in Minkowski space are no longer present in curved spacetime; it is easy to see that global Poincaré symmetry is meaningless in a general curved background, but more revealing is the fact that preferred vacuum states — and even a global particle interpretation — are impossible in general curved spacetimes [63].

However, it is not satisfactory to regard each possible spacetime background as an independent and distinguished object when constructing our theories. In order for us to be able to make meaningful predictions, we should not need knowledge of the geometrical structure of our universe beyond the regions we are observing. In particular, the result of an experiment should not be affected by, or affect, any event that exists within its causal complement. This idea is known as *locality*, and it asserts that the localized properties of any quantum theory in a given region of spacetime should depend only on the local properties of the spacetime in that region.

The loss of Poincaré symmetry also presents a problem; while the absence of a preferred vacuum state or particle interpretation in general are expected in curved spacetimes, other results traditionally relying on Poincaré symmetry such as the spin-statistics theorem [51, 44, 13] should still hold in curved spacetimes. It was shown by Verch in [61] that rather than depending on global symmetries of spacetimes, it is in fact possible to prove a spin-statistics theorem for general curved-spacetime theories by invoking *general covariance*, which requires only local symmetries that exist for all spacetimes.

Following this work, a set of axioms for quantum field theory in curved spacetime have been set out by Brunetti, Fredenhagen and Verch [11] that are consistent with the locality principle and build in the idea of general covariance. A theory obeying these axioms is called a *locally covariant theory* (or LCT; definitions are given in Chapter 2). The basic idea is to use the tools of category theory to describe a given physical theory as a covariant functor from a category of spacetimes to a category of physical systems; the desired properties of locality and covariance come directly from the functorial status of the theory, although extra causality conditions are usually applied.

We now come to the question of physicality, or 'what makes a particular locally covariant theory physically relevant' (or realistic)? In [25], Fewster and Verch argue that the condition of local covariance alone is insufficient for physicality, and support this by constructing a number of locally covariant theories that display certain types of behaviour that appear to be pathological.

However, the construction of such pathological theories alone does not go a long way towards answering the question of what really does make a theory physical. Indeed, even heuristically and informally it is difficult to come up with a useful answer to this question; the naïve approach of saying that physical theories are precisely those that model real, observable fields to a high degree of accuracy is circular at best, and at worst useless, since it automatically discounts any theory that models an existing, but as yet unobserved type of physical field.

The approach of Fewster and Verch is to attempt to come up with conditions that are necessary for physicality, in the sense that they preclude the type of pathologies that have been observed to exist in the LCT framework. In particular, the *SPASs* (Same Physics in All Spacetimes) condition is an axiom that applies to classes of LCTs, which prevents a natural transformation existing between two theories in the class that is an isomorphism on some spacetimes but not on others (again, full definitions are given in Chapter 2). The SPASs condition is not particularly tractable, in the sense that it is difficult to test without additional machinery. The particular machinery that is proposed by Fewster and Verch in [25] is the axiom of *dynamical locality*, which applies to individual theories, is often relatively easy to test, and is sufficient, when applied globally to a given class, to ensure SPASs.

It is perhaps a point of contention whether the discovery of any number of necessary conditions for physical realism comes anywhere close to answering what physicality actually *is*, rather than *is not*. Furthermore, although dynamical locality is sufficient for SPASs, it is not necessary; and although it is reasonable to consider SPASs necessary for physicality, it is certainly not clear whether it should be sufficient. Therefore, current understanding allows no definite logical link between physicality and dynamical locality, in either direction. Nevertheless, the difficulty inherent in obtaining a more formal definition of physicality suggests that any inroads into the question are worthwhile.

We are therefore forced in this situation to resort to empiricism, and attempt to gather evidence for dynamical locality to be a sensible axiom by testing it against the small number of rigorously defined locally covariant theories that we consider should be physical under any reasonable definition. This approach for testing our axioms is slightly dangerous; it would be tempting, if dynamical locality proved too strong, to tweak it until it fit our list of physical theories. This would, of course, make us guilty of the Texas sharpshooter fallacy, where we adjust our hypothesis to fit our data.

However, it is unfair to suggest that the axiom of dynamical locality stands or falls on its performance under testing; its central idea (that two complementary definitions of the local algebras of observables must coincide) has merit on its own, and so we should not simply discard it if it fails on a particular theory. In fact, it has been shown in [26] to fail for the locally covariant theory of the scalar field in the case where the mass of the field is zero and the field equation is not coupled to the curvature of the spacetime, and this particular failure can be interpreted in a meaningful way (as resulting from a rigid gauge symmetry in the Lagrangian), without having to abandon either the idea that this particular theory is physical, or that dynamical locality is an appropriate axiom to impose. Dynamical locality has been shown to hold for the minimally coupled scalar field theory with all other masses [26]. In addition, the principle of dynamical locality seems to be intimately related to the behaviour of the stress tensor in the theories that have been examined so far; since the stress tensor contains the information about the energy density of a given field, which is the surest indicator of whether something physical is actually present, this bodes well for dynamical locality being some link to physical realism. It is also possible to show that any non-trivial dynamically local theory does not admit a natural state [25], i.e. a family of states of the algebras of observables, indexed by spacetimes, that interacts naturally (in the strict categorical sense) with the algebra homomorphisms arising from local diffeomorphisms of the spacetimes.

In this thesis, we examine three different aspects of dynamical locality. The first is concerned with demonstrating that the nonminimally coupled scalar field is dynamically local at all masses — this is given in Chapter 4, along with the proof that dynamical locality holds in two particular cases of the theory of the enlarged algebra of observables for the scalar field. We also give a general argument for the conditions under which we can extend dynamical locality of a given classical LCT to the corresponding quantum theory quantized according to the canonical anticommutation relations (CAR) in Chapter 5, following similar work for the quantization according to canonical commutation relations in [26]. Finally, we demonstrate that the classical and quantum Dirac theories are dynamically local, giving a simplified locally covariant construction in the process. We begin with an overview of the basic facts regarding globally hyperbolic spacetimes, which are the suitable background objects for our locally covariant theories.

1.1 Mathematical preliminaries

In order to understand canonical quantization in a locally covariant framework, as well as the concept of dynamical locality, we must have a good understanding of the underlying categorical notions. A complete introduction to category theory is beyond the remit of this thesis, so we assume that the reader has a working knowledge of the basics; [45] provides a suitable introductory text. However, some additional definitions (including some not contained in the cited reference) are presented in Appendix A.

1.1.1 Globally hyperbolic spacetimes

The fundamental objects to which we apply locally covariant (classical or quantum) theories are *globally hyperbolic spacetimes.* The definitions contained in this section are mostly covered in [25].

Definition 1.1.1. *A* spacetime *is a collection* $(\mathcal{M}, \boldsymbol{g}, \mathfrak{o}, \mathfrak{t})$ *where*

- $\mathcal{M}$ *is a smooth orientable paracompact manifold of dimension* $n \geq 2$ *with finitely many connected components,*
- $\boldsymbol{g}$ *is a smooth time-orientable metric on* $\mathcal{M}$ *of signature* $+ - \cdots -$ *,*
- $\mathfrak{o}$ *is one of the two components of the set of nowhere-zero smooth* n*-form fields on* $\mathcal{M}$*, constituting a choice of orientation for* $\mathcal{M}$*,*

- $\mathfrak{t}$ *is one of the two components of the set of nowhere-zero smooth 1-form fields on* $\mathcal{M}$ *which are timelike with respect to* $\boldsymbol{g}$*, constituting a choice of time-orientation for* $\boldsymbol{g}$*.*

In general, we may safely blur the distinction between a spacetime and its underlying manifold; given a spacetime $\boldsymbol{M} = (\mathcal{M}, \boldsymbol{g}, \mathfrak{o}, \mathfrak{t})$, to avoid unnecessary clutter we may for example use notation such as "$p \in \boldsymbol{M}$" to indicate that a point p lies in the underlying spacetime, rather than the technically correct "$p \in \mathcal{M}$".

Given a point $p \in \boldsymbol{M}$ we denote by $J^{\pm}_{\boldsymbol{M}}(p) \subset \boldsymbol{M}$ the causal future (past) of p; that is the set of points, including p, which lie on some future- (past-) directed causal curve beginning at p (a *causal curve* in this context is a piecewise differentiable curve in $\boldsymbol{M}$ whose tangent is everywhere non-zero and either timelike or null). We also have $J_{\boldsymbol{M}}(p) := J^{+}_{\boldsymbol{M}}(p) \cup J^{-}_{\boldsymbol{M}}(p)$. For any subset $S \subset \boldsymbol{M}$ we denote $J^{(\pm)}_{\boldsymbol{M}}(S) := \bigcup_{p \in S} J^{(\pm)}_{\boldsymbol{M}}(p)$.

Definition 1.1.2. *A spacetime* $\boldsymbol{M}$ *is* globally hyperbolic *if it contains no closed causal curves and for every* $p, q \in \boldsymbol{M}$ *the intersection* $J^{+}_{\boldsymbol{M}}(p) \cap J^{-}_{\boldsymbol{M}}(q)$ *is compact.*

The topic of globally hyperbolic spacetimes is covered extensively in [3, 47]. The most relevant results are as follows:

Definition 1.1.3. *Let* $\boldsymbol{M}$ *be a globally hyperbolic spacetime. A* Cauchy surface *is a subset* $\Sigma \subset \boldsymbol{M}$ *which has precisely one point of intersection with every inextendible timelike curve in* $\boldsymbol{M}$*. A* spacelike *Cauchy surface is a* C^1 *Cauchy surface in which all tangent vectors to the surface are spacelike. Note that any* C^1 *Cauchy surface of* $\boldsymbol{M}$ *is necessarily a submanifold of codimension* 1*.*

- A spacetime $\boldsymbol{M}$ is globally hyperbolic if and only if it admits a Cauchy surface [47, Cor. 14.39],
- All Cauchy surfaces of a globally hyperbolic spacetime are homeomorphic [47, Cor. 14.27],
- A globally hyperbolic spacetime $\boldsymbol{M}$ admits a smooth spacelike Cauchy surface Σ; moreover, the underlying manifold $\mathcal{M}$ is diffeomorphic to

$\mathbb{R} \times \Sigma$ [3, Thm. 1], and it is possible to construct a diffeomorphism $\rho : \mathbb{R}\times\Sigma \to \boldsymbol{M}$ such that $\rho^*\boldsymbol{g}_{\boldsymbol{M}} = \beta\, dt \otimes dt - \boldsymbol{h}_t$, where $\beta : \mathbb{R}\times\Sigma \to \mathbb{R}^+$ is smooth, and $(\boldsymbol{h}_t)_{t\in\mathbb{R}}$ is a smoothly varying family of smooth Riemannian metrics for Σ [4, Thm. 1.1].

- As a consequence, all smooth Cauchy surfaces of a globally hyperbolic spacetime are diffeomorphic.

The following definitions are taken from [25]:

Definition 1.1.4. *The category* Loc *contains as its objects all globally hyperbolic spacetimes. An arrow from* $(\mathcal{M}, \boldsymbol{g}_{\mathcal{M}}, \mathfrak{o}_{\mathcal{M}}, \mathfrak{t}_{\mathcal{M}})$ *to* $(\mathcal{N}, \boldsymbol{g}_{\mathcal{N}}, \mathfrak{o}_{\mathcal{N}}, \mathfrak{t}_{\mathcal{N}})$ *is a smooth embedding* $\psi : \mathcal{M} \to \mathcal{N}$ *that is isometric (i.e.* $\psi^* g_{\mathcal{N}} = g_{\mathcal{M}}$*), and orientation- and time-orientation-preserving (i.e.* $\psi^*\mathfrak{o}_{\mathcal{N}} = \mathfrak{o}_{\mathcal{M}}$ *and* $\psi^*\mathfrak{t}_{\mathcal{N}} = \mathfrak{t}_{\mathcal{M}}$*). It must also respect the causal structure: the image* $\psi(\mathcal{M}) \subset \mathcal{N}$ *must be* causally convex *in* $\mathcal{N}$*, i.e. any finite causal curve in* $\mathcal{N}$ *with both endpoints in* $\psi(\mathcal{M})$ *must be entirely contained in* $\psi(\mathcal{M})$*. This has the effect that no pair of points in* $\psi(\mathcal{M})$ *can be causally related unless their inverse images in* $\mathcal{M}$ *are causally related.*

An example of a map which does not respect the causal structure in this way is given in figure 1.[1]

Definition 1.1.5. *Let* $\boldsymbol{M}, \boldsymbol{N}$ *be objects of* Loc*. An arrow* $\psi : \boldsymbol{M} \to \boldsymbol{N}$ *is a* Cauchy arrow *if* $\psi(\boldsymbol{M})$ *contains a Cauchy surface for* $\boldsymbol{N}$*.*

Now, suppose that $\psi : \boldsymbol{M} \to \boldsymbol{N}$ is an arrow in Loc, and that $\Sigma_{\boldsymbol{M}}$ is an arbitrary Cauchy surface of $\boldsymbol{M}$. We might imagine that the image $\psi(\Sigma_{\boldsymbol{M}})$ can always be extended to a Cauchy surface of $\boldsymbol{N}$; however, this is not the case, as we can see in the following example.

Example 1.1.6. Let $\boldsymbol{N} = \mathbb{R}^{1,1}$ be 2-dimensional Minkowski space, and let $\boldsymbol{M}$ be the open wedge $J^-_{\boldsymbol{N}}(\mathbf{0})$, as illustrated in figure 2. Then the canonical embedding $\iota : \boldsymbol{M} \to \boldsymbol{N}$ is an arrow in Loc,

[1]This 'helical strip' example comes from Kay ([40]), and was originally suggested by Stephen Hawking.

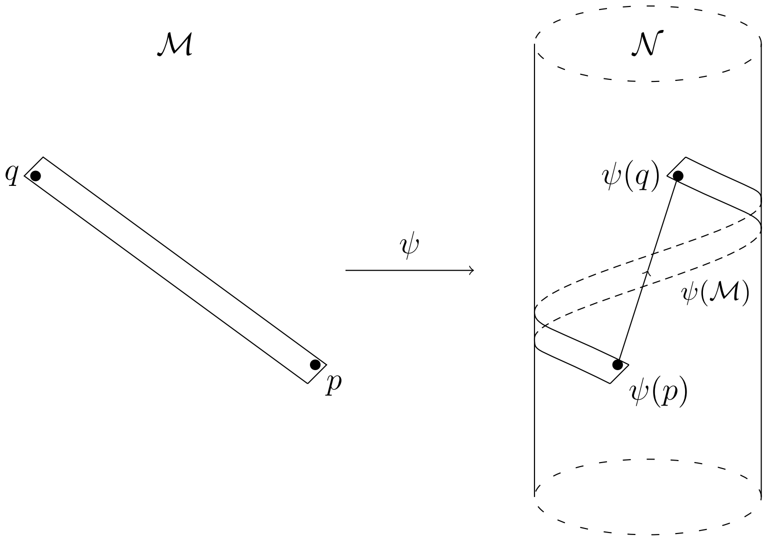


Figure 1: An example of an isometric embedding $\psi : \mathcal{M} \to \mathcal{N}$ that does not respect the causal structure: p and q are spacelike separated in $\mathcal{M}$, but $\psi(p)$ and $\psi(q)$ are connected by a causal curve in $\mathcal{N}$.

and the half-hyperbola $\Sigma_{\boldsymbol{M}}$ defined by $x^a x_a = -1$, $x^0 < 0$ is a Cauchy surface for $\boldsymbol{M}$. However, $\Sigma_{\boldsymbol{M}}$ cannot be extended to a Cauchy surface for $\boldsymbol{N}$.

While Loc is the simplest and most general category of spacetimes that we will use, it is necessary for some constructions to use different categories whose objects are subject to additional conditions or are defined in more detail.

Definition 1.1.7.

(a). *For $d \in \mathbb{N} \setminus \{1\}$, the category Loc_d is the full subcategory of Loc whose objects are spacetimes of dimension d.*

(b). *The category $\mathsf{Loc}^c_{(d)}$ is the full subcategory of $\mathsf{Loc}_{(d)}$ whose objects are connected.*

(c). *The category $\mathsf{Loc}^{sc}_{(d)}$ is the full subcategory of $\mathsf{Loc}_{(d)}$ whose objects are simply connected (but not necessarily connected).*

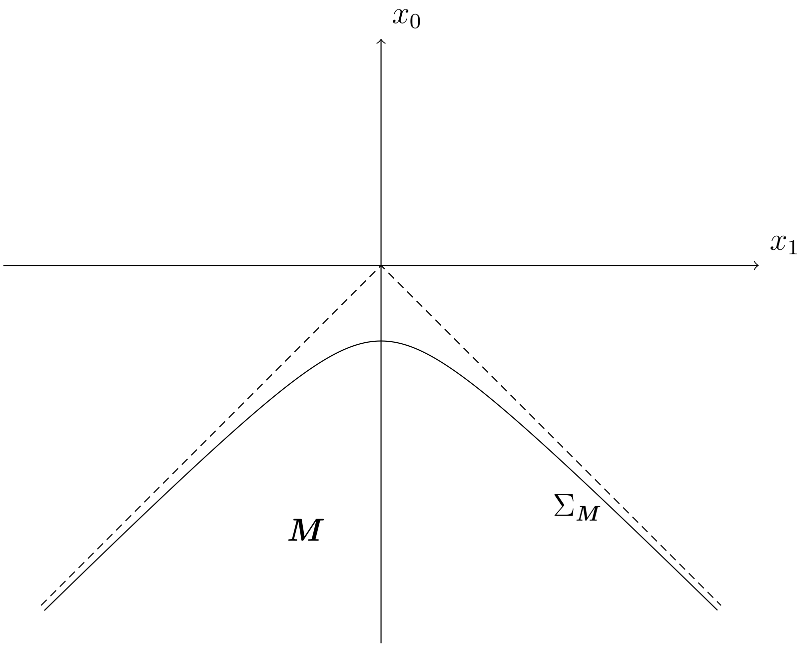


Figure 2: Example of Loc-embedding $\iota : \boldsymbol{M} \to \boldsymbol{N}$ and Cauchy surface $\Sigma_{\boldsymbol{M}}$ that cannot be extended to a Cauchy surface of $\boldsymbol{N}$

A number of other categories of spacetimes based on these will be required for the construction of the theory of the Dirac field; these will be defined in due course.

1.1.2 Additional categories

The categorical nature of the constructions contained within this thesis will require the definition of a number of different categories. The full definitions of most of these categories are best given in context, and so we will not give an exhaustive list of categories and their definitions here. For the purpose of reference only, there is a table of categories with short definitions given in Appendix E. We do however give the definitions of two categories here that are used throughout this thesis.

Definition 1.1.8. *We denote by* $\mathsf{Vect}_{\mathbb{C}}$ *the category whose objects are vector spaces over* $\mathbb{C}$ *and whose arrows are injective* $\mathbb{C}$*-linear maps. The equalizer of two arrows* $f, g : V \to W$ *is given by the inclusion of* $\ker(f - g)$ *in* V*. The* *categorical intersection and union of two subobjects are given by the subobjects corresponding respectively to the usual intersection and linear span of the associated subspaces.*

Let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be an arrow in Loc. If $f \in C_0^\infty(\boldsymbol{M})$ then we define the push-forward $\psi_* : C_0^\infty(\boldsymbol{M}) \to C_0^\infty(\boldsymbol{N})$ by

$$(\psi_* f)(x) := \begin{cases} f(\psi^{-1}(x)), & x \in \psi(\boldsymbol{M}), \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

We therefore have $\psi^* \circ \iota_{\boldsymbol{N}} \circ \psi_* = \iota_{\boldsymbol{M}}$ where $\psi^* : C^\infty(\boldsymbol{N}) \to C^\infty(\boldsymbol{M})$ is the pullback map and $\iota_{\boldsymbol{M}} : C_0^\infty(\boldsymbol{M}) \hookrightarrow C^\infty(\boldsymbol{M})$, $\iota_{\boldsymbol{N}} : C_0^\infty(\boldsymbol{N}) \hookrightarrow C^\infty(\boldsymbol{N})$ are the canonical embeddings. We may also view C_0^∞ as a covariant functor from Loc to $\mathsf{Vect}_{\mathbb{C}}$, and C^∞ as a contravariant functor from Loc to $\mathsf{Vect}_{\mathbb{C}}$, if we define $C_0^\infty(\psi) := \psi_*$ and $C^\infty(\psi) := \psi^*$.

Definition 1.1.9. *The category* Alg *has as its objects all unital $*$-algebras; that is to say, algebras over $\mathbb{C}$ with a unit element*[2] $\mathbf{1}$ *satisfying* $\mathbf{1}A = A = A\mathbf{1}$ *for all elements A, and an antilinear involution * satisfying $(AB)^* = B^*A^*$ for all elements A, B. An arrow in* Alg *is an injective $*$-homomorphism that preserves the unit.*

Definition 1.1.10. *An object $\mathfrak{A} \in \mathsf{Alg}$ is* weakly graded *if it may be presented in the form $\mathfrak{A} = \bigoplus_{i=0}^\infty \mathfrak{A}_i$, such that $A_iA_j \in \bigoplus_{0\le 2k\le i+j} \mathfrak{A}_{i+j-2k}$ for all $A_i \in \mathfrak{A}_i$ and $A_j \in \mathfrak{A}_j$. If $A \in \bigoplus_{k=0}^\infty \mathfrak{A}_{2k}$ then A is an* even *element, and if $A \in \bigoplus_{k=0}^\infty \mathfrak{A}_{2k+1}$ then A is an* odd *element.*

This is a generalization of the usual concept of a graded algebra, in which the rule for products is that $A_iA_j \in \mathfrak{A}_{i+j}$ for $A_i \in \mathfrak{A}_i$ and $A_j \in \mathfrak{A}_j$. Note that in a graded unital algebra, the unit must be an element of $\mathfrak{A}_0$, and in fact we often have $\mathfrak{A}_0 = \mathbb{C}\mathbf{1}$. This will also generally be the case for our weakly graded algebras. An example of a graded algebra is the tensor algebra $\mathscr{T}(V) = \bigoplus_{n=0}^\infty V^{\otimes n}$ of a complex vector space V, where the product of two elements is given by their tensor product.

[2] We insist that a unit element should be different from zero; this prevents the trivial single-element algebra $\{0\}$ from being classed as an object of Alg.

Note that since the product of two even elements in a weakly graded algebra $\mathfrak{A}$ is itself even, it follows that the even elements form a proper subalgebra of $\mathfrak{A}$.

Chapter 2

Locally covariant theories and dynamical locality

In this chapter, we will review in detail the concept of local covariance, for the most part following [11] and [25]. We discuss the ways in which fundamental solutions, fields and states may be translated into a locally covariant setting, and give definitions of the timeslice axiom, relative Cauchy evolution and the SPASs condition before examining dynamical locality in detail, again following [25].

2.1 The generally covariant locality principle

2.1.1 The Haag-Kastler axioms

Algebraic quantum field theory has its roots in the principle of locality, embodied by the tenet that some observables of a theory (on a particular spacetime) may be considered to reside in a certain subregion of that spacetime, and that observables in regions which are causally disjoint are compatible [31]. In the cited paper Haag and Kastler consider only Minkowski space as the background spacetime, but the statement is meaningful as applied to any globally hyperbolic background.

Haag and Kastler contend that the content of a quantum field theory on Minkowski space is entirely determined by an assignment $\mathscr{A} : \mathrm{Op}(\mathbb{R}^{1,3}) \to$

Obj(Alg), where Op($\boldsymbol{M}$) is the set of open subsets of a spacetime $\boldsymbol{M}$ with a compact closure. This assignment is defined to be *causal* if

$$[\mathscr{A}(O), \mathscr{A}(O')] = \{0\} \tag{2.1}$$

whenever $O, O' \subset \mathbb{R}^{1,3}$ are causally disjoint.[1] For $\mathscr{A}$ to describe a quantum field theory, it is also necessary that it satisfy further conditions:

- $\mathscr{A}(O) \subset \mathscr{A}(O')$ whenever $O \subset O'$ (*isotony*),
- The union $\mathscr{A} := \bigcup_{O \in \mathrm{Op}(\mathbb{R}^{1,3})} \mathscr{A}(O)$ exists, and (possibly after completion) gives the *quasilocal algebra* containing all observables of interest,
- The proper orthochronous Poincaré group $\mathcal{P}(1,3)$ generates a group of automorphisms, denoted $\cdot_\Lambda : \mathscr{A} \to \mathscr{A}$, such that $\mathscr{A}(O)_\Lambda = \mathscr{A}(\Lambda O)$ for all $\Lambda \in \mathcal{P}(1,3)$, $O \in \mathrm{Op}(\mathbb{R}^{1,3})$ (*Lorentz covariance*).

There are some additional technical assumptions that we do not mention here.

Note that using this model entails that the content of a theory is independent of any particular interpretation given to the observables. In fact, two theories $\mathscr{A}$ and $\mathscr{A}'$ for which there exists an isomorphism of the nets $O \mapsto \mathscr{A}(O)$ and $O \mapsto \mathscr{A}'(O)$ (where $O \in \mathrm{Op}(\mathbb{R}^{1,3})$) are indistinguishable under this framework. The Haag-Kastler axioms therefore represent a more abstract scheme for the construction of quantum field theories, and indeed the phrase 'field theory' could be taken to be somewhat misleading in this framework, since the quantum fields are not the fundamental objects of the theory, and are not even necessary for the construction.

We may in fact extend this idea to *classical* field theories, in order to foreshadow the ideas developed by Fewster and Verch in [25, 26]: a classical field theory on $\mathbb{R}^{1,3}$ is defined, at the most basic level, by a space of solutions to some Poincaré-invariant field equation. Although nonlinear field theories can certainly be constructed in this framework, here we consider almost

[1] For any algebra $\mathfrak{A}$, the commutator of subalgebras $\mathfrak{B}, \mathfrak{B}'$ is the subalgebra $[\mathfrak{B}, \mathfrak{B}'] := \{[B, B'] : B \in \mathfrak{B},\ B' \in \mathfrak{B}'\}$.

exclusively *linear* classical theories, in which the space of solutions is a vector space, and which often admit a bilinear or sesquilinear form with good physical properties.[2] We may assign a solution space $\mathscr{L}(O)$ (in the classical case, usually the vector space of solutions along with additional structure, including the bi-/sesquilinear form) to each $O \in \mathrm{Op}(\mathbb{R}^{1,3})$, and in parallel to the quantum case, require that:

- $\mathscr{L}(O) \subset \mathscr{L}(O')$ whenever $O \subset O'$ (*isotony*),
- The union $\mathscr{L} := \bigcup_{O \in \mathrm{Op}(\mathbb{R}^{1,3})} \mathscr{L}(O)$ exists, and (possibly after completion) gives the *quasilocal solution space* containing all solutions of interest,
- The proper orthochronous Poincaré group $\mathcal{P}(1,3)$ generates a group of automorphisms, denoted $\cdot_\Lambda : \mathscr{L} \to \mathscr{L}$, such that $\mathscr{L}(O)_\Lambda = \mathscr{L}(\Lambda O)$ for all $\Lambda \in \mathcal{P}(1,3)$, $O \in \mathrm{Op}(\mathbb{R}^{1,3})$ (*Lorentz covariance*).

Denoting the aforementioned bilinear or sesquilinear form by $b : \mathscr{L} \times \mathscr{L} \to \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), we define a linear classical theory to be *causal* if

$$b(\mathscr{L}(O), \mathscr{L}(O')) = \{0\} \tag{2.2}$$

whenever O, O' are causally disjoint.

The principle of local covariance described in [11] is an extension of these ideas to curved spacetimes, using the tools of category theory.

2.1.2 Locally covariant theories

In the spirit of the Haag-Kastler axioms, Brunetti, Fredenhagen and Verch consider a quantum field theory to be an assignment of an algebra of observables to each of a suitable class of globally hyperbolic spacetimes. This is done by defining a covariant functor from a category of spacetimes denoted

[2]For example, solutions to the field equation typically solve the Cauchy problem for an arbitrary Cauchy surface in $\mathbb{R}^{1,3}$, and we find that the bi-/sesquilinear form of two solutions may then be expressed in terms of their Cauchy data on an arbitrary Cauchy surface.

Sp (typically Loc, or a subcategory thereof) to a suitable target category; in [11] only the category C*-Alg of C^*-algebras is considered, but following [25] we will consider a much wider range of categories, requiring that their objects may represent some physical system, and that their arrows correspond to embeddings of these systems. We therefore let Phys be such a category of physical systems, and require that it possess an initial object $\mathcal{I}$, that all its arrows are monic and that it has categorical equalizers, intersections and unions.

Definition 2.1.1. *A* locally covariant theory *(LCT) is a covariant functor from a category* Sp *of spacetimes to a category* Phys *of physical systems.*

A quantum theory will typically be denoted $\mathscr{A}$, in which case Phys will generally be some category of algebras. A classical theory, on the other hand, will generally be denoted $\mathscr{L}$; in the linear case, which we consider almost exclusively, Phys will then be some category of vector spaces admitting bi-/sesquilinear forms. Definitions of causality are entirely dependent on the theory in question; for a linear classical theory $\mathscr{L}$ with bi-/sesquilinear form $b_{\boldsymbol{N}}$ on the spacetime $\boldsymbol{N}$, we typically require that

$$b_{\boldsymbol{N}}\Big(\mathscr{L}(\iota_{\boldsymbol{M}_1,\boldsymbol{N}})(\mathscr{L}(\boldsymbol{M}_1)), \mathscr{L}(\iota_{\boldsymbol{M}_2,\boldsymbol{N}})(\mathscr{L}(\boldsymbol{M}_2))\Big) = \{0\} \tag{2.3}$$

whenever the two spacetimes $\boldsymbol{M}_1, \boldsymbol{M}_2$ may be embedded into a spacetime $\boldsymbol{N}$ via the arrows $\iota_{\boldsymbol{M}_i,\boldsymbol{N}}$, $i = 1, 2$, such that $\iota_{\boldsymbol{M}_1,\boldsymbol{N}}(\boldsymbol{M}_1)$ and $\iota_{\boldsymbol{M}_2,\boldsymbol{N}}(\boldsymbol{M}_2)$ are causally disjoint. For a quantum theory $\mathscr{A}$ we may use the analogue to (2.2) and say that $\mathscr{A}$ is causal if

$$[\mathscr{A}(\iota_{\boldsymbol{M}_1,\boldsymbol{N}})(\mathscr{A}(\boldsymbol{M}_1)), \mathscr{A}(\iota_{\boldsymbol{M}_2,\boldsymbol{N}})(\mathscr{A}(\boldsymbol{M}_2))] = \{0\} \tag{2.4}$$

when $\boldsymbol{M}_1, \boldsymbol{M}_2$ are defined as above. This relation is certainly expected when $\mathscr{A}$ represents a bosonic field theory; in the case of a fermionic theory, however, we find that on generators, the commutator in (2.4) is usually replaced by the anticommutator. In a theory that contains both Bose and Fermi fields, the algebras assigned to given spacetimes have a $\mathbb{Z}_2$ grading, and the causality property that gives the correct (anti)commutation relations is described using *twisted locality* (see e.g. [22]).

The anticommutation at spacelike separation for Fermi fields is not particularly physical in nature, since the vanishing of the commutator relates to the lack of a canonical time-ordering of spacelike-separated events. Therefore, even in the fermionic case, we should regard an algebra that does not obey (2.4) to be fundamentally unphysical. However, we will see that physicality may be recovered by considering the subtheory of a fermionic theory whose algebras are generated by pairs of generators of the full algebra.

Clearly any LCT defined as a functor from Sp to Phys may also be regarded as a functor from Sp' to Phys, whenever Sp' is a subcategory of Sp. Hereafter in this section we take $\mathscr{A}$ to be a quantum LCT, with the objects of Phys being algebras; all subsequent definitions may also be applied in an obvious way to arbitrary LCTs.

Definition 2.1.2. *For any spacetime $\boldsymbol{M}$ in Sp, the notation $\mathscr{O}(\boldsymbol{M})$ denotes as in [25] the set of all nonempty globally hyperbolic subregions of $\boldsymbol{M}$ with finitely many connected components, all of which are causally disjoint, and such that for any $O \in \mathscr{O}(\boldsymbol{M})$, the restriction $\boldsymbol{M}|_O$ is an object of Sp in its own right.*[3] *For any $O \in \mathscr{O}(\boldsymbol{M})$, we denote by $\iota_{\boldsymbol{M};O}$ the canonical embedding of $\boldsymbol{M}|_O$ in $\boldsymbol{M}$. This immediately gives us a candidate for the local algebra of a region of $\boldsymbol{M}$; again following [25], for any $O \in \mathscr{O}(\boldsymbol{M})$, we define the* kinematic algebra *of O in $\boldsymbol{M}$ to be the algebra $\mathscr{A}^{\rm kin}(\boldsymbol{M};O) := \mathscr{A}(\boldsymbol{M}|_O)$. For each $O \in \mathscr{O}(\boldsymbol{M})$ we may also define the map $\alpha^{\rm kin}_{\boldsymbol{M};O} := \mathscr{A}(\iota_{\boldsymbol{M};O})$. The assignment $O \mapsto \alpha^{\rm kin}_{\boldsymbol{M};O}$ is called the* kinematic net. *For convenience, we use the notation*

$$\hat{\mathscr{A}}^{\rm kin}(\boldsymbol{M};O) := \alpha^{\rm kin}_{\boldsymbol{M};O}(\mathscr{A}^{\rm kin}(\boldsymbol{M};O)), \tag{2.5}$$

as we will frequently wish to refer to the image of the kinematic algebra in $\mathscr{A}(\boldsymbol{M})$.

We may now examine the definition of a locally covariant theory to see whether the conditions of isotony and Lorentz covariance, and the definition of the quasilocal algebra/solution space survive intact when we use the kinematic

[3] The latter condition is satisfied automatically when $\mathsf{Sp} = \mathsf{Loc}$.

definition of locality given above. This discussion closely follows [11, §2.4] and [25, §3.3].

The analogue to isotony comes immediately from the fact that the arrows in Phys are monic; given an LCT $\mathscr{A}$, and any $O \in \mathscr{O}(\boldsymbol{M})$, the arrow $\alpha^{\mathrm{kin}}_{\boldsymbol{M};O} : \mathscr{A}^{\mathrm{kin}}(\boldsymbol{M};O) \to \mathscr{A}(\boldsymbol{M})$ is necessarily an embedding. If $O, O' \in \mathscr{O}(\boldsymbol{M})$ with $O \subset O'$, then we also have $O \in \mathscr{O}(\boldsymbol{M}|_{O'})$, and $\iota_{\boldsymbol{M};O} = \iota_{\boldsymbol{M};O'} \circ \iota_{\boldsymbol{M}|_{O'};O}$. Therefore $\hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M};O) = \alpha^{\mathrm{kin}}_{\boldsymbol{M};O'}(\hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M}|_{O'};O)) \subset \hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M};O')$, and so isotony is satisfied automatically by the conditions imposed on the categories used in the construction.

While the condition of Lorentz covariance relies explicitly on the Poincaré symmetries of Minkowski space, and thus cannot transfer directly into a more general framework where no such symmetry is guaranteed, we may nevertheless note the following covariance property of the kinematic net proved in [25, §3.3]. Let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be an arrow in Sp, and consider a nonempty $O \in \mathscr{O}(\boldsymbol{M})$. We then have $\alpha^{\mathrm{kin}}_{\boldsymbol{N};\psi(O)} \circ \mathscr{A}(\widetilde{\psi}_O) = \mathscr{A}(\psi) \circ \alpha^{\mathrm{kin}}_{\boldsymbol{M};O}$, where $\widetilde{\psi}_O : \boldsymbol{M}|_O \to \boldsymbol{N}|_{\psi(O)}$ is the diffeomorphism induced by ψ. Since $\mathscr{A}(\widetilde{\psi}_O)$ is therefore an isomorphism, it follows immediately that $\hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{N};\psi(O)) = \mathscr{A}(\psi)(\hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M};O))$. We may consider the special case where $\boldsymbol{N} = \boldsymbol{M}$ and ψ is an automorphism to note that the group of automorphisms in $\mathsf{Sp}(\boldsymbol{M},\boldsymbol{M})$ generates a subgroup of the automorphism group of $\mathscr{A}(\boldsymbol{M})$ whose elements satisfy the relation

$$\mathscr{A}(\psi)(\hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M};O)) = \hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M};\psi(O)). \tag{2.6}$$

In the case where $\boldsymbol{M} = \mathbb{R}^{1,3}$ and $\psi \in \mathcal{P}(1,3)$, we therefore recover the condition of Lorentz covariance.

There is no direct analogue of the Haag-Kastler quasilocal algebra in the LCT framework, since there is generally no final object of the category Sp. Instead, we give the following definition:

Definition 2.1.3. *Let $\mathscr{A}$ be an LCT and $\boldsymbol{M}$ an object in Sp. The* quasilocal algebra *generated by $\mathscr{A}$ on $\boldsymbol{M}$ is given by the categorical union*

$$\mathscr{A}^{\mathrm{q.l.}}(\boldsymbol{M}) := \bigvee_{\substack{O \in \mathscr{O}(\boldsymbol{M}) \\ \mathrm{cl}(O)\ \mathrm{compact}}} \hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M};O), \tag{2.7}$$

where $\mathrm{cl}(O)$ *denotes the closure of* O *in* $\boldsymbol{M}$*; in other words, the union is taken over all relatively compact elements of* $\mathscr{O}(\boldsymbol{M})$*. There is generally no guarantee that* $\mathscr{A}^{\mathrm{q.l.}}(\boldsymbol{M}) = \mathscr{A}(\boldsymbol{M})$*. However, we expect this to hold for many theories of interest, and in this case* $\mathscr{A}$ *is called an* additive *theory.*

2.1.3 The timeslice axiom and relative Cauchy evolution

We have not yet addressed the question of under what circumstances an LCT may be said to represent a physical system. The purpose of a physical theory is to make predictions, and therefore at the very least we should expect it to obey some dynamical rule. In terms of the covariant locality principle, we demand that the content of such a theory may be reconstructed from the algebras generated by neighbourhoods of Cauchy surfaces, as in [11]; we may put this more precisely as follows:

Definition 2.1.4. *An LCT* $\mathscr{A}$ *obeys the* timeslice axiom *if every Cauchy arrow* $\psi : \boldsymbol{M} \to \boldsymbol{N}$ *in* Sp *is mapped to an isomorphism* $\mathscr{A}(\psi) : \mathscr{A}(\boldsymbol{M}) \to \mathscr{A}(\boldsymbol{N})$ *(recall that a Cauchy arrow is a map whose image contains a Cauchy surface for the target spacetime).*[4]

Given an LCT which obeys the timeslice axiom, we are interested in what happens to the algebra of a spacetime $\boldsymbol{M}$ when we perturb the metric in a compact region. Again, these next definitions follow [11, 25]; we adopt the notation used in the latter.

Definition 2.1.5. *Let* $\boldsymbol{M} = (\mathcal{M}, \boldsymbol{g}, \mathfrak{o}, \mathfrak{t})$ *be an object of* Loc*. A* metric perturbation *of* $\boldsymbol{M}$ *is a compactly supported symmetric tensor field* $\boldsymbol{h} \in C_0^\infty(T_2^0\mathcal{M})$ *with the property that* $\boldsymbol{g} + \boldsymbol{h}$ *is a time-orientable Lorentz metric on* $\mathcal{M}$*. There is a unique choice of time orientation* $\mathfrak{t}_{\boldsymbol{h}}$ *that agrees with* $\mathfrak{t}$ *outside* $\operatorname{supp}\boldsymbol{h}$*; we denote* $\boldsymbol{M}[\boldsymbol{h}] := (\mathcal{M}, \boldsymbol{g} + \boldsymbol{h}, \mathfrak{o}, \mathfrak{t}_{\boldsymbol{h}})$*. The set of all metric perturbations of* $\boldsymbol{M}$ *is denoted* $H(\boldsymbol{M})$*, and for any* $O \subset \mathscr{O}(\boldsymbol{M})$*, the subset of* $H(\boldsymbol{M})$ *comprising perturbations with support in* O *is denoted* $H(\boldsymbol{M}; O)$*.*

[4]As remarked in [25], the definition in [11] only requires surjectivity of $\mathscr{A}(\psi)$; however, since arrows are monic, this definition is equivalent.

Definition 2.1.6. *Let $\mathscr{A}$ be an LCT, $\boldsymbol{M}$ be an object in* Loc*, and $\boldsymbol{h} \in H(\boldsymbol{M})$. The* relative Cauchy evolution *(r.c.e.) of $\mathscr{A}(\boldsymbol{M})$ generated by $\boldsymbol{h}$ is an automorphism $\mathrm{rce}^{(\mathscr{A})}_{\boldsymbol{M}}[\boldsymbol{h}] : \mathscr{A}(\boldsymbol{M}) \to \mathscr{A}(\boldsymbol{M})$ defined as follows: let $\mathcal{N}^{\pm} \in \mathscr{O}(\boldsymbol{M})$ be subregions of $\boldsymbol{M}$ with the properties that each contains a Cauchy surface for $\boldsymbol{M}$, and that $\mathcal{N}^{+}$ lies strictly to the future, and $\mathcal{N}^{-}$ to the past, of* $\operatorname{supp}\boldsymbol{h}$*. We find that $\mathcal{N}^{\pm} \in \mathscr{O}(\boldsymbol{M}[\boldsymbol{h}])$; since $\boldsymbol{M}|_{\mathcal{N}^{\pm}} = \boldsymbol{M}[\boldsymbol{h}]|_{\mathcal{N}^{\pm}}$, we may unambiguously write $\boldsymbol{M}|_{\mathcal{N}^{\pm}} = \boldsymbol{N}^{\pm} = \boldsymbol{M}[\boldsymbol{h}]|_{\mathcal{N}^{\pm}}$. We denote the canonical embeddings by $\iota^{\pm} : \boldsymbol{N}^{\pm} \to \boldsymbol{M}$ and $\iota^{\pm}[\boldsymbol{h}] : \boldsymbol{N}^{\pm} \to \boldsymbol{M}[\boldsymbol{h}]$. Since these are Cauchy morphisms, the* Phys*-arrows $\alpha^{\pm} := \mathscr{A}(\iota^{\pm})$ and $\alpha^{\pm}[\boldsymbol{h}] := \mathscr{A}(\iota^{\pm}[\boldsymbol{h}])$ are isomorphisms. We define*

$$\mathrm{rce}^{(\mathscr{A})}_{\boldsymbol{M}}[\boldsymbol{h}] := \alpha^{-} \circ (\alpha^{-}[\boldsymbol{h}])^{-1} \circ \alpha^{+}[\boldsymbol{h}] \circ (\alpha^{+})^{-1}. \tag{2.8}$$

A pictorial representation of the spacetime objects and arrows involved is given in figure 3, along with the corresponding objects and arrows in Alg.

The explicit labelling of the r.c.e. with the theory to which it applies will be dropped for the sake of clarity, unless needed to avoid ambiguity. We also denote by $\mathbf{S}(\boldsymbol{M};\boldsymbol{h})$ the set of all pairs $(\mathcal{N}^{+},\mathcal{N}^{-})$ satisfying the properties given above.

The relative Cauchy evolutions may be regarded as a family of transformations that measures the reaction of an LCT to a perturbation of the background metric. We first discuss some general properties of the r.c.e., starting with the following lemma:

Lemma 2.1.7. *Let $\mathscr{A}$ be an LCT and $\boldsymbol{M}$ be a spacetime in* Sp*, and consider some $\boldsymbol{h} \in H(\boldsymbol{M})$. The definition of the r.c.e. given in* (2.8) *is independent of the choice of subregions $(\mathcal{N}^{+},\mathcal{N}^{-}) \in \mathbf{S}(\boldsymbol{M};\boldsymbol{h})$.*

This lemma may be proved using for example [25, Prop. 3.3]; we include an explicit proof here for completeness.

Proof. Let $\mathcal{N}_1^{\pm},\mathcal{N}_2^{\pm} \in \mathbf{S}(\boldsymbol{M};\boldsymbol{h})$ for $i = 1, 2$. We define $\boldsymbol{N}_i^{\pm}$, $\iota_i^{\pm}$, $\iota_i^{\pm}[\boldsymbol{h}]$, $\alpha_i^{\pm}$ and $\alpha_i^{\pm}[\boldsymbol{h}]$ as in Definition 2.1.6. We wish to show that

$$\alpha_1^{-} \circ (\alpha_1^{-}[\boldsymbol{h}])^{-1} \circ \alpha_1^{+}[\boldsymbol{h}] \circ (\alpha_1^{+})^{-1} = \alpha_2^{-} \circ (\alpha_2^{-}[\boldsymbol{h}])^{-1} \circ \alpha_2^{+}[\boldsymbol{h}] \circ (\alpha_2^{+})^{-1}. \tag{2.9}$$

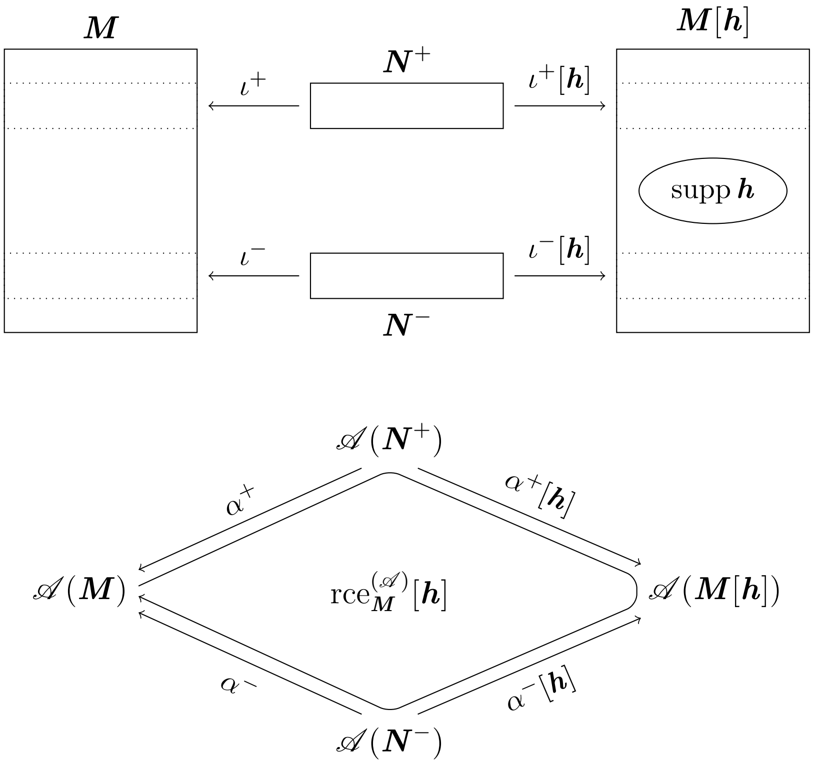


Figure 3: Diagrams of the objects and arrows involved in the definition of the relative Cauchy evolution

To do this we define $\mathcal{N}_0^\pm \in \mathbf{S}(\boldsymbol{M}; h)$, with the additional property that $\mathcal{N}_i^\pm \subset \mathcal{N}_0^\pm$, $i = 1, 2$. Writing $\boldsymbol{N}_0^\pm := \boldsymbol{M}|_{\mathcal{N}_0^\pm} = \boldsymbol{M}[\boldsymbol{h}]|_{\mathcal{N}_0^\pm}$ and the canonical embeddings as $\iota_0^\pm : \boldsymbol{N}_0^\pm \to \boldsymbol{M}$, $\iota_0^\pm[\boldsymbol{h}] : \boldsymbol{N}_0^\pm \to \boldsymbol{M}[\boldsymbol{h}]$ and $\iota_{i,0}^\pm : \boldsymbol{N}_i^\pm \to \boldsymbol{N}_0^\pm$, we have

$$\iota_i^\pm = \iota_0^\pm \circ \iota_{i,0}^\pm, \qquad \iota_i^\pm[\boldsymbol{h}] = \iota_0^\pm[\boldsymbol{h}] \circ \iota_{i,0}^\pm, \tag{2.10}$$

for $i = 1, 2$. Note that these are all still Cauchy morphisms, so we may obtain isomorphisms $\alpha_0^\pm$, $\alpha_0^\pm[\boldsymbol{h}]$ and $\alpha_{i,0}^\pm$ by applying the functor $\mathscr{A}$ as above. It follows from (2.10) that $\alpha_i^\pm \circ (\alpha_i^\pm[\boldsymbol{h}])^{-1} = \alpha_0^\pm \circ (\alpha_0^\pm[\boldsymbol{h}])^{-1}$, and so

$$\alpha_i^- \circ (\alpha_i^-[\boldsymbol{h}])^{-1} \circ \alpha_i^+[\boldsymbol{h}] \circ (\alpha_i^+)^{-1} = \alpha_0^- \circ (\alpha_0^-[\boldsymbol{h}]) \circ \alpha_0^+[\boldsymbol{h}] \circ (\alpha_0^+)^{-1} \tag{2.11}$$

for $i = 1, 2$. Therefore (2.9) holds. □

Alternatively, as shown in [25], it is possible to define the regions $\mathcal{N}^{\pm}$ canonically in terms of the support of $\boldsymbol{h}$.

We may use this result to prove the following factorization property of the relative Cauchy evolution, which is an original result:

Proposition 2.1.8. *Let $\mathscr{A}$ be an LCT and $\boldsymbol{M}$ be a spacetime in Sp. Suppose that $\boldsymbol{h}_1, \boldsymbol{h}_2 \in H(\boldsymbol{M})$ are separated by a region $\mathcal{N} \in \mathscr{O}(\boldsymbol{M})$ such that $\mathcal{N}$ contains a Cauchy surface for $\boldsymbol{M}$, and $\operatorname{supp} \boldsymbol{h}_1$ lies strictly to the future of $\mathcal{N}$, with $\operatorname{supp} \boldsymbol{h}_2$ to the past. Then*

$$\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}_1 + \boldsymbol{h}_2] = \mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}_2] \circ \mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}_1]. \tag{2.12}$$

Proof. Let $\boldsymbol{N}^0 := \boldsymbol{M}|_{\mathcal{N}}$, with embedding $\iota^0 : \boldsymbol{N}^0 \to \boldsymbol{M}$, and define sub-spacetimes $\boldsymbol{N}^{\pm} := \boldsymbol{M}|_{\mathcal{N}^{\pm}}$ with embeddings $\iota^{\pm} : \boldsymbol{N}^{\pm} \to \boldsymbol{M}$, where $\mathcal{N}^{\pm} \in \mathbf{S}(\boldsymbol{M}; \boldsymbol{h}_1 + \boldsymbol{h}_2)$. Denoting $\alpha^{\pm/0} := \mathscr{A}(\iota^{\pm/0})$, and as before denoting arrows into (algebras of) perturbed spacetimes $\boldsymbol{M}[\boldsymbol{h}]$ by affixing $[\boldsymbol{h}]$, we have

$$\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}_1 + \boldsymbol{h}_2] = \alpha^- \circ (\alpha^-[\boldsymbol{h}_1 + \boldsymbol{h}_2])^{-1} \circ \alpha^+[\boldsymbol{h}_1 + \boldsymbol{h}_2] \circ (\alpha^+)^{-1}. \tag{2.13}$$

Note that we may consider $\boldsymbol{M}[\boldsymbol{h}_2]$ as the background spacetime in its own right, and form the relative Cauchy evolution with respect to the perturbation $\boldsymbol{h}_1$; we have

$$\mathrm{rce}_{\boldsymbol{M}[\boldsymbol{h}_2]}[\boldsymbol{h}_1] = \alpha^-[\boldsymbol{h}_2] \circ (\alpha^-[\boldsymbol{h}_1 + \boldsymbol{h}_2]) \circ \alpha^+[\boldsymbol{h}_1 + \boldsymbol{h}_2] \circ (\alpha^+[\boldsymbol{h}_2])^{-1}. \tag{2.14}$$

But since the image of $\boldsymbol{N}^0$ also lies to the past of $\operatorname{supp} \boldsymbol{h}_1$, we have

$$\mathrm{rce}_{\boldsymbol{M}[\boldsymbol{h}_2]}[\boldsymbol{h}_1] = \alpha^0[\boldsymbol{h}_2] \circ (\alpha^0[\boldsymbol{h}_1 + \boldsymbol{h}_2]) \circ \alpha^+[\boldsymbol{h}_1 + \boldsymbol{h}_2] \circ (\alpha^+[\boldsymbol{h}_2])^{-1}; \tag{2.15}$$

these definitions are equivalent by Lemma 2.1.7, and consequently

$$\alpha^-[\boldsymbol{h}_2] \circ (\alpha^-[\boldsymbol{h}_1 + \boldsymbol{h}_2])^{-1} = \alpha^0[\boldsymbol{h}_2] \circ (\alpha^0[\boldsymbol{h}_1 + \boldsymbol{h}_2])^{-1}. \tag{2.16}$$

A similar calculation using $\mathrm{rce}_{\boldsymbol{M}[\boldsymbol{h}_1]}[\boldsymbol{h}_2]$ gives us

$$\alpha^+[\boldsymbol{h}_1 + \boldsymbol{h}_2] \circ (\alpha^+[\boldsymbol{h}_1])^{-1} = \alpha^0[\boldsymbol{h}_1 + \boldsymbol{h}_2] \circ (\alpha^0[\boldsymbol{h}_1])^{-1} \tag{2.17}$$

Substituting these into (2.13) yields

$$\begin{aligned}\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}_1 + \boldsymbol{h}_2] &= \alpha^- \circ (\alpha^-[\boldsymbol{h}_2])^{-1} \circ \alpha^0[\boldsymbol{h}_2] \circ (\alpha^0[\boldsymbol{h}_1])^{-1} \circ \alpha^+[\boldsymbol{h}_1] \circ (\alpha^+)^{-1} \\ &= \alpha^- \circ (\alpha^-[\boldsymbol{h}_2])^{-1} \circ \alpha^0[\boldsymbol{h}_2] \circ (\alpha^0)^{-1} \\ &\qquad\qquad \circ \alpha^0 \circ (\alpha^0[\boldsymbol{h}_1])^{-1} \circ \alpha^+[\boldsymbol{h}_1] \circ (\alpha^+)^{-1} \\ &= \mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}_2] \circ \mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}_1]. \end{aligned} \tag{2.18}$$

□

A simple corollary of this is the following:

Corollary 2.1.9. *Let $\mathscr{A}$ be an LCT and $\boldsymbol{M}$ a spacetime in* Loc. *Suppose that $\boldsymbol{h}_1, \boldsymbol{h}_2 \in H(\boldsymbol{M})$ with* $\operatorname{supp} \boldsymbol{h}_1$ *causally disjoint from* $\operatorname{supp} \boldsymbol{h}_2$. *Then*

$$\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}_1] \circ \mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}_2] = \mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}_1 + \boldsymbol{h}_2] = \mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}_2] \circ \mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}_1]. \tag{2.19}$$

Proof. If $\operatorname{supp} \boldsymbol{h}_1$ and $\operatorname{supp} \boldsymbol{h}_2$ are causally disjoint, then we can find a subregion $\mathcal{N} \in \mathscr{O}(\boldsymbol{M})$ containing a Cauchy surface for $\boldsymbol{M}$ such that $\operatorname{supp} \boldsymbol{h}_1$ lies to the future of $\mathcal{N}$ and $\operatorname{supp} \boldsymbol{h}_2$ to the past, giving the second equality by Proposition 2.1.8; conversely, we can find a subregion $\mathcal{N}' \in \mathscr{O}(\boldsymbol{M})$ containing a Cauchy surface for $\boldsymbol{M}$ such that $\operatorname{supp} \boldsymbol{h}_1$ lies to the past of $\mathcal{N}'$ and $\operatorname{supp} \boldsymbol{h}_2$ to the future, giving the first equality. □

As a consequence of this, we find that the relative Cauchy evolutions generated by spacelike separated perturbations commute.

We now examine a second notion of locality that derives from the dynamics of the theory (as encapsulated in the relative Cauchy evolution), following [25].

2.2 Dynamical locality and SPASs

2.2.1 The dynamical net

Recall that the definitions of the kinematic algebras and nets given above do not require the theory to obey the timeslice axiom. This suggests that the kinematic notion of locality cannot be directly related to the physics of the theory, if we consider the physics of a theory on a given spacetime to be embodied in the properties of its dynamical laws. An alternative definition of locality, which takes into account the dynamics of the theory, is given in [25], as follows.

Firstly, we note that for any compact subregion K of a spacetime $\boldsymbol{M}$, the set of all subobjects α of $\mathscr{A}(\boldsymbol{M})$ satisfying

$$\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}] \circ \alpha = \alpha \tag{2.20}$$

for all $\boldsymbol{h} \in H(\boldsymbol{M}; K^{\perp})$ has a maximal element in the subobject lattice, which we denote $\alpha^{\bullet}_{\boldsymbol{M};K}$ [25, Lemma 5.1]; here $K^{\perp}$ is the causal complement $\mathcal{M} \setminus J_{\boldsymbol{M}}(K)$. To define the dynamical net and algebra, we need the following definition, based on [12]:

Definition 2.2.1. *Consider a spacetime $\boldsymbol{M}$ in* Loc *of dimension n, containing a Cauchy surface Σ. A region $B \subset \Sigma$ is called a* Cauchy ball *if there exists a chart (U, ϕ) of Σ such that $\phi(B)$ is a nonempty open ball in $\mathbb{R}^{n-1}$ with closure contained in $\phi(U)$. We say that an open, relatively compact $O \subset \boldsymbol{M}$ is a* diamond *region in $\boldsymbol{M}$ if $O = D_{\boldsymbol{M}}(B)$ for some Cauchy ball B, where $D_{\boldsymbol{M}}(B)$ is the domain of determinacy of B in $\boldsymbol{M}$, i.e. the set of points $p \in \boldsymbol{M}$ for which any inextendible causal curve through p intersects B. The ball B is called the* base *of O.*

A finite collection of causally disjoint diamonds is called a multi-diamond. *We are interested in the set of compact subregions of $\boldsymbol{M}$ which have multi-diamond neighbourhoods; as in [25] we denote by $\mathscr{K}(\boldsymbol{M})$ the set of such subregions, and by $\mathscr{K}(\boldsymbol{M}; O)$ those elements of $\mathscr{K}(\boldsymbol{M})$ that are contained in an open region $O \subset \boldsymbol{M}$.*

We may now give the explicit definition of the dynamical net.

Definition 2.2.2. *For any LCT $\mathscr{A}$, spacetime $\boldsymbol{M}$ in* Sp *and $O \in \mathscr{O}(\boldsymbol{M})$, we define*

$$\alpha^{\mathrm{dyn}}_{\boldsymbol{M};O} := \bigvee_{K \in \mathscr{K}(\boldsymbol{M};O)} \alpha^{\bullet}_{\boldsymbol{M};K}. \tag{2.21}$$

(The simpler definition $\alpha^{\bullet}_{\boldsymbol{M};\mathrm{cl}(O)}$ is not used as this does not give desirable results when compared to the kinematic algebra [25]). The dynamical net *is then defined to be the assignment $O \mapsto \alpha^{\mathrm{dyn}}_{\boldsymbol{M};O}$.*

The domain of the arrow $\alpha^{\mathrm{dyn}}_{\boldsymbol{M};O}$ is unique (up to isomorphism) and is denoted $\mathscr{A}^{\mathrm{dyn}}(\boldsymbol{M};O)$; as before, we denote $\hat{\mathscr{A}}^{\mathrm{dyn}}(\boldsymbol{M};O) := \alpha^{\mathrm{dyn}}_{\boldsymbol{M};O}(\mathscr{A}^{\mathrm{dyn}}(\boldsymbol{M};O))$.

It is sometimes convenient to deal explicitly with the subalgebra of all elements of $\mathscr{A}(\boldsymbol{M})$ that are unchanged by relative Cauchy evolutions generated by any $\boldsymbol{h} \in H(\boldsymbol{M};K)$ for a given $K \in \mathscr{K}(\boldsymbol{M})$, so we define

$$\mathscr{A}^{\bullet}(\boldsymbol{M};K) := \{A \in \mathscr{A}(\boldsymbol{M}) : \mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}]A = A \text{ for all } \boldsymbol{h} \in H(\boldsymbol{M};K^{\perp})\}. \tag{2.22}$$

We usually find that $\mathscr{A}^{\bullet}(\boldsymbol{M};K)$ coincides with the image of $\alpha^{\bullet}_{\boldsymbol{M};K}$, and that

$$\mathscr{A}^{\mathrm{dyn}}(\boldsymbol{M};O) \cong \bigvee_{K \in \mathscr{K}(\boldsymbol{M};O)} \mathscr{A}^{\bullet}(\boldsymbol{M};K) \tag{2.23}$$

for any $O \in \mathscr{O}(\boldsymbol{M})$; this will be the case for all theories defined here explicitly.

As discussed in [25], the dynamical definition of locality can also be shown to display the properties of isotony and covariance that the kinematic definition has; isotony may easily be seen from the definition, since $\mathscr{K}(\boldsymbol{M};O_1) \subset \mathscr{K}(\boldsymbol{M};O_2)$ whenever $O_1 \subset O_2$, and any automorphism $\psi : \boldsymbol{M} \to \boldsymbol{M}$ in Loc has the property that

$$\mathscr{A}(\psi)(\hat{\mathscr{A}}^{\mathrm{dyn}}(\boldsymbol{M};O)) = \hat{\mathscr{A}}^{\mathrm{dyn}}(\boldsymbol{M};\psi(O)) \tag{2.24}$$

for $O \in \mathscr{O}(\boldsymbol{M})$, as proved in [25, Theorem 5.4(b)].

Definition 2.2.3. *An LCT $\mathscr{A}$ is* dynamically local *if it obeys the timeslice axiom and if $\hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M};O) = \hat{\mathscr{A}}^{\mathrm{dyn}}(\boldsymbol{M};O)$ (alternatively, if $\alpha^{\mathrm{dyn}}_{\boldsymbol{M};O} \cong \alpha^{\mathrm{kin}}_{\boldsymbol{M};O}$) for each $\boldsymbol{M} \in$ Sp and $O \in \mathscr{O}(\boldsymbol{M})$.*

2.2.2 SPASs

The question of what, if anything, can be said to represent 'physical realism' in an LCT was discussed at length in [25]. While no absolute conclusions have been made in this direction (and it is unlikely that this is possible or even desirable), it was suggested in the aforementioned paper that physically realistic theories (that is, assignments of Phys-objects to spacetimes) should be constructible in the LCT framework, obey the timeslice axiom and belong to a class of theories obeying the following condition, known as the *SPASs* (Same Physics in All Spacetimes) condition. Before we give the definition, note that locally covariant theories from a given category Sp to Phys comprise the objects of a category $\mathsf{LCT}(\mathsf{Sp}, \mathsf{Phys})$ whose arrows are natural transformations between theories. If a natural $\eta : \mathscr{A} \stackrel{.}{\longrightarrow} \mathscr{B}$ exists, then we regard it as an embedding of $\mathscr{A}$ as a subtheory of $\mathscr{B}$.

Definition 2.2.4. *Let $\mathbf{T}$ be a class of LCTs obeying the timeslice axiom. A natural transformation η between theories $\mathscr{A}$ and $\mathscr{B}$ of $\mathbf{T}$ is a* partial isomorphism *if at least one of its components $\eta_{\boldsymbol{M}} : \mathscr{A}(\boldsymbol{M}) \to \mathscr{B}(\boldsymbol{M})$ is an isomorphism. The class $\mathbf{T}$ obeys the SPASs condition if every partial isomorphism in $\mathbf{T}$ is a natural isomorphism: in other words, if $\eta_{\boldsymbol{M}}$ is an isomorphism for some $\boldsymbol{M} \in \mathsf{Sp}$, then every component of η is an isomorphism.*

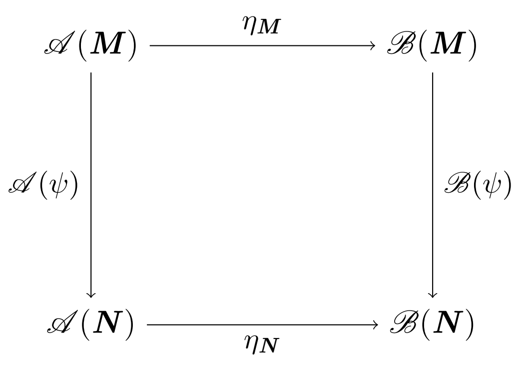


Figure 4: A commuting square for a natural transformation η that embeds a theory $\mathscr{A}$ as a subtheory of $\mathscr{B}$. If $\mathscr{A}$ and $\mathscr{B}$ both lie in some class of theories with the SPASs property, then there can be no spacetimes $\boldsymbol{M}$, $\boldsymbol{N}$ such that exactly one of $\eta_{\boldsymbol{M}}, \eta_{\boldsymbol{N}}$ is an isomorphism, for any such natural transformation η.

Therefore, as indicated in Figure 4, any arrow in $\mathsf{LCT}(\mathsf{Sp},\mathsf{Phys})$ between objects $\mathscr{A}, \mathscr{B}$ of $\mathbf{T}$ is either an isomorphism (in which case $\mathscr{A}$ is equivalent to $\mathscr{B}$), or has no isomorphisms as components (in which case $\mathscr{A}$ is unambiguously a strict subtheory of $\mathscr{B}$).

It was shown in [25, §4] that it is possible to construct examples of LCTs that obey the timeslice axiom but manifestly do not display the same behaviour on all spacetimes, and therefore cannot be said to be physically realistic theories. It is also possible to construct an LCT which cannot be in any class obeying the SPASs property, since it admits a natural endomorphism that is an isomorphism on some spacetimes but not others. The utility of dynamical locality is demonstrated in the following result [25, Theorem 6.10]:

Theorem 2.2.5. *The class of dynamically local theories obeys the SPASs property.*

2.3 Locally covariant fields and solutions

The principle of local covariance also gives rise to a description of quantum fields as natural transformations [11]. The starting point is a covariant functor $\mathscr{D}$ from Sp to some suitable category Test of vector spaces that constructs a space of test functions $\mathscr{D}(\boldsymbol{M})$ for each spacetime $\boldsymbol{M}$. We then consider a locally covariant theory $\mathscr{A} : \mathsf{Sp} \to \mathsf{Phys}$, where Phys is taken to be some category of algebras. We may consider a quantum field for the algebra $\mathscr{A}(\boldsymbol{M})$ to be a map $\Phi_{\boldsymbol{M}} : \mathscr{D}(\boldsymbol{M}) \to \mathscr{A}(\boldsymbol{M})$; given such a definition for each spacetime $\boldsymbol{M}$, the field should satisfy

$$\Phi_{\boldsymbol{N}} \circ \mathscr{D}(\psi) = \mathscr{A}(\psi) \circ \Phi_{\boldsymbol{M}} \tag{2.25}$$

for each Sp-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$.

If we regard the target category of $\mathscr{D}$ and the category Phys as subcategories of some single larger category, we may consider the field Φ to be a natural transformation from $\mathscr{D}$ to $\mathscr{A}$; (2.25) is then precisely the condition of naturality for Φ. Note that we do not require that the maps $\Phi_{\boldsymbol{M}}$ satisfy any further properties, such as linearity, beyond those dictated by the definition of Test and Phys.

We may extend this idea to the case of a classical theory $\mathscr{L} : \mathsf{Sp} \to \mathsf{Phys}$, where $\mathscr{L}(\boldsymbol{M})$ represents the space of solutions to a particular field equation, and in many cases there is a fundamental solution $G_{\boldsymbol{M}} : \mathscr{D}(\boldsymbol{M}) \to \mathscr{L}(\boldsymbol{M})$ for some test space functor $\mathscr{D}$. Typically this solution satisfies

$$G_{\boldsymbol{N}} \circ \mathscr{D}(\psi) = \mathscr{L}(\psi) \circ G_{\boldsymbol{M}} \tag{2.26}$$

for any Loc-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$. Formally, we give the following definition:

Definition 2.3.1. *Let $\mathscr{D} : \mathsf{Sp} \to \mathsf{Test}$ be a covariant functor that constructs some space of test functions for each $\boldsymbol{M} \in \mathsf{Sp}$. Suppose that $\mathscr{A} : \mathsf{Sp} \to \mathsf{Phys}_Q$ is a quantum LCT and $\mathscr{L} : \mathsf{Sp} \to \mathsf{Phys}_C$ is a classical LCT. Moreover, let C_Q and C_C be categories with the property that both Test and Phys_Q (resp. Phys_C) may be regarded as subcategories of C_Q (resp. C_C). We may then construct both $\mathscr{D}$ and $\mathscr{A}$ as functors from Sp to C_Q, and both $\mathscr{D}$ and $\mathscr{L}$ as functors from Sp to C_C.*

A locally covariant quantum field *of the theory $\mathscr{A}$ is a natural transformation $\Phi : \mathscr{D} \stackrel{\cdot}{\longrightarrow} \mathscr{A}$. A* locally covariant solution *for the theory $\mathscr{L}$ is a natural transformation $G : \mathscr{D} \stackrel{\cdot}{\longrightarrow} \mathscr{L}$. In the latter case, since $G_{\boldsymbol{M}}$ is generally surjective (and linear, in the case of a linear theory), we require that the components of a locally covariant solution be surjective (resp. linear).*

Note that G and Φ are commonly not injective, so (in contrast to Sp, Phys_C and Phys_Q in most circumstances), the arrows of C_C and C_Q will typically not all be monic.

2.3.1 Quantization functors

The construction of many quantum LCTs involves first the construction of a classical theory, followed by the use of the resulting solution spaces to generate algebras of observables. We will see specific examples of this approach in subsequent chapters, but it is a natural question to ask whether the step from classical theory to quantum theory may be made functorial. In other words, if we are given target categories Phys_Q and Phys_C for some general classes of quantum and classical theories respectively, we wish to know whether there exists a functor $\mathscr{Q} : \mathsf{Phys}_C \to \mathsf{Phys}_Q$ that gives the quantization of a classical theory $\mathscr{L} : \mathsf{Sp} \to \mathsf{Phys}_C$.

It turns out to be possible to construct such functors (although the precise form depends on the type of quantization to be performed, as we will see later). For now, we will suppose that we have constructed a quantum theory $\mathscr{A} : \mathsf{Sp} \to \mathsf{Phys}_Q$ and a classical theory $\mathscr{L} : \mathsf{Sp} \to \mathsf{Phys}_C$ such that $\mathscr{A} = \mathscr{Q}\mathscr{L}$ for some quantization functor $\mathscr{Q}$: for the following definition and results, $G : \mathscr{D} \stackrel{.}{\longrightarrow} \mathscr{L}$ is a locally covariant solution and $\Phi : \mathscr{D} \stackrel{.}{\longrightarrow} \mathscr{A}$ is a locally covariant field.

Definition 2.3.2. Φ factors through G *if there exists a family of maps* $\hat{\Phi}_{\boldsymbol{M}} : \mathscr{L}(\boldsymbol{M}) \to \mathscr{A}(\boldsymbol{M})$ *indexed by spacetimes such that* $\Phi_{\boldsymbol{M}} = \hat{\Phi}_{\boldsymbol{M}} \circ G_{\boldsymbol{M}}$.

Proposition 2.3.3. *If* Φ *factors through* G *via* $(\hat{\Phi}_{\boldsymbol{M}})_{\boldsymbol{M}\in\mathsf{Loc}}$, *and each* $\hat{\Phi}_{\boldsymbol{M}}$ *can be regarded as an arrow in some larger category* Phys' *containing* Phys_Q *and* Phys_C, *then* $\hat{\Phi}$ *is a natural transformation from* $\mathscr{L}$ *to* $\mathscr{A}$.

Proof. For any Sp-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$ we have

$$\begin{aligned}\mathscr{A}(\psi) \circ \hat{\Phi}_{\boldsymbol{M}} \circ G_{\boldsymbol{M}} &= \mathscr{A}(\psi) \circ \Phi_{\boldsymbol{M}} = \Phi_{\boldsymbol{N}} \circ \mathscr{D}(\psi) \\ &= \hat{\Phi}_{\boldsymbol{N}} \circ G_{\boldsymbol{N}} \circ \mathscr{D}(\psi) = \hat{\Phi}_{\boldsymbol{N}} \circ \mathscr{L}(\psi) \circ G_{\boldsymbol{M}}. \end{aligned} \tag{2.27}$$

Therefore $\mathscr{A}(\psi) \circ \hat{\Phi}_{\boldsymbol{M}} = \hat{\Phi}_{\boldsymbol{N}} \circ \mathscr{L}(\psi)$ by the surjectivity of $G_{\boldsymbol{M}}$, and so $\hat{\Phi}$ is a natural transformation. □

Lemma 2.3.4. *Suppose that* $\mathscr{L}$ *is linear, that the components of* Φ *are linear maps, and that* Φ *factors through* G *via* $\hat{\Phi} : \mathscr{L} \stackrel{.}{\longrightarrow} \mathscr{A}$. *It follows that the components of* $\hat{\Phi}$ *are also linear maps.*

Proof. For a given $\boldsymbol{M} \in \mathsf{Sp}$, we pick some $\varphi_1, \varphi_2 \in \mathscr{L}(\boldsymbol{M})$; since $G_{\boldsymbol{M}}$ is surjective, we may pick some $f_1, f_2 \in \mathscr{D}(\boldsymbol{M})$ such that $\varphi_i = G_{\boldsymbol{M}} f_i$, and it follows that for any $\lambda, \mu \in \mathbb{K}$, where $\mathbb{K}$ is the base field of $\mathscr{L}(\boldsymbol{M})$, we have

$$\begin{aligned}\hat{\Phi}_{\boldsymbol{M}}(\lambda\varphi_1 + \mu\varphi_2) &= \hat{\Phi}_{\boldsymbol{M}}(G_{\boldsymbol{M}}(\lambda f_1 + \mu f_2)) \\ &= \lambda\Phi_{\boldsymbol{M}}(f_1) + \mu\Phi_{\boldsymbol{M}}(f_2) = \lambda\hat{\Phi}_{\boldsymbol{M}}(\varphi_1) + \mu\hat{\Phi}_{\boldsymbol{M}}(\varphi_2),\end{aligned} \tag{2.28}$$

so $\hat{\Phi}_{\boldsymbol{M}}$ is also linear. □

Lemma 2.3.5. *Suppose that $\mathscr{L}$ is linear and that the components of Φ are linear maps; it follows that a necessary and sufficient condition for Φ to factor through G is that $\ker G_{\boldsymbol{M}} \subset \ker \Phi_{\boldsymbol{M}}$ for all spacetimes $\boldsymbol{M}$.*

Proof. Firstly, suppose that Φ factors through G, so that there exists a natural transformation $\hat{\Phi} : \mathscr{L} \stackrel{.}{\longrightarrow} \mathscr{A}$ with $\Phi = \hat{\Phi} \circ G$. As shown in Lemma 2.3.4, the components of $\hat{\Phi}$ are linear maps, so for any $\boldsymbol{M} \in \mathsf{Sp}$ and $f \in \ker G_{\boldsymbol{M}}$, we have

$$\Phi_{\boldsymbol{M}}(f) = \hat{\Phi}_{\boldsymbol{M}}(0) = 0. \tag{2.29}$$

Conversely, suppose that $\ker G_{\boldsymbol{M}} \subset \ker \Phi_{\boldsymbol{M}}$ for all $\boldsymbol{M} \in \mathsf{Sp}$; it follows that for each $\varphi \in \mathscr{L}(\boldsymbol{M})$ we may pick some $f \in \mathscr{D}(\boldsymbol{M})$ such that $\varphi = G_{\boldsymbol{M}} f$, and then define $\hat{\Phi}_{\boldsymbol{M}} : \mathscr{L}(\boldsymbol{M}) \to \mathscr{A}(\boldsymbol{M})$ by

$$\hat{\Phi}_{\boldsymbol{M}}(\varphi) := \Phi_{\boldsymbol{M}}(f). \tag{2.30}$$

This is well defined since $G_{\boldsymbol{M}} f = G_{\boldsymbol{M}} f'$ implies that $\Phi_{\boldsymbol{M}}(f) = \Phi_{\boldsymbol{M}}(f')$ by assumption, and it may easily be checked that Φ then factors through G via $\hat{\Phi}$. □

Chapter 3

Locally covariant scalar field theories

The construction of the theory of the free scalar field $\mathscr{A}$ in a locally covariant way is well understood [11, 32, 26]. Dynamical locality has already been shown to hold in the massive minimally coupled case, and to fail in the massless minimally coupled case owing to a rigid gauge symmetry in the Lagrangian [26]. Our main aim in this section is to construct an enlarged theory $\mathscr{W}$ which, in addition to containing the theory $\mathscr{A}$, also contains normally ordered Wick powers of the locally covariant field Φ.

The construction of Wick powers on a single arbitrary spacetime is described in [10, 8]; an enlarged algebra containing Wick powers as well as some additional elements is constructed in [36], where in addition the Wick polynomials are constructed as locally covariant fields (see also a more general result in [46]). An alternative prescription for constructing the enlarged algebra on a given spacetime $\boldsymbol{M}$ is given in [9, 14], although a fully locally covariant description of the theory is not given.

We will give precise definitions of the arrows of the theory, and we will also demonstrate explicitly how the locally covariant construction of the Wick powers given in [36] may be applied in this setting. The material of this and the following chapter is mostly contained within a previously published paper by the author [24]. The construction of the enlarged algebra in [9, 14] may equally be applied to the original theory of the free scalar field, so we will first outline how this may be done.

3.1 The locally covariant scalar field

Definition 3.1.1. *The* Klein-Gordon operator *on a spacetime* $\boldsymbol{M} \in \mathsf{Loc}$ *is* $P_{\boldsymbol{M}} := \Box_{\boldsymbol{g}} + \xi R_{\boldsymbol{g}} + m^2$. *We call any* $\phi \in C^\infty(\boldsymbol{M})$ *that solves the field equation* $P_{\boldsymbol{M}}\phi = 0$ *a* classical solution. *The coupling constant* $\xi \in \mathbb{R}$ *and the mass* $m \geq 0$ *are held constant over all spacetimes.*

The Klein-Gordon operator has associated with it two unique continuous linear operators $E^{\pm}_{\boldsymbol{M}} : C_0^\infty(\boldsymbol{M}) \to C^\infty(\boldsymbol{M})$ with the properties[1]

$$
\begin{aligned}
E^{\pm}_{\boldsymbol{M}} \circ P_{\boldsymbol{M}} = \iota = P_{\boldsymbol{M}} \circ E^{\pm}_{\boldsymbol{M}}, \\
\mathrm{supp}(E^{\pm}_{\boldsymbol{M}} f) \subset J^{\pm}_{M}(\mathrm{supp}(f))
\end{aligned}
\tag{3.1}
$$

for any $f \in C_0^\infty(\boldsymbol{M})$ [63], where $\iota : C_0^\infty(\boldsymbol{M}) \hookrightarrow C^\infty(\boldsymbol{M})$ is the canonical embedding. The operator $E_{\boldsymbol{M}} := E^-_{\boldsymbol{M}} - E^+_{\boldsymbol{M}}$ is the *(advanced-minus-retarded) fundamental solution* for the Klein-Gordon field on $\boldsymbol{M}$, and any classical solution ϕ with compact support on Cauchy surfaces is of the form $\phi = E_{\boldsymbol{M}} f$ for some $f \in C_0^\infty(\boldsymbol{M})$. We denote by $E_{\boldsymbol{M}}(x, y)$ the antisymmetric bidistribution on test functions satisfying

$$
\int_{\boldsymbol{M}} dy\, E_{\boldsymbol{M}}(x, y) f(y) = (E_{\boldsymbol{M}} f)(x) \tag{3.2}
$$

for each $f \in C_0^\infty(\boldsymbol{M})$.[2] Furthermore, we write

$$
E_{\boldsymbol{M}}(f, f') := \int_{\boldsymbol{M}} dx\, f(x)(E_{\boldsymbol{M}} f')(x) = \int_{\boldsymbol{M}^{\times 2}} dx\, dy\, f(x) E_{\boldsymbol{M}}(x, y) f'(y), \tag{3.3}
$$

[1]The two instances of the operator $P_{\boldsymbol{M}}$ in (3.1) act on different function spaces and are technically therefore different operators, but as they act pointwise and have the same action we denote them by the same symbol.

[2]Here, and throughout this thesis, the notation dy or similar is shorthand for $dvol_{\boldsymbol{M}}(y)$; in the case where we integrate across a sequence of variables such as $x_1, \ldots, x_n$, we write $d^n x = dx_1 \cdots dx_n$.

for $f, f' \in C_0^\infty(\boldsymbol{M})$. Note that this entails

$$\int_{\boldsymbol{M}} dx\, f(x)(E_{\boldsymbol{M}} f')(x) = -\int_{\boldsymbol{M}} dx\, (E_{\boldsymbol{M}} f)(x) f'(x). \tag{3.4}$$

Since any Loc-arrow is isometric, it follows that $\psi_* \circ P_{\boldsymbol{M}} = P_{\boldsymbol{N}} \circ \psi_*$ and $P_{\boldsymbol{M}} \circ \psi^* = \psi^* \circ P_{\boldsymbol{N}}$. We then see that $\psi^* \circ E^{\pm}_{\boldsymbol{N}} \circ \psi_*$ satisfies (3.1), so by uniqueness

$$\psi^* \circ E^{(\pm)}_{\boldsymbol{N}} \circ \psi_* = E^{(\pm)}_{\boldsymbol{M}}. \tag{3.5}$$

When treated as a bidistribution, as in (3.2), the analogous relation is

$$E_{\boldsymbol{N}} \circ (\psi_* \otimes \psi_*) = E_{\boldsymbol{M}}. \tag{3.6}$$

3.1.1 The algebra of functionals

Given a fixed spacetime $\boldsymbol{M}$, the algebra of the Klein-Gordon quantum field theory is usually constructed directly as the unital $*$-algebra generated by elements $\Phi_{\boldsymbol{M}}(t)$, $t \in C_0^\infty(\boldsymbol{M})$ satisfying the following four conditions:

$$\text{The assignment } t \mapsto \Phi_{\boldsymbol{M}}(t) \text{ is linear,} \tag{3.7a}$$

$$\Phi_{\boldsymbol{M}}(t)^* = \Phi_{\boldsymbol{M}}(\bar{t}), \tag{3.7b}$$

$$[\Phi_{\boldsymbol{M}}(t), \Phi_{\boldsymbol{M}}(t')] = iE_{\boldsymbol{M}}(t, t')\mathbf{1}, \tag{3.7c}$$

$$\Phi_{\boldsymbol{M}}(P_{\boldsymbol{M}} t) = 0. \tag{3.7d}$$

The field $\Phi = (\Phi_{\boldsymbol{M}})_{\boldsymbol{M} \in \mathsf{Loc}}$ is locally covariant [11]; in addition, the quantization attained through this process can be regarded as the image of a covariant CCR-quantization functor in a similar fashion to the construction of CAR-quantization functors in Chapter 5 [26].

In this section, we construct an alternative presentation of the above algebra, following [9]. While this algebra is well understood and does not necessarily need another admittedly less elegant presentation purely to describe the free field, the advantage of this alternative approach is that it provides a convenient framework for the construction of the extended algebra of Wick polynomials.

We begin by applying a deformation quantization process to an algebra of nonlinear functionals that is isomorphic to the freest unital $*$-algebra satisfying (3.7a)–(3.7c), and then quotient this algebra further to impose (3.7d). The underlying vector space of this algebra comprises those functionals on $C^\infty(\boldsymbol{M})$ of the form

$$F[f] = \sum_{n=0}^{N} \int_{\boldsymbol{M}^{\times n}} d^n x\, t_n(x_1, \ldots, x_n) f(x_1) \cdots f(x_n), \tag{3.8}$$

where each t_n is a totally symmetric finite sum of products of test functions in one variable:

$$t_n(x_1, \ldots, x_n) = \mathbf{S} \sum_{j \text{ finite}} \prod_{k=1}^{n} \varphi_{jk}(x_k) \tag{3.9}$$

for some $\varphi_{jk} \in C_0^\infty(\boldsymbol{M})$, where $\mathbf{S}$ denotes symmetrisation over the x_k. We denote the set of all such t_n as $\mathscr{F}^n(\boldsymbol{M})$; we define $\mathscr{F}^0(\boldsymbol{M}) = \mathbb{C}$, and we may note that $\mathscr{F}^1(\boldsymbol{M}) = C_0^\infty(\boldsymbol{M})$. We denote by $\mathbf{1}$ the constant functional that maps every f to 1. We will use the shorthand notation

$$t_n[f] := \int_{\boldsymbol{M}^{\times n}} d^n x\, t_n(x_1, \ldots, x_n) f(x_1) \cdots f(x_n). \tag{3.10}$$

For each $F = \sum_{n=0}^{N} t_n$ with $t_N \neq 0$ we denote the order of F by $O(F) := N < \infty$.

Addition and scalar multiplication in this algebra are given by the obvious pointwise operations on the functionals, and the product is defined below.

The k^{th} functional derivative of $F = \sum_{n=0}^{N} t_n$ is given by

$$F^{(k)}[f](x_1, \ldots, x_k) := \sum_{n=k}^{N} t_n^{(k)}[f](x_1, \ldots, x_k), \tag{3.11}$$

where for $k \leq n$,

$$t_n^{(k)}[f](x_1, \ldots, x_k) := \frac{n!}{(n-k)!} \int_{\boldsymbol{M}^{\times(n-k)}} dx_{k+1} \cdots dx_n \Big(t_n(x_1, \ldots, x_n) \\ f(x_{k+1}) \cdots f(x_n) \Big). \tag{3.12}$$

For $k > n$ we use the convention that $t_n^{(k)} = 0$. For any $f \in C_0^\infty(\boldsymbol{M})$ and $F \in \mathscr{F}(\boldsymbol{M})$, we may regard the functional derivative $F^{(k)}[f](x_1, \ldots, x_k)$ as an element of $\mathscr{F}^k(\boldsymbol{M})$ for $k \leq O(F)$. The product of two elements in $\mathscr{F}(\boldsymbol{M})$ is defined by

$$F \star F' = \boldsymbol{\mu} \exp\left(\frac{i}{2}\mathcal{E}_{\boldsymbol{M}}\right)(F \otimes F'), \tag{3.13}$$

where the linear operator $\mathcal{E}_{\boldsymbol{M}} : \mathscr{F}(\boldsymbol{M}) \otimes \mathscr{F}(\boldsymbol{M}) \to \mathscr{F}(\boldsymbol{M}) \otimes \mathscr{F}(\boldsymbol{M})$, which restricts for particular $m, n > 0$ to a map from $\mathscr{F}^m(\boldsymbol{M}) \otimes \mathscr{F}^n(\boldsymbol{M})$ to $\mathscr{F}^{m-1}(\boldsymbol{M}) \otimes \mathscr{F}^{n-1}(\boldsymbol{M})$, is defined for $t_m \in \mathscr{F}^m(\boldsymbol{M})$, $t_n \in \mathscr{F}^n(\boldsymbol{M})$ by

$$\begin{aligned}&\mathcal{E}_{\boldsymbol{M}}(t_m \otimes t_n)(x_1, \ldots, x_{m-1}; y_1, \ldots, y_{n-1}) \\ &\quad := mn \int_{\boldsymbol{M}^{\times 2}} d^2z\, E_{\boldsymbol{M}}(z_1, z_2) t_m(z_1, x_1, \ldots, x_{m-1}) t_n(z_2, y_1, \ldots, y_{n-1}),\end{aligned} \tag{3.14}$$

and for $\alpha \in \mathscr{F}^0(\boldsymbol{M})$ by $\mathcal{E}_{\boldsymbol{M}}(\alpha, F) = 0 = \mathcal{E}_{\boldsymbol{M}}(F, \alpha)$. Here we also define $\boldsymbol{\mu} : \mathscr{F}(\boldsymbol{M}) \otimes \mathscr{F}(\boldsymbol{M}) \to \mathscr{F}(\boldsymbol{M})$ to be the commutative multiplication map $\boldsymbol{\mu}(F \otimes F')[f] := F[f]F'[f]$, so that in terms of the symmetric integral kernels we have

$$\boldsymbol{\mu}(t_m \otimes t_n)(x_1, \ldots, x_{m+n}) = \mathbf{S}(t_m(x_1, \ldots, x_m) t_n(x_{m+1}, \ldots, x_{m+n})). \tag{3.15}$$

Note that there are no issues with convergence in (3.13), since the properties of $\mathcal{E}_{\boldsymbol{M}}$ entail that for every $F, F' \in \mathscr{F}(\boldsymbol{M})$ we have $\mathcal{E}_{\boldsymbol{M}}^k(F \otimes F') = 0$ for all $k > \max(O(F), O(F'))$. We may therefore give an alternative power series expansion for the product, as

$$F \star F' := \sum_{k=0}^{\min(O(F), O(F'))} \frac{i^k}{2^k k!} \boldsymbol{\mu} \mathcal{E}_{\boldsymbol{M}}^k (F \otimes F'). \tag{3.16}$$

In particular, where $t, t' \in \mathscr{F}_1(\boldsymbol{M})$, note that

$$(t \star t')[f] = \int_{\boldsymbol{M}^{\times 2}} dx\, dy\, t(x) t'(y) f(x) f(y) + \frac{i}{2} E_{\boldsymbol{M}}(t, t')\mathbf{1}, \tag{3.17}$$

so $t \star t' - t' \star t = iE_{\boldsymbol{M}}(t, t')\mathbf{1}$. We prove that the product $\star$ is associative in Lemma B.1.

Since $\mathbf{1} \in \mathscr{F}^0(\boldsymbol{M})$ we have $(F \star \mathbf{1})[f] = F[f]\mathbf{1}[f] = F[f]$ for all $f \in C^\infty(\boldsymbol{M})$. Similarly $(\mathbf{1} \star F)[f] = F[f]$, and so $\mathbf{1}$ is the identity element. We may define an involution of $F = \sum_{n=0}^N t_n \in \mathscr{F}(\boldsymbol{M})$ by $F^* := \sum_{n=0}^N \bar{t}_n$, so that $F^*[f] = \overline{F[\overline{f}]}$, as proved in Lemma B.2.

We denote by $\mathscr{J}(\boldsymbol{M})$ the set of elements in $\mathscr{F}(\boldsymbol{M})$ satisfying $F[E_{\boldsymbol{M}} f] = 0$ for all $f \in C_0^\infty(\boldsymbol{M})$. This will turn out to be the ideal of the Klein-Gordon operator $P_{\boldsymbol{M}}$, in the sense that it is the two-sided $*$-ideal generated by elements $P_{\boldsymbol{M}} t \in \mathscr{F}^1(\boldsymbol{M})$; therefore, quotienting $\mathscr{F}(\boldsymbol{M})$ by $\mathscr{J}(\boldsymbol{M})$ will have the effect of reestablishing the field equation as in (3.7d).

Lemma 3.1.2. *Let $F = \sum_{n=0}^N t_n \in \mathscr{J}(\boldsymbol{M})$ for some spacetime $\boldsymbol{M}$, where $t_n \in \mathscr{F}^n(\boldsymbol{M})$ for each $n = 0, 1, \ldots, N$. Then we have the polarization result*

$$(E_{\boldsymbol{M}}{}^{\otimes n} t_n)(x_1, \ldots, x_n) = 0. \tag{3.18}$$

Moreover, $\mathscr{J}(\boldsymbol{M})$ is a two-sided $*$-ideal of $\mathscr{F}(\boldsymbol{M})$.

Proof. For any $f \in C_0^\infty(\boldsymbol{M})$ and $\kappa \in \mathbb{R}$ we have

$$0 = F[E_{\boldsymbol{M}}(\kappa f)] = \sum_{n=0}^N \kappa^n t_n[E_{\boldsymbol{M}} f]. \tag{3.19}$$

Consequently $t_n[E_{\boldsymbol{M}} f] = 0$ for each n, and so by (3.8), and using the fact that for any $g, g' \in C_0^\infty(\boldsymbol{M})$ we have $\int_{\boldsymbol{M}} dx\, g(x) E_{\boldsymbol{M}} g'(x) = -\int_{\boldsymbol{M}} dx\, g'(x) E_{\boldsymbol{M}} g(x)$, it follows that

$$(E_{\boldsymbol{M}}{}^{\otimes n} t_n)[f] = (-1)^n t_n[E_{\boldsymbol{M}} f] = 0. \tag{3.20}$$

But if also then follows that $(E_{\boldsymbol{M}}{}^{\otimes n} t_n)[f + \kappa f'] = 0$ for all f, f'; if we differentiate this equation with respect to κ and evaluate at $\kappa = 0$, we see from the total symmetry of t_n that

$$\int_{\boldsymbol{M}^{\times n}} d^n x\, t_n(x_1, \ldots, x_n) f(x_1) \cdots f(x_{n-1}) f'(x_n) = 0. \tag{3.21}$$

We may repeat this argument to see that in fact

$$\int_{\boldsymbol{M}^{\times n}} d^n x\, t_n(x_1, \ldots, x_n) f_1(x_1) \cdots f_n(x_n) = 0 \tag{3.22}$$

for any $f_1, \ldots, f_n \in C_0^\infty(\boldsymbol{M})$, and therefore ${E_{\boldsymbol{M}}}^{\otimes n} t_n$ is identically zero.

From this we see that if $F[E_{\boldsymbol{M}} f] = 0$ for all $f \in C_0^\infty(\boldsymbol{M})$, then

$$\left(\boldsymbol{\mu} \mathcal{E}_{\boldsymbol{M}}^k (F \otimes F')\right) [E_{\boldsymbol{M}} f] = 0 \tag{3.23}$$

for all $f \in C_0^\infty(\boldsymbol{M})$ and $k \geq 0$, and therefore $(F \star F')[E_{\boldsymbol{M}} f] = 0$. Similarly $(F' \star F)[E_{\boldsymbol{M}} f] = 0$, and since $F^*[E_{\boldsymbol{M}} f] = \overline{F[E_{\boldsymbol{M}} \overline{f}]}$ we also have $F^*[E_{\boldsymbol{M}} f] = 0$ for all $f \in C_0^\infty(\boldsymbol{M})$. Therefore $\mathscr{J}(\boldsymbol{M})$ is indeed a two-sided $*$-ideal. □

Definition 3.1.3. *For a spacetime $\boldsymbol{M} \in \mathsf{Loc}$, the algebra $\mathscr{A}(\boldsymbol{M})$ is defined to be the quotient $\mathscr{F}(\boldsymbol{M})/\mathscr{J}(\boldsymbol{M})$.*

The ideal $\mathscr{J}(\boldsymbol{M})$ generates an equivalence relation $\sim_{\boldsymbol{M}}$; i.e. for any $F, F' \in \mathscr{F}(\boldsymbol{M})$, $F \sim_{\boldsymbol{M}} F'$ if and only if $F - F' \in \mathscr{J}(\boldsymbol{M})$, or equivalently $F[E_{\boldsymbol{M}} f] = F'[E_{\boldsymbol{M}} f]$ for all $f \in C_0^\infty(\boldsymbol{M})$. For any $F \in \mathscr{F}(\boldsymbol{M})$, the equivalence class of F under $\sim_{\boldsymbol{M}}$ is denoted $[F]_{\boldsymbol{M}}$; the elements of the algebra $\mathscr{A}(\boldsymbol{M})$ constitute the set of equivalence classes $[F]_{\boldsymbol{M}}$ with $F \in \mathscr{F}(\boldsymbol{M})$.

Since this quotienting has the effect of once again establishing (3.7d), it follows that $\mathscr{A}(\boldsymbol{M})$ is isomorphic to the algebra obtained from the quantization procedure set out at the beginning of this section. Given any $t \in C_0^\infty(\boldsymbol{M})$, the field $\Phi_{\boldsymbol{M}}(t)$ is defined on the algebra $\mathscr{A}(\boldsymbol{M})$ by $\Phi_{\boldsymbol{M}}(t) := [\varphi_{\boldsymbol{M}}(t)]_{\boldsymbol{M}}$, where $\varphi_{\boldsymbol{M}}(t)[f] := \int_{\boldsymbol{M}} dx\, t(x) f(x)$ for $f \in C^\infty(\boldsymbol{M})$.

Definition 3.1.4. *Consider a Loc-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$, and a functional $F = \sum_{n=0}^N t_n \in \mathscr{F}(\boldsymbol{N})$, with $t_n \in \mathscr{F}^n(\boldsymbol{N})$. The pullback of F by ψ is defined by*

$$\psi^* F := \sum_{n=0}^N (\psi^{\otimes n})^* t_n. \tag{3.24}$$

Note that ψ^* is only a partial function on $\mathscr{F}(\boldsymbol{N})$; since $\psi(\boldsymbol{M})$ is open in $\boldsymbol{N}$ and therefore $\psi^* t_n$ is not compactly supported in general, we define $\psi^* F$ only when $\mathrm{supp}(t_n) \in \psi(\boldsymbol{M})^{\times n}$ for each n. This entails that $(\psi^* F)[f] = F[\psi_* f]$ when $f \in C_0^\infty(\boldsymbol{M})$. More generally, for $f \in C^\infty(\boldsymbol{M})$ we have $(\psi^* F)[f] = F[\hat{f}]$ whenever $\hat{f} \in C^\infty(\boldsymbol{N})$ satisfies $\psi^* \hat{f} = f$.

We also denote the pushforward of $F \in \mathscr{F}(\boldsymbol{M})$ by $\psi_* F \in \mathscr{F}(\boldsymbol{N})$, where

$$\psi_* \left(\sum_{n=0}^{N} t_n \right) := \sum_{n=0}^{n} (\psi^{\otimes n})_* t_n \tag{3.25}$$

for all $f \in C^\infty(\boldsymbol{N})$; it follows that $(\psi_* F)[f] = F[\psi^* f]$.

Lemma 3.1.5. *Let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be an arrow in* Loc*. Then $\psi_* : \mathscr{F}(\boldsymbol{M}) \to \mathscr{F}(\boldsymbol{N})$ is an arrow in* Alg*, and $\psi^* : \mathscr{F}(\boldsymbol{N}) \to \mathscr{F}(\boldsymbol{M})$ is an injective partial $*$-homomorphism, i.e. has the properties of an* Alg*-arrow when restricted to its domain of definition.*

Proof. If $(\psi_* F)[f] = 0$ for all $f \in C^\infty(\boldsymbol{N})$ then $F[\psi^* f] = 0$ for all such f, and therefore $F = 0$; therefore ψ_* is injective. ψ_* respects the involution because $(\psi_* F)^*[f] = \overline{(\psi_* F)[\overline{f}]} = \overline{F[\overline{\psi^* f}]} = (\psi_* F^*)[f]$. Moreover, (3.6) entails that $\mathcal{E}_{\boldsymbol{N}} \circ (\psi_* \otimes \psi_*) = (\psi_* \otimes \psi_*) \circ \mathcal{E}_{\boldsymbol{M}}$; it is then easy to see that $\psi_*(F \star F') = \psi_* F \star \psi_* F'$ as required. Linearity is obvious, so ψ_* is a homomorphism and consequently an arrow in Alg.

Since the domain of ψ^* is precisely the range of ψ_*, and $\psi^* \circ \psi_* = \mathbf{1}_{\mathscr{F}(\boldsymbol{M})}$, it follows that ψ^* is the inverse of ψ_* considered as a map into $\psi_*(\mathscr{F}(\boldsymbol{M}))$. Therefore ψ^* must be an injective $*$-homomorphism on its domain. □

3.1.2 The locally covariant scalar field theory

While the arrows of the theory were not constructed explicitly in [9], nevertheless the construction of $\mathscr{A}(\boldsymbol{M})$ is equivalent to the standard method, and so it is trivial to use the equivalence to construct $\mathscr{A}(\psi)$ for a given Loc-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$. We first define the map $\mathscr{F}(\psi) : \mathscr{F}(\boldsymbol{M}) \to \mathscr{F}(\boldsymbol{N})$ to be given by the pushforward ψ_* defined above, so that $\mathscr{F}(\psi) F := F \circ \psi^*$. We have already shown that this map is an arrow in Alg, and it is easy to see that $\mathscr{F} : \mathsf{Loc} \to \mathsf{Alg}$ is a covariant functor. Moreover, we have the following result:

Lemma 3.1.6. *Let $\boldsymbol{M}, \boldsymbol{N} \in$ Loc, and $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be a* Loc*-arrow. Then, for any $F, F' \in \mathscr{F}(\boldsymbol{M})$ we have $F \sim_{\boldsymbol{M}} F'$ if and only if $\mathscr{F}(\psi) F \sim_{\boldsymbol{N}} \mathscr{F}(\psi) F'$.*

This is a special case of Lemma 3.2.6, so we will not prove it here. We proceed to define the map

$$\begin{aligned}\mathscr{A}(\psi):\mathscr{A}(\boldsymbol{M})&\to\mathscr{A}(\boldsymbol{N})\\ [F]_{\boldsymbol{M}}&\mapsto[\mathscr{F}(\psi)F]_{\boldsymbol{N}}.\end{aligned}\tag{3.26}$$

Lemma 3.1.7. *The map $\mathscr{A}(\psi)$ is a well defined injective $*$-homomorphism, and $\mathscr{A}:\mathsf{Loc}\to\mathsf{Alg}$ is a covariant functor.*

Proof. $\mathscr{A}(\psi)$ is injective and well-defined owing to Lemma 3.1.6, and inherits both the $*$-homomorphism property and functoriality from $\mathscr{F}(\psi)$. □

A consequence of this is that the transformation $[\,\cdot\,]:\mathscr{F}\to\mathscr{A}$ that maps $F\in\mathscr{F}(\boldsymbol{M})$ to $[F]_{\boldsymbol{M}}$ is natural.

Lemma 3.1.8. *Let $\psi:\boldsymbol{M}\to\boldsymbol{N}$ in Loc. Then $A\in\mathscr{A}(\psi)(\mathscr{A}(\boldsymbol{M}))$ if and only if every $F\in\mathscr{F}(\boldsymbol{N})$ with $A=[F]_{\boldsymbol{N}}$ satisfies $F[E_{\boldsymbol{N}}f]=F[0]$ for all $f\in C_0^\infty(\boldsymbol{N})$ such that $\mathrm{supp}(f)\cap J_{\boldsymbol{N}}(\psi(\boldsymbol{M}))=\emptyset$. Moreover, the theory $\mathscr{A}$ is causal.*

Proof. Note that $\mathscr{A}(\psi)(\mathscr{A}(\boldsymbol{M}))$ comprises elements $A\in\mathscr{A}(\boldsymbol{N})$ that can be represented by some $F=\psi_*(\mathscr{F}(\boldsymbol{M}))$. But these are precisely those $F=\sum_{n=0}^N t_n$ for which $\mathrm{supp}(t_n)\subset\psi(\boldsymbol{M})^{\times n}$ for $n\geq1$, and so $F\in\psi_*(\mathscr{F}(\boldsymbol{M}))$ if and only if $F[f]=F[0]$ for all $f\in C_0^\infty(\boldsymbol{N})$ with $\mathrm{supp}(f)\cap\psi(\boldsymbol{M})=\emptyset$.

Since $F\sim_{\boldsymbol{N}}F'$ if and only if $F[E_{\boldsymbol{N}}f]=F'[E_{\boldsymbol{N}}f]$ for all $f\in C_0^\infty(\boldsymbol{N})$, it follows that F represents an element of $\mathscr{A}(\psi)(\mathscr{A}(\boldsymbol{M}))$ if and only if $F[E_{\boldsymbol{N}}f]=F[0]$ for all $f\in C_0^\infty(\boldsymbol{N})$ with $\mathrm{supp}(f)\cap J_{\boldsymbol{N}}(\psi(\boldsymbol{M}))=\emptyset$.

Now suppose that $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are spacetimes embedded in $\boldsymbol{N}$ by Loc-arrows ψ_1,ψ_2 respectively, and that $\psi_1(\boldsymbol{M}_1)$ and $\psi_2(\boldsymbol{M}_2)$ are causally disjoint in $\boldsymbol{N}$. It follows that if $A_i\in\mathscr{A}(\psi_i)(\mathscr{A}(\boldsymbol{M}_i))$, $i=1,2$, we may pick $F_1,F_2\in\mathscr{F}(\boldsymbol{N})$ such that $[F_i]_{\boldsymbol{N}}=A_i$, and that the n^{th} component of F_i is supported in $\psi_i(\boldsymbol{M}_i)^{\times n}$. It is then clear from (3.13),(3.14) that $(F\star F')[f]=F[f]F'[f]$ for any $f\in C_0^\infty(\boldsymbol{N})$. It follows that $[A_1,A_2]=0$, and therefore the theory is causal. □

As a final note on this construction, we remark that while its main advantage is the immediate extension to the theory containing the Wick polynomials as described in the following section, it also has an additional advantage over the standard construction; namely, that it is easy to work with arbitrary elements of the algebra $\mathscr{A}(\boldsymbol{M})$ rather than the generators only. In particular, this makes it easy to compute the relative Cauchy evolution of an arbitrary element, which will be useful when we prove dynamical locality.

3.2 The enlarged algebra of Wick polynomials

The construction of the Klein-Gordon theory can be extended as in [9] and [14] to a larger theory containing the Wick polynomials. The general aim is to include in the algebras of functionals previously denoted $\mathscr{F}(\boldsymbol{M})$ a greater range of distributions, including (but not limited to) elements representing smearings of covariantly constructed fields corresponding to normal ordered Wick powers of the field Φ. The following description directly follows [9].

3.2.1 The fundamental solution and Klein-Gordon operator on distributions

We first need to establish the behaviour of the fundamental solution $E_{\boldsymbol{M}}$ and the Klein-Gordon operator $P_{\boldsymbol{M}}$ on distributions. For a distribution $t \in \mathcal{D}'(\boldsymbol{M})$ (resp. $\mathcal{E}'(\boldsymbol{M})$, i.e. compactly supported distributions), and arbitrary $f \in C_0^\infty(\boldsymbol{M})$ (resp. $C^\infty(\boldsymbol{M})$), we simply define

$$\langle P_{\boldsymbol{M}} t, f \rangle := \langle t, P_{\boldsymbol{M}} f \rangle . \tag{3.27}$$

Since $P_{\boldsymbol{M}}$ is a formally self-adjoint linear differential operator, the restriction of the map $P_{\boldsymbol{M}} : \mathcal{D}'(\boldsymbol{M}) \to \mathcal{D}'(\boldsymbol{M})$ to $C^\infty(\boldsymbol{M})$ is compatible with the previous definition of $P_{\boldsymbol{M}}$ on smooth functions.

Now, analogously to the case for smooth functions, we wish to construct maps $\overline{E^{\pm}_{\boldsymbol{M}}} : \mathcal{E}'(\boldsymbol{M}) \to \mathcal{D}'(\boldsymbol{M})$ satisfying

$$\overline{E^{\pm}_{\boldsymbol{M}}} P_{\boldsymbol{M}} t = t = P_{\boldsymbol{M}} \overline{E^{\pm}_{\boldsymbol{M}}} t \tag{3.28a}$$

$$\operatorname{supp}(\overline{E^{\pm}_{\boldsymbol{M}}} t) \subset J^{\pm}_{\boldsymbol{M}}(\operatorname{supp}(t)). \tag{3.28b}$$

We therefore let $\overline{E^{\pm}_{\boldsymbol{M}}} t := (E^{\mp}_{\boldsymbol{M}})' t$: this expression is clearly a well-defined element of $\mathcal{D}'(\boldsymbol{M})$ for any $t \in \mathcal{E}'(\boldsymbol{M})$, and this definition ensures that (3.28a) is satisfied. Moreover, we may see that (3.28b) is satisfied by noting that for any $t \in \mathcal{E}'(\boldsymbol{M})$, $f \in C_0^\infty(\boldsymbol{M})$, we have $J^{\pm}_{\boldsymbol{M}}(\operatorname{supp}(t)) \cap \operatorname{supp}(f) = \emptyset$ if and only if $\operatorname{supp}(t) \cap J^{\mp}_{\boldsymbol{M}}(\operatorname{supp}(f)) = \emptyset$.

Lemma 3.2.1. *The operators $\overline{E^{\pm}_{\boldsymbol{M}}} : \mathcal{E}'(\boldsymbol{M}) \to \mathcal{D}'(\boldsymbol{M})$ satisfying the conditions in* (3.28a)*,* (3.28b) *are unique.*

Proof. Suppose that $G^{\pm}_{\boldsymbol{M}} : \mathcal{E}'(\boldsymbol{M}) \to \mathcal{D}'(\boldsymbol{M})$ also satisfy these conditions. Clearly $\overline{E^{+}_{\boldsymbol{M}}} P_{\boldsymbol{M}} G^{+}_{\boldsymbol{M}} t = \overline{E^{+}_{\boldsymbol{M}}} t$ for any $t \in \mathcal{E}'(\boldsymbol{M})$. However, for any $x \in \boldsymbol{M}$, as $\operatorname{supp}(G^{+}_{\boldsymbol{M}} t) \subset J^{+}_{\boldsymbol{M}}(\operatorname{supp} t)$, we can decompose $G^{+}_{\boldsymbol{M}} t$ into a sum $\chi G^{+}_{\boldsymbol{M}} t + (1 - \chi) G^{+}_{\boldsymbol{M}} t$, where $\chi \in C^\infty(\boldsymbol{M})$ is identically 1 in $J^{-}_{\boldsymbol{M}}(x)$ and identically 0 to the future of some Cauchy surface lying to the future of x. Consequently $\chi G^{+}_{\boldsymbol{M}} t$ is compactly supported, and $(1 - \chi) G^{+}_{\boldsymbol{M}} t$ is supported strictly to the future of x, and so

$$(\overline{E^{+}_{\boldsymbol{M}}} P_{\boldsymbol{M}} G^{+}_{\boldsymbol{M}} t)(x) = \chi(x)(G^{+}_{\boldsymbol{M}} t)(x) + \overline{E^{+}_{\boldsymbol{M}}} P_{\boldsymbol{M}} \left((1 - \chi) G^{+}_{\boldsymbol{M}} t \right)(x) = G^{+}_{\boldsymbol{M}} t(x). \tag{3.29}$$

Therefore $\overline{E^{+}_{\boldsymbol{M}}} t = \overline{E^{+}_{\boldsymbol{M}}} P_{\boldsymbol{M}} G^{+}_{\boldsymbol{M}} t = G^{+}_{\boldsymbol{M}} t$. Similarly $\overline{E^{-}_{\boldsymbol{M}}} t = G^{-}_{\boldsymbol{M}} t$, and therefore $\overline{E^{\pm}_{\boldsymbol{M}}}$ are unique. □

We know that the corresponding maps $E^{\pm}_{\boldsymbol{M}} : C_0^\infty(\boldsymbol{M}) \to C^\infty(\boldsymbol{M})$ are also unique, so the restrictions of $\overline{E^{\pm}_{\boldsymbol{M}}}$ to $C_0^\infty(\boldsymbol{M})$ must coincide with $E^{\pm}_{\boldsymbol{M}}$. As before, we let $\overline{E_{\boldsymbol{M}}} := \overline{E^{-}_{\boldsymbol{M}}} - \overline{E^{+}_{\boldsymbol{M}}}$, and therefore $\overline{E_{\boldsymbol{M}}} = -(E_{\boldsymbol{M}})'$, as would be expected from the relation (3.4). From now on, we will drop the bar from the notation and simply write $E^{(\pm)}_{\boldsymbol{M}} t$ for a distribution $t \in \mathcal{E}'(\boldsymbol{M})$.

3.2.2 The enlarged algebra of functionals

Recall that for any spacetime $\boldsymbol{M}$, the algebra of functionals $\mathscr{F}(\boldsymbol{M})$ consists of elements of the form $F = \sum_{n=0}^{N} t_n$, with each $t_n \in \mathscr{F}^n(\boldsymbol{M})$ being a finite sum of finite products of test functions of one variable. We wish to include a much wider range of allowed distributions into the new theory $\mathscr{W}$, but we must apply enough restrictions to ensure that the resulting expressions are well defined. Again, the construction of the following algebras closely follows [9]. We might naïvely assume that we can use the same product as defined in (3.13) for distributions, but this is not the case. For example, consider two elements $t, t' \in \mathscr{F}^1(\boldsymbol{M})$; recall from (3.17) that for any $f \in C^\infty(\boldsymbol{M})$,

$$(t \star t')[f] = \int_{\boldsymbol{M}^{\times 2}} dx\, dy\, t(x) t'(y) \left(f(x) f(y) + \frac{i}{2} E_{\boldsymbol{M}}(x, y) \right); \tag{3.30}$$

again, for $t \in \mathcal{E}'(\boldsymbol{M}^{\times n})$ we use the notation

$$t[f] := \left\langle t, f^{\otimes n} \right\rangle = \int_{\boldsymbol{M}^{\times n}} d^{n-1}x\, t(x_1, \ldots, x_n) f(x_1) \cdots f(x_n), \tag{3.31}$$

so for any $f \in C_0^\infty(\boldsymbol{M})$ we have $t[E_{\boldsymbol{M}} f] = (-1)^n ({E_{\boldsymbol{M}}}^{\otimes n} t)[f]$. When t and t' are test functions the second term on the right hand side of (3.30) is well defined, but pointwise products of distributions are not always so, and we require both a condition on the existence of such pointwise products and a deformation of the product to ensure that all the expressions are well defined. The solution is given in [9]: we can find a suitable condition for existence of pointwise products in [38], relying on the wavefront sets of the distributions concerned.

Definition 3.2.2. *Let u be a distribution in $\mathcal{D}'(\boldsymbol{M}^{\times n})$ (resp. $\mathcal{E}'(\boldsymbol{M}^{\times n})$). A pair $(\boldsymbol{x}, \boldsymbol{k}) \in T^*(\boldsymbol{M}^{\times n})$ is a* regular direction *for u if there exists a smooth function $\phi \in C_0^\infty(\boldsymbol{M}^{\times n})$ with $\phi(\boldsymbol{x}) \neq 0$, a conic neighbourhood V of $\boldsymbol{k}$ and a sequence of constants $(C_N)_{N \in \mathbb{N}}$ such that the localized Fourier transform (i.e. the transform as computed locally in some local coordinates, the choice* *of which does not affect regularity) of the pointwise product ϕu satisfies*

$$\left|\widehat{\phi u}(\boldsymbol{k}')\right| < \frac{C_N}{1+|\boldsymbol{k}'|^N} \tag{3.32}$$

for all $\boldsymbol{k}' \in V$ and $N \in \mathbb{N}$.

The wavefront set *of u, denoted $WF(u)$ is the set of all $(\boldsymbol{x}, \boldsymbol{k}) \in T^*(\boldsymbol{M}^{\times n})$ with $\boldsymbol{k} \neq 0$ that are not regular directions for u.*

Lemma 3.2.3 (Hörmander's criterion)**.** *If t and t' are distributions, then the pointwise product $t(\boldsymbol{x})t'(\boldsymbol{x})$ is a well-defined distribution if the set*

$$\{(\boldsymbol{x}, \boldsymbol{k}+\boldsymbol{k}') : (\boldsymbol{x}, \boldsymbol{k}) \in WF(t),\ (\boldsymbol{x}, \boldsymbol{k}') \in WF(t')\} \tag{3.33}$$

contains no element of the form $(\boldsymbol{x}, 0)$.

It is well known (see e.g. [23]) that the wavefront set of the bidistribution $E_{\boldsymbol{M}}(x,y)$ satisfies

$$WF(E_{\boldsymbol{M}}) \subset \bigcup_{\substack{x,y\in \boldsymbol{M}\\ x\leftrightarrow y}} (V^+_{\boldsymbol{M};x} \times V^-_{\boldsymbol{M};y}) \cup (V^-_{\boldsymbol{M};x} \times V^+_{\boldsymbol{M};y}), \tag{3.34}$$

where $V^{\pm}_{\boldsymbol{M};x} \subset T^*_x\boldsymbol{M}$ is the forward/backward light cone at x, excluding zero, and $x \leftrightarrow y$ indicates that x and y coincide or are connected by a null geodesic. We denote by $V^{\pm}_{\boldsymbol{M}}$ the union $\bigcup_{x\in\boldsymbol{M}} V^{\pm}_{\boldsymbol{M};x}$. We then define for $n \geq 1$ (cf. [14])

$$\begin{aligned}\mathscr{T}^n(\boldsymbol{M}) = \{t \in \mathcal{E}'(\boldsymbol{M}^{\times n}) : t \text{ totally symmetric},\\ WF(t) \cap \overline{(V^+_{\boldsymbol{M}})^{\times n} \cup (V^-_{\boldsymbol{M}})^{\times n}} = \emptyset\}.\end{aligned} \tag{3.35}$$

As before we also define $\mathscr{T}^0(\boldsymbol{M}) = \mathbb{C}$. Such a definition ensures that the expression $\int_{\boldsymbol{M}^{\times 2}} dx\, dy\, t(x)t'(y)E_{\boldsymbol{M}}(x,y)$ for $t, t' \in \mathscr{T}^1(\boldsymbol{M})$ is well defined (and more generally, that

$$\int_{\boldsymbol{M}^{\times 2}} dx_1\, dy\, t_n(x_1, \ldots, x_n)t'(y)E_{\boldsymbol{M}}(x_1, y) \tag{3.36}$$

for $t_n \in \mathscr{T}^n(\boldsymbol{M})$, $t \in \mathscr{T}^1(\boldsymbol{M})$ is always a well defined element of $\mathscr{T}^{n-1}(\boldsymbol{M})$).

Analogously to the previous case, we wish to define an algebra $\mathscr{T}(\boldsymbol{M})$ comprising elements of the form

$$T = \sum_{n=0}^{N} t_n \tag{3.37}$$

with $t_n \in \mathscr{T}^n(\boldsymbol{M})$, so that at the level of vector spaces we have $\mathscr{T}(\boldsymbol{M}) = \bigoplus_{n=0}^{\infty} \mathscr{T}^n(\boldsymbol{M})$. For any $f \in C^\infty(\boldsymbol{M})$ and T of the above form we define the functional derivative $T^{(k)}[f]$ in the same way as detailed in (3.11) and (3.12). It may be shown using [38, Thm. 8.2.12] that the functional derivative $T^{(k)}[f]$ is an element of $\mathscr{T}^k(\boldsymbol{M})$.

It is shown in [14] that for any $t \in \mathscr{T}^n(\boldsymbol{M})$, the wavefront set of

$$(E^{\pm}_{\boldsymbol{M}})_k t := \mathbf{1}^{\otimes k-1} \otimes E^{\pm}_{\boldsymbol{M}} \otimes \mathbf{1}^{\otimes n-k} t \tag{3.38}$$

has the property that $WF((E^{\pm}_{\boldsymbol{M}})_k t) \cap \overline{(V^+_{\boldsymbol{M}})^{\times n} \cup (V^-_{\boldsymbol{M}})^{\times n}} = \emptyset$. Since differential operators and multiplication by smooth functions cannot enlarge the wavefront set of a distribution, it follows that any element of $\mathcal{E}'(\boldsymbol{M}^{\times n})$ which is obtained via application of any such operators and $(E^{\pm}_{\boldsymbol{M}})_k$ on an element of $\mathscr{T}^n(\boldsymbol{M})$ must itself be an element of $\mathscr{T}^n(\boldsymbol{M})$.

Unfortunately, the restriction on elements of $\mathscr{T}^n(\boldsymbol{M})$ alone does not solve the problem of ill-defined distributions. Note that for any $g \in C_0^\infty(\boldsymbol{M})$, the distribution $t_2(x, y) = g(x)\delta(x - y)$ has empty wavefront set, and is therefore an element of $\mathscr{T}^2(\boldsymbol{M})$; however

$$(t_2 \star t_2)[f] = \int_{\boldsymbol{M}^{\times 2}} dx\, dy\, t(x)t(y) \bigg(f(x)^2 f(y)^2 + 2iE_{\boldsymbol{M}}(x, y)f(x)g(y) - \frac{1}{2} E_{\boldsymbol{M}}(x, y)^2 \bigg), \tag{3.39}$$

and the distribution $E_{\boldsymbol{M}}(x, y)^2$ is ill-defined since it does not obey Hörmander's criterion. A solution to this problem is given in [9]: on each spacetime $\boldsymbol{M}$, it is possible to find a real symmetric 'Hadamard'[3] bidistribution H which

[3]The terminology here relates to the *Hadamard condition* on states ω of the algebra $\mathscr{A}(\boldsymbol{M})$, which is shown in [52] to be equivalent to the microlocal spectrum condition, that

satisfies the property

$$WF(E_{\boldsymbol{M}} + 2iH) = WF(E_{\boldsymbol{M}}) \cap (V^+_{\boldsymbol{M}} \times V^-_{\boldsymbol{M}}). \tag{3.40}$$

There is no canonical choice for such an H on a spacetime $\boldsymbol{M} \in \mathsf{Loc}$, but for any pair H, H' of possible candidates, the difference $H - H'$ is smooth [9, Theorem 6]. For reasons that will become clear, we also demand that H be a bisolution; in other words, that

$$H(P_{\boldsymbol{M}} f, f') = 0 = H(f, P_{\boldsymbol{M}} f') \tag{3.41}$$

for all $f, f' \in C_0^\infty(\boldsymbol{M})$. We denote by $\mathscr{H}(\boldsymbol{M})$ the set of all possible such H on a spacetime $\boldsymbol{M}$.

The k-fold pointwise product $(E_{\boldsymbol{M}} + 2iH)^k$ is well defined for any $k \geq 1$ and $H \in \mathscr{H}(\boldsymbol{M})$, and consequently we define a new product $\star_H$ that acts on distributions as

$$T \star_H T' = \boldsymbol{\mu} \exp\left(\frac{i}{2} \mathcal{E}_{\boldsymbol{M};H}\right) (T \otimes T'), \tag{3.42}$$

where in exact analogue to the $\star$ product we define, for $t_m \in \mathscr{T}^m(\boldsymbol{M})$ and $t_n \in \mathscr{T}^n(\boldsymbol{M})$,

$$\begin{aligned}\mathcal{E}_{\boldsymbol{M};H}&(t_m \otimes t_n)(x_1, \ldots, x_{m-1}; y_1, \ldots, y_{n-1}) \\ &:= mn \int_{\boldsymbol{M}^{\times 2}} d^2 z\, (E_{\boldsymbol{M}}(z_1, z_2) + 2iH(z_1, z_2)) t_m(z_1, \boldsymbol{x}) t_n(z_2, \boldsymbol{y}),\end{aligned} \tag{3.43}$$

with $\mathcal{E}_{\boldsymbol{M};H}(\alpha, T) = 0 = \mathcal{E}_{\boldsymbol{M};H}(T, \alpha)$ for $\alpha \in \mathscr{T}^0(\boldsymbol{M})$. We then denote by $\mathscr{T}_H(\boldsymbol{M})$ the algebra whose underlying vector space is $\bigoplus_{n=0}^\infty \mathscr{T}^n(\boldsymbol{M})$, with product $\star_H$. We may again write

$$T \star_H T' := \sum_{k=0}^{\min(O(T), O(T'))} \frac{i^k}{2^k k!} \mathcal{E}^k_{\boldsymbol{M};H}(T \otimes T'), \tag{3.44}$$

states that the wavefront set of the 2-point function satisfies $WF(\omega_2) \subset V^+_{\boldsymbol{M}} \times V^-_{\boldsymbol{M}}$ (other axioms imply that $WF(\omega_2)$ is also in fact a subset of $WF(E_{\boldsymbol{M}})$). We do not, however, require H to be the 2-point function of a state.

and associativity may be shown with an exactly analogous argument to that in Lemma B.1. Addition and involution on $\mathscr{T}_H(\boldsymbol{M})$ are again given by addition and complex conjugation of distributions respectively.

The canonical embedding of $C_0^\infty(\boldsymbol{M})$ in $\mathcal{E}'(\boldsymbol{M})$ induces an embedding $\iota_H : \mathscr{F}(\boldsymbol{M}) \to \mathscr{T}_H(\boldsymbol{M})$. For any $H \in \mathscr{H}(\boldsymbol{M})$, there is an Alg-arrow $\lambda_H : \mathscr{F}(\boldsymbol{M}) \to \mathscr{T}_H(\boldsymbol{M})$ defined by

$$\lambda_H = \boldsymbol{\iota}_H \exp\left(-\frac{1}{2}\eta_H\right), \tag{3.45}$$

where $\eta_H : \mathscr{F}(\boldsymbol{M}) \to \mathscr{F}(\boldsymbol{M})$ is defined for $t_n \in \mathscr{F}^n(\boldsymbol{M})$, $n \geq 2$, by

$$\eta_H(t_n)(x_1, \ldots, x_{n-2}) := n(n-1) \int_{\boldsymbol{M}^{\times 2}} H(y_1, y_2) t_n(y_1, y_2, x_1, \ldots, x_{n-2}), \tag{3.46}$$

with $\eta_H|_{\mathscr{F}^0(\boldsymbol{M}) \cup \mathscr{F}^1(\boldsymbol{M})} = 0$. Note that every element of $\mathscr{F}(\boldsymbol{M})$ is therefore annihilated by a sufficiently high power of η_H, so there are no convergence issues with (3.45); we may alternatively write[4]

$$(\lambda_H T)[f] := \sum_{k=0}^{\lfloor O(T)/2 \rfloor} \frac{(-1)^k}{2^k k!} \boldsymbol{\iota}_H \eta_H^k(T) \tag{3.47}$$

Then λ_H is an arrow in Alg, i.e. an injective unit-preserving $*$-homomorphism; this result is proved in Lemma B.3. We may also show that the choice of H does not affect the algebra, in the sense that for any pair $H, H' \in \mathscr{H}(\boldsymbol{M})$, the algebras $\mathscr{T}_H(\boldsymbol{M})$ and $\mathscr{T}_{H'}(\boldsymbol{M})$ are isomorphic [9]. We define the map $\lambda_{H,H'} : \mathscr{T}_H(\boldsymbol{M}) \to \mathscr{T}_{H'}(\boldsymbol{M})$ by

$$\lambda_{H,H'} T := \sum_{k=0}^{\lfloor O(T)/2 \rfloor} \frac{1}{2^k k!} \boldsymbol{\iota}_{H,H'} (\eta_H - \eta_{H'})^k (T), \tag{3.48}$$

where $\boldsymbol{\iota}_{H,H'} : \mathscr{T}_H(\boldsymbol{M}) \to \mathscr{T}_{H'}(\boldsymbol{M})$ is the identity map on the underlying vector spaces of $\mathscr{T}_H(\boldsymbol{M})$ and $\mathscr{T}_{H'}(\boldsymbol{M})$, which are equal. Note that $\eta_H - \eta_{H'}$, defined according to (3.46), is well defined on elements of $\mathscr{T}_H(\boldsymbol{M}) \otimes \mathscr{T}_H(\boldsymbol{M})$ because as already mentioned, the difference $H - H'$ is smooth. Again, we

[4] Note the missing factor of 2^{-k} from (3.45), (3.48) in [9, 24].

may write

$$\lambda_{H,H'} = \boldsymbol{\iota}_{H,H'} \exp\left(\frac{1}{2}(\eta_H - \eta_{H'})\right);\tag{3.49}$$

since η_H and $\eta_{H'}$ themselves commute on their domain of definition we may alternatively write this as

$$\lambda_{H,H'} = \boldsymbol{\iota}_{H,H'} \exp\left(\frac{1}{2}\eta_H\right) \exp\left(-\frac{1}{2}\eta_{H'}\right),\tag{3.50}$$

where the exponentiated operators are defined as formal series only (η_H is not well defined, on its own, on $\mathscr{T}_H(\boldsymbol{M})$). It is clear that if we suppress the embedding maps, it holds formally in terms of the defining expressions for λ_H that

$$\lambda_{H,H'} = \lambda_{H'}\lambda_H^{-1}.\tag{3.51}$$

Lemma 3.2.4. *The maps $\lambda_{H,H'}$ are $*$-isomorphisms in* Alg*, and*

$$\lambda_{H,H} = \mathrm{id}_{\mathscr{T}_H(\boldsymbol{M})}, \quad \lambda_{H,H'} = \lambda_{H',H}^{-1}, \quad \lambda_{H',H''} \circ \lambda_{H,H'} = \lambda_{H,H''}\tag{3.52}$$

for any $H, H', H'' \in \mathscr{H}(\boldsymbol{M})$.

Proof. The relations in (3.52) are evident from (3.51); the $*$-homomorphism property may be deduced using an identical argument to that in Lemma B.3, and the existence of an inverse guarantees that $\lambda_{H,H'}$ is an isomorphism. □

Notice in particular that the collection $(\lambda_{H,H'})_{H,H'\in\mathscr{H}(\boldsymbol{M})}$ is a cocycle, and each $\lambda_{H,H'}$ is unambiguously defined (unlike the spin cocycles introduced in Section 6.2.4).

In exactly the same way that the set $\mathscr{J}(\boldsymbol{M})$ is an ideal for $\mathscr{F}(\boldsymbol{M})$, it also holds that the analogous set

$$\widetilde{\mathscr{J}}(\boldsymbol{M}) = \{T \in \mathscr{T}_H(\boldsymbol{M}) : T[E_{\boldsymbol{M}} f] = 0 \text{ for all } f \in C_0^\infty(\boldsymbol{M})\}\tag{3.53}$$

(which is independent of the choice of $H \in \mathscr{H}(\boldsymbol{M})$) is an ideal for $\mathscr{T}_H(\boldsymbol{M})$.[5] We therefore define the algebra $\mathscr{W}_H(\boldsymbol{M}) = \mathscr{T}_H(\boldsymbol{M})/\widetilde{\mathscr{J}}(\boldsymbol{M})$. Since the equiv-

[5]This only holds if H is a bisolution; this introduces certain problems with the construction of covariant fields, which will be discussed below.

alence class of an element $T \in \mathscr{T}_H(\boldsymbol{M})$ does not depend on H, we will denote it unambiguously by $[T]_{\boldsymbol{M}}$, and if $T - T' \in \widetilde{\mathscr{J}}(\boldsymbol{M})$ we will write $T \sim_{\boldsymbol{M}} T'$ as before. It follows from (3.41) that $T \sim_{\boldsymbol{M}} T'$ if and only if $\lambda_{H,H'}T \sim_{\boldsymbol{M}} \lambda_{H,H'}T'$, so the isomorphism

$$\begin{aligned} \widetilde{\lambda}_{H,H'} : \mathscr{W}_H(\boldsymbol{M}) &\to \mathscr{W}_{H'}(\boldsymbol{M}) \\ [T]_{\boldsymbol{M}} &\mapsto [\lambda_{H,H'}T]_{\boldsymbol{M}} \end{aligned} \tag{3.54}$$

is well defined. We also note that the reasoning used to prove Lemma 3.1.2 can be similarly used to show the corresponding result; that if $T \in \mathscr{T}_H(\boldsymbol{M})$ can be written $T = \sum_{n=0}^{N} t_n$ with $t_n \in \mathscr{T}^n(\boldsymbol{M})$ for each n, then $T \in \widetilde{\mathscr{J}}(\boldsymbol{M})$ if and only if $t_0 = 0$ and

$$E_{\boldsymbol{M}}{}^{\otimes n} t_n = 0 \tag{3.55}$$

for all $n = 1, \ldots, N$.

Since there is no preferred method of uniquely specifying some $H \in \mathscr{H}(\boldsymbol{M})$ for each spacetime $\boldsymbol{M}$, the above construction does not constitute a locally covariant theory, as we have not yet defined a unique algebra for each $\boldsymbol{M}$. We therefore wish to construct an algebra $\mathscr{W}(\boldsymbol{M})$ which is independent of the choice of H. Again following [9], we do this by letting $\mathscr{W}(\boldsymbol{M})$ comprise families of elements indexed by choice of $H \in \mathscr{H}(\boldsymbol{M})$, as follows:

$$\mathscr{W}(\boldsymbol{M}) = \{(W_H)_{H \in \mathscr{H}(\boldsymbol{M})} : \widetilde{\lambda}_{H,H'} W_H = W_{H'} \text{ for all } H, H' \in \mathscr{H}(\boldsymbol{M})\}. \tag{3.56}$$

Given $W = (W_H)_{H \in \mathscr{H}(\boldsymbol{M})}$, $W' = (W'_H)_{H \in \mathscr{H}(\boldsymbol{M})}$, we define $(W + W')_H = W_H + W'_H$, $(W \star W')_H = W_H \star_H W'_H$ and $(W^*)_H = W^*_H$. These operations are clearly consistent with the compatibility condition $\widetilde{\lambda}_{H,H'} W_H = W_{H'}$. Since this condition also ensures that each family $W = (W_H)_{H \in \mathscr{H}(\boldsymbol{M})} \in \mathscr{W}(\boldsymbol{M})$ is completely defined by any single entry W_H, it follows that $\mathscr{W}(\boldsymbol{M}) \cong \mathscr{W}_H(\boldsymbol{M})$ for any $H \in \mathscr{H}(\boldsymbol{M})$; in fact, it may easily be seen that the map $W \mapsto W_H$ is an arrow in Alg for each H.

3.2.3 The locally covariant theory of the enlarged algebra of Wick polynomials

The construction of the arrows of the enlarged theory is not given in [9]; there are a number of slight subtleties involved, so we will give a full description here. We use the same definitions as in (3.24) and (3.25) for the pullback $\psi^* : \mathscr{T}^n(\boldsymbol{N}) \to \mathscr{T}^n(\boldsymbol{M})$ and push-forward $\psi_* : \mathscr{T}^n(\boldsymbol{M}) \to \mathscr{T}^n(\boldsymbol{M})$ given a Loc-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$; again, the pullback is defined as a partial function whose domain coincides with the image of ψ_*. There are, however, slight impediments to defining the pullback and push-forward directly on the algebras $\mathscr{T}_H(\boldsymbol{M})$, due to the requirement that some H be chosen on each spacetime. We may also extend the definitions (3.24) and (3.25) to (total) maps $\psi^* : \mathcal{D}'(\boldsymbol{N}) \to \mathcal{D}'(\boldsymbol{M})$ and $\psi_* : \mathcal{E}'(\boldsymbol{M}) \to \mathcal{E}'(\boldsymbol{N})$. Note that (3.6) is then equivalent to $\psi^* E_{\boldsymbol{N}} = E_{\boldsymbol{M}}$.

Lemma 3.2.5. *Let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be an arrow in Loc. Then for any $H \in \mathscr{H}(\boldsymbol{N})$, we have $\psi^* H \in \mathscr{H}(\boldsymbol{M})$.*

Proof. We have $WF(\phi^* t) \subset \phi^* WF(t)$ for any smooth $\phi : \boldsymbol{M} \to \boldsymbol{N}$ and distribution t on $\boldsymbol{N}$ [37, Theorem 2.5.11′]. It is a clear consequence that we have equality whenever ϕ is a local diffeomorphism; this entails that when $\psi : \boldsymbol{M} \to \boldsymbol{N}$ is an arrow in Loc, we have $WF(\psi^* T) = \psi^* WF(T)$ for any $T \in \mathcal{D}'(\boldsymbol{N}^{\times n})$. Therefore

$$\begin{aligned} WF(E_{\boldsymbol{M}} + 2i\psi^* H) &= WF(\psi^*(E_{\boldsymbol{N}} + 2iH)) \\ &= \psi^* WF(E_{\boldsymbol{N}} + 2iH) \\ &= WF(E_{\boldsymbol{M}}) \cap (V^+_{\boldsymbol{M}} \times V^-_{\boldsymbol{M}}). \end{aligned} \tag{3.57}$$

Moreover, if $H(P_{\boldsymbol{N}} f, f') = 0$ for all $f, f' \in C_0^\infty(\boldsymbol{N})$, it follows that

$$\psi^* H(P_{\boldsymbol{N}} f, f') = H(P_{\boldsymbol{N}} \psi_* f, \psi_* f') = 0 \tag{3.58}$$

for all $f, f' \in C_0^\infty(\boldsymbol{M})$. Therefore $\psi^* H \in \mathscr{H}(\boldsymbol{M})$. □

Note that for any Loc-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$, we also have $WF(\psi_* U) =$

$\psi_* WF(U)$ for $U \in \mathcal{E}'(\boldsymbol{M}^{\times n})$.[6]

Now, for any $H \in \mathscr{H}(\boldsymbol{N})$ we define $\mathscr{T}_H(\psi)$ to be a map from $\mathscr{T}_{\psi^* H}(\boldsymbol{M})$ to $\mathscr{T}_H(\boldsymbol{N})$ with the same action as ψ_*; i.e. for any $\sum_{n=0}^N t_n \in \mathscr{T}_{\psi^* H}(\boldsymbol{M})$ such that $t_n \in \mathscr{T}^n(\boldsymbol{M})$, we define $\mathscr{T}_H(\psi)T := \sum_{n=0}^N \psi_* t_n$. Since t_n is compactly supported for each $n \geq 1$, we have $WF(\psi_* t_n) = \psi_* WF(t_n)$. Thus $\mathscr{T}_H(\psi)T$ is an element of $\mathscr{T}_H(\boldsymbol{N})$ as required. An alternative and equivalent definition is given by

$$(\mathscr{T}_H(\psi)T)[f] = T[\psi^* f]. \tag{3.59}$$

It is easy to see that $\mathscr{T}_H(\psi)$ is then a $*$-monomorphism.

Lemma 3.2.6. *Let $\boldsymbol{M}, \boldsymbol{N}$ be objects in* Loc *with $H \in \mathscr{H}(\boldsymbol{N})$, and let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be a* Loc*-arrow. Then, for any $T, T' \in \mathscr{T}_{\psi^* H}(\boldsymbol{M})$ it holds that $T \sim_{\boldsymbol{M}} T'$ if and only if $\mathscr{T}_H(\psi)T \sim_{\boldsymbol{N}} \mathscr{T}_H(\psi)T'$.*

Proof. If $\mathscr{T}_H(\psi)T \sim_{\boldsymbol{N}} \mathscr{T}_H(\psi)T'$ then for every $g \in C_0^\infty(\boldsymbol{N})$ we have

$$(\mathscr{T}_H(\psi)T)[E_{\boldsymbol{N}} g] = (\mathscr{T}_H(\psi)T')[E_{\boldsymbol{N}} g]. \tag{3.60}$$

Now, for every $f \in C_0^\infty(\boldsymbol{M})$ it holds that $E_{\boldsymbol{M}} f = \psi^* E_{\boldsymbol{N}} \psi_* f$; since $\psi_* f \in C_0^\infty(\boldsymbol{N})$, it follows that

$$T[E_{\boldsymbol{M}} f] = (\mathscr{T}_H(\psi)T)[E_{\boldsymbol{N}} \psi_* f] = (\mathscr{T}_H(\psi)T')[E_{\boldsymbol{N}} \psi_* f] = T'[E_{\boldsymbol{M}} f]. \tag{3.61}$$

Therefore $T \sim_{\boldsymbol{M}} T'$.

Now suppose that $T \sim_{\boldsymbol{M}} T'$. Since $O(T)$, $O(T')$ are finite, it follows that there is a compact region $K \subset \boldsymbol{M}$ with the property that the support of the n^{th} components of both T and T' lie within $K^{\times n}$ for $1 \leq n \leq \max(O(T), O(T'))$. Let $\Sigma_{\boldsymbol{M}}$ be a Cauchy surface for $\boldsymbol{M}$, and consider the intersection $S = J_{\boldsymbol{M}}(K) \cap \Sigma_{\boldsymbol{M}}$; for any classical solution $E_{\boldsymbol{N}} f$, $f \in C_0^\infty(\boldsymbol{N})$, it will always be possible to pick a smooth pair of functions (φ_f, π_f) on $\Sigma_{\boldsymbol{M}}$ which are compactly supported and coincide with the Cauchy data for $\psi^* E_{\boldsymbol{N}} f$ on S (even if $\psi(\Sigma_{\boldsymbol{M}})$ cannot be extended to a Cauchy surface for $\boldsymbol{N}$). But since

[6] We require compact support of U here; if $U \in \mathcal{D}'(\boldsymbol{M})$, then we might not have equality, although $(x_1, \ldots, x_n; k_1, \ldots, k_n) \in WF(\psi_* U) \setminus \psi_* WF(U)$ only if $x_k \in \partial(\psi(\boldsymbol{M}))$ for each k.

(φ_f, π_f) are compactly supported they provide data for a solution $E_{\boldsymbol{M}} g$, for some $g \in C_0^\infty(\boldsymbol{M})$. It then holds that $E_{\boldsymbol{M}} g$ must coincide with $\psi^* E_{\boldsymbol{N}} f$ on the domain of determinacy of S; since this region contains K, it holds that $(E_{\boldsymbol{M}} g)|_K = (\psi^* E_{\boldsymbol{N}} f)|_K$. It follows that

$$\begin{aligned}(\mathscr{T}_H(\psi)T)[E_{\boldsymbol{N}} f] &= (\mathscr{T}_H(\psi)T)[\psi_* E_{\boldsymbol{M}} g] = T[E_{\boldsymbol{M}} g] \\ &= T'[E_{\boldsymbol{M}} g] = (\mathscr{T}_H(\psi)T')[\psi_* E_{\boldsymbol{M}} g] = (\mathscr{T}_H(\psi)T')[E_{\boldsymbol{N}} f]. \quad (3.62)\end{aligned}$$

Since the choice of $f \in C_0^\infty(\boldsymbol{N})$ was arbitrary, we conclude that $\mathscr{T}_H(\psi)T \sim_{\boldsymbol{N}} \mathscr{T}_H(\psi)T'$. □

Definition 3.2.7. *Let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be an arrow in* Loc*, and $H \in \mathscr{H}(\boldsymbol{N})$. We define*

$$\begin{aligned}\mathscr{W}_H(\psi) : \mathscr{W}_{\psi^* H}(\boldsymbol{M}) &\to \mathscr{W}_H(\boldsymbol{N}) \\ [T]_{\boldsymbol{M}} &\mapsto [\mathscr{T}_H(\psi)T]_{\boldsymbol{N}}. \quad (3.63)\end{aligned}$$

This is a well defined Alg*-arrow, due to the previous lemma and the fact that $\mathscr{T}_H(\psi)$ is also an arrow in* Alg*. We also define the map $\mathscr{W}(\psi) : \mathscr{W}(\boldsymbol{M}) \to \mathscr{W}(\boldsymbol{N})$ by*

$$(\mathscr{W}(\psi)W)_H := \mathscr{W}_H(\psi) W_{\psi^* H}, \quad (3.64)$$

where $H \in \mathscr{H}(\boldsymbol{N})$.

Lemma 3.2.8. *The definition* (3.64) *is consistent with the compatibility condition on $\mathscr{W}(\boldsymbol{M})$ and $\mathscr{W}(\boldsymbol{N})$, and $\mathscr{W}$:* Loc → Alg *is a covariant functor.*

Proof. First, we must show that for any pair $H, H' \in \mathscr{H}(\boldsymbol{N})$, we have $\widetilde{\lambda}_{H,H'}(\mathscr{W}(\psi)W)_H = (\mathscr{W}(\psi)W)_{H'}$. If $W = (W_H)_{H \in \mathscr{H}(\boldsymbol{M})} \in \mathscr{W}(\boldsymbol{M})$ with $W_H = [T_H]_{\boldsymbol{M}}$, then

$$\widetilde{\lambda}_{H,H'}(\mathscr{W}(\psi)W)_H = \widetilde{\lambda}_{H,H'}[\mathscr{T}_H(\psi)T_{\psi^* H}]_{\boldsymbol{N}} = [\lambda_{H,H'}\mathscr{T}_H(\psi)T_{\psi^* H}]_{\boldsymbol{N}}. \quad (3.65)$$

On the other hand, $(\mathscr{W}(\psi)W)_{H'} = [\mathscr{T}_{H'}(\psi)\lambda_{\psi^* H, \psi^* H'} T_{\psi^* H}]_{\boldsymbol{N}}$, so we need only show that $\lambda_{H,H'} \circ \mathscr{T}_H(\psi) = \mathscr{T}_{H'}(\psi) \circ \lambda_{\psi^* H, \psi^* H'}$; this is easy to see once we note that $(\eta_H - \eta_{H'})(\psi_* t_n) = \psi_*((\eta_{\psi^* H} - \eta_{\psi^* H'})(t_n))$ which is clear from the definition (3.46).

Moving to functoriality of $\mathscr{W}$, it is trivial to show that for any spacetime $\boldsymbol{M}$, we have $\mathscr{W}(\mathrm{id}_{\boldsymbol{M}}) = \mathrm{id}_{\mathscr{W}(\boldsymbol{M})}$. It remains to show that for any Loc-arrows $\psi_1 : \boldsymbol{L} \to \boldsymbol{M}$, $\psi_2 : \boldsymbol{M} \to \boldsymbol{N}$, it holds that $\mathscr{W}(\psi_2) \circ \mathscr{W}(\psi_1) = \mathscr{W}(\psi_2 \circ \psi_1)$. For any $T \in \mathscr{T}_{\psi_1^* \psi_2^* H}(\boldsymbol{L})$ and $H \in \mathscr{H}(\boldsymbol{N})$, we have

$$\mathscr{T}_H(\psi_2)\mathscr{T}_{\psi_2^* H}(\psi_1)T = T \circ \psi_1^* \circ \psi_2^* = \mathscr{T}_H(\psi_2 \circ \psi_1)T. \tag{3.66}$$

The desired result follows by (3.63), (3.64). □

The covariant functor $\mathscr{W}$ is thus a locally covariant theory which represents the extended algebra of Wick polynomials. We also have the corresponding result to Lemma 3.1.8:

Lemma 3.2.9. *Let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ in Loc. Then $W \in \mathscr{W}(\psi)(\mathscr{W}(\boldsymbol{M}))$ if and only if any $T \in \mathscr{T}_H(\boldsymbol{N})$ satisfying $W_H = [T]_{\boldsymbol{N}}$ for some $H \in \mathscr{H}(\boldsymbol{N})$ has the property that $T[E_{\boldsymbol{M}} f] = T[0]$ for every $f \in C_0^\infty(\boldsymbol{N})$ such that $\mathrm{supp}(f) \cap J_{\boldsymbol{N}}(\psi(\boldsymbol{M})) = \emptyset$. Moreover, the theory $\mathscr{W}$ is causal.*

Proof. $W \in \mathscr{W}(\psi)(\mathscr{W}(\boldsymbol{N}))$ if and only if we have $W_H \in \mathscr{W}_H(\psi)(\mathscr{W}_{\psi^* H}(\boldsymbol{M}))$ for some (and consequently every) $H \in \mathscr{H}(\boldsymbol{N})$; the required results then follow using an analogous argument to that given in the proof of Lemma 3.1.8. □

3.3 Locally covariant Wick powers

We will now discuss the construction of the local Wick powers of the field Φ in a locally covariant way. This has already been done in [36, 46], albeit with a different construction of the extended algebras $\mathscr{W}(\boldsymbol{M})$. For a given $\psi : \boldsymbol{M} \to \boldsymbol{N}$ in Loc, recall that $C_0^\infty : \mathsf{Loc} \to \mathsf{Vect}_{\mathbb{C}}$ may be regarded as a covariant functor if we let $C_0^\infty(\psi) := \psi_*$. It is then easy to see that the transformation $\varphi : C_0^\infty \stackrel{.}{\longrightarrow} \mathscr{F}$ given by

$$\varphi_{\boldsymbol{M}}(t) : f \mapsto \int_{\boldsymbol{M}} dx\, t(x) f(x) \tag{3.67}$$

is natural; consequently $\Phi : C_0^\infty \stackrel{\cdot}{\longrightarrow} \mathscr{A}$ given by $\Phi_{\boldsymbol{M}}(t) := [\varphi_{\boldsymbol{M}}(t)]_{\boldsymbol{M}}$ is also natural, since it is the composition of φ with the natural transformation $[\,\cdot\,]$. We therefore have a locally covariant field Φ that generates the algebra $\mathscr{A}(\boldsymbol{M})$ on a given spacetime $\boldsymbol{M}$. We may similarly define a locally covariant field on $\mathscr{W}$, although this is slightly more complicated due to the construction of $\mathscr{W}(\boldsymbol{M})$.

Firstly, note that for any $H, H' \in \mathscr{H}(\boldsymbol{M})$, the restriction of $\lambda_{H,H'}$ to $\mathscr{T}^1(\boldsymbol{M})$ acts as the identity map. If $t \in \mathscr{T}^1(\boldsymbol{M})$, we therefore have $(W_H)_{H\in\mathscr{H}(\boldsymbol{M})} \in \mathscr{W}(\boldsymbol{M})$ when $W_H = [t]_{\boldsymbol{M}}$ for all H. Recall that $\mathscr{F}(\boldsymbol{M})$ has a canonical embedding $\boldsymbol{\iota}_H$ into $\mathscr{T}_H(\boldsymbol{M})$, and therefore $\boldsymbol{\iota}_H\varphi_{\boldsymbol{M}}(t) \in \mathscr{T}_H(\boldsymbol{M})$ for any $t \in C_0^\infty(\boldsymbol{M})$ and $H \in \mathscr{H}(\boldsymbol{M})$. We define $\widetilde{\Phi}_{\boldsymbol{M}} : C_0^\infty(\boldsymbol{M}) \to \mathscr{W}(\boldsymbol{M})$ by $(\widetilde{\Phi}_{\boldsymbol{M}}(t))_H := [\boldsymbol{\iota}_H\varphi_{\boldsymbol{M}}(t)]_{\boldsymbol{M}}$ for each H; it is then easy to see that $\widetilde{\Phi} : C_0^\infty \stackrel{\cdot}{\longrightarrow} \mathscr{W}$ is a locally covariant field.

However, we may define further locally covariant fields on $\mathscr{W}$ that correspond to the traditional Wick powers of $\widetilde{\Phi}$. Given a spacetime $\boldsymbol{M}$, the Wick powers of the field $\widetilde{\Phi}_{\boldsymbol{M}}$ involve in their construction a specific choice of Hadamard bidistribution satisfying the wavefront set condition (3.40). As a first attempt, we use a bisolution $H \in \mathscr{H}(\boldsymbol{M})$ in the construction (although as we will see, this is not sufficient for local covariance); it turns out that for any $t \in C_0^\infty(\boldsymbol{M})$ and $n \geq 2$, the H-normal ordered Wick power $:\widetilde{\Phi}^n_{\boldsymbol{M}}(t):_H$ corresponds in the algebra $\mathscr{W}_H(\boldsymbol{M})$ to the equivalence class $[:\varphi^n_{\boldsymbol{M}}(t):_H]_{\boldsymbol{M}}$, where $:\varphi^n_{\boldsymbol{M}}(t):_H \in \mathscr{T}_H(\boldsymbol{M})$ is defined by

$$\begin{aligned} :\varphi^n_{\boldsymbol{M}}(t):_H[f] &:= \int_{\boldsymbol{M}} dx\, t(x)(f(x))^n \\ &= \int_{\boldsymbol{M}} d^n x\, t(x_1)\delta(x_1,\ldots,x_n)f(x_1)\cdots f(x_n). \end{aligned} \tag{3.68}$$

However, there is of course no requirement that the distribution used to construct the Wick power must coincide with the bisolution used to construct the algebra; by (3.56), the Wick power $:\widetilde{\Phi}^n_{\boldsymbol{M}}(t):_H$ must correspond in the algebra $\mathscr{W}_{H'}(\boldsymbol{M})$ to $\widetilde{\lambda}_{H,H'}([:\varphi^n_{\boldsymbol{M}}(t):_H]_{\boldsymbol{M}}) = [\lambda_{H,H'}:\varphi^n_{\boldsymbol{M}}(t):_H]_{\boldsymbol{M}}$. We denote $:\varphi^n_{\boldsymbol{M}}(t):_{H,H'} = \lambda_{H,H'}:\varphi^n_{\boldsymbol{M}}(t):_H$. In the case of the Wick square, for example, we may calculate explicitly

$$(:\varphi^2_{\boldsymbol{M}}(t):_{H,H'})[f] = \int_{\boldsymbol{M}} dx\, dy\, t(x)\delta(x,y)\left(f(x)f(y) + H(x,y) - H'(x,y)\right). \tag{3.69}$$

In [36], it was observed that any definition of the Wick square of this type will conflict with local covariance. We find the same limitation here; using a bisolution for the construction requires a specific choice $H_{\boldsymbol{M}}$ for each spacetime, and for any Loc-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$ it holds that

$$:\Phi^2_{\boldsymbol{N}}(\psi_* t):_{H_{\boldsymbol{N}}} - \mathscr{W}(\psi) :\Phi^2_{\boldsymbol{M}}(t):_{H_{\boldsymbol{M}}} = D_\psi(t)\mathbf{1}_{\mathscr{W}(\boldsymbol{N})}, \tag{3.70}$$

where

$$D_\psi(t) = \int_{\boldsymbol{M}} dx\, t(x)\Big((\psi^* H_{\boldsymbol{N}})(x,x) - H_{\boldsymbol{M}}(x,x)\Big). \tag{3.71}$$

Naturality then requires that $\psi^* H_{\boldsymbol{N}} = H_{\boldsymbol{M}}$ (at least on the diagonal); it was shown in [36] that it is not possible to satisfy this simultaneously for all possible $\psi, \boldsymbol{M}$ and $\boldsymbol{N}$.

The solution, suggested by Bernard Kay and detailed in [36], is to use a covariantly constructed Hadamard *parametrix* which we denote $H^{\mathrm{par}}_{\boldsymbol{M}}$. This is a particular bidistribution satisfying (3.40), defined uniquely up to an arbitrary length scale (which is fixed and constant over all spacetimes),[7] but which is not a bisolution (so not an element of $\mathscr{H}(\boldsymbol{M})$). In non-analytic spacetimes $\boldsymbol{M}$, this parametrix can only be defined in a neighbourhood of the diagonal $\Delta_2(\boldsymbol{M})$, where $\Delta_n(\boldsymbol{M}) = \{(x,\ldots,x) : x \in \boldsymbol{M}\} \subset \boldsymbol{M}^{\times n}$. In our setting, the construction runs as follows: for any $H, H' \in \mathscr{H}(\boldsymbol{M})$, we may regard $:\varphi^n_{\boldsymbol{M}}(t):_{H,H'}$ simply as a formal expression in H, H' and t, and then replace every instance of H by $H^{\mathrm{par}}_{\boldsymbol{M}}$. This process yields an expression $:\varphi^n_{\boldsymbol{M}}(t):_{H^{\mathrm{par}}_{\boldsymbol{M}},H'}$ which turns out to be a well-defined element of $\mathscr{T}_{H'}(\boldsymbol{M})$; the restrictions in defining $H^{\mathrm{par}}_{\boldsymbol{M}}$ are not problematic since the distribution $t(x_1)\delta(x_1,\ldots,x_n)$ has support contained in $\Delta_n(\boldsymbol{M})$.

Having fixed the arbitrary length scale, we then define the field $\widetilde{\Phi}^n_{\boldsymbol{M}}$:

[7]Note that the reliance of the definition of $H^{\mathrm{par}}_{\boldsymbol{M}}$ on the value of a universal length scale introduces a renormalization ambiguity that cannot be resolved in the setting of locally covariant QFT. For more details, see [63, 62].

$C_0^\infty(\boldsymbol{M}) \to \mathscr{W}(\boldsymbol{M})$ by $(\widetilde{\Phi}^n_{\boldsymbol{M}}(t))_H := [:\varphi^n_{\boldsymbol{M}}(t):_{H^{\rm par}_{\boldsymbol{M}},H}]_{\boldsymbol{M}}$; since the covariance condition $\psi^* H^{\rm par}_{\boldsymbol{M}} = H^{\rm par}_{\boldsymbol{N}}$ can be satisfied in a suitable neighbourhood of $\Delta_2(\boldsymbol{N})$ simultaneously for all $\boldsymbol{N}, \boldsymbol{M}$ and $\psi : \boldsymbol{M} \to \boldsymbol{N}$ [36], it follows that this construction is locally covariant. However, note that the algebra $\mathscr{W}(\boldsymbol{M})$ is not generated by $\widetilde{\Phi}^n_{\boldsymbol{M}}(t)$ for $t \in C_0^\infty(\boldsymbol{M})$ and $n \geq 1$; for example, for any vector field $v \in \Gamma^\infty(T\boldsymbol{M})$ and smooth $t \in \mathscr{F}^1(\boldsymbol{M})$, the functional $T(f) = \int_{\boldsymbol{M}^{\times 2}} dx\, dy\, \tau(x,y) f(x) f(y)$ where

$$\tau(x,y) = t(x)((\mathbf{1} \otimes \nabla_v^2)\delta)(x,y) \tag{3.72}$$

is an element of $\mathscr{T}^2(\boldsymbol{M})$, yet cannot be expressed as a finite sum of products of $\widetilde{\Phi}_{\boldsymbol{M}}$ and its Wick powers.

3.4 Spaces of smooth functions on spacetimes

Before we consider the timeslice axiom and dynamical locality of the two theories, we discuss the following spaces of smooth functions on $\boldsymbol{M}$, in addition to $C_0^\infty(\boldsymbol{M})$ and $C^\infty(\boldsymbol{M})$. We define

$$\begin{aligned} C_s^\infty(\boldsymbol{M}) &= \{f \in C^\infty(\boldsymbol{M}) : \mathrm{supp}(f) \subset J_{\boldsymbol{M}}(K) \text{ for some compact } K \subset \boldsymbol{M}\}, \\ C_{s,\pm}^\infty(\boldsymbol{M}) &= \{f \in C_s^\infty(\boldsymbol{M}) : \mathrm{supp}(f) \subset J^\pm_{\boldsymbol{M}}(K) \text{ for some compact } K \subset \boldsymbol{M}\}. \end{aligned} \tag{3.73}$$

We also introduce the following notation for the canonical embeddings

$$\begin{aligned} \iota_{0,\pm} &: C_0^\infty(\boldsymbol{M}) \hookrightarrow C_{s,\pm}^\infty(\boldsymbol{M}), \\ \iota_{\pm,s} &: C_{s,\pm}^\infty(\boldsymbol{M}) \hookrightarrow C_s^\infty(\boldsymbol{M}), \\ \iota_{s,\infty} &: C_s^\infty(\boldsymbol{M}) \hookrightarrow C^\infty(\boldsymbol{M}). \end{aligned} \tag{3.74}$$

We wish to demonstrate that there exist continuous maps $\hat{E}^\pm_{\boldsymbol{M}} : C_0^\infty(\boldsymbol{M}) \to C_{s,\pm}^\infty(\boldsymbol{M})$ that satisfy $E^\pm_{\boldsymbol{M}} = \iota_{s,\infty} \circ \iota_{\pm,s} \circ \hat{E}^\pm_{\boldsymbol{M}}$. As it is clear that for any $f \in C_0^\infty(\boldsymbol{M})$, the function $E^\pm_{\boldsymbol{M}} f$ lies within the range of $\iota_{s,\infty} \circ \iota_{\pm,s}$, we may unambiguously let $\hat{E}^\pm_{\boldsymbol{M}} = (\iota_{s,\infty} \circ \iota_{\pm,s})^{-1} \circ E^\pm_{\boldsymbol{M}}$.

To establish continuity we must first define the topologies on each of these spaces of functions. The spaces $C^\infty(\boldsymbol{M})$ and $C_0^\infty(\boldsymbol{M})$ can be constructed as convex topological spaces, following [64, 54]. A *compact exhausting sequence* for $\boldsymbol{M}$ is a sequence $(K_n)_{n\in\mathbb{N}}$ of compact submanifolds of $\boldsymbol{M}$ such that $K_n \subset \mathring{K}_{n+1}$ for each n, and for every point $p \in \boldsymbol{M}$ there exists $N \in \mathbb{N}$ such that $p \in K_n$ for all $n > N$. Any space of smooth functions on a smooth manifold can be endowed with the C^∞ topology; we do not need to go into details here, except to say that the topology on $C^\infty(\boldsymbol{M})$ is generated by seminorms $p_{K_n,k}$, $k, n \in \mathbb{N}$, where $(K_n)_{n\in\mathbb{N}}$ is a compact exhausting sequence for $\boldsymbol{M}$, and $p_{K_n,k}(f)$ is given by the supremum over K_n of the norms of all covariant derivatives of f of order no greater than k (using a Riemannian metric to induce the norms of the derivatives). The C^∞ topology on a space of smooth functions on $\boldsymbol{M}$ is then defined as the subspace topology induced from $C^\infty(\boldsymbol{M})$.

The topology of $C_0^\infty(\boldsymbol{M})$, on the other hand, is constructed as an inductive limit of the topological spaces $C_{K_n}^\infty(\boldsymbol{M})$ (that is, the finest topology such that each embedding $\iota_n : C_{K_n}^\infty(\boldsymbol{M}) \hookrightarrow C_0^\infty(\boldsymbol{M})$ is continuous), where $(K_n)_{n\in\mathbb{N}}$ is again a compact exhausting sequence for $\boldsymbol{M}$, and $C_K^\infty(\boldsymbol{M})$ is the space $\{f \in C^\infty(\boldsymbol{M}) : \mathrm{supp}(f) \subset K\}$ endowed with the C^∞ topology. Now, for any inductive limit X of locally convex spaces $(X_n)_{n\in\mathbb{N}}$, and locally convex space Y, a map $T : X \to Y$ is continuous if and only if each restriction $T|_{X_n} : X_N \to Y$ is continuous [54, Theorem V.16]. Since the space $C_{K_n}^\infty(\boldsymbol{M})$ inherits the subspace topology induced from $C^\infty(\boldsymbol{M})$, it follows that the embedding $C_0^\infty(\boldsymbol{M}) \hookrightarrow C^\infty(\boldsymbol{M})$ is continuous.

For a given spacetime $\boldsymbol{M} \in \mathsf{Loc}$ we wish to endow $C_s^\infty(\boldsymbol{M})$ and $C_{s,\pm}^\infty(\boldsymbol{M})$ with topologies in a similar way to that given for $C_0^\infty(\boldsymbol{M})$ in [64, 54]; starting with a compact exhausting sequence $(K_n)_{n\in\mathbb{N}}$ for $\boldsymbol{M}$, we consider the topological spaces $C_{J_{\boldsymbol{M}}(K_n)}^\infty(\boldsymbol{M})$ and $C_{J_{\boldsymbol{M}}^\pm(K_n)}^\infty(\boldsymbol{M})$ defined analogously to $C_{K_n}^\infty(\boldsymbol{M})$, and let $C_s^\infty(\boldsymbol{M})$ and $C_{s,\pm}^\infty(\boldsymbol{M})$ be the inductive limit of $C_{J_{\boldsymbol{M}}(K_n)}^\infty(\boldsymbol{M})$ and $C_{J_{\boldsymbol{M}}^\pm(K_n)}^\infty(\boldsymbol{M})$ respectively as $n \to \infty$. We then have:

Lemma 3.4.1. *Let $\boldsymbol{M} \in \mathsf{Loc}$. The embeddings $\iota_{0,\pm} : C_0^\infty(\boldsymbol{M}) \hookrightarrow C_{s,\pm}^\infty(\boldsymbol{M})$, $\iota_{\pm,s} : C_{s,\pm}^\infty(\boldsymbol{M}) \hookrightarrow C_s^\infty(\boldsymbol{M})$ and $\iota_{s,\infty} : C_s^\infty(\boldsymbol{M}) \hookrightarrow C^\infty(\boldsymbol{M})$ are all continuous in the relevant topologies.*

Proof. For the sake of readable notation, we denote $X_n = C^\infty_{K_n}(\boldsymbol{M})$, $Y_n^\pm = C^\infty_{J^\pm_{\boldsymbol{M}}(K_n)}(\boldsymbol{M})$ and $Z_n = C^\infty_{J_{\boldsymbol{M}}(K_n)}(\boldsymbol{M})$. First, we consider $\iota_{s,\infty}$: for any $n \in \mathbb{N}$, the space Z_n is endowed with the subspace topology induced from $C^\infty(\boldsymbol{M})$, so the embedding must be continuous; therefore $\iota_{s,\infty}|_{Z_n} : Z_n \hookrightarrow C^\infty(\boldsymbol{M})$ is continuous for all n, which is sufficient for continuity of $\iota_{s,\infty}$. Now, for each n we may factorise $\iota_{\pm,s}|_{Y_n}$ as the composition of the embeddings of $Y_n^\pm \hookrightarrow Z_n$ and $Z_n \hookrightarrow C^\infty_s(\boldsymbol{M})$; the former is continuous as $Y_n^\pm$ has the subspace topology induced from Z_n, and the latter is continuous by definition of $C^\infty_s(\boldsymbol{M})$. Therefore $\iota_{\pm,s}$ is continuous. Similarly, we may factorise $\iota_{0,\pm}|_{X_n}$ as the composition of the embeddings of $X_n \hookrightarrow Y_n^\pm$ and $Y_n^\pm \hookrightarrow C^\infty_{s,\pm}(\boldsymbol{M})$, both of which are continuous. Therefore $\iota_{0,\pm}$ is continuous. □

This also allows us to prove:

Lemma 3.4.2. *The maps $\hat{E}^\pm_{\boldsymbol{M}} : C^\infty_0(\boldsymbol{M}) \to C^\infty_{s,\pm}(\boldsymbol{M})$ are continuous.*

Proof. We recall that if a topological space Y is endowed with the subspace topology from a space Z, and the embedding is denoted $\iota : Y \hookrightarrow Z$, then a map $T : X \to Y$ is continuous if and only if $\iota \circ T$ is continuous. We note that $X_n = C^\infty_{K_n}(\boldsymbol{M})$ has the subspace topology induced from $C^\infty(\boldsymbol{M})$; since $E^\pm_{\boldsymbol{M}} : C^\infty_0(\boldsymbol{M}) \to C^\infty(\boldsymbol{M})$ is continuous, it follows that the restrictions $E^\pm_{\boldsymbol{M}}|_{X_n} : X_n \to C^\infty(\boldsymbol{M})$ are all continuous. Denoting the canonical embedding by $\iota_n : X_n \hookrightarrow C^\infty(\boldsymbol{M})$, it is clear that we may factorise $E^\pm_{\boldsymbol{M}}|_{X_n} = \iota_n \circ \hat{E}^\pm_{\boldsymbol{M}}|_{X_n}$, so each $\hat{E}^\pm_{\boldsymbol{M}}|_{X_n}$ is continuous. Therefore $\hat{E}^\pm_{\boldsymbol{M}}$ is continuous. □

We define

$$\begin{aligned} \hat{E}_{\boldsymbol{M}} : C^\infty_0(\boldsymbol{M}) &\to C^\infty_s(\boldsymbol{M}) \\ f &\mapsto \iota_{s,-}(\hat{E}^-_{\boldsymbol{M}} f) - \iota_{s,+}(\hat{E}^+_{\boldsymbol{M}} f), \end{aligned} \tag{3.75}$$

which is clearly continuous; we also define $\check{E}_{\boldsymbol{M}} : (C^\infty_s(\boldsymbol{M}))' \to \mathcal{D}'(\boldsymbol{M})$ by $\check{E}_{\boldsymbol{M}} = -(\hat{E}_{\boldsymbol{M}})'$. The map $P_{\boldsymbol{M}}$ may be considered to act on elements of $C^\infty_s(\boldsymbol{M})$ and $C^\infty_{s,\pm}(\boldsymbol{M})$ in the obvious way, from which we see that strictly speaking $P_{\boldsymbol{M}} \hat{E}^\pm_{\boldsymbol{M}} f = \iota_{0,\pm} = \hat{E}^\pm_{\boldsymbol{M}} P_{\boldsymbol{M}} f$ for any $f \in C^\infty_0(\boldsymbol{M})$.

We say that a distribution $t \in \mathcal{D}'(\boldsymbol{M}^{\times n})$ is *time-compact* if there exist spacelike Cauchy surfaces $\Sigma^\pm \subset \boldsymbol{M}$ such that $\mathrm{supp}(t) \subset (J^-_{\boldsymbol{M}}(\Sigma^+) \cap$

$J^-_{\boldsymbol{M}}(\Sigma^-))^{\times n}$. Note that the action of a time-compact distribution t is well-defined on $f \in C^\infty_s(\boldsymbol{M}^{\times n})$, since the intersection $\mathrm{supp}(t) \cap \mathrm{supp}(f)$ is compact. Therefore any time-compact distribution can be considered to be an element of $(C^\infty_s(\boldsymbol{M}^{\times n}))'$. We also say that a distribution t is *future-compact* if there exists a Cauchy surface $\Sigma \subset \boldsymbol{M}$ such that $\mathrm{supp}(t) \subset (J^-_{\boldsymbol{M}}(\Sigma))^{\times n}$, and *past-compact* if there exists a Cauchy surface $\Sigma \subset \boldsymbol{M}$ such that $\mathrm{supp}(t) \subset (J^+_{\boldsymbol{M}}(\Sigma))^{\times n}$. We may similarly see that a future-/past-compact distribution can be considered to be an element of $(C^\infty_{s,\pm}(\boldsymbol{M}^{\times n}))'$.

We now state the following result, which will be important later:

Lemma 3.4.3. *Let $u \in \mathcal{D}'(\boldsymbol{M})$, with $u[P_{\boldsymbol{M}} f] = 0$ for all $f \in C^\infty_0(\boldsymbol{M})$. Then there exists a distribution $t \in (C^\infty_s(\boldsymbol{M}))'$ such that $u = \check{E}_{\boldsymbol{M}} t$.*

Before we prove this, we give a useful definition:

Definition 3.4.4. *Let $\boldsymbol{M}$ be an object of* Loc*. A* Cauchy partition *for $\boldsymbol{M}$ is a triple $(\Sigma^{\mathrm{adv}}, \Sigma^{\mathrm{ret}}, \chi)$ where $\Sigma^{\mathrm{adv/ret}}$ are disjoint Cauchy surfaces for $\boldsymbol{M}$ such that $\Sigma^{\mathrm{ret}} \subset J^+_{\boldsymbol{M}}(\Sigma^{\mathrm{adv}})$, and where $\chi \in C^\infty(\boldsymbol{M})$ satisfies $\chi(x) = 0$ for $x \in J^+_{\boldsymbol{M}}(\Sigma^{\mathrm{ret}})$ and $\chi(x) = 1$ for $x \in J^-_{\boldsymbol{M}}(\Sigma^{\mathrm{adv}})$. A* Cauchy partition function *(c.p.f.) is such a $\chi \in C^\infty(\boldsymbol{M})$ that arises as part of a Cauchy partition.*

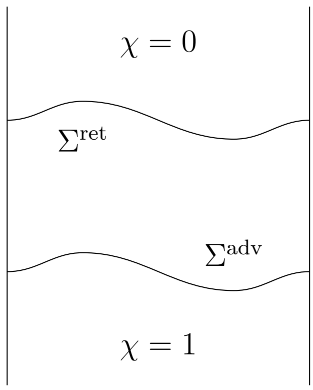


Figure 5: Diagram of a Cauchy partition

Proof of Lemma 3.4.3. Let $(\Sigma^{\mathrm{adv}}, \Sigma^{\mathrm{ret}}, \chi^{\mathrm{adv}})$ be a Cauchy partition for $\boldsymbol{M}$, and denote $\chi^{\mathrm{ret}} = 1 - \chi^{\mathrm{adv}}$. Now, let $\tau \in C^\infty(\boldsymbol{M})$ be time-compact, and defined such that $\tau(x) = 1$ for all $x \in J^+_{\boldsymbol{M}}(\widetilde{\Sigma}^{\mathrm{adv}}) \cap J^-_{\boldsymbol{M}}(\widetilde{\Sigma}^{\mathrm{ret}})$, where $\widetilde{\Sigma}^{\mathrm{adv/ret}} \subset \boldsymbol{M}$

are further Cauchy surfaces disjoint from $\Sigma^{\mathrm{adv/ret}}$ with $\Sigma^{\mathrm{adv/ret}} \subset J^{\pm}_{\boldsymbol{M}}(\widetilde{\Sigma}^{\mathrm{adv/ret}})$ (so that $\Sigma^{\mathrm{adv/ret}}$ lie within the region enclosed by $\widetilde{\Sigma}^{\mathrm{adv/ret}}$). We define the map $\tau_s : C^\infty_s(\boldsymbol{M}) \to C^\infty_0(\boldsymbol{M})$ to be the action of pointwise multiplication by τ. We also consider $\chi^{\mathrm{adv/ret}} : C^\infty_0(\boldsymbol{M}) \to C^\infty_0(\boldsymbol{M})$ as defined by action of multiplication. The operator $P_{\boldsymbol{M}}$ can be considered as an endomorphism acting on any of the spaces of functions we defined above; we may similarly consider it as an endomorphism on any of the dual spaces in question, by

$$\langle P_{\boldsymbol{M}} u, f\rangle = \langle u, P_{\boldsymbol{M}} f\rangle. \tag{3.76}$$

We may then show that $u = \check{E}_{\boldsymbol{M}}\tau'_s P_{\boldsymbol{M}}(\chi^{\mathrm{adv}})'u$, as follows. Let $f \in C^\infty_0(\boldsymbol{M})$ be arbitrary, then

$$\begin{aligned}
&(\check{E}_{\boldsymbol{M}}\tau'_s P_{\boldsymbol{M}}(\chi^{\mathrm{adv}})'u)[f] && (3.77)\\
&\quad = -(\tau'_s P_{\boldsymbol{M}}(\chi^{\mathrm{adv}})'u)[\hat{E}_{\boldsymbol{M}} f]\\
&\quad = (\tau'_s P_{\boldsymbol{M}}(\chi^{\mathrm{adv}})'u)[\iota_{+,s}\hat{E}^{+}_{\boldsymbol{M}} f] - (\tau'_s P_{\boldsymbol{M}}(\chi^{\mathrm{adv}})'u)[\iota_{-,s}\hat{E}^{-}_{\boldsymbol{M}} f]\\
&\quad = u[\chi^{\mathrm{adv}} P_{\boldsymbol{M}}\tau_s\iota_{+,s}\hat{E}^{+}_{\boldsymbol{M}} f] - u[\chi^{\mathrm{adv}} P_{\boldsymbol{M}}\tau_s\iota_{-,s}\hat{E}^{-}_{\boldsymbol{M}} f]. && (3.78)
\end{aligned}$$

Since $u[P_{\boldsymbol{M}}\tau_s\iota_{-,s}\hat{E}^{-}_{\boldsymbol{M}} f] = 0$ by assumption, we may use $\chi^{\mathrm{adv}} = \mathbf{1} - \chi^{\mathrm{ret}}$ to see that

$$(\check{E}_{\boldsymbol{M}}\tau'_s P_{\boldsymbol{M}}(\chi^{\mathrm{adv}})'u)[f] \;=\; u[\chi^{\mathrm{adv}} P_{\boldsymbol{M}}\tau_s\iota_{+,s}\hat{E}^{+}_{\boldsymbol{M}} f] + u[\chi^{\mathrm{ret}} P_{\boldsymbol{M}}\tau_s\iota_{-,s}\hat{E}^{-}_{\boldsymbol{M}} f]. \tag{3.79}$$

Now any $g \in C^\infty_{s,+}(\boldsymbol{M})$ can be split into a sum of three smooth functions g_-, g_0 and g_+, with the properties that $\mathrm{supp}(g_\pm) \subset J^{\pm}_{\boldsymbol{M}}(\Sigma^{\mathrm{ret/adv}})$ and $\mathrm{supp}(g_0) \subset J^{-}_{\boldsymbol{M}}(\widetilde{\Sigma}^{\mathrm{ret}}) \cap J^{+}_{\boldsymbol{M}}(\widetilde{\Sigma}^{\mathrm{adv}})$. We may note that $\mathrm{supp}(g_-)$ and $\mathrm{supp}(g_0)$ are both compact, so we can consider g_0 and g_- as elements of $C^\infty_0(\boldsymbol{M})$, whereupon

$$g = \iota_{0,+}g_- + \iota_{0,+}g_0 + g_+. \tag{3.80}$$

By construction, we have $\tau_s\iota_{+,s}\iota_{0,+}g_0 = g_0$; the definition of χ^{adv} also shows that $\chi^{\mathrm{adv}}T_1 g_- = T_1 g_-$ and $\chi^{\mathrm{adv}}T_2 g_+ = 0$ for any operators $T_1 : C^\infty_0(\boldsymbol{M}) \to C^\infty_0(\boldsymbol{M})$, $T_2 : C^\infty_{s,+}(\boldsymbol{M}) \to C^\infty_0(\boldsymbol{M})$ such that $\mathrm{supp}(T_i f) \subset \mathrm{supp}(f)$ for all f,

$i = 1, 2$. It follows directly from these that if we let $g = \hat{E}^+_{\boldsymbol{M}} f$ and split as described, then

$$\begin{aligned}\chi^{\mathrm{adv}} P_{\boldsymbol{M}} \tau_s \iota_{+,s} \hat{E}^+_{\boldsymbol{M}} f \\ &= \chi^{\mathrm{adv}} P_{\boldsymbol{M}} \tau_s \iota_{+,s} \iota_{0,+} g_- + \chi^{\mathrm{adv}} P_{\boldsymbol{M}} \tau_s \iota_{+,s} \iota_{0,+} g_0 + \chi^{\mathrm{adv}} P_{\boldsymbol{M}} \tau_s \iota_{+,s} g_+ \\ &= P_{\boldsymbol{M}} \tau_s \iota_{+,s} \iota_{0,+} g_- + \chi^{\mathrm{adv}} P_{\boldsymbol{M}} g_0. \end{aligned} \tag{3.81}$$

But $f = (\iota_{0,+})^{-1} P_{\boldsymbol{M}} \hat{E}^+_{\boldsymbol{M}} f = P_{\boldsymbol{M}} g_0 + P_{\boldsymbol{M}} g_- + (\iota_{0,+})^{-1} P_{\boldsymbol{M}} g_+$, so by the properties of χ^{adv} we have

$$\begin{aligned}\chi^{\mathrm{adv}} P_{\boldsymbol{M}} \tau_s \iota_{+,s} \hat{E}^+_{\boldsymbol{M}} f \\ &= P_{\boldsymbol{M}} \tau_s \iota_{+,s} \iota_{0,+} g_- + \chi^{\mathrm{adv}} f - \chi^{\mathrm{adv}} P_{\boldsymbol{M}} g_- - \chi^{\mathrm{adv}} (\iota_{0,+})^{-1} P_{\boldsymbol{M}} g_+ \\ &= P_{\boldsymbol{M}} (\tau_s \iota_{+,s} \iota_{0,+} - \mathbf{1}) g_- + \chi^{\mathrm{adv}} f. \end{aligned} \tag{3.82}$$

Since u is a weak solution and $(\tau_s \iota_{+,s} \iota_{0,+} - \mathbf{1}) g_-$ is compactly supported, we have

$$u[\chi^{\mathrm{adv}} P_{\boldsymbol{M}} \tau_s \iota_{+,s} \hat{E}^+_{\boldsymbol{M}} f] = u[\chi^{\mathrm{adv}} f]. \tag{3.83}$$

We may similarly conclude that $u[\chi^{\mathrm{ret}} P_{\boldsymbol{M}} \tau_s \iota_{-,s} \hat{E}^-_{\boldsymbol{M}} f] = u[\chi^{\mathrm{ret}} f]$. It follows from (3.79) that

$$\begin{aligned}(\check{E}_{\boldsymbol{M}} \tau'_s P_{\boldsymbol{M}} (\chi^{\mathrm{adv}})' u)[f] &= u[\chi^{\mathrm{adv}} f] + u[\chi^{\mathrm{ret}} f] \\ &= u[f]. \end{aligned} \tag{3.84}$$

This proves the required result, and also gives us an explicit example of a distribution $t \in (C^\infty_s(\boldsymbol{M}))'$ satisfying $u = \check{E}_{\boldsymbol{M}} t$. □

While we have been very careful with our definitions in this section, in the following chapter we will not need to be so exact with our notation. Firstly, we make the observation that since any multiplication operator μ between spaces of smooth functions is formally self-adjoint, it makes sense to write $\mu' t = \mu t$ for a distribution t and formally regard μt as the pointwise product of t with the underlying function $\mu \in C^\infty(\boldsymbol{M})$. We will particularly use this convention when a distributional solution u is of the form $u = E_{\boldsymbol{M}} t$, where $t \in \mathcal{E}'(\boldsymbol{M})$.

Lemma 3.4.3 tells us that $E_{\boldsymbol{M}}t = \check{E}_{\boldsymbol{M}}\tau_s' P_{\boldsymbol{M}}(\chi^{\mathrm{adv}})' E_{\boldsymbol{M}}t = \check{E}_{\boldsymbol{M}}\tau P_{\boldsymbol{M}}\chi^{\mathrm{adv}} E_{\boldsymbol{M}}t$; however, regarding $\chi^{\mathrm{adv}} E_{\boldsymbol{M}}t$ as a pointwise product allows us to see that in fact the distribution $P_{\boldsymbol{M}}\chi^{\mathrm{adv}} E_{\boldsymbol{M}}t$ must be supported within the region $J^-_{\boldsymbol{M}}(\Sigma^{\mathrm{ret}}) \cap J^+_{\boldsymbol{M}}(\Sigma^{\mathrm{adv}})$ where χ^{adv} is non-constant, by the properties of $P_{\boldsymbol{M}}$ and $E_{\boldsymbol{M}}$. Moreover, the support of $P_{\boldsymbol{M}}\chi^{\mathrm{adv}} E_{\boldsymbol{M}}t$ lies within $J_{\boldsymbol{M}}(\mathrm{supp}(t))$, which has compact intersection with $J^-_{\boldsymbol{M}}(\Sigma^{\mathrm{ret}}) \cap J^+_{\boldsymbol{M}}(\Sigma^{\mathrm{adv}})$, so the support of $P_{\boldsymbol{M}}\chi^{\mathrm{adv}} E_{\boldsymbol{M}}t$ is compact. Since $\tau = 1$ everywhere within $\mathrm{supp}(P_{\boldsymbol{M}}\chi^{\mathrm{adv}} E_{\boldsymbol{M}}t)$, we may suppress τ and instead regard $P_{\boldsymbol{M}}\chi^{\mathrm{adv}} E_{\boldsymbol{M}}t$ itself as an element of $\mathcal{E}'(\boldsymbol{M})$, writing

$$E_{\boldsymbol{M}}P_{\boldsymbol{M}}\chi^{\mathrm{adv}} E_{\boldsymbol{M}}t = E_{\boldsymbol{M}}t. \tag{3.85}$$

Moreover $P_{\boldsymbol{M}}E_{\boldsymbol{M}}t = 0$ for any $t \in \mathcal{E}'(\boldsymbol{M})$, so we also have

$$E_{\boldsymbol{M}}P_{\boldsymbol{M}}\chi^{\mathrm{ret}} E_{\boldsymbol{M}}t = -E_{\boldsymbol{M}}t. \tag{3.86}$$

Chapter 4

Dynamical locality of scalar field theories

In this chapter we will explore the concept of dynamical locality for the theories defined in the previous chapter. In order to do so, we will provide proofs that both theories obey the timeslice axiom; this is already known to hold for both $\mathscr{A}$ [11] and $\mathscr{W}$ [14], albeit with different constructions; we include proofs both for completeness and in order to obtain explicit expressions for the inverse maps $\mathscr{A}(\psi)^{-1}$ and $\mathscr{W}(\psi)^{-1}$ when $\psi : \boldsymbol{M} \to \boldsymbol{N}$ is a Cauchy arrow. We will then give expressions for the relative Cauchy evolutions on $\mathscr{A}(\boldsymbol{M})$ and $\mathscr{W}(\boldsymbol{M})$ induced by a given metric perturbation in $H(\boldsymbol{M})$; this is relatively straightforward for $\mathscr{A}$ but involves some subtleties for $\mathscr{W}$ due to the underlying choices of bisolution $H \in \mathscr{H}(\boldsymbol{M})$. Finally, we demonstrate dynamical locality for the theory $\mathscr{A}$ in all cases and for the theory $\mathscr{W}$ in some special cases.

4.1 The timeslice axiom

4.1.1 The scalar field theory

In order to compute the relative Cauchy evolution for either $\mathscr{A}$ or $\mathscr{W}$, we must first demonstrate that they obey the timeslice axiom. It is worth asking first whether the theory $\mathscr{F}$ obeys the timeslice axiom; since the construction for $\mathscr{F}$ contains no condition relating to the field equation, we should not expect $\mathscr{F}$ to obey the axiom, and indeed this is the case. Let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be a Cauchy arrow in Loc, and suppose that $\psi(\boldsymbol{M}) \neq \boldsymbol{N}$; then, pick some nonzero $t \in \mathscr{F}^1(\boldsymbol{N})$ whose support lies within $\boldsymbol{N} \setminus \psi(\boldsymbol{M})$. Clearly $t[\bar{t}] = \int_{\boldsymbol{N}} dx\, |t(x)|^2 \neq 0$, but as $\psi^* \bar{t} = 0$, we have $(\mathscr{F}(\psi)F)[\bar{t}] = 0$ for all $F \in \mathscr{F}(\boldsymbol{M})$. Therefore $\mathscr{F}(\psi)$ is not surjective, and consequently cannot be invertible; hence $\mathscr{F}$ does not satisfy the timeslice axiom.

To demonstrate that $\mathscr{A}$, on the other hand, does obey the timeslice axiom, we use the following lemma, which is proved in [19] (and can also be seen to be a consequence of Lemma 3.4.3: see (3.85)). This is a standard result, but given its importance we include a proof for the sake of completeness.

Lemma 4.1.1. *Let χ^{adv} be a Cauchy partition function on $\boldsymbol{M}$ with $\chi^{\mathrm{ret}} = 1 - \chi^{\mathrm{adv}}$. Then $P_{\boldsymbol{M}} \chi^{\mathrm{adv/ret}} E_{\boldsymbol{M}} f \in C_0^\infty(\boldsymbol{M})$ for all $f \in C_0^\infty(\boldsymbol{M})$, and*

$$\begin{aligned} E_{\boldsymbol{M}} P_{\boldsymbol{M}} \chi^{\mathrm{adv}} E_{\boldsymbol{M}} f &= E_{\boldsymbol{M}} f, \\ E_{\boldsymbol{M}} P_{\boldsymbol{M}} \chi^{\mathrm{ret}} E_{\boldsymbol{M}} f &= -E_{\boldsymbol{M}} f. \end{aligned} \tag{4.1}$$

Proof. We deal with compact support first: as both χ^{adv} and χ^{ret} are constant outside some time-compact slice of $\boldsymbol{M}$, the function $P_{\boldsymbol{M}} \chi^{\mathrm{adv/ret}} E_{\boldsymbol{M}} f$ must be identically zero outside this slice. Since its support is also contained within $J_{\boldsymbol{M}}(\mathrm{supp}(f))$, which has compact intersection with a time-compact slice, we must have $P_{\boldsymbol{M}} \chi^{\mathrm{adv/ret}} E_{\boldsymbol{M}} f \in C_0^\infty(\boldsymbol{M})$.

Now, fix $f \in C_0^\infty(\boldsymbol{M})$, and let χ be a second c.p.f. such that $\chi(x) = 1$ for all $x \in J^-_{\boldsymbol{M}}(\mathrm{supp}(f))$. Then both $(\chi^{\mathrm{adv}} - \chi)(E_{\boldsymbol{M}} f)$ and $\chi E^+_{\boldsymbol{M}} f$ are compactly supported, and are therefore annihilated by $E_{\boldsymbol{M}} P_{\boldsymbol{M}}$. Consequently

$$\begin{aligned} E_{\boldsymbol{M}} P_{\boldsymbol{M}} \chi^{\mathrm{adv}} E_{\boldsymbol{M}} f = E_{\boldsymbol{M}} P_{\boldsymbol{M}} \chi E_{\boldsymbol{M}} f &= E_{\boldsymbol{M}} P_{\boldsymbol{M}} \chi E^-_{\boldsymbol{M}} f \\ &= E_{\boldsymbol{M}} P_{\boldsymbol{M}} E^-_{\boldsymbol{M}} f = E_{\boldsymbol{M}} f. \end{aligned} \tag{4.2}$$

The remaining equality is obtained by noting that $E_{\boldsymbol{M}} P_{\boldsymbol{M}} (\chi^{\mathrm{adv}} + \chi^{\mathrm{ret}}) E_{\boldsymbol{M}} f = E_{\boldsymbol{M}} P_{\boldsymbol{M}} E_{\boldsymbol{M}} f = 0$. □

Now, if we fix some Cauchy partition $(\Sigma^{\mathrm{adv}}, \Sigma^{\mathrm{ret}}, \chi^{\mathrm{adv}})$ for $\boldsymbol{M}$ and define

$\zeta t := P_{\boldsymbol{M}}\chi^{\mathrm{adv}}E_{\boldsymbol{M}}t$ for $t \in \mathscr{F}^1(\boldsymbol{M})$, it follows directly that for any $t_n \in \mathscr{F}^n(\boldsymbol{M})$, $n \geq 1$, we have

$$\mathrm{supp}(\zeta^{\otimes n}t_n) \subset (J^+_{\boldsymbol{M}}(\Sigma^{\mathrm{adv}}) \cap J^-_{\boldsymbol{M}}(\Sigma^{\mathrm{ret}}))^{\times n} \cap \mathrm{supp}(E_{\boldsymbol{M}}{}^{\otimes n}t_n). \tag{4.3}$$

Clearly $\zeta^{\otimes n}$ maps elements of $\mathscr{F}^n(\boldsymbol{M})$ to elements of $\mathscr{F}^n(\boldsymbol{M})$. We also note that by Lemma 4.1.1, we have

$$\begin{aligned}\zeta^{\otimes n}t[E_{\boldsymbol{M}}f] &= (-1)^n(E_{\boldsymbol{M}}\zeta)^{\otimes n}t[f]\\ &= (-1)^n E_{\boldsymbol{M}}{}^{\otimes n}t[f]\\ &= t[E_{\boldsymbol{M}}f]\end{aligned} \tag{4.4}$$

for any $t \in \mathscr{F}^n(\boldsymbol{M})$, $n \geq 1$ and $f \in C_0^\infty(\boldsymbol{M})$. It follows that if we define

$$\begin{aligned} Z : \mathscr{F}(\boldsymbol{M}) &\to \mathscr{F}(\boldsymbol{M})\\ \sum_{n=0}^{N} t_n &\mapsto \sum_{n=0}^{N}\zeta^{\otimes n}t_n \qquad (t_n \in \mathscr{F}^n(\boldsymbol{M})),\end{aligned} \tag{4.5}$$

we have $ZF \sim_{\boldsymbol{M}} F$ for all $F \in \mathscr{F}(\boldsymbol{M})$.

Lemma 4.1.2. *The theory $\mathscr{A}$ obeys the timeslice axiom.*

Proof. Suppose that $\psi : \boldsymbol{M} \to \boldsymbol{N}$ is a Cauchy arrow in Loc. We will always be able to find two disjoint Cauchy surfaces for $\boldsymbol{N}$ in $\psi(\boldsymbol{M})$; we denote the Cauchy surface to the past by Σ^{adv} and the one to the future by Σ^{ret}, and define the operator Z as above using these Cauchy surfaces for the Cauchy partition; it follows that for any $F \in \mathscr{F}(\boldsymbol{N})$, the n^{th} component of ZF is supported in $\psi(\boldsymbol{M})^{\times n}$ for each $n \geq 1$. We may then define

$$\begin{aligned}\mathscr{G}(\psi) : \mathscr{F}(\boldsymbol{N}) &\to \mathscr{F}(\boldsymbol{M})\\ F &\mapsto \psi^* ZF.\end{aligned} \tag{4.6}$$

For any $F \in \mathscr{F}(\boldsymbol{N})$ and $f \in C^\infty(\boldsymbol{N})$, we have

$$(\mathscr{F}(\psi)\mathscr{G}(\psi)F)[f] = (\mathscr{F}(\psi)\psi^* ZF)[f] = (\psi_*\psi^* ZF)[f] = (ZF)[f], \tag{4.7}$$

since the n^{th} component of ZF must be supported in $\psi(\boldsymbol{M})^{\times n}$. Therefore $\mathscr{F}(\psi)\mathscr{G}(\psi)F = ZF$. Now suppose that $F \in \mathscr{F}(\boldsymbol{M})$ and $f \in C^\infty(\boldsymbol{M})$; then,

$$(\mathscr{G}(\psi)\mathscr{F}(\psi)F)[f] = \mathscr{G}(\psi)(\psi_* F)[f] = \psi^* Z(\psi_* F)[f]. \tag{4.8}$$

Writing $F = \sum_{n=0}^{N} t_n$, with $t_n \in \mathscr{F}^n(\boldsymbol{M})$, we have

$$\psi^* Z(\psi_* F) = \sum_{n=0}^{N} \psi^* \zeta^{\otimes n} \psi_* t_n. \tag{4.9}$$

But notice that for any $t \in \mathscr{F}^1(\boldsymbol{M})$, $f \in C_0^\infty(\boldsymbol{M})$, we have

$$\begin{aligned}(\psi^* \zeta \psi_* t)[E_{\boldsymbol{M}} f] &= (P_{\boldsymbol{M}} \psi^* (\chi^{\text{adv}} E_{\boldsymbol{N}} \psi_* t))[E_{\boldsymbol{M}} f] \\ &= (P_{\boldsymbol{M}}((\psi^* \chi^{\text{adv}}) E_{\boldsymbol{M}} t))[E_{\boldsymbol{M}} f] = t[E_{\boldsymbol{M}} f]\end{aligned} \tag{4.10}$$

by (4.4) and Lemma 4.1.1. We have therefore shown that $\mathscr{F}(\psi)\mathscr{G}(\psi)F \sim_{\boldsymbol{N}} F$ for all $F \in \mathscr{F}(\boldsymbol{N})$, and $\mathscr{G}(\psi)\mathscr{F}(\psi)F \sim_{\boldsymbol{M}} F$ for all $F \in \mathscr{F}(\boldsymbol{M})$.

Next, we observe that if $F, F' \in \mathscr{F}(\boldsymbol{N})$ with $F \sim_{\boldsymbol{N}} F'$, then we have $\mathscr{F}(\psi)\mathscr{G}(\psi)F \sim_{\boldsymbol{N}} \mathscr{F}(\psi)\mathscr{G}(\psi)F'$; a simple consequence of this is that $\mathscr{G}(\psi)F \sim_{\boldsymbol{M}} \mathscr{G}(\psi)F'$. This means that the map

$$\begin{aligned}\mathscr{B}(\psi) : \mathscr{A}(\boldsymbol{N}) &\to \mathscr{A}(\boldsymbol{M}) \\ [F]_{\boldsymbol{N}} &\mapsto [\mathscr{G}(\psi)F]_{\boldsymbol{M}}\end{aligned} \tag{4.11}$$

is well defined, and we can conclude that $\mathscr{B}(\psi) \circ \mathscr{A}(\psi) = \text{id}_{\mathscr{A}(\boldsymbol{M})}$, and $\mathscr{A}(\psi) \circ \mathscr{B}(\psi) = \text{id}_{\mathscr{A}(\boldsymbol{N})}$. Therefore $\mathscr{A}(\psi)$ is invertible, and so $\mathscr{A}$ obeys the timeslice axiom. □

4.1.2 The enlarged algebra

We now proceed to the timeslice axiom for $\mathscr{W}$, adapting the proof given for an equivalent construction in [14] for the formulation used here.

Lemma 4.1.3. *The theory $\mathscr{W}$ obeys the timeslice axiom.*

Proof. Suppose that $\psi : \boldsymbol{M} \to \boldsymbol{N}$ is a Cauchy arrow in Loc. We choose some c.p.f. χ^{adv} for $\boldsymbol{N}$, and again define $\zeta t = P_{\boldsymbol{N}}\chi^{\mathrm{adv}}E_{\boldsymbol{N}}t$ for any $t \in \mathscr{T}^1(\boldsymbol{N})$. For any $H \in \mathscr{H}(\boldsymbol{N})$, we let

$$\begin{aligned} Z_H : \mathscr{T}_H(\boldsymbol{N}) &\to \mathscr{T}_H(\boldsymbol{N}) \\ \sum_{n=0}^{N} t_n &\mapsto \sum_{n=0}^{N} \zeta^{\otimes n} t_n \qquad (t_n \in \mathscr{T}^n(\boldsymbol{N})), \end{aligned} \tag{4.12}$$

then by (3.85), $Z_H T \sim_{\boldsymbol{N}} T$ for all $T \in \mathscr{T}_H(\boldsymbol{N})$, and T is compactly supported in $\psi(\boldsymbol{M})$. Moreover, since Z_H is constructed from differential operators, multiplication by smooth functions and applications of $E^{\pm}_{\boldsymbol{N}}$, we recall from our previous observation that Z_H must indeed map elements of $\mathscr{T}_H(\boldsymbol{N})$ to elements of $\mathscr{T}_H(\boldsymbol{N})$.

Therefore, if we define

$$\begin{aligned} \mathscr{S}_H(\psi) : \mathscr{T}_H(\boldsymbol{N}) &\to \mathscr{T}_{\psi^* H}(\boldsymbol{M}) \\ T &\mapsto \psi^* Z_H T, \end{aligned} \tag{4.13}$$

then the same argument as used in the proof of Lemma 4.1.2 shows that $\mathscr{T}_H(\psi)\mathscr{S}_H(\psi)T \sim_{\boldsymbol{N}} T$ for all $T \in \mathscr{T}_H(\boldsymbol{N})$ and $\mathscr{S}_H(\psi)\mathscr{T}_H(\psi)T \sim_{\boldsymbol{M}} T$ for all $T \in \mathscr{T}_{\psi^* H}(\boldsymbol{M})$.

Now, if $\psi(\boldsymbol{M})$ contains a Cauchy surface for $\boldsymbol{N}$ then for each $H \in \mathscr{H}(\boldsymbol{M})$ there is precisely one $H' \in \mathscr{H}(\boldsymbol{N})$ with $\psi^* H' = H$, as a result of the condition (3.41). We will denote this extension by $\psi_\bullet H$. Now suppose that $W = (W_H)_{H \in \mathscr{H}(\boldsymbol{N})} \in \mathscr{W}(\boldsymbol{N})$ with $W_H = [T_H]_{\boldsymbol{M}}$. We then define

$$\begin{aligned} \mathscr{U}_H(\psi) : \mathscr{W}_H(\boldsymbol{N}) &\to \mathscr{W}_{\psi^* H}(\boldsymbol{M}) \\ [T]_{\boldsymbol{N}} &\mapsto [\mathscr{S}_H(\psi)T]_{\boldsymbol{M}}. \end{aligned} \tag{4.14}$$

This then gives us a map $\mathscr{U}(\psi) : \mathscr{W}(\boldsymbol{N}) \to \mathscr{W}(\boldsymbol{M})$ with the property that for any $H \in \mathscr{H}(\boldsymbol{M})$, we have

$$(\mathscr{U}(\psi)W)_H = \mathscr{U}_{\psi_\bullet H}(\psi) W_{\psi_\bullet H}. \tag{4.15}$$

It is easy to show that $\mathscr{W}(\psi) \circ \mathscr{U}(\psi) = \mathrm{id}_{\mathscr{W}(\boldsymbol{N})}$, and $\mathscr{U}(\psi) \circ \mathscr{W}(\psi) = \mathrm{id}_{\mathscr{W}(\boldsymbol{M})}$.

Therefore $\mathscr{U}(\psi) = \mathscr{W}(\psi)^{-1}$ and so $\mathscr{W}$ obeys the timeslice axiom. □

4.1.3 Relative Cauchy evolutions

In order to demonstrate (or rule out) dynamical locality for $\mathscr{A}$ or $\mathscr{W}$, we must first compute the relative Cauchy evolution of an arbitrary element; this has already been done for the scalar Klein-Gordon theory in [11] for a different construction, and we will derive a similar expression in our formalism. We begin with the theory $\mathscr{A}$; we fix $\boldsymbol{h} \in H(\boldsymbol{M})$ and choose two subspacetimes $\boldsymbol{N}^{\pm} \subset \boldsymbol{M}$, such that:

- each $\boldsymbol{N}^{\pm}$ is an object of Loc, and their embeddings into $\boldsymbol{M}$ are arrows in Loc,
- there are Cauchy partition functions $\chi_{\pm}^{\mathrm{adv}}$ for $\boldsymbol{M}$ that are non-constant only within $\boldsymbol{N}^{\pm}$,
- each $\boldsymbol{N}^{\pm}$ is disjoint from the support of $\boldsymbol{h}$, and $\boldsymbol{N}^{\pm} \subset J^{\pm}_{\boldsymbol{M}}(\mathrm{supp}(\boldsymbol{h}))$.

As before we define

$$\begin{aligned} \zeta^{\pm} : \mathscr{F}^1(\boldsymbol{M}) &\to \mathscr{F}^1(\boldsymbol{M}) \\ t &\mapsto P_{\boldsymbol{M}} \chi_{\pm}^{\mathrm{adv}} E_{\boldsymbol{M}} t, \end{aligned} \tag{4.16}$$

and let

$$\begin{aligned} Z^{\pm} : \mathscr{F}(\boldsymbol{M}) &\to \mathscr{F}(\boldsymbol{M}) \\ \sum_{n=0}^{N} t_n &\mapsto \sum_{n=0}^{N} (\zeta^{\pm})^{\otimes n} t_n \qquad (t_n \in \mathscr{F}^n(\boldsymbol{M})). \end{aligned} \tag{4.17}$$

Additionally, we define

$$\begin{aligned} \zeta^{\pm}[\boldsymbol{h}] : \mathscr{F}^1(\boldsymbol{M}[\boldsymbol{h}]) &\to \mathscr{F}^1(\boldsymbol{M}[\boldsymbol{h}]) \\ t &\mapsto P_{\boldsymbol{M}[\boldsymbol{h}]} \chi_{\pm}^{\mathrm{adv}} E_{\boldsymbol{M}[\boldsymbol{h}]} t, \end{aligned} \tag{4.18}$$

and define $Z^{\pm}[\boldsymbol{h}] : \mathscr{F}(\boldsymbol{M}[\boldsymbol{h}]) \to \mathscr{F}(\boldsymbol{M}[\boldsymbol{h}])$ in an analagous way to $Z^{\pm}$.

Now, if we denote by $\iota^\pm$, $\iota^\pm[\boldsymbol{h}]$ the embeddings of $\boldsymbol{N}^\pm$ into $\boldsymbol{M}$ and $\boldsymbol{M}[\boldsymbol{h}]$ respectively, it is clear that the Alg-arrows $\mathscr{A}(\iota^\pm)$, $\mathscr{A}(\iota^\pm[\boldsymbol{h}])$ act as

$$\begin{aligned} \mathscr{A}(\iota^\pm)[F]_{\boldsymbol{N}^\pm} &= [\mathscr{F}(\iota^\pm)F]_{\boldsymbol{M}}, \\ \mathscr{A}(\iota^\pm[\boldsymbol{h}])[F]_{\boldsymbol{N}^\pm} &= [\mathscr{F}(\iota^\pm[\boldsymbol{h}])F]_{\boldsymbol{M}[\boldsymbol{h}]}, \end{aligned} \tag{4.19}$$

and for any $F \in \mathscr{F}(\boldsymbol{N}^\pm)$, $f \in C^\infty(\boldsymbol{M})$ we have

$$(\mathscr{F}(\iota^\pm)F)[f] = F\left[f|_{\boldsymbol{N}^\pm}\right] = (\mathscr{F}(\iota^\pm[\boldsymbol{h}])F)[f]. \tag{4.20}$$

Moreover, from Lemma 4.1.2 we can see that the inverse arrows $\mathscr{A}(\iota^\pm)^{-1}$, $\mathscr{A}(\iota^\pm[\boldsymbol{h}])^{-1}$ act as

$$\begin{aligned} \mathscr{A}(\iota^\pm)^{-1}[F]_{\boldsymbol{M}} &= [\mathscr{G}(\iota^\pm)F]_{\boldsymbol{N}^\pm} \\ \mathscr{A}(\iota^\pm[\boldsymbol{h}])^{-1}[F]_{\boldsymbol{M}[\boldsymbol{h}]} &= [\mathscr{G}(\iota^\pm[\boldsymbol{h}])F]_{\boldsymbol{N}^\pm}, \end{aligned} \tag{4.21}$$

where for any $f \in C^\infty(\boldsymbol{N}^\pm)$, $F \in \mathscr{F}(\boldsymbol{M})$ and $F' \in \mathscr{F}(\boldsymbol{M}[\boldsymbol{h}])$, we see from (4.6) that

$$\begin{aligned} (\mathscr{G}(\iota^\pm)F)[f] &= (Z^\pm F)[\iota^\pm_* f], \\ (\mathscr{G}(\iota^\pm[\boldsymbol{h}])F')[f] &= (Z^\pm[\boldsymbol{h}]F')[\iota^\pm[\boldsymbol{h}]_* f]. \end{aligned} \tag{4.22}$$

It follows that for any $A = [F]_{\boldsymbol{M}} \in \mathscr{A}(\boldsymbol{M})$, we have

$$\begin{aligned} \mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}]A &= \mathscr{A}(\iota^-)\mathscr{A}(\iota^-[\boldsymbol{h}])^{-1}\mathscr{A}(\iota^+[\boldsymbol{h}])\mathscr{A}(\iota^+)^{-1}A \\ &= \left[\mathscr{F}(\iota^-)\mathscr{G}(\iota^-[\boldsymbol{h}])\mathscr{F}(\iota^+[\boldsymbol{h}])\mathscr{G}(\iota^+)F\right]_{\boldsymbol{M}}. \end{aligned} \tag{4.23}$$

Now, for any $f \in C^\infty(\boldsymbol{M})$ and $F \in \mathscr{F}(\boldsymbol{M})$ we have

$$(\mathscr{F}(\iota^+[\boldsymbol{h}])\mathscr{G}(\iota^+)F)[f] = (Z^+F)|_{\boldsymbol{N}^+}\left[f|_{\boldsymbol{N}^+}\right], \tag{4.24}$$

but since the range of Z^+ is contained in $\iota^+(\boldsymbol{N}^+)$, it holds that

$$\mathscr{F}(\iota^+[\boldsymbol{h}]) \circ \mathscr{G}(\iota^+) = \iota^+[\boldsymbol{h}]_* \circ (\iota^+)^* \circ Z^+, \tag{4.25}$$

and similarly

$$\mathscr{F}(\iota^-) \circ \mathscr{G}(\iota^-[\boldsymbol{h}]) = \iota^-_* \circ \iota^-[\boldsymbol{h}]^* \circ Z^-[\boldsymbol{h}]. \tag{4.26}$$

Explicitly, the relative Cauchy evolution of $A = [F]_{\boldsymbol{M}}$ is therefore given by $\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}]A = [B[\boldsymbol{h}]F]_{\boldsymbol{M}}$, where

$$\begin{aligned} B[\boldsymbol{h}] : \mathscr{F}(\boldsymbol{M}) &\to \mathscr{F}(\boldsymbol{M}) \\ \sum_{n=0}^{N} t_n &\mapsto \sum_{n=0}^{N} \beta[\boldsymbol{h}]^{\otimes n} t_n \qquad (t_n \in \mathscr{F}^n(\boldsymbol{M})), \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} \beta[\boldsymbol{h}] : \mathscr{F}^1(\boldsymbol{M}) &\to \mathscr{F}^1(\boldsymbol{M}) \\ t &\mapsto P_{\boldsymbol{M}[\boldsymbol{h}]} \chi_-^{\mathrm{adv}} E_{\boldsymbol{M}[\boldsymbol{h}]} P_{\boldsymbol{M}} \chi_+^{\mathrm{adv}} E_{\boldsymbol{M}} t. \end{aligned} \tag{4.28}$$

Lemma 2.1.7 entails that the definition of $\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}]$ is independent of the choice of $\chi_\pm^{\mathrm{adv}}$, provided that the regions $\boldsymbol{N}^\pm$ in which they are non-constant lie strictly to the future/past of $\mathrm{supp}(\boldsymbol{h})$.

We now calculate the relative Cauchy evolution of an element $W \in \mathscr{W}(\boldsymbol{M})$ generated by a perturbation $\boldsymbol{h} \in H(\boldsymbol{M})$. While the calculation is largely similar to the process for calculating the r.c.e. of an element of $\mathscr{A}(\boldsymbol{M})$, there are some subtleties introduced by the need to specify an $H \in \mathscr{H}(\boldsymbol{M})$ to form the algebras $\mathscr{T}_H(\boldsymbol{M})$. We will proceed as before, fixing some $\boldsymbol{h} \in H(\boldsymbol{M})$ and defining $\boldsymbol{N}^\pm$, $\chi_\pm^{\mathrm{adv}}, \chi_\pm^{\mathrm{ret}}$ and $\iota^\pm$ and $\iota^\pm[\boldsymbol{h}]$ as in the previous subsection. The relative Cauchy evolution of an element $W \in \mathscr{W}(\boldsymbol{M})$ is given by

$$\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}]W = \mathscr{W}(\iota^-)\mathscr{U}(\iota^-[\boldsymbol{h}])\mathscr{W}(\iota^+[\boldsymbol{h}])\mathscr{U}(\iota^+)W. \tag{4.29}$$

But when we calculate the component corresponding to $H \in \mathscr{H}(\boldsymbol{M})$, we see that

$$\begin{aligned} (\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}]W)_H &= \Big(\mathscr{W}(\iota^-)\mathscr{U}(\iota^-[\boldsymbol{h}])\mathscr{W}(\iota^+[\boldsymbol{h}])\mathscr{U}(\iota^+)W\Big)_H \\ &= \mathscr{W}_H(\iota^-)\mathscr{U}_{\tilde{H}_{\boldsymbol{h}}}(\iota^-[\boldsymbol{h}])\mathscr{W}_{\tilde{H}_{\boldsymbol{h}}}(\iota^+[\boldsymbol{h}])\mathscr{U}_{\check{H}_{\boldsymbol{h}}}(\iota^+)W_{\check{H}_{\boldsymbol{h}}} \end{aligned} \tag{4.30}$$

where for any $H \in \mathscr{H}(\boldsymbol{M})$, the distributions $\tilde{H}_{\boldsymbol{h}} \in \mathscr{H}(\boldsymbol{M}[\boldsymbol{h}])$ and $\check{H}_{\boldsymbol{h}} \in \mathscr{H}(\boldsymbol{M})$ are defined by

$$\begin{aligned} \tilde{H}_{\boldsymbol{h}} &= \iota^-[\boldsymbol{h}]_\bullet(\iota^-)^* H \\ \check{H}_{\boldsymbol{h}} &= \iota^+_\bullet \iota^+[\boldsymbol{h}]^* \tilde{H}_{\boldsymbol{h}}. \end{aligned} \tag{4.31}$$

This definition is independent of the choice of $\boldsymbol{N}^\pm$, as a consequence of (3.41).

Note that in general, $\check{H}_{\boldsymbol{h}} \neq H$; this is closely related to the fact that it is impossible to make a choice $H_{\boldsymbol{M}} \in \mathscr{H}(\boldsymbol{M})$ for each spacetime $\boldsymbol{M}$ such that $\psi^* H_{\boldsymbol{N}} = H_{\boldsymbol{M}}$ for each Loc-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$, as remarked in the discussion of the construction of Wick powers in Section 3.3. We may in fact calculate $\check{H}_{\boldsymbol{h}}$ explicitly, as follows.

Lemma 4.1.4. *Let $\boldsymbol{M}$ be a spacetime, and $\boldsymbol{h} \in H(\boldsymbol{M})$ a metric perturbation on $\boldsymbol{M}$. Suppose that $H \in \mathscr{H}(\boldsymbol{M})$, and let $\check{H}_{\boldsymbol{h}}, \boldsymbol{N}^\pm$ and $\chi^{\mathrm{adv}}_\pm$ be defined as above. Then*

$$\check{H}_{\boldsymbol{h}} = \left(\check{E}_{\boldsymbol{M}}(\tau_s^+)' P_{\boldsymbol{M}[\boldsymbol{h}]}(\chi^{\mathrm{adv}}_+)' \check{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\tau_s^-)' P_{\boldsymbol{M}}(\chi^{\mathrm{adv}}_-)'\right)^{\otimes 2} H, \tag{4.32}$$

where $\chi^{\mathrm{adv}}_\pm : C_0^\infty(\boldsymbol{M}) \to C_0^\infty(\boldsymbol{M})$ are the multiplication operators induced by the functions $\chi^{\mathrm{adv}}_\pm \in C^\infty(\boldsymbol{M})$, and $\tau_s^\pm : C_s^\infty(\boldsymbol{M}) \to C_0^\infty(\boldsymbol{M})$ are defined as multiplication by time-compact smooth functions $\tau^\pm$ that are supported in $\boldsymbol{N}^\pm$, such that $\tau^\pm \equiv 1$ in the region in which $\chi^{\mathrm{adv}}_\pm$ is non-constant.

Proof. Since H is a bisolution, we see from the proof of Lemma 3.4.3 that

$$\left(\check{E}_{\boldsymbol{M}}(\tau_s^\pm)' P_{\boldsymbol{M}}(\chi^{\mathrm{adv}}_\pm)'\right)^{\otimes 2} H = H. \tag{4.33}$$

Since $\tau^\pm$ is supported in $\boldsymbol{N}^\pm$, it follows that $\left((\tau_s^\pm)' P_{\boldsymbol{M}}(\chi^{\mathrm{adv}}_\pm)'\right)^{\otimes 2} H$ is supported in $(\boldsymbol{N}^\pm)^{\times 2}$, and therefore

$$\tilde{H}_{\boldsymbol{h}}|_{\boldsymbol{N}^-} = H|_{\boldsymbol{N}^-} = \check{E}^{\otimes 2}_{\boldsymbol{N}^-} \left(\left((\tau_s^-)' P_{\boldsymbol{M}}(\chi^{\mathrm{adv}}_-)'\right)^{\otimes 2} H\right)\Big|_{\boldsymbol{N}^-}. \tag{4.34}$$

Since the action of our multiplication operators does not depend on the metric of the underlying manifold, we may also consider them as maps on the corresponding function spaces on $\boldsymbol{M}[\boldsymbol{h}]$; since $\tilde{H}_{\boldsymbol{h}}$ is a bisolution on $\boldsymbol{M}[\boldsymbol{h}]$ and $\check{E}_{\boldsymbol{M}[\boldsymbol{h}]}|_{\boldsymbol{N}^-} = \check{E}_{\boldsymbol{N}^-}$, it follows that

$$\tilde{H}_{\boldsymbol{h}} = \left(\check{E}_{\boldsymbol{M}[\boldsymbol{h}]}(\tau_s^-)' P_{\boldsymbol{M}}(\chi_-^{\mathrm{adv}})'\right)^{\otimes 2} H. \tag{4.35}$$

A similar argument yields $\check{H}_{\boldsymbol{h}} = \left(\check{E}_{\boldsymbol{M}}(\tau_s^+)' P_{\boldsymbol{M}[\boldsymbol{h}]}(\chi_+^{\mathrm{adv}})'\right)^{\otimes 2} \tilde{H}_{\boldsymbol{h}}$, and so (4.32) is satisfied.[1] □

Lemma 4.1.5. *Let $\boldsymbol{M}$ be a spacetime, with a metric perturbation $\boldsymbol{h} \in H(\boldsymbol{M})$. Suppose also that $H \in \mathscr{H}(\boldsymbol{M})$, and let $\check{H}_{\boldsymbol{h}}$ be defined as above. Then* $\mathrm{supp}(H - \check{H}_{\boldsymbol{h}}) \subset (J_{\boldsymbol{M}}(\mathrm{supp}(\boldsymbol{h}))^{\times 2})$.

Proof. Let $x \in \boldsymbol{M}$, with $x \notin J^+_{\boldsymbol{M}}(\mathrm{supp}(\boldsymbol{h}))$. Since $\mathrm{supp}(\boldsymbol{h})$ is compact, we can find a choice for $\boldsymbol{N}^-$ with $x \in \boldsymbol{N}^-$. It follows that $H(x,y) = \tilde{H}_{\boldsymbol{h}}(x,y)$ for all $x \notin J^+_{\boldsymbol{M}}(\mathrm{supp}(\boldsymbol{h}))$. Similarly, if $x \notin J^-_{\boldsymbol{M}}(\mathrm{supp}(\boldsymbol{h}))$ then we can find a choice for $\boldsymbol{N}^+$ with $x \in \boldsymbol{N}^+$. Therefore $\check{H}_{\boldsymbol{h}}(x,y) = \tilde{H}_{\boldsymbol{h}}(x,y)$ for all $x \notin J^-_{\boldsymbol{M}}(\mathrm{supp}(\boldsymbol{h}))$. Consequently, if $x \in \mathrm{supp}(\boldsymbol{h})^\perp$ then $H(x,y) = \check{H}_{\boldsymbol{h}}(x,y)$. The required result follows by symmetry of H. □

The coherency condition on elements of $\mathscr{W}(\boldsymbol{M})$ given in (3.56) tells us that (4.30) can be expressed as

$$(\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}]W)_H = \mathscr{W}_H(\iota^-)\mathscr{U}_{\tilde{H}_{\boldsymbol{h}}}(\iota^-[\boldsymbol{h}])\mathscr{W}_{\tilde{H}_{\boldsymbol{h}}}(\iota^+[\boldsymbol{h}])\mathscr{U}_{\check{H}_{\boldsymbol{h}}}(\iota^+)\widetilde{\lambda}_{H,\check{H}_{\boldsymbol{h}}} W_H. \tag{4.36}$$

Explicitly, we can then see from (4.13),(4.14) that the relative Cauchy evolution of an element $W = (W_H)_{H \in \mathscr{H}(\boldsymbol{M})} \in \mathscr{W}(\boldsymbol{M})$, where each W_H can be represented by $T_H \in \mathscr{T}_H(\boldsymbol{M})$, is given by

$$(\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}]W)_H = [B_H[\boldsymbol{h}]\lambda_{H,\check{H}_{\boldsymbol{h}}} T_H]_{\boldsymbol{M}}, \tag{4.37}$$

[1] Note that (4.32) strongly resembles the action of the map $\beta[\boldsymbol{h}]$ defined in (4.28), albeit with $\boldsymbol{N}^+$ and $\boldsymbol{N}^-$ interchanged; indeed, if we consider the subcategory of Loc containing only Cauchy arrows, we can regard $\mathscr{H}$ as a functor from this subcategory to a suitable category of distribution spaces, with $\mathscr{H}(\psi)H = \psi_\bullet H$. This functor can be seen to be covariant; the resemblance remarked above can be explained by noting that we may define the relative Cauchy evolution of the functor $\mathscr{H}$ in the same way as for a locally covariant theory; this then satisfies $\mathrm{rce}^{(\mathscr{H})}_{\boldsymbol{M}}[\boldsymbol{h}]\check{H}_{\boldsymbol{h}} = H$.

where

$$B_H[\boldsymbol{h}] : \mathscr{T}_{\check{H}_{\boldsymbol{h}}}(\boldsymbol{M}) \to \mathscr{T}_H(\boldsymbol{M})$$
$$\sum_{n=0}^{N} t_n \mapsto \sum_{n=0}^{N} \beta[\boldsymbol{h}]^{\otimes n} t_n \qquad (t_n \in \mathscr{T}^n(\boldsymbol{M})), \tag{4.38}$$

with

$$\beta[\boldsymbol{h}] : \mathscr{T}^1(\boldsymbol{M}) \to \mathscr{T}^1(\boldsymbol{M})$$
$$t \mapsto P_{\boldsymbol{M}[\boldsymbol{h}]}\chi_-^{\mathrm{adv}} E_{\boldsymbol{M}[\boldsymbol{h}]} P_{\boldsymbol{M}} \chi_+^{\mathrm{adv}} E_{\boldsymbol{M}} t \tag{4.39}$$

as before.

Before we proceed to the dynamical locality of $\mathscr{A}$ and $\mathscr{W}$ we will need the following results, which are proved in Appendix B.

Lemma 4.1.6. *Let $\boldsymbol{M} \in \mathsf{Loc}$, and let $t \in \mathscr{T}_H^1(\boldsymbol{M})$ for some $H \in \mathscr{H}(\boldsymbol{M})$. For any $\boldsymbol{h} \in H(\boldsymbol{M})$ and $f \in C_0^\infty(\boldsymbol{M})$, we have*

$$\frac{d}{ds}(\beta[s\boldsymbol{h}]t)[E_{\boldsymbol{M}} f]\bigg|_{s=0} = \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{\mu\nu} T^{\mu\nu}[E_{\boldsymbol{M}} t, E_{\boldsymbol{M}} f], \tag{4.40}$$

where

$$T^{\mu\nu}[u,\phi] = (\nabla^{(\mu} u)(\nabla^{\nu)}\phi) - \frac{1}{2} g^{\mu\nu}(\nabla^\rho u)(\nabla_\rho \phi)$$
$$+ \frac{1}{2} m^2 g^{\mu\nu} u\phi + \xi(g^{\mu\nu}\Box_{\boldsymbol{g}} - \nabla^\mu\nabla^\nu - G^{\mu\nu})(u\phi) \tag{4.41}$$

for $u \in E_{\boldsymbol{M}}\mathscr{T}^1(\boldsymbol{M})$, $\phi \in E_{\boldsymbol{M}} C_0^\infty(\boldsymbol{M})$.

Note that the above expression is closely linked to the classical stress-energy tensor for the Klein-Gordon theory, which we may recover from the polarized form via $T^{\mu\nu}[\phi] = T^{\mu\nu}[\phi, \overline{\phi}]$ for a smooth classical solution ϕ.

This result leads directly to the following:

Corollary 4.1.7. *Let $t_n \in \mathscr{T}_H^n(\boldsymbol{M})$ for some $H \in \mathscr{H}(\boldsymbol{M})$ and $f \in C_0^\infty(\boldsymbol{M})$.*

Then

$$\frac{d}{ds}(\beta[s\boldsymbol{h}])^{\otimes n} t_n[E_{\boldsymbol{M}}f]\bigg|_{s=0} = n\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{\mu\nu}T^{\mu\nu}\left[E_{\boldsymbol{M}}\tau^n_f, E_{\boldsymbol{M}}f\right], \tag{4.42}$$

where

$$\tau^n_f(x) = \int_{\boldsymbol{M}^{\times(n-1)}} d^{n-1}y\, t_n(x, y_1, \ldots, y_{n-1})E_{\boldsymbol{M}}f(y_1)\cdots E_{\boldsymbol{M}}f(y_{n-1}) \tag{4.43}$$

for $n \geq 2$*, and* $\tau^1_f(x) = t_1(x)$*.*

Note that the previous two results also apply to the elements of $\mathscr{F}^1(\boldsymbol{M})$ and $\mathscr{F}^n(\boldsymbol{M})$ respectively, by invoking the canonical embeddings of $\mathscr{F}(\boldsymbol{M})$ in $\mathscr{T}_H(\boldsymbol{M})$ for $H \in \mathscr{H}(\boldsymbol{M})$.

4.2 Dynamical locality

4.2.1 Dynamical locality of the $\xi \neq 0$ scalar field theory

It has already been shown in [26] that the locally covariant scalar field theory is dynamically local in the case when $\xi = 0$ and $m \neq 0$, and that it is not dynamically local when $\xi = 0$ and $m = 0$. We wish to show that the theory $\mathscr{A}$ obeys the axiom of dynamical locality in the nonminimally coupled case (i.e. when $\xi \neq 0$), for both $m = 0$ and $m > 0$. Throughout this section, we consider some fixed $\boldsymbol{M} \in \mathsf{Loc}$ and $O \in \mathscr{O}(\boldsymbol{M})$.

It is easy to construct the kinematic algebra $\hat{\mathscr{A}}^{\text{kin}}(\boldsymbol{M}; O)$, since the elements of this algebra are precisely those $A \in \mathscr{A}(\boldsymbol{M})$ that can be represented by $F = \sum_{n=0}^{N} t_n \in \mathscr{F}(\boldsymbol{M})$ with $\text{supp}(t_n) \in O^{\times n}$. However, the task of constructing the algebras $\mathscr{A}^\bullet(\boldsymbol{M}; K)$ for a given $K \in \mathscr{K}(\boldsymbol{M}; O)$, and hence the dynamical algebras, is much more complicated.

Suppose that $A \in \mathscr{A}^\bullet(\boldsymbol{M}; K)$; from (2.22) it follows that $\text{rce}_{\boldsymbol{M}}[\boldsymbol{h}]A = A$ for all $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$. Now, suppose that A is represented by a functional $F \in \mathscr{F}(\boldsymbol{M})$. This means that $B[\boldsymbol{h}]F \sim_{\boldsymbol{M}} F$ for all $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$, and consequently $B[s\boldsymbol{h}]F - F \in \mathscr{J}(\boldsymbol{M})$ for all $s \in \mathbb{R}$ sufficiently small that $s\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$. Writing $F = \sum_{n=0}^{N} t_n$, with each $t_n \in \mathscr{F}^n(\boldsymbol{M})$, we can refer to Lemma 3.1.2 to see that for $n = 1, \ldots, N$, we have

$$(B[s\boldsymbol{h}]t_n)[E_{\boldsymbol{M}}f] = \left((\beta[s\boldsymbol{h}])^{\otimes n}t_n\right)[E_{\boldsymbol{M}}f] = t_n[E_{\boldsymbol{M}}f] \tag{4.44}$$

for all $f \in C_0^\infty(\boldsymbol{M})$ and for all $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$. The algebra $\mathscr{A}^\bullet(\boldsymbol{M}; K)$ may then be characterized by the following lemma, which we prove in Appendix C.

Lemma 4.2.1. *Let $\boldsymbol{M} \in \mathsf{Loc}$, and $t_n \in \mathscr{F}^n(\boldsymbol{M})$, $n \geq 1$. If $O \in \mathscr{O}(\boldsymbol{M})$, $K \in \mathscr{K}(\boldsymbol{M}; O)$ and $\left((\beta[s\boldsymbol{h}])^{\otimes n}t_n\right)[E_{\boldsymbol{M}}f] = t_n[E_{\boldsymbol{M}}f]$ for all $f \in C_0^\infty(\boldsymbol{M})$ and for all $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$, then*

$$\operatorname{supp}({E_{\boldsymbol{M}}}^{\otimes n}t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}. \tag{4.45}$$

Consequently $A \in \mathscr{A}^\bullet(\boldsymbol{M}; K)$ if and only if it may be represented by some $F = \sum_{n=0}^N t_n \in \mathscr{F}(\boldsymbol{M})$ with the property that $\operatorname{supp}({E_{\boldsymbol{M}}}^{\otimes n}t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}$ for all $n \geq 1$.

The following result applies equally to the enlarged algebra of functionals, so we prove it in the more general case; it does, of course, still hold on $\mathscr{F}^n(\boldsymbol{M})$.

Lemma 4.2.2. *Let $t_n \in \mathscr{T}^n(\boldsymbol{M})$, with* $\operatorname{supp}({E_{\boldsymbol{M}}}^{\otimes n}t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}$. *Furthermore, let S be any open globally hyperbolic neighbourhood of an arbitrary Cauchy surface $\Sigma \subset \boldsymbol{M}$. Then there exist $s, u_k \in \mathscr{T}^n(\boldsymbol{M})$, $k = 1, \ldots, n$, such that*

$$t_n = s + \sum_{k=1}^{n}(P_{\boldsymbol{M}})_k u_k, \tag{4.46}$$

where we define $(P_{\boldsymbol{M}})_k = \mathbf{1}^{\otimes k-1} \otimes P_{\boldsymbol{M}} \otimes \mathbf{1}^{\otimes n-k}$, and such that $\operatorname{supp}(s) \subset (J_{\boldsymbol{M}}(K) \cap S)^{\times n}$. *Moreover, if $K \in \mathscr{K}(\boldsymbol{M}; O)$ for some $O \in \mathscr{O}(\boldsymbol{M})$, then S can be chosen in such a way that* $\operatorname{supp}(s) \subset O^{\times k}$.

Proof. To prove this, we will need the result of Lemma C.3: namely, that

$$\ker {E_{\boldsymbol{M}}}^{\otimes n} = \left\{\sum_{k=1}^{n}(P_{\boldsymbol{M}})_k u_k : u_k \in \mathscr{T}^n(\boldsymbol{M})\right\}. \tag{4.47}$$

Now, if S is an open globally hyperbolic neighbourhood of a Cauchy surface, then we can find a Cauchy partition function χ^{adv} for $\boldsymbol{M}$ that is nonconstant only within S. We let $s = (P_{\boldsymbol{M}}\chi^{\mathrm{adv}}E_{\boldsymbol{M}})^{\otimes n}t_n$; by (3.85) we have ${E_{\boldsymbol{M}}}^{\otimes n}s = {E_{\boldsymbol{M}}}^{\otimes n}t_n$, so by Lemma C.3 it follows that

$$t_n - s = \sum_{k=1}^{n} (P_{\boldsymbol{M}})_k u_k \tag{4.48}$$

for some $u_k \in \mathscr{T}^n(\boldsymbol{M})$, $k = 1, \ldots, n$. The required support properties of s follow from the support of ${E_{\boldsymbol{M}}}^{\otimes n}t_n$ and the fact that χ^{adv} is constant outside S.

Furthermore, if $K \in \mathscr{K}(\boldsymbol{M}; O)$ then K has a multi-diamond neigbourhood based in O, so there exists a Cauchy surface $\Sigma \subset \boldsymbol{M}$ and a finite collection $B_i \subset \Sigma \cap O$ of Cauchy balls such that $K \subset \bigcup_i D_{\boldsymbol{M}}(B_i)$. Since K is compact we may also demand that $\mathrm{cl}(B_i) \subset O$; we can then find an open neigbourhood $S \supset \Sigma$ that is small enough that both $(S \cap J_{\boldsymbol{M}}(K)) \subset \bigcup_i D_{\boldsymbol{M}}(B_i)$ and $D_{\boldsymbol{M}}(B_i) \cap S \subset O$. It then follows that $(S \cap J_{\boldsymbol{M}}(K)) \subset O$, and therefore $\mathrm{supp}(s) \subset O^{\times n}$. □

The previous two lemmas give us the following:

Corollary 4.2.3. *For any $\boldsymbol{M} \in \mathsf{Loc}$, $O \in \mathscr{O}(\boldsymbol{M})$ and $K \in \mathscr{K}(\boldsymbol{M}; O)$, we have $\mathscr{A}^{\bullet}(\boldsymbol{M}; K) \subset \hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M}; O)$. Consequently $\hat{\mathscr{A}}^{\mathrm{dyn}}(\boldsymbol{M}; O) \subset \hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M}; O)$.*

We may finally then prove:

Proposition 4.2.4. *The locally covariant scalar field theory is dynamically local in the nonminimally coupled case, for all $m \geq 0$.*

Proof. Given the result of the previous corollary, it remains only to show that $\hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M}; O) \subset \hat{\mathscr{A}}^{\mathrm{dyn}}(\boldsymbol{M}; O)$ for any $O \in \mathscr{O}(\boldsymbol{M})$. We adapt the proof of [26, Lemma 3.3].

Given an arbitrary $A \in \hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M}; O)$, there is some $F = \sum_{n=0}^{N} t_n \in \mathscr{F}(\boldsymbol{M})$ with $[F]_{\boldsymbol{M}} = A$ and $\mathrm{supp}(t_n) \subset O^{\times n}$. Now, t_n is a finite sum of finite products of test functions, and we may take each test function φ to have support in O. However, since $\mathrm{supp}(\varphi)$ is compact for each φ, there is a compact $K \subset O$ such that $\mathrm{supp}(\varphi) \subset K$ for each test function φ used to construct the t_n.

We take a cover of K by open diamonds based in O, extract a finite subcover O_i, $i = 1, \ldots, K$, and then choose a smooth partition of unity χ_i for K such that $\mathrm{supp}(\chi_i\varphi) \subset O_i$ for each i, φ. There is, of course, a compact K_i for each i such that $\mathrm{supp}(\chi_i\varphi) \subset K_i$; by definition we have $K_i \in \mathscr{K}(\boldsymbol{M}; O)$, and it follows that $\varphi = \sum_{i=1}^{K} \chi_i\varphi$ with $[\chi_i\varphi]_{\boldsymbol{M}} \in \mathscr{A}^{\bullet}(\boldsymbol{M}; K_i)$. Since F is generated by these φ, it follows that $A \in \hat{\mathscr{A}}^{\mathrm{dyn}}(\boldsymbol{M}; O)$. Therefore $\hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M}; O) = \hat{\mathscr{A}}^{\mathrm{dyn}}(\boldsymbol{M}; O)$, and consequently $\mathscr{A}$ is dynamically local. □

4.2.2 Dynamical locality of the algebra of Wick Polynomials

We now proceed to examine the cases in which we can demonstrate dynamical locality for the theory $\mathscr{W}$. We begin by looking at the minimally coupled massless case. The corresponding case for the Klein-Gordon theory is not dynamically local, and so one would not expect dynamical locality to hold here. Indeed, this is the case; when $\xi = m = 0$, any constant function is a classical solution to the Klein-Gordon equation. Therefore, in any spacetime $\boldsymbol{M}$ with compact Cauchy surfaces, the function $\phi(x) = 1$ is an element of $E_{\boldsymbol{M}} C_0^\infty(\boldsymbol{M})$. However, if we pick $t \in C_0^\infty(\boldsymbol{M})$ such that $E_{\boldsymbol{M}} t \equiv 1$, and an element $W \in \mathscr{W}(\boldsymbol{M})$ such that $W_H = [t]_{\boldsymbol{M}}$ for some $H \in \mathscr{H}(\boldsymbol{M})$, then it may be shown, using the fact that $\phi \equiv 1$ is also an element of $E_{\boldsymbol{M}[\boldsymbol{h}]} C_0^\infty(\boldsymbol{M}[\boldsymbol{h}])$, that $\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}] W = W$ for all $\boldsymbol{h} \in H(\boldsymbol{M})$. Therefore $W \in \hat{\mathscr{W}}^{\mathrm{dyn}}(\boldsymbol{M}; O)$ for any $O \in \mathscr{O}(\boldsymbol{M})$. But it is also the case that if we pick $f \in C_0^\infty(\boldsymbol{M})$ with $\mathrm{supp}(f) \cap J_{\boldsymbol{M}}(O) = \emptyset$ and $\int_{\boldsymbol{M}} dx\, f(x) \neq 0$, then $t[E_{\boldsymbol{M}} f] \neq 0$; therefore, $W \notin \hat{\mathscr{W}}^{\mathrm{kin}}(\boldsymbol{M}; O)$ as long as $O' := (\mathrm{cl}(O))^\perp$ is nonempty.

We may, however, demonstrate dynamical locality in two cases. To do this, we need the following results. The first is roughly analogous to Lemma 4.2.1: again, we defer the proof to Appendix C.

Lemma 4.2.5. *Let $\boldsymbol{M} \in \mathsf{Loc}$, $O \in \mathscr{O}(\boldsymbol{M})$ and $K \in \mathscr{K}(\boldsymbol{M}; O)$. Let $t_n \in \mathscr{T}^n(\boldsymbol{M})$ for some $n \geq 1$, and suppose that for all $f \in C_0^\infty(\boldsymbol{M})$ and $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$ we have*

$$\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{\mu\nu} T^{\mu\nu}[E_{\boldsymbol{M}} \tau_f^n, E_{\boldsymbol{M}} f] = 0, \tag{4.49}$$

where τ_f^n *is defined as in* (4.43)*. Then, in the massive minimally coupled case* ($m \neq 0$, $\xi = 0$) *and massive conformally coupled case* ($m \neq 0$, $\xi = \frac{d-2}{4(d-1)}$, *where* d *is the dimension of* $\boldsymbol{M}$), *we have* $\operatorname{supp}(E_{\boldsymbol{M}}{}^{\otimes n} t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}$.

We now wish to prove that $\mathscr{W}$ is dynamically local in these two cases: the appearance of $\lambda_{H,\check{H}_{\boldsymbol{h}}}$ in the relative Cauchy evolution is a complicating factor, so we cannot use a directly analogous argument to the proof of dynamical locality for $\mathscr{A}$. However, it is equally easy to construct the kinematic algebras; given some $O \in \mathscr{O}(\boldsymbol{M})$, it is clear that $W \in \hat{\mathscr{W}}^{\mathrm{kin}}(\boldsymbol{M}; O)$ if and only if there exists for a given $H \in \mathscr{H}(\boldsymbol{M})$ some $T_H = \sum_{n=0}^{N} t_n \in \mathscr{T}_H(\boldsymbol{M})$ such that $[T_H]_{\boldsymbol{M}} = W_H$ and $\operatorname{supp}(t_n) \subset O^{\times n}$.

We now wish to construct the algebras $\mathscr{W}^\bullet(\boldsymbol{M}; K)$ for a given $K \in \mathscr{K}(\boldsymbol{M}; O)$. Let $W = (W_H)_{H \in \mathscr{H}(\boldsymbol{M})}$ with $W_H = [T_H]_{\boldsymbol{M}}$ for each H; for a particular fixed $H \in \mathscr{H}(\boldsymbol{M})$ we write $T_H = \sum_{n=0}^{N} t_n$ with $t_n \in \mathscr{T}^n(\boldsymbol{M})$. Then it is clear from (4.37) that $W \in \mathscr{W}^\bullet(\boldsymbol{M}; K)$ if and only if

$$B_H[\boldsymbol{h}]\lambda_{H,\check{H}_{\boldsymbol{h}}} T_H \sim_{\boldsymbol{M}} T_H \tag{4.50}$$

for each $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$. Using (3.48), relabelling and interchanging sums, we may write

$$\begin{aligned}
\lambda_{H,\check{H}_{\boldsymbol{h}}} T_H &= \boldsymbol{\iota}_{H,\check{H}_{\boldsymbol{h}}} \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{1}{2^k k!} \sum_{n=2k}^{N} (\eta_H - \eta_{\check{H}_{\boldsymbol{h}}})^k (t_n) \\
&= \boldsymbol{\iota}_{H,\check{H}_{\boldsymbol{h}}} \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{1}{2^k k!} \sum_{n=0}^{N-2k} (\eta_H - \eta_{\check{H}_{\boldsymbol{h}}})^k (t_{n+2k}) \\
&= \boldsymbol{\iota}_{H,\check{H}_{\boldsymbol{h}}} \sum_{n=0}^{N} \sum_{k=0}^{\lfloor \frac{N-n}{2} \rfloor} \frac{1}{2^k k!} (\eta_H - \eta_{\check{H}_{\boldsymbol{h}}})^k (t_{n+2k}).
\end{aligned} \tag{4.51}$$

Note that in the latter expression, the inner sum for each n consists only of elements of $\mathscr{T}^n(\boldsymbol{M})$; we write

$$\mathscr{T}^n(\boldsymbol{M}) \ni \tilde{t}_{n;\boldsymbol{h}} := \sum_{k=0}^{\lfloor \frac{N-n}{2} \rfloor} \frac{1}{2^k k!} (\eta_H - \eta_{\check{H}_{\boldsymbol{h}}})^k (t_{n+2k}) \tag{4.52}$$

for $n = 0, \ldots, N$, and may express the condition (4.50) as

$$\left((\beta[\boldsymbol{h}])^{\otimes n}\tilde{t}_{n;\boldsymbol{h}}\right)[E_{\boldsymbol{M}}f] = t_n[E_{\boldsymbol{M}}f] \tag{4.53}$$

for all $f \in C_0^\infty(\boldsymbol{M})$ and for each $1 \leq n \leq N$. We note that the $n = 0$ term in (4.50) requires

$$\sum_{k=1}^{\lfloor N/2 \rfloor} \frac{1}{2^k k!}(\eta_H - \eta_{\check{H}_{\boldsymbol{h}}})^k(t_{2k}) = 0 \tag{4.54}$$

for all $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$. We have therefore proved the following lemma:

Lemma 4.2.6. *Let $K \in \mathscr{K}(\boldsymbol{M})$ be arbitrary. If $W \in \mathscr{W}^\bullet(\boldsymbol{M}; K)$, then for any given $H \in \mathscr{H}(\boldsymbol{M})$ there exists a representative $T_H = \sum_{n=0}^N t_n \in \mathscr{T}_H(\boldsymbol{M})$ of W_H satisfying* (4.53) *and* (4.54) *for all $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$.*

The final step before we demonstrate dynamical locality for $\mathscr{W}$ in the massive minimally coupled and massive conformally coupled cases is to prove the following lemma. Again, the proof is quite long-winded, so we include it in Appendix C.

Lemma 4.2.7. *Let $K \in \mathscr{K}(\boldsymbol{M})$ and $W \in \mathscr{W}^\bullet(\boldsymbol{M}; K)$, with W_H represented by $T_H = \sum_{n=0}^N t_n \in \mathscr{T}_H(\boldsymbol{M})$ for some fixed $H \in \mathscr{H}(\boldsymbol{M})$. Then, in the massive minimally coupled and massive conformally coupled cases,*

(a). $\tilde{t}_{n;\boldsymbol{h}} \sim_{\boldsymbol{M}} t_n$ *for all $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$ and $n \geq 0$,*

(b). $\mathrm{supp}({E_{\boldsymbol{M}}}^{\otimes n} t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}$ *for each $n \geq 1$.*

Corollary 4.2.8. *Suppose that $O \in \mathscr{O}(\boldsymbol{M})$, $K \in \mathscr{K}(\boldsymbol{M}; O)$, and $W \in \mathscr{W}^\bullet(\boldsymbol{M}; K)$, and fix $H \in \mathscr{H}(\boldsymbol{M})$. In the massive minimally/conformally coupled cases, there exists some representative $T_H = \sum_{n=0}^N t_n \in \mathscr{T}_H(\boldsymbol{M})$ of W_H such that* $\mathrm{supp}(t_n) \subset O^{\times n}$ *for each $n \geq 1$.*

Therefore $\mathscr{W}^\bullet(\boldsymbol{M}; K) \subset \hat{\mathscr{W}}^{\mathrm{kin}}(\boldsymbol{M}; O)$ for all $K \in \mathscr{K}(\boldsymbol{M}; O)$, and as a consequence $\hat{\mathscr{W}}^{\mathrm{dyn}}(\boldsymbol{M}; O) \subset \hat{\mathscr{W}}^{\mathrm{kin}}(\boldsymbol{M}; O)$.

Proof. This is an immediate result of lemmas 4.2.7 and 4.2.2. □

Finally, we may prove the following:

Proposition 4.2.9. *The theory $\mathscr{W}$ is dynamically local in the massive minimally coupled and massive conformally coupled cases, and not dynamically local in the massless minimally coupled case.*

Proof. The failure in the massless minimally coupled case has already been demonstrated; for the massive minimally coupled and massive conformally coupled cases, in light of the previous corollary it remains only to prove that $\hat{\mathscr{A}}^{\mathrm{kin}}(\boldsymbol{M};O) \subset \hat{\mathscr{A}}^{\mathrm{dyn}}(\boldsymbol{M};O)$ for each $O \in \mathscr{O}(\boldsymbol{M})$. We may not use exactly the same argument as in Proposition 4.2.4, since the elements of $\mathscr{T}^n(\boldsymbol{M})$ are not finite sums of products of test distributions, so we need an alternative strategy.

The important fact to note here is that any $t_n \in \mathscr{T}^n(\boldsymbol{M})$, supported in $O^{\times n}$ for some $O \in \mathscr{O}(\boldsymbol{M})$, is equivalent under the relation $\sim_{\boldsymbol{M}}$ to some finite sum $\sum_{r=1}^{R} u_r$, where each $u_r \in \mathscr{T}^n(\boldsymbol{M})$ is supported in $K_r^{\times n}$ for some $K_r \in \mathscr{K}(\boldsymbol{M};O)$. The proof of this statement is too lengthy to be contained here, so it is given in Appendix C as Lemma C.6.

It follows that given some $H \in \mathscr{H}(\boldsymbol{M})$, each t_n is a sum of functionals, each of which represents in $\mathscr{W}_H(\boldsymbol{M})$ an element of $\mathscr{W}^{\bullet}_{\boldsymbol{M}}(\boldsymbol{M};K_r)$ for some $K_r \in \mathscr{K}(\boldsymbol{M};O)$. Therefore whenever $W_H \in \mathscr{W}_H(\boldsymbol{M})$ is represented by a functional $T_H = \sum_{n=0}^{N} t_n$ with $\mathrm{supp}(t_n) \in O^{\times n}$, we have $W_H \in \hat{\mathscr{W}}^{\mathrm{dyn}}(\boldsymbol{M};O)$. □

Chapter 5

Quantization functors

In this section, we provide concrete representations of the CAR-quantized algebras of a number of types of locally covariant linear classical theory. These algebras will be formed by applying covariant 'quantization' functors to the solution spaces obtained on a particular spacetime, and we therefore follow [7] to a certain extent; however, we go further in providing a more explicit construction, before examining the circumstances in which dynamical locality of the quantized theories may be shown. The material in this chapter follows similar work done for CCR quantization in [26]. The main goal of this chapter is to provide a framework in which to understand the quantization of the locally covariant Dirac theories developed in subsequent chapters.

5.1 Hermitian spaces and adjoint structures

In order to construct the classical Dirac field as a locally covariant theory, we must specify a particular category of vector spaces for the theory to take values in. We also wish to provide a more general framework for CAR quantization, and to do this we must establish some preliminary categorical concepts. Unless otherwise specified, we will always consider vector spaces over $\mathbb{C}$; these will generally be infinite-dimensional, although we do not insist upon this point. In the following subsection we introduce categories of Hermitian spaces and adjoint structures, which will turn out to be the best candidates for the target categories of our classical CAR-quantizable theories.

5.1.1 Hermitian spaces

Definition 5.1.1. *A* Hermitian form *over a complex vector space V is a map $s : V \times V \to \mathbb{C}$ which is sesquilinear (i.e. antilinear in the first argument and linear in the second) with the property that for all $v, w \in V$, we have $s(v, w) = \overline{s(w, v)}$. The pair (V, s) is a* Hermitian space. *A sesquilinear form s (and by association, the pair (V, s)) is* weakly nondegenerate *if for every nonzero $v \in V$ there exists some $w \in V$ with $s(v, w) \neq 0$, and* degenerate *otherwise.*

Additionally, we use the usual terminology that s is *positive definite* if $s(v, v) > 0$ for all nonzero $v \in V$ and *positive semi-definite* if $s(v, v) \geq 0$ for all $v \in V$; a positive definite Hermitian form is an inner product.

Before we continue, we must describe Hermitian spaces from a categorical perspective.

Definition 5.1.2. Herm *is a category whose objects are Hermitian spaces. The arrows in $\mathsf{Herm}((V, s), (V', s'))$ are injective linear maps $f : V \to V'$ satisfying*

$$s(v, w) = s'(f(v), f(w)) \tag{5.1}$$

for all $v, w \in V$. There is a forgetful functor from Herm to the category $\mathsf{Vect}_{\mathbb{C}}$.

The subobjects of an object (V, s) in Herm correspond to complex vector subspaces $U \subset V$ along with the restriction $s|_{U \times U}$. The equalizer of two arrows $f, g \in \mathsf{Herm}((V, s), (V', s'))$ is given by the canonical inclusion in (V, s) of the subspace $\ker(f - g)$. The intersection and union of two or more subobjects of (V, s) are given by the intersection and linear span respectively of the corresponding vector subspaces along with the relevant restriction of s. Note that the restriction of a weakly nondegenerate form to a subspace might possibly be degenerate, and vice versa. For example, consider $\mathbb{C}^2$ with the forms $s_1(\langle w, z\rangle, \langle w', z'\rangle) := \overline{w}z' + \overline{z}w'$, which is weakly nondegenerate, and $s_2(\langle w, z\rangle, \langle w', z'\rangle) := \overline{w}w'$, which is degenerate; when restricted to $\mathbb{C} \oplus \{0\}$, s_1 becomes degenerate (in fact zero) and s_2 becomes nondegenerate.

Definition 5.1.3. *Let V be a complex vector space. We may consider V explicitly as a triple $(\mathcal{V}, \alpha, \mu)$ where $\mathcal{V}$ is the underlying set of elements and $\alpha : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and $\mu : \mathbb{C} \times \mathcal{V} \to \mathcal{V}$ are respectively the addition and scalar multiplication maps. The covariant functor $\overline{\cdot} : \mathsf{Vect}_{\mathbb{C}} \to \mathsf{Vect}_{\mathbb{C}}$ maps a complex vector space V to its* complex conjugate*, that is the vector space $\overline{V} = (\mathcal{V}, \alpha, \overline{\mu})$ where*

$$\overline{\mu}(z, v) = \mu(\overline{z}, v) \tag{5.2}$$

for $v \in V$ and $z \in \mathbb{C}$. The identity map on $\mathcal{V}$ lifts to a canonical antilinear map $j_V : V \to \overline{V}$. The functor $\overline{\cdot}$ maps an arrow $f : V \to V'$ to the corresponding arrow $\overline{f} : \overline{V} \to \overline{V'}$ whose action on elements is identical to that of f; in other words, we have $\overline{f} \circ j_V = j_{V'} \circ f$.

From now on, we will use the notation $\overline{v} := j_V(v)$ for any $v \in V$. This should not be confused with the standard notation for complex conjugation, as we do not identify the vector spaces V and $\overline{V}$. Note that for any complex vector space V, we have $\overline{\overline{V}} = V$ and $\overline{j_V} \circ j_V = \mathrm{id}_V$, so that $\overline{\overline{v}} = v$. The reason for utilising these structures is in order to accommodate the fact that many of the maps representing physical operations on our theories will be antilinear (for example, charge conjugation and Dirac adjoint maps); since the morphisms in our categories are all *linear* maps we must regard these physical operations instead as linear maps from a vector space to its complex conjugate space, in much the same way that a contravariant functor from C to D may be considered as a covariant functor from C into D^{op}. Note that as j_V itself is antilinear, it is not an arrow in $\mathsf{Vect}_{\mathbb{C}}$.

The functor $\overline{\cdot}$ lifts to a functor from Herm to itself, mapping (V, s) to $(\overline{V}, \overline{s})$ where $\overline{s}(\overline{v}, \overline{w}) := \overline{s(v, w)} = s(w, v)$ for $v, w \in V$.

5.1.2 Adjoint structures and charge conjugations

Definition 5.1.4. *A complex vector space V is* adjointable *if there exists a second complex vector space W with a linear isomorphism $A_V : V \to \overline{W}$, the* adjoint operation*. An* adjoint structure *is a quadruple (V, W, A_V, A_W) where V, W are complex vector spaces and $A_V : V \to \overline{W}$ and $A_W : W \to \overline{V}$ are* *isomorphisms. We also define a* Hermitian adjoint structure *to be a collection* $(V, s_V, W, s_W, A_V, A_W)$ *where* (V, s_V) *and* (W, s_W) *are objects of* Herm*,* A_V *is an isomorphism in* $\mathsf{Herm}((V, s_V), \overline{(W, s_W)})$ *and* A_W *is an isomorphism in* $\mathsf{Herm}((W, s_W), \overline{(V, s_V)})$*. Consequently*

$$s_V(v, v') = s_V(\overline{A_W} A_V v, \overline{A_W} A_V v') \tag{5.3}$$

for all $v, v' \in V$*. A Hermitian adjoint structure is weakly nondegenerate if one of its* Herm*-components is weakly nondegenerate (in which case both are).*

Lemma 5.1.5. *Every complex vector space* V *is adjointable.*

Proof. Let $W = \overline{V}$ and $A_V = \mathrm{id}_V$. □

Definition 5.1.6. *The category* $\mathsf{HermAdj}$ *comprises as its objects all possible Hermitian adjoint structures. An arrow from* $(V, s_V, W, s_W, A_V, A_W)$ *to* $(V', s_{V'}, W', s_{W'}, A_{V'}, A_{W'})$ *is a pair* (f, g) *such that* $f \in \mathsf{Herm}((V, s_V), (V', s_{V'}))$ *and* $g \in \mathsf{Herm}((W, s_W), (W', s_{W'}))$*, and the following diagram commutes.*

$$\begin{array}{ccccc} V & \xrightarrow{\;A_V\;} & \overline{W} & \xrightarrow{\;\overline{A_W}\;} & V \\ \Big\downarrow f & & \Big\downarrow \overline{g} & & \Big\downarrow f \\ V' & \xrightarrow[\;A_{V'}\;]{} & \overline{W}' & \xrightarrow[\;\overline{A_{W'}}\;]{} & V' \end{array} \tag{5.4}$$

Note that g is therefore determined by f (and vice versa) by $g = \overline{A_{V'}} \circ \overline{f} \circ A_V^{-1}$; there are also compatibility conditions for f and g given by

$$\begin{aligned} \overline{A_{W'}} \circ A_{V'} \circ f &= f \circ \overline{A_W} \circ A_V, \\ \overline{A_{V'}} \circ A_{W'} \circ g &= g \circ \overline{A_V} \circ A_W. \end{aligned} \tag{5.5}$$

The category of Hermitian adjoint spaces provides a second candidate for the target category of a classical CAR-quantizable theory. Note that the underlying vector space of a Hermitian adjoint structure is always isomorphic to the direct sum of the underlying vector space of a Hermitian structure with itself, and so the process of passing from a Hermitian space to a corresponding Hermitian adjoint structure involves, in some sense, a 'doubling' of the content. We will see that the CAR quantization of a theory valued in Herm also involves a similar doubling, which is not present in the quantization of a theory valued in $\mathsf{HermAdj}$.

In fact, we will show that the quantization of a $\mathsf{HermAdj}$-object is isomorphic to the quantization of either of its undoubled parts, and so the choice of whether to use Herm and double during quantization, or to use $\mathsf{HermAdj}$ and double before quantization, is effectively irrelevant to the physical content of the quantized theory. We will see that the locally covariant Dirac theory is most naturally described using Hermitian adjoint structures.

There are two forgetful functors $\mathscr{F}_1, \mathscr{F}_2 : \mathsf{HermAdj} \to \mathsf{Herm}$ given by

$$\begin{aligned} \mathscr{F}_1(V, s_V, W, s_W, A_V, A_W) &:= (V, s_V), & \mathscr{F}_1(f, g) &:= f, \\ \mathscr{F}_2(V, s_V, W, s_W, A_V, A_W) &:= (W, s_W), & \mathscr{F}_2(f, g) &:= g. \end{aligned} \tag{5.6}$$

There are also endofunctors $\mathscr{R}, \overline{\cdot}$ of $\mathsf{HermAdj}$ which act on objects and arrows by

$$\begin{aligned} \mathscr{R}(V, s_V, W, s_W, A_V, A_W) &:= (W, s_W, V, s_V, \overline{A_V^{-1}}, \overline{A_W^{-1}}), \\ \mathscr{R}(f, g) &:= (g, f), \\ \overline{(V, s_V, W, s_W, A_V, A_W)} &:= (\overline{V}, \overline{s_V}, \overline{W}, \overline{s_W}, A_W^{-1}, A_V^{-1}), \\ \overline{(f, g)} &:= (\overline{f}, \overline{g}). \end{aligned} \tag{5.7}$$

It is easy to check that $\mathscr{R}$ and $\overline{\cdot}$ obey the required conditions to be functors. Moreover, $\mathscr{R} \circ \mathscr{R} = \mathrm{id} = \overline{\cdot} \circ \overline{\cdot}$ and $\mathscr{R} \circ \overline{\cdot} = \overline{\cdot} \circ \mathscr{R}$. Note that A_W^{-1} and A_V^{-1} are used rather than $\overline{A_V}$ and $\overline{A_W}$, as this gives the correct relations when we come to define charge conjugations on a Hermitian adjoint structure.

A subobject of a Hermitian adjoint structure $(V, s_V, W, s_W, A_V, A_W)$ corresponds to a pair of subspaces $U_V \subset V$, $U_W \subset W$ such that $A_V(U_V) = \overline{U_W}$ and $A_W(U_W) = \overline{U_V}$, with the restrictions $s_V|_{U_V \times U_V}$, $s_W|_{U_W \times U_W}$, $A_V|_{U_V}$ and $A_W|_{U_W}$.

We wish to be able to deal with physical theories that admit a charge conjugation map. Traditionally, at the classical level this comes in the form of an antilinear involution on a vector space. However, in the categories introduced above, the arrows are restricted to linear maps; in order to incorporate charge conjugation into our model we introduce the following terminology:

Definition 5.1.7. *A* charge conjugation *on a complex vector space V is a linear isomorphism $C_V : V \to \overline{V}$ with the property that $\overline{C_V} \circ C_V = \mathrm{id}_V$. In the case where $\mathbf{V} = (V, s)$ is an object of* Herm*, a charge conjugation on $\mathbf{V}$ must also be an arrow in* Herm*. A charge conjugation on a Hermitian adjoint structure is an arrow $(C_V, C_W) : (\mathbf{V}, \mathbf{W}, A_V, A_W) \to \overline{(\mathbf{V}, \mathbf{W}, A_V, A_W)}$, where C_V and C_W are charge conjugations on $\mathbf{V}$ and $\mathbf{W} = (W, s_W)$ respectively.*

We do not impose any particular relation between A_V and A_W beyond (5.3); in general, it is certainly not the case that $\overline{A_V} = A_W^{-1}$. However, not all Hermitian adjoint structures admit a charge conjugation. (5.5) requires that

$$\overline{A_V^{-1}} \circ A_W^{-1} \circ C_V = C_V \circ \overline{A_W} \circ A_V, \tag{5.8}$$

and such a C_V can only be found for certain A_V, A_W. The category $\mathsf{HermAdj_C}$ is defined to be the full subcategory of HermAdj whose objects admit charge conjugations. If V and V' are two objects of $\mathsf{Vect}_{\mathbb{C}}$ (respectively Herm) with charge conjugations $C_V, C_{V'}$, and $f : V \to V'$ is an arrow in the relevant category, then $C_V, C_{V'}$ are *compatible* with f if $C_{V'} \circ f = \overline{f} \circ C_V$.

Examining (5.4), we see that the condition of (C_V, C_W) being an arrow from $(\mathbf{V}, \mathbf{W}, A_V, A_W)$ to $\overline{(\mathbf{V}, \mathbf{W}, A_V, A_W)}$ may be restated as the condition that

$$A_W^{-1} \circ C_V = \overline{C_W} \circ A_V, \qquad \overline{A_V^{-1}} \circ \overline{C_W} = C_V \circ \overline{A_W}. \tag{5.9}$$

Note also that (5.8) is a consequence of the above condition, and since $\overline{C_V} \circ C_V = \mathrm{id}_V$, $\overline{C_W} \circ C_W = \mathrm{id}_W$, we can see that the two above equations are in fact equivalent by taking inverses.

5.1.3 Squared adjoint structures

Before we describe the quantization of these structures to CAR algebras, we will define one further category of structures that will be needed when we define a classical Dirac theory that does not rely on an unphysical choice of reference frame for the spacetime. The basic definition looks rather contrived and unnatural, but we will in fact find that it provides exactly the right structure to generate theories whose physical content may be described in terms of bilinear combinations of an underlying unphysical field.

Definition 5.1.8. *A* squared adjoint structure *is a vector space* $U \in \mathsf{Vect}_{\mathbb{C}}$ *equipped with:*

- *A linear form* $\omega : U \to \mathbb{C}$*, the* trace map
- *An antilinear involution* * *called the* adjoint map*, satisfying* $\omega(u^*) = \overline{\omega(u)}$,
- *A linear involution* Φ *called the* internal swap map*, satisfying* $^* \circ \Phi = \Phi \circ ^*$ *and* $\omega \circ \Phi = \omega$,
- *A map* $Z : U \otimes U \to U \otimes U$*, the* exchange map*, satisfying* $Z^2 = \mathrm{id}$, $Z \circ (\Phi \otimes \Phi) \circ Z = X$*, and* $Z \circ (^* \otimes ^*) = (^* \otimes ^*) \circ \tilde{Z}$*, where* X *is the external swap map on* $U \otimes U$ *that maps* $u_1 \otimes u_2$ *to* $u_2 \otimes u_1$*, and* $\tilde{Z} := (\Phi \otimes \Phi) \circ Z \circ (\Phi \otimes \Phi)$.[1]

Squared adjoint structures form the objects of the category SAdj*. An arrow from* $\mathbf{U} = (U, \omega, ^*, \Phi, Z)$ *to* $\mathbf{U}' = (U', \omega', ^\star, \Phi', Z')$ *is an injective linear map* $f : U \to U'$ *satisfying*

$$\omega' \circ f = \omega, \qquad ^\star \circ f = f \circ ^*, \qquad \Phi' \circ f = f \circ \Phi,$$
$$Z' \circ (f \otimes f) = (f \otimes f) \circ Z. \tag{5.10}$$

The reason for most of this nomenclature will become clear. The most important squared adjoint structures are those arising from the action of the following functor.

[1]The condition that $(^* \otimes ^*)$ must intertwine Z and $\tilde{Z}$ is equivalent to the condition that Z commutes with $(^* \otimes ^*) \circ (\Phi \otimes \Phi)$.

Definition 5.1.9. *Let $\mathcal{A} = (\mathbf{V}, \mathbf{W}, A_V, A_W) \in \mathsf{HermAdj}$, where $\mathbf{V} = (V, s_V)$ and $\mathbf{W} = (W, s_W)$ are Hermitian spaces. The squared adjoint structure $\mathfrak{S}(\mathcal{A})$ is given by $(U, \omega, {}^*, \Phi, Z)$, where*

$$\begin{aligned} U &:= (V \oplus W) \otimes (V \oplus W), \\ \omega(\langle v, w\rangle \otimes \langle v', w'\rangle) &:= s_W(\overline{A_V v}, w') + s_W(\overline{A_V v'}, w) \\ (\langle v, w\rangle \otimes \langle v', w'\rangle)^* &:= \left(\langle A_V^{-1}\overline{w'}, \overline{A_V v'}\rangle \otimes \langle A_V^{-1}\overline{w}, \overline{A_V v}\rangle\right), \\ \Phi(\boldsymbol{u}_1 \otimes \boldsymbol{u}_2) &:= \boldsymbol{u}_2 \otimes \boldsymbol{u}_1, \\ Z((\boldsymbol{u}_1 \otimes \boldsymbol{u}_2) \otimes (\boldsymbol{u}_3 \otimes \boldsymbol{u}_4)) &:= (\boldsymbol{u}_1 \otimes \boldsymbol{u}_3) \otimes (\boldsymbol{u}_2 \otimes \boldsymbol{u}_4), \end{aligned} \tag{5.11}$$

for $\boldsymbol{u}_i \in V \oplus W$. For a $\mathsf{HermAdj}$-arrow $(f, g) : \mathcal{A} \to \mathcal{A}' = (\mathbf{V}', \mathbf{W}', A_{V'}, A_{W'})$, we define $\mathfrak{S}(f, g) := (f \oplus g) \otimes (f \oplus g)$.

Lemma 5.1.10. *$\mathfrak{S}$ is a covariant functor.*

Proof. Let $\mathcal{A} = (\mathbf{V}, \mathbf{W}, A_V, A_W)$ and $\mathcal{A}' = (\mathbf{V}', \mathbf{W}', A_{V'}, A_{W'})$, and $(f, g) : \mathcal{A} \to \mathcal{A}'$ be an arrow in $\mathsf{HermAdj}$. It may easily (but laboriously) be checked that $\mathfrak{S}(\mathcal{A})$ obeys all of the conditions for it to be a squared adjoint structure, and that $\mathfrak{S}(f, g)$ is indeed an arrow in SAdj; for example,

$$\begin{aligned} \omega'(\mathfrak{S}(f, g)(\langle v, w\rangle \otimes \langle v', w'\rangle)) &= \omega'(\langle f(v), g(w)\rangle \otimes \langle f(v'), g(w')\rangle) \\ &= s_{W'}(\overline{A_{V'} f(v)}, g(w')) + s_{W'}(\overline{A_{V'} f(v')}, g(w)) \\ &= s_{W'}(g(\overline{A_V v}), g(w')) + s_{W'}(g(\overline{A_V v'}), g(w)) \\ &= s_W(\overline{A_V v}, w') + s_W(\overline{A_V v'}, w) \\ &= \omega(\langle v, w\rangle \otimes \langle v', w'\rangle), \end{aligned} \tag{5.12}$$

where we have used the commutativity of (5.4) and the fact that for all $w, w' \in \mathbf{W}$, we have $s_{W'}(g(w), g(w')) = s_W(w, w')$.

We also have $\mathfrak{S}(\mathrm{id}_{\mathbf{V}}, \mathrm{id}_{\mathbf{W}}) = (\mathrm{id}_{\mathbf{V}} \oplus \mathrm{id}_{\mathbf{W}}) \otimes (\mathrm{id}_{\mathbf{V}} \oplus \mathrm{id}_{\mathbf{W}}) = \mathrm{id}_{\mathfrak{S}(\mathcal{A})}$ and $\mathfrak{S}((f', g') \circ (f, g)) = \mathfrak{S}(f' \circ f, g' \circ g) = \mathfrak{S}(f', g') \circ \mathfrak{S}(f, g)$, which concludes the proof. ∎

In particular, note that for any charge conjugation (C_V, C_W) of $\mathcal{A} = (\mathbf{V}, \mathbf{W}, A_V, A_W)$, there is a SAdj-arrow $\mathfrak{S}(C_V, C_W) : \mathfrak{S}(\mathcal{A}) \to \mathfrak{S}(\overline{\mathcal{A}})$.

5.2 The CAR algebra of Herm

Before we discuss the quantization of the classical Dirac field, we will first set out a more general CAR quantization framework of which the Dirac quantization is a special case. This formalism is closely analogous to the treatment of CCR quantization in [26]. To start with we must give a description of an algebra generated by the elements of a given object of Herm, and satisfying the canonical anticommutation relations given by its Hermitian form.

5.2.1 Construction

Given any object $\mathbf{V} = (V, s)$ in Herm, we wish to define the *CAR (canonical anticommutation relation) algebra* of $\mathbf{V}$ as a $*$-algebra generated by the elements $\mathbf{1}$ and $B_{\mathbf{V}}(v), B^*_{\mathbf{V}}(v), v \in V$ satisfying the following relations:

$$B_{\mathbf{V}}(v)^* = B^*_{\mathbf{V}}(v), \tag{5.13a}$$

$$B_{\mathbf{V}}(\lambda v + \mu w) = \lambda B_{\mathbf{V}}(v) + \mu B_{\mathbf{V}}(w) \tag{5.13b}$$

$$\{B_{\mathbf{V}}(v), B_{\mathbf{V}}(w)\} = 0 = \{B^*_{\mathbf{V}}(v), B^*_{\mathbf{V}}(w)\} \tag{5.13c}$$

$$\{B^*_{\mathbf{V}}(v), B_{\mathbf{V}}(w)\} = s(v, w)\mathbf{1}, \tag{5.13d}$$

for $v, w \in V$ and $\lambda, \mu \in \mathbb{C}$. Note that $v \mapsto B^*_{\mathbf{V}}(v)$ is antilinear. While we might simply define the CAR algebra as the freest $*$-algebra generated by the above relations, as in [2], or give a definition in terms of Clifford algebras, as in [7], we wish to provide a more concrete definition of the algebra. Let W be a complex vector space equipped with a symmetric bilinear (not Hermitian) form S and an antilinear involution $^* : W \to W$ satisfying $S(v^*, w^*) = \overline{S(v, w)}$. We define the *deformed exterior algebra* $\Lambda_S(W, {}^*)$ as the $*$-algebra whose underlying vector space coincides with that of the exterior algebra $\Lambda(W)$, and whose product $\diamond$ is defined in the following way, relative to S: for any $u_1, \ldots, u_m, v_1, \ldots, v_n \in W$, the product $(u_1 \wedge \cdots \wedge u_m) \diamond (v_1 \wedge \cdots \wedge v_n)$ is given by

$$\sum_{\substack{\sigma\in S_m\\ \sigma'\in S_n}}\sum_{k=0}^{\min(m,n)}\frac{\operatorname{sgn}\sigma\operatorname{sgn}\sigma'\operatorname{sgn}\sigma_{m,n}}{2^k k!(m-k)!(n-k)!}P_S^{m,n,k}(u_{\sigma(1)},\ldots,u_{\sigma(m)};v_{\sigma'(1)},\ldots,v_{\sigma'(n)}),\tag{5.14}$$

where $\sigma_{m,n}\in S_{m+n}$ is the permutation which maps $(u_1,\ldots,u_m,v_1,\ldots,v_n)$ to $(u_1,v_1,\ldots,u_m,v_m,v_{m+1},\ldots,v_n)$ if $m\le n$, or $(u_1,v_1,\ldots,u_n,v_n,u_{n+1},\ldots,u_m)$ if $m>n$, and $P_S^{m,n,k}(u_1,\ldots,u_m;v_1,\ldots,v_n)\in\Lambda_{m+n-2k}(W)$ is defined to be

$$S(u_1,v_1)\cdots S(u_k,v_k)u_{k+1}\wedge v_{k+1}\wedge\cdots\wedge u_m\wedge v_m\wedge v_{m+1}\wedge\ldots\wedge v_n\tag{5.15}$$

when $m\le n$, and

$$S(u_1,v_1)\cdots S(u_k,v_k)u_{k+1}\wedge v_{k+1}\wedge\cdots\wedge u_n\wedge v_n\wedge u_{n+1}\wedge\ldots\wedge u_m\tag{5.16}$$

when $m>n$. This product may be shown to be associative.

Thankfully, it is sufficient to know the definition of products of the form $w\diamond(u_1\wedge\cdots\wedge u_n)$, which is significantly simpler:

$$w\diamond(u_1\wedge\cdots\wedge u_n)=w\wedge u_1\wedge\cdots\wedge u_n+\frac{1}{2}\sum_{k=1}^{n}(-1)^{k+1}S(w,u_k)u_1\wedge\cdots\widehat{u}_k\cdots\wedge u_n\tag{5.17}$$

for $w,u_1,\ldots,u_n\in W$, where $\widehat{u}_k$ indicates omission. We also have $w\diamond\alpha=\alpha w$ for $\alpha\in\Lambda^0(W)=\mathbb{C}$. We note that the identity in this algebra is the element $1\in\Lambda^0(W)$. The $*$-operation on $\Lambda_S(W,{}^*)$ is given on each grade as

$$(w_1\wedge\cdots\wedge w_n)^*=w_n^*\wedge\cdots\wedge w_1^*.\tag{5.18}$$

It is possible to show that the product $\diamond$ respects the $*$-operation on the algebra, i.e. $(A\diamond B)^*=B^*\diamond A^*$ for any elements $A,B\in\Lambda_S(W,{}^*)$. It follows that $\Lambda_S(W,{}^*)$ is indeed a $*$-algebra. This allows us to represent the CAR algebra of a Hermitian space as follows:

Theorem 5.2.1. *Let (V,s) be an object in* Herm*. The freest $*$-algebra generated by $\mathbf{1}$ and elements $B_{\mathbf{V}}(v),B^*_{\mathbf{V}}(v),v\in V$ satisfying the relations* (5.13a)–

(5.13d) *is then isomorphic under the map* $B_{\mathbf{V}}(v) \mapsto \langle \overline{0}, v\rangle$ *to* $\Lambda_S(\overline{V} \oplus V, {}^*)$, *where*

$$S\Big(\langle \overline{v}_1, v_2\rangle, \langle \overline{w}_1, w_2\rangle\Big) := s(v_1, w_2) + s(w_1, v_2), \tag{5.19}$$

and $\langle \overline{v}, w\rangle^* := \langle \overline{w}, v\rangle$.

Proof. The free $*$-algebra generated by $\mathbf{1}$ and $B_{\mathbf{V}}(v), B^*_{\mathbf{V}}(v), v \in V$ with $B_{\mathbf{V}}(v)^* = B^*_{\mathbf{V}}(v)$ and $B_{\mathbf{V}}(\lambda v + \mu w) = \lambda B_{\mathbf{V}}(v) + \mu B_{\mathbf{V}}(w)$ is isomorphic to the tensor algebra

$$\mathscr{T}(\overline{V} \oplus V) = \bigoplus_{n=0}^{\infty} (\overline{V} \oplus V)^{\otimes n} \tag{5.20}$$

where $(\overline{V} \oplus V)^{\otimes 0} := \mathbb{C}$, with $*$-operation given by

$$\begin{aligned}(\langle \overline{v}_1, w_1\rangle \otimes \cdots \otimes \langle \overline{v}_n, w_n\rangle)^* &:= \langle \overline{w}_n, v_n\rangle \otimes \cdots \otimes \langle \overline{w}_1, v_1\rangle \\ &= \langle \overline{v}_n, w_n\rangle^* \otimes \cdots \otimes \langle \overline{v}_1, w_1\rangle^*,\end{aligned} \tag{5.21}$$

under the mapping $B_{\mathbf{V}}(v) \mapsto \langle \overline{0}, v\rangle$ (whereupon $B^*_{\mathbf{V}}(v) = \langle \overline{v}, 0\rangle$). In order that our algebra may also satisfy the relations (5.13c) and (5.13d), we quotient by the two-sided $*$-ideal $\mathscr{I}$ generated by elements of the form

$$\begin{aligned}&\langle \overline{0}, v\rangle \otimes \langle \overline{0}, w\rangle + \langle \overline{0}, w\rangle \otimes \langle \overline{0}, v\rangle \\ &\langle \overline{v}, 0\rangle \otimes \langle \overline{0}, w\rangle + \langle \overline{0}, w\rangle \otimes \langle \overline{v}, 0\rangle - s(v, w)\mathbf{1},\end{aligned} \tag{5.22}$$

for $v, w \in V$. Note that since $\mathscr{I}$ is a $*$-ideal, it will also contain elements of the form $\langle \overline{v}, 0\rangle \otimes \langle \overline{w}, 0\rangle + \langle \overline{w}, 0\rangle \otimes \langle \overline{v}, 0\rangle$.

We may alternatively describe $\mathscr{I}$ as the two-sided $*$-ideal generated by elements

$$\mathbf{v} \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{v} - S(\mathbf{v}, \mathbf{w})\mathbf{1} \tag{5.23}$$

for $\mathbf{v}, \mathbf{w} \in \overline{V} \oplus V$.[2] Therefore every element $A \in \mathscr{T}(\overline{V} \oplus V)$ is equivalent to an element of the antisymmetric subspace $\mathscr{T}_{as}(\overline{V} \oplus V) \subset \mathscr{T}(\overline{V} \oplus V)$ (comprising elements which are totally antisymmetric with respect to the $\otimes$-product in

[2]This can be made even simpler, by noting that $\mathscr{I}$ may even be defined as the two-sided $*$-ideal generated by elements of the form $\mathbf{v} \otimes \mathbf{v} - \frac{1}{2}S(\mathbf{v}, \mathbf{v}), \mathbf{v} \in \overline{V} \oplus V$, by a polarization argument.

each grade). Since the antisymmetric subspace has trivial intersection with $\mathscr{I}$, it follows that this representative is unique, and we will denote it A_{as}.

At the level of underlying vector spaces, $\mathscr{T}_{as}(\overline{V} \oplus V)$ is isomorphic to $\Lambda_S(\overline{V} \oplus V, {}^*)$ under the map $\pi := \bigoplus_{n=0}^{\infty} \pi_n$, where

$$\begin{aligned} \pi_n : \Lambda^n(\overline{V} \oplus V) &\to \mathscr{T}_{as}(\overline{V} \oplus V) \\ \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n &\mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \mathbf{v}_{\sigma(1)} \otimes \cdots \otimes \mathbf{v}_{\sigma(n)}, \end{aligned} \tag{5.24}$$

for $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \overline{V} \oplus V$. It remains to show that the product defined in (5.17) is equivalent to the product on $\mathscr{T}(\overline{V} \oplus V) \big/ \mathscr{I}$, or alternatively that

$$\pi_n(\mathbf{v}_1 \diamond \cdots \diamond \mathbf{v}_n) = (\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n)_{as}. \tag{5.25}$$

For any $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \overline{V} \oplus V$, consider the finite-dimensional subspace $U = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \overline{V} \oplus V$. We may split this as $U = U_0 \oplus U_1$ where $U_0 = \{\mathbf{u} \in U : S(\mathbf{u}, \cdot) = 0\}$ and $S|_{U_1 \times U_1}$ is weakly nondegenerate. U_1 has a basis $\mathbf{e}_1, \ldots, \mathbf{e}_k$ which is orthonormal with respect to S, that is to say $S(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$; we also pick a basis $\mathbf{e}_{k+1}, \ldots, \mathbf{e}_m$ for U_0. It suffices to show that

$$\pi_n(\mathbf{e}_{i_1} \diamond \cdots \diamond \mathbf{e}_{i_n}) = (\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n})_{as} \tag{5.26}$$

for all possible $i_1, \ldots, i_n \in \{1, \ldots, m\}$.

We consider first the case in which the i_r are all distinct. In this circumstance $\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n}$ can easily be seen to be equivalent in the quotient algebra to $\pi_n(\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_n})$; however, we also have $\mathbf{e}_{i_1} \diamond \cdots \diamond \mathbf{e}_{i_n} = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_n}$, so (5.26) is satisfied. Now suppose that at least two of the i_r are the same. We will denote by j_r the r^{th} distinct element of the sequence $(i_1, \ldots, i_n)$, and by p_r the multiplicity of each vector $\mathbf{e}_{i_r}$; since the effect of transposing two distinct consecutive elements in $\mathbf{e}_{i_1} \diamond \cdots \diamond \mathbf{e}_{i_n}$ is to introduce a factor of -1, we have

$$\mathbf{e}_{i_1} \diamond \cdots \diamond \mathbf{e}_{i_n} = (-1)^N \mathbf{e}_{j_1}^{\diamond p_1} \diamond \cdots \diamond \mathbf{e}_{j_M}^{\diamond p_M}, \tag{5.27}$$

where M is the number of distinct basis vectors and N is the number of transpositions needed to manipulate the product into this form. If $p_r \geq 2$ for some $i_r > k$, then the above expression vanishes, since $\mathbf{e}_i^{\diamond 2} = \mathbf{e}_i \wedge \mathbf{e}_i = 0$ for $i > k$; otherwise, since $\mathbf{e}_i \diamond \mathbf{e}_i = \frac{1}{2}$ for $i \leq k$, we may write this as

$$\mathbf{e}_{i_1} \diamond \cdots \diamond \mathbf{e}_{i_n} = 2^{-\sum_{r=1}^{M} \lfloor p_r/2 \rfloor} (-1)^N \prod_{\substack{1 \leq r \leq M \\ p_r \text{ odd}}} \mathbf{e}_{j_r}, \tag{5.28}$$

where $\prod$ is taken using the $\diamond$-product. We may therefore refer to the previous case to see that

$$\pi_n(\mathbf{e}_{i_1} \diamond \cdots \diamond \mathbf{e}_{i_n}) = 2^{-\sum_{f=1}^{M} \lfloor p_r/2 \rfloor} (-1)^N \left(\bigotimes_{\substack{1 \leq r \leq M \\ p_r \text{ odd}}} \mathbf{e}_{j_r} \right)_{as}. \tag{5.29}$$

On the other hand, the effect of transposing two distinct consecutive elements in the expression $\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n}$ is to introduce a factor of -1 to its antisymmetric representative, so we have

$$(\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n})_{as} = (-1)^N (\mathbf{e}_{j_1}^{\otimes p_1} \otimes \cdots \otimes \mathbf{e}_{j_M}^{\otimes p_M})_{as}, \tag{5.30}$$

where M and N are defined as before. If $p_r \geq 2$ for some $i_r > k$, then the above expression vanishes, since $\mathbf{e}_i \otimes \mathbf{e}_i \in \mathscr{I}$ for $i > k$; otherwise, since $\mathbf{e}_i \otimes \mathbf{e}_i - \frac{1}{2} \in \mathscr{I}$ for $i \leq k$, it follows that

$$(\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_n})_{as} = 2^{-\sum_{r=1}^{M} \lfloor p_r/2 \rfloor} (-1)^N \left(\bigotimes_{\substack{1 \leq r \leq M \\ p_r \text{ odd}}} \mathbf{e}_{j_r} \right)_{as}. \tag{5.31}$$

Therefore π is an algebra $*$-isomorphism. □

We now wish to make the assignment $(V, s) \mapsto \Lambda_S(\overline{V} \oplus V, {}^*)$ functorial; that is, to define a covariant functor $\mathscr{Q}_{CAR} : \mathsf{Herm} \to \mathsf{Alg}$ satisfying $\mathscr{Q}_{CAR}(V, s) = \Lambda_S(\overline{V} \oplus V, {}^*)$ where S and * are defined as in Theorem 5.2.1. To do so we must define the action of $\mathscr{Q}_{CAR}$ on an arrow $f : (V, s) \to (V', s')$, and check that $\mathscr{Q}_{CAR}$ satisfies the required properties to be a covariant functor. To this end we define

$$\mathscr{Q}_{CAR}(f) := \mathscr{T}(f_2) = \bigoplus_{n=0}^{\infty} f_2{}^{\otimes n}, \tag{5.32}$$

where

$$f_2\left(\langle \overline{v}, w\rangle\right) := \left\langle \overline{f(v)}, f(w) \right\rangle. \tag{5.33}$$

This clearly maps the identity of $\mathscr{Q}_{CAR}(V,s)$ to the identity of $\mathscr{Q}_{CAR}(V',s')$, and it is easy to check that $\mathscr{Q}_{CAR}(f)$ is a $*$-homomorphism from $\mathscr{Q}_{CAR}(V,s)$ to $\mathscr{Q}_{CAR}(V',s')$. We still require injectivity; this may be proved via the following lemma, which is proved as part of Lemma A.1 in [26].

Lemma 5.2.2. *Consider vector spaces $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$, for $n \geq 1$, with linear maps $S_i : X_i \to Y_i$ for each i. Then, if each S_i is injective then so is $S_1 \otimes \cdots \otimes S_n$.*

We therefore have the following:

Proposition 5.2.3. *$\mathscr{Q}_{CAR}$ is a covariant functor.*

Proof. We have shown that $\mathscr{Q}_{CAR}(f)$ is always an arrow in Alg; the identity arrow $\mathbf{1}_V$ on (V,s) is mapped by $\mathscr{Q}_{CAR}$ to the identity map $\mathbf{1}_{\mathscr{Q}_{CAR}(V,s)}$, and for $f:(V,s)\to(V',s')$, $f':(V',s')\to(V'',s'')$ we have

$$\begin{aligned}\mathscr{Q}_{CAR}(f'\circ f) = \mathscr{T}((f'\circ f)_2) = \mathscr{T}(f_2'\circ f_2)&\\ = \mathscr{T}(f_2')\circ\mathscr{T}(f_2) = \mathscr{Q}_{CAR}(f')\circ\mathscr{Q}_{CAR}(f).&\end{aligned} \tag{5.34}$$

Therefore $\mathscr{Q}_{CAR}$ is indeed a covariant functor. □

Lemma 5.2.4. *For any Herm-object $\mathbf{V}$ and (possibly infinite) collection of subobjects $\mathbf{U}_k$ we have*

$$\bigvee_k \mathscr{Q}_{CAR}(\mathbf{U}_k) = \mathscr{Q}_{CAR}\Big(\bigvee_k \mathbf{U}_k\Big). \tag{5.35}$$

Proof. Let $\mathbf{U} = \bigvee_k \mathbf{U}_k$; an arbitrary element $U \in \mathscr{Q}_{CAR}(\mathbf{U})$ is a finite sum of finite products of generators of $\mathscr{Q}_{CAR}(\mathbf{U})$, and each generator $B_{\mathbf{U}}(v)$ of $\mathscr{Q}_{CAR}(\mathbf{U})$ may be written as $\sum_k B_{\mathbf{U}}(v_k)$, where each $v_k \in \mathbf{U}_k$ and all but finitely many of the v_k are zero. Therefore U may be written as a finite sum of finite products of generators of the subalgebras $\mathscr{Q}_{CAR}(\mathbf{U}_k)$; consequently $U \in \bigvee_k \mathscr{Q}_{CAR}(\mathbf{U}_k)$.

Conversely, suppose that $U \in \bigvee_k \mathscr{Q}_{CAR}(\mathbf{U}_k)$; then U is a finite sum of finite products of generators of the algebras $\mathscr{Q}_{CAR}(\mathbf{U}_k)$, which clearly lies within $\mathscr{Q}_{CAR}(\mathbf{U})$. □

Proposition 5.2.5. *Let* $\mathbf{V}$ *be an object of* Herm*. The algebra* $\mathscr{Q}_{CAR}(\mathbf{V})$ *is simple if and only if* $\mathbf{V}$ *is weakly nondegenerate.*

Proof. Recall that an algebra $\mathfrak{A}$ is *simple* if it possesses no proper two-sided ideals. Let $\mathbf{V} = (V, s)$ be degenerate; in other words, suppose that there exists a nonzero element $d \in V$ such that $s(d, v) = 0$ for all $v \in V$. We consider the two-sided $*$-ideal $\mathscr{I} \subset \mathscr{Q}_{CAR}(\mathbf{V})$ generated by the element $\mathbf{d} := \langle \overline{d}, d\rangle$. Let $A \in \mathscr{I}$ be arbitrary; note that since $S(\mathbf{d}, \mathbf{v}) = 0$ for all $\mathbf{v} \in \overline{V} \oplus V$, it follows from (5.17) that the n^{th} grade part of A can be written as a sum of terms of the form $\mathbf{d} \wedge \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{n-1}$ for some $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \in \overline{V} \oplus V$. Therefore $\mathscr{I} = \mathbf{d} \diamond \mathscr{Q}_{CAR}(\mathbf{V})$. Consequently $\mathscr{I}$ is a proper ideal (for example, the identity element does not lie in $\mathscr{I}$, since $\mathbf{d} \diamond A = \mathbf{1}$ implies that $\mathbf{d} = (\mathbf{d} \diamond \mathbf{d}) \diamond A = 0$, contradicting $d \neq 0$); $\mathscr{Q}_{CAR}(\mathbf{V})$ is therefore not simple.

Conversely, suppose that $\mathbf{V}$ is weakly nondegenerate. Then the (suitably modified) argument of [57, Lemma 1] shows that $\mathscr{Q}_{CAR}(\mathbf{V})$ is simple. □

Recall that the algebra $\mathscr{Q}_{CAR}(\mathbf{V})$ is generated by the elements $B_{\mathbf{V}}(v) = \langle \overline{0}, v\rangle$ and $B^*_{\mathbf{V}}(v) = \langle \overline{v}, 0\rangle$, for $v \in \mathbf{V}$, and the identity element. For the following results it will generally be sufficient and more convenient to work with generators than arbitrary elements. Note that (5.32) and (5.33) entail that for any $f : \mathbf{V} \to \mathbf{V}'$, we have $\mathscr{Q}_{CAR}(f)(B_{\mathbf{V}}(v)) = B_{\mathbf{V}'}(f(v))$ and $\mathscr{Q}_{CAR}(f)(B^*_{\mathbf{V}}(v)) = B^*_{\mathbf{V}'}(f(v))$, or more abstractly,

$$\mathscr{Q}_{CAR}(f) \circ B^{(*)}_{\mathbf{V}} = B^{(*)}_{\mathbf{V}'} \circ f. \tag{5.36}$$

The following result is needed for the definition of charge conjugations on the CAR algebras.

Proposition 5.2.6. *Let* $\mathbf{V} = (V, s)$ *be an object of* Herm. *The map* $\Xi_{\mathbf{V}} : \mathscr{Q}_{CAR}(\mathbf{V}) \to \mathscr{Q}_{CAR}(\overline{\mathbf{V}})$ *defined on generators by*

$$\Xi_{\mathbf{V}} B_{\mathbf{V}}(v) := B^*_{\overline{\mathbf{V}}}(\overline{v}), \qquad \Xi_{\mathbf{V}} B^*_{\mathbf{V}}(v) := B_{\overline{\mathbf{V}}}(\overline{v}), \qquad \Xi_{\mathbf{V}} \mathbf{1} = \mathbf{1} \tag{5.37}$$

is an Alg*-isomorphism. Moreover, the components* $\Xi_{\mathbf{V}}$ *together make up a natural isomorphism* $\Xi : \mathscr{Q}_{CAR} \stackrel{\cdot}{\longrightarrow} \mathscr{Q}_{CAR} \circ \overline{\cdot}$.

Proof. $\Xi_{\mathbf{V}}$ clearly preserves the involution, and must be an isomorphism since $\Xi_{\mathbf{V}} \circ \Xi_{\mathbf{V}} = \mathrm{id}_{\mathscr{Q}_{CAR}(\mathbf{V})}$. We may also easily check that $\Xi_{\mathbf{V}}$ is compatible with (5.13a)–(5.13d). It is therefore an arrow in Alg by definition. For Ξ to be a natural isomorphism, we must show that $\mathscr{Q}_{CAR}(\overline{f}) \circ \Xi_{\mathbf{V}} = \Xi_{\mathbf{V}'} \circ \mathscr{Q}_{CAR}(f)$ for any Herm-arrow $f : \mathbf{V} \to \mathbf{V}'$; this may be straightforwardly done on generators using (5.36), and therefore extends to the whole of $\mathscr{Q}_{CAR}(\mathbf{V})$. □

5.2.2 The CAR algebra of a Hermitian adjoint structure

We may define a functor $\mathscr{Q}_{\mathrm{adj}} : \mathsf{HermAdj} \to \mathsf{Alg}$ by assigning to a Hermitian adjoint structure $(\mathbf{V}, \mathbf{W}, A_V, A_W)$ (where $\mathbf{V} = (V, s_V)$ and $\mathbf{W} = (W, s_W)$ are objects in Herm) the unital $*$-algebra generated by elements $D_{\mathbf{V}}(v), E_{\mathbf{W}}(w)$ with $v \in V$, $w \in W$ satisfying the following relations:

$$D_{\mathbf{V}}(v)^* = E_{\mathbf{W}}(\overline{A_V v}) \tag{5.38a}$$

$$D_{\mathbf{V}}(\lambda v + \mu v') = \lambda D_{\mathbf{V}}(v) + \mu D_{\mathbf{V}}(v') \tag{5.38b}$$

$$\{D_{\mathbf{V}}(v), D_{\mathbf{V}}(v')\} = 0 = \{E_{\mathbf{W}}(w), E_{\mathbf{W}}(w')\} \tag{5.38c}$$

$$\{E_{\mathbf{W}}(w), D_{\mathbf{V}}(v)\} = s_W(\overline{A_V v}, w)\mathbf{1}, \tag{5.38d}$$

for $v, v' \in V$, $w, w' \in W$ and $\lambda, \mu \in \mathbb{C}$. Note that (5.38a) and (5.38b) together imply that the assignment $w \mapsto E_{\mathbf{W}}(w)$ is also complex linear. For an arrow $(f, g) : (\mathbf{V}, \mathbf{W}, A_V, A_W) \to (\mathbf{V}', \mathbf{W}', A_{V'}, A_{W'})$, the action of $\mathscr{Q}_{\mathrm{adj}}(f, g)$ on generators is given by

$$\mathscr{Q}_{\mathrm{adj}}(f, g)(D_{\mathbf{V}}(v) + E_{\mathbf{W}}(w)) = D_{\mathbf{V}'}(f(v)) + E_{\mathbf{W}'}(g(w)), \tag{5.39}$$

where $v \in V$, $w \in W$.

Note that the inherent symmetry of the Hermitian adjoint structure is not, in fact, violated by this definition; while it appears that A_V is preferred over A_W in (5.38a) and (5.38d), we may replace $\overline{A_V}$ by A_W^{-1} and obtain an equivalent representation of the same algebra (up to isomorphism[3]).

Proposition 5.2.7. *Let $\mathscr{A} = (\mathbf{V}, \mathbf{W}, A_V, A_W)$ be a Hermitian adjoint structure. The maps $\Omega_{\mathscr{A}} : \mathscr{Q}_{\mathrm{adj}}(\mathscr{A}) \to \mathscr{Q}_{\mathrm{adj}}(\mathscr{R}(\mathscr{A}))$ and $\Upsilon^i_{\mathscr{A}} : \mathscr{Q}_{\mathrm{adj}}(\mathscr{A}) \to \mathscr{Q}_{\mathrm{adj}}(\overline{\mathscr{A}})$, $i = 1, 2$, defined by*

$$\begin{aligned} \Omega_{\mathscr{A}} D_{\mathbf{V}}(v) &= E_{\mathbf{V}}(v), & \Omega_{\mathscr{A}} E_{\mathbf{W}}(w) &= D_{\mathbf{W}}(w), & \Omega_{\mathscr{A}} \mathbf{1} &= \mathbf{1} \\ \Upsilon^1_{\mathscr{A}} D_{\mathbf{V}}(v) &= E_{\overline{\mathbf{W}}}(\overline{A_W^{-1} v}), & \Upsilon^1_{\mathscr{A}} E_{\mathbf{W}}(w) &= D_{\overline{\mathbf{V}}}(\overline{A_V^{-1} w}), & \Upsilon^1_{\mathscr{A}} \mathbf{1} &= \mathbf{1} \\ \Upsilon^2_{\mathscr{A}} D_{\mathbf{V}}(v) &= E_{\overline{\mathbf{W}}}(A_V v), & \Upsilon^2_{\mathscr{A}} E_{\mathbf{W}}(w) &= D_{\overline{\mathbf{V}}}(A_W w), & \Upsilon^2_{\mathscr{A}} \mathbf{1} &= \mathbf{1} \end{aligned} \tag{5.40}$$

are Alg*-isomorphisms. There are also natural isomorphisms $\Omega : \mathscr{Q}_{\mathrm{adj}} \stackrel{.}{\longrightarrow} \mathscr{Q}_{\mathrm{adj}} \circ \mathscr{R}$ and $\Upsilon^i : \mathscr{Q}_{\mathrm{adj}} \to \mathscr{Q}_{\mathrm{adj}} \circ \overline{\cdot}$, $i = 1, 2$, with components $\Omega_{\mathscr{A}}$ and $\Upsilon^i_{\mathscr{A}}$ respectively.*

Proof. We recall from (5.7) that $\mathscr{R}(\mathscr{A}) = (\mathbf{W}, \mathbf{V}, \overline{A_V^{-1}}, \overline{A_W^{-1}})$ and $\overline{\mathscr{A}} = (\overline{\mathbf{V}}, \overline{\mathbf{W}}, A_W^{-1}, A_V^{-1})$. It is easy to check that $\Omega_{\mathscr{A}}, \Upsilon^i_{\mathscr{A}}$ are isomorphisms in Alg, and compatible with (5.38a)–(5.38d). We also see that for any $\mathsf{HermAdj}$-arrow $(f, g) : \mathscr{A} \to \mathscr{A}'$, we have $\mathscr{Q}_{\mathrm{adj}}(\mathscr{R}(f, g)) \circ \Omega_{\mathscr{A}} = \Omega_{\mathscr{A}'} \circ \mathscr{Q}_{\mathrm{adj}}(f, g)$ and $\mathscr{Q}_{\mathrm{adj}}\overline{(f, g)} \circ \Upsilon^i_{\mathscr{A}} = \Upsilon^i_{\mathscr{A}'} \circ \mathscr{Q}_{\mathrm{adj}}(f, g)$, $i = 1, 2$, and so Ω and Υ^i are natural isomorphisms. □

Proposition 5.2.8. *Recall that the forgetful functors $\mathscr{F}_1, \mathscr{F}_2$:* $\mathsf{HermAdj} \to \mathsf{Herm}$ *map $\mathscr{A} := (\mathbf{V}, \mathbf{W}, A_V, A_W)$ to $\mathbf{V}$ and $\mathbf{W}$ respectively. There is a pair of natural isomorphisms $\eta^i : \mathscr{Q}_{\mathrm{adj}} \stackrel{.}{\longrightarrow} \mathscr{Q}_{CAR} \mathscr{F}_i$, $i = 1, 2$, defined on generators by*

$$\begin{aligned} \eta^1_{\mathscr{A}} D_{\mathbf{V}}(v) &= B_{\mathbf{V}}(v), & \eta^1_{\mathscr{A}} E_{\mathbf{W}}(w) &= B^*_{\mathbf{V}}(A_V^{-1} \overline{w}), & \eta^1_{\mathscr{A}} \mathbf{1} &= \mathbf{1}, \\ \eta^2_{\mathscr{A}} D_{\mathbf{V}}(v) &= B^*_{\mathbf{W}}(\overline{A_V v}), & \eta^2_{\mathscr{A}} E_{\mathbf{W}}(w) &= B_{\mathbf{W}}(w), & \eta^2_{\mathscr{A}} \mathbf{1} &= \mathbf{1}. \end{aligned} \tag{5.41}$$

[3]Explicitly, the isomorphism that leaves $D_{\mathbf{V}}(v)$ unchanged and maps $E_{\mathbf{W}}(w) \mapsto E_{\mathbf{W}}(\overline{A_V} A_W w)$ takes the algebra defined by (5.38a)–(5.38d) to an algebra with the same relations but with $\overline{A_V}$ replaced by A_W^{-1}.

Proof. The components of η^i are clearly isomorphisms, and it is easy to show that $\eta^i_{\mathcal{A}}$ converts the relations (5.38a)–(5.38d) to (5.13a)–(5.13d). To prove naturality of η^1, we must show that the following diagram commutes for all arrows $(f,g) : \mathcal{A} \to \mathcal{A}' := (\mathbf{V}', \mathbf{W}', A_{V'}, A_{W'})$:

$$\begin{array}{ccc} \mathcal{Q}_{\mathrm{adj}}(\mathcal{A}) & \xrightarrow{\eta^1_{\mathcal{A}}} & \mathcal{Q}_{CAR}(\mathbf{V}) \\ {\scriptstyle \mathcal{Q}_{\mathrm{adj}}(f,g)}\downarrow & & \downarrow{\scriptstyle \mathcal{Q}_{CAR}(f)} \\ \mathcal{Q}_{\mathrm{adj}}(\mathcal{A}') & \xrightarrow[\eta^1_{\mathcal{A}'}]{} & \mathcal{Q}_{CAR}(\mathbf{V}') \end{array}$$

We may check this on generators: for any $v \in V, w \in W$ we have

$$\begin{aligned} \eta^1_{\mathcal{A}'}\mathcal{Q}_{\mathrm{adj}}(f,g)(D_{\mathbf{V}}(v) + E_{\mathbf{W}}(w)) &= B^*_{\mathbf{V}}\Big(A^{-1}_{V'}\overline{g(w)}\Big) + B_{\mathbf{V}}(f(v)) \\ \mathcal{Q}_{CAR}(f)\eta^1_{\mathcal{A}}(D_{\mathbf{V}}(v) + E_{\mathbf{W}}(w)) &= B^*_{\mathbf{V}}\Big(f(A^{-1}_{V}\overline{w})\Big) + B_{\mathbf{V}}(f(v)). \end{aligned} \tag{5.42}$$

But we recall from (5.4) that $A_{V'} \circ f = \overline{g} \circ A_V$, and so we also have $f \circ A^{-1}_V = A^{-1}_{V'} \circ \overline{g}$, which completes the proof for η^1. The proof for η^2 is similar. $\square$

We now consider a charge conjugation on a Hermitian space $\mathbf{V} = (V, s)$. The functor $\mathcal{Q}_{CAR}$ lifts the linear isomorphism $C_V : V \to \overline{V}$ to an Alg-isomorphism $\mathcal{Q}_{CAR}(C_V) : \mathcal{Q}_{CAR}(\mathbf{V}) \to \mathcal{Q}_{CAR}(\overline{\mathbf{V}})$; this in turn induces a linear involution $\mathcal{C}_{\mathbf{V}} := \Xi^{-1}_{\mathbf{V}} \circ \mathcal{Q}_{CAR}(C_V)$ of $\mathcal{Q}_{CAR}(\mathbf{V})$, where $\Xi_{\mathbf{V}} : \mathcal{Q}_{CAR}(\mathbf{V}) \to \mathcal{Q}_{CAR}(\overline{\mathbf{V}})$ is defined as in Proposition 5.2.6. We may compute the action of $\mathcal{C}_{\mathbf{V}}$ explicitly; on generators we have

$$\begin{aligned} \mathcal{C}_{\mathbf{V}} B_{\mathbf{V}}(v) &= \Xi^{-1}_{\mathbf{V}} B_{\overline{\mathbf{V}}}(C_V v) = B^*_{\mathbf{V}}(\overline{C_V v}), \\ \mathcal{C}_{\mathbf{V}} B^*_{\mathbf{V}}(v) &= \Xi^{-1}_{\mathbf{V}} B^*_{\overline{\mathbf{V}}}(C_V v) = B_{\mathbf{V}}(\overline{C_V v}), \end{aligned} \tag{5.43}$$

where $v \in V$. More abstractly we have

$$\begin{aligned} \mathcal{C}_{\mathbf{V}} \circ B_{\mathbf{V}} &= B^*_{\mathbf{V}} \circ \overline{\cdot} \circ C_V, \\ \mathcal{C}_{\mathbf{V}} \circ B^*_{\mathbf{V}} &= B_{\mathbf{V}} \circ \overline{\cdot} \circ C_V. \end{aligned} \tag{5.44}$$

Similarly, if $(C_V, C_W) : \mathcal{A} \to \overline{\mathcal{A}}$ is a charge conjugation on $\mathcal{A} = (\mathbf{V}, \mathbf{W}, A_V, A_W) \in \mathsf{HermAdj_C}$, then there are linear involutions of $\mathscr{Q}_{\mathrm{adj}}(\mathcal{A})$, given by $\mathcal{C}^i_{\mathcal{A}} := (\Upsilon^i_{\mathcal{A}})^{-1} \circ \mathscr{Q}_{\mathrm{adj}}(C_V, C_W)$, $i = 1, 2$, which act as

$$\begin{aligned} \mathcal{C}^1_{\mathcal{A}}(D_{\mathbf{V}}(v) + E_{\mathbf{W}}(w)) &= D_{\mathbf{V}}(\overline{A_W} C_W w) + E_{\mathbf{W}}(\overline{A_V} C_V v), \\ \mathcal{C}^2_{\mathcal{A}}(D_{\mathbf{V}}(v) + E_{\mathbf{W}}(w)) &= D_{\mathbf{V}}(A_V^{-1} C_W w) + E_{\mathbf{W}}(A_W^{-1} C_V v), \end{aligned} \tag{5.45}$$

where $v \in V$, $w \in W$, or more abstractly,

$$\begin{aligned} \mathcal{C}^1_{\mathcal{A}} \circ D_{\mathbf{V}} &= E_{\mathbf{W}} \circ \overline{A_V} \circ C_V, & \mathcal{C}^1_{\mathcal{A}} \circ E_{\mathbf{W}} &= D_{\mathbf{V}} \circ \overline{A_W} \circ C_W, \\ \mathcal{C}^2_{\mathcal{A}} \circ D_{\mathbf{V}} &= E_{\mathbf{W}} \circ A_W^{-1} \circ C_V, & \mathcal{C}^2_{\mathcal{A}} \circ E_{\mathbf{W}} &= D_{\mathbf{V}} \circ A_V^{-1} \circ C_W. \end{aligned} \tag{5.46}$$

It may easily be checked that

$$\mathcal{C}_{\mathbf{V}} \circ \eta^1_{\mathcal{A}} = \eta^1_{\mathcal{A}} \circ \mathcal{C}^1_{\mathcal{A}}, \qquad \mathcal{C}_{\mathbf{W}} \circ \eta^2_{\mathcal{A}} = \eta^2_{\mathcal{A}} \circ \mathcal{C}^2_{\mathcal{A}}. \tag{5.47}$$

Definition 5.2.9. *Let* $\mathbf{V} = (V, s_V)$ *and* $\mathbf{W} = (W, s_W)$ *be objects in* Herm*, and* $\mathcal{A} = (\mathbf{V}, \mathbf{W}, A_V, A_W)$ *an object in* $\mathsf{HermAdj_C}$ *with a charge conjugation* (C_V, C_W)*. The charge conjugation on* $\mathscr{Q}_{CAR}(\mathbf{V})$ *induced by* C_V *is defined to be the map* $\mathcal{C}_{\mathbf{V}}$ *described above; similarly the maps* $\mathcal{C}^1_{\mathcal{A}}, \mathcal{C}^2_{\mathcal{A}}$ *are the charge conjugations on* $\mathscr{Q}_{\mathrm{adj}}(\mathcal{A})$ *induced by* (C_V, C_W)*.*

While it may seem odd to have two charge conjugations on the algebra $\mathscr{Q}_{\mathrm{adj}}(\mathcal{A})$, rather than a single canonical conjugation, it is necessary in order to retain the symmetry of the adjoint structure. However, it will be convenient from now on to regard $\mathbf{V}$ as the *principal* vector space of $\mathcal{A}$ and $\mathbf{W}$ as the *auxiliary* space (without forgetting that interchanging the roles of $\mathbf{V}$ and $\mathbf{W}$ gives an equivalent algebra); we call $\mathcal{C}^1_{\mathcal{A}}$ the *principal* charge conjugation of $\mathscr{Q}_{\mathrm{adj}}(\mathcal{A})$ (or simply *the* charge conjugation), and $\mathcal{C}^2_{\mathcal{A}}$ the *auxiliary* charge conjugation.

5.2.3 The even subalgebra

Recall that for any weakly graded unital $*$-algebra $\mathfrak{A} = \bigoplus_{n=0}^{\infty} \mathfrak{A}_n$, the even ordered grades together form a unital sub-$*$-algebra, since the product of any two even elements is itself even. In the case of the algebras $\mathscr{Q}_{CAR}(\mathbf{V})$ or $\mathscr{Q}_{\mathrm{adj}}(\mathcal{A})$, where $\mathbf{V} \in \mathsf{Herm}$ and $\mathcal{A} = (\mathbf{V}, \mathbf{W}, A_V, A_W) \in \mathsf{HermAdj}_{\mathsf{C}}$, note that the non-unital generators of the subalgebra are of the form $B_{\mathbf{V}}(v)B_{\mathbf{V}}(v')$ for $v, v' \in \mathbf{V}$, or $(D_{\mathbf{V}}(v) + E_{\mathbf{W}}(w))(D_{\mathbf{V}}(v') + E_{\mathbf{W}}(w'))$ for $v, v' \in \mathbf{V}$ and $w, w' \in \mathbf{W}$. The full list of relations that these generators satisfy is not so tractable, however. We will deal only with the case of the functor $\mathscr{Q}_{\mathrm{adj}}$, as the situation for $\mathscr{Q}_{CAR}$ is similar, using the squared adjoint structures defined in the previous section.

Proposition 5.2.10. *Let $\mathcal{A} = (\mathbf{V}, \mathbf{W}, A_V, A_W) \in \mathsf{HermAdj}_{\mathsf{C}}$. Then the even subalgebra $\mathscr{Q}_{\mathrm{ev}}(\mathcal{A})$ of $\mathscr{Q}_{\mathrm{adj}}(\mathcal{A})$ is generated, under a suitable identification, by elements $\mathbf{1}$ and $F_{\mathcal{A}}(\boldsymbol{a})$, $\boldsymbol{a} \in \mathfrak{S}(\mathcal{A})$, subject to:*

$$F_{\mathcal{A}}(\boldsymbol{a})^* = F_{\mathcal{A}}(\boldsymbol{a}^*) \tag{5.48a}$$

$$F_{\mathcal{A}}(\lambda \boldsymbol{a} + \mu \boldsymbol{a}') = \lambda F_{\mathcal{A}}(\boldsymbol{a}) + \mu F_{\mathcal{A}}(\boldsymbol{a}') \tag{5.48b}$$

$$F_{\mathcal{A}}(\boldsymbol{a} + \Phi \boldsymbol{a}) = \omega(\boldsymbol{a})\mathbf{1} \tag{5.48c}$$

$$\left(F_{\mathcal{A}}{}^{\otimes 2}\right)\left((\mathbf{1} + Z)(\boldsymbol{a} \otimes \boldsymbol{a}')\right) = \left(\omega \otimes F_{\mathcal{A}}\right)\left(Z(\Phi \boldsymbol{a} \otimes \boldsymbol{a}')\right) \tag{5.48d}$$

for $\boldsymbol{a}, \boldsymbol{a}' \in \mathfrak{S}(\mathcal{A})$ and $\lambda, \mu \in \mathbb{C}$.

Proof. We will show that the above generators and relations generate $\mathscr{Q}_{\mathrm{ev}}(\mathcal{A})$, under the identification

$$F_{\mathcal{A}}(\langle v, w\rangle \otimes \langle v', w'\rangle) = (D_{\mathbf{V}}(v) + E_{\mathbf{W}}(w))(D_{\mathbf{V}}(v') + E_{\mathbf{W}}(w')). \tag{5.49}$$

To do this we must show that the above relations are indeed satisfied given this assignment; then we must demonstrate that these relations are sufficient to define the algebra completely.

For $v, v' \in \mathbf{V}$ and $w, w' \in \mathbf{W}$ we refer to Definition 5.1.9 to see that

$$\begin{aligned} F_{\mathcal{A}}(\langle v, w\rangle \otimes \langle v', w'\rangle)^* &= (D_{\mathbf{V}}(v') + E_{\mathbf{W}}(w'))^*(D_{\mathbf{V}}(v) + E_{\mathbf{W}}(w))^* \\ &= (E_{\mathbf{W}}(\overline{A_V v'}) + D_{\mathbf{V}}(A_V^{-1}\overline{w'}))(E_{\mathbf{W}}(\overline{A_V v}) + D_{\mathbf{V}}(A_V^{-1}\overline{w})) \\ &= F_{\mathcal{A}}(\langle A_V^{-1}\overline{w'}, \overline{A_V v'}\rangle \otimes \langle A_V^{-1}\overline{w'}, \overline{A_V v'}\rangle) \\ &= F_{\mathcal{A}}((\langle v, w\rangle \otimes \langle v', w'\rangle)^*). \end{aligned} \tag{5.50}$$

$F_{\mathscr{A}}$ is clearly $\mathbb{C}$-linear by definition, and

$$\begin{aligned} F_{\mathscr{A}}\Big(\langle v,w\rangle \otimes \langle v',w'\rangle + \Phi(\langle v,w\rangle \otimes \langle v',w'\rangle)\Big) \\ &= \{D_{\mathbf{V}}(v)+E_{\mathbf{W}}(w), D_{\mathbf{V}}(v')+E_{\mathbf{W}}(w')\} \\ &= \{D_{\mathbf{V}}(v), E_{\mathbf{W}}(w')\} + \{D_{\mathbf{V}}(v'), E_{\mathbf{W}}(w)\} \\ &= (s_W(\overline{A_V v}, w') + s_W(\overline{A_V v'}, w))\mathbf{1} \\ &= \omega(\langle v,w\rangle \otimes \langle v',w'\rangle)\mathbf{1}. \end{aligned} \tag{5.51}$$

Now, consider $\boldsymbol{u}_1 \ldots, \boldsymbol{u}_4 \in V \oplus W$ with $\boldsymbol{u}_i = \langle v_i, w_i\rangle$, and let $A_i = D_{\mathbf{V}}(v_i) + E_{\mathbf{W}}(w_i)$. We have $(\mathbf{1}+Z)((\boldsymbol{u}_1\otimes\boldsymbol{u}_2)\otimes(\boldsymbol{u}_3\otimes\boldsymbol{u}_4)) = (\boldsymbol{u}_1\otimes\boldsymbol{u}_2)\otimes(\boldsymbol{u}_3\otimes\boldsymbol{u}_4) + (\boldsymbol{u}_1\otimes\boldsymbol{u}_3)\otimes(\boldsymbol{u}_2\otimes\boldsymbol{u}_4)$, so

$$\begin{aligned} ({F_{\mathscr{A}}}^{\otimes 2})((1+Z)(\boldsymbol{u}_1\otimes\boldsymbol{u}_2)\otimes(\boldsymbol{u}_3\otimes\boldsymbol{u}_4)) &= A_1\{A_2,A_3\}A_4 \\ &= \omega(\boldsymbol{u}_2\otimes\boldsymbol{u}_3)F_{\mathscr{A}}(\boldsymbol{u}_1\otimes\boldsymbol{u}_4), \end{aligned} \tag{5.52}$$

where we have used (5.11) and (5.38c), (5.38d). Since $Z(\Phi(\boldsymbol{u}_1\otimes\boldsymbol{u}_2)\otimes(\boldsymbol{u}_3\otimes\boldsymbol{u}_4)) = (\boldsymbol{u}_2\otimes\boldsymbol{u}_3)\otimes(\boldsymbol{u}_1\otimes\boldsymbol{u}_4)$, it follows that (5.48d) is satisfied.

In order to show that the relations in (5.48a)–(5.48d) are, in fact, sufficient to generate the even subalgebra, we show that the relations on even elements that follow from (5.38a)–(5.38d) may be recovered.

Since even elements are all sums of terms of the form $A_1 \cdots A_{2k}$, where $A_i = D_{\mathbf{V}}(v_i) + E_{\mathbf{W}}(w_i)$ for $\langle v_i, w_i\rangle = \boldsymbol{u}_i \in V \oplus W$, we need only check the relations on these terms. The first relation, (5.38a) is concerned with the action of the star operation; for even elements, the relation that must be recovered is precisely that

$$(A_1 \cdots A_{2k})^* = A_{2k}^* \cdots A_1^*, \tag{5.53}$$

where $A_i^* = D_{\mathbf{V}}(A_V^{-1}\overline{w}) + E_{\mathbf{W}}(\overline{A_V v})$. Using the relation (5.48a), we have

$$\begin{aligned}(A_1 \cdots A_{2k})^* &= (A_{2k-1}A_{2k})^* \cdots (A_1 A_2)^* \\ &= F_{\mathcal{A}}(\boldsymbol{u}_{2k-1} \otimes \boldsymbol{u}_{2k})^* \cdots F_{\mathcal{A}}(\boldsymbol{u}_1 \otimes \boldsymbol{u}_2)^* \\ &= (A_{2k}^* A_{2k-1}^*) \cdots (A_2^* A_1^*) \end{aligned} \tag{5.54}$$

as required, where we have used the first two lines of (5.50).

The second relation (5.38b) is concerned with linearity of the maps $v \mapsto D_{\mathbf{V}}(v)$ and $w \mapsto E_{\mathbf{W}}(w)$. On even elements, it suffices to show that

$$\begin{aligned} AD_{\mathbf{V}}(\lambda v_1 + \mu v_2) &= \lambda AD_{\mathbf{V}}(v_1) + \mu AD_{\mathbf{V}}(v_2), \\ D_{\mathbf{V}}(\lambda v_1 + \mu v_2)A &= \lambda D_{\mathbf{V}}(v_1)A + \mu D_{\mathbf{V}}(v_2)A, \end{aligned} \tag{5.55}$$

where $A = (D_{\mathbf{V}}(v) + E_{\mathbf{W}}(w))$, with similar expressions for $E_{\mathbf{W}}$ (which in fact follow from the above via the adjoint property already established). This may be shown to hold using (5.48b).

The anticommutation relations (5.38c), (5.38d) give us precisely the information we need to calculate the effect of transposing two generators in a pure product. We must therefore be able to derive from the relations in (5.48) expressions of the form

$$A_1 \cdots \{A_r, A_{r+1}\} \cdots A_{2k} = S(A_r, A_{r+1}) A_1 \cdots A_{r-1} A_{r+2} \cdots A_{2k}, \tag{5.56}$$

for some $1 \leq r < 2k$, where

$$S(A_r, A_{r+1}) = s_W(\overline{A_V v_r}, w_{r+1}) + s_W(\overline{A_V v_{r+1}}, w_r) = \omega(\boldsymbol{u}_r \otimes \boldsymbol{u}_{r+1}). \tag{5.57}$$

This may be recovered via (5.48c) or (5.48d), depending on whether r is odd or even; when it is odd, we have

$$\begin{aligned}
A_1 \cdots \{A_r{,}A_{r+1}\} \cdots A_{2k}&\\
&= A_1 \cdots A_{r-1} F_{\mathscr{A}}(\boldsymbol{u}_r \otimes \boldsymbol{u}_{r+1} + \boldsymbol{u}_{r+1} \otimes \boldsymbol{u}_r) A_{r+2} \cdots A_{2k}\\
&= A_1 \cdots A_{r-1} F_{\mathscr{A}}((\mathbf{1}+\Phi)\boldsymbol{u}_r \otimes \boldsymbol{u}_{r+1}) A_{r+2} \cdots A_{2k}\\
&= \omega(\boldsymbol{u}_r \otimes \boldsymbol{u}_{r+1}) A_1 \cdots A_{r-1} A_{r+2} \cdots A_{2k}\\
&= S(A_r, A_{r+1}) A_1 \cdots A_{r-1} A_{r+2} \cdots A_{2k}.
\end{aligned} \tag{5.58}$$

On the other hand, when r is even, we use a similar process with (5.48d), noting that

$$\begin{aligned}
A_{r-1}\{A_r, A_{r+1}\}A_{r+2} &= (F_{\mathscr{A}}{}^{\otimes 2})[(\mathbf{1}+Z)(\boldsymbol{u}_{r-1} \otimes \boldsymbol{u}_r) \otimes (\boldsymbol{u}_{r+1} \otimes \boldsymbol{u}_{r+2})]\\
&= \omega(\boldsymbol{u}_r \otimes \boldsymbol{u}_{r+1}) A_{r-1} A_{r+2}.
\end{aligned} \tag{5.59}$$

Therefore all properties deriving from the relations (5.38a)–(5.38d) are recoverable, and so (5.48a)–(5.48d) are sufficient to define $\mathscr{Q}_{ev}(\mathscr{A})$. □

Note in particular that the relations in the above proposition are sufficient to define the commutator of two generators $F_{\mathscr{A}}(\boldsymbol{a})$ and $F_{\mathscr{A}}(\boldsymbol{a}')$ in terms of a linear combination of single generators, as follows: if X is the map that sends $\boldsymbol{a} \otimes \boldsymbol{a}'$ to $\boldsymbol{a}' \otimes \boldsymbol{a}$, then $[F_{\mathscr{A}}(\boldsymbol{a}), F_{\mathscr{A}}(\boldsymbol{a}')] = (F_{\mathscr{A}}{}^{\otimes 2})((\mathbf{1}-X)(\boldsymbol{a} \otimes \boldsymbol{a}'))$. Writing $\Phi_1 = \Phi \otimes \mathbf{1}$ and $\Phi_2 = \mathbf{1} \otimes \Phi$, we use the defining properties of Z given in Definition 5.1.8 to see that

$$\begin{aligned}
1 - X &= (\mathbf{1}+Z) - (Z + \Phi_2 Z) + (\Phi_2 Z + \Phi^{\otimes 2} Z) - (\Phi^{\otimes 2} Z + X)\\
&= (\mathbf{1}+Z) - (\mathbf{1}+\Phi_2)Z + (\mathbf{1}+\Phi_1)\Phi^{\otimes 2} Z - (ZX + X)\\
&= (\mathbf{1}+Z) - (\mathbf{1}+\Phi_2)Z + (\mathbf{1}+\Phi_1)ZX - (\mathbf{1}+Z)X.
\end{aligned} \tag{5.60}$$

Now, from (5.48c) and (5.48d) we see that $F_{\mathscr{A}}(\mathbf{1}+\Phi) = \omega$ and $F_{\mathscr{A}}{}^{\otimes 2}(\mathbf{1}+Z) = (\omega \otimes F_{\mathscr{A}})Z\Phi_1$; we also have $(F_{\mathscr{A}} \otimes \omega)X = \omega \otimes F_{\mathscr{A}}$, so

$$\begin{aligned}
F_{\mathscr{A}}{}^{\otimes 2}(1 - X) &= (\omega \otimes F_{\mathscr{A}})(Z\Phi_1 + ZX - Z\Phi_1 X) - (F_{\mathscr{A}} \otimes \omega)Z\\
&= (\omega \otimes F_{\mathscr{A}})(Z\Phi_1 + ZX - ZX\Phi_2 - XZ).
\end{aligned} \tag{5.61}$$

When applied to $\boldsymbol{a} \otimes \boldsymbol{a}'$, this operator produces a sum of multiples of single generators.

Definition 5.2.11. *For any object* $\mathbf{U} \in \mathsf{SAdj}$*, let* $\mathscr{Q}_{SA}(\mathbf{U})$ *be the* $*$*-algebra generated by* $\mathbf{1}$ *and* $F_{\mathbf{U}}(\boldsymbol{a})$*,* $\boldsymbol{a} \in \mathbf{U}$*, subject to the relations* (5.48a)–(5.48d)*. For any* SAdj*-arrow* $f : \mathbf{U} \to \mathbf{U}'$*, define* $\mathscr{Q}_{SA}(f)$ *to act on generators by* $\mathbf{1}_{\mathscr{Q}_{SA}(\mathbf{U})} \mapsto \mathbf{1}_{\mathscr{Q}_{SA}(\mathbf{U}')}$ *and* $F_{\mathbf{U}}(\boldsymbol{a}) \mapsto F_{\mathbf{U}'}(f(\boldsymbol{a}))$*.*

Proposition 5.2.12. *For any* $f : \mathbf{U} \to \mathbf{U}'$ *in* SAdj*,* $\mathscr{Q}_{SA}(f)$ *may be extended uniquely to a well-defined arrow in* Alg*. Moreover,* $\mathscr{Q}_{SA}$ *is a covariant functor from* SAdj *to* Alg*.*

Proof. Since $\mathscr{Q}_{SA}(f)$ is defined on generators, uniqueness of extension is obvious; we still need to show well-definedness. This is a matter of checking that $\mathscr{Q}_{SA}(f)$ is compatible with the relations (5.48a)–(5.48d), which can be easily done by referring to the definition of an arrow in SAdj given in (5.10). We still need to show that $\mathscr{Q}_{SA}(f)$ is injective for any f; this can be done via Lemma 5.2.2.

For functoriality, note that the identity arrow $\mathrm{id}_{\mathbf{U}}$ on $\mathbf{U}$ is clearly mapped by $\mathscr{Q}_{SA}$ to the identity arrow on $\mathscr{Q}_{SA}(\mathbf{U})$, and if $f : \mathbf{U} \to \mathbf{U}'$, $f' : \mathbf{U}' \to \mathbf{U}''$ in SAdj then $\mathscr{Q}_{SA}(f' \circ f) \circ F_{\mathbf{U}} = F_{\mathbf{U}''} \circ f' \circ f = \mathscr{Q}_{SA}(f') \circ \mathscr{Q}_{SA}(f) \circ F_{\mathbf{U}}$. As $\mathscr{Q}_{SA}(f' \circ f)$ and $\mathscr{Q}_{SA}(f') \circ \mathscr{Q}_{SA}(f)$ have identical action on generators they are equal. □

Therefore the even subalgebra of $\mathscr{Q}_{\mathrm{adj}}(\mathcal{A})$ may be obtained functorially by applying $\mathscr{Q}_{SA}\mathfrak{S}$ to $\mathcal{A}$. Moreover, if we denote the embedding of $\mathscr{Q}_{SA}(\mathfrak{S}(\mathcal{A}))$ in $\mathscr{Q}_{\mathrm{adj}}(\mathcal{A})$ obtained from (5.49) by $\epsilon_{\mathcal{A}}$, then $\epsilon = (\epsilon_{\mathcal{A}})_{\mathcal{A} \in \mathsf{HermAdj}_{\mathsf{C}}}$ is a natural transformation: if $\mathcal{A} = (\mathbf{V}, \mathbf{W}, A_V, A_W)$ and $\mathcal{A}' = (\mathbf{V}', \mathbf{W}', A_{V'}, A_{W'})$ with an arrow $(f, g) : \mathcal{A} \to \mathcal{A}'$, then

$$\begin{aligned}
\Big[\epsilon_{\mathcal{A}'} \circ \mathscr{Q}_{SA}(\mathfrak{S}(f,g)) \circ F_{\mathcal{A}}\Big]&(\langle v, w\rangle \otimes \langle v', w'\rangle) \\
&= \epsilon_{\mathcal{A}'}\Big(F_{\mathcal{A}'}(\langle f(v), g(w)\rangle \otimes \langle f(v'), g(w')\rangle)\Big) \\
&= (D_{\mathbf{V}'}(f(v)) + E_{\mathbf{W}'}(g(w)))(D_{\mathbf{V}'}(f(v')) + E_{\mathbf{W}'}(g(w'))) \\
&= \mathscr{Q}_{\mathrm{adj}}(f, g)\left((D_{\mathbf{V}}(v) + E_{\mathbf{W}}(w))(D_{\mathbf{V}}(v') + E_{\mathbf{W}}(w'))\right) \\
&= \Big[\mathscr{Q}_{\mathrm{adj}}(f, g) \circ \epsilon_{\mathcal{A}} \circ F_{\mathcal{A}}\Big](\langle v, w\rangle \otimes \langle v', w'\rangle). \qquad (5.62)
\end{aligned}$$

Therefore $\epsilon_{\mathscr{A}'} \circ \mathscr{Q}_{SA}(\mathfrak{S}(f,g)) = \mathscr{Q}_{\text{adj}}(f,g) \circ \epsilon_{\mathscr{A}}$.

5.3 Quantization functors

We expect to see CAR algebras arise as the algebras of observables assigned to spacetimes by a locally covariant fermionic quantum theory. Consequently we will refer to $\mathscr{Q}_{CAR}$ and $\mathscr{Q}_{\text{adj}}$ as *quantization functors*. We wish to examine cases in which such a quantum theory $\mathscr{A} : \mathsf{Loc} \to \mathsf{Alg}$ can be expressed as the composition of a classical theory with a quantization functor; either when $\mathscr{A} = \mathscr{Q}_{CAR}\mathscr{L}$, where $\mathscr{L}$ is a covariant functor from Loc to Herm, or when $\mathscr{A} = \mathscr{Q}_{\text{adj}}\mathscr{L}_{\text{adj}}$, where $\mathscr{L}_{\text{adj}}$ is a covariant functor from Loc to $\mathsf{HermAdj}_{\mathsf{C}}$. Where an explicit distinction is necessary, we will refer to theories from Loc to Herm as *undoubled* theories, and to theories from Loc to $\mathsf{HermAdj}_{\mathsf{C}}$ as *doubled* theories. Note that any doubled theory $\mathscr{L}_{\text{adj}}$ induces two undoubled theories $\mathscr{F}_1\mathscr{L}_{\text{adj}}$ and $\mathscr{F}_2\mathscr{L}_{\text{adj}}$; as a consequence of Proposition 5.2.8, the resulting quantized theories $\mathscr{Q}_{CAR}\mathscr{F}_i\mathscr{L}_{\text{adj}}$ are naturally isomorphic to $\mathscr{Q}_{\text{adj}}\mathscr{L}_{\text{adj}}$. Moreover, we also have the following result:

Lemma 5.3.1. *Let $\mathscr{L}_{\text{adj}} : \mathsf{Loc} \to \mathsf{HermAdj}_{\mathsf{C}}$ be a doubled classical theory, with $\mathscr{L}_{\text{adj}}(\boldsymbol{M}) := (\mathbf{V}_{\boldsymbol{M}}, \mathbf{W}_{\boldsymbol{M}}, A^1_{\boldsymbol{M}}, A^2_{\boldsymbol{M}})$ for each spacetime $\boldsymbol{M}$. Then the transformations $A^1 : \mathscr{F}_1\mathscr{L}_{\text{adj}} \stackrel{.}{\longrightarrow} \overline{\mathscr{F}_2\mathscr{L}_{\text{adj}}}$ and $A^2 : \mathscr{F}_2\mathscr{L}_{\text{adj}} \stackrel{.}{\longrightarrow} \overline{\mathscr{F}_1\mathscr{L}_{\text{adj}}}$ with components $A^1_{\boldsymbol{M}}, A^2_{\boldsymbol{M}}$ respectively are natural isomorphisms.*

Proof. We need to show that for any Loc-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$, we have $\overline{\mathscr{F}_2(\mathscr{L}_{\text{adj}}(\psi))} \circ A^1_{\boldsymbol{M}} = A^1_{\boldsymbol{N}} \circ \mathscr{F}_1(\mathscr{L}_{\text{adj}}(\psi))$ and $\overline{\mathscr{F}_1(\mathscr{L}_{\text{adj}}(\psi))} \circ A^2_{\boldsymbol{M}} = A^2_{\boldsymbol{N}} \circ \mathscr{F}_2(\mathscr{L}_{\text{adj}}(\psi))$. This follows immediately from the commutativity of (5.4), and (5.6). □

Proposition 5.3.2. *Consider an undoubled theory $\mathscr{L}$, and suppose that $\mathscr{D} : \mathsf{Loc} \to \mathsf{Test}$ is a functor which constructs some space of test functions for each spacetime. Given a locally covariant solution $G : \mathscr{D} \stackrel{.}{\longrightarrow} \mathscr{L}$ defined as in Section 2.3, we may define locally covariant fields $\Phi, \Phi^* : \mathscr{D} \stackrel{.}{\longrightarrow} \mathscr{A} := \mathscr{Q}_{CAR}\mathscr{L}$ by $\Phi^{(*)}_{\boldsymbol{M}} := B^{(*)}_{\mathscr{L}(\boldsymbol{M})} \circ G_{\boldsymbol{M}}$.*

Proof. Let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be an arrow in Loc. We must show that $\mathscr{A}(\psi)\circ\Phi^{(*)}_{\boldsymbol{M}} = \Phi^{(*)}_{\boldsymbol{N}} \circ \mathscr{D}(\psi)$; this follows from (5.36) and (2.26). □

While the nature of a locally covariant solution for an undoubled theory is clear, it is not so clear what happens for a doubled theory $\mathscr{L}_{\mathrm{adj}}$. For a given spacetime $\boldsymbol{M}$ we might wish to consider a single test function space $\mathscr{D}(\boldsymbol{M})$ as for the undoubled case, and a pair of maps $G^i_{\boldsymbol{M}} : \mathscr{D}(\boldsymbol{M}) \to \mathscr{F}_i(\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M}))$, $i = 1, 2$, which would then satisfy the naturality condition $G^i_{\boldsymbol{N}} \circ \mathscr{D}(\psi) = \mathscr{F}_i(\mathscr{L}_{\mathrm{adj}}(\psi)) \circ G^i_{\boldsymbol{M}}$ for any $\psi : \boldsymbol{M} \to \boldsymbol{N}$. Alternatively, we might wish to consider a doubled test functor which assigns to each $\boldsymbol{M}$ a pair of test spaces $\mathscr{D}_1(\boldsymbol{M})$ and $\mathscr{D}_2(\boldsymbol{M})$ and to each $\psi : \boldsymbol{M} \to \boldsymbol{N}$ a pair of arrows $(\mathscr{D}_1(\psi), \mathscr{D}_2(\psi)) : \mathscr{D}_1(\boldsymbol{M})\oplus\mathscr{D}_2(\boldsymbol{M}) \to \mathscr{D}_1(\boldsymbol{N})\oplus\mathscr{D}_2(\boldsymbol{N})$, and then define $G^i_{\boldsymbol{M}} : \mathscr{D}_i(\boldsymbol{M}) \to \mathscr{F}_i(\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M}))$ satisfying $G^i_{\boldsymbol{N}}\circ\mathscr{D}_i(\psi) = \mathscr{F}_i(\mathscr{L}_{\mathrm{adj}}(\psi))\circ G^i_{\boldsymbol{M}}$. We will insist in this case that $\mathscr{D}_1(\boldsymbol{M})$ is isomorphic to $\mathscr{D}_2(\boldsymbol{M})$, so in fact the two alternatives are equivalent, but we find that the latter is more intuitive, so this is the approach we will use. We write $\mathscr{D}_{\mathrm{adj}} := \mathscr{D}_1 \oplus \mathscr{D}_2$ and $G^{\mathrm{adj}} := (G^1, G^2)$, so the naturality condition may be written

$$G^{\mathrm{adj}}_{\boldsymbol{N}} \circ \mathscr{D}_{\mathrm{adj}}(\psi) = \mathscr{L}_{\mathrm{adj}}(\psi) \circ G^{\mathrm{adj}}_{\boldsymbol{M}}. \tag{5.63}$$

Proposition 5.3.3. *Consider a doubled theory $\mathscr{L}_{\mathrm{adj}}$, and suppose that $\mathscr{D}_{\mathrm{adj}}$ and G^{adj} are defined as above. We may define locally covariant fields $\Psi^i : \mathscr{D}_i \stackrel{.}{\longrightarrow} \mathscr{A}_{\mathrm{adj}} := \mathscr{Q}_{\mathrm{adj}}\mathscr{L}_{\mathrm{adj}}$, $i = 1, 2$, by*

$$\begin{aligned}\Psi^1_{\boldsymbol{M}} &:= D_{\mathscr{F}_1(\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M}))} \circ G^1_{\boldsymbol{M}},\\ \Psi^2_{\boldsymbol{M}} &:= E_{\mathscr{F}_2(\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M}))} \circ G^2_{\boldsymbol{M}}.\end{aligned} \tag{5.64}$$

Alternatively, we may consider this to be a single locally covariant field, by $\Psi^d := \Psi^1 \oplus \Psi^2 : \mathscr{D}_{\mathrm{adj}} \stackrel{.}{\longrightarrow} \mathscr{A}_{\mathrm{adj}}$ (i.e. $\Psi^d_{\boldsymbol{M}}(v, w) = \Psi^1_{\boldsymbol{M}}(v) + \Psi^2_{\boldsymbol{M}}(w)$).

Proof. Similarly to the previous result, this follows from (5.39) and (5.63). □

Definition 5.3.4. *A* charge conjugation *on an undoubled theory $\mathscr{L}$ is a natural isomorphism $C : \mathscr{L} \stackrel{.}{\longrightarrow} \overline{\mathscr{L}}$ with the property that $\overline{C} \circ C = \mathrm{id}_{\mathscr{L}}$, where*

$\overline{\mathscr{L}}$ *is the composition* $\bar{\cdot} \circ \mathscr{L}$ *and* $\overline{C} : \overline{\mathscr{L}} \stackrel{.}{\longrightarrow} \mathscr{L}$ *is the natural transformation with components* $\overline{C_{\boldsymbol{M}}}$.

A charge conjugation *on a doubled theory* $\mathscr{L}_{\text{adj}}$ *is a natural isomorphism* $(C^1, C^2) : \mathscr{L}_{\text{adj}} \stackrel{.}{\longrightarrow} \overline{\mathscr{L}_{\text{adj}}}$ *with the property that* $\overline{(C^1, C^2)} \circ (C^1, C^2) = \text{id}_{\mathscr{L}_{\text{adj}}}$, *where* $\overline{\mathscr{L}_{\text{adj}}}$ *is the composition* $\bar{\cdot} \circ \mathscr{L}_{\text{adj}}$ *and* $\overline{(C^1, C^2)} : \overline{\mathscr{L}_{\text{adj}}} \stackrel{.}{\longrightarrow} \mathscr{L}_{\text{adj}}$ *is the natural transformation with components* $\overline{(C^1_{\boldsymbol{M}}, C^2_{\boldsymbol{M}})}$.

Note that this definition entails that each component of a charge conjugation on an undoubled theory is itself a charge conjugation at the level of Hermitian spaces, and that for every arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$ in Loc, the components $C_{\boldsymbol{M}}$ and $C_{\boldsymbol{N}}$ are compatible with $\mathscr{L}(\psi)$ in the sense of Definition 5.1.7. Similarly, each component of a charge conjugation on a doubled theory is a charge conjugation at the level of Hermitian adjoint structures, and for every arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$ in Loc, the components $C^i_{\boldsymbol{M}}$ and $C^i_{\boldsymbol{N}}$ are compatible with $\mathscr{F}_i(\mathscr{L}_{\text{adj}}(\psi))$ in the sense of Definition 5.1.7, for $i = 1, 2$.

Proposition 5.3.5. *Let* $\mathscr{L}$ *be an undoubled theory, and* $\mathscr{L}_{\text{adj}}$ *a doubled theory. Every charge conjugation* C *on* $\mathscr{L}$ *induces a natural automorphism* χ *of the quantized theory* $\mathscr{A}\mathscr{Q}_{CAR}\mathscr{L}$ *with components* $\chi_{\boldsymbol{M}} = \mathcal{C}_{\mathscr{L}(\boldsymbol{M})}$ *as defined in* (5.44), *and every charge conjugation* (C^1, C^2) *on* $\mathscr{L}_{\text{adj}}$ *induces natural automorphisms* χ^1, χ^2 *of the quantized theory* $\mathscr{A}_{\text{adj}} := \mathscr{Q}_{\text{adj}}\mathscr{L}_{\text{adj}}$ *with components* $\chi^i_{\boldsymbol{M}} = \mathcal{C}^i_{\mathscr{L}_{\text{adj}}(\boldsymbol{M})}$, $i = 1, 2$, *as defined in* (5.46).

Proof. The components of χ and χ^i have already been shown to be automorphisms, so we need only show naturality. Let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be an arrow in Loc; since C is natural we have $\overline{\mathscr{L}(\psi)} \circ C_{\boldsymbol{M}} = C_{\boldsymbol{N}} \circ \mathscr{L}(\psi)$, so

$$\begin{aligned} \mathscr{A}(\psi) \circ \chi_{\boldsymbol{M}} \circ B_{\mathscr{L}(\boldsymbol{M})} &= \mathscr{Q}_{CAR}(\mathscr{L}(\psi)) \circ B^*_{\mathscr{L}(\boldsymbol{M})} \circ \bar{\cdot} \circ C_{\boldsymbol{M}} \\ &= B^*_{\mathscr{L}(\boldsymbol{N})} \circ \bar{\cdot} \circ \overline{\mathscr{L}(\psi)} \circ C_{\boldsymbol{M}} \\ &= B^*_{\mathscr{L}(\boldsymbol{N})} \circ \bar{\cdot} \circ C_{\boldsymbol{N}} \circ \mathscr{L}(\psi) \\ &= \chi_{\boldsymbol{N}} \circ B_{\mathscr{L}(\boldsymbol{N})} \circ \mathscr{L}(\psi) \\ &= \chi_{\boldsymbol{N}} \circ \mathscr{A}(\psi) \circ B_{\mathscr{L}(\boldsymbol{M})}. \end{aligned} \tag{5.65}$$

Similarly $\mathscr{A}(\psi) \circ \chi_{\boldsymbol{M}} \circ B^*_{\mathscr{L}(\boldsymbol{M})} = \chi_{\boldsymbol{N}} \circ \mathscr{A}(\psi) \circ B^*_{\mathscr{L}(\boldsymbol{M})}$; since the ranges of $B_{\mathscr{L}(\boldsymbol{M})}$

and $B^*_{\mathscr{L}(\boldsymbol{M})}$ generate $\mathscr{A}(\boldsymbol{M})$ it follows that χ is a natural automorphism of $\mathscr{A}$.

The result for the doubled theory then follows immediately from the undoubled case, using (5.47). □

Since the maps $\mathcal{C}_{\mathscr{L}(\boldsymbol{M})}$ and $\mathcal{C}^i_{\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M})}$ are involutions, it follows that the natural transformations χ and χ^i are also involutions; i.e. $\chi \circ \chi = \mathrm{id}_{\mathscr{Q}_{CAR}\mathscr{L}}$ and $\chi^i \circ \chi^i = \mathrm{id}_{\mathscr{A}}$, $i = 1, 2$.

5.4 Dynamical locality of quantized theories

It was shown in [26] that while dynamical locality of a classical theory does not immediately confer dynamical locality on the CCR-quantized theory, it is possible to find a small number of additional conditions on a classical theory which, if satisfied, are sufficient for the quantized theory to be dynamically local. We claim that the following similar conditions perform an analogous role for CAR-quantized theories. We consider a weakly nondegenerate classical theory $\mathscr{L} : \mathsf{Loc} \to \mathsf{Herm}$ which obeys the timeslice axiom, and denote the Hermitian space $\mathscr{L}(\boldsymbol{M})$ for $\boldsymbol{M} \in \mathsf{Loc}$ by $(V_{\boldsymbol{M}}, s_{\boldsymbol{M}})$. The relative Cauchy evolution induced by a perturbation $\boldsymbol{h} \in H(\boldsymbol{M})$ is denoted $R_{\boldsymbol{M}}[\boldsymbol{h}]$. The conditions are:

(H1) $\mathscr{L}$ has a smooth stress-energy tensor: the relative Cauchy evolution is differentiable in the weak topology induced by $s_{\boldsymbol{M}}$ and the derivative $F_{\boldsymbol{M}}[\boldsymbol{h}] = \frac{d}{ds} R_{\boldsymbol{M}}[s\boldsymbol{h}]\Big|_{s=0}$ satisfies

$$s_{\boldsymbol{M}}(v, F_{\boldsymbol{M}}[\boldsymbol{h}]v) = \frac{i}{2} \int_{\boldsymbol{M}} h_{\mu\nu} T^{\mu\nu}_{\boldsymbol{M}}[v] \, dvol_{\boldsymbol{M}}, \tag{5.66}$$

where $\boldsymbol{T}_{\boldsymbol{M}}[v] \in C^\infty(T^2_0(\boldsymbol{M}))$ is a smooth conserved real symmetric tensor field for each $v \in V_{\boldsymbol{M}}$.

(H2) For each $O \in \mathscr{O}(\boldsymbol{M})$ and $\boldsymbol{h} \in H(\boldsymbol{M}; O)$, we have

$$\mathrm{im}\, F_{\boldsymbol{M}}[\boldsymbol{h}] \subset \hat{\mathscr{L}}^{\mathrm{kin}}(\boldsymbol{M}; O). \tag{5.67}$$

(H3) $\mathscr{L}$ obeys *extended locality*: for any spacetimes $\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{N}$ with arrows $\psi_i : \boldsymbol{M}_i \to \boldsymbol{N}$ such that $\psi_1(\boldsymbol{M}_1)$ and $\psi_2(\boldsymbol{M}_2)$ are spacelike separated, we have

$$\mathscr{L}(\psi_1)(\mathscr{L}(\boldsymbol{M}_1)) \cap \mathscr{L}(\psi_2)(\mathscr{L}(\boldsymbol{M}_2)) = \{0\}. \tag{5.68}$$

(H4) For every $K \in \mathscr{K}(\boldsymbol{M})$, we have

$$\mathscr{L}^\bullet(\boldsymbol{M}; K) = \bigcap_{\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)} \ker F_{\boldsymbol{M}}[\boldsymbol{h}]. \tag{5.69}$$

Note that the existence of $\boldsymbol{T}_{\boldsymbol{M}}[v]$ as defined in (H1) is equivalent to the existence of a polarized stress-energy tensor, i.e. for each $v, w \in V_{\boldsymbol{M}}$ there is a smooth symmetric tensor field $\boldsymbol{T}_{\boldsymbol{M}}[v, w] \in C^\infty(T^2_0(\boldsymbol{M}))$ satisfying

$$s_{\boldsymbol{M}}(v, F_{\boldsymbol{M}}[\boldsymbol{h}]w) = \frac{i}{2} \int_{\boldsymbol{M}} h_{\mu\nu} T^{\mu\nu}_{\boldsymbol{M}}[v, w] \, dvol_{\boldsymbol{M}} \tag{5.70}$$

for each $\boldsymbol{h} \in H(\boldsymbol{M})$.

The conditions (H1)–(H4) are closely analogous to the conditions (L1)–(L4) in [26] for CCR-quantizable theories. The only real difference is in (H1), where the Hermitian form is used in place of a presymplectic form.

The main result of this section is to demonstrate that these conditions are indeed sufficient to ensure that the quantized theory is dynamically local; we consider a weakly nondegenerate classical theory $\mathscr{L} : \mathsf{Loc} \to \mathsf{Herm}$ obeying the timeslice axiom and the conditions (H1)–(H4) and its CAR-quantization $\mathscr{A} := \mathscr{Q}_{CAR}\mathscr{L}$. First, we state the following result; it is directly analogous to the corresponding result [26, Prop. 3.7], and the proof is identical, so we will not include it here.

Lemma 5.4.1. *Let* $\mathscr{L} : \mathsf{Loc} \to \mathsf{Herm}$ *be a weakly nondegenerate theory obeying timeslice and conditions* (H1)–(H4)*, and* $\boldsymbol{M} \in \mathsf{Loc}$*,* $O \in \mathscr{O}(\boldsymbol{M})$ *be arbitrary. If* Y *is a finite dimensional subspace of* $\mathscr{L}(\boldsymbol{M})$ *which is invariant* *under* $F_{\boldsymbol{M}}[\boldsymbol{h}]$ *for all* $\boldsymbol{h} \in H(\boldsymbol{M}; O)$ *then*

$$Y \subset \bigcap_{\boldsymbol{h} \in H(\boldsymbol{M};O)} \ker F_{\boldsymbol{M}}[\boldsymbol{h}]. \tag{5.71}$$

Consequently, for each $v \in Y$ *the stress tensor* $\boldsymbol{T}_{\boldsymbol{M}}[v]$ *vanishes in* O*. If* $O = K^{\perp}$ *for some* $K \in \mathscr{K}(\boldsymbol{M})$*, then* $Y \subset \mathscr{L}^{\bullet}(\boldsymbol{M}; K)$*.*

We will need the following proposition:

Proposition 5.4.2. *Let* $\mathscr{L} : \mathsf{Loc} \to \mathsf{Herm}$ *be a weakly nondegenerate classical theory obeying the timeslice axiom and* (H1)–(H4)*. The corresponding classical theory* $\overline{\mathscr{L}} : \mathsf{Loc} \to \mathsf{Herm}$ *is also weakly nondegenerate, and obeys the timeslice axiom and* (H1)–(H4)*. Moreover, if* $\mathscr{L}' : \mathsf{Loc} \to \mathsf{Herm}$ *is a classical theory and* $\eta : \mathscr{L} \stackrel{\cdot}{\longrightarrow} \mathscr{L}'$ *is a natural isomorphism, then* $\mathscr{L}'$ *is weakly nondegenerate, and obeys the timeslice axiom and* (H1)–(H4)*.*

Proof. Showing that $\overline{\mathscr{L}}$ is weakly nondegenerate is easy, and since functors preserve isomorphisms it follows immediately that it also obeys the timeslice axiom. We denote the relative Cauchy evolution for $\mathscr{L}$ and $\overline{\mathscr{L}}$ generated by a perturbation $\boldsymbol{h}$ on a spacetime $\boldsymbol{M}$ respectively by $R^{(\mathscr{L})}_{\boldsymbol{M}}[\boldsymbol{h}]$ and $R^{(\overline{\mathscr{L}})}_{\boldsymbol{M}}[\boldsymbol{h}]$, and their functional derivatives by $F^{(\mathscr{L})}_{\boldsymbol{M}}[\boldsymbol{h}]$ and $F^{(\overline{\mathscr{L}})}_{\boldsymbol{M}}[\boldsymbol{h}]$; we note that $R^{(\overline{\mathscr{L}})}_{\boldsymbol{M}}[\boldsymbol{h}] = \overline{R^{(\mathscr{L})}_{\boldsymbol{M}}[\boldsymbol{h}]}$ and $F^{(\overline{\mathscr{L}})}_{\boldsymbol{M}}[\boldsymbol{h}] = \overline{F^{(\mathscr{L})}_{\boldsymbol{M}}[\boldsymbol{h}]}$, so for any $\overline{v} \in \overline{\mathscr{L}(\boldsymbol{M})} = \overline{(V_{\boldsymbol{M}}, s_{\boldsymbol{M}})}$ we have

$$\begin{aligned} \overline{s_{\boldsymbol{M}}}(\overline{v}, F^{(\overline{\mathscr{L}})}_{\boldsymbol{M}}[\boldsymbol{h}]\overline{v}) &= \overline{s_{\boldsymbol{M}}(v, F^{(\mathscr{L})}_{\boldsymbol{M}}[\boldsymbol{h}]v)} \\ &= -\frac{i}{2}\int_{\boldsymbol{M}} h_{\mu\nu} T^{\mu\nu}_{\boldsymbol{M}}[v] \, dvol_{\boldsymbol{M}} \\ &= \frac{i}{2}\int_{\boldsymbol{M}} h_{\mu\nu} \overline{T}^{\mu\nu}_{\boldsymbol{M}}[\overline{v}] \, dvol_{\boldsymbol{M}}, \end{aligned} \tag{5.72}$$

where $\overline{\boldsymbol{T}}_{\boldsymbol{M}}[\overline{v}] \in C^{\infty}(T^2_0(\boldsymbol{M}))$ is defined by $\overline{\boldsymbol{T}}_{\boldsymbol{M}}[\overline{v}] := -\boldsymbol{T}_{\boldsymbol{M}}[v]$. Therefore $\overline{\mathscr{L}}$ obeys (H1).

For $O \in \mathscr{O}(\boldsymbol{M})$ and $\boldsymbol{h} \in H(\boldsymbol{M}; O)$, we have

$$\operatorname{im} F^{(\overline{\mathscr{L}})}_{\boldsymbol{M}}[\boldsymbol{h}] = \overline{\operatorname{im} F^{\mathscr{L}}_{\boldsymbol{M}}[\boldsymbol{h}]} \subset \overline{\mathscr{L}^{\mathrm{kin}}(\boldsymbol{M}; O)} = \overline{\mathscr{L}}^{\mathrm{kin}}(\boldsymbol{M}; O), \tag{5.73}$$

so $\overline{\mathscr{L}}$ obeys (H2). The properties (H3) and (H4) may be proved for $\overline{\mathscr{L}}$ similarly.

The results for $\mathscr{L}'$ are easy to show. We will prove (H3); the rest are similar. Suppose that $\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{N}$ are spacetimes and that there exist arrows $\psi_i : \boldsymbol{M}_i \to \boldsymbol{N}$ with $\psi_1(\boldsymbol{M}_1)$ and $\psi_2(\boldsymbol{M}_2)$ spacelike separated in $\boldsymbol{N}$. Let $A \in \mathscr{L}'(\psi_1)(\mathscr{L}'(\boldsymbol{M}_1)) \cap \mathscr{L}'(\psi_2)(\mathscr{L}'(\boldsymbol{M}_2))$; in other words, $A = \mathscr{L}'(\psi_i)A_i$ for some $A_i \in \mathscr{L}'(\boldsymbol{M}_i)$, $i = 1, 2$. But since $\mathscr{L}'(\psi) = \eta_{\boldsymbol{N}} \circ \mathscr{L}(\psi) \circ \eta_{\boldsymbol{M}}^{-1}$, it follows that $\eta_{\boldsymbol{N}}^{-1} A = \mathscr{L}(\psi)(\eta_{\boldsymbol{M}}^{-1} A_i) \in \mathscr{L}(\psi)(\mathscr{L}(\boldsymbol{M}_i))$ for $i = 1, 2$, and therefore $\eta_{\boldsymbol{N}}^{-1} A = 0$. Consequently $\mathscr{L}'(\psi_1)(\mathscr{L}'(\boldsymbol{M}_1)) \cap \mathscr{L}'(\psi_2)(\mathscr{L}'(\boldsymbol{M}_2)) = \{0\}$ as required. □

A trivial consequence of this result and Lemma 5.3.1 is that if $\mathscr{L}_{\text{adj}}$ is a doubled theory, then the undoubled theory $\mathscr{F}_1\mathscr{L}_{\text{adj}}$ is weakly nondegenerate and obeys timeslice and (H1)–(H4) if and only if $\mathscr{F}_2\mathscr{L}_{\text{adj}}$ has the same properties.

We may observe that since $\mathscr{L}$ obeys the timeslice axiom, $\mathscr{L}(\psi)$ is an isomorphism whenever ψ is a Cauchy morphism in Loc; it follows immediately that $\mathscr{A}(\psi)$ is also an isomorphism, and therefore the quantized theory $\mathscr{A}$ also obeys the timeslice axiom. Given the relative Cauchy evolution $R_{\boldsymbol{M}}[\boldsymbol{h}] : \mathscr{L}(\boldsymbol{M}) \to \mathscr{L}(\boldsymbol{M})$ in $\mathscr{L}$, we may immediately calculate the relative Cauchy evolution in $\text{rce}_{\boldsymbol{M}}[\boldsymbol{h}] : \mathscr{A}(\boldsymbol{M}) \to \mathscr{A}(\boldsymbol{M})$ by

$$\text{rce}_{\boldsymbol{M}}[\boldsymbol{h}] = \mathscr{Q}_{CAR}(R_{\boldsymbol{M}}[\boldsymbol{h}]) = \bigoplus_{n=0}^{\infty} R_{\boldsymbol{M}}[\boldsymbol{h}]_2, \tag{5.74}$$

where as in (5.33) we define $R_{\boldsymbol{M}}[\boldsymbol{h}]_2\langle\overline{v}, w\rangle := \langle\overline{R_{\boldsymbol{M}}[\boldsymbol{h}]v}, R_{\boldsymbol{M}}[\boldsymbol{h}]w\rangle$.

We now move to the computation of the kinematic and dynamical nets for $\mathscr{Q}_{CAR}\mathscr{L}$. For any $O \in \mathscr{O}(\boldsymbol{M})$, we clearly have $\alpha^{\text{kin}}_{\boldsymbol{M};O} = \mathscr{Q}_{CAR}(\lambda^{\text{kin}}_{\boldsymbol{M};O})$, where $\alpha^{\text{kin}}_{\boldsymbol{M};O}$ and $\lambda^{\text{kin}}_{\boldsymbol{M};O}$ are the kinematic nets for $\mathscr{A}$ and $\mathscr{L}$ respectively. We may alternatively express this as $\mathscr{A}^{\text{kin}}(\boldsymbol{M};O) = \mathscr{Q}_{CAR}(\mathscr{L}^{\text{kin}}(\boldsymbol{M};O))$. However, as in [26] it is much harder to compute the dynamical algebras; we use a similar strategy, and note that every element $A \in \mathscr{A}(\boldsymbol{M})$ has associated with it a subspace $Y_A \subset \mathscr{L}(\boldsymbol{M})$, called its *support space*: for each $n \geq 1$, we take the n^{th}-grade component $A_n \in \Lambda^n(\overline{\mathscr{L}(\boldsymbol{M})} \oplus \mathscr{L}(\boldsymbol{M}))$ of A in the concrete description of the CAR algebra given in Subsection 5.2.1, and regard it as the linear map

$$A_n : \left(\Lambda^{n-1}(\overline{\mathscr{L}(\boldsymbol{M})} \oplus \mathscr{L}(\boldsymbol{M}))\right)^* \to \overline{\mathscr{L}(\boldsymbol{M})} \oplus \mathscr{L}(\boldsymbol{M}) \tag{5.75}$$

that acts as

$$\omega \mapsto \frac{1}{n} \sum_{j=1}^{J} \sum_{k=1}^{n} (-1)^{k+1} \omega(\mathbf{v}_{j1} \wedge \cdots \widehat{\mathbf{v}}_{jk} \cdots \wedge \mathbf{v}_{jn}) \mathbf{v}_{jk}, \tag{5.76}$$

where $A_n = \sum_{j=1}^{J} \mathbf{v}_{j1} \wedge \cdots \wedge \mathbf{v}_{jn}$ and $\mathbf{v}_{jk} \in \overline{\mathscr{L}(\boldsymbol{M})} \oplus \mathscr{L}(\boldsymbol{M})$. The image of this map is of the form $\mathrm{im}(A_n) = \overline{W_1} \oplus W_2$, where W_1, W_2 are finite-dimensional subspaces of $\mathscr{L}(\boldsymbol{M})$: the *support subspace* Y_n of A_n is defined to be the span of W_1 and W_2. The support space Y_A is defined as the span of the support subspaces; it is finite-dimensional since all but finitely many of the support subspaces are trivial.

Lemma 5.4.3. *For any $A \in \mathscr{A}(\boldsymbol{M})$, we have $A \in \mathscr{Q}_{CAR}(\mathscr{L}(\boldsymbol{M})|_{Y_A})$.*

Proof. Let $A = \sum_{n=0}^{N} A_n$ with each $A_n \in \Lambda^n(\overline{\mathscr{L}(\boldsymbol{M})} \oplus \mathscr{L}(\boldsymbol{M}))$, and fix $n \geq 1$. As above we write $A_n = \sum_{j=1}^{J} \mathbf{v}_{j1} \wedge \cdots \wedge \mathbf{v}_{jn}$ with $\mathbf{v}_{jk} \in \overline{\mathscr{L}(\boldsymbol{M})} \oplus \mathscr{L}(\boldsymbol{M})$. We may assume without loss of generality that for any j the components $\mathbf{v}_{j1}, \ldots \mathbf{v}_{jn}$ are linearly independent; we may also assume that for any $1 \leq j_1 < j_2 \leq J$, we have

$$\mathrm{span}\{\mathbf{v}_{j_1 1}, \ldots, \widehat{\mathbf{v}}_{j_1 k_1}, \ldots, \mathbf{v}_{j_1 n}\} \neq \mathrm{span}\{\mathbf{v}_{j_2 1}, \ldots, \widehat{\mathbf{v}}_{j_2 k_2}, \ldots, \mathbf{v}_{j_2 n}\} \tag{5.77}$$

for any $1 \leq k_i \leq n$, $i = 1, 2$, otherwise $\mathbf{v}_{j_1 1} \wedge \cdots \wedge \mathbf{v}_{j_1 n}$ and $\mathbf{v}_{j_2 1} \wedge \cdots \wedge \mathbf{v}_{j_2 n}$ could be combined into a single term. It follows that the vectors $\mathbf{v}_{j1} \wedge \cdots \widehat{\mathbf{v}}_{jk} \cdots \wedge \mathbf{v}_{jn} \in \Lambda^{n-1}(\overline{\mathscr{L}(\boldsymbol{M})} \oplus \mathscr{L}(\boldsymbol{M}))$ are linearly independent; therefore for every $1 \leq j \leq J$ and $1 \leq k \leq n$ there exist functionals ω_{jk} satisfying

$$\omega_{jk}(\mathbf{v}_{j'1} \wedge \cdots \widehat{\mathbf{v}}_{j'k'} \cdots \wedge \mathbf{v}_{j'n'}) = \delta_{jj'} \delta_{kk'}. \tag{5.78}$$

From (5.76), it follows that $\mathbf{v}_{jk} \in \mathrm{im}(A_n)$ for each j, k; writing $\mathbf{v}_{jk} = \langle \overline{v_{jk}}, w_{jk} \rangle$, we have $v_{jk}, w_{jk} \in Y_n \subset Y_A$. This holds for all $n \geq 1$, so consequently $A \in \mathscr{Q}_{CAR}(\mathscr{L}(\boldsymbol{M})|_{Y_A})$ as required. □

As in [26, Lemma 5.2], we have

Lemma 5.4.4. *Let* $K \in \mathscr{K}(\boldsymbol{M})$*, and* $A \in \mathscr{A}^\bullet(\boldsymbol{M}; K)$*. Then* Y_A *is invariant under* $R_{\boldsymbol{M}}[\boldsymbol{h}]$ *for all* $\boldsymbol{h} \in H(\boldsymbol{M}; K)$*. Moreover,*

$$\mathscr{A}^\bullet(\boldsymbol{M}; K) \subset \mathscr{Q}_{CAR}(\mathscr{L}^\bullet(\boldsymbol{M}; K)). \tag{5.79}$$

Proof. The proof runs along the same lines as in [26]; if $A \in \mathscr{A}^\bullet(\boldsymbol{M}; K)$ then for all $\boldsymbol{h} \in H(\boldsymbol{M}; K)$ we have $\mathscr{Q}_{CAR}(R_{\boldsymbol{M}}[\boldsymbol{h}])A = \mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}]A = A$. This entails that for $n \geq 1$, we have $R_{\boldsymbol{M}}[\boldsymbol{h}]_2{}^{\otimes n} A_n = (\overline{R_{\boldsymbol{M}}[\boldsymbol{h}]} \oplus R_{\boldsymbol{M}}[\boldsymbol{h}])^{\otimes n} A_n = A_n$, where $A_n \in \Lambda^n(\overline{\mathscr{L}(\boldsymbol{M})} \oplus \mathscr{L}(\boldsymbol{M}))$ is the n^{th}-grade component of A. By [26, Lemma A.1], if we regard A_n as a linear map as in (5.76) then the image $\mathrm{im}(A_n)$ is invariant under the isomorphism $R_{\boldsymbol{M}}[\boldsymbol{h}]_2$. Consequently the support subspace Y_n is invariant under $R_{\boldsymbol{M}}[\boldsymbol{h}]$ for all $\boldsymbol{h} \in H(\boldsymbol{M}; K)$, and we may conclude that the support space Y_A is also invariant.

It immediately follows that Y_A is invariant under $F_{\boldsymbol{M}}[\boldsymbol{h}]$ for all $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$; by Lemma 5.4.1 we therefore have $Y_A \subset \mathscr{L}^\bullet(\boldsymbol{M}; K)$. Therefore $A \in \mathscr{Q}_{CAR}(\mathscr{L}^\bullet(\boldsymbol{M}; K))$, by Lemma 5.4.3. □

Finally, corresponding to [26, Theorem 5.3], we have

Theorem 5.4.5. *Let* $\mathscr{L} : \mathsf{Loc} \to \mathsf{Herm}$ *be a weakly nondegenerate theory obeying the timeslice axiom and* (H1)–(H4)*, and let* $\mathscr{A} = \mathscr{Q}_{CAR}\mathscr{L}$*. For any* $\boldsymbol{M} \in \mathsf{Loc}$ *and* $K \in \mathscr{K}(\boldsymbol{M})$*,*

$$\mathscr{A}^\bullet(\boldsymbol{M}; K) = \mathscr{Q}_{CAR}(\mathscr{L}^\bullet(\boldsymbol{M}; K)), \tag{5.80}$$

i.e. $\alpha^\bullet_{\boldsymbol{M};K} \cong \mathscr{Q}_{CAR}(\lambda^\bullet_{\boldsymbol{M};K})$*. Moreover, for any* $O \in \mathscr{O}(\boldsymbol{M})$*,*

$$\mathscr{A}^{\mathrm{dyn}}(\boldsymbol{M}; O) = \mathscr{Q}_{CAR}(\mathscr{L}^{\mathrm{dyn}}(\boldsymbol{M}; O)), \tag{5.81}$$

i.e. $\alpha^{\mathrm{dyn}}_{\boldsymbol{M};O} \cong \mathscr{Q}_{CAR}(\lambda^{\mathrm{dyn}}_{\boldsymbol{M};O})$*. Consequently if* $\mathscr{L}$ *is dynamically local then so is* $\mathscr{A}$*.*

Proof. The inclusion of the left hand side of (5.80) in the right hand side is established by Lemma 5.4.4. Conversely, for any $A \in \mathscr{Q}_{CAR}(\mathscr{L}^\bullet(\boldsymbol{M}; K))$ and

$\boldsymbol{h} \in H(\boldsymbol{M}; K^{\perp})$, we have $\mathrm{rce}_{\boldsymbol{M}}[\boldsymbol{h}]A = A$ by (5.74), so the reverse inclusion holds. (5.81) follows from (5.80) by Lemma 5.2.4. □

Due to the equivalence between quantized doubled theories and the corresponding quantized undoubled theories noted earlier, this result leads immediately to a corresponding result regarding theories of the form $\mathscr{A}_{\mathrm{adj}} = \mathscr{Q}_{\mathrm{adj}}\mathscr{L}_{\mathrm{adj}}$, where $\mathscr{L}_{\mathrm{adj}} : \mathsf{Loc} \to \mathsf{HermAdj}_{\mathsf{C}}$ is a classical theory obeying the timeslice axiom such that for any $\boldsymbol{M} \in \mathsf{Loc}$, we have $\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M}) = (\mathscr{L}_1(\boldsymbol{M}), \mathscr{L}_2(\boldsymbol{M}), A^1_{\boldsymbol{M}}, A^2_{\boldsymbol{M}})$. For $\boldsymbol{h} \in H(\boldsymbol{M})$, we have

$$\mathrm{rce}^{(\mathscr{L}_{\mathrm{adj}})}_{\boldsymbol{M}}[\boldsymbol{h}] = (R^1_{\boldsymbol{M}}[\boldsymbol{h}], R^2_{\boldsymbol{M}}[\boldsymbol{h}]), \tag{5.82}$$

where $R^i_{\boldsymbol{M}}[\boldsymbol{h}]$ is the relative Cauchy evolution on $\mathscr{F}_i(\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M}))$. The definition we use for the Hermitian adjoint structure $\mathscr{L}^{\bullet}_{\mathrm{adj}}(\boldsymbol{M}; K)$ is

$$\mathscr{L}^{\bullet}_{\mathrm{adj}}(\boldsymbol{M}; K) := (\mathscr{L}^{\bullet}_1(\boldsymbol{M}; K), \mathscr{L}^{\bullet}_2(\boldsymbol{M}; K), A^1_{\boldsymbol{M};K}, A^2_{\boldsymbol{M};K}) \tag{5.83}$$

for $K \in \mathscr{K}(\boldsymbol{M})$, where $A^i_{\boldsymbol{M};K} := A^i_{\boldsymbol{M}}|_{(\mathscr{F}_i\mathscr{L}_{\mathrm{adj}})^{\bullet}(\boldsymbol{M};K)}$. Note that this is a well-defined subobject of $\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M})$, since $\overline{R^2_{\boldsymbol{M}}[\boldsymbol{h}]} \circ A^1_{\boldsymbol{M}} = A^1_{\boldsymbol{M}} \circ R^1_{\boldsymbol{M}}[\boldsymbol{h}]$ and therefore $A^1_{\boldsymbol{M}}(\mathscr{L}^{\bullet}_1(\boldsymbol{M}; K)) = \overline{\mathscr{L}^{\bullet}_2(\boldsymbol{M}; K)}$ (and similarly, $A^2_{\boldsymbol{M}}(\mathscr{L}^{\bullet}_2(\boldsymbol{M}; K)) = \overline{\mathscr{L}^{\bullet}_1(\boldsymbol{M}; K)}$) as required. A consequence of this is that

$$\mathscr{L}^{\mathrm{dyn}}_{\mathrm{adj}}(\boldsymbol{M}; O) = ((\mathscr{F}_1\mathscr{L}_{\mathrm{adj}})^{\mathrm{dyn}}(\boldsymbol{M}; O), (\mathscr{F}_2\mathscr{L}_{\mathrm{adj}})^{\mathrm{dyn}}(\boldsymbol{M}; O), A^1_{\boldsymbol{M};O}, A^2_{\boldsymbol{M};O}), \tag{5.84}$$

where $O \in \mathscr{O}(\boldsymbol{M})$ and $A^i_{\boldsymbol{M};O} := A^i_{\boldsymbol{M}}|_{(\mathscr{F}_1\mathscr{L}_{\mathrm{adj}})^{\mathrm{dyn}}(\boldsymbol{M};O)}$. We have:

Theorem 5.4.6. *Let $\mathscr{L}_{\mathrm{adj}}$ be a doubled theory, and let $\mathscr{A}_{\mathrm{adj}} := \mathscr{Q}_{\mathrm{adj}}\mathscr{L}_{\mathrm{adj}}$. If $\mathscr{F}_1\mathscr{L}_{\mathrm{adj}}$ is a weakly nondegenerate theory obeying the timeslice axiom and* (H1)–(H4)*, then for any $\boldsymbol{M} \in \mathsf{Loc}$ and $K \in \mathscr{K}(\boldsymbol{M})$,*

$$\mathscr{A}^{\bullet}_{\mathrm{adj}}(\boldsymbol{M}; K) = \mathscr{Q}_{\mathrm{adj}}(\mathscr{L}^{\bullet}_{\mathrm{adj}}(\boldsymbol{M}; K)). \tag{5.85}$$

Additionally, for any $O \in \mathscr{O}(\boldsymbol{M})$,

$$\mathscr{A}^{\mathrm{dyn}}_{\mathrm{adj}}(\boldsymbol{M}; O) = \mathscr{Q}_{\mathrm{adj}}(\mathscr{L}^{\mathrm{dyn}}_{\mathrm{adj}}(\boldsymbol{M}; O)). \tag{5.86}$$

Consequently, if $\mathscr{L}_{\mathrm{adj}}$ *is dynamically local then so is* $\mathscr{A}_{\mathrm{adj}}$.

Proof. Suppose that $\mathscr{F}_1\mathscr{L}_{\mathrm{adj}}$ is weakly nondegenerate and obeys the timeslice axiom and (H1)–(H4); we let $\mathscr{A} := \mathscr{Q}_{CAR}\mathscr{F}_1\mathscr{L}_{\mathrm{adj}}$. Since η^1 as defined in Proposition 5.2.8 is a natural isomorphism from $\mathscr{Q}_{\mathrm{adj}}$ to $\mathscr{Q}_{CAR}\mathscr{F}_1$, it follows that $\eta^1_{\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M})} \circ \mathrm{rce}^{\mathscr{A}_{\mathrm{adj}}}_{\boldsymbol{M}}[\boldsymbol{h}] = \mathrm{rce}^{\mathscr{A}}_{\boldsymbol{M}}[\boldsymbol{h}] \circ \eta^1_{\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M})}$. Therefore $\mathscr{A}^\bullet_{\mathrm{adj}}(\boldsymbol{M}; K) = (\eta^1_{\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M})})^{-1}\mathscr{A}^\bullet(\boldsymbol{M}; K)$ for any $K \in \mathscr{K}(\boldsymbol{M})$, but by Theorem 5.4.5 and (5.83) we have

$$
\begin{aligned}
(\eta^1_{\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M})})^{-1}\mathscr{A}^\bullet(\boldsymbol{M}; K) &= (\eta^1_{\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M})})^{-1}\mathscr{Q}_{CAR}((\mathscr{F}_1\mathscr{L}_{\mathrm{adj}})^\bullet(\boldsymbol{M}; K)) \\
&= (\eta^1_{\mathscr{L}_{\mathrm{adj}}(\boldsymbol{M})})^{-1}\mathscr{Q}_{CAR}(\mathscr{F}_1(\mathscr{L}^\bullet_{\mathrm{adj}}(\boldsymbol{M}; K))) \\
&= \mathscr{Q}_{\mathrm{adj}}(\mathscr{L}^\bullet_{\mathrm{adj}}(\boldsymbol{M}; K)). \qquad (5.87)
\end{aligned}
$$

Similarly, we may use the same theorem and (5.84) to show that

$$
\mathscr{A}^{\mathrm{dyn}}_{\mathrm{adj}}(\boldsymbol{M}; O) = \mathscr{Q}_{\mathrm{adj}}(\mathscr{L}^{\mathrm{dyn}}_{\mathrm{adj}}(\boldsymbol{M}; O)) \qquad (5.88)
$$

for any $O \in \mathscr{O}(\boldsymbol{M})$.

□

Chapter 6

Spin geometry

We now present a construction of the Dirac field theory in a locally covariant way. While this has already been set out in [56], based on work in [20, 27, 17], we are able to give a significantly simplified version of the theory resulting from observations by Geroch [29] and Isham [39] regarding the nature of possible spin structures on four-dimensional globally hyperbolic spacetimes. In this section we set out some preliminary geometrical material that is necessary for the construction.

6.1 Spin structures

6.1.1 Categories of vector bundles

First we establish some further categorical notions which will be needed for the construction of the Dirac field.

Definition 6.1.1. *The category* Bund *has as its objects all smooth fibre bundles* (B, π) *over a base space* $\boldsymbol{M} \in \mathsf{Loc}$*. An arrow* $\beta : (B, \pi) \to (B', \pi')$ *is a smooth map from* B *to* B' *with the property that there exists a (necessarily unique)* Loc*-arrow* $\psi : \pi(B) \to \pi'(B')$ *satisfying* $\pi' \circ \beta = \psi \circ \pi$*. The functor* $\mathscr{b} : \mathsf{Bund} \to \mathsf{Loc}$ *maps a fibre bundle to its base space and a bundle morphism to its associated* Loc*-arrow. An arrow* $\beta : (B, \pi) \to (B', \pi')$ *is called* base-point-preserving *if* $\mathscr{b}(B) = \mathscr{b}(B')$ *and* $\mathscr{b}(\beta) = \mathrm{id}_{\mathscr{b}(B)}$*.*

There are covariant functors T, T^* : $\mathsf{Loc} \to \mathsf{Bund}$ which assign to any $\boldsymbol{M} \in \mathsf{Loc}$ the (co-)tangent bundle $T^{(*)}\boldsymbol{M}$. A Loc-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$ is mapped by T to the derivative $D\psi$, and by T^* to the push-forward ψ_* defined by

$$\langle \psi_* \xi, x \rangle := \langle \xi, D\psi_p^{-1} x \rangle, \tag{6.1}$$

where $\xi \in T_p^* \boldsymbol{M}$ and $x \in T_{\psi(p)} \boldsymbol{M}$.

Definition 6.1.2. *The category RPBund has as its objects all smooth principal G-bundles (B, π, G, R) where $(B, \pi) \in \mathsf{Bund}$, G is a group and the group action $R_g : B \to B$, $g \in G$ acts from the right. An arrow $(\beta, \alpha) : (B, \pi, G, R) \to (B', \pi', G', R')$ consists of a Bund-arrow from $\beta : (B, \pi) \to (B', \pi')$ and a group homomorphism $\alpha : G \to G'$ with the property that β intertwines the group actions, i.e.*

$$R_{\alpha(g)} \circ \beta = \beta \circ R_g \tag{6.2}$$

for all $g \in G$. The composition of two RPBund-arrows is given by

$$(\beta', \alpha') \circ (\beta, \alpha) := (\beta' \circ \beta, \alpha' \circ \alpha). \tag{6.3}$$

The covariant functor $\boldsymbol{p} : \mathsf{RPBund} \to \mathsf{Bund}$ is defined by

$$\boldsymbol{p}(B, \pi, G, R) := (B, \pi), \qquad \boldsymbol{p}(\beta, \alpha) = \beta. \tag{6.4}$$

For any group G there is a covariant functor $\mathscr{T}_G : \mathsf{Loc} \to \mathsf{RPBund}$ that maps $\boldsymbol{M} \in \mathsf{Loc}$ to the trivial G-bundle, that is to say $(\boldsymbol{M} \times G, \pi_0, G, R)$ where $\pi_0(p, g) := p$ and $R_g(p, g') := (p, g'g)$. For an arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$ we have $\mathscr{T}_G(\psi) := (\boldsymbol{\psi}, \mathrm{id}_G)$ where $\boldsymbol{\psi}(p, g) := (\psi(p), g)$.

More generally, a given principal G-bundle $\mathcal{B} = (B, \pi, G, R)$ over a spacetime $\boldsymbol{M}$ is *trivial* if there is a base-point-preserving RPBund-isomorphism between $\mathcal{B}$ and $\mathscr{T}_G \boldsymbol{M}$ of the form (β, id_G); we call such a map a *trivialising morphism*. To distinguish between principal bundles that are trivial in this sense and the trivial G-bundle $\mathscr{T}_G \boldsymbol{M}$, we will describe $\mathscr{T}_G \boldsymbol{M}$ as the *product bundle* of $\boldsymbol{M}$ with G. Moreover, a functor $\mathscr{F} : \mathsf{Loc} \to \mathsf{RPBund}$ is called *G-trivial* if $\mathscr{F}(\boldsymbol{M})$ is a trivial G-bundle for each $\boldsymbol{M} \in \mathsf{Loc}$, and *naturally*

G*-trivial* if there is a natural isomorphism $\tau : \mathscr{F} \stackrel{\cdot}{\longrightarrow} \mathscr{T}_G$ whose components are trivialising morphisms, which we call a *natural trivialisation*. Clearly all naturally G-trivial functors are G-trivial, but the converse is not true, as we will see later.

The categories Bund and RPBund have full subcategories $\mathsf{Bund}_{(d)}^{(c/sc)}$ and $\mathsf{RPBund}_{(d)}^{(c/sc)}$ whose objects have base spaces residing in $\mathsf{Loc}_{(d)}^{(c/sc)}$.

Definition 6.1.3. *Let* $\boldsymbol{M} = (\mathcal{M}, \boldsymbol{g}, \mathfrak{o}, \mathfrak{t}) \in \mathsf{Loc}_4$ *be a 4-dimensional spacetime. The* oriented orthonormal frame bundle $F_+^\uparrow \boldsymbol{M} \in \mathsf{RPBund}_4$ *is the principal* $\mathcal{L}_+^\uparrow$*-bundle of (time-)oriented orthonormal frames of* $\boldsymbol{M}$*, i.e. the bundle whose fibre at* $p \in \boldsymbol{M}$ *consists of all ordered bases* $\boldsymbol{e} = (e_0, \ldots, e_3)$ *of the tangent space* $T_p\boldsymbol{M}$ *that are oriented and time-oriented according to* $\mathfrak{o}$ *and* $\mathfrak{t}$*, and satisfy* $e_a^\mu e_b^\nu g_{\mu\nu}(x) = \eta_{ab}$*.*[1] *Here* $\mathcal{L}_+^\uparrow$ *is the proper orthochronous Lorentz group sometimes denoted* $SO^+(1,3)$ *or* $SO_0(1,3)$*. The right action* R *of* $\mathcal{L}_+^\uparrow$ *on* $F_+^\uparrow \boldsymbol{M}$ *is given by* $R_\Lambda(p, \boldsymbol{e}) := (p, \boldsymbol{e}\Lambda)$*, where* $(\boldsymbol{e}\Lambda)_a := e_b \Lambda^b{}_a$*. We denote the canonical projection of* $F_+^\uparrow \boldsymbol{M}$ *onto* $\boldsymbol{M}$ *by* $\pi_{\boldsymbol{M}}$*.*

Lemma 6.1.4. *There is a covariant functor* $F_+^\uparrow : \mathsf{Loc}_4 \to \mathsf{RPBund}_4$ *that assigns the frame bundle to each 4-dimensional spacetime* $\boldsymbol{M} \in \mathsf{Loc}_4$*, and to each arrow* $\psi : \boldsymbol{M} \to \boldsymbol{N}$ *the pair* $(\psi_*, \mathrm{id}_{\mathcal{L}_+^\uparrow}) : F_+^\uparrow \boldsymbol{M} \to F_+^\uparrow \boldsymbol{N}$*. The push-forward* ψ_* *is defined in this case by* $\psi_*(p, \boldsymbol{e}) := (\psi(p), D\psi_p\, \boldsymbol{e})$*, where* $(D\psi_p\, \boldsymbol{e})_a := D\psi_p\, e_a$*.*

Proof. It is easy to see that $\pi_{\boldsymbol{N}} \circ \psi_* = \psi \circ \pi_{\boldsymbol{M}}$, so ψ_* is an arrow in Bund_4 as required. To see that ψ_* intertwines the group action, note that for any $\Lambda \in \mathcal{L}_+^\uparrow$ and $(p, \boldsymbol{e}) \in F_+^\uparrow \boldsymbol{M}$ we have $R_\Lambda \psi_*(p, \boldsymbol{e}) = (\psi(p), (D\psi_p\, \boldsymbol{e})\Lambda)$ and $\psi_*(R_\Lambda(p, \boldsymbol{e})) = (\psi(p), D\psi_p(\boldsymbol{e}\Lambda))$; these are equal because $[D\psi_p(\boldsymbol{e}\Lambda)]_a = D\psi_p\, e_b \Lambda^b{}_a = [(D\psi_p\, \boldsymbol{e})\Lambda]_a$. □

Note that the relation $\pi_{\boldsymbol{N}} \circ \psi_* = \psi \circ \pi_{\boldsymbol{M}}$ entails that the projections $\pi_{\boldsymbol{M}}$ form the components of a natural transformation $\pi : F_+^\uparrow \stackrel{\cdot}{\longrightarrow} \mathscr{T}_I$, where $I = \{\mathbf{1}\}$ is the trivial group and we identify $\boldsymbol{M}$ with $\mathscr{T}_I \boldsymbol{M}$.

A basis $\boldsymbol{e} = (e_0, \ldots, e_3) \in F_+^\uparrow \boldsymbol{M}_p$ has associated with it the dual basis $(e^0, \ldots, e^3)$ of the cotangent space $T_p^*\boldsymbol{M}$ defined by $e_\mu^a := g_{\mu\nu}(p)\eta^{ab}e_b^\nu$. We

[1] Some authors use $\eta^{ab} e_a^\mu e_b^\nu = g^{\mu\nu}(x)$; the two definitions are equivalent.

write $\boldsymbol{e}^\flat := (e^0, \ldots, e^3)$. Given any vector $x \in T_p\boldsymbol{M}$ and covector $u \in T_p^*\boldsymbol{M}$ we use the notation $u(x) := u_\mu x^\mu$, so that in particular $e^a(e_b) = \delta^a_b$ for any $\boldsymbol{e} \in F_+^\uparrow \boldsymbol{M}_p$. For any pair of bases $\boldsymbol{e}, \boldsymbol{e}' \in F_+^\uparrow \boldsymbol{M}_p$, we define $\boldsymbol{e}'^\flat(\boldsymbol{e}) := [e'^a(e_b)]_{a,b=0}^3 \in M_4(\mathbb{R})$. As it happens, we always have $\boldsymbol{e}'^\flat(\boldsymbol{e}) \in \mathcal{L}_+^\uparrow$: if $\Lambda^a{}_b := e'^a(e_b)$ then $e_b = e'_a \Lambda^a{}_b$, and so $\Lambda \in \mathcal{L}_+^\uparrow$ by definition.

Finally, given an established choice of frame $\boldsymbol{e} \in F_+^\uparrow \boldsymbol{M}_p$, and a vector $x \in T_p\boldsymbol{M}$ or covector $u \in T_p^*\boldsymbol{M}$, we use the notation

$$x^a := x^\mu e^a_\mu, \qquad u_a := u_\mu e^\mu_a. \tag{6.5}$$

It may be checked that this definition is compatible with the raising and lowering of indices, in the sense that $(x^\flat)_a = x_\mu e^\mu_a = x_\mu e^b_\nu \eta_{ab} g^{\mu\nu}(p) = \eta_{ab} x^b$, where $x^\flat \in T_p^*\boldsymbol{M}$ is the covector with components $(x^\flat)_\mu := x^\nu g_{\mu\nu}(p)$.

6.1.2 Spin bundles

Definition 6.1.5. *The* Dirac algebra $\mathfrak{D}$ *is the unital algebra over* $\mathbb{R}$ *that is generated by an orthonormal basis*[2] g_0, g_1, g_2, g_3 *of Minkowski space* $\mathbb{R}^{1,3}$ *and satisfying the relation*

$$g_a g_b + g_b g_a = 2\eta_{ab}\mathbf{1}. \tag{6.6}$$

We fix an irreducible complex representation $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3 \in GL_4(\mathbb{C})$ of $\mathfrak{D}$; the structure of the theory does not depend on this choice, since any two such representations are equivalent [50],[3] but since we wish to make our construction of the Dirac theory as concrete as possible, we will use the chiral representation $\rho_0(g_a) = \boldsymbol{\gamma}_a$ defined by

$$\boldsymbol{\gamma}_0 := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma}_i := \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0, \end{pmatrix} \tag{6.7}$$

[2]That is to say, a basis g_0, g_1, g_2, g_3 satisfying $g^\mu_a g^\nu_b \eta_{\mu\nu} = \eta_{ab}$.

[3]'Equivalence' here means that for any two irreducible complex representations $\rho, \rho' : \mathfrak{D} \to GL_4(\mathbb{C})$ there exists a matrix $L \in GL_4(\mathbb{C})$, which is unique up to multiplication by a non-zero complex number, that satisfies $\rho'(\mathfrak{d})L = L\rho(\mathfrak{d})$ for all $\mathfrak{d} \in \mathfrak{D}$.

where as usual

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.8}$$

are the Pauli matrices. The proper subalgebra $\mathfrak{D}^0 \subset \mathfrak{D}$ is generated by elements of the form $g_a g_b$. The groups $\mathrm{Pin}_{1,3}$ and $\mathrm{Spin}_{1,3}$ may be defined as in [56] as

$$\begin{aligned} \mathrm{Pin}_{1,3} &:= \{\mathfrak{d} \in \mathfrak{D} : \mathfrak{d} = u_1 \cdots u_k,\ u_i \in \mathbb{R}^{1,3},\ u_i^2 = \pm\mathbf{1}\}, \\ \mathrm{Spin}_{1,3} &:= \mathrm{Pin}_{1,3} \cap \mathscr{D}^0; \end{aligned} \tag{6.9}$$

We may safely blur the distinction between these groups and their images in $GL_4(\mathbb{C})$ under ρ_0; $\mathrm{Pin}_{1,3}$ may then be classified as the group of all $S \in \rho_0(\mathfrak{D})$ with $\det S = 1$ and

$$S\boldsymbol{\gamma}_a S^{-1} = \boldsymbol{\gamma}_b \Lambda(S)^b{}_a \tag{6.10}$$

for some $\Lambda(S) \in M_4(\mathbb{R})$. It is well known that $\Lambda(S)$ must be an element of the Lorentz group $\mathcal{L}$, and that the assignment $\mathrm{Pin}_{1,3} \ni S \mapsto \Lambda(S)$ is a group homomorphism given by the above relation is a double covering of the Lorentz group (indeed, the universal double covering) [43, Thm. 2.10]. We denote by $\mathrm{Spin}^0_{1,3}$ the connected component of $\mathrm{Spin}_{1,3}$ containing the identity; the map Λ restricts to a double covering of $\mathcal{L}^\uparrow_+$ by $\mathrm{Spin}^0_{1,3}$. The following result is proved in [56, Thm. 3.5] (see also [30]):

Lemma 6.1.6. *For any irreducible complex representation ρ of $\mathfrak{D}$ in $GL_4(\mathbb{C})$, with $\boldsymbol{\gamma}_a = \rho(g_a)$, there are matrices $A, C \in GL_4(\mathbb{C})$ satisfying*

$$\begin{aligned} A &= A^\dagger, & \boldsymbol{\gamma}_a^\dagger &= A\boldsymbol{\gamma}_a A^{-1}, & A\rho(n) &> 0 \\ \overline{C}C &= \mathbf{1}, & -\overline{\boldsymbol{\gamma}_a} &= C\boldsymbol{\gamma}_a C^{-1} \end{aligned} \tag{6.11}$$

where n is any future-pointing timelike vector in $\mathbb{R}^{1,3}$. In the chiral representation, we may take $A := \boldsymbol{\gamma}_0$ and $C := \boldsymbol{\gamma}_2$. For these matrices it holds that

$$A = -C^\dagger A^T C, \qquad S^\dagger A S = A, \qquad CS = \overline{S}C \tag{6.12}$$

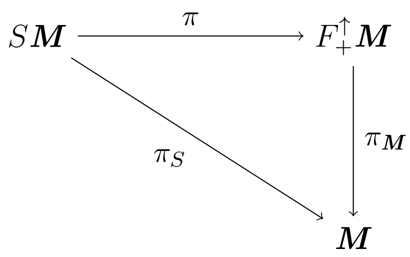
for any $S \in \mathrm{Spin}^0_{1,3}$ *(taken in the chiral representation, although the same holds for any representation).*

In order to construct the theory classical Dirac field, we will require the definition of a *spin structure* on a given 4-dimensional spacetime $\boldsymbol{M}$. In [20, 27, 17, 56] a spin structure on a spacetime $\boldsymbol{M} \in \mathsf{Loc}_4$ was defined to be any principal $\mathrm{Spin}^0_{1,3}$-bundle $\boldsymbol{S} = (\mathcal{S}, p, \mathrm{Spin}^0_{1,3}, R)$ over $\boldsymbol{M}$ (the *spin bundle*), along with a RPBund_4-morphism $(\pi, \Lambda) : \boldsymbol{S} \to F^{\uparrow}_{+}\boldsymbol{M}$ that preserves base points. The map $\pi : (\mathcal{S}, p) \to p(F^{\uparrow}_{+}\boldsymbol{M})$ is called the *spin frame projection.* In the above references the structure of the spin bundle is not specified further, and in [56] a locally covariant theory is constructed from a category whose objects are spacetimes with a choice of spin structure. However, it is proved in [39] that a spin bundle over a 4-dimensional globally hyperbolic spacetime is necessarily trivial; in summary, the structure group of a $\mathrm{Spin}^0_{1,3}$-bundle may be reduced to $SU(2)$, and it may be shown that there is a one-to-one correspondence between $SU(2)$-bundles over $\boldsymbol{M}$ with the homotopy classes of maps from $\boldsymbol{M}$ to S^4. These in turn are in one-to-one correspondence with the elements of the cohomology group $H^4(\boldsymbol{M}; \mathbb{Z})$, which is itself trivial for any spacetime $\boldsymbol{M} \in \mathsf{Loc}_4$.

This allows for a much simpler presentation than in the cited references. We emphasise that this does not preclude the existence of distinct and inequivalent spin structures, but rather that any nontriviality in the choice of spin structure is located in the freedom in defining π and not in the choice of spin bundle; we cannot simply define the classical Dirac theory as a functor whose domain is Loc_4.

For the sake of neatness we use the notation $S := \mathscr{T}_{\mathrm{Spin}^0_{1,3}}$ for the trivial spin bundle functor, and write $\pi_S(\boldsymbol{M}) : S\boldsymbol{M} \to \boldsymbol{M}$ as the canonical projection onto a given $\boldsymbol{M} \in \mathsf{Loc}_4$ (again writing π_S if the spacetime is clear from context).

A spin structure on $\boldsymbol{M}$ may be expressed as the following commuting diagram.



Since $\pi_{\boldsymbol{M}}$ and π_S are canonically defined the only freedom of choice here is in the definition of π. Therefore we may denote a spin structure over $\boldsymbol{M}$ with spin frame projection π unambiguously by $(S\boldsymbol{M}, \pi)$. Note that for any $p \in \boldsymbol{M}$ and $S \in \mathrm{Spin}^0_{1,3}$, we have $\pi(p, S) = R_{\Lambda(S)}\pi(p, \mathbf{1})$ by definition, and so π is completely defined by its action on elements of the form $(p, \mathbf{1})$. In addition π is smooth and preserves base points, so we are restricted to maps of the form $\pi(p, \mathbf{1}) = (p, \boldsymbol{\varepsilon}(p))$ where $\boldsymbol{\varepsilon}$ is an element of the space $\Gamma^\infty(F^{\uparrow}_{+}\boldsymbol{M})$ of smooth sections of the frame bundle. For any such $\boldsymbol{\varepsilon}$ we write $\boldsymbol{\varepsilon}(p) = (\varepsilon_0(p), \ldots, \varepsilon_3(p))$, and define $\pi_{\boldsymbol{\varepsilon}}(p, \mathbf{1}) := (p, \boldsymbol{\varepsilon}(p))$; this entails that

$$\pi_{\boldsymbol{\varepsilon}}(p, S) = R_{\Lambda(S)}(p, \boldsymbol{\varepsilon}(p)) \tag{6.13}$$

for any $S \in \mathrm{Spin}^0_{1,3}$.

The existence of smooth global sections of the frame bundle in an arbitrary object of Loc_4 is not obvious: it is clear that such sections exist if and only if the manifold is parallelizable, and due to [4, Thm. 1.1], a globally hyperbolic spacetime is parallelizable only if its Cauchy surfaces are. A Cauchy surface Σ of $\boldsymbol{M} \in \mathsf{Loc}_4$ is an orientable 3-manifold, and it is a well-known theorem of Stiefel [59] that compact orientable 3-manifolds are parallelizable. There is a generalization to the non-compact case in [49], and an independent proof was kindly provided by Christian Baer (private communication). Note that the existence of global frames on all objects of Loc_4 implies that the frame bundle of any $\boldsymbol{M} \in \mathsf{Loc}_4$ is trivial, and therefore $F^{\uparrow}_{+}$ is $\mathcal{L}^{\uparrow}_{+}$-trivial. The following lemma classifies its trivialising morphisms:

Lemma 6.1.7. *Given any fixed global frame* $\mathring{\boldsymbol{\varepsilon}} \in \Gamma^\infty(F^{\uparrow}_{+}\boldsymbol{M})$ *we may define a* *trivialising morphism* $(\phi_{\mathring{\boldsymbol{\varepsilon}}}, \mathrm{id}_{\mathcal{L}_+^\uparrow}) : F_+^\uparrow \boldsymbol{M} \to \mathscr{T}_{\mathcal{L}_+^\uparrow} \boldsymbol{M}$ *by*

$$\phi_{\mathring{\boldsymbol{\varepsilon}}}(p, \boldsymbol{e}) := (p, \mathring{\boldsymbol{\varepsilon}}(p)^\flat(\boldsymbol{e})). \tag{6.14}$$

Moreover every trivialising morphism $\phi : F_+^\uparrow \boldsymbol{M} \to \mathscr{T}_{\mathcal{L}_+^\uparrow} \boldsymbol{M}$ *is of this form.*

Proof. $\phi_{\mathring{\boldsymbol{\varepsilon}}}$ is obviously smooth, and $(\phi_{\mathring{\boldsymbol{\varepsilon}}}, \mathrm{id}_{\mathcal{L}_+^\uparrow})$ can then easily be shown to be a RPBund_4-arrow into $\mathscr{T}_{\mathcal{L}_+^\uparrow} \boldsymbol{M}$ by the properties of the frame bundle. $(\phi_{\mathring{\boldsymbol{\varepsilon}}}, \mathrm{id}_{\mathcal{L}_+^\uparrow})$ has a two-sided inverse $(\phi_{\mathring{\boldsymbol{\varepsilon}}}^{-1}, \mathrm{id}_{\mathcal{L}_+^\uparrow})$ where $\phi_{\mathring{\boldsymbol{\varepsilon}}}^{-1}(p, \Lambda) = R_\Lambda(p, \mathring{\boldsymbol{\varepsilon}}(p))$. Therefore $(\phi_{\mathring{\boldsymbol{\varepsilon}}}, \mathrm{id}_{\mathcal{L}_+^\uparrow})$ is a trivialising morphism.

On the other hand, if $(\phi, \mathrm{id}_{\mathcal{L}_+^\uparrow}) : F_+^\uparrow \boldsymbol{M} \to \mathscr{T}_{\mathcal{L}_+^\uparrow} \boldsymbol{M}$ is a trivialising morphism then $\phi(p, \boldsymbol{e}) = (p, \tilde{\phi}(p, \boldsymbol{e}))$ for some $\tilde{\phi} : F_+^\uparrow \boldsymbol{M} \to \mathcal{L}_+^\uparrow$. For fixed $p \in \boldsymbol{M}$, the map $\tilde{\phi}(p, \cdot\,) : F_+^\uparrow \boldsymbol{M}_p \to \mathcal{L}_+^\uparrow$ is an isomorphism, so there exists a unique $\boldsymbol{e}_p \in F_+^\uparrow \boldsymbol{M}_p$ such that $\tilde{\phi}(p, \boldsymbol{e}_p) = \mathbf{1}$. We write $\mathring{\boldsymbol{\varepsilon}}(p) := \boldsymbol{e}_p$, and since ϕ has a smooth inverse we have $\mathring{\boldsymbol{\varepsilon}} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$. Furthermore, since $R_\Lambda \circ \phi = \phi \circ R_\Lambda$ for all $\Lambda \in \mathcal{L}_+^\uparrow$, we have $\tilde{\phi}(p, \boldsymbol{e})\Lambda = \tilde{\phi}(p, \boldsymbol{e}\Lambda)$ for all $\Lambda \in \mathcal{L}_+^\uparrow$. If we choose $\Lambda := \mathring{\boldsymbol{\varepsilon}}(p)^\flat(\boldsymbol{e})$, where $\boldsymbol{e} \in F_+^\uparrow \boldsymbol{M}_p$ is arbitrary, then $(\mathring{\boldsymbol{\varepsilon}}(p)\Lambda)_a = \mathring{\varepsilon}_b(p)[\mathring{\varepsilon}^b(p)(e_a)] = e_a$, so

$$\tilde{\phi}(p, \boldsymbol{e}) = \tilde{\phi}(p, \mathring{\boldsymbol{\varepsilon}}(p)\Lambda) = \tilde{\phi}(p, \mathring{\boldsymbol{\varepsilon}}(p))\Lambda = \Lambda = \mathring{\boldsymbol{\varepsilon}}(p)^\flat(\boldsymbol{e}). \tag{6.15}$$

Therefore $\phi(p, \boldsymbol{e}) = (p, \mathring{\boldsymbol{\varepsilon}}(p)^\flat(\boldsymbol{e}))$. ☐

Since $F_+^\uparrow$ is $\mathcal{L}_+^\uparrow$-trivial, the obvious question to ask is whether it can be shown to be naturally $\mathcal{L}_+^\uparrow$-trivial. The above lemma shows that a choice of section $\mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$ for each $\boldsymbol{M}$ gives a family of trivialising morphisms $\Phi_{\boldsymbol{M}} := (\phi_{\mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}}}, \mathrm{id}_{\mathcal{L}_+^\uparrow}) : F_+^\uparrow \boldsymbol{M} \to \mathscr{T}_{\mathcal{L}_+^\uparrow} \boldsymbol{M}$, over $\boldsymbol{M} \in \mathsf{Loc}$, and any natural trivialisation of $F_+^\uparrow$ must be of this form. However, we have the following lemma:

Lemma 6.1.8. $F_+^\uparrow$ *is not naturally* $\mathcal{L}_+^\uparrow$*-trivial.*

Proof. Suppose that $\Phi : F_+^\uparrow \stackrel{\cdot}{\longrightarrow} \mathscr{T}_{\mathcal{L}_+^\uparrow}$ has components made up of trivialising morphisms, so that $\Phi_{\boldsymbol{M}} = (\phi_{\mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}}}, \mathrm{id}_{\mathcal{L}_+^\uparrow})$ for some choice of $\mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$ for each $\boldsymbol{M} \in \mathsf{Loc}$. Naturality of Φ requires that for any $(p, \boldsymbol{e}) \in F_+^\uparrow \boldsymbol{M}$ and

$\psi \in \mathsf{Loc}(\boldsymbol{M}, \boldsymbol{N})$, we have

$$
\begin{aligned}
(\psi(p), \mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}}(p)^\flat(\boldsymbol{e})) = \boldsymbol{\psi}(\phi_{\mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}}}(p, \boldsymbol{e})) &= \phi_{\mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{N}}}(\psi_*(p, \boldsymbol{e})) \\
&= (\psi(p), \mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{N}}(\psi(p))^\flat(D\psi_p\, \boldsymbol{e})). \qquad (6.16)
\end{aligned}
$$

Therefore we have $\mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}}(p)^\flat(\boldsymbol{e}) = \mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{N}}(\psi(p))^\flat(D\psi_p\, \boldsymbol{e})$; letting $\boldsymbol{e} = \mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}}(p)$ yields $\mathbf{1} = \mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{N}}(\psi(p))^\flat(D\psi_p\, \mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}}(p))$, and therefore $\mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{N}}(\psi(p)) = D\psi_p\, \mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}}(p)$. This is impossible to satisfy in general; if we consider $\boldsymbol{N} = \boldsymbol{M} = \mathbb{R}^{1,3}$ and let ψ be some rotation of $\boldsymbol{M}$ around an axis passing through p, then this relation becomes $\mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}}(p) = D\psi_p\, \mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}}(p)$, which entails that $\mathring{\boldsymbol{\varepsilon}}_{\boldsymbol{M}}(p)$ is conserved under any such rotation. This is clearly impossible; therefore no natural trivialisation exists, and so $F_+^\uparrow$ is not naturally $\boldsymbol{\mathcal{L}}_+^\uparrow$-trivial. □

6.2 Global frames

As previously mentioned, it will not turn out to be possible to define the locally covariant classical Dirac field theory as a functor with domain Loc. We wish to give the most general formulation possible, and to do this we will need a number of additional categories based on Loc, but with the additional properties necessary to construct the theory. In this section we will examine some of the properties of the global frames $\boldsymbol{\varepsilon} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$, and give definitions of further spacetime categories based on these properties. We will first give the following definition and result:

Definition 6.2.1. *A* right G-torsor *is a triple (X, G, R) where X is a non-empty set, G is a group and R is a right action of G on X.* RTor *is the category of right torsors; an arrow from (X, G, R) to (X', G', R') is a pair (f, α) where $\alpha : G \to G'$ is a group homomorphism and $f : X \to X'$ is a function that intertwines the group actions, i.e. $R_{\alpha(g)} \circ f = f \circ R_g$ for all $g \in G$.*

Clearly the fibre at any point of a principal G-bundle may be regarded as a G-torsor, and any bundle morphism in RPBund consists of a fibrewise family of RTor-arrows. Furthermore, the space $C^\infty(\boldsymbol{M}; \boldsymbol{\mathcal{L}}_+^\uparrow)$ of smooth $\boldsymbol{\mathcal{L}}_+^\uparrow$-valued functions over $\boldsymbol{M}$ can be given a group structure via pointwise multiplication; the space $\Gamma^\infty(F_+^\uparrow \boldsymbol{M})$ of global frames of a spacetime $\boldsymbol{M} \in \mathsf{Loc}_4$ is then clearly a right $C^\infty(\boldsymbol{M}; \mathcal{L}_+^\uparrow)$-torsor, inheriting the group action pointwise from $F_+^\uparrow \boldsymbol{M}$.

Lemma 6.2.2. *Consider the category $F_+^\uparrow \mathsf{Loc}_4 \subset \mathsf{RPBund}_4$, that is the image of Loc_4 under the functor $F_+^\uparrow$. There is a contravariant functor Γ^∞ from $F_+^\uparrow \mathsf{Loc}_4$ to RTor that maps $F_+^\uparrow \boldsymbol{M}$ to $\Gamma^\infty(F_+^\uparrow \boldsymbol{M})$ and $F_+^\uparrow(\psi) = (\psi_*, \mathrm{id}_{\mathcal{L}_+^\uparrow})$ to $(\psi^\star, \psi^*)$, where for arbitrary $\boldsymbol{\varepsilon} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{N})$ and $p \in \boldsymbol{M}$ we define*

$$(\psi^\star(\boldsymbol{\varepsilon}))_a\,(p) := D\psi_p^{-1}\,\varepsilon_a(\psi(p)). \tag{6.17}$$

Proof. Firstly, (6.17) is well defined since ψ is a local diffeomorphism. This definition of $\Gamma^\infty(F_+^\uparrow(\psi))$ can then easily be seen to make Γ^∞ a contravariant functor; in particular, for Loc_4 arrows $\psi : \boldsymbol{L} \to \boldsymbol{M}$ and $\psi' : \boldsymbol{M} \to \boldsymbol{N}$ we have

$$\begin{aligned}\left(\psi^\star\psi'^\star(\boldsymbol{\varepsilon})\right)_a(p) &= D\psi_p^{-1}\left(\psi'^\star(\boldsymbol{\varepsilon})\right)_a(\psi(p))\\ &= D\psi_p^{-1}\,D\psi'^{-1}_{\psi(p)}\,\varepsilon_a(\psi'(\psi(p)))\\ &= D(\psi'\circ\psi)_p^{-1}\,\varepsilon_a(\psi'(\psi(p)))\\ &= ((\psi'\circ\psi)^\star(\boldsymbol{\varepsilon}))_a\,(p).\end{aligned} \tag{6.18}$$

Finally, we show that $\psi^\star$ intertwines the group action. Let $\lambda \in C^\infty(\boldsymbol{N}; \mathcal{L}_+^\uparrow)$. We have

$$\begin{aligned}(R_{\psi^*\lambda}\psi^\star(\boldsymbol{\varepsilon}))_a\,(p) &= \left(D\psi_p^{-1}\,\varepsilon_b(\psi(p))\right)\lambda^b{}_a(\psi(p))\\ &= D\psi_p^{-1}\,(R_\lambda\boldsymbol{\varepsilon})_a(\psi(p))\\ &= (\psi^\star(R_\lambda\boldsymbol{\varepsilon}))_a\,(p).\end{aligned} \tag{6.19}$$

Therefore $R_{\psi^*\lambda} \circ \psi^\star = \psi^\star \circ R_\lambda$, and so $(\psi^\star, \psi^*)$ is an arrow in RTor as required. ☐

If $\boldsymbol{\varepsilon}'(p) = \boldsymbol{\varepsilon}(p)\lambda(p)$ for some smooth $\lambda : \boldsymbol{M} \to \mathcal{L}_+^\uparrow$, i.e. $\boldsymbol{\varepsilon}' = R_\lambda\boldsymbol{\varepsilon}$, then by (6.13) we have $\pi_{\boldsymbol{\varepsilon}'}(p, \mathbf{1}) = R_\lambda(p, \boldsymbol{\varepsilon}(p)) = R_\lambda\pi_{\boldsymbol{\varepsilon}}(p, \mathbf{1})$, so

$$\pi_{R_\lambda\boldsymbol{\varepsilon}} = R_\lambda \circ \pi_{\boldsymbol{\varepsilon}} \tag{6.20}$$

for all $\boldsymbol{\varepsilon} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$.

6.2.1 Equivalence of spin structures and global frames

Lemma 6.2.3. *Let $\boldsymbol{M}, \boldsymbol{N} \in \mathsf{Loc}_4$. Suppose that there exists an RPBund_4-arrow $(\sigma, \mathrm{id}_{\mathrm{Spin}^0_{1,3}}) : S\boldsymbol{M} \to S\boldsymbol{N}$. Then there exists a smooth map $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$ such that $\sigma(p, S) = (\mathscr{b}(\sigma)(p), s(p)S)$, where $\mathscr{b}(\sigma)$ is defined as in Definition 6.1.1.*

Proof. Since $S\boldsymbol{M}$ and $S\boldsymbol{N}$ are product bundles, the fibre bundle $\mathscr{p}(S\boldsymbol{M})$ is simply $(\boldsymbol{M} \times \mathrm{Spin}^0_{1,3}, \pi_0)$. Therefore for any $p \in \boldsymbol{M}$, we have $\sigma(p, \mathbf{1}) = (\mathscr{b}(\sigma)(p), s(p))$ for some smooth $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$. Moreover, in order for $(\sigma, \mathrm{id}_{\mathrm{Spin}^0_{1,3}})$ to be an RPBund_4-arrow we must have $R_S \circ \sigma = \sigma \circ R_S$ for any $S \in \mathrm{Spin}^0_{1,3}$, and accordingly

$$\sigma(p, S) = R_S(\mathscr{b}(\sigma)(p), s(p)) = (\mathscr{b}(\sigma)(p), s(p)S). \tag{6.21}$$

□

Definition 6.2.4. *Following [39], we say that the spin structures $(S\boldsymbol{M}, \pi)$ and $(S\boldsymbol{M}, \pi')$ over $\boldsymbol{M} \in \mathsf{Loc}_4$ are* equivalent *if there exists an RPBund_4 isomorphism $(\sigma, \mathrm{id}_{\mathrm{Spin}^0_{1,3}}) : (S\boldsymbol{M}, \pi) \to (S\boldsymbol{M}, \pi')$ satisfying $\pi = \pi' \circ \sigma$. We then write $(S\boldsymbol{M}, \pi) \sim (S\boldsymbol{M}, \pi')$. It is clear that the map σ may be chosen to be base-point preserving, and is then of the form $\sigma : (p, S) \mapsto (p, s(p)S)$ for some smooth map $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$, by the above lemma.*

Lemma 6.2.5. *Suppose that we have spin structures $(S\boldsymbol{M}, \pi_{\boldsymbol{\varepsilon}})$ and $(S\boldsymbol{M}, \pi_{\boldsymbol{\varepsilon}'})$ over $\boldsymbol{M} \in \mathsf{Loc}_4$, where we now specify the global frames $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$ that define the spin frame projections as in* (6.13)*. Then $(S\boldsymbol{M}, \pi_{\boldsymbol{\varepsilon}}) \sim (S\boldsymbol{M}, \pi_{\boldsymbol{\varepsilon}'})$ if and only if there exists a smooth map $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$ such that $\boldsymbol{\varepsilon}(p) = \boldsymbol{\varepsilon}'(p)\Lambda(s(p))$.*

Proof. We choose a base-point-preserving isomorphism $\sigma : (S\boldsymbol{M}, \pi_{\boldsymbol{\varepsilon}}) \to (S\boldsymbol{M}, \pi_{\boldsymbol{\varepsilon}'})$ such that $\pi_{\boldsymbol{\varepsilon}} = \pi_{\boldsymbol{\varepsilon}'} \circ \sigma$, which must act as $\sigma(p, S) = (p, s(p)S)$ for some smooth map $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$. For arbitary $p \in \boldsymbol{M}$ and $S \in \mathrm{Spin}^0_{1,3}$, (6.13) entails that

$$(p, \boldsymbol{\varepsilon}(p)) = \pi_{\boldsymbol{\varepsilon}}(p, \mathbf{1}_{\mathrm{Spin}^0_{1,3}}) = \pi_{\boldsymbol{\varepsilon}'}(p, s(p))$$
$$= \pi_{\boldsymbol{\varepsilon}'} R_{s(p)}(p, \mathbf{1}_{\mathrm{Spin}^0_{1,3}}) = R_{\Lambda(s(p))}\pi_{\boldsymbol{\varepsilon}'}(p, \mathbf{1}_{\mathrm{Spin}^0_{1,3}}) = (p, \boldsymbol{\varepsilon}'(p)\Lambda(s(p))). \quad (6.22)$$

Conversely suppose that $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$ is smooth, and $\boldsymbol{\varepsilon} \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$ is arbitrary; then we may define a new global frame $\boldsymbol{\varepsilon}' \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$ by $\boldsymbol{\varepsilon}(p) = \boldsymbol{\varepsilon}'(p)\Lambda(s(p))$. There is then an equivalence $(S\boldsymbol{M}, \pi_{\boldsymbol{\varepsilon}}) \sim (S\boldsymbol{M}, \pi_{\boldsymbol{\varepsilon}'})$ via $\sigma : (S\boldsymbol{M}, \pi_{\boldsymbol{\varepsilon}}) \to (S\boldsymbol{M}, \pi_{\boldsymbol{\varepsilon}'})$ defined by $\sigma(p, S) := (p, s(p)S)$. □

Definition 6.2.6. *The frames $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$ are equivalent if $\boldsymbol{\varepsilon}(p) = \boldsymbol{\varepsilon}'(p)\Lambda(s(p))$ for some smooth $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$. We then write $\boldsymbol{\varepsilon} \sim \boldsymbol{\varepsilon}'$.*

6.2.2 Equivalence classes of global frames

We might be tempted to assume in fact that $\boldsymbol{\varepsilon} \sim \boldsymbol{\varepsilon}'$ whenever $\boldsymbol{\varepsilon}(p) = \boldsymbol{\varepsilon}'(p)\lambda(p)$ for some smooth $\lambda : \boldsymbol{M} \to \mathcal{L}^\uparrow_+$. However, although such a map exists for all $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$,[4] there may be examples of such maps which cannot be decomposed as $\lambda = \Lambda \circ s$ for any smooth $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$. This is a consequence of a more general property of covering spaces (see e.g. [33, Prop. 1.33]):

Lemma 6.2.7 (The lifting criterion)**.** *Consider a topological space X and a point $x_0 \in X$. Let C be a covering space of X with cover $p : C \to X$ and let $c_0 \in C$ satisfy $p(c_0) = x_0$. Furthermore, consider a continuous map $f : Y \to X$ from a path-connected and locally path-connected space Y with $y_0 \in Y$ chosen such that $f(y_0) = x_0$. Then a continuous map $\tilde{f} : Y \to C$ satisfying $f = p \circ \tilde{f}$ and $\tilde{f}(y_0) = c_0$ exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(C, c_0))$.*

A map $\tilde{f}$ of this form is called a *lift* of f. Note that in the case that p is the universal cover, C is simply connected; therefore the condition becomes simply $f_*(\pi_1(Y, y_0)) = \{0\}$. Moreover, since $\Lambda : \mathrm{Spin}^0_{1,3} \to \mathcal{L}^\uparrow_+$ is a local

[4] For any $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$ and $p \in \boldsymbol{M}$ we define $\lambda(p) := \boldsymbol{\varepsilon}'(p)^\flat(\boldsymbol{\varepsilon}(p)) \in \mathcal{L}^\uparrow_+$; as $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ are smooth so is the map $\lambda : \boldsymbol{M} \to \mathcal{L}^\uparrow_+$, and we have $\boldsymbol{\varepsilon}(p) = \boldsymbol{\varepsilon}'(p)\lambda(p)$.

diffeomorphism, it has a smooth local inverse; therefore if $\lambda : \boldsymbol{M} \to \mathcal{L}_+^\uparrow$ is smooth, then any lift of λ must also be smooth.

We may apply this criterion to the situation of a continuous map $\lambda : \boldsymbol{M} \to \mathcal{L}_+^\uparrow$ where $\boldsymbol{M}$ is connected, letting $Y = \boldsymbol{M}$, $C = \mathrm{Spin}_{1,3}^0$ and $X = \mathcal{L}_+^\uparrow$ with x_0 arbitrary; this gives the following corollary:

Corollary 6.2.8. *Let $\boldsymbol{M} \in \mathsf{Loc}_4^c$, and let $\lambda : \boldsymbol{M} \to \mathcal{L}_+^\uparrow$ be smooth. Then λ has a smooth lift $\tilde{\lambda} : \boldsymbol{M} \to \mathrm{Spin}_{1,3}^0$ if and only if $\lambda_*(\pi_1(\boldsymbol{M})) = \{0\}$.*

This result is explicitly stated in [39], where the following result is also proved.

Lemma 6.2.9. *Let $\boldsymbol{M} \in \mathsf{Loc}_4^c$. The equivalence classes of global frames (and hence spin structures) are in one-to-one correspondence with the homomorphisms from $\pi_1(\boldsymbol{M})$ to $\pi_1(\mathcal{L}_+^\uparrow) = \mathbb{Z}_2$, or alternatively, the elements of the singular cohomology group $H^1(\boldsymbol{M}; \mathbb{Z}_2)$.*

Using the convention that when $\boldsymbol{M} \in \mathsf{Loc}_4$ is composed of two or more disconnected components $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_N \in \mathsf{Loc}_4^c$, we take

$$\pi_1(\boldsymbol{M}) := \bigtimes_{i=1}^{N} \pi_1(\boldsymbol{M}_i), \tag{6.23}$$

the above lemma extends directly to the whole of Loc_4.

Another immediate consequence of corollary 6.2.8, which is also stated in [39], is the following:

Lemma 6.2.10. *Two frames $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$ are equivalent if there exists a homotopy between them; in other words, if there exists a continuous function $E : \boldsymbol{M} \times [0,1] \to F_+^\uparrow \boldsymbol{M}$ such that $E(\,\cdot\,, 0) = \boldsymbol{\varepsilon}$, $E(\,\cdot\,, 1) = \boldsymbol{\varepsilon}'$ and $E(\,\cdot\,, \tau) \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$ for all $\tau \in [0,1]$.*

Proof. The existence of such a homotopy E is equivalent to the existence of a continuous function $L : \boldsymbol{M} \times [0,1] \to \mathcal{L}_+^\uparrow$ such that $L(\,\cdot\,, 0) \equiv \mathbf{1}_{\mathcal{L}_+^\uparrow}$ and $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon} L(\,\cdot\,, 1)$, so that E and L are related via $E(p,\tau) = \boldsymbol{\varepsilon}(p) L(p,\tau)$. In other words, if $\lambda : \boldsymbol{M} \to \mathcal{L}_+^\uparrow$ is defined via $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\lambda$, then $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ are homotopic if and only if λ is homotopic to the constant map $\mathbf{1}_{\mathcal{L}_+^\uparrow}$.

Suppose that such an L exists, and that $f : S^1 \to \boldsymbol{M}$ is a loop in $\boldsymbol{M}$. We denote by $g := \lambda \circ f$ the induced loop in $\mathcal{L}_+^\uparrow$. Then $G : S^1 \times [0,1] \to \mathcal{L}_+^\uparrow$ defined by $G(e^{i\theta}, \tau) := L(f(e^{i\theta}), \tau)$ is continuous and satisfies $G(\,\cdot\,, 0) = \mathbf{1}_{\mathcal{L}_+^\uparrow}$ and $G(\,\cdot\,, 1) = g$, so g is null homotopic. Since f was arbitrary it follows that $\lambda_* = 0$, and therefore λ has a lift into $\mathrm{Spin}^0_{1,3}$. Consequently $\boldsymbol{\varepsilon} \sim \boldsymbol{\varepsilon}'$. □

The converse to this lemma is not true; since $\mathrm{Spin}^0_{1,3}$ is not contractible, there may be a map $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$ that is not null homotopic. In this case a frame $\boldsymbol{\varepsilon}$ on $\boldsymbol{M}$ would not be homotopic to $\boldsymbol{\varepsilon}\Lambda(s)$.

Since any spacetime $\boldsymbol{M}$ is diffeomorphic to $\mathbb{R} \times \Sigma$ where Σ is a Cauchy surface of $\boldsymbol{M}$, it holds that $\pi_1(\boldsymbol{M}) = \pi_1(\mathbb{R}) \times \pi_1(\Sigma) = \pi_1(\Sigma)$. We would therefore expect equivalence or otherwise of global frames to be visible when we restrict only to a given Cauchy surface of a spacetime, and indeed we find this to be the case.

Lemma 6.2.11. *Let $\boldsymbol{M} \in \mathsf{Loc}_4$ have a Cauchy surface Σ, and consider $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$. The restrictions $\boldsymbol{\varepsilon}|_\Sigma, \boldsymbol{\varepsilon}'|_\Sigma$ are equivalent (in other words, are related by $\boldsymbol{\varepsilon}'|_\Sigma = \boldsymbol{\varepsilon}|_\Sigma \Lambda(s_\Sigma)$ for some smooth $s_\Sigma : \Sigma \to \mathrm{Spin}^0_{1,3}$) if and only if $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ are themselves equivalent.*

Proof. Suppose that $\boldsymbol{\varepsilon} \sim \boldsymbol{\varepsilon}'$, with $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\Lambda(s)$. Then $\boldsymbol{\varepsilon}'|_\Sigma = \boldsymbol{\varepsilon}|_\Sigma \Lambda(s|_\Sigma)$, so $\boldsymbol{\varepsilon}|_\Sigma \sim \boldsymbol{\varepsilon}'|_\Sigma$. Conversely, suppose that $\boldsymbol{\varepsilon}'|_\Sigma = \boldsymbol{\varepsilon}'|_\Sigma \Lambda(s_\Sigma)$ for some smooth $s_\Sigma : \Sigma \to \mathrm{Spin}^0_{1,3}$. There exists a Loc-arrow $\psi : \mathbb{R} \times \Sigma \to \boldsymbol{M}$ with $\psi(\{0\} \times \Sigma) = \Sigma \subset \boldsymbol{M}$; if $\lambda : \boldsymbol{M} \to \mathcal{L}_+^\uparrow$ is defined by $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\lambda$, then λ may be pulled back by ψ to a map $\widetilde{\lambda} : \mathbb{R} \times \Sigma \to \mathcal{L}_+^\uparrow$. We define $\widetilde{\lambda}_t := \widetilde{\lambda}|_{\{t\} \times \Sigma}$, so that $\widetilde{\lambda}_0 = \lambda|_\Sigma = \Lambda(s_\Sigma)$.

Since $\widetilde{\lambda}$ is smooth it follows that $\widetilde{\lambda}_t$ is homotopic to $\widetilde{\lambda}_0$ for all $t \in \mathbb{R}$, and so there exists some $s_t : \Sigma \to \mathrm{Spin}^0_{1,3}$ for each t such that $\widetilde{\lambda}_t = \widetilde{\lambda}_0 \Lambda(s_t)$. But since $\mathbb{R}$ is contractible it follows that s_t may be chosen to vary smoothly with respect to t, so that there exists some smooth $\tilde{s} : \mathbb{R} \times \Sigma \to \mathrm{Spin}^0_{1,3}$ satisfying $\tilde{s}(t, x) = s_t(x)$. Then $\widetilde{\lambda}(t, x) = \widetilde{\lambda}_0(x)\Lambda(\tilde{s}(t, x)) = \Lambda(s_\Sigma(x)\tilde{s}(t, x))$, so $\widetilde{\lambda}$ is liftable to $\mathrm{Spin}^0_{1,3}$. Hence $\lambda = \Lambda(s)$ for some smooth $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$, and therefore $\boldsymbol{\varepsilon} \sim \boldsymbol{\varepsilon}'$. □

Clearly, the global frames of a spacetime $\boldsymbol{M} \in \mathsf{Loc}_4$ are *all* equivalent if[5] (each disconnected component of) $\boldsymbol{M}$ is simply connected, in which case $\pi_1(\boldsymbol{M})$ — and hence $H^1(\boldsymbol{M};\mathbb{Z}_2)$ — are trivial. If $H^1(\boldsymbol{M};\mathbb{Z}_2)$ is non-trivial, we wish to ask whether we can find an equivalence class of frames that is 'preferred' in some way over the others; the existence of spacetimes in which there is no preferred class (or more than one) provides an obstacle to defining the full Dirac theory as a functor with domain Loc_4. One way of approaching this problem is to look at *extendibility* of global frames under a Loc_4-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$.

6.2.3 Extendibility of global frames

Lemma 6.2.12. *Let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be an arrow in Loc_4. If two global frames $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Gamma^\infty(F^\uparrow_+\boldsymbol{N})$ are equivalent, then the frames $\psi^\star(\boldsymbol{\varepsilon}), \psi^\star(\boldsymbol{\varepsilon}') \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$, as defined in* (6.17)*, are also equivalent.*

Proof. If $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ are equivalent then there is a smooth map $s : \boldsymbol{N} \to \mathrm{Spin}^0_{1,3}$ such that $\boldsymbol{\varepsilon}'(p) = \boldsymbol{\varepsilon}\Lambda(s(p))$, or equivalently $\boldsymbol{\varepsilon}' = R_{\Lambda\circ s}\boldsymbol{\varepsilon}$. Applying the RTor-arrow $\Gamma^\infty(\psi) = (\psi^\star, \psi^*)$, we find that

$$\psi^\star(\boldsymbol{\varepsilon}') = \psi^\star(R_{\Lambda\circ s}\boldsymbol{\varepsilon}) = R_{\psi^*(\Lambda\circ s)}\psi^\star(\boldsymbol{\varepsilon}) = R_{\Lambda\circ(\psi^* s)}\psi^\star(\boldsymbol{\varepsilon}). \tag{6.24}$$

Since $\psi^* s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$ is smooth, it follows that $\psi^\star(\boldsymbol{\varepsilon}) \sim \psi^\star(\boldsymbol{\varepsilon}')$. ☐

Definition 6.2.13. *Let $\psi : \boldsymbol{M} \to \boldsymbol{N}$ be an arrow in Loc_4. A global frame $\boldsymbol{\varepsilon} \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$ is* extendible under ψ *if there exists $\boldsymbol{\varepsilon}' \in \Gamma^\infty(F^\uparrow_+\boldsymbol{N})$ such that $\psi^\star(\boldsymbol{\varepsilon}') = \boldsymbol{\varepsilon}$.*

The converse of Lemma 6.2.12, that given $\psi : \boldsymbol{M} \to \boldsymbol{N}$, two equivalent global frames $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$ have equivalent extensions under ψ, fails on two levels. First, there may exist frames in $\Gamma^\infty(F^\uparrow_+\boldsymbol{M})$ that are not extendible under ψ; and secondly, equivalent frames in $\Gamma^\infty(F^\uparrow_+\boldsymbol{M})$ may have extensions in $\Gamma^\infty(F^\uparrow_+\boldsymbol{N})$ that are not equivalent.

[5] But not *only* if; for example, we might construct a spacetime in Loc_4 with Cauchy surfaces homeomorphic to a spherical manifold of the form $S^3/\mathbb{Z}_p$ for p odd. This spacetime then has fundamental group $\mathbb{Z}_p$, and by the first isomorphism theorem, this group admits only the trivial (zero) homomorphism into $\mathbb{Z}_2$.

In the first case, consider the situation where $\boldsymbol{M}$ has more than one equivalence class of frames, and where $\boldsymbol{N}$ is simply connected[6]. By Lemma 6.2.12, the image of $\Gamma^\infty(F_+^\uparrow \boldsymbol{N})$ under $\psi^\star$ is a subset of only one of the two equivalence classes of frames of $\boldsymbol{M}$, and so only these frames may have extensions with respect to ψ. In the second case, consider the reverse case where $\boldsymbol{M}$ is simply connected but $\boldsymbol{N}$ has multiple equivalence classes of global frames. We may choose $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Gamma^\infty(F_+^\uparrow \boldsymbol{N})$ that are inequivalent; however it must hold that $\psi^\star(\boldsymbol{\varepsilon}) \sim \psi^\star(\boldsymbol{\varepsilon}')$.

Even if two frames $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$ are equivalent, extendibility of $\boldsymbol{\varepsilon}$ under $\psi : \boldsymbol{M} \to \boldsymbol{N}$ does not guarantee extendibility of $\boldsymbol{\varepsilon}'$ under ψ. The following examples demonstrate why this is the case:

Example 6.2.14. Let $\boldsymbol{Z}_0 = \{(0,0,0,z) \in \mathbb{R}^{1,3}\}$ and $\boldsymbol{M} = \mathbb{R}^{1,3} \setminus J_{\mathbb{R}^{1,3}}(\boldsymbol{Z}_0)$. Denote the canonical inclusion of $\boldsymbol{M}$ into $\mathbb{R}^{1,3}$ by $\iota_{\boldsymbol{M}}$. The standard frame $\mathring{\boldsymbol{\varepsilon}}$ on $\boldsymbol{M}$ is defined by $\mathring{\varepsilon}_a^\mu(p) = \delta_a^\mu$; this is clearly extendible under $\iota_{\boldsymbol{M}}$. Now, define a nonconstant function $\phi : [0, 2\pi] \to \mathbb{R}$ satisfying $\phi(0) = \phi(2\pi)$. Define a second frame $\boldsymbol{\varepsilon}[\phi]$ by

$$\begin{aligned}
\varepsilon[\phi]_0(p) &:= \mathring{\varepsilon}_0(p), \\
\varepsilon[\phi]_1(p) &:= \cos(\phi(\theta_p))\mathring{\varepsilon}_1(p) - \sin(\phi(\theta_p))\mathring{\varepsilon}_2(p), \\
\varepsilon[\phi]_2(p) &:= \sin(\phi(\theta_p))\mathring{\varepsilon}_1(p) + \cos(\phi(\theta_p))\mathring{\varepsilon}_2(p), \\
\varepsilon[\phi]_3(p) &:= \mathring{\varepsilon}_3(p),
\end{aligned} \tag{6.25}$$

where $\theta_p \in [0, 2\pi)$ is the angle in the x-y plane obtained by expressing p in cylindrical polar coordinates. This frame is a well defined element of $\Gamma^\infty(F_+^\uparrow \boldsymbol{M})$, and is equivalent to $\mathring{\boldsymbol{\varepsilon}}$ under the homotopy $E(\,\cdot\,, \tau) = \boldsymbol{\varepsilon}[\tau\phi]$, since $\boldsymbol{\varepsilon}[0] = \mathring{\boldsymbol{\varepsilon}}$. However, it is not extendible under $\iota_{\boldsymbol{M}}$; since ϕ is nonconstant, there is no well-defined choice of $\boldsymbol{\varepsilon}[\phi](p)$ for $p \in \boldsymbol{Z}_0$ that would retain continuity, let alone smoothness of $\boldsymbol{\varepsilon}[\phi]$.

[6] For example, let $\boldsymbol{N}$ be Minkowski space $\mathbb{R}^{1,3}$, and let $\boldsymbol{M}$ be defined as in the following example, so that $\boldsymbol{M}$ has fundamental group $\mathbb{Z}$ and $H^1(\boldsymbol{M}; \mathbb{Z}_2) = \mathbb{Z}_2$.

Example 6.2.15. Consider $\boldsymbol{N} \in \mathsf{Loc}_4^{sc}$ consisting of the 'future half' of Minkowski space $\mathbb{R}^{1,3}$, i.e. with underlying manifold $\{(t,x,y,z) \in \mathbb{R}^4 : t > 0\}$, and the standard flat metric $\boldsymbol{\eta}$. As in the previous example, we denote the canonical inclusion of $\boldsymbol{N}$ in $\mathbb{R}^{1,3}$ by $\iota_{\boldsymbol{N}}$, and denote the standard frame for $\boldsymbol{N}$ by $\mathring{\boldsymbol{\varepsilon}}$; this is again extendible under $\iota_{\boldsymbol{N}}$. We now fix $u \in \mathbb{R} \setminus \{0\}$ and define a new frame $\boldsymbol{\varepsilon}[u] \in \Gamma^\infty(F_+^\uparrow \boldsymbol{N})$ by

$$\begin{aligned} \varepsilon[u]_0(p) &:= \cosh(u/t_p)\mathring{\varepsilon}_0(p) + \sinh(u/t_p)\mathring{\varepsilon}_1(p), \\ \varepsilon[u]_1(p) &:= \sinh(u/t_p)\mathring{\varepsilon}_0(p) + \cosh(u/t_p)\mathring{\varepsilon}_1(p), \\ \varepsilon[u]_2(p) &:= \mathring{\varepsilon}_2(p), \\ \varepsilon[u]_3(p) &:= \mathring{\varepsilon}_3(p), \end{aligned} \tag{6.26}$$

where t_p is the time coordinate of p. Clearly this frame is well defined and equivalent to $\mathring{\boldsymbol{\varepsilon}}$ under the homotopy $[0,1] \ni \tau \mapsto \boldsymbol{\varepsilon}[\tau u]$, since $\boldsymbol{\varepsilon}[0] = \mathring{\boldsymbol{\varepsilon}}$. Again, however, $\boldsymbol{\varepsilon}[u]$ is not extendible under $\iota_{\boldsymbol{N}}$ since there is no possible choice of $\boldsymbol{\varepsilon}[u](p)$ for $t_p = 0$ that would retain continuity of $\boldsymbol{\varepsilon}[u]$.

Therefore, it is not even guaranteed that a well-defined frame on a simply connected spacetime is extendible to a second simply connected spacetime under a Cauchy arrow, which we might expect to be the case. However, we do have the following result:

Lemma 6.2.16. *Let $\boldsymbol{M} \in \mathsf{Loc}_4$, and $\Sigma \subset \boldsymbol{M}$ be a smooth Cauchy surface. Let $\boldsymbol{\varepsilon}_\Sigma$ be a smooth frame defined only on Σ. Then there is an extension $\boldsymbol{\varepsilon} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$ such that $\boldsymbol{\varepsilon}|_\Sigma = \boldsymbol{\varepsilon}_\Sigma$.*

Proof. The spacetime $\boldsymbol{M}$ is diffeomorphic to $\mathbb{R} \times \Sigma$; we choose a diffeomorphism $\psi : \mathbb{R} \times \Sigma \to \boldsymbol{M}$. We also pick an arbitrary global frame $\mathring{\boldsymbol{\varepsilon}} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$; then there is a unique map $\lambda_\Sigma : \Sigma \to \mathcal{L}_+^\uparrow$ such that $\boldsymbol{\varepsilon}_\Sigma = \mathring{\boldsymbol{\varepsilon}}|_\Sigma \lambda_\Sigma$. We then define a smooth map $\lambda : \boldsymbol{M} \to \mathcal{L}_+^\uparrow$ by $\lambda(\psi(t,p)) := \lambda_\Sigma(p)$ for $p \in \Sigma$, and define $\boldsymbol{\varepsilon} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$ by $\boldsymbol{\varepsilon} := \mathring{\boldsymbol{\varepsilon}}\lambda$. It follows that $\boldsymbol{\varepsilon}|_\Sigma = \boldsymbol{\varepsilon}_\Sigma$. □

The property of extendibility clearly depends on the Loc-arrow; indeed, there are examples of pairs of Loc-arrows $\psi_1, \psi_2 : \boldsymbol{M} \to \boldsymbol{N}$ where $\boldsymbol{M}$ has more than one equivalence class of frames and $\boldsymbol{N}$ is simply connected, and yet for any frame $\boldsymbol{\varepsilon} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$, the pulled-back frames $\psi_1^\star \boldsymbol{\varepsilon}$ and $\psi_2^\star \boldsymbol{\varepsilon}$ on $\boldsymbol{M}$ are inequivalent. This suggests that there is no way of canonically specifying a preferred class of frames for each spacetime $\boldsymbol{M}$.

In the following chapter we will show that any sensible construction of the Dirac theory must depend on the equivalence class of frames chosen, and so we abandon hope of constructing the Dirac theory as a functor whose domain is Loc_4, and concentrate instead on making the formulation as general as we can. To this end we define the following additional categories of spacetimes:

Definition 6.2.17. *The category* FLoc_4 *has as its objects all pairs* $(\boldsymbol{M}, \boldsymbol{\varepsilon})$ *where* $\boldsymbol{M} \in \mathsf{Loc}_4$ *and* $\boldsymbol{\varepsilon} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$. *An arrow* $\boldsymbol{\psi} : (\boldsymbol{M}, \boldsymbol{\varepsilon}_{\boldsymbol{M}}) \to (\boldsymbol{N}, \boldsymbol{\varepsilon}_{\boldsymbol{N}})$ *covers a* Loc*-arrow* $\psi : \boldsymbol{M} \to \boldsymbol{N}$ *that satisfies* $\psi^\star(\boldsymbol{\varepsilon}_{\boldsymbol{N}}) = \boldsymbol{\varepsilon}_{\boldsymbol{M}}$.

We also define the category $[\mathsf{F}]\mathsf{Loc}_4$*, whose objects are pairs* $(\boldsymbol{M}, \mathcal{E})$ *where* $\boldsymbol{M} \in \mathsf{Loc}_4$ *and* $\mathcal{E}$ *is an equivalence class of global frames on* $\boldsymbol{M}$. *An arrow* $[\psi]$ *in* $[\mathsf{F}]\mathsf{Loc}_4((\boldsymbol{M}, \mathcal{E}_{\boldsymbol{M}}), (\boldsymbol{N}, \mathcal{E}_{\boldsymbol{N}}))$ *consists of an arrow* $\psi : \boldsymbol{M} \to \boldsymbol{N}$ *such that there exists* $\boldsymbol{\varepsilon} \in \mathcal{E}_{\boldsymbol{N}}$ *with* $\psi^\star(\boldsymbol{\varepsilon}) \in \mathcal{E}_{\boldsymbol{M}}$. *(By Lemma 6.2.12, it then follows that* $\psi^\star(\boldsymbol{\varepsilon}') \in \mathcal{E}_{\boldsymbol{M}}$ *for any* $\boldsymbol{\varepsilon}' \in \mathcal{E}_{\boldsymbol{N}}$*.)*

There is a forgetful functor $\mathscr{E} : \mathsf{FLoc}_4 \to [\mathsf{F}]\mathsf{Loc}_4$ *that maps* $(\boldsymbol{M}, \boldsymbol{\varepsilon})$ *to* $(\boldsymbol{M}, \mathcal{E})$ *where* $\mathcal{E} \ni \boldsymbol{\varepsilon}$*, and maps an arrow* $\boldsymbol{\psi}$ *to the arrow* $[\psi]$ *that covers the same underlying* Loc_4*-arrow.*

We find that the simplest construction of the classical Dirac field theory is given by a functor from FLoc_4 to the category $\mathsf{HermAdj}_\mathsf{C}$ of Hermitian adjoint structures, defined in the previous chapter, and this may be made into a quantum theory based in Alg by applying the quantization functor $\mathscr{Q}_{\text{adj}}$. We will also show that it is possible to successfully define classical and quantum Dirac theories whose domain is $[\mathsf{F}]\mathsf{Loc}_4$, although these theories are not isomorphic to the aforementioned theories from FLoc_4; in fact, in the quantum case, we find that the algebra arising from the FLoc_4 theory is a weakly graded algebra, and the corresponding $[\mathsf{F}]\mathsf{Loc}_4$ algebra is isomorphic to its even subalgebra. We will see that this relationship arises due to a sign ambiguity in the transformation law for spinor fields between different spacetimes, resulting from the need to define a way of converting between each pair of global frames in an equivalence class for a particular spacetime. We will therefore discuss the difficulties inherent in this particular problem before we begin to construct the classical Dirac theory in any form.

6.2.4 Conversion between equivalent frames

Suppose that we have two equivalent frames $\boldsymbol{\varepsilon} \sim \boldsymbol{\varepsilon}' \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$ for some $\boldsymbol{M} \in \mathsf{Loc}_4$. According to Definition 6.2.6, this entails that there is some smooth map $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$ such that $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\Lambda(s)$. However, $\Lambda : \mathrm{Spin}^0_{1,3} \to \mathsf{Loc}$ is, of course, a double cover; there are precisely two options for such a map, given by s and $-s$. There is no canonical choice of which sign to take (unless $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}$; and even then, we argue, the constant map $-\mathbf{1}_{\mathrm{Spin}^0_{1,3}}$ in this context should be no less preferred than $\mathbf{1}_{\mathrm{Spin}^0_{1,3}}$). This sign ambiguity does not usually present a problem; after all, for most purposes we are concerned only with the value of $\Lambda(s)$ and not with s itself. However, we will see that when we attempt to define the transformation law for spinors under a change of frame, we do need to deal directly with s, and a choice must therefore be made.

Suppose we choose an equivalence class $\mathcal{E}$ of frames for $\boldsymbol{M}$, and make such a choice of map into $\mathrm{Spin}^0_{1,3}$ for each pair of frames in $\mathcal{E}$, so that $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\Lambda(s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})$ for all $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \mathcal{E}$. We clearly then have

$$\Lambda(s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}}) = \mathbf{1}_{\mathcal{L}^\uparrow_+}, \quad \Lambda(s^{\boldsymbol{\varepsilon}'}_{\boldsymbol{\varepsilon}}) = \Lambda(s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})^{-1}, \quad \Lambda(s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})\Lambda(s^{\boldsymbol{\varepsilon}'}_{\boldsymbol{\varepsilon}''}) = \Lambda(s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}''}) \tag{6.27}$$

for all $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'' \in \mathcal{E}$. If the $s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'}$ satisfy the corresponding identities, i.e.

$$s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}} = \mathbf{1}_{\mathrm{Spin}^0_{1,3}}, \quad s^{\boldsymbol{\varepsilon}'}_{\boldsymbol{\varepsilon}} = (s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})^{-1}, \quad s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'} s^{\boldsymbol{\varepsilon}'}_{\boldsymbol{\varepsilon}''} = s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}''} \tag{6.28}$$

then we call the collection $(s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}' \in \mathcal{E}}$ a *spin cocycle* for $\mathcal{E}$.

Lemma 6.2.18. *Let $\mathcal{E}$ be an equivalence class of frames for a spacetime $\boldsymbol{M} \in \mathsf{Loc}_4$. Then $\mathcal{E}$ admits a spin cocycle.*

Proof. Let $\mathring{\boldsymbol{\varepsilon}} \in \mathcal{E}$ be fixed; we can certainly make a choice $s^{\mathring{\boldsymbol{\varepsilon}}}_{\boldsymbol{\varepsilon}}$ for each $\boldsymbol{\varepsilon} \in \mathcal{E}$

such that $\boldsymbol{\varepsilon} = \mathring{\boldsymbol{\varepsilon}}\Lambda(s^{\mathring{\boldsymbol{\varepsilon}}}_{\boldsymbol{\varepsilon}})$. We then define $s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'} := {s^{\mathring{\boldsymbol{\varepsilon}}}_{\boldsymbol{\varepsilon}}}^{-1} s^{\mathring{\boldsymbol{\varepsilon}}}_{\boldsymbol{\varepsilon}'}$ for each pair $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \mathcal{E}$. This choice clearly satisfies (6.28). □

While spin cocycles will turn out to be useful, they should not be thought of as 'physical', in the sense that they are merely a non-canonical way of choosing a lift to $\mathrm{Spin}^0_{1,3}$ for each possible transformation between global frames. There is also no natural concept of the restriction of a spin cocycle to a particular region; for every proper subregion $O \in \mathscr{O}(\boldsymbol{M})$, there are an infinite number of distinct smooth functions $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$ that are identically $\mathbf{1}_{\mathrm{Spin}^0_{1,3}}$ in O. Therefore, given some frame $\mathring{\boldsymbol{\varepsilon}} \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$, there are an infinite number of equivalent frames $\boldsymbol{\varepsilon}$ whose restriction to O coincides with $\mathring{\boldsymbol{\varepsilon}}|_O$; namely, any frame $\boldsymbol{\varepsilon} = \mathring{\boldsymbol{\varepsilon}}\Lambda(s)$ where s is one of the maps whose value is the identity everywhere in O. Moreover, as we observed earlier, there are frames on $\boldsymbol{M}|_O$ that are not extendible to frames on $\boldsymbol{M}$. Therefore, if $\boldsymbol{s}$ is a spin cocycle for $\boldsymbol{M}$, the restriction $\boldsymbol{s}|_O = (s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'}|_O)_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}' \in \mathcal{E}}$ cannot be a spin cocycle for $\boldsymbol{M}|_O$.

Chapter 7

Locally covariant Dirac theories

In this chapter, we provide a two constructions of the locally covariant Dirac field, both in the classical and quantum case. The overwhelming majority of the work is concerned with the classical construction, in view of the quantization procedure that has already been set out in Chapter 5. We will first provide a locally covariant construction of the classical Dirac field that has domain FLoc_4, and therefore depends on a particular choice of global frame as well as a spacetime, and then proceed to define a theory with domain $[\mathsf{F}]\mathsf{Loc}_4$, which therefore no longer depends on the individual frames, but rather on the equivalence classes thereof.

7.1 The classical Dirac field

We start with the construction of the classical Dirac field on a given object of FLoc_4. We follow [20, 27, 17, 56] for the most part, although the observations made in the previous chapter provide a number of simplifications. We will find that in fact many of the structures involved in the classical Dirac theory may be defined without any reference to the choice of spin frame projection. This is in contrast to the aforementioned references, where the choice of spin structure affected the construction to a much larger degree.

Given a spacetime $\boldsymbol{M} \in \mathsf{Loc}_4$, the spinor bundle on a given spin structure with spin bundle $\mathcal{S}$ has traditionally been constructed as the space $D_{\mathcal{S}} :=$

$\mathcal{S} \times_{\mathrm{Spin}^0_{1,3}} \mathbb{C}^4$, that is the vector bundle whose fibre at $p \in \boldsymbol{M}$ consists of equivalence classes

$$[(p, \boldsymbol{s}), x] := \{(R_S(p, \boldsymbol{s}), S^{-1}x) \in \mathcal{S} \times \mathbb{C}^4 : S \in \mathrm{Spin}^0_{1,3}\}, \tag{7.1}$$

where x is taken to be a column vector. However, we recall from Section 6.1 that the only spin structures (up to equivalence) are those for which the spin bundle is $S\boldsymbol{M}$; an element of the fibre of $D_{S\boldsymbol{M}}$ at $p \in \boldsymbol{M}$ satisfies

$$[(p, S), x] = [(p, \mathbf{1}), Sx], \tag{7.2}$$

and so the spinor bundle for $S\boldsymbol{M}$ is equivalent to (and may indeed be defined to be) the trivial bundle $\boldsymbol{M} \times \mathbb{C}^4$:

Definition 7.1.1. *Let $\boldsymbol{M} \in \mathsf{Loc}_4$. The* spinor bundle *on $\boldsymbol{M}$ is defined to be $\boldsymbol{M} \times \mathbb{C}^4$ and denoted $D\boldsymbol{M}$. Analogously, we define the* cospinor bundle *$D^*\boldsymbol{M}$ to be the dual bundle to $D\boldsymbol{M}$, given by $D^*\boldsymbol{M} = \boldsymbol{M} \times (\mathbb{C}^4)^*$. We denote the spaces of smooth spinor and cospinor fields by $\mathscr{S}(\boldsymbol{M}) := \Gamma^\infty(D\boldsymbol{M})$ and $\mathscr{S}^*(\boldsymbol{M}) := \Gamma^\infty(D^*\boldsymbol{M})$. We also denote $\mathscr{S}_0(\boldsymbol{M}) := \Gamma^\infty_0(D\boldsymbol{M}) \cong C^\infty_0(\boldsymbol{M}, \mathbb{C}^4)$ and $\mathscr{S}^*_0(\boldsymbol{M}) := \Gamma^\infty_0(D^*\boldsymbol{M}) \cong C^\infty_0(\boldsymbol{M}, (\mathbb{C}^4)^*)$; elements of these spaces are called* test spinors/cospinors *respectively.*

There is a canonical bilinear pairing $\langle \cdot\,, \cdot \rangle_p : D^*\boldsymbol{M}_p \times D\boldsymbol{M}_p \to \mathbb{C}$ given by

$$\langle (p, w), (p, x) \rangle_p := w(x), \tag{7.3}$$

from which we define the pairing $\langle \cdot\,, \cdot \rangle : \mathscr{S}^*_{(0)}(\boldsymbol{M}) \times \mathscr{S}_{(0)}(\boldsymbol{M}) \to C^\infty_{(0)}(\boldsymbol{M})$, by

$$\langle h, f \rangle(p) := \langle h(p), f(p) \rangle_p. \tag{7.4}$$

There is an obvious notion of the pullback of a smooth (co)spinor field for a Loc-arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$, given by

$$(\psi^* u)(p) := u(\psi(p)) \tag{7.5}$$

where $u \in \mathscr{S}^{(*)}(\boldsymbol{N})$, and a similar notion of the pushforward of a test (co)spinor, namely

$$(\psi_* v)(p) := \begin{cases} v(\psi^{-1}(p)), & p \in \psi(\boldsymbol{M}), \\ 0, & \text{otherwise}, \end{cases} \tag{7.6}$$

where $v \in \mathscr{S}^{(*)}(\boldsymbol{M})$. For any arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$ in Loc_4, we define

$$\begin{aligned} \mathscr{S}_0^{(*)}(\psi) : \mathscr{S}_0^{(*)}(\boldsymbol{M}) &\to \mathscr{S}_0^{(*)}(\boldsymbol{N}) \\ u &\mapsto \psi_* u, \end{aligned} \tag{7.7}$$

and

$$\begin{aligned} \mathscr{S}^{(*)}(\psi) : \mathscr{S}^{(*)}(\boldsymbol{N}) &\to \mathscr{S}^{(*)}(\boldsymbol{M}) \\ v &\mapsto \psi^* v. \end{aligned} \tag{7.8}$$

It is trivial to check that these define $\mathscr{S}_0^{(*)}$ as covariant functors from Loc_4 to $\mathsf{Vect}_{\mathbb{C}}$, and $\mathscr{S}^{(*)}$ as contravariant functors from Loc_4 to $\mathsf{Vect}_{\mathbb{C}}$.

We now construct some maps on $D^{(*)}\boldsymbol{M}$ that will turn out to be the adjoint maps and charge conjugations for our theories.[1] For any $p \in \boldsymbol{M}$ we may form the linear maps $A_p : D\boldsymbol{M}_p \to \overline{D^*\boldsymbol{M}_p}$, $C_p : D\boldsymbol{M}_p \to \overline{D\boldsymbol{M}_p}$ and $C_p^* : D^*\boldsymbol{M}_p \to \overline{D^*\boldsymbol{M}_p}$, by

$$\begin{aligned} A_p(p, x) &:= \overline{(p, x^\dagger A)}, \\ C_p(p, x) &:= \overline{(p, \overline{C}\overline{x})}, \\ C_p^*(p, w) &:= \overline{(p, \overline{w}C)}, \end{aligned} \tag{7.9}$$

where $x \in \mathbb{C}^4$, $w \in (\mathbb{C}^4)^*$ and A, C are the matrices defined in Lemma 6.1.6. It is easy to see that the properties of A and C entail $\langle \overline{A_p \boldsymbol{x}}, \boldsymbol{x'} \rangle_p = \overline{\langle \overline{A_p \boldsymbol{x'}}, \boldsymbol{x} \rangle_p}$

[1]These maps are the same as those used in previous constructions ([20, 27, 17, 15, 56]), except for the fact that here they are defined as linear maps (by taking the formal complex conjugate of the target space), in order that they be consistent with the categorical description given in Section 5.1.

and $\langle\overline{C_p^*\boldsymbol{w}}, \overline{C_p\boldsymbol{x}}\rangle_p = \overline{\langle\boldsymbol{w},\boldsymbol{x}\rangle_p}$ for $\boldsymbol{x},\boldsymbol{x}' \in D\boldsymbol{M}_p$ and $\boldsymbol{w} \in D^*\boldsymbol{M}_p$; we also have

$$\begin{aligned}\overline{C_p^*}A_p(p,x) = \overline{C_p^*(p,x^\dagger A)} = (p, x^T\overline{A}C) \\ = -(p, x^TC^TA) = -\overline{A_p(p,\overline{C}\overline{x})} = -\overline{A_p}C_p(p,x),\end{aligned} \tag{7.10}$$

where we have used (6.12).

We may use the above maps to construct linear isomorphisms $A_{\boldsymbol{M}} : \mathscr{S}_{(0)}(\boldsymbol{M}) \to \overline{\mathscr{S}^*_{(0)}(\boldsymbol{M})}$, $C_{\boldsymbol{M}} : \mathscr{S}_{(0)}(\boldsymbol{M}) \to \overline{\mathscr{S}_{(0)}(\boldsymbol{M})}$, and $C^*_{\boldsymbol{M}} : \mathscr{S}^*_{(0)}(\boldsymbol{M}) \to \overline{\mathscr{S}^*_{(0)}(\boldsymbol{M})}$, by applying A_p, C_p and C_p^* pointwise. Note that we then have

$$\overline{C^*_{\boldsymbol{M}}} \circ A_{\boldsymbol{M}} = -\overline{A_{\boldsymbol{M}}} \circ C_{\boldsymbol{M}}, \tag{7.11}$$

as well as

$$\langle\overline{A_{\boldsymbol{M}}f}, f'\rangle = \overline{\langle\overline{A_{\boldsymbol{M}}f'}, f\rangle} \qquad \langle\overline{C^*_{\boldsymbol{M}}h}, \overline{C_{\boldsymbol{M}}f}\rangle = \overline{\langle h, f\rangle} \tag{7.12}$$

for $f, f' \in \mathscr{S}(\boldsymbol{M})$ and $h \in \mathscr{S}^*(\boldsymbol{M})$.

7.1.1 Transformation of spinor fields

In defining the spinor bundle $D\boldsymbol{M}$ as in Definition 7.1.1, we lose the explicit relation between $D\boldsymbol{M}$ and the spin group that is obvious from previous definitions, i.e. as given in (7.1). To recover this, note that according to (7.1) and (7.2) we may associate the element $(p,x) \in D\boldsymbol{M}$ with $[(p,\mathbf{1}),x] \in S\boldsymbol{M} \times_{\mathrm{Spin}^0_{1,3}} \mathbb{C}^4$. This gives us the rule for transforming a spinor field under a change in spin frame; according to (6.13), if $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\Lambda(s)$ for some $\boldsymbol{\varepsilon} \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$ and smooth $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$, then $\pi_{\boldsymbol{\varepsilon}}(p,\mathbf{1}) = \pi_{\boldsymbol{\varepsilon}'}(p, \pm s(p)^{-1})$ for any $p \in \boldsymbol{M}$. Therefore a change in spin frame from $\boldsymbol{\varepsilon}$ to $\boldsymbol{\varepsilon}'$ must be accompanied by a transformation of $(p,\mathbf{1}) \in S\boldsymbol{M}$ to $(p, \pm s(p)^{-1})$, and consequently a change of $[(p,\mathbf{1}),x]$ to $[(p,\mathbf{1}), \pm s(p)^{-1}x]$. Finally, we see that under the aforementioned equivalence, $(p,x) \in D\boldsymbol{M}$ must transform to $(p, \pm s(p)^{-1}x)$.

As a result, if a spinor field $f \in \mathscr{S}_{(0)}(\boldsymbol{M})$ is given by $f(p) = (p, f_p)$, then f must transform to $\pm s^{-1}f$ under a change of spin frame from $\boldsymbol{\varepsilon}$ to $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\Lambda(s)$, where $(s^{-1}f)(p) := (p, s(p)^{-1}f_p)$. We must therefore regard $f \in \mathscr{S}_{(0)}(\boldsymbol{M})$

as merely the component expression for an underlying unobservable field,[2] since f itself changes under a change of spin frame, and we do not expect the underlying fields to depend on a particular non-canonical choice of spin frame (beyond perhaps a choice of equivalence class). Since $\langle h, f\rangle$ is a scalar field for any $h \in \mathscr{S}^*(\boldsymbol{M})$, it must remain constant under changes of spin frame; therefore $h : p \mapsto (p, h_p)$ must transform to $\pm hs$, where $(hs)(p) := (p, h_p s)$.

Unfortunately, it is clearly impossible to give an unambiguous formula for the rule under which (co)spinor fields themselves transform under a change of spin frame, due to the inherent sign ambiguity. We present a way of partially resolving this problem in Subsection 7.1.5, by passing to a squared adjoint structure.

Recall that for any frame $\boldsymbol{\varepsilon} \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$, a vector field $v \in \Gamma^\infty(T\boldsymbol{M})$ may be written in component form as $v = \varepsilon_a v^a$, where $v^a = \varepsilon^a(v) = \varepsilon^a_\mu v^\mu$. It follows that the transformed components under a change $\boldsymbol{\varepsilon} \mapsto \boldsymbol{\varepsilon}\Lambda(s)$ are given by $v'^{a'} = \Lambda(s^{-1})^{a'}{}_a v^a$. Similarly, the components w_a of a covector field transform to $w'_{a'} = w_a \Lambda(s)^a{}_{a'}$.

We may also write the components of a spinor field $f \in \mathscr{S}(\boldsymbol{M})$ explicitly as f^A, $A = 1, \ldots, 4$, where for fixed A and $p \in \boldsymbol{M}$, the complex number $f^A(p)$ is simply the A^{th} component of $f_p \in \mathbb{C}^4$; similarly, we write $h \in \mathscr{S}^*(\boldsymbol{M})$ in components as h_A, where $h_A(p)$ is the A^{th} component of $h_p \in (\mathbb{C}^4)^*$. Note that since $D\boldsymbol{M}$ is now defined as the trivial $\mathrm{Spin}^0_{1,3}$ bundle, we have global 'basis' sections $E_A \in \mathscr{S}(\boldsymbol{M})$, for $A = 1, \ldots, 4$, given by $E_A(p) := (p, b_A)$ where b_A is the A^{th} standard $\mathbb{C}^4$ basis vector. Similarly there are basis sections E^A for $\mathscr{S}^*(\boldsymbol{M})$ defined by the property that $\langle E^A, E_B\rangle(p) = \delta^A_B$ for all $p \in \boldsymbol{M}$. Alternatively, $E^A(p) = (p, b^A)$ where b^A is the dual basis to b_A.

This entails that using the summation convention, $f = E_A f^A$ and $h = h_A E^A$. We see that the components f^A transform to $f'^{A'} = \pm(s^{-1})^{A'}{}_A f^A$ under a change of spin frame from $\boldsymbol{\varepsilon}$ to $\boldsymbol{\varepsilon}\Lambda(s)$, where $(s^{-1})^A{}_B$ are the matrix elements of s^{-1}. Similarly, the components h_A transform to $h'_{A'} = \pm h_A s^A{}_{A'}$. Note that this change in components could also be seen as the result of

[2]In the same way, a vector field $v \in \Gamma^\infty(T\boldsymbol{M})$ may be described in a given frame by its four components, which hold information corresponding to an underlying geometrical object and transform in a particular way under a change of frame. Unfortunately, there is no similar geometrical picture of a spinor field.

a change of basis at each point p from $(b_A)_{A=1}^4$ to $(b_A s(p)^A{}_{A'})_{A'=1}^4$, which underlines the fact that the spinor field $f \in \mathscr{S}(\boldsymbol{M})$ is really just a list of components in a particular basis rather than the field itself.

We may now write down the transformation rule for the components of an arbitrary spinor-tensor $T \in \Gamma^\infty(T\boldsymbol{M}^{\otimes j} \otimes T^*\boldsymbol{M}^{\otimes k} \otimes D\boldsymbol{M}^{\otimes m} \otimes D^*\boldsymbol{M}^{\otimes n})$ under such a change of frame, namely

$$\begin{aligned} T'^{a'_1\cdots a'_j}{}_{b'_1\cdots b'_k}{}^{A'_1\cdots A'_m}{}_{B'_1\cdots B'_n} &= (\pm 1)^{m+n}\Lambda(s^{-1})^{a'_1}{}_{a_1}\cdots\Lambda(s^{-1})^{a'_j}{}_{a_j}(s^{-1})^{A'_1}{}_{A_1}\cdots(s^{-1})^{A'_m}{}_{A_m} \\ &\quad T^{a_1\cdots a_j}{}_{b_1\cdots b_k}{}^{A_1\cdots A_m}{}_{B_1\cdots B_n}\,\Lambda(s)^{b_1}{}_{b'_1}\cdots\Lambda(s)^{b_k}{}_{b'_k}s^{B_1}{}_{B'_1}\cdots s^{B_n}{}_{B'_n}. \end{aligned} \tag{7.13}$$

In particular, note that this and (6.10) entail that the spinor-tensor $\boldsymbol{\gamma} \in \Gamma^\infty(T\boldsymbol{M} \otimes D^*\boldsymbol{M} \otimes D\boldsymbol{M})$, whose components in some frame $\boldsymbol{\varepsilon}$ are the matrix elements $\gamma_a{}^A{}_B$ of the a^{th} gamma matrix, has an identical component expression in *any* equivalent frame.

7.1.2 The spin connection

We will now cover the basic facts behind the spin connection and Dirac operator. These are generally well known, and more details of their construction are given in [55, §4.1.4]. We fix a spin frame $\boldsymbol{\varepsilon} \in \Gamma^\infty(F_+^\uparrow\boldsymbol{M})$. Since the vectors $\varepsilon_a(p)$, $a = 0, \ldots, 3$ span $T_p\boldsymbol{M}$ it follows that we may write

$$\nabla_{\varepsilon_b}\varepsilon_c = \Gamma^a{}_{bc}\varepsilon_a \tag{7.14}$$

for some real numbers $\Gamma^a{}_{bc}$.[3] These have the form of connection coefficients for the covariant derivative, in the following sense: for any vector field $v \in \Gamma^\infty(T\boldsymbol{M})$, we have $v = v^a\varepsilon_a$, where for each a the component v^a is a

[3]Writing spacetime indices explicitly, we may calculate

$$\Gamma^a{}_{bc} = \varepsilon^a_\mu(\nabla_{\varepsilon_b}\varepsilon_c)^\mu = \varepsilon^a_\mu\varepsilon^\nu_b\nabla_\nu\varepsilon^\mu_c = \varepsilon^a_\mu\varepsilon^\nu_b\partial_\nu\varepsilon^\mu_c + \varepsilon^a_\mu\varepsilon^\nu_b\varepsilon^\rho_c\Gamma^\mu{}_{\nu\rho},$$

where $\Gamma^\mu{}_{\nu\rho}$ are the usual Christoffel symbols.

scalar field, so

$$\begin{aligned}\nabla_{\varepsilon_b} v &= \nabla_{\varepsilon_b}(v^a \varepsilon_a) \\ &= (D_{\varepsilon_b} v^a)\varepsilon_a + v^a \nabla_{\varepsilon_b} \varepsilon_a \\ &= (D_{\varepsilon_b} v^a + \Gamma^a{}_{bc} v^c)\, \varepsilon_a, \end{aligned} \tag{7.15}$$

where $D_{\varepsilon_b} v^a$ denotes the contraction of the gradient of the scalar field v^a with ε_b.

For any covector field $w \in \Gamma^\infty(T^*\boldsymbol{M})$ and vector field $v \in \Gamma^\infty(T\boldsymbol{M})$ we have $w(v) = w_a v^a$. Accordingly, we may use the Leibniz rule to see that $w_a D_{\varepsilon_b} v^a + (D_{\varepsilon_b} w_a) v^a = w(\nabla_{\varepsilon_b} v) + (\nabla_{\varepsilon_b} w)(v)$. Hence $(\nabla_{\varepsilon_b} w)(v) = ((D_{\varepsilon_b} w_a)\varepsilon^a)(v) - (w_a \Gamma^a{}_{bc} \varepsilon^c)(v)$, and since v was arbitrary, it follows that

$$\nabla_{\varepsilon_b} w = \varepsilon^c \left(D_{\varepsilon_b} w_c - \Gamma^a{}_{bc} w_a \right). \tag{7.16}$$

As usual, we may then express the covariant derivative of any tensor by appending the relevant term for each index. For example, since $\boldsymbol{g} = (\varepsilon^c \otimes \varepsilon^d)\eta_{cd}$, we have

$$0 = \nabla_{\varepsilon_b} \boldsymbol{g} = (\varepsilon^c \otimes \varepsilon^d) \left(D_{\varepsilon_b} \eta_{cd} - \eta_{ad} \Gamma^d{}_{bc} - \eta_{ac} \Gamma^a{}_{bd} \right). \tag{7.17}$$

Since $D_{\varepsilon_b} \eta_{cd} = 0$, it follows that

$$\eta_{ad} \Gamma^d{}_{bc} + \eta_{ac} \Gamma^a{}_{bd} = 0. \tag{7.18}$$

Associated with the connection $\Gamma^a{}_{bc}$ on $T\boldsymbol{M}$ is a system of coefficients $\sigma_b{}^A{}_B$ that define a connection on $D\boldsymbol{M}$, the *spin connection*.[4] These coefficients are given by[5]

$$\sigma_b{}^A{}_B := \frac{1}{4} \Gamma^a{}_{bc} \gamma_a{}^A{}_C \gamma^{cC}{}_B, \tag{7.19}$$

and we may then extend the covariant derivative to $D\boldsymbol{M}$ via

$$\nabla_{\varepsilon_b} E_B = \sigma_b{}^A{}_B E_A. \tag{7.20}$$

[4] Details are given in [55, §4.1.4].

[5] Note the sign error in [20, 27, 17], pointed out in [55].

Since the relationship between $S\boldsymbol{M}$ and $F_+^\uparrow\boldsymbol{M}$ depends explicitly on the spin frame $\boldsymbol{\varepsilon}$, it follows that the connection on $D\boldsymbol{M}$ also depends on $\boldsymbol{\varepsilon}$, as does the covariant derivative. This dependence is in fact required, since the spinor fields $f \in \mathscr{S}(\boldsymbol{M})$ are only component expressions of an underlying object that require a choice of spin frame for their definition. The covariant derivative must therefore respect the transformation rule for spinor fields, so that if ∇ and ∇' denote the derivatives defined with relation to $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\Lambda(s)$ respectively, then

$$\nabla'_v(s^{-1}f) = s^{-1}\nabla_v f \tag{7.21}$$

for any vector field v.

As before, we see that for any $f = E_A f^A$, we have

$$\begin{aligned}\nabla_{\varepsilon_b} f &= \nabla_{\varepsilon_b}(E_A f^A) \\ &= E_A D_{\varepsilon_b} f^A + (\nabla_{\varepsilon_b} E_A) f^A \\ &= E_A \left(D_{\varepsilon_b} f^A + {\sigma_b}^A{}_B f^B \right).\end{aligned} \tag{7.22}$$

Just as the coefficients ${\gamma_a}^A{}_B$ may be regarded as the matrix elements of a matrix $\boldsymbol{\gamma}_a \in GL_4(\mathbb{C})$, we may regard ${\sigma_b}^A{}_B$ as the matrix elements of a matrix $\boldsymbol{\sigma}_b$; we then have $\boldsymbol{\sigma}_b = \frac{1}{4}\Gamma^a{}_{bc}\boldsymbol{\gamma}_a\boldsymbol{\gamma}^c$ and

$$\nabla_{\varepsilon_b} f = D_{\varepsilon_b} f + \boldsymbol{\sigma}_b f, \tag{7.23}$$

where $D_{\varepsilon_b} f := E_A D_{\varepsilon_b} f^A$.

Again, we may use the Leibniz rule and the property that $\langle h, f\rangle = h_A f^A$ for $h \in \mathscr{S}^*(\boldsymbol{M})$ and $f \in \mathscr{S}(\boldsymbol{M})$ to see that for any cospinor field $h \in \mathscr{S}^*(\boldsymbol{M})$ we have $\nabla_b h = D_{\varepsilon_b} h - h\boldsymbol{\sigma}_b$. It must also hold that $\nabla'_v(hs) = (\nabla_v h)s$ for any vector field v, where ∇' again represents the covariant derivative defined with relation to $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\Lambda(s)$.

The action of the covariant derivative on mixed spinor-tensors may then be computed by appending the relevant term for each index, so for example

$$\nabla_{\varepsilon_b}\boldsymbol{\gamma} = \left(D_{\varepsilon_b}\boldsymbol{\gamma}_a - \Gamma^c{}_{ba}\boldsymbol{\gamma}_c + \boldsymbol{\sigma}_b\boldsymbol{\gamma}_a - \boldsymbol{\gamma}_a\boldsymbol{\sigma}_b\right)\varepsilon^a. \tag{7.24}$$

In this case one may show that in fact $\nabla_{\varepsilon_b}\boldsymbol{\gamma} = 0$ as expected: following [55], we use (7.18) and the fact that the components of $\boldsymbol{\gamma}$ are constant in any frame to see that

$$\begin{aligned}(\nabla_{\varepsilon_b}\boldsymbol{\gamma})_a &= \Gamma^c{}_{bd}\left(-\boldsymbol{\gamma}_c\delta^d_a + \frac{1}{4}[\boldsymbol{\gamma}_c\boldsymbol{\gamma}^d, \boldsymbol{\gamma}_a]\right) \\ &= \Gamma^c{}_{bd}\left(-\boldsymbol{\gamma}_c\delta^d_a + \frac{1}{4}\boldsymbol{\gamma}_c\{\boldsymbol{\gamma}^d, \boldsymbol{\gamma}_a\} - \frac{1}{4}\{\boldsymbol{\gamma}_c, \boldsymbol{\gamma}_b\}\boldsymbol{\gamma}^d\right) \\ &= \Gamma^c{}_{bd}\left(-\frac{1}{2}\boldsymbol{\gamma}_c\delta^d_a - \frac{1}{2}\boldsymbol{\gamma}^d\eta_{ac}\right) \\ &= 0. \end{aligned} \tag{7.25}$$

Given $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$, we now define operators $\nabla\!\!\!/^s_{\mathcal{M}} : \mathscr{S}_{(0)}(\boldsymbol{M}) \to \mathscr{S}_{(0)}(\boldsymbol{M})$ and $\nabla\!\!\!/^c_{\mathcal{M}} : \mathscr{S}^*_{(0)}(\boldsymbol{M}) \to \mathscr{S}^*_{(0)}(\boldsymbol{M})$ by

$$\nabla\!\!\!/^s_{\mathcal{M}} f := \boldsymbol{\gamma}^a \nabla_{\varepsilon_a} f = E_A \gamma^{aA}{}_B (\nabla_{\varepsilon_a} f)^B, \tag{7.26}$$

$$\nabla\!\!\!/^c_{\mathcal{M}} h := (\nabla_{\varepsilon_a} h)\boldsymbol{\gamma}^a = (\nabla_{\varepsilon_a} h)_B \gamma^{aB}{}_A E^A, \tag{7.27}$$

for spinor fields f and cospinor fields h. From now on it will be convenient to use the notation $\nabla_b := \nabla_{\varepsilon_b}$ and $D_b := D_{\varepsilon_b}$; however, we should take care to remember that for example $\nabla_b v^a$ is the derivative along a particular vector field ε_b of a scalar field v^a, and not an abstract tensor field with indices used merely as placeholders.

Lemma 7.1.2. (cf. [55, Lem. 4.1.22])**.** *Let* $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$. *Then*

$$A_{\boldsymbol{M}} \circ \nabla\!\!\!/^s_{\mathcal{M}} = \overline{\nabla\!\!\!/^c_{\mathcal{M}}} \circ A_{\boldsymbol{M}}, \quad C_{\boldsymbol{M}} \circ \nabla\!\!\!/^s_{\mathcal{M}} = -\overline{\nabla\!\!\!/^s_{\mathcal{M}}} \circ C_{\boldsymbol{M}}, \quad C^*_{\boldsymbol{M}} \circ \nabla\!\!\!/^c_{\mathcal{M}} = -\overline{\nabla\!\!\!/^c_{\mathcal{M}}} \circ C^*_{\boldsymbol{M}}. \tag{7.28}$$

Proof. This can easily be checked by using (7.19), (7.9) and Lemma 6.1.6. For example, we have $\boldsymbol{\gamma}^{a\dagger} A = A\boldsymbol{\gamma}^a$ and

$$\boldsymbol{\sigma}^\dagger_a \boldsymbol{\gamma}^{a\dagger} A = \frac{1}{4}\Gamma^c{}_{ab}\boldsymbol{\gamma}^{b\dagger}\boldsymbol{\gamma}^\dagger_c \boldsymbol{\gamma}^{a\dagger} A = \frac{1}{4}\Gamma^c{}_{ab} A \boldsymbol{\gamma}^b \boldsymbol{\gamma}_c \boldsymbol{\gamma}^a = -A\boldsymbol{\sigma}_a\boldsymbol{\gamma}^a, \tag{7.29}$$

so for a spinor field f,

$$\begin{aligned}[A_{\boldsymbol{M}}(\nabla\!\!\!/^{s}_{\mathcal{M}} f)](p) &= A_p(p, \boldsymbol{\gamma}^a (D_a f)_p + \boldsymbol{\gamma}^a \boldsymbol{\sigma}_a f_p) \\ &= \overline{(p, (D_a f)^\dagger_p \boldsymbol{\gamma}^{a\dagger} A + f^\dagger_p \boldsymbol{\sigma}^\dagger_a \boldsymbol{\gamma}^\dagger_a A)} \\ &= \overline{(p, (D_a f)^\dagger_p A \boldsymbol{\gamma}^a - f^\dagger_p A \boldsymbol{\sigma}_a \boldsymbol{\gamma}^a)} \\ &= \overline{[\nabla\!\!\!/^{c}_{\mathcal{M}}(A_{\boldsymbol{M}} f)](p)}. \end{aligned} \tag{7.30}$$

The other two statements can be proved in a similar way. □

7.1.3 The Dirac equation and solution spaces

Definition 7.1.3. *Let* $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$. *The* Dirac operators $P^s_{\mathcal{M}} : \mathscr{S}_{(0)}(\boldsymbol{M}) \to \mathscr{S}_{(0)}(\boldsymbol{M})$ *and* $P^c_{\mathcal{M}} : \mathscr{S}^*_{(0)}(\boldsymbol{M}) \to \mathscr{S}^*_{(0)}(\boldsymbol{M})$ *for a fixed mass* $m \geq 0$ *are given by*

$$P^s_{\mathcal{M}} := -i\nabla\!\!\!/^{s}_{\mathcal{M}} + m\mathbf{1}, \qquad P^c_{\mathcal{M}} := i\nabla\!\!\!/^{c}_{\mathcal{M}} + m\mathbf{1}. \tag{7.31}$$

The Dirac equations for spinor fields $f \in \mathscr{S}(\boldsymbol{M})$ *and cospinor fields* $h \in \mathscr{S}^*(\boldsymbol{M})$ *respectively are*

$$P^s_{\mathcal{M}} f = 0, \qquad P^c_{\mathcal{M}} h = 0. \tag{7.32}$$

Lemma 7.1.4. *For any* $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$,

$$A_{\boldsymbol{M}} \circ P^s_{\mathcal{M}} = \overline{P^c_{\mathcal{M}}} \circ A_{\boldsymbol{M}}, \quad C_{\boldsymbol{M}} \circ P^s_{\mathcal{M}} = \overline{P^s_{\mathcal{M}}} \circ C_{\boldsymbol{M}}, \quad C^*_{\boldsymbol{M}} \circ P^c_{\mathcal{M}} = \overline{P^c_{\mathcal{M}}} \circ C^*_{\boldsymbol{M}}. \tag{7.33}$$

Proof. This is an immediate consequence of Lemma 7.1.2. □

Therefore if $f \in \mathscr{S}(\boldsymbol{M})$, $h \in \mathscr{S}^*(\boldsymbol{M})$ are solutions to the Dirac equations then so are $\overline{A_{\boldsymbol{M}} f}$, $\overline{C_{\boldsymbol{M}} f}$, $A^{-1}_{\boldsymbol{M}} \overline{h}$ and $\overline{C^*_{\boldsymbol{M}} h}$. Moreover, whenever $f \in \mathscr{S}(\boldsymbol{M})$, $h \in \mathscr{S}^*(\boldsymbol{M})$ satisfy the condition that $h(f) \in C^\infty(\boldsymbol{M})$ is compactly supported, we have

$$\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \langle h, P^s_{\mathcal{M}} f \rangle = \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \langle P^c_{\mathcal{M}} h, f \rangle; \tag{7.34}$$

this may easily be proved by noting that $\nabla_a \langle h, \boldsymbol{\gamma}^a f \rangle = \langle \nabla\!\!\!/^{c}_{\mathcal{M}} h, f \rangle + \langle h, \nabla\!\!\!/^{s}_{\mathcal{M}} f \rangle$

and integrating over $\boldsymbol{M}$ (cf. [20]). Note that since any $\boldsymbol{\psi} : \mathcal{M} \to \mathcal{N}$ in FLoc_4 covers $\psi : \boldsymbol{M} \to \boldsymbol{N}$ in Loc_4 which is isometric, it follows that

$$P^{s/c}_{\mathcal{M}} \circ \psi^* = \psi^* \circ P^{s/c}_{\mathcal{N}}, \qquad P^{s/c}_{\mathcal{N}} \circ \psi_* = \psi_* \circ P^{s/c}_{\mathcal{M}}, \tag{7.35}$$

in analogue to the scalar case.

Lemma 7.1.5. *Let $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon})$ and $\mathcal{M}' = (\boldsymbol{M}, \boldsymbol{\varepsilon}')$ be objects of FLoc_4 where $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\Lambda(s)$ for smooth $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$. For any $f \in \mathscr{S}(\boldsymbol{M})$ and $h \in \mathscr{S}^*(\boldsymbol{M})$, we have*

$$P^s_{\mathcal{M}'}(s^{-1}f) = s^{-1}P^s_{\mathcal{M}}f, \qquad P^c_{\mathcal{M}'}(hs) = (P^c_{\mathcal{M}}h)s. \tag{7.36}$$

In particular, if $P^s_{\mathcal{M}}f = 0$ and $P^c_{\mathcal{M}}h = 0$, then $P^s_{\mathcal{M}'}(s^{-1}f) = 0$ and $P^c_{\mathcal{M}'}(hs) = 0$.

Proof. We denote by ∇ and ∇' the covariant derivatives defined with relation to $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ respectively. We see from (7.21) that

$$\begin{aligned} \nabla\!\!\!\!/^{\,s}_{\mathcal{M}'}(s^{-1}f) &= \boldsymbol{\gamma}^a \nabla'_{\varepsilon'_a}(s^{-1}f) \\ &= \boldsymbol{\gamma}^a s^{-1} \nabla_{\varepsilon'_a} f \\ &= \boldsymbol{\gamma}^a s^{-1} \Lambda(s)^b{}_a \nabla_{\varepsilon_b} f \\ &= s^{-1}\boldsymbol{\gamma}^b \nabla_{\varepsilon_b} f \\ &= s^{-1} \nabla\!\!\!\!/^{\,s}_{\mathcal{M}} f, \end{aligned} \tag{7.37}$$

where we have used (6.10) in the penultimate step. We may similarly show that $\nabla\!\!\!\!/^{\,c}_{\mathcal{M}'}(hs) = (\nabla\!\!\!\!/^{\,c}_{\mathcal{M}}h)s$.

It then follows that $P^s_{\mathcal{M}'}(s^{-1}f) = s^{-1}(-i\nabla\!\!\!\!/^{\,s}_{\mathcal{M}}f + mf) = s^{-1}P^s_{\mathcal{M}}f$, and $P^c_{\mathcal{M}'}(hs) = (i\nabla\!\!\!\!/^{\,c}_{\mathcal{M}}h + mh)s = (P^c_{\mathcal{M}}h)s$. □

For each $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$ there exist maps $(S^s_{\mathcal{M}})^{\pm} : \mathscr{S}_0(\boldsymbol{M}) \to \mathscr{S}(\boldsymbol{M})$, $(S^c_{\mathcal{M}})^{\pm} : \mathscr{S}^*_0(\boldsymbol{M}) \to \mathscr{S}^*(\boldsymbol{M})$ that satisfy

$$P^{s/c}_{\mathcal{M}} \circ (S^{s/c}_{\mathcal{M}})^{\pm} = \iota^{s/c}_{\boldsymbol{M}} = (S^{s/c}_{\mathcal{M}})^{\pm} \circ P^{s/c}_{\mathcal{M}}, \tag{7.38}$$

$$\mathrm{supp}\left((S^{s/c}_{\mathcal{M}})^{\pm}u\right) \subseteq J^{\pm}_{\boldsymbol{M}}(\mathrm{supp}(u)), \tag{7.39}$$

where u is an arbitrary test (co)spinor and $\iota^s_{\boldsymbol{M}}$ (resp. ι^c) is the canonical embedding of $\mathscr{S}_0(\boldsymbol{M})$ into $\mathscr{S}(\boldsymbol{M})$ (resp. $\mathscr{S}^*_0(\boldsymbol{M})$ into $\mathscr{S}^*(\boldsymbol{M})$); these maps are unique [20, Thm. 2.1]. This uniqueness property, along with (7.35), entails that

$$\psi^* \circ S^{s/c}_{\mathcal{N}} \circ \psi_* = S^{s/c}_{\mathcal{M}} \tag{7.40}$$

for any $\boldsymbol{\psi} : \mathcal{M} \to \mathcal{N}$ in FLoc_4; it also allows us to show that

$$A_{\boldsymbol{M}} \circ (S^s_{\mathcal{M}})^{\pm} = \overline{(S^c_{\mathcal{M}})^{\pm}} \circ A_{\boldsymbol{M}}, \qquad C_{\boldsymbol{M}} \circ (S^s_{\mathcal{M}})^{\pm} = \overline{(S^s_{\mathcal{M}})^{\pm}} \circ C_{\boldsymbol{M}},$$
$$C^*_{\boldsymbol{M}} \circ (S^c_{\mathcal{M}})^{\pm} = \overline{(S^c_{\mathcal{M}})^{\pm}} \circ C^*_{\boldsymbol{M}}. \tag{7.41}$$

For example, since $A^{\pm 1}_{\boldsymbol{M}}$ and $P^{s/c}_{\mathcal{M}}$ do not enlarge the support of a (co)spinor field, we have $\mathrm{supp}(\overline{A_{\boldsymbol{M}}(S^s_{\mathcal{M}})^{\pm}A^{-1}_{\boldsymbol{M}}h}) \subset J^{\pm}_{\boldsymbol{M}}(\mathrm{supp}(h))$ for all $h \in \mathscr{S}^*_0(\boldsymbol{M})$, and

$$\begin{aligned} P^c_{\mathcal{M}} \circ \overline{A_{\boldsymbol{M}}} \circ \overline{(S^s_{\mathcal{M}})^{\pm}} \circ \overline{A^{-1}_{\boldsymbol{M}}} &= \overline{A_{\boldsymbol{M}}} \circ \overline{P^s_{\mathcal{M}}} \circ \overline{(S^s_{\mathcal{M}})^{\pm}} \circ \overline{A^{-1}_{\boldsymbol{M}}} \\ &= \iota^c_{\boldsymbol{M}} = P^c_{\mathcal{M}} \circ (S^c_{\mathcal{M}})^{\pm}, \\ \overline{A_{\boldsymbol{M}}} \circ \overline{(S^s_{\mathcal{M}})^{\pm}} \circ \overline{A^{-1}_{\boldsymbol{M}}} \circ P^c_{\mathcal{M}} &= \overline{A_{\boldsymbol{M}}} \circ \overline{(S^s_{\mathcal{M}})^{\pm}} \circ \overline{P^s_{\mathcal{M}}} \circ \overline{A^{-1}_{\boldsymbol{M}}} \\ &= \iota^c_{\boldsymbol{M}} = (S^c_{\mathcal{M}})^{\pm} \circ P^c_{\mathcal{M}}, \end{aligned} \tag{7.42}$$

hence $\overline{A_{\boldsymbol{M}}} \circ \overline{(S^s_{\mathcal{M}})^{\pm}} \circ \overline{A^{-1}_{\boldsymbol{M}}} = (S^c_{\mathcal{M}})^{\pm}$. Moreover, for any test spinor $f \in \mathscr{S}_0(\boldsymbol{M})$ and cospinor $h \in \mathscr{S}^*_0(\boldsymbol{M})$, we have

$$\begin{aligned} \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, \langle h, (S^s_{\mathcal{M}})^{\pm} f\rangle &= \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, \langle P^c_{\mathcal{M}}(S^c_{\mathcal{M}})^{\mp} h, (S^s_{\mathcal{M}})^{\pm} f\rangle \\ = \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, &\langle (S^c_{\mathcal{M}})^{\mp} h, P^s_{\mathcal{M}}(S^s_{\mathcal{M}})^{\pm} f\rangle = \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, \langle (S^c_{\mathcal{M}})^{\mp} h, f\rangle, \end{aligned} \tag{7.43}$$

where we have used (7.34). As for the scalar field theory, we define the *advanced-minus-retarded fundamental solution* for (co)spinor fields by $S^{s/c}_{\mathcal{M}} := (S^{s/c}_{\mathcal{M}})^- - (S^{s/c}_{\mathcal{M}})^+$. Clearly $P^{s/c}_{\mathcal{M}} \circ S^{s/c}_{\mathcal{M}} = 0 = S^{s/c}_{\mathcal{M}} \circ P^{s/c}_{\mathcal{M}}$; we state the standard results that $\ker S^{s/c}_{\mathcal{M}} = P^{s/c}_{\mathcal{M}} \mathscr{S}^{(*)}_0(\boldsymbol{M})$, and that every (co)spinor field u with compact support on Cauchy surfaces satisfying $P^{s/c}_{\mathcal{M}} u = 0$ is of the form $u = S^{s/c}_{\mathcal{M}} v$, where v is a test (co)spinor. For $f_1, f_2 \in \mathscr{S}_0(\boldsymbol{M})$ we define

$$(f_1, f_2)^s := -i \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, \langle \overline{A_{\boldsymbol{M}} f_1}, S^s_{\mathcal{M}} f_2\rangle. \tag{7.44}$$

This is a Hermitian form on $\mathscr{S}_0(\boldsymbol{M})$: it is clearly sesquilinear, and (7.12), (7.41) and (7.43) entail that

$$\begin{aligned}(f_1, f_2)^s &= i\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, \langle S^c_{\mathcal{M}} \overline{A_{\boldsymbol{M}} f_1}, f_2\rangle \\ &= i\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, \langle \overline{A_{\boldsymbol{M}} S^s_{\mathcal{M}} f_1}, f_2\rangle \\ &= \overline{i\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, \langle \overline{A_{\boldsymbol{M}} f_2}, S^s_{\mathcal{M}} f_1\rangle} \\ &= \overline{(f_2, f_1)^s}. \end{aligned} \tag{7.45}$$

We also define a Hermitian form on $\mathscr{S}^*_0(\boldsymbol{M})$ by

$$(h_1, h_2)^c := (A^{-1}_{\boldsymbol{M}}\overline{h_2}, A^{-1}_{\boldsymbol{M}}\overline{h_1})^s. \tag{7.46}$$

Note that

$$(P^s_{\mathcal{M}} f_1, f_2)^s = 0 = (P^c_{\mathcal{M}} h_1, h_2)^c. \tag{7.47}$$

7.1.4 The framed-spacetime classical Dirac theory

We are finally, then, in a position to define the vector spaces that will comprise the content of the classical Dirac theory in a given spacetime. We begin with $\mathcal{M} \in \mathsf{FLoc}_4$, and define

$$\mathrm{Sol}^s(\mathcal{M}) := S^s_{\mathcal{M}} \mathscr{S}_0(\boldsymbol{M}), \qquad \mathrm{Sol}^c(\mathcal{M}) := S^c_{\mathcal{M}} \mathscr{S}^*_0(\boldsymbol{M}). \tag{7.48}$$

Therefore $\mathrm{Sol}^{s/c}(\mathcal{M})$ is the vector space of all (co)spinor solutions to the Dirac equations (7.32) with compact support on Cauchy surfaces. We equip $\mathrm{Sol}^{s/c}(\mathcal{M})$ with Hermitian forms defined by

$$s^{s/c}_{\mathcal{M}}(S^{s/c}_{\mathcal{M}} u_1, S^{s/c}_{\mathcal{M}} u_2) := (u_1, u_2)^{s/c}; \tag{7.49}$$

these are well defined owing to (7.47). We denote the corresponding Hermitian spaces by

$$\mathscr{L}^{s/c}(\mathcal{M}) := (\mathrm{Sol}^{s/c}(\mathcal{M}), s^{s/c}_{\mathcal{M}}). \tag{7.50}$$

Lemma 7.1.6. *The spaces $\mathscr{L}^{s/c}(\mathcal{M})$ are weakly nondegenerate.*

Proof. If $u \in D_0(\boldsymbol{M})$ has the property that $s^s_{\mathcal{M}}(S^s_{\mathcal{M}}u', S^s_{\mathcal{M}}u) = 0$ for all $u' \in D_0(\boldsymbol{M})$, then $(u', u)^s = 0$ for all u', so it is clear from (7.44) that $S_{\mathcal{M}}u = 0$. The result for cospinors follows since $A_{\boldsymbol{M}}$ is an isomorphism. □

Proposition 7.1.7. *For every $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$,*

$$\mathscr{L}(\mathcal{M}) := (\mathscr{L}^s(\mathcal{M}), \mathscr{L}^c(\mathcal{M}), A_{\boldsymbol{M}}, -\overline{A^{-1}_{\boldsymbol{M}}}) \tag{7.51}$$

is a Hermitian adjoint structure (see Definition 5.1.4).

Proof. $A_{\boldsymbol{M}}$ is clearly an isomorphism from $\mathrm{Sol}^s(\mathcal{M})$ to $\overline{\mathrm{Sol}^c(\mathcal{M})}$, and $-\overline{A^{-1}_{\boldsymbol{M}}}$ is likewise an isomorphism from $\mathrm{Sol}^c(\mathcal{M})$ to $\overline{\mathrm{Sol}^s(\mathcal{M})}$. It therefore remains to show that they are also isomorphisms in Herm. Firstly, for any $h_1, h_2 \in \mathscr{S}^*_0(\boldsymbol{M})$ we have

$$\begin{aligned}
\overline{s^s_{\mathcal{M}}}(-\overline{A^{-1}_{\boldsymbol{M}}}S^c_{\mathcal{M}}h_1, -\overline{A^{-1}_{\boldsymbol{M}}}S^c_{\mathcal{M}}h_2) &= s^s_{\mathcal{M}}(A^{-1}_{\boldsymbol{M}}\overline{S^c_{\mathcal{M}}h_2}, A^{-1}_{\boldsymbol{M}}\overline{S^c_{\mathcal{M}}h_1}) \\
&= s^s_{\mathcal{M}}(S^s_{\mathcal{M}}A^{-1}_{\boldsymbol{M}}\overline{h_2}, S^s_{\mathcal{M}}A^{-1}_{\boldsymbol{M}}\overline{h_1}) \\
&= (A^{-1}_{\boldsymbol{M}}\overline{h_2}, A^{-1}_{\boldsymbol{M}}\overline{h_1})^s \\
&= (h_1, h_2)^c \\
&= s^c_{\mathcal{M}}(S^c_{\mathcal{M}}h_1, S^c_{\mathcal{M}}h_2).
\end{aligned} \tag{7.52}$$

Secondly, for any $f_1, f_2 \in \mathscr{S}_0(\boldsymbol{M})$, it follows that

$$\begin{aligned}
\overline{s^c_{\mathcal{M}}}(A_{\boldsymbol{M}}S^s_{\mathcal{M}}f_1, A_{\boldsymbol{M}}S^s_{\mathcal{M}}f_2) &= s^c_{\mathcal{M}}(\overline{A_{\boldsymbol{M}}S^s_{\mathcal{M}}f_2}, \overline{A_{\boldsymbol{M}}S^s_{\mathcal{M}}f_1}) \\
&= \overline{s^s_{\mathcal{M}}}(\overline{S^s_{\mathcal{M}}f_2}, \overline{S^s_{\mathcal{M}}f_1}) \\
&= s^s_{\mathcal{M}}(S^s_{\mathcal{M}}f_1, S^s_{\mathcal{M}}f_2).
\end{aligned} \tag{7.53}$$

Therefore $A_{\boldsymbol{M}}$ and $-\overline{A^{-1}_{\boldsymbol{M}}}$ are indeed Herm-isomorphisms. □

Proposition 7.1.8. *For every $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$, $(C_{\boldsymbol{M}}, C^*_{\boldsymbol{M}})$ is a charge conjugation of $\mathscr{L}(\mathcal{M})$.*

Proof. In order that $(C_{\boldsymbol{M}}, C^*_{\boldsymbol{M}})$ be a charge conjugation, we must show that

$$\overline{s^s_{\mathcal{M}}}(C_{\boldsymbol{M}}u_1, C_{\boldsymbol{M}}u_2) = s^s_{\mathcal{M}}(u_1, u_2) \tag{7.54}$$

for all $u_1, u_2 \in \mathrm{Sol}^s(\mathcal{M})$, that

$$\overline{s^c_{\mathcal{M}}}(C^*_{\boldsymbol{M}} v_1, C^*_{\boldsymbol{M}} v_2) = s^c_{\mathcal{M}}(v_1, v_2) \tag{7.55}$$

for all $v_1, v_2 \in \mathrm{Sol}^c(\mathcal{M})$, and that in line with (5.9), we have $-\overline{A_{\boldsymbol{M}}} \circ C_{\boldsymbol{M}} = \overline{C^*_{\boldsymbol{M}}} \circ A_{\boldsymbol{M}}$. This last condition is seen to be true from (7.11); as for (7.54), if $u_i = S^s_{\mathcal{M}} f_i$ with $f_1, f_2 \in D_0(\boldsymbol{M})$, we calculate directly

$$\begin{aligned}
\overline{s^s_{\mathcal{M}}}(C_{\boldsymbol{M}} S^s_{\mathcal{M}} f_1, C_{\boldsymbol{M}} S^s_{\mathcal{M}} f_2) &= s^s_{\mathcal{M}}(\overline{C_{\boldsymbol{M}} S^s_{\mathcal{M}} f_1}, \overline{C_{\boldsymbol{M}} S^s_{\mathcal{M}} f_2}) \\
&= s^s_{\mathcal{M}}(S^s_{\mathcal{M}} \overline{C_{\boldsymbol{M}} f_1}, S^s_{\mathcal{M}} \overline{C_{\boldsymbol{M}} f_2}) \\
&= -i \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \, \langle \overline{A_{\boldsymbol{M}}} C_{\boldsymbol{M}} f_2, S^s_{\mathcal{M}} \overline{C_{\boldsymbol{M}} f_1} \rangle \\
&= i \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \, \langle \overline{C^*_{\boldsymbol{M}}} A_{\boldsymbol{M}} f_2, \overline{C_{\boldsymbol{M}} S^s_{\mathcal{M}} f_1} \rangle \\
&= i \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \, \overline{\langle \overline{A_{\boldsymbol{M}} f_2}, S^s_{\mathcal{M}} f_1 \rangle} \\
&= \overline{(f_2, f_1)^s} \\
&= s^s_{\boldsymbol{M}}(S^s_{\mathcal{M}} f_1, S^s_{\mathcal{M}} f_2),
\end{aligned} \tag{7.56}$$

where we have used (6.12). Finally, we have

$$\begin{aligned}
\overline{s^c_{\mathcal{M}}}(C^*_{\boldsymbol{M}} v_1, C^*_{\boldsymbol{M}} v_2) &= s^c_{\mathcal{M}}(\overline{C^*_{\boldsymbol{M}} v_2}, \overline{C^*_{\boldsymbol{M}} v_1}) = s^s_{\mathcal{M}}(A^{-1}_{\boldsymbol{M}} C^*_{\boldsymbol{M}} v_1, A^{-1}_{\boldsymbol{M}} C^*_{\boldsymbol{M}} v_2) \\
&= s^s_{\mathcal{M}}(\overline{C_{\boldsymbol{M}}} \overline{A^{-1}_{\boldsymbol{M}}} v_1, \overline{C_{\boldsymbol{M}}} \overline{A^{-1}_{\boldsymbol{M}}} v_2) = s^s_{\mathcal{M}}(A^{-1}_{\boldsymbol{M}} \overline{v_2}, A^{-1}_{\boldsymbol{M}} \overline{v_1}) \\
&= s^c_{\mathcal{M}}(v_1, v_2).
\end{aligned} \tag{7.57}$$

This concludes the proof. □

In order to construct $\mathscr{L}$ as a functor from FLoc_4 to $\mathsf{HermAdj}_{\mathsf{C}}$ we must define the arrows $\mathscr{L}(\boldsymbol{\psi})$, where $\boldsymbol{\psi} : \mathcal{M} \to \mathcal{N}$ for some $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}_{\boldsymbol{M}})$ and $\mathcal{N} = (\boldsymbol{N}, \boldsymbol{\varepsilon}_{\boldsymbol{N}})$. To this end we define

$$\begin{aligned}
\mathscr{L}^s(\boldsymbol{\psi}) S^s_{\mathcal{M}} f &:= S^s_{\mathcal{N}} \psi_* f, \\
\mathscr{L}^c(\boldsymbol{\psi}) S^c_{\mathcal{M}} h &:= S^c_{\mathcal{N}} \psi_* h, \\
\mathscr{L}(\boldsymbol{\psi}) &:= (\mathscr{L}^s(\boldsymbol{\psi}), \mathscr{L}^c(\boldsymbol{\psi})).
\end{aligned} \tag{7.58}$$

These are well defined: for example, $S^s_{\mathcal{M}} f = S^s_{\mathcal{M}} f'$ if and only if $f' = f + P^s_{\mathcal{M}} g$ for some $g \in \mathscr{S}_0(\boldsymbol{M})$, and $S^s_{\mathcal{N}} \psi_* f' = S^s_{\mathcal{N}} \psi_* (f + P^s_{\mathcal{M}} g) = S^s_{\mathcal{N}} \psi_* f$, by (7.35).

Lemma 7.1.9. *For every* FLoc_4*-arrow* $\boldsymbol{\psi} : \mathcal{M} \to \mathcal{N}$*, the map* $\mathscr{L}(\boldsymbol{\psi})$ *is an arrow in* $\mathsf{HermAdj}_{\mathsf{C}}$*.*

Proof. First, we must show that $\mathscr{L}^{s/c}(\boldsymbol{\psi})$ are Herm-arrows from $\mathscr{L}^{s/c}(\mathcal{M})$ to $\mathscr{L}^{s/c}(\mathcal{N})$. They are clearly linear maps, and are injective since they have a left inverse (namely ψ^*, by (7.40)). Moreover, for any $S^s_{\mathcal{M}} f_1, S^s_{\mathcal{M}} f_2 \in \mathscr{L}^s(\mathcal{M})$ we have

$$\begin{aligned} s^s_{\mathcal{N}}(\mathscr{L}(\boldsymbol{\psi}) S^s_{\mathcal{M}} f_1, \mathscr{L}(\boldsymbol{\psi}) S^s_{\mathcal{M}} f_2) &= s^s_{\mathcal{N}}(S^s_{\mathcal{N}} \psi_* f_1, S^s_{\mathcal{N}} \psi_* f_2) \\ &= -i \int_{\boldsymbol{N}} dvol_{\boldsymbol{N}} \, \langle \overline{A_{\boldsymbol{N}} \psi_* f_1}, S^s_{\mathcal{N}} \psi_* f_2 \rangle \\ &= -i \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \, \langle \psi^* \overline{A_{\boldsymbol{N}} \psi_* f_1}, \psi^* S^s_{\mathcal{N}} \psi_* f_2 \rangle \\ &= -i \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \, \langle \overline{A_{\boldsymbol{M}} f_1}, S^s_{\mathcal{M}} f_2 \rangle \\ &= s^s_{\mathcal{M}}(S^s_{\mathcal{M}} f_1, S^s_{\mathcal{M}} f_2), \end{aligned} \tag{7.59}$$

where we have used the fact that $A_{\boldsymbol{N}} \psi_* f_1$ is supported in $\psi(\boldsymbol{M})$ in the third equality. The analogous statement for $\mathscr{L}^c(\boldsymbol{\psi})$ follows from previous results and the fact that $A_{\boldsymbol{N}} \circ \mathscr{L}^s(\boldsymbol{\psi}) = \overline{\mathscr{L}^c(\boldsymbol{\psi})} \circ A_{\boldsymbol{M}}$. This, along with the statement that $A_{\boldsymbol{N}} \circ \overline{\mathscr{L}^c(\boldsymbol{\psi})} = \mathscr{L}^s(\boldsymbol{\psi}) \circ A_{\boldsymbol{M}}$, may be seen to follow from the properties of $A_{\boldsymbol{M}}$, $A_{\boldsymbol{N}}$, $S^{s/c}_{\mathcal{M}}$ and $S^{s/c}_{\mathcal{N}}$. These last two results show that (5.4) commutes in this case, and therefore $\mathscr{L}(\boldsymbol{\psi})$ is indeed an arrow in $\mathsf{HermAdj}_{\mathsf{C}}$. □

Proposition 7.1.10. *The map* $\mathscr{L} : \mathsf{FLoc}_4 \to \mathsf{HermAdj}_{\mathsf{C}}$ *defines a covariant functor, and is hence a locally covariant theory (the* framed-spacetime classical Dirac theory*).*

Proof. As we have already shown that $\mathscr{L}(\boldsymbol{\psi})$ is a $\mathsf{HermAdj}_{\mathsf{C}}$-arrow from $\mathscr{L}(\mathcal{M})$ to $\mathscr{L}(\mathcal{N})$ whenever $\boldsymbol{\psi} : \mathcal{M} \to \mathcal{N}$ is an arrow in FLoc_4, it remains only to show that $\mathscr{L}(\boldsymbol{\psi}_2 \circ \boldsymbol{\psi}_1) = \mathscr{L}(\boldsymbol{\psi}_2) \circ \mathscr{L}(\boldsymbol{\psi}_1)$ for any $\boldsymbol{\psi}_1 : \boldsymbol{L} \to \boldsymbol{M}$ and $\boldsymbol{\psi}_2 : \boldsymbol{M} \to \boldsymbol{N}$, and that $\mathscr{L}(\boldsymbol{\iota}_{\mathcal{M}}) = \mathrm{id}_{\mathscr{L}(\mathcal{M})}$ where $\boldsymbol{\iota}_{\mathcal{M}}$ is the identity arrow on $\mathcal{M}$. Note that $\mathscr{L}(\boldsymbol{\psi}_2) \circ \mathscr{L}(\boldsymbol{\psi}_1) = (\mathscr{L}^s(\boldsymbol{\psi}_2) \circ \mathscr{L}^s(\boldsymbol{\psi}_1), \mathscr{L}^c(\boldsymbol{\psi}_2) \circ \mathscr{L}^c(\boldsymbol{\psi}_1))$ by definition, and for any $f \in \mathscr{S}_0^{(*)}(\boldsymbol{L})$, it holds that

$$\mathscr{L}^{s/c}(\boldsymbol{\psi}_2)\mathscr{L}^{s/c}(\boldsymbol{\psi}_1)S^{s/c}_{\boldsymbol{L}} f = \mathscr{L}^{s/c}(\boldsymbol{\psi}_2)S^{s/c}_{\mathcal{M}}(\psi_1)_* f = S^{s/c}_{\mathcal{N}}(\psi_2)_*(\psi_1)_* f \\ = S^{s/c}_{\mathcal{N}}(\psi_2 \circ \psi_1)_* f = \mathscr{L}^{s/c}(\boldsymbol{\psi}_2 \circ \boldsymbol{\psi}_1) f. \quad (7.60)$$

Moreover, $\mathscr{L}(\boldsymbol{\iota}_{\mathcal{M}}) = \mathrm{id}_{\mathscr{L}(\mathcal{M})}$ follows immediately from the observation that $(\iota_{\mathcal{M}})_* f = f$ for all $f \in \mathscr{S}_0^{(*)}(\boldsymbol{M})$. □

Note that $\mathscr{L}$ is a doubled classical theory, in the sense of Section 5.3.

Proposition 7.1.11. *The family $(C, C^*) = (C_{\boldsymbol{M}}, C^*_{\boldsymbol{M}})_{(\boldsymbol{M},\boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4}$ is a natural transformation between $\mathscr{L}$ and $\overline{\mathscr{L}}$; that is to say, (C, C^*) is a charge conjugation on $\mathscr{L}$.*

Proof. We need to show that for any map $\boldsymbol{\psi} : \mathcal{M} \to \mathcal{N}$ in FLoc_4, we have $C^{(*)}_{\boldsymbol{N}} \circ \mathscr{L}^{s/c}(\boldsymbol{\psi}) = \overline{\mathscr{L}^{s/c}(\boldsymbol{\psi})} \circ C^{(*)}_{\boldsymbol{M}}$. This follows immediately from the properties of $C^{(*)}_{\boldsymbol{M}}$, $C^{(*)}_{\boldsymbol{N}}$, $S^{s/c}_{\mathcal{M}}$ and $S^{s/c}_{\mathcal{N}}$ listed in (7.41). □

For each $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$, we define

$$S_{\mathcal{M}} : \mathscr{S}_0(\boldsymbol{M}) \oplus \mathscr{S}_0^*(\boldsymbol{M}) \to \mathscr{L}^s(\mathcal{M}) \oplus \mathscr{L}^c(\mathcal{M}) \\ (f, h) \mapsto (S^s_{\mathcal{M}} f, S^c_{\mathcal{M}} h). \quad (7.61)$$

Proposition 7.1.12. *$S = (S_{\mathcal{M}})_{\mathcal{M} \in \mathsf{FLoc}_4}$ is a locally covariant solution for $\mathscr{L}$, in the sense of Section 5.3.*

Proof. We recall that the condition for S to be a locally covariant solution in this case is that $S_{\mathcal{M}}$ is surjective for each $\mathcal{M}$, and that

$$S_{\mathcal{N}} \circ \mathscr{S}_{\mathrm{adj}}(\boldsymbol{\psi}) = \mathscr{L}(\boldsymbol{\psi}) \circ S_{\mathcal{M}} \quad (7.62)$$

for each FLoc_4-arrow $\boldsymbol{\psi} : \mathcal{M} \to \mathcal{N}$, or equivalently that

$$S^{s/c}_{\mathcal{N}} \circ \psi_* = \mathscr{L}^{s/c}(\boldsymbol{\psi}) \circ S^{s/c}_{\mathcal{M}}. \quad (7.63)$$

Both conditions follow immediately from the definition of $\mathscr{L}$; the former from (7.48), and the latter from (7.58). □

7.1.5 The frame-independent classical Dirac theory

We have seen that it is possible to construct a classical Dirac theory valued in $\mathsf{HermAdj}_\mathsf{C}$ with a charge conjugation and locally covariant solutions; however, so far we have required the choice of a global frame $\boldsymbol{\varepsilon} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M})$ in order to define the Dirac operators (and hence the solution spaces). This choice has been at least implicitly required in the previous locally covariant formulation [56], but we regard this as an obstacle to a completely general theory, since the physics and observable quantities of the system should not depend on the choice of frame (at least within a given equivalence class). We will therefore attempt to describe a classical Dirac theory that does not display such a dependence.

Note that this situation is very similar to the situation of defining the algebra of Wick polynomials for the scalar field, where a choice of the symmetric bidistribution H satisfying the Hadamard wavefront set condition (3.40) and the bisolution condition (3.41) was required for the explicit construction. In that situation, as here, there was no covariant, canonical choice of H for each spacetime, but there was a canonical isomorphism between the theories arising from each pair H, H', and so we were able to define a covariant H-independent theory $\mathscr{W}$ by constructing the elements of the algebra as families of elements of the individual H-dependent algebras $\mathscr{W}_H(\boldsymbol{M})$, with a compatibility condition that ensured that the resulting algebra $\mathscr{W}(\boldsymbol{M})$ was isomorphic to each of the individual algebras $\mathscr{W}_H(\boldsymbol{M})$.

We might attempt a similar approach here to define a covariant theory based in $[\mathsf{F}]\mathsf{Loc}_4$; for each object $(\boldsymbol{M}, \mathcal{E})$ we would like to define a theory around a vector space whose elements are families of solutions to the Dirac equation on $(\boldsymbol{M}, \boldsymbol{\varepsilon})$, indexed by $\boldsymbol{\varepsilon} \in \mathcal{E}$, with a suitable compatibility condition. However, the existence of such a condition relies on the existence of isomorphisms between the individual solution spaces, and although we have shown that such isomorphisms exist, we have also shown that there exists a sign ambiguity in each map that prevents us from making a canonical, covariant choice of compatibility condition. We will see that this makes it impossible to construct a covariant theory based in $[\mathsf{F}]\mathsf{Loc}_4$ that is isomorphic on a given

$(\boldsymbol{M}, \mathcal{E}) \in [\mathsf{F}]\mathsf{Loc}_4$ to $\mathscr{L}(\boldsymbol{M}, \boldsymbol{\varepsilon})$ for any $\boldsymbol{\varepsilon} \in \mathcal{E}$; rather than a failing of the theory, this is a result of the fact that even in a classical context, individual spinor fields are fundamentally unobservable.

We first need to check how the solution spaces $\mathrm{Sol}^{s/c}(\boldsymbol{M}, \boldsymbol{\varepsilon})$ react to a change of frame $\boldsymbol{\varepsilon} \mapsto \boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}\Lambda(s)$. Recall from Lemma 7.1.5 that if $f \in \mathrm{Sol}^s(\boldsymbol{M}, \boldsymbol{\varepsilon})$ and $h \in \mathrm{Sol}^c(\boldsymbol{M}, \boldsymbol{\varepsilon})$ then $s^{-1}f \in \mathrm{Sol}^s(\boldsymbol{M}, \boldsymbol{\varepsilon}')$ and $hs \in \mathrm{Sol}^c(\boldsymbol{M}, \boldsymbol{\varepsilon}')$. By swapping the roles of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ we may prove the reverse implication, and so for every pair $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'$ in a given equivalence class of frames, we have a pair of isomorphisms between $\mathrm{Sol}^{s/c}(\boldsymbol{M}, \boldsymbol{\varepsilon})$ and $\mathrm{Sol}^{s/c}(\boldsymbol{M}, \boldsymbol{\varepsilon}')$, given by $f \mapsto \pm s^{-1}f$, $h \mapsto \pm hs$.

As before neither of these two isomorphisms is preferred above the other. However, we find a solution to this problem in Subsection 6.2.4: given an object $(\boldsymbol{M}, \mathcal{E}) \in [\mathsf{F}]\mathsf{Loc}_4$ and spin cocycle $\boldsymbol{s} = (s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}' \in \mathcal{E}}$, we define $\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} : \mathrm{Sol}^{s/c}(\boldsymbol{M}, \boldsymbol{\varepsilon}) \to \mathrm{Sol}^{s/c}(\boldsymbol{M}, \boldsymbol{\varepsilon}')$ by

$$\alpha^s_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} f := (s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})^{-1} f, \qquad \alpha^c_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} h := h s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'}. \tag{7.64}$$

We may clearly define these maps $\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}$ in exactly the same way as automorphisms of $\mathscr{S}^{(*)}_{(0)}(\boldsymbol{M})$, and we will use the same symbol for the maps on both spaces without ambiguity; which is meant will be obvious from context. Note that by Lemma 7.1.5 we have $P^{s/c}_{(\boldsymbol{M},\boldsymbol{\varepsilon}')} \circ \alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} = \alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} \circ P^{s/c}_{(\boldsymbol{M},\boldsymbol{\varepsilon})}$ for all $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'$.

Lemma 7.1.13. *Let $(\boldsymbol{M}, \mathcal{E}) \in [\mathsf{F}]\mathsf{Loc}_4$, with a spin cocycle $\boldsymbol{s} = (s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}' \in \mathcal{E}}$, and consider $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}'' \in \mathcal{E}$. Then*

(a). *$\alpha_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} := (\alpha^s_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}, \alpha^c_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'})$ is a $\mathsf{HermAdj}_{\mathsf{C}}$-isomorphism between $\mathscr{L}(\boldsymbol{M}, \boldsymbol{\varepsilon})$ and $\mathscr{L}(\boldsymbol{M}, \boldsymbol{\varepsilon}')$,*

(b). *$C^{(*)}_{\boldsymbol{M}} \circ \alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} = \alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} \circ C^{(*)}_{\boldsymbol{M}}$, and*

(c). *$\alpha_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}} = \mathrm{id}_{\mathscr{L}(\boldsymbol{M},\boldsymbol{\varepsilon})}$, $\alpha_{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}} = {\alpha_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}}^{-1}$, and $\alpha_{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}''} \circ \alpha_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} = \alpha_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}''}$.*

Proof.

(a). Since $\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}$ are isomorphisms between the individual vector spaces, we need only show that $\alpha_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}$ is an arrow in $\mathsf{HermAdj}_{\mathsf{C}}$. Writing $\mathcal{M} =$

$(\boldsymbol{M},\boldsymbol{\varepsilon})$ and $\mathcal{M}'=(\boldsymbol{M},\boldsymbol{\varepsilon}')$, we must then show that $s^{s/c}_{\mathcal{M}'}(\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}u,\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}u')=s^{s/c}_{\mathcal{M}}(u,u')$ for all $u\in\mathrm{Sol}^{s/c}(\mathcal{M})$; in addition, we must have $A_{\boldsymbol{M}}\circ\alpha^{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}=\overline{\alpha^{c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}}\circ A_{\boldsymbol{M}}$ (the analogue to the right hand box of (5.4) being redundant in this situation). For the latter condition, given a spinor field f we have

$$(A_{\boldsymbol{M}}(\alpha^{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}f))(p)=\overline{(p,((s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})^{-1}f_p)^{\dagger}A)}=\overline{(p,f_p^{\dagger}(s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'}{}^{\dagger})^{-1}A)} \tag{7.65}$$

and

$$(\overline{\alpha^{c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}}(A_{\boldsymbol{M}}f))(p)=\overline{(p,f_p^{\dagger}As^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})}, \tag{7.66}$$

and these may be seen to be equal via (6.12). For the former, note that by Lemma 7.1.5 we have

$$\begin{aligned} P^{s/c}_{\mathcal{M}'}\circ\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}\circ(S^{s/c}_{\mathcal{M}})^{\pm}\circ(\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'})^{-1}&=\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}\circ P^{s/c}_{\mathcal{M}}\circ(S^{s/c}_{\mathcal{M}})^{\pm}\circ(\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'})^{-1}\\ &=\iota^{s/c}_{\mathcal{M}'} \end{aligned} \tag{7.67}$$

when acting on test (co)spinors, and similarly

$$\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}\circ(S^{s/c}_{\mathcal{M}})^{\pm}\circ(\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'})^{-1}\circ P^{s/c}_{\mathcal{M}'}=\iota^{s/c}_{\mathcal{M}'}. \tag{7.68}$$

Since $(\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'})^{\pm1}$ cannot alter the support properties of (co)spinor fields, it follows by uniqueness that

$$\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}\circ(S^{s/c}_{\mathcal{M}})^{\pm}=(S^{s/c}_{\mathcal{M}'})^{\pm}\circ\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}. \tag{7.69}$$

Consequently, for any $S^{s}_{\mathcal{M}}f,S^{s}_{\mathcal{M}}f'\in\mathrm{Sol}^{s}(\mathcal{M})$ we have

$$\begin{aligned} s^{s}_{\mathcal{M}'}(\alpha^{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}S^{s}_{\mathcal{M}}f,\alpha^{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}S^{s}_{\mathcal{M}}f')&=-i\int_{\boldsymbol{M}}dvol_{\boldsymbol{M}}\,\langle\overline{A_{\boldsymbol{M}}\alpha^{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}f},\alpha^{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}S^{s}_{\mathcal{M}}f'\rangle\\ &=-i\int_{\boldsymbol{M}}dvol_{\boldsymbol{M}}\,\langle\alpha^{c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}\overline{A_{\boldsymbol{M}}f},\alpha^{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}S^{s}_{\mathcal{M}}f'\rangle\\ &=-i\int_{\boldsymbol{M}}dvol_{\boldsymbol{M}}\,\langle\overline{A_{\boldsymbol{M}}f},S^{s}_{\mathcal{M}}f'\rangle\\ &=s^{s}_{\mathcal{M}}(S^{s}_{\mathcal{M}}f,S^{s}_{\mathcal{M}}f'). \end{aligned} \tag{7.70}$$

The result for cospinors may be proved in a similar way.

(b). Let f be a spinor field. Then

$$\begin{aligned}(C_{\boldsymbol{M}}(\alpha^s_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}f))(p) &= \overline{(p, \overline{C(s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})^{-1} f_p})} \\ &= \overline{(p, (s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})^{-1} \overline{C f_p})} \\ &= (\alpha^s_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}(C_{\boldsymbol{M}} f))(p), \end{aligned} \tag{7.71}$$

where we have again used (6.12). The result for cospinors may be proved analogously.

(c). These relations are a direct consequence of the condition that $(s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'})_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'\in\mathcal{E}}$ be a spin cocycle.

□

We have therefore shown that whenever $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$ are equivalent for a given spacetime $\boldsymbol{M} \in \mathsf{Loc}_4$, the $\mathsf{HermAdj}_\mathsf{C}$ objects $\mathscr{L}(\boldsymbol{M},\boldsymbol{\varepsilon})$ and $\mathscr{L}(\boldsymbol{M},\boldsymbol{\varepsilon}')$ are isomorphic, though we require a spin cocycle to give an explicit set of compatible isomorphisms for each such pair of frames. We finally move to the definition of a locally covariant theory whose domain is $[\mathsf{F}]\mathsf{Loc}_4$, and is therefore independent of both the choice of a particular spin frame, and the requirement for an explicit spin cocycle. As previously mentioned, we will not attempt to define a locally covariant theory from $[\mathsf{F}]\mathsf{Loc}_4$ to $\mathsf{HermAdj}_\mathsf{C}$ that is isomorphic for each $\mathscr{M} = (\boldsymbol{M},\mathcal{E})$ to $\mathscr{L}(\boldsymbol{M},\boldsymbol{\varepsilon})$ for any $\boldsymbol{\varepsilon} \in \mathcal{E}$. We will, however, use the machinery developed in this subsection, along with the squared adjoint structures defined in Section 5.1.3, to construct a theory that is able to deal with the sign ambiguities present in the choice of spin cocycle.

Let $\mathscr{M} = (\boldsymbol{M},\mathcal{E})$ be an object of $[\mathsf{F}]\mathsf{Loc}_4$, and pick some $\boldsymbol{\varepsilon} \in \mathcal{E}$. Now define the squared adjoint structure $\mathscr{L}_{\boldsymbol{\varepsilon}}(\mathscr{M}) := \mathfrak{S}(\mathscr{L}(\boldsymbol{M},\boldsymbol{\varepsilon}))$; recall that the underlying vector space of this object is then

$$(\mathrm{Sol}^s(\boldsymbol{M},\boldsymbol{\varepsilon}) \oplus \mathrm{Sol}^c(\boldsymbol{M},\boldsymbol{\varepsilon})) \otimes (\mathrm{Sol}^s(\boldsymbol{M},\boldsymbol{\varepsilon}) \oplus \mathrm{Sol}^c(\boldsymbol{M},\boldsymbol{\varepsilon})). \tag{7.72}$$

We will show that these spaces are canonically isomorphic for different $\boldsymbol{\varepsilon}$, and then use them to define a further squared adjoint structure that does not depend on $\boldsymbol{\varepsilon}$, denoted by $\mathscr{L}_{FI}(\mathscr{M})$.

First define $\mathrm{Sol}_{FI}(\mathscr{M})$ as the set whose elements are families $(\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}}$ such that $\boldsymbol{a}_{\boldsymbol{\varepsilon}} \in \mathscr{L}_{\boldsymbol{\varepsilon}}(\mathscr{M})$ and such that there exists a spin cocycle $\boldsymbol{s}$ for $\mathcal{E}$ satisfying

$$\boldsymbol{a}_{\boldsymbol{\varepsilon}'} = (\alpha^{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} \oplus \alpha^{c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}) \otimes (\alpha^{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} \oplus \alpha^{c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'})\, \boldsymbol{a}_{\boldsymbol{\varepsilon}} \tag{7.73}$$

for all $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \mathcal{E}$, where $\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} : \mathrm{Sol}^{s/c}(\boldsymbol{M}, \boldsymbol{\varepsilon}) \to \mathrm{Sol}^{s/c}(\boldsymbol{M}, \boldsymbol{\varepsilon}')$ are defined as in (7.64). Then:

Lemma 7.1.14. *If* $(\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}}$ *satisfies* (7.73) *for some spin cocycle* $\boldsymbol{s}$*, then it satisfies it for any other spin cocycle* $\boldsymbol{s}'$*.*

Proof. For each pair $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \mathcal{E}$, we must have $s'^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'} = \pm s^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'}$ by definition. Hence $\alpha'^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} = \pm\alpha'^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}$, and therefore

$$(\alpha'^{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} \oplus \alpha'^{c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'})^{\otimes 2} = (\pm\alpha^{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} \oplus \pm\alpha^{c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'})^{\otimes 2} = (\alpha^{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} \oplus \alpha^{c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'})^{\otimes 2}, \tag{7.74}$$

and so (7.64) is satisfied. □

This immediately leads to the following:

Corollary 7.1.15. *Any element* $\boldsymbol{a} = (\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}} \in \mathrm{Sol}_{FI}(\mathscr{M})$ *is uniquely defined by a single component* $\boldsymbol{a}_{\boldsymbol{\varepsilon}}$*.*

Since the maps $\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}$ are linear, it follows that we may define Sol_{FI} to be a vector space, with vector operations acting pointwise.

We also define the trace map $\omega_{FI} : \mathrm{Sol}_{FI}(\mathscr{M}) \to \mathbb{C}$ by

$$\omega_{FI}((\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}}) := \omega_{\boldsymbol{\varepsilon}}(\boldsymbol{a}_{\boldsymbol{\varepsilon}}), \tag{7.75}$$

where $\boldsymbol{\varepsilon} \in \mathcal{E}$ is arbitrary, an adjoint map

$$[(\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}}]^* := (\boldsymbol{a}^*_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}}, \tag{7.76}$$

an internal swap map

$$\Phi_{FI}[(\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}}] := (\Phi_{\boldsymbol{\varepsilon}}\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}}, \tag{7.77}$$

and an exchange map

$$Z_{FI}[(\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}}] \otimes [(\boldsymbol{a}'_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}}] := (Z_{FI}(\boldsymbol{a}_{\boldsymbol{\varepsilon}} \otimes \boldsymbol{a}'_{\boldsymbol{\varepsilon}}))_{\boldsymbol{\varepsilon}\in\mathcal{E}}, \tag{7.78}$$

where $\omega_{\boldsymbol{\varepsilon}}$, $\Phi_{\boldsymbol{\varepsilon}}$ and $Z_{\boldsymbol{\varepsilon}}$ are respectively the trace map, internal swap map and exchange map on $\mathfrak{S}(\mathscr{L}(\boldsymbol{M}, \boldsymbol{\varepsilon}))$.

Lemma 7.1.16. *The above objects are well defined, and*

$$\mathscr{L}_{FI}(\mathscr{M}) := (\mathrm{Sol}_{FI}(\mathscr{M}), \omega_{FI}, {}^*, \Phi_{FI}, Z_{FI}) \tag{7.79}$$

is a squared adjoint structure.

Proof. We may prove that the structures given by (7.75)–(7.78) are well defined by using the fact that for any spin cocycle $\boldsymbol{s}$ for $\mathcal{E}$ and its associated maps $\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}$, the map $\mathfrak{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} := \mathfrak{S}(\alpha^s_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}, \alpha^c_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'})$ is an arrow in SAdj. Since $\boldsymbol{a}_{\boldsymbol{\varepsilon}'} = \mathfrak{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}\boldsymbol{a}_{\boldsymbol{\varepsilon}}$ for any $(\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}}$, it follows from (5.10) that

$$\begin{aligned} \omega_{\boldsymbol{\varepsilon}'}(\boldsymbol{a}_{\boldsymbol{\varepsilon}'}) &= \omega_{\boldsymbol{\varepsilon}'}(\mathfrak{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}\boldsymbol{a}_{\boldsymbol{\varepsilon}}) = \omega_{\boldsymbol{\varepsilon}}(\boldsymbol{a}_{\boldsymbol{\varepsilon}}), \\ (\boldsymbol{a}_{\boldsymbol{\varepsilon}'})^* &= (\mathfrak{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}\boldsymbol{a}_{\boldsymbol{\varepsilon}})^* = \mathfrak{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}(\boldsymbol{a}^*_{\boldsymbol{\varepsilon}}), \\ \Phi_{\boldsymbol{\varepsilon}'}\boldsymbol{a}_{\boldsymbol{\varepsilon}'} &= \Phi_{\boldsymbol{\varepsilon}'}(\mathfrak{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}\boldsymbol{a}_{\boldsymbol{\varepsilon}}) = \Phi_{\boldsymbol{\varepsilon}}\boldsymbol{a}_{\boldsymbol{\varepsilon}}, \\ Z_{\boldsymbol{\varepsilon}'}(\boldsymbol{a}_{\boldsymbol{\varepsilon}'} \otimes \boldsymbol{a}'_{\boldsymbol{\varepsilon}'}) &= Z_{\boldsymbol{\varepsilon}'}((\mathfrak{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} \otimes \mathfrak{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'})(\boldsymbol{a}_{\boldsymbol{\varepsilon}} \otimes \boldsymbol{a}'_{\boldsymbol{\varepsilon}})) = Z_{\boldsymbol{\varepsilon}}(\boldsymbol{a}_{\boldsymbol{\varepsilon}} \otimes \boldsymbol{a}'_{\boldsymbol{\varepsilon}}). \end{aligned} \tag{7.80}$$

It is then easy to show that $\mathscr{L}_{FI}(\mathscr{M})$ is a squared adjoint structure by looking at the definitions of its constituent parts on components in a particular frame $\boldsymbol{\varepsilon}$. ◻

Definition 7.1.17. *For an arrow* $[\psi] : (\boldsymbol{M}, \mathcal{E}) \to (\boldsymbol{N}, \mathcal{E}')$ *in* $[\mathsf{F}]\mathsf{Loc}_4$*, define*

$$\begin{aligned} \mathscr{L}_{FI}([\psi]) : \mathscr{L}_{FI}(\boldsymbol{M}, \mathcal{E}) &\to \mathscr{L}_{FI}(\boldsymbol{N}, \mathcal{E}') \\ (\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}} &\mapsto (\mathscr{L}_{\boldsymbol{\varepsilon}'}(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'})\boldsymbol{a}_{\psi^\star(\boldsymbol{\varepsilon}')})_{\boldsymbol{\varepsilon}'\in\mathcal{E}'} \end{aligned} \tag{7.81}$$

where $\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'} : (\boldsymbol{M}, \psi^\star(\boldsymbol{\varepsilon}')) \to (\boldsymbol{N}, \boldsymbol{\varepsilon}')$ *is the* FLoc_4*-arrow covering* ψ *and the map* $\mathscr{L}_{\boldsymbol{\varepsilon}'}(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'}) : \mathscr{L}_{\psi^\star(\boldsymbol{\varepsilon}')}(\boldsymbol{M}, \mathcal{E}) \to \mathscr{L}_{\boldsymbol{\varepsilon}}(\boldsymbol{N}, \mathcal{E}')$ *is given by*

$$\mathscr{L}_{\boldsymbol{\varepsilon}'}(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'}) := (\mathscr{L}^s(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'}) \oplus \mathscr{L}^c(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'}))^{\otimes 2}, \tag{7.82}$$

which we note from Definition 5.1.9 to be just $\mathfrak{S}(\mathscr{L}(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'}))$.

Proposition 7.1.18. $\mathscr{L}_{FI}$ *defines a covariant functor from* $[\mathsf{F}]\mathsf{Loc}_4$ *to* SAdj*, and is therefore a locally covariant theory (the* frame-independent classical Dirac theory*).*

Proof. Let $\mathscr{M} = (\boldsymbol{M}, \mathcal{E})$, $\mathscr{M}' = (\boldsymbol{M}', \mathcal{E}')$ and $\mathscr{M}'' = (\boldsymbol{M}'', \mathcal{E}'')$ be objects of $[\mathsf{F}]\mathsf{Loc}_4$ with arrows $[\psi] : \mathscr{M} \to \mathscr{M}'$ and $[\psi'] : \mathscr{M}' \to \mathscr{M}''$, and let $[\iota_{\mathscr{M}}]$ be the identity map on $\mathscr{M}$. For $\boldsymbol{a} = (\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}}$, we have

$$(\mathscr{L}_{FI}([\iota_{\mathscr{M}}])(\boldsymbol{a}))_{\boldsymbol{\varepsilon}} = \mathscr{L}_{\boldsymbol{\varepsilon}}(\boldsymbol{\iota}_{\boldsymbol{\varepsilon}})(\boldsymbol{a}_{\boldsymbol{\varepsilon}}) = \boldsymbol{a}_{\boldsymbol{\varepsilon}} \tag{7.83}$$

for all $\boldsymbol{\varepsilon} \in \mathcal{E}$, so $\mathscr{L}_{FI}([\iota_{\mathscr{M}}]) = \mathrm{id}_{\mathscr{L}_{FI}(\mathscr{M})}$; moreover, for any $\boldsymbol{\varepsilon} \in \mathcal{E}''$,

$$\begin{aligned}(\mathscr{L}_{FI}([\psi'] \circ [\psi])(\boldsymbol{a}))_{\boldsymbol{\varepsilon}} &= \mathscr{L}_{\boldsymbol{\varepsilon}}((\boldsymbol{\psi}' \circ \boldsymbol{\psi})_{\boldsymbol{\varepsilon}})(\boldsymbol{a}_{(\psi'\circ\psi)^\star(\boldsymbol{\varepsilon})}) \\ &= \mathscr{L}_{\boldsymbol{\varepsilon}}(\boldsymbol{\psi}'_{\boldsymbol{\varepsilon}} \circ \boldsymbol{\psi}_{\psi'^\star(\boldsymbol{\varepsilon})})(\boldsymbol{a}_{\psi^\star(\psi'^\star(\boldsymbol{\varepsilon}))}) \\ &= \mathscr{L}_{\boldsymbol{\varepsilon}}(\boldsymbol{\psi}'_{\boldsymbol{\varepsilon}}) \left(\mathscr{L}_{\psi'^\star(\boldsymbol{\varepsilon})}(\boldsymbol{\psi}_{\psi'^\star(\boldsymbol{\varepsilon})})(\boldsymbol{a}_{\psi^\star(\psi'^\star(\boldsymbol{\varepsilon}))})\right) \\ &= \mathscr{L}_{\boldsymbol{\varepsilon}}(\boldsymbol{\psi}'_{\boldsymbol{\varepsilon}}) \left((\mathscr{L}_{FI}([\psi])(\boldsymbol{a}))_{\psi'^\star(\boldsymbol{\varepsilon})}\right) \\ &= \left(\mathscr{L}_{FI}([\psi'])(\mathscr{L}_{FI}([\psi])(\boldsymbol{a}))\right)_{\boldsymbol{\varepsilon}}. \end{aligned} \tag{7.84}$$

Therefore $\mathscr{L}_{FI}([\psi'] \circ [\psi]) = \mathscr{L}_{FI}([\psi']) \circ \mathscr{L}_{FI}([\psi])$ as required. □

Lemma 7.1.19. *There is a natural isomorphism* $p : \mathscr{L}_{FI}\mathscr{E} \stackrel{\cdot}{\longrightarrow} \mathfrak{S}\mathscr{L}$ *that maps an element* $\boldsymbol{a} \in \mathscr{L}_{FI}(\mathscr{E}(\boldsymbol{M}, \boldsymbol{\varepsilon}))$ *to* $p_{\boldsymbol{M},\boldsymbol{\varepsilon}}(\boldsymbol{a}) := \boldsymbol{a}_{\boldsymbol{\varepsilon}} \in \mathfrak{S}(\mathscr{L}(\boldsymbol{M}, \boldsymbol{\varepsilon}))$.[6]

Proof. Naturality of p requires that for any arrow $\boldsymbol{\psi} : (\boldsymbol{M}, \boldsymbol{\varepsilon}) \to (\boldsymbol{N}, \boldsymbol{\varepsilon}')$, we have $p_{\boldsymbol{N},\boldsymbol{\varepsilon}'} \circ \mathscr{L}_{FI}(\mathscr{E}(\boldsymbol{\psi})) = \mathfrak{S}(\mathscr{L}(\boldsymbol{\psi})) \circ p_{\boldsymbol{M},\boldsymbol{\varepsilon}}$; this is evident from Definition 7.1.17. The isomorphism property is a consequence of corollary 7.1.15. □

Thus we have defined a locally covariant theory for the Dirac field that depends only on the spacetime $\boldsymbol{M}$ and a choice of equivalence class for the global frames of $\boldsymbol{M}$. Since the subcategory Loc_4^{sc} of simply connected 4-dimensional spacetimes contains spacetimes with only one equivalence class of frames, it follows that the restriction of $\mathscr{L}_{FI}$ to simply connected spacetimes

[6]The description of the functor $\mathscr{E}$ is given in Definition 6.2.17.

may actually be taken to act directly on Loc_4^{sc}; this is the closest we come to a theory on Loc_4 itself.

However, we do not think it will be possible to remove the dependency of the frame-independent theory on the frame classes in non-simply-connected spacetimes. It is not clear whether the sign change in a spinor field resulting from a 2π rotation would be in any way *observable* in this setting. Some, such as Aharonov and Susskind [1], have suggested that it might be possible to design an experiment that is sensitive to such a rotation, and indeed experiments demonstrating similar behaviour have actually been performed [6, 53, 65]. Nevertheless, there is significant criticism of the argument of Aharonov and Susskind (e.g. [35]), and it is not clear whether the phenomenon observed in the experiments in question is indeed exactly analogous to the sign change of a spinor field under a 2π rotation. However, this sign change is an important feature of the spinorial description of the Dirac field, and thus the description of observed phenomena, whether or not it is itself directly observable.

For example, consider two smooth timelike paths $u_1, u_2 : [0,1] \to F_+^\uparrow \boldsymbol{M}$ through the frame bundle of a topologically nontrivial spacetime $\boldsymbol{M}$ with $u_1(0) = u_2(0)$ and $u_1(1) = u_2(1)$. If the underlying paths through $\boldsymbol{M}$ are homotopic then the rotation of u_1 relative to u_2, up to 4π, can be deduced from $\boldsymbol{M}$ itself. However, if the underlying paths are not homotopic (for example, if they pass to either side of some topological obstruction in $\boldsymbol{M}$) then the possibility remains that the relative rotation of u_1 and u_2 can only be fixed by a choice of equivalence class of global frames on $\boldsymbol{M}$. Since the underlying spinor fields are sensitive to this relative rotation, it follows that the choice of equivalence class is necessary to adequately define the fields in this setting.

We therefore claim that the choice of a global frame in the construction is physically justified, and leave the question of the observability of this frame in a topologically nontrivial spacetime for future work.

7.2 Dynamical locality for the classical Dirac theories

We now move to the question of whether the theories defined in the above section obey the axiom of dynamical locality. We begin by the demonstration of the timeslice axiom and formulation of the relative Cauchy evolution for each theory, following results already shown in [56] for the full quantized theory.

7.2.1 The timeslice axiom

The aim of this subsection is to demonstrate that the theories defined in the previous section ($\mathscr{L}$ and $\mathscr{L}_{FI}$) both obey the timeslice axiom. To do this we will need the following lemma, which is analogous to Lemma 4.1.1 for the scalar field theory.

Lemma 7.2.1. *Let $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$ and consider $u \in \mathrm{Sol}^{s/c}(\mathcal{M})$. If $(\Sigma^{\mathrm{adv}}, \Sigma^{\mathrm{ret}}, \chi^{\mathrm{adv}})$ is a Cauchy partition for $\boldsymbol{M}$, then $P^{s/c}_{\mathcal{M}}(\chi^{\mathrm{adv}} u)$ is compactly supported in $J^+_{\boldsymbol{M}}(\Sigma^{\mathrm{ret}}) \cap J^-_{\boldsymbol{M}}(\Sigma^{\mathrm{adv}})$, and*

$$u = S^{s/c}_{\mathcal{M}} P^{s/c}_{\mathcal{M}}(\chi^{\mathrm{adv}} u). \tag{7.85}$$

Proof. We will provide the proof for spinors; the result for cospinors is exactly analogous. For the support properties note that since $\chi^{\mathrm{adv}} u = 0$ in $J^+_{\boldsymbol{M}}(\Sigma^+)$, we have $(P^s_{\mathcal{M}}(\chi^{\mathrm{adv}} u))(p) = 0$ for $p \in J^+_{\boldsymbol{M}}(\Sigma^+)$; on the other hand, for $p \in J^-_{\boldsymbol{M}}(\Sigma^-)$ we have $(P^s_{\mathcal{M}}(\chi^{\mathrm{adv}} u))(p) = P^s_{\mathcal{M}} u = 0$. Since $P^s_{\mathcal{M}}(\chi^{\mathrm{adv}} u)$ is also supported within $J_{\boldsymbol{M}}(\mathrm{supp}(f))$ the result follows.

Now consider $f \in \mathscr{S}_0(\boldsymbol{M})$ such that $u = S^s_{\mathcal{M}} f$. We have

$$S^s_{\mathcal{M}} P^s_{\mathcal{M}}(\chi^{\mathrm{adv}} u) = S^s_{\mathcal{M}} P^s_{\mathcal{M}}(\chi^{\mathrm{adv}} (S^s_{\mathcal{M}})^- f - \chi^{\mathrm{adv}} (S^s_{\mathcal{M}})^+ f), \tag{7.86}$$

and since $\mathrm{supp}(\chi^{\mathrm{adv}} (S^s_{\mathcal{M}})^+ f) \subset J^-_{\boldsymbol{M}}(\Sigma^{\mathrm{ret}}) \cap J^+_{\boldsymbol{M}}(\mathrm{supp}(f))$, which is compact, we have $S^s_{\mathcal{M}} P^s_{\mathcal{M}}(\chi^{\mathrm{adv}} (S^s_{\mathcal{M}})^+ f) = 0$. Now consider a further Cauchy partition function χ satisfying $\chi(p) = 1$ for all $p \in \mathrm{supp}(f)$. It follows that $\mathrm{supp}(\chi^{\mathrm{adv}} -$

$\chi) \cap J^-_{\boldsymbol{M}}(\mathrm{supp}(f))$ is compact and $\chi(S^s_{\mathcal{M}})^- f = (S^s_{\mathcal{M}})^- f$, and therefore

$$\begin{aligned} S^s_{\mathcal{M}} P^s_{\mathcal{M}}(\chi^{\mathrm{adv}} u) &= S^s_{\mathcal{M}} P^s_{\mathcal{M}}(\chi^{\mathrm{adv}}(S^s_{\mathcal{M}})^- f) \\ &= S^s_{\mathcal{M}} P^s_{\mathcal{M}}((\chi^{\mathrm{adv}} - \chi)(S^s_{\mathcal{M}})^- f + \chi(S^s_{\mathcal{M}})^- f) \\ &= S^s_{\mathcal{M}} P^s_{\mathcal{M}}(S^s_{\mathcal{M}})^- f \\ &= S^s_{\mathcal{M}} f = u. \end{aligned} \tag{7.87}$$

□

Corollary 7.2.2. *Suppose that $O \in \mathscr{O}(\boldsymbol{M})$, and that $u \in \mathscr{L}^{s/c}(\mathcal{M})$ is supported within $J_{\boldsymbol{M}}(K)$ for some $K \in \mathscr{K}(\boldsymbol{M}; O)$. It is then possible to find $f \in \mathscr{S}^{(*)}_0(\boldsymbol{M})$ supported within O such that $u = S^{s/c}_{\mathcal{M}} f$.*

Proof. We pick some multi-diamond set $\tilde{O}$ with base $B \subset O$, such that $K \subset \tilde{O}$; since $J_{\boldsymbol{M}}(K) \subset J_{\boldsymbol{M}}(\tilde{O}) = J_{\boldsymbol{M}}(B)$, it follows that we may find some open neighbourhood $O' \subset \boldsymbol{M}$ of $B \cap J_{\boldsymbol{M}}(K)$ that lies within O. We then define a smooth function χ on $J_{\boldsymbol{M}}(O')$ such that $\chi = 1$ to the past of O', and $\chi = 0$ to the future of O'; we may extend this to a Cauchy partition function on the whole of $\boldsymbol{M}$, and by the previous lemma, we have $u = S^{s/c}_{\mathcal{M}} f$ where $f = P^{s/c}_{\mathcal{M}} \chi u$. But $P^{s/c}_{\mathcal{M}} \chi u = 0$ in any region where χ is constant or $u = 0$, so therefore f is supported within $O' \subset O$. □

We now move directly to the proof of the timeslice axiom for the theory $\mathscr{L} : \mathsf{FLoc}_4 \to \mathsf{HermAdj}_{\mathsf{C}}$.

Proposition 7.2.3. *$\mathscr{L}$ obeys the timeslice axiom.*

Proof. Let $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon})$ and $\mathcal{N} = (\boldsymbol{N}, \boldsymbol{\varepsilon}')$, with an arrow $\boldsymbol{\psi} : \mathcal{M} \to \mathcal{N}$ that covers a Cauchy arrow $\psi : \boldsymbol{M} \to \boldsymbol{N}$. We define maps $\mathscr{K}^{s/c}(\boldsymbol{\psi}) : \mathrm{Sol}^{s/c}(\mathcal{N}) \to \mathrm{Sol}^{s/c}(\mathcal{M})$ by choosing a c.p.f. χ^{adv} for $\boldsymbol{N}$ such that the region in which χ^{adv} is non-constant is contained within $\psi(\boldsymbol{M})$, and letting

$$\mathscr{K}^{s/c}(\boldsymbol{\psi}) u := S^{s/c}_{\mathcal{M}} \psi^*(P^{s/c}_{\mathcal{N}}(\chi^{\mathrm{adv}} u)). \tag{7.88}$$

We claim then that $\mathscr{K}^{s/c}(\boldsymbol{\psi})$ is an inverse to $\mathscr{L}^{s/c}(\boldsymbol{\psi})$ in Herm, and that $\mathscr{K}(\boldsymbol{\psi}) := (\mathscr{K}^s(\boldsymbol{\psi}), \mathscr{K}^c(\boldsymbol{\psi})) = \mathscr{L}(\boldsymbol{\psi})^{-1}$ in $\mathsf{HermAdj}_{\mathsf{C}}$, and consequently, $\mathscr{L}$ obeys the timeslice axiom.

Firstly, $P^{s/c}_{\mathcal{N}}(\chi^{\mathrm{adv}}u)$ is compactly supported within $\psi(\boldsymbol{M})$ by the choice of χ^{adv}, and therefore $\psi^*(P^{s/c}_{\mathcal{N}}(\chi^{\mathrm{adv}}u)) \in \mathscr{S}^{(*)}_0(\boldsymbol{M})$; consequently $\mathscr{K}^{s/c}(\boldsymbol{\psi})$ does map into $\mathrm{Sol}^{s/c}(\boldsymbol{M})$ as required. Next, note that for any $u \in \mathrm{Sol}^{s/c}(\boldsymbol{N})$,

$$\begin{aligned}\mathscr{L}^{s/c}(\boldsymbol{\psi})\mathscr{K}^{s/c}(\boldsymbol{\psi})u &= \mathscr{L}^{s/c}(\boldsymbol{\psi})\left(S^{s/c}_{\mathcal{M}}\psi^*(P^{s/c}_{\mathcal{N}}(\chi^{\mathrm{adv}}u))\right)\\ &= S^{s/c}_{\mathcal{N}}\psi_*\psi^*(P^{s/c}_{\mathcal{N}}(\chi^{\mathrm{adv}}u))\end{aligned} \tag{7.89}$$

by definition, and since $\psi_*\psi^*g = g$ whenever $\mathrm{supp}(g) \subset \psi(\boldsymbol{M})$ we have

$$\mathscr{L}^{s/c}(\boldsymbol{\psi})\mathscr{K}^{s/c}(\boldsymbol{\psi})u = S^{s/c}_{\mathcal{N}}P^{s/c}_{\mathcal{N}}(\chi^{\mathrm{adv}}u) = u. \tag{7.90}$$

We also therefore have $\mathscr{L}^{s/c}(\boldsymbol{\psi})\circ\mathscr{K}^{s/c}(\boldsymbol{\psi})\circ\mathscr{L}^{s/c}(\boldsymbol{\psi}) = \mathscr{L}^{s/c}(\boldsymbol{\psi}) = \mathscr{L}^{s/c}(\boldsymbol{\psi})\circ \mathrm{id}$. Since $\mathscr{L}^{s/c}(\boldsymbol{\psi})$ is injective, and therefore monic, it follows that $\mathscr{K}^{s/c}(\boldsymbol{\psi})\circ \mathscr{L}^{s/c}(\boldsymbol{\psi}) = \mathrm{id}$. Therefore $\mathscr{K}^{s/c}(\boldsymbol{\psi}) = \mathscr{L}^{s/c}(\boldsymbol{\psi})^{-1}$, and consequently $\mathscr{K}(\boldsymbol{\psi}) = \mathscr{L}(\boldsymbol{\psi})^{-1}$. Since $\mathscr{L}^{s/c}(\boldsymbol{\psi})$ is an arrow in Herm and $\mathscr{L}(\boldsymbol{\psi})$ is an arrow in $\mathsf{HermAdj}_{\mathsf{C}}$, the same is true for $\mathscr{K}^{s/c}(\boldsymbol{\psi})$ and $\mathscr{K}(\boldsymbol{\psi})$ respectively. □

Thus $\mathscr{L}$ obeys the timeslice axiom, and it does not require much extra work to prove the same for $\mathscr{L}_{FI}$, since its arrows are built componentwise out of $\mathscr{L}$ arrows. However, before we prove this, we will show that the theory $\mathscr{L}$ is causal:

Proposition 7.2.4. *$\mathscr{L}$ is causal — by which we mean that each of $\mathscr{L}^{s/c}$ is causal, in the sense of* (2.3)*.*

Proof. Suppose that $\boldsymbol{\psi}_1 : \mathcal{M}_1 \to \mathcal{N}$, $\boldsymbol{\psi}_2 : \mathcal{M}_2 \to \mathcal{N}$ are arrows in FLoc_4, where $\mathcal{M}_i = (\boldsymbol{M}_i, \boldsymbol{\varepsilon})$ and $\mathcal{N} = (\boldsymbol{N}, \boldsymbol{\varepsilon}')$. Suppose also that $\psi_1(\boldsymbol{M}_1)$ and $\psi_2(\boldsymbol{M}_2)$ are causally disjoint in $\boldsymbol{N}$. Consider spinor solutions $u_i = S^s_{\mathcal{M}_i}f_i \in \mathscr{L}^s(\mathcal{M}_i)$; writing $f'_i = (\psi_i)_*f_i$, we have $\mathscr{L}(\boldsymbol{\psi}_i)u_i = S^s_{\mathcal{N}}f'_i$, and

$$s^s_{\mathcal{N}}(\mathscr{L}(\boldsymbol{\psi}_1)u_1, \mathscr{L}(\boldsymbol{\psi}_2)u_2) = i\int_{\boldsymbol{N}} dvol_{\boldsymbol{N}}\,\langle\overline{A_{\boldsymbol{N}}f'_1}, S^s_{\mathcal{N}}f'_2\rangle = 0, \tag{7.91}$$

since $\mathrm{supp}(\overline{A_{\boldsymbol{N}}f'_1}) \cap \mathrm{supp}(S^s_{\mathcal{N}}f'_2) \subset \psi(\boldsymbol{M}_1) \cap J_{\boldsymbol{N}}(\psi(\boldsymbol{M}_2)) = \emptyset$. The result for cospinors may be proved in a similar way. □

Proposition 7.2.5. *$\mathscr{L}_{FI}$ obeys the timeslice axiom.*

Proof. Let $\mathscr{M} = (\boldsymbol{M}, \mathcal{E})$ and $\mathscr{N} = (\boldsymbol{N}, \mathcal{E}')$ be objects of $\mathsf{[F]Loc}_4$, with a Cauchy arrow $[\psi] : \mathscr{M} \to \mathscr{N}$. Let $\boldsymbol{a} = (\boldsymbol{a}_{\boldsymbol{\varepsilon}'})_{\boldsymbol{\varepsilon}' \in \mathcal{E}'} \in \mathscr{L}_{FI}(\mathscr{N})$; we pick some $\boldsymbol{\varepsilon}' \in \mathcal{E}'$ and define

$$\boldsymbol{b}_{\psi^\star(\boldsymbol{\varepsilon}')} := (\mathscr{K}^s(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'}) \oplus \mathscr{K}^c(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'}))^{\otimes 2} \boldsymbol{a}_{\boldsymbol{\varepsilon}'}. \tag{7.92}$$

Now, for an arbitrary $\boldsymbol{\varepsilon} \in \mathcal{E}$, we define

$$\boldsymbol{b}_{\boldsymbol{\varepsilon}} := \mathfrak{s}_{\psi^\star(\boldsymbol{\varepsilon}'),\boldsymbol{\varepsilon}} \boldsymbol{b}_{\psi^\star(\boldsymbol{\varepsilon}')}, \tag{7.93}$$

where we recall that $\mathfrak{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'} := \mathfrak{S}(\alpha^s_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}, \alpha^c_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'})$, where $\alpha^{s/c}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}$ is defined according to (7.64) with relation to some spin cocycle $\boldsymbol{s}$ (although $\mathfrak{s}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}$ is independent of this choice). Then the family $(\boldsymbol{b}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon} \in \mathcal{E}}$ is an element of $\mathscr{L}_{FI}(\mathscr{M})$, and moreover is independent of the choice of $\boldsymbol{\varepsilon}'$; if we instead chose $\boldsymbol{\varepsilon}'' \in \mathcal{E}'$, we would have

$$\begin{aligned}
\boldsymbol{b}_{\boldsymbol{\varepsilon}} &= \mathfrak{s}_{\psi^\star(\boldsymbol{\varepsilon}''),\boldsymbol{\varepsilon}} (\mathscr{K}^s(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}''}) \oplus \mathscr{K}^c(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}''}))^{\otimes 2} \boldsymbol{a}_{\boldsymbol{\varepsilon}''} \\
&= \mathfrak{s}_{\psi^\star(\boldsymbol{\varepsilon}''),\boldsymbol{\varepsilon}} (\mathscr{K}^s(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}''}) \oplus \mathscr{K}^c(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}''}))^{\otimes 2} \mathfrak{s}_{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}''} \boldsymbol{a}_{\boldsymbol{\varepsilon}'} \\
&= \mathfrak{s}_{\psi^\star(\boldsymbol{\varepsilon}''),\boldsymbol{\varepsilon}} \mathfrak{s}_{\psi^\star(\boldsymbol{\varepsilon}'),\psi^\star(\boldsymbol{\varepsilon}'')} (\mathscr{K}^s(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'}) \oplus \mathscr{K}^c(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'}))^{\otimes 2} \boldsymbol{a}_{\boldsymbol{\varepsilon}'} \\
&= \mathfrak{s}_{\psi^\star(\boldsymbol{\varepsilon}'),\boldsymbol{\varepsilon}} (\mathscr{K}^s(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'}) \oplus \mathscr{K}^c(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'}))^{\otimes 2} \boldsymbol{a}_{\boldsymbol{\varepsilon}'}
\end{aligned} \tag{7.94}$$

as required, where we have used the fact that since $\mathscr{K}^{s/c}(\boldsymbol{\psi}) = \mathscr{L}^{s/c}(\boldsymbol{\psi})^{-1}$, we have $\mathscr{K}^{s/c}(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}''}) \circ \alpha^{s/c}_{\boldsymbol{\varepsilon}',\boldsymbol{\varepsilon}''} = \alpha^{s/c}_{\psi^\star(\boldsymbol{\varepsilon}'),\psi^\star(\boldsymbol{\varepsilon}'')} \circ \mathscr{K}^{s/c}(\boldsymbol{\psi}_{\boldsymbol{\varepsilon}'})$. We define

$$\mathscr{K}_{FI}([\psi])\boldsymbol{a} := (\boldsymbol{b}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon} \in \mathcal{E}}, \tag{7.95}$$

whereupon it is clear that $\mathscr{K}_{FI}([\psi]) = \mathscr{L}_{FI}([\psi])^{-1}$. Therefore $\mathscr{L}_{FI}$ obeys the timeslice axiom. □

Causality properties similar to that proved for $\mathscr{L}$ may be analogously demonstrated for $\mathscr{L}_{FI}$.

7.2.2 The relative Cauchy evolution

We proceed to the calculation of the relative Cauchy evolution for the classical Dirac theories we have defined. The calculation for an already-quantized version of this theory is done in [55, 56], where the expression in (7.96) is derived as part of the process. We will unsurprisingly find a similarity when we derive the r.c.e. for our theories, but here we do examine a previously unnoticed subtlety in defining a perturbed frame on $\boldsymbol{M}[\boldsymbol{h}]$ that potentially causes an ambiguity in the relative Cauchy evolution under certain circumstances.

Throughout this subsection, $\boldsymbol{M}$ will be a spacetime in Loc_4, $\boldsymbol{h}$ will be a smooth metric perturbation in $H(\boldsymbol{M})$ and $\boldsymbol{N}^\pm \subset \boldsymbol{M}$ will be subspacetimes of $\boldsymbol{M}$ such that:

- each $\boldsymbol{N}^\pm$ is an object of Loc, and their embeddings $\iota^\pm$ into $\boldsymbol{M}$ are arrows in Loc,
- each $\boldsymbol{N}^\pm$ admits a Cauchy partition $(\Sigma^{\mathrm{adv}}_\pm, \Sigma^{\mathrm{ret}}_\pm, \chi^{\mathrm{adv}}_\pm)$ for $\boldsymbol{M}$.

As usual we denote the embeddings of $\boldsymbol{N}^\pm$ into $\boldsymbol{M}[\boldsymbol{h}]$ by $\iota^\pm[\boldsymbol{h}]$. Let $\boldsymbol{\varepsilon} \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$, and denote $\boldsymbol{\varepsilon}_\pm = \boldsymbol{\varepsilon}|_{\boldsymbol{N}^\pm}$, so that $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon})$, $\mathcal{N}^\pm = (\boldsymbol{N}^\pm, \boldsymbol{\varepsilon}_\pm)$ are objects of FLoc_4. We need to define the perturbed FLoc_4-spacetime $\mathcal{M}[\boldsymbol{h}]$, and this proves slightly more difficult than in Loc, since an object of FLoc_4 includes a choice of global frames. Of course, global frames for $\boldsymbol{M}$ are not global frames for $\boldsymbol{M}[\boldsymbol{h}]$ unless $\boldsymbol{h} = 0$, but we may always choose a frame $\tilde{\boldsymbol{\varepsilon}}$ for $\boldsymbol{M}[\boldsymbol{h}]$ that coincides with $\boldsymbol{\varepsilon}$ outside $\mathrm{supp}(\boldsymbol{h})$[7] (although there is unfortuately no canonical way of doing this). In most cases, it is enough to pick one such frame $\tilde{\boldsymbol{\varepsilon}}$, define $\mathcal{M}[\boldsymbol{h}] = (\boldsymbol{M}[\boldsymbol{h}], \tilde{\boldsymbol{\varepsilon}})$, and define the relative Cauchy evolution accordingly. However, in some particularly pathological cases, this introduces a sign ambiguity into the definition of the r.c.e., as we will show shortly.

[7]The vector field ε_0 is always future-pointing and normalized according to the metric $\boldsymbol{g}$ of $\boldsymbol{M}$. This is not necessarily the case for $\boldsymbol{g}+\boldsymbol{h}$; however, since $\boldsymbol{h} = 0$ on $\partial(\mathrm{supp}(\boldsymbol{h}))$ and the future light cone at any point is open it follows that there is an open neighbourhood $O \supset \partial(\mathrm{supp}(\boldsymbol{h}))$ on which ε_0 is at least future pointing. We choose a normalized future-pointing vector field a on $\boldsymbol{M}[\boldsymbol{h}]$ that is linearly independent of $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and let $\xi : \boldsymbol{M} \to [0, 1]$ be a smooth function with $\xi = 1$ on $\mathrm{supp}(\boldsymbol{h}) \setminus O$ and $\xi = 0$ outside $\mathrm{supp}(\boldsymbol{h})$. We then define $\tilde{\varepsilon}_0 := v/\|v\|$, where

$$v := (1 - \xi)\varepsilon_0 + \xi a.$$

Since $\tilde{\varepsilon}_0 = a$ in $\mathrm{supp}(\boldsymbol{h})\setminus O$ and $\tilde{\varepsilon}_0 = \varepsilon_0$ outside $\mathrm{supp}(\boldsymbol{h})$, and the convex linear combination of two future-pointing vectors is future-pointing, it follows that $\tilde{\varepsilon}_0$ is future-pointing and normalized as required. We then apply a Gram-Schmidt-type process to $\tilde{\varepsilon}_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$, which will produce a global frame for $\boldsymbol{M}[\boldsymbol{h}]$ that coincides with $\boldsymbol{\varepsilon}$ outside $\mathrm{supp}(\boldsymbol{h})$.

Happily, this issue may be resolved by imposing a slightly stronger condition on the choice of $\tilde{\boldsymbol{\varepsilon}}$.

We begin by calculating the expression for the relative Cauchy evolution generated by the perturbation $\boldsymbol{h}$ and a particular choice of $\tilde{\boldsymbol{\varepsilon}} \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M}[\boldsymbol{h}])$ that coincides with $\boldsymbol{\varepsilon}$ outside $\mathrm{supp}(\boldsymbol{h})$, denoting $\mathcal{M}[\boldsymbol{h}] = (\boldsymbol{M}[\boldsymbol{h}], \tilde{\boldsymbol{\varepsilon}})$. Note that the embeddings $\boldsymbol{\iota}^\pm$, $\boldsymbol{\iota}^\pm[\boldsymbol{h}]$ covering $\iota^\pm$, $\iota^\pm[\boldsymbol{h}]$ respectively are arrows in FLoc_4. It is then a simple task to calculate $\mathrm{rce}_{\mathcal{M}}[\boldsymbol{h}] = (\mathrm{rce}^s_{\mathcal{M}}[\boldsymbol{h}], \mathrm{rce}^c_{\mathcal{M}}[\boldsymbol{h}]) : \mathscr{L}(\mathcal{M}) \to \mathscr{L}(\mathcal{M})$; note that $\mathscr{L}^{s/c}(\boldsymbol{\iota}^+[\boldsymbol{h}])\mathscr{K}^{s/c}(\boldsymbol{\iota}^+)u = S^{s/c}_{\mathcal{M}[\boldsymbol{h}]} P^{s/c}_{\mathcal{M}} \chi^{\mathrm{adv}}_+ u$ and $\mathscr{L}^{s/c}(\boldsymbol{\iota}^-)\mathscr{K}^{s/c}(\boldsymbol{\iota}^-[\boldsymbol{h}])v = S^{s/c}_{\mathcal{M}} P^{s/c}_{\mathcal{M}[\boldsymbol{h}]} \chi^{\mathrm{adv}}_- v$, so

$$\mathrm{rce}^{s/c}_{\mathcal{M}}[\boldsymbol{h}]\, u = S^{s/c}_{\mathcal{M}} P^{s/c}_{\mathcal{M}[\boldsymbol{h}]} \chi^{\mathrm{adv}}_- S^{s/c}_{\mathcal{M}[\boldsymbol{h}]} P^{s/c}_{\mathcal{M}} \chi^{\mathrm{adv}}_+ u \tag{7.96}$$

for $u \in \mathscr{L}^{s/c}(\mathcal{M})$ (cf. [56, Prop. 4.15]; note the difference in sign convention for $\boldsymbol{N}^\pm$).

It is easy to show that while keeping $\tilde{\boldsymbol{\varepsilon}}$ fixed, the expression in (7.96) is independent of the choice of $\boldsymbol{N}^\pm$ and $\chi^{\mathrm{adv}}_\pm$, as long as $\boldsymbol{N}^\pm$ are kept disjoint from $\mathrm{supp}(\boldsymbol{h})$, by Lemma 2.1.7. However, for the relative Cauchy evolution to be well-defined, (7.96) must also be independent of the choice of $\tilde{\boldsymbol{\varepsilon}}$, subject to the restriction that it coincides with $\boldsymbol{\varepsilon}$ outside $\mathrm{supp}(\boldsymbol{h})$. Clearly, (7.96) seems to be dependent, through the appearance of $P^{s/c}_{\mathcal{M}[\boldsymbol{h}]}$ and $S^{s/c}_{\mathcal{M}[\boldsymbol{h}]}$, on the choice of $\tilde{\boldsymbol{\varepsilon}}$; as mentioned before, we will show that there are circumstances where this creates a potential sign ambiguity.

Suppose that $\boldsymbol{M}$ has compact Cauchy surfaces, and that the interior of $\mathrm{supp}(\boldsymbol{h})$ contains a Cauchy surface for $\boldsymbol{M}$. It follows that there is a map $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$ which is identically $\mathbf{1}_{\mathrm{Spin}^0_{1,3}}$ to the future of $\mathrm{supp}(\boldsymbol{h})$, and identically $-\mathbf{1}_{\mathrm{Spin}^0_{1,3}}$ to the past of $\mathrm{supp}(\boldsymbol{h})$. Consequently $\tilde{\boldsymbol{\varepsilon}}' = \tilde{\boldsymbol{\varepsilon}}\Lambda(s)$ coincides with $\tilde{\boldsymbol{\varepsilon}}$, and therefore $\boldsymbol{\varepsilon}$, outside $\mathrm{supp}(\boldsymbol{h})$, and so we may equally use $\tilde{\boldsymbol{\varepsilon}}'$ for the definition of the relative Cauchy evolution. Using this map s, we construct the map $\alpha^{s/c}_{\tilde{\boldsymbol{\varepsilon}},\tilde{\boldsymbol{\varepsilon}}'}$ as in (7.64); this allows us to compare the results of using $\tilde{\boldsymbol{\varepsilon}}$ and $\tilde{\boldsymbol{\varepsilon}}'$ to construct the relative Cauchy evolution. We write $f = P^{s/c}_{\mathcal{M}} \chi^{\mathrm{adv}}_+ u$, which is supported within $\boldsymbol{N}^+$; since $\alpha^{s/c}_{\tilde{\boldsymbol{\varepsilon}},\tilde{\boldsymbol{\varepsilon}}'}$ acts pointwise we have $\alpha^{s/c}_{\tilde{\boldsymbol{\varepsilon}},\tilde{\boldsymbol{\varepsilon}}'} f = f$. We also write $\mathcal{M}[\boldsymbol{h}]' = (\boldsymbol{M}[\boldsymbol{h}], \tilde{\boldsymbol{\varepsilon}}')$ and invoke Lemma 7.1.5 and (7.69) to see that

$$\begin{aligned}
S^{s/c}_{\mathcal{M}} P^{s/c}_{\mathcal{M}[\boldsymbol{h}]'} \chi^{\mathrm{adv}}_{-} S^{s/c}_{\mathcal{M}[\boldsymbol{h}]'} f &= S^{s/c}_{\mathcal{M}} P^{s/c}_{\mathcal{M}[\boldsymbol{h}]'} \chi^{\mathrm{adv}}_{-} S^{s/c}_{\mathcal{M}[\boldsymbol{h}]'} \alpha^{s/c}_{\tilde{\boldsymbol{\varepsilon}},\tilde{\boldsymbol{\varepsilon}}'} f \\
&= S^{s/c}_{\mathcal{M}} P^{s/c}_{\mathcal{M}[\boldsymbol{h}]'} \alpha^{s/c}_{\tilde{\boldsymbol{\varepsilon}},\tilde{\boldsymbol{\varepsilon}}'} \chi^{\mathrm{adv}}_{-} S^{s/c}_{\mathcal{M}[\boldsymbol{h}]} f \\
&= S^{s/c}_{\mathcal{M}} \alpha^{s/c}_{\tilde{\boldsymbol{\varepsilon}},\tilde{\boldsymbol{\varepsilon}}'} P^{s/c}_{\mathcal{M}[\boldsymbol{h}]} \chi^{\mathrm{adv}}_{-} S^{s/c}_{\mathcal{M}[\boldsymbol{h}]} f. \qquad (7.97)
\end{aligned}$$

But since $P^{s/c}_{\mathcal{M}[\boldsymbol{h}]} \chi^{\mathrm{adv}}_{-} S^{s/c}_{\mathcal{M}[\boldsymbol{h}]} f$ is supported within $\boldsymbol{N}^-$, it changes sign by the action of $\alpha^{s/c}_{\tilde{\boldsymbol{\varepsilon}},\tilde{\boldsymbol{\varepsilon}}'}$, and so

$$S^{s/c}_{\mathcal{M}} P^{s/c}_{\mathcal{M}[\boldsymbol{h}]'} \chi^{\mathrm{adv}}_{-} S^{s/c}_{\mathcal{M}[\boldsymbol{h}]'} f = -S^{s/c}_{\mathcal{M}} P^{s/c}_{\mathcal{M}[\boldsymbol{h}]} \chi^{\mathrm{adv}}_{-} S^{s/c}_{\mathcal{M}[\boldsymbol{h}]} f. \qquad (7.98)$$

It is not enough, therefore, to simply demand that $\tilde{\boldsymbol{\varepsilon}}$ coincides with $\boldsymbol{\varepsilon}$ outside $\mathrm{supp}(\boldsymbol{h})$, given the possibility of compactly supported metric perturbations that nevertheless contain a Cauchy surface within their support.[8] Instead, we demand that there exists a homotopy between $\boldsymbol{\varepsilon}$ and $\tilde{\boldsymbol{\varepsilon}}$, regarded as ordered bases of $T\boldsymbol{M}$, that is fixed and equal to $\boldsymbol{\varepsilon}$ outside $\mathrm{supp}(\boldsymbol{h})$.

Lemma 7.2.6. *Let $\boldsymbol{h} \in H(\boldsymbol{M})$, and consider $\boldsymbol{\varepsilon} \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M})$ and $\tilde{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\varepsilon}}' \in \Gamma^\infty(F^\uparrow_+\boldsymbol{M}[\boldsymbol{h}])$. Suppose that there exist homotopies between $\boldsymbol{\varepsilon}$ and both $\tilde{\boldsymbol{\varepsilon}}$ and $\tilde{\boldsymbol{\varepsilon}}'$ that are fixed and equal to $\boldsymbol{\varepsilon}$ outside* $\mathrm{supp}(\boldsymbol{h})$. *Then, denoting $\mathcal{M}[\boldsymbol{h}] = (\boldsymbol{M}[\boldsymbol{h}], \tilde{\boldsymbol{\varepsilon}})$ and $\mathcal{M}[\boldsymbol{h}]' = (\boldsymbol{M}[\boldsymbol{h}], \tilde{\boldsymbol{\varepsilon}}')$,*

$$S^{s/c}_{\mathcal{M}} P^{s/c}_{\mathcal{M}[\boldsymbol{h}]} \chi^{\mathrm{adv}}_{-} S^{s/c}_{\mathcal{M}[\boldsymbol{h}]} P^{s/c}_{\mathcal{M}} \chi^{\mathrm{adv}}_{+} u = S^{s/c}_{\mathcal{M}} P^{s/c}_{\mathcal{M}[\boldsymbol{h}]'} \chi^{\mathrm{adv}}_{-} S^{s/c}_{\mathcal{M}[\boldsymbol{h}]'} P^{s/c}_{\mathcal{M}} \chi^{\mathrm{adv}}_{+} u \qquad (7.99)$$

for any $u \in \mathscr{L}^{s/c}(\mathcal{M})$, where $\chi^{\mathrm{adv}}_{\pm}$ are Cauchy partition functions that are nonconstant in some subspactimes $\boldsymbol{N}^\pm \subset \boldsymbol{M}$ defined as in the beginning of this subsection.

Proof. Clearly $\tilde{\boldsymbol{\varepsilon}}$ and $\tilde{\boldsymbol{\varepsilon}}'$ are themselves homotopic, so there is a continuous map $L : \boldsymbol{M} \times [0,1] \to \mathcal{L}^\uparrow_+$ with $L(\,\cdot\,, 0) = \mathbf{1}_{\mathcal{L}^\uparrow_+}$ and $\tilde{\boldsymbol{\varepsilon}}'(p) = \tilde{\boldsymbol{\varepsilon}}(p) L(p,1)$. By the defining properties of $\tilde{\boldsymbol{\varepsilon}}$ and $\tilde{\boldsymbol{\varepsilon}}'$ we may take this L to be fixed and equal to $\mathbf{1}_{\mathcal{L}^\uparrow_+}$ outside $\mathrm{supp}(\boldsymbol{h})$.

Since the constant map $\mathbf{1}_{\mathcal{L}^\uparrow_+}$ is trivially liftable to $\mathrm{Spin}^0_{1,3}$ under Λ, and any covering map has the homotopy lifting property [58, Thm. 2.2.3], there

[8] This point seems to have been overlooked in [56].

must be a homotopy $S : \boldsymbol{M} \times [0,1] \to \mathrm{Spin}^0_{1,3}$ satisfying $L = \Lambda \circ S$. Now $L(\,\cdot\,,0) = \mathbf{1}_{\mathcal{L}^\uparrow_+}$, so we may take $S(\,\cdot\,,0) = \mathbf{1}_{\mathrm{Spin}^0_{1,3}}$. But since $\Lambda(p,\tau) = \mathbf{1}_{\mathcal{L}^\uparrow_+}$ for all $\tau \in [0,1]$ and $p \notin \mathrm{supp}(\boldsymbol{h})$, it follows by continuity that S is fixed and equal to $\mathbf{1}_{\mathrm{Spin}^0_{1,3}}$ outside $\mathrm{supp}(\boldsymbol{h})$. Consequently there exists a smooth $s : \boldsymbol{M} \to \mathrm{Spin}^0_{1,3}$, given by $s = S(\,\cdot\,,1)$, such that $\tilde{\boldsymbol{\varepsilon}}' = \tilde{\boldsymbol{\varepsilon}}\Lambda(s)$; outside $\mathrm{supp}(\boldsymbol{h})$, we must have $s = \mathbf{1}_{\mathrm{Spin}^0_{1,3}}$.

Again, we use this s to construct the map $\alpha^{s/c}_{\tilde{\boldsymbol{\varepsilon}},\tilde{\boldsymbol{\varepsilon}}'}$ as in (7.64), and write $f = P^{s/c}_{\mathcal{M}} \chi^{\mathrm{adv}}_+ u$. The argument used to show (7.97) still holds here, but this time $\alpha^{s/c}_{\tilde{\boldsymbol{\varepsilon}},\tilde{\boldsymbol{\varepsilon}}'}$ must act as the identity outside $\mathrm{supp}(\boldsymbol{h})$, and so (7.99) holds. □

This gives us an expression for the r.c.e. for $\mathscr{L}$; we use this to find the r.c.e. for $\mathscr{L}_{FI}$. For $\mathscr{M} = (\boldsymbol{M}, \mathcal{E}) \in \mathsf{[F]Loc}_4$ we may denote by $\mathcal{E}|_{\boldsymbol{N}^\pm}$ the unique equivalence class of frames on $\boldsymbol{N}^\pm$ that contains the restriction of frames in $\mathcal{E}$; we have $\mathscr{N}^\pm := (\boldsymbol{N}^\pm, \mathcal{E}|_{\boldsymbol{N}^\pm}) = (\boldsymbol{N}^\pm, \mathcal{E}[\boldsymbol{h}]|_{\boldsymbol{N}^\pm})$, and may form the relative Cauchy evolution by

$$\mathrm{rce}_{\mathscr{M}}[\boldsymbol{h}] := \mathscr{L}_{FI}([\iota^-]) \circ \mathscr{K}_{FI}([\iota^-[\boldsymbol{h}]]) \circ \mathscr{L}_{FI}([\iota^+[\boldsymbol{h}]]) \circ \mathscr{K}_{FI}([\iota^+]). \tag{7.100}$$

Since $p : \mathscr{L}_{FI}\mathscr{E} \stackrel{\cdot}{\longrightarrow} \mathfrak{S}\mathscr{L}$ is natural, we have

$$p_{\boldsymbol{M},\boldsymbol{\varepsilon}} \circ \mathrm{rce}_{\mathscr{M}}[\boldsymbol{h}] = \mathfrak{S}(\mathrm{rce}^s_{(\boldsymbol{M},\boldsymbol{\varepsilon})}[\boldsymbol{h}], \mathrm{rce}^c_{(\boldsymbol{M},\boldsymbol{\varepsilon})}[\boldsymbol{h}]) \circ p_{\boldsymbol{M},\boldsymbol{\varepsilon}}, \tag{7.101}$$

so for any $\boldsymbol{a} = (\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon}\in\mathcal{E}}$ we have

$$(\mathrm{rce}_{\mathscr{M}}[\boldsymbol{h}]\,\boldsymbol{a})_{\boldsymbol{\varepsilon}} = (\mathrm{rce}^s_{(\boldsymbol{M},\boldsymbol{\varepsilon})}[\boldsymbol{h}] \oplus \mathrm{rce}^c_{(\boldsymbol{M},\boldsymbol{\varepsilon})}[\boldsymbol{h}])^{\otimes 2} \boldsymbol{a}_{\boldsymbol{\varepsilon}}. \tag{7.102}$$

7.2.3 The dynamical solution spaces

We are now, finally, in a position to calculate the dynamical solution spaces for the classical Dirac theories. We start with the theory $\mathscr{L}$; given $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$, an open subregion $O \in \mathscr{O}(\boldsymbol{M})$ and compact $K \in \mathscr{K}(\boldsymbol{M}; O)$, the space $\mathscr{L}^\bullet(\mathcal{M}; K) = \mathscr{L}^\bullet_s(\mathcal{M}; K) \oplus \mathscr{L}^\bullet_c(\mathcal{M}; K)$ comprises solutions $u \in \mathrm{Sol}^s(\mathcal{M})$ and $v \in \mathrm{Sol}^c(\mathcal{M})$ such that

$$\mathrm{rce}^s_{\mathcal{M}}[\boldsymbol{h}]u = u, \qquad \mathrm{rce}^c_{\mathcal{M}}[\boldsymbol{h}]v = v, \tag{7.103}$$

for any $\boldsymbol{h} \in H(\boldsymbol{M}; K^{\perp})$. We now show that the (co)spinor solutions supported within $J_{\boldsymbol{M}}(K)$ include all those that comprise the spaces $\mathscr{L}^{\bullet}_{s/c}(\mathcal{M}; K)$. In exact parallel to the argument given in the proof of Lemma 4.1.6 in Appendix C, we may see that for sufficiently small $s \in \mathbb{R}$

$$\mathrm{rce}^{s/c}_{\mathcal{M}}[s\boldsymbol{h}]u - u = -S^{s/c}_{\mathcal{M}}(P^{s/c}_{\mathcal{M}[s\boldsymbol{h}]} - P^{s/c}_{\mathcal{M}})u + \mathcal{O}(s^2), \tag{7.104}$$

(cf. [56, Prop. 4.16]), and so if $u \in \mathscr{L}^{\bullet}_{s/c}(\mathcal{M}; K)$, we have

$$F^{s/c}_{\mathcal{M}}[\boldsymbol{h}]u := \frac{d}{ds}\mathrm{rce}^{s/c}_{\mathcal{M}}[s\boldsymbol{h}]u\bigg|_{s=0} = -\frac{d}{ds}S^{s/c}_{\mathcal{M}}P^{s/c}_{\mathcal{M}[s\boldsymbol{h}]}u\bigg|_{s=0} = 0. \tag{7.105}$$

The proof of the following lemma is tedious and long-winded, and has therefore been relegated to Appendix D; there are also proofs of very similar results contained in [55, §4.3.2].

Lemma 7.2.7. *For any $u \in \mathscr{L}^s(\mathcal{M})$ and $v \in \mathscr{L}^c(\mathcal{M})$, we have*

$$\begin{aligned}\frac{d}{ds}S^s_{\mathcal{M}}P^s_{\mathcal{M}[s\boldsymbol{h}]}u\bigg|_{s=0} &= S^s_{\mathcal{M}}\left(\frac{i}{2}h^{ab}\boldsymbol{\gamma}_a\nabla_b u - \frac{i}{4}(\delta_{\boldsymbol{h}}\Gamma^{\mu}{}_{\nu\rho})\varepsilon^b_{\mu}\varepsilon^{\nu}_a\varepsilon^{\rho}_c\,\boldsymbol{\gamma}^a\boldsymbol{\gamma}_b\boldsymbol{\gamma}^c u\right),\\ \frac{d}{ds}S^c_{\mathcal{M}}P^c_{\mathcal{M}[s\boldsymbol{h}]}v\bigg|_{s=0} &= S^c_{\mathcal{M}}\left(-\frac{i}{2}h^{ab}(\nabla_b v)\boldsymbol{\gamma}_a - \frac{i}{4}(\delta_{\boldsymbol{h}}\Gamma^{\mu}{}_{\nu\rho})\varepsilon^b_{\mu}\varepsilon^{\nu}_a\varepsilon^{\rho}_c\,v\boldsymbol{\gamma}^a\boldsymbol{\gamma}_b\boldsymbol{\gamma}^c\right),\end{aligned} \tag{7.106}$$

where $\delta_{\boldsymbol{h}}\Gamma^{\mu}{}_{\nu\rho} := \frac{d}{ds}(\Gamma_{\boldsymbol{g}+s\boldsymbol{h}})^{\mu}{}_{\nu\rho}\big|_{s=0}$. Moreover, for $u' \in \mathscr{L}^s(\mathcal{M})$ and $v' \in \mathscr{L}^c(\mathcal{M})$, it also holds that

$$\begin{aligned}s^s_{\mathcal{M}}(u', F^s_{\mathcal{M}}[\boldsymbol{h}]u) &= -\frac{1}{4}\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{ab}\left(\langle\overline{A_{\boldsymbol{M}}u'}, \boldsymbol{\gamma}^{(a}\nabla^{b)}u\rangle - \langle\overline{A_{\boldsymbol{M}}\nabla^{(a}u'}, \boldsymbol{\gamma}^{b)}u\rangle\right),\\ s^c_{\mathcal{M}}(v', F^c_{\mathcal{M}}[\boldsymbol{h}]v) &= -\frac{1}{4}\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{ab}\left(\langle\nabla^{(a}v\boldsymbol{\gamma}^{b)}, A^{-1}_{\boldsymbol{M}}\overline{v'}\rangle - \langle v\boldsymbol{\gamma}^{(a}, A^{-1}_{\boldsymbol{M}}\overline{\nabla^{b)}v'}\rangle\right).\end{aligned} \tag{7.107}$$

This allows us to prove the following property of $\mathscr{L}^{\bullet}_{s/c}(\mathcal{M}; K)$.

Proposition 7.2.8. *Let $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$, and $K \in \mathscr{K}(\boldsymbol{M})$. If $u \in \mathscr{L}^{\bullet}_{s/c}(\mathcal{M}; K)$ then* $\mathrm{supp}(u) \subset J_{\boldsymbol{M}}(K)$.

Proof. Let $u \in \mathscr{L}_s^\bullet(\mathcal{M}; K)$. As remarked earlier, we know that $F^s_{\boldsymbol{M}}[\boldsymbol{h}]u = 0$ for all $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$, and therefore by the above lemma we have

$$\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{ab} \left(\langle \overline{A_{\boldsymbol{M}} u'}, \boldsymbol{\gamma}^{(a} \nabla^{b)} u \rangle - \langle \overline{A_{\boldsymbol{M}} \nabla^{(a} u'}, \boldsymbol{\gamma}^{b)} u \rangle \right) = 0 \qquad (7.108)$$

for all such $\boldsymbol{h}$ and all $u' \in \mathscr{L}^s(\mathcal{M})$. It follows that within $K^\perp$, the integrand is identically zero, and in particular,

$$\langle \overline{A_{\boldsymbol{M}} u'}, \boldsymbol{\gamma}^0 \nabla^0 u \rangle - \langle \overline{A_{\boldsymbol{M}} \nabla^0 u'}, \boldsymbol{\gamma}^0 u \rangle = 0. \qquad (7.109)$$

We pick some $p \in K^\perp$; since u' can be chosen such that $u'(p) = 0$ and $\nabla^0 u'(p)$ is arbitrary, it follows that $u(p) = 0$. Therefore $u = 0$ everywhere in $K^\perp$, so $\mathrm{supp}(u) \subset J_{\boldsymbol{M}}(K)$. This shows that $\mathscr{L}_s^\bullet(\mathcal{M}; K) \subset \{u \in \mathscr{L}^s(\mathcal{M}) : \mathrm{supp}(u) \subset J_{\boldsymbol{M}}(K)\}$. The proof for cospinors is similar. □

Corollary 7.2.9. *Let $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$, and $O \in \mathscr{O}(\boldsymbol{M})$. Then $\hat{\mathscr{L}}^{\mathrm{dyn}}_{s/c}(\mathcal{M}; O) \subset \hat{\mathscr{L}}^{\mathrm{kin}}_{s/c}(\mathcal{M}; O)$.*

Proof. It is easy to see that $u \in \hat{\mathscr{L}}^{\mathrm{kin}}_{s/c}(\mathcal{M}; O)$ if and only if $u = S^{s/c}_{\mathcal{M}} f$ for some $f \in \mathscr{S}_0^{(*)}(\boldsymbol{M})$ with support in O. Therefore the previous proposition, along with corollary 7.2.2, shows that $\mathscr{L}^\bullet_{s/c}(\mathcal{M}; K) \subset \hat{\mathscr{L}}^{\mathrm{kin}}_{s/c}(\mathcal{M}; O)$ for all $K \in \mathscr{K}(\boldsymbol{M}; O)$. The required result follows. □

Proposition 7.2.10. *The theory $\mathscr{L}$ is dynamically local, and both $\mathscr{L}^s$ and $\mathscr{L}^c$ obey the conditions* (H1)–(H4)*.*

Proof. Let $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$, and $O \in \mathscr{O}(\boldsymbol{M})$. For dynamical locality, it remains only to show that $\hat{\mathscr{L}}^{\mathrm{kin}}_{s/c}(\mathcal{M}; O) \subset \hat{\mathscr{L}}^{\mathrm{dyn}}_{s/c}(\mathcal{M}; O)$; once again, we adapt the proof of [26, Lemma 3.3]. Any $u \in \hat{\mathscr{L}}^{\mathrm{kin}}_{s/c}(\mathcal{M}; O)$ may be decomposed into a finite sum $u = \sum_i u_i$, where $u_i \in \mathscr{L}^\bullet_{s/c}(\mathcal{M}; K_i)$ for some $K_i \in \mathscr{K}(\boldsymbol{M}; O)$, by writing $u = S^{s/c}_{\mathcal{M}} f$, finding a finite cover of $\mathrm{supp}(f)$ by diamonds O_i based in O, and defining $u_i = S^{s/c}_{\mathcal{M}}(\chi_i f)$ and $K_i = \mathrm{supp}(\chi_i f)$, where χ_i is a smooth partition of unity satisfying $\mathrm{supp}(\chi_i f) \subset O_i$. Therefore $u \in \hat{\mathscr{L}}^{\mathrm{dyn}}(\boldsymbol{M}; O)$, and so $\mathscr{L}$ is dynamically local.

For the conditions (H1)–(H4), we prove that they are satisfied for $\mathscr{L}^s$; the proof for $\mathscr{L}^c$ is similar (and follows from Proposition 5.4.2). By Lemma 7.2.7, we have

$$s^s_{\mathscr{M}}(u, F^s_{\mathscr{M}}[\boldsymbol{h}]) = \frac{i}{2}\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{ab} T^{ab}_{\mathscr{M}}[u], \tag{7.110}$$

where

$$T^{ab}_{\mathscr{M}}[u] := \frac{i}{2}\left(\langle \overline{A_{\boldsymbol{M}} u'}, \boldsymbol{\gamma}^{(a}\nabla^{b)} u\rangle - \langle \overline{A_{\boldsymbol{M}} \nabla^{(a} u'}, \boldsymbol{\gamma}^{b)} u\rangle\right). \tag{7.111}$$

The tensor field with these components can be seen to be real by (7.12), and conserved; in fact, it is the usual classical stress tensor for the Dirac theory. Therefore $\mathscr{L}^s$ satisfies (H1).

Now suppose that $u \in \mathscr{L}^s(\mathscr{M})$, $O \in \mathscr{O}(\boldsymbol{M})$ and $\boldsymbol{h} \in H(\boldsymbol{M}; O)$; since $F^s_{\mathscr{M}}[\boldsymbol{h}]u = -\frac{d}{ds}\, S^s_{\mathscr{M}} P^s_{\mathscr{M}[\boldsymbol{h}]} u\Big|_{s=0}$, we see from Lemma 7.2.7 that $F^s_{\mathscr{M}}[\boldsymbol{h}]u$ may be written $S^s_{\mathscr{M}} f$ where $\mathrm{supp}(f) \subset \mathrm{supp}(\boldsymbol{h}) \subset O$, and therefore $u \in \hat{\mathscr{L}}^{\mathrm{kin}}_s(\mathscr{M}; O)$. Therefore (H2) is satisfied.

For (H3), suppose that $\psi_i : \mathscr{M}_i \to \mathscr{N}$ are arrows in FLoc_4, for $i = 1, 2$, such that $\psi_1(\boldsymbol{M}_1)$ and $\psi_2(\boldsymbol{M}_2)$ are spacelike separated, and consider $u \in \mathscr{L}^s(\psi_1)(\mathscr{L}^s(\mathscr{M}_1)) \cap \mathscr{L}^s(\psi_2)(\mathscr{L}^s(\mathscr{M}_2))$. Then $\mathrm{supp}(u) \subset J_{\boldsymbol{N}}(\psi_1(\mathscr{M}_1)) \cap J_{\boldsymbol{N}}(\psi_2(\mathscr{M}_2))$, which means that u must be zero within both $\psi_1(\mathscr{M}_1)$ and $\psi_2(\mathscr{M}_2)$. However, this is only possible if $u = 0$.

Finally, (H4) is evident from the proof of Proposition 7.2.8. □

In the concluding part of this section, we show that the theory $\mathscr{L}_{FI}$ is also dynamically local.

Proposition 7.2.11. *$\mathscr{L}_{FI}$ is a dynamically local theory.*

Proof. Consider $\mathscr{M} = (\boldsymbol{M}, \mathcal{E}) \in [\mathsf{F}]\mathsf{Loc}_4$, and an element $\boldsymbol{a} = (\boldsymbol{a}_{\boldsymbol{\varepsilon}})_{\boldsymbol{\varepsilon} \in \mathcal{E}}$; it is clear that $\boldsymbol{a} \in \hat{\mathscr{L}}^{\mathrm{kin}}_{FI}(\mathscr{M}; O)$ if and only if each $\boldsymbol{a}_{\boldsymbol{\varepsilon}}$ is an element of $(\hat{\mathscr{L}}^{\mathrm{kin}}_s(\boldsymbol{M}, \boldsymbol{\varepsilon}) \oplus \hat{\mathscr{L}}^{\mathrm{kin}}_c(\boldsymbol{M}, \boldsymbol{\varepsilon}))^{\otimes 2}$. This is the case for every $\boldsymbol{\varepsilon} \in \mathcal{E}$ if and only if it is the case for at least one $\boldsymbol{\varepsilon}$, since the isomorphisms $\alpha^{s/c}_{\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'}$ defined in (7.64) do not change the support of (co)spinor solutions.

Now $\boldsymbol{a}_{\boldsymbol{\varepsilon}} \in (\mathrm{Sol}^s(\boldsymbol{M}, \boldsymbol{\varepsilon}) \oplus \mathrm{Sol}^c(\boldsymbol{M}, \boldsymbol{\varepsilon}))^{\otimes 2}$, and following [26], may therefore be regarded as a linear map from $(\mathrm{Sol}^s(\boldsymbol{M}, \boldsymbol{\varepsilon}))^* \oplus (\mathrm{Sol}^c(\boldsymbol{M}, \boldsymbol{\varepsilon}))^*$ to $\mathrm{Sol}^s(\boldsymbol{M}, \boldsymbol{\varepsilon}) \oplus \mathrm{Sol}^c(\boldsymbol{M}, \boldsymbol{\varepsilon})$ in two ways, by contracting in either the first or second slot. We denote these induced maps by ρ_1 and ρ_2 respectively, and write $Y_i := \mathrm{im}\, \rho_i$ for the associated support spaces. From [26, Appx. A], we see that $\boldsymbol{a}_{\boldsymbol{\varepsilon}} \in Y_1 \otimes Y_2$, and if

$$(\mathrm{rce}_{\mathscr{M}}[\boldsymbol{h}]\,\boldsymbol{a})_{\boldsymbol{\varepsilon}} = (\mathrm{rce}^s_{(\boldsymbol{M},\boldsymbol{\varepsilon})}[\boldsymbol{h}] \oplus \mathrm{rce}^c_{(\boldsymbol{M},\boldsymbol{\varepsilon})}[\boldsymbol{h}])^{\otimes 2}\boldsymbol{a}_{\boldsymbol{\varepsilon}} = \boldsymbol{a}_{\boldsymbol{\varepsilon}} \tag{7.112}$$

for some $\boldsymbol{h} \in H(\boldsymbol{M})$ then Y_1, Y_2 are invariant under $\mathrm{rce}^s_{(\boldsymbol{M},\boldsymbol{\varepsilon})}[\boldsymbol{h}] \oplus \mathrm{rce}^c_{(\boldsymbol{M},\boldsymbol{\varepsilon})}[\boldsymbol{h}]$. Consequently, by Proposition 7.2.10 and Lemma 5.4.1, if $\boldsymbol{a} \in \mathscr{L}^\bullet_{FI}(\mathscr{M}; K)$ for some $K \in \mathscr{K}(\boldsymbol{M})$ then $\boldsymbol{a}_{\boldsymbol{\varepsilon}} \in Y_1 \otimes Y_2 \subset (\mathscr{L}^\bullet_s(\boldsymbol{M},\boldsymbol{\varepsilon}; K) \oplus \mathscr{L}^\bullet_c(\boldsymbol{M},\boldsymbol{\varepsilon}; K))^{\otimes 2}$.

Conversely, if $\boldsymbol{a}_{\boldsymbol{\varepsilon}} \in (\mathscr{L}^\bullet_s(\boldsymbol{M},\boldsymbol{\varepsilon}; K) \oplus \mathscr{L}^\bullet_c(\boldsymbol{M},\boldsymbol{\varepsilon}; K))^{\otimes 2}$ for some $K \in \mathscr{K}(\boldsymbol{M})$ and $\boldsymbol{\varepsilon} \in \mathcal{E}$ then

$$(\mathrm{rce}^s_{(\boldsymbol{M},\boldsymbol{\varepsilon}')}[\boldsymbol{h}] \oplus \mathrm{rce}^c_{(\boldsymbol{M},\boldsymbol{\varepsilon}')}[\boldsymbol{h}])^{\otimes 2}\boldsymbol{a}_{\boldsymbol{\varepsilon}'} = \boldsymbol{a}_{\boldsymbol{\varepsilon}'} \tag{7.113}$$

for all $\boldsymbol{\varepsilon}' \in \mathcal{E}$.

Consequently $\boldsymbol{a} \in \mathscr{L}^\bullet_{FI}(\mathscr{M}; K)$ if and only if $\boldsymbol{a}_{\boldsymbol{\varepsilon}} \in (\mathscr{L}^\bullet_s(\boldsymbol{M},\boldsymbol{\varepsilon}; K) \oplus \mathscr{L}^\bullet_c(\boldsymbol{M},\boldsymbol{\varepsilon}; K))^{\otimes 2}$ for some $\boldsymbol{\varepsilon} \in \mathcal{E}$, and therefore $\boldsymbol{a} \in \hat{\mathscr{L}}^{\mathrm{dyn}}_{FI}(\mathscr{M}; O)$ if and only if $\boldsymbol{a}_{\boldsymbol{\varepsilon}} \in (\hat{\mathscr{L}}^{\mathrm{dyn}}_s(\boldsymbol{M},\boldsymbol{\varepsilon}; O) \oplus \hat{\mathscr{L}}^{\mathrm{dyn}}_c(\boldsymbol{M},\boldsymbol{\varepsilon}; O))^{\otimes 2}$ for some $\boldsymbol{\varepsilon} \in \mathcal{E}$. Then the property $\hat{\mathscr{L}}^{\mathrm{dyn}}_{FI}(\mathscr{M}; O) = \hat{\mathscr{L}}^{\mathrm{kin}}_{FI}(\mathscr{M}; O)$ follows directly from $\hat{\mathscr{L}}^{\mathrm{dyn}}(\boldsymbol{M},\boldsymbol{\varepsilon}) = \hat{\mathscr{L}}^{\mathrm{kin}}(\boldsymbol{M},\boldsymbol{\varepsilon})$, and so the theory $\mathscr{L}_{FI}$ is dynamically local. □

7.3 The locally covariant quantum Dirac theories

We now finally come to the quantized theory of the Dirac field. While the full description of the classical theories of the Dirac field were quite involved and technical, we will find that the machinery developed in Chapter 5 makes the subsequent definition of the quantized equivalents very straightforward.

7.3.1 The framed-spacetime quantum Dirac theory

The functor $\mathscr{L}$ is a weakly nondegenerate theory from FLoc_4 to $\mathsf{HermAdj}_{\mathsf{C}}$, with charge conjugation $C = (C_{\boldsymbol{M}}, C^*_{\boldsymbol{M}})_{(\boldsymbol{M},\boldsymbol{\varepsilon})\in\mathsf{FLoc}_4}$. Therefore there is a CAR-quantized theory $\mathscr{A}_D := \mathscr{Q}_{\mathrm{adj}}\mathscr{L} : \mathsf{FLoc}_4 \to \mathsf{Alg}$ and a charge conjugation

$\chi := (\chi_{\mathcal{M}})_{\mathcal{M}\in\mathsf{FLoc}_4}$ with $\chi_{\mathcal{M}} = (\Upsilon^1_{\mathscr{L}(\mathcal{M})})^{-1} \circ \mathscr{Q}_{\mathrm{adj}}(C_{\boldsymbol{M}}, C^*_{\boldsymbol{M}})$ (definitions are given in Section 5.2.2). Explicitly, the algebra $\mathscr{A}_D(\mathcal{M})$ is generated by a unit $\mathbf{1}$ and elements $D_{\mathcal{M}}(u)$, $E_{\mathcal{M}}(v)$, with $u \in \mathscr{L}^s(\mathcal{M})$ and $v \in \mathscr{L}^c(\mathcal{M})$, subject to the relations

$$D_{\mathcal{M}}(u)^* = E_{\mathcal{M}}(\overline{A_{\boldsymbol{M}}u}) \tag{7.114a}$$

$$D_{\mathcal{M}}(\lambda u + \mu u') = \lambda D_{\mathcal{M}}(u) + \mu D_{\mathcal{M}}(u') \tag{7.114b}$$

$$\{D_{\mathcal{M}}(u), D_{\mathcal{M}}(u')\} = 0 = \{E_{\mathcal{M}}(v), E_{\mathcal{M}}(v')\} \tag{7.114c}$$

$$\{E_{\mathcal{M}}(v), D_{\mathcal{M}}(u)\} = s^c_{\mathcal{M}}(\overline{A_{\boldsymbol{M}}u}, v)\mathbf{1}, \tag{7.114d}$$

for $\lambda, \mu \in \mathbb{C}$; note that this entails

$$\{E_{\mathcal{M}}(S^c_{\mathcal{M}}h), D_{\mathcal{M}}(S^s_{\mathcal{M}}f)\} = -i\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, \langle h, S^s_{\mathcal{M}}f\rangle\, \mathbf{1}. \tag{7.115}$$

We see that on any particular framed spacetime $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon})$, our quantized theory is equivalent to the algebra defined in [20, 27, 17, 56] for the quantum Dirac theory, with the fields $\Psi_{\mathcal{M}} : \mathscr{S}^*_0(\boldsymbol{M}) \to \mathscr{A}_D(\mathcal{M})$, $\Psi^+_{\mathcal{M}} : \mathscr{S}_0(\boldsymbol{M}) \to \mathscr{A}_D(\mathcal{M})$ in these formulations being linked to our generators by

$$\Psi_{\mathcal{M}}(h) = E_{\mathcal{M}}(S^c_{\mathcal{M}}h), \qquad \Psi^+_{\mathcal{M}}(f) = D_{\mathcal{M}}(S^s_{\mathcal{M}}f). \tag{7.116}$$

It follows from propositions 5.3.3 and 7.1.12 that $\Psi^+ : \mathscr{S}_0 \stackrel{.}{\longrightarrow} \mathscr{A}_D$, $\Psi : \mathscr{S}^*_0 \stackrel{.}{\longrightarrow} \mathscr{A}_D$ and $\Psi^+ \oplus \Psi : \mathscr{S}_0 \oplus \mathscr{S}^*_0 \stackrel{.}{\longrightarrow} \mathscr{A}_D$ are locally covariant fields. By following the relevant definitions, we see that the charge conjugation acts as

$$\chi_{\mathcal{M}} D_{\mathcal{M}}(u) = E_{\mathcal{M}}(\overline{A_{\boldsymbol{M}}}C_{\boldsymbol{M}}u), \qquad \chi_{\mathcal{M}} E_{\mathcal{M}}(v) = D_{\mathcal{M}}(\overline{C_{\boldsymbol{M}}\overline{A^{-1}_{\boldsymbol{M}}}}v). \tag{7.117}$$

The theory $\mathscr{A}_D$ clearly inherits the timeslice axiom from $\mathscr{L}$, since $\mathscr{Q}_{\mathrm{adj}}$ maps isomorphisms to isomorphisms. Moreover, suppose that $\mathcal{M}_i = (\boldsymbol{M}_i, \boldsymbol{\varepsilon}_i) \in \mathsf{FLoc}_4$ for $i = 1, 2$, and that $\boldsymbol{\psi}_i : \mathcal{M}_i \to \boldsymbol{N}$ are arrows such that $\psi_1(\boldsymbol{M}_1)$ is spacelike separated from $\psi_2(\boldsymbol{M}_2)$. Then, for $S^s_{\mathcal{N}}f \in \mathscr{L}^s(\boldsymbol{\psi}_1)(\mathscr{L}^s(\mathcal{M}_1))$ and

$S^c_{\mathcal{N}}h \in \mathscr{L}^c(\boldsymbol{\psi}_2)(\mathscr{L}^c(\mathcal{M}_2))$, we have

$$\{E_{\mathcal{N}}(S^c_{\mathcal{N}}h), D_{\mathcal{N}}(S^s_{\mathcal{N}}f)\} = -i \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \langle h, S^s_{\mathcal{N}}f \rangle \, \mathbf{1} = 0. \tag{7.118}$$

Therefore all generators of $\mathscr{L}(\boldsymbol{\psi}_1)(\mathscr{L}(\mathcal{M}_1))$ anticommute with all generators of $\mathscr{L}(\boldsymbol{\psi}_2)(\mathscr{L}(\mathcal{M}_2))$. As mentioned previously, we do not therefore consider the theory $\mathscr{A}_D$ to be truly causal. We will see that this unphysical behaviour is remedied when we pass to the frame-independent theory.

Proposition 7.3.1. *The framed-spacetime Dirac quantum theory $\mathscr{A}_D$ is dynamically local.*

Proof. This may be seen by combining the results of Theorem 5.4.6, Lemma 7.1.6 and Proposition 7.2.10. □

7.3.2 The frame-independent Dirac quantum theory

It remains only to describe the quantization of the frame-independent theory $\mathscr{L}_{FI}$, and establish dynamical locality. We compose the functor $\mathscr{Q}_{SA}$ with $\mathscr{L}_{FI}$ to get a frame-independent theory $\mathscr{A}_{FI} = \mathscr{Q}_{SA}\mathscr{L}_{FI} : \mathsf{[F]Loc}_4 \to \mathsf{Alg}$; for a given $\mathscr{M} = (\boldsymbol{M}, \mathcal{E}) \in \mathsf{[F]Loc}_4$, the algebra $\mathscr{A}_{FI}(\mathscr{M})$ is generated by $\mathbf{1}$ and elements $F_{\mathscr{M}}(\boldsymbol{a})$, $\boldsymbol{a} \in \mathscr{L}_{FI}(\mathscr{M})$, subject to

$$F_{\mathscr{M}}(\boldsymbol{a})^* = F_{\mathscr{M}}(\boldsymbol{a}^*) \tag{7.119a}$$

$$F_{\mathscr{M}}(\lambda\boldsymbol{a} + \mu\boldsymbol{a}') = \lambda F_{\mathscr{M}}(\boldsymbol{a}) + \mu F_{\mathscr{M}}(\boldsymbol{a}') \tag{7.119b}$$

$$F_{\mathscr{M}}(\boldsymbol{a} + \Phi_{FI}\boldsymbol{a}) = \omega_{FI}(\boldsymbol{a})\mathbf{1} \tag{7.119c}$$

$$[F_{\mathscr{M}}(\boldsymbol{a}), F_{\mathscr{M}}(\boldsymbol{a}')] = 2\,\omega \otimes F_{\mathscr{M}}\,\left(Z_{FI}((\boldsymbol{a} - \Phi_{FI}\boldsymbol{a}) \otimes \boldsymbol{a}')\right), \tag{7.119d}$$

where $\lambda, \mu \in \mathbb{C}$, and *, Φ_{FI}, ω_{FI} and Z_{FI} are defined according to (7.75)–(7.78).

Proposition 7.3.2. *For any $\mathscr{M} = (\boldsymbol{M}, \mathcal{E}) \in \mathsf{[F]Loc}_4$ and $\boldsymbol{\varepsilon} \in \mathcal{E}$, the algebra $\mathscr{A}_{FI}(\mathscr{M})$ is isomorphic to the even subalgebra of $\mathscr{A}_D(\boldsymbol{M}, \boldsymbol{\varepsilon})$.*

Proof. Lemma 7.1.19 and (7.75)–(7.78) make it clear that $\mathscr{L}_{FI}(\mathscr{M})$ is isomorphic to $\mathfrak{S}(\mathscr{L}(\boldsymbol{M}, \boldsymbol{\varepsilon}))$ as a squared adjoint structure, and therefore the algebras

$\mathscr{A}_{FI}(\mathscr{M})$ and $\mathscr{Q}_{SA}(\mathfrak{S}(\mathscr{L}(\boldsymbol{M},\boldsymbol{\varepsilon})))$ are isomorphic. But by Proposition 5.2.10, the algebra $\mathscr{Q}_{SA}(\mathfrak{S}(\mathscr{L}(\boldsymbol{M},\boldsymbol{\varepsilon})))$ can be identified with the even subalgebra of $\mathscr{Q}_{\mathrm{adj}}(\mathscr{L}(\boldsymbol{M},\boldsymbol{\varepsilon})) = \mathscr{A}_D(\boldsymbol{M},\boldsymbol{\varepsilon})$. □

More explicitly, there is a natural transformation $\rho : \mathscr{A}_{FI} \stackrel{.}{\longrightarrow} \mathscr{Q}_{SA}\mathfrak{S}\mathscr{L}$ defined by $\rho_{\boldsymbol{M},\boldsymbol{\varepsilon}} := \mathscr{Q}_{SA}(p_{\boldsymbol{M},\boldsymbol{\varepsilon}})$, which acts on generators by $\rho_{\boldsymbol{M},\boldsymbol{\varepsilon}}\, F_{\mathscr{M}}(\boldsymbol{a}) := F_{\mathscr{L}(\boldsymbol{M},\boldsymbol{\varepsilon})}(\boldsymbol{a}_{\boldsymbol{\varepsilon}})$. From Proposition 5.2.10, we see that for a 'pure' element $\boldsymbol{a}_{\boldsymbol{\varepsilon}} = \langle u_1, v_1\rangle \otimes \langle u_2, v_2\rangle$,[9] we have

$$F_{\mathscr{L}(\boldsymbol{M},\boldsymbol{\varepsilon})}(\boldsymbol{a}_{\boldsymbol{\varepsilon}}) = (D_{\mathcal{M}}(u_1) + E_{\mathcal{M}}(v_1))(D_{\mathcal{M}}(u_2) + E_{\mathcal{M}}(v_2)). \tag{7.120}$$

Proposition 7.3.3. *The theory $\mathscr{A}_{FI}$ obeys the timeslice axiom and is causal.*

Proof. Once again, $\mathscr{A}_{FI}$ inherits the timeslice axiom from $\mathscr{L}_{FI}$. For causality, suppose that $\mathscr{M}_i = (\boldsymbol{M}_i, \mathcal{E}_i) \in [\mathsf{F}]\mathsf{Loc}_4$ for $i = 1, 2$, and $[\psi_i] : \mathscr{M}_i \to \mathscr{N} = (\boldsymbol{N}, \mathcal{E})$ are $[\mathsf{F}]\mathsf{Loc}_4$-arrows such that $\psi_1(\boldsymbol{M}_1)$ is spacelike separated from $\psi_2(\boldsymbol{M}_2)$. Now, we pick some $\boldsymbol{\varepsilon} \in \mathcal{E}$, and choose $\boldsymbol{a} \in \mathscr{L}_{FI}([\psi_1])(\mathscr{L}_{FI}(\mathscr{M}_1))$ and $\boldsymbol{a}' \in \mathscr{L}_{FI}([\psi_2])(\mathscr{L}_{FI}(\mathscr{M}_2))$ such that $\boldsymbol{a}_{\boldsymbol{\varepsilon}} = \langle u_1, v_1\rangle \otimes \langle u_2, v_2\rangle$ and $\boldsymbol{a}'_{\boldsymbol{\varepsilon}} = \langle u'_1, v'_1\rangle \otimes \langle u'_2, v'_2\rangle$. It follows that

$$\{D(u_i) + E(v_i), D(u'_j) + E(v'_j)\} = 0, \tag{7.121}$$

and since $[AB, A'B'] = A\{B, A'\}B' - \{A, A'\}BB' + A'A\{B, B'\} - A'\{A, B'\}B$, it holds that $[F_{\mathscr{L}(\boldsymbol{N},\boldsymbol{\varepsilon})}(\boldsymbol{a}_{\boldsymbol{\varepsilon}}), F_{\mathscr{L}(\boldsymbol{N},\boldsymbol{\varepsilon})}(\boldsymbol{a}'_{\boldsymbol{\varepsilon}})] = 0$. Consequently

$$[F_{\mathscr{N}}(\boldsymbol{a}), F_{\mathscr{N}}(\boldsymbol{a}')] = 0, \tag{7.122}$$

and by linearity this holds for any $\boldsymbol{a} \in \mathscr{L}_{FI}([\psi_1])(\mathscr{L}_{FI}(\mathscr{M}_1))$ and $\boldsymbol{a}' \in \mathscr{L}_{FI}([\psi_2])(\mathscr{L}_{FI}(\mathscr{M}_2))$. But since the commutator of any finite product of generators in $\mathscr{L}_{FI}([\psi_1])(\mathscr{L}_{FI}(\mathscr{M}_1))$ with any other finite product of generators in $\mathscr{L}_{FI}([\psi_2])(\mathscr{L}_{FI}(\mathscr{M}_2))$ may be decomposed into terms each of which contains a commutator of single generators, which vanishes, it follows that $[A, A'] = 0$ for any $A \in \mathscr{L}_{FI}([\psi_1])(\mathscr{L}_{FI}(\mathscr{M}_1))$, $A' \in \mathscr{L}_{FI}([\psi_2])(\mathscr{L}_{FI}(\mathscr{M}_2))$. Therefore $\mathscr{A}_{FI}$

[9]The term $\langle u_i, v_i\rangle$ here represents an element of the direct sum rather than the complex scalar field obtained from pairing v_i with u_i.

is causal. □

Therefore the theory $\mathscr{A}_{FI}$, in addition to relying only on measurable properties of a spacetime rather than a non-canonical choice of frame, also displays better causality properties than the theory $\mathscr{A}_D$. It remains only to show demonstrate dynamical locality.

Proposition 7.3.4. *The theory $\mathscr{A}_{FI}$ is dynamically local.*

Proof. Let $\mathscr{M} = (\boldsymbol{M}, \mathcal{E}) \in [\mathsf{F}]\mathsf{Loc}_4$ and $O \in \mathscr{O}(\boldsymbol{M})$, and choose $\boldsymbol{\varepsilon} \in \mathcal{E}$. It is clear that $A \in \hat{\mathscr{A}}_{FI}^{\text{kin}}(\mathscr{M}; O)$ if and only if $\rho_{\boldsymbol{M},\boldsymbol{\varepsilon}}(A) \in \hat{\mathscr{A}}_D^{\text{kin}}(\boldsymbol{M}, \boldsymbol{\varepsilon}; O)$. Moreover, since ρ is natural we have

$$\rho_{\boldsymbol{M},\boldsymbol{\varepsilon}} \circ \operatorname{rce}_{\mathscr{M}}^{(\mathscr{A}_{FI})}[\boldsymbol{h}] = \operatorname{rce}_{(\boldsymbol{M},\boldsymbol{\varepsilon})}^{(\mathscr{A}_D)}[\boldsymbol{h}] \circ \rho_{\boldsymbol{M},\boldsymbol{\varepsilon}}. \tag{7.123}$$

Therefore the image of $\hat{\mathscr{A}}_{FI}^{\text{dyn}}(\mathscr{M}; O)$ under $\rho_{\boldsymbol{M},\boldsymbol{\varepsilon}}$ coincides with the even subalgebra of $\hat{\mathscr{A}}_D^{\text{dyn}}(\boldsymbol{M}, \boldsymbol{\varepsilon}; O)$. Consequently the dynamical locality of $\mathscr{A}_{FI}$ follows from the dynamical locality of $\mathscr{A}_D$ itself. □

Chapter 8

Discussion

It was shown in [26] that the locally covariant scalar field theory was dynamically local in the minimally coupled massive case, and failed to be dynamically local in the minimally coupled massless case. In this thesis we have shown that in the nonminimally coupled case, the theory is dynamically local for all masses. We have also shown that the theory of the extended algebra of Wick polynomials is dynamically local in the massive minimally coupled and massive conformally coupled cases, and fails to be dynamically local in the massless minimally coupled case.

This has been done by demonstrating in Chapter 3 how the functional formalism of [9] may be established in a locally covariant setting, then in Chapter 4 by constructing the kinematic and dynamical algebras for the aforementioned cases of the two theories. We were unable, however, to use our approach to generate useful presentations of the dynamical algebras for the general nonminimally coupled formulation of the enlarged algebra theory, which prevented us from obtaining a complete characterization of the extent to which the enlarged theory obeys dynamical locality. We conjecture that as for the basic scalar field theory, dynamical locality should hold in all cases but the massless minimally coupled case.

In Chapter 5, we have provided a concrete representation of the CAR quantizations of certain classical solution spaces as deformed exterior algebras, in a categorical context. This follows similar work in [26] for the process of CCR quantization, in which a list of sufficient conditions for dynamical locality of the CCR quantization of a classical theory was obtained. We found a similar list of conditions for CAR quantization, and proved that they were again sufficient for dynamical locality of the quantized theory to hold.

In Chapter 6, we have reviewed the geometrical constructions necessary for the construction of the Dirac field on a curved spacetime, incorporating some simplifications made possible by material in [29, 39]. This was followed in Chapter 7 by a construction of the locally covariant classical Dirac theory, first in a way that depended on the choice of a particular global frame, and then in a way that depended only on an equivalence class of frames — although these two constructions were not isomorphic. Finally, we showed that these classical theories were dynamically local, and that their quantizations under the functors constructed in Chapter 5 were also dynamically local.

The construction of the locally covariant classical and quantum Dirac theories that depend only on the spacetime (that is to say, topological structure and metric) and an equivalence class of global frames is new; in the past, constructions have relied on unphysical choices of spin structure so cannot be said to be locally covariant in the true spirit of the term. In passing to this frame-independent setting we have essentially lost information, since the frame-independent quantum theory on a particular $[\mathsf{F}]\mathsf{Loc}_4$ object is always isomorphic to the even subalgebra of the frame-dependent theory of an associated FLoc_4 object. However, we feel that this loss of information is not undesirable when compared to the simplifications made, and is even advantageous; since the physically observable quantities should not rely on a particular unphysical choice, the full algebra obtained from our FLoc_4 theory cannot be a true algebra of observables.

On the other hand, we do not think it possible to construct a simpler Dirac theory (that is to say, dependent on fewer pieces of information) that still contains all physically observable quantities, since the nature of spinorial objects entails that the choice of equivalence class of frames will have an effect on the theory.

We have thus found that dynamical locality has held up well to scrutiny, in the sense that we have managed to considerably enlarge the number of theories known to obey the axiom, and apart from the minimally coupled massless enlarged algebra for the scalar field (in which failure was expected and explainable for the same reason as for the plain scalar field theory), it has not failed to hold for any of the theories for which we have definite results. In terms of the discussion of dynamical locality in the introduction, we have found significant evidence to support dynamical locality as a realistic measure of physicality, and none to oppose it.

While there are few models of QFT in curved spacetime that have been formulated rigorously, let alone in a locally covariant way, there are still a number of cases in which dynamical locality may be checked. We feel that a result totally classifying the cases in which the enlarged algebra of the scalar field obeys dynamical locality is within reach, and as mentioned before, we suspect that the axiom will be satisfied in all cases apart from the massless minimally coupled case. Another obvious extension of our work on the enlarged algebra of observables for the scalar field would be to test whether dynamical locality holds in the perturbatively constructed interacting scalar field theory, which is well-understood on curved spacetimes (see e.g. [9]). It will also be interesting to explore what effect our formulation of the Dirac theory that has domain $[\mathsf{F}]\mathsf{Loc}_4$, and hence depends only on observable factors, has on the construction of the enlarged algebra of observables for the Dirac field, as in [15], and the interacting theory, as in [66]. In both cases it should be possible to examine the extent to which dynamical locality holds. For the enlarged algebra, as for the scalar field, we conjecture that dynamical locality will hold for all masses, but we do not make any firm prediction for the interacting theory.

Appendix A

Some categorical definitions

In this appendix we give a very brief overview of some categorical concepts. A good basic reference to category theory is contained in [45], and material more specific to the categorical unions and intersections defined below can be found in [18]. The presentation here closely follows [25, Appx. B]. We assume that the reader is aware of the definitions of a category, objects, arrows (or morphisms), functors, and natural transformations.

A *concrete* category is a category with a forgetful functor to the category Set of sets with functions as arrows. In other words, a category is concrete if its objects can be described as sets with some extra structure, and its arrows are functions between those sets satisfying certain properties. The vast majority of categories we use in this thesis are concrete categories.

An arrow m is *monic* if it is left-cancellable, i.e. it has the property that $m \circ f = m \circ g$ implies $f = g$. Note that in a concrete category, all injective functions between objects that are also arrows are monic (however, the converse is not necessarily true). Whenever an arrow f factors through a monic m in the sense that $f = m \circ g$ for some arrow g, the left-cancelling property implies that g is unique.

Definition A.1. *A* subobject *of an object A in a category C is a diagram of the form $M \xrightarrow{m} A$, where m is monic. If $M' \xrightarrow{m'} A$ is also a subobject and $m = m' \circ f$ for some (unique) arrow f, then we write $m \leq m'$. If $m \leq m'$ and $m' \leq m$, then we say that m and m' are* isomorphic.

Note that if $m = m' \circ f$ and $m' = m \circ g$ for some arrows f, g, then $m \circ \mathrm{id}_M = m \circ g \circ f$, which entails that $g \circ f = \mathrm{id}_M$. Similarly $f \circ g = \mathrm{id}_{M'}$, and so f is an isomorphism. The subobject $M \xrightarrow{m} A$ is of course entirely defined by the arrow m.

In the case where C is a concrete category, we may associate each subobject $M \xrightarrow{m} A$ with the image of m in A, which coincides with the more usual understanding of the term 'subobject'. In the case where all monic arrows are additionally injective functions (which is not always the case for concrete categories, but will hold in most circumstances here, since we often define arrows to be injective anyway) then it holds that two subobjects of A are isomorphic if and only if the associated images in A coincide.

Definition A.2. *An* equalizer *of two arrows $f, g : A \to B$ in a category C is an arrow eq satisfying $f \circ eq = g \circ eq$, with the property that if $f \circ h = g \circ h$ for some other arrow h then $h = eq \circ b$ for some unique b.*

A category has equalizers *if every pair of arrows with common domain and codomain have an equalizer.*

It is easy to see that an equalizer is necessarily monic, by the uniqueness of factorization.

Definition A.3. *Consider a collection $\left[M_i \xrightarrow{m_i} A\right]_{i \in I}$ of subobjects of an object A, where I is some indexing class. An* intersection *of this collection is a subobject $M \xrightarrow{m} A$ such that for each $i \in I$ there exists a (necessarily unique) arrow $j_i : M \to M_i$ satisfying $m_i \circ j_i = m$; it must also satisfy the additional property that if there exist arrows $f : B \to A$ and $f_i : B \to M_i$ for all $i \in I$ satisfying $f = m_i \circ f_i$, then there exists a unique arrow $g : B \to M$ such that $j_i \circ g = f_i$ for all $i \in I$.*

A category has intersections *if an intersection exists for every such class of subobjects.*

Note that the intersection is hence an extension of the idea of a (categorical) pullback to an arbitary class of arrows, which bears some relation to the categorical product. The intersection is uniquely defined up to isomorphism, and therefore, in a concrete category with injective monics, is associated with a unique image in A. In Set, the intersection of a given collection of subobjects, each of which is associated with a subset $A_i \subset A$, has image equal to the usual intersection $\bigcap_{i \in I} A_i$.

Definition A.4. *Consider a collection* $\left[M_i \xrightarrow{m_i} A\right]_{i \in I}$ *of subobjects of an object* A*, where* I *is some indexing class. A* union *of this collection is a subobject* $M \xrightarrow{m} A$ *such that for each* $i \in I$ *there exists a (necessarily unique) arrow* $u_i : M_i \to M$ *satisfying* $m_i = m \circ u_i$*; it must also satisfy the additional property that if there exists an arrow* $f : A \to B$ *and a subobject* $N \xrightarrow{n} B$ *such that for each* $i \in I$*, we have* $f \circ m_i = n \circ v_i$ *for some* $v_i : M_i \to N$*, then there exists a unique* $g : M \to N$ *such that* $n \circ g = f \circ m$*. (Note that this also implies that* $g \circ u_i = v_i$ *for each* $i \in I$*).*

A category has unions *if a union exists for every such class of subobjects.*

Again, the union is uniquely defined up to isomorphism. In the same way that the intersection can be related to the idea of the categorical product, a similar relation exists between the union and the coproduct. However, these two ideas should not be confused — in Set, as for intersections, categorical unions can be understood to be equivalent to unions in the usual set-theoretic sense. However, the coproduct of two sets is always the *disjoint* union.

If a category C admits unions and intersections, then there is a lattice structure over subobjects of a given object A, the *subobject lattice*, with the ordering $\leq$ defined as in Definition A.1, meets given by intersections and joins given by unions.

Appendix B

Properties of algebras of functionals

In this appendix we prove a number of properties of the algebras of functionals defined in Sections 3.1 and 3.2.

Lemma B.1. *The product defined in* (3.13) *is associative.*

Proof. Throughout this proof, composition is maps is denoted by concatenation for the sake of readability.

Recall that we may write $F \star F' = \boldsymbol{\mu} \exp\left(\frac{i}{2}\mathcal{E}_{\boldsymbol{M}}\right)(F \otimes F')$, where $\mathcal{E}_{\boldsymbol{M}}$ is defined in (3.14) and $\boldsymbol{\mu}$ is the pointwise multiplication map. The product is therefore associative if and only if

$$\boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_M}\left(\mathbf{1} \otimes \boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_M}\right) = \boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_M}\left(\boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_M} \otimes \mathbf{1}\right). \tag{B.1}$$

Now, consider $t_m \in \mathscr{F}^m(\boldsymbol{M})$, $t_n \in \mathscr{F}^n(\boldsymbol{M})$ and $t_p \in \mathscr{F}^p(\boldsymbol{M})$. We have $\boldsymbol{\mu}(t_n \otimes t_p)(x_1, \ldots, x_{n+p}) = \boldsymbol{S}(t_n(x_1, \ldots, x_n)t_p(x_1, \ldots, x_p))$, so by a combinatorical argument we have

$$\mathcal{E}_{\boldsymbol{M}}(t_m \otimes \boldsymbol{\mu}(t_n \otimes t_p)) = (\mathbf{1} \otimes \boldsymbol{\mu})(\mathcal{E}_{\boldsymbol{M}}(t_m \otimes t_n) \otimes t_p) + (\mathbf{1} \otimes \boldsymbol{\mu})(\mathcal{E}_{\boldsymbol{M}}(t_m \otimes t_p) \otimes t_n). \tag{B.2}$$

Therefore, if we define $\boldsymbol{\phi}_{23} : \mathscr{F}(\boldsymbol{M})^{\otimes 3} \to \mathscr{F}(\boldsymbol{M})^{\otimes 3}$ by $\boldsymbol{\phi}_{23}(F_1 \otimes F_2 \otimes F_3) =$

$(F_1 \otimes F_3 \otimes F_2)$, then

$$\mathcal{E}_{\boldsymbol{M}}(\mathbf{1} \otimes \boldsymbol{\mu}) = (\mathbf{1} \otimes \boldsymbol{\mu})(\mathcal{E}_{\boldsymbol{M}} \otimes \mathbf{1} + \boldsymbol{\phi}_{23}(\mathcal{E}_{\boldsymbol{M}} \otimes \mathbf{1})\boldsymbol{\phi}_{23}). \tag{B.3}$$

Since $[\mathcal{E}_{\boldsymbol{M}} \otimes \mathbf{1}, \phi_{23}(\mathcal{E}_{\boldsymbol{M}} \otimes \mathbf{1})\boldsymbol{\phi}_{23}] = 0$, we have

$$\boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_M}\left(\mathbf{1} \otimes \boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_M}\right) = \boldsymbol{\mu}(\mathbf{1} \otimes \boldsymbol{\mu}) e^{\frac{i}{2}\mathcal{E}_M \otimes \mathbf{1}} e^{\frac{i}{2}\boldsymbol{\phi}_{23}(\mathcal{E}_M \otimes \mathbf{1})\boldsymbol{\phi}_{23}} e^{\frac{i}{2}\mathbf{1} \otimes \mathcal{E}_M}. \tag{B.4}$$

Similarly

$$\mathcal{E}_{\boldsymbol{M}}(\boldsymbol{\mu} \otimes \mathbf{1}) = (\boldsymbol{\mu} \otimes \mathbf{1})(\mathbf{1} \otimes \mathcal{E}_{\boldsymbol{M}} + \boldsymbol{\phi}_{23}(\mathcal{E}_{\boldsymbol{M}} \otimes \mathbf{1})\boldsymbol{\phi}_{23}), \tag{B.5}$$

and $[\mathbf{1} \otimes \mathcal{E}_{\boldsymbol{M}}, \boldsymbol{\phi}_{23}(\mathcal{E}_{\boldsymbol{M}} \otimes \mathbf{1})\boldsymbol{\phi}_{23}] = 0$. We also have $[\mathbf{1} \otimes \mathcal{E}_{\boldsymbol{M}}, \mathcal{E}_{\boldsymbol{M}} \otimes \mathbf{1}] = 0$, so

$$\begin{aligned} \boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_M}\left(\boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_M} \otimes \mathbf{1}\right) &= \boldsymbol{\mu}(\boldsymbol{\mu} \otimes \mathbf{1}) e^{\frac{i}{2}\mathbf{1} \otimes \mathcal{E}_M} e^{\frac{i}{2}\boldsymbol{\phi}_{23}(\mathcal{E}_M \otimes \mathbf{1})\boldsymbol{\phi}_{23}} e^{\frac{i}{2}\mathcal{E}_M \otimes \mathbf{1}} \\ &= \boldsymbol{\mu}(\boldsymbol{\mu} \otimes \mathbf{1}) e^{\frac{i}{2}\mathcal{E}_M \otimes \mathbf{1}} e^{\frac{i}{2}\boldsymbol{\phi}_{23}(\mathcal{E}_M \otimes \mathbf{1})\boldsymbol{\phi}_{23}} e^{\frac{i}{2}\mathbf{1} \otimes \mathcal{E}_M}. \end{aligned} \tag{B.6}$$

The equality (B.1) follows from the observation that $\boldsymbol{\mu}(\mathbf{1} \otimes \boldsymbol{\mu}) = \boldsymbol{\mu}(\boldsymbol{\mu} \otimes \mathbf{1})$. □

Lemma B.2. *The map* $\left(\sum_{n=0}^{N} t_N\right)^* = \sum_{n=0}^{N} \overline{t_n}$ *is an involution on* $\mathscr{F}(\boldsymbol{M})$*.*

Proof. Since the bidistribution $E_{\boldsymbol{M}}$ is real and antisymmetric we have for $t_m \in \mathscr{F}^m(\boldsymbol{M})$, $t_n \in \mathscr{F}^n(\boldsymbol{M})$

$$\begin{aligned} t_m^* \star t_n^* &= \boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_M} \overline{t_m} \otimes \overline{t_n} \\ &= \overline{\boldsymbol{\mu} e^{-\frac{i}{2}\mathcal{E}_M} t_m \otimes t_n} \\ &= \overline{\boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_M} t_n \otimes t_m} \\ &= (t_n \star t_m)^*. \end{aligned} \tag{B.7}$$

The map $F \mapsto F^*$ is clearly antilinear, and $F^{**} = F$, so it is an involution. □

Lemma B.3. *For any* $H \in \mathscr{H}(\boldsymbol{M})$*, the map* $\lambda_H = \boldsymbol{\iota}_H \exp\left(-\frac{1}{2}\eta_H\right)$ *from* $\mathscr{F}(\boldsymbol{M})$ *to* $\mathscr{T}_H(\boldsymbol{M})$ *is an arrow in* Alg*.*

Proof. λ_H is clearly linear, preserves the unit and has a left inverse given by $\exp\left(\frac{1}{2}\eta_H\right)\boldsymbol{\iota}_H^{-1}$, so it is injective. It only remains to prove the homomorphism property $\lambda_H(F \star F') = \lambda_H F \star_H \lambda_H F'$. This is equivalent to the condition that

$$\boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_{\boldsymbol{M};H}} \left(\boldsymbol{\iota}_H e^{-\frac{1}{2}\eta_H}\right)^{\otimes 2} = \boldsymbol{\iota}_H e^{-\frac{1}{2}\eta_H} \boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_{\boldsymbol{M}}}. \tag{B.8}$$

To prove this, we consider $t_m \in \mathscr{F}^m(\boldsymbol{M})$, $t_n \in \mathscr{F}^n(\boldsymbol{M})$. Recalling the symmetrizing properties of $\boldsymbol{\mu}$, note that the action of η_H on the pointwise product of t_m with t_n gives a sum in which every possible pair of variables is integrated against H. Therefore $\eta_H \boldsymbol{\mu}(t_m \otimes t_n)$ consists of three terms, one in which both of the integrated variables come from t_m, one in which both come from t_n, and one in which a variable from t_m is paired with a variable from t_n. Explicitly, we have

$$\eta_H \boldsymbol{\mu}(t_m \otimes t_n) = \boldsymbol{\mu}\left(\eta_H t_m \otimes t_n + t_m \otimes \eta_H t_n + 2\theta_H(t_m \otimes t_n)\right) \tag{B.9}$$

where

$$\theta_H := \frac{1}{2i}\left(\mathcal{E}_{\boldsymbol{M};H} - \mathcal{E}_{\boldsymbol{M}}\right) \tag{B.10}$$

(which is well-defined on $\mathscr{F}(\boldsymbol{M}) \otimes \mathscr{F}(\boldsymbol{M})$) has the effect of pairing a variable from t_m with one from t_n in the integration against H. We also have $[\eta_H \otimes \mathbf{1}, \theta_H] = 0 = [\mathbf{1} \otimes \eta_H, \theta_H]$, therefore

$$\begin{aligned}
\boldsymbol{\iota}_H e^{-\frac{1}{2}\eta_H} \boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_{\boldsymbol{M}}} &= \boldsymbol{\iota}_H \boldsymbol{\mu} e^{-\frac{1}{2}\eta_H \otimes \mathbf{1}} e^{-\frac{1}{2}\mathbf{1}\otimes \eta_H} e^{-\theta_H} e^{\frac{i}{2}\mathcal{E}_{\boldsymbol{M}}} \\
&= \boldsymbol{\mu}(\boldsymbol{\iota}_H \otimes \boldsymbol{\iota}_H) e^{-\frac{1}{2}\eta_H \otimes \mathbf{1}} e^{-\frac{1}{2}\mathbf{1}\otimes \eta_H} e^{\frac{i}{2}\mathcal{E}_{\boldsymbol{M};H}} \\
&= \boldsymbol{\mu} e^{\frac{i}{2}\mathcal{E}_{\boldsymbol{M};H}} \left(\boldsymbol{\iota}_H e^{-\frac{1}{2}\eta_H}\right)^{\otimes 2}.
\end{aligned} \tag{B.11}$$

This completes the proof. □

Appendix C

Construction of dynamical algebras for the scalar field theories

This appendix contains proofs of various results stated in Chapter 4, concerned with the construction of the dynamical algebras $\mathscr{A}^{\mathrm{dyn}}(\boldsymbol{M};O)$ and $\mathscr{W}^{\mathrm{dyn}}(\boldsymbol{M};O)$ for a given $\boldsymbol{M} \in \mathsf{Loc}$ and $O \in \mathscr{O}(\boldsymbol{M})$. Lemma 4.1.6 was stated as follows:

Lemma C.1. *Let $\boldsymbol{M} \in \mathsf{Loc}$, and let $t \in \mathscr{T}^1_H(\boldsymbol{M})$ for some $H \in \mathscr{H}(\boldsymbol{M})$. For any $\boldsymbol{h} \in H(\boldsymbol{M})$ and $f \in C_0^\infty(\boldsymbol{M})$, we have*

$$\frac{d}{ds}(\beta[s\boldsymbol{h}]t)[E_{\boldsymbol{M}}f]\bigg|_{s=0} = \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{\mu\nu} T^{\mu\nu}[E_{\boldsymbol{M}}t, E_{\boldsymbol{M}}f], \tag{C.1}$$

where

$$\begin{aligned} T^{\mu\nu}[u,\phi] = (\nabla^{(\mu}u)(\nabla^{\nu)}\phi) - \frac{1}{2}g^{\mu\nu}(\nabla^\rho u)(\nabla_\rho \phi) \\ + \frac{1}{2}m^2 g^{\mu\nu}u\phi + \xi(g^{\mu\nu}\Box_{\boldsymbol{g}} - \nabla^\mu\nabla^\nu - G^{\mu\nu})(u\phi) \end{aligned} \tag{C.2}$$

for $u \in E_{\boldsymbol{M}}\mathscr{T}^1(\boldsymbol{M})$, $\phi \in E_{\boldsymbol{M}}C_0^\infty(\boldsymbol{M})$.

The proof of this lemma follows the strategy used in Appendix B of [26].

Proof. Let $\boldsymbol{h} \in H(\boldsymbol{M})$, and consider the metric perturbation $s\boldsymbol{h}$ where $s \in \mathbb{R}$

is sufficiently small to ensure that $s\boldsymbol{h} \in H(\boldsymbol{M})$. We have $E_{\boldsymbol{M}}\beta[s\boldsymbol{h}]t = E_{\boldsymbol{M}}\zeta^-[s\boldsymbol{h}]\zeta^+t$, therefore

$$\begin{aligned} E_{\boldsymbol{M}}\beta[s\boldsymbol{h}]t - E_{\boldsymbol{M}}t = E_{\boldsymbol{M}}(P_{\boldsymbol{M}[s\boldsymbol{h}]} - P_{\boldsymbol{M}})\chi_-^{\mathrm{adv}} E_{\boldsymbol{M}[s\boldsymbol{h}]}\zeta^+t \\ + E_{\boldsymbol{M}}P_{\boldsymbol{M}}\chi_-^{\mathrm{adv}}(E_{\boldsymbol{M}[s\boldsymbol{h}]} - E_{\boldsymbol{M}})\zeta^+t. \end{aligned} \quad \text{(C.3)}$$

Since $P_{\boldsymbol{M}}$ is a differential operator it follows that the support of $(P_{\boldsymbol{M}[s\boldsymbol{h}]} - P_{\boldsymbol{M}})f$ lies within $\mathrm{supp}(\boldsymbol{h}) \cap \mathrm{supp}(f)$ for any $f \in C^\infty(\boldsymbol{M})$. The support of χ_-^{adv} lies strictly to the past of $\mathrm{supp}(\boldsymbol{h})$, so the first term above vanishes. Moreover, note that the support of $E^+_{\boldsymbol{M}[s\boldsymbol{h}]}f - E^+_{\boldsymbol{M}}f$ is contained within $J^+_{\boldsymbol{M}}(\mathrm{supp}(\boldsymbol{h}))$ for any $f \in C_0^\infty(\boldsymbol{M})$, and is also therefore disjoint from $\mathrm{supp}(\chi_-^{\mathrm{adv}})$; it follows that

$$E_{\boldsymbol{M}}\beta[s\boldsymbol{h}]t - E_{\boldsymbol{M}}t = E_{\boldsymbol{M}}P_{\boldsymbol{M}}\chi_-^{\mathrm{adv}}(E^-_{\boldsymbol{M}[s\boldsymbol{h}]} - E^-_{\boldsymbol{M}})\zeta^+t. \quad \text{(C.4)}$$

Similarly, $(E^-_{\boldsymbol{M}[s\boldsymbol{h}]} - E^-_{\boldsymbol{M}})f$ must be supported in $J^-_{\boldsymbol{M}}(\mathrm{supp}(\boldsymbol{h}))$ for any $f \in C_0^\infty(\boldsymbol{M})$; it follows that the support of $\chi_-^{\mathrm{ret}}(E^-_{\boldsymbol{M}[s\boldsymbol{h}]} - E^-_{\boldsymbol{M}})f$ is compact. Therefore

$$E_{\boldsymbol{M}}\beta[s\boldsymbol{h}]t - E_{\boldsymbol{M}}t = E_{\boldsymbol{M}}P_{\boldsymbol{M}}(E^-_{\boldsymbol{M}[s\boldsymbol{h}]} - E^-_{\boldsymbol{M}})\zeta^+t. \quad \text{(C.5)}$$

We use $P_{\boldsymbol{M}[s\boldsymbol{h}]}E^-_{\boldsymbol{M}[s\boldsymbol{h}]}\zeta^+t = \zeta^+t = P_{\boldsymbol{M}}E^-_{\boldsymbol{M}}\zeta^+t$ to see that

$$\begin{aligned} E_{\boldsymbol{M}}\beta[s\boldsymbol{h}]t - E_{\boldsymbol{M}}t &= -E_{\boldsymbol{M}}(P_{\boldsymbol{M}[s\boldsymbol{h}]} - P_{\boldsymbol{M}})E^-_{\boldsymbol{M}[s\boldsymbol{h}]}\zeta^+t \\ &= E_{\boldsymbol{M}}(P_{\boldsymbol{M}[s\boldsymbol{h}]} - P_{\boldsymbol{M}})E^-_{\boldsymbol{M}[s\boldsymbol{h}]}(P_{\boldsymbol{M}[s\boldsymbol{h}]} - P_{\boldsymbol{M}})E^-_{\boldsymbol{M}}\zeta^+t \\ &\quad - E_{\boldsymbol{M}}(P_{\boldsymbol{M}[s\boldsymbol{h}]} - P_{\boldsymbol{M}})E^-_{\boldsymbol{M}}\zeta^+t, \end{aligned} \quad \text{(C.6)}$$

where we have used the fact that $E^-_{\boldsymbol{M}[\boldsymbol{h}]}P_{\boldsymbol{M}[\boldsymbol{h}]}E^-_{\boldsymbol{M}}u = E^-_{\boldsymbol{M}}u$ for any $u \in \mathcal{E}'(\boldsymbol{M})$ and $\boldsymbol{h} \in H(\boldsymbol{M})$; this is proved below.

Finally, we note that $\mathrm{supp}(\boldsymbol{h}) \cap \mathrm{supp}(E^+_{\boldsymbol{M}}\zeta^+t) = \emptyset$, so

$$\begin{aligned} E_{\boldsymbol{M}}\beta[s\boldsymbol{h}]t - E_{\boldsymbol{M}}t = E_{\boldsymbol{M}}(P_{\boldsymbol{M}[s\boldsymbol{h}]} - P_{\boldsymbol{M}})E^-_{\boldsymbol{M}[s\boldsymbol{h}]}(P_{\boldsymbol{M}[s\boldsymbol{h}]} - P_{\boldsymbol{M}})E^-_{\boldsymbol{M}}\zeta^+t \\ - E_{\boldsymbol{M}}(P_{\boldsymbol{M}[s\boldsymbol{h}]} - P_{\boldsymbol{M}})E_{\boldsymbol{M}}\zeta^+t. \end{aligned} \quad \text{(C.7)}$$

Now, for any $\phi \in C^\infty(\boldsymbol{M})$ we wish to calculate the value of $\frac{d}{ds}\Box_{\boldsymbol{g}+s\boldsymbol{h}}\phi\Big|_{s=0}$. In order to calculate this quantity, it is first convenient to consider the functional derivative of the contracted Levi-Civita connection ${\Gamma^{\mu}}_{\mu\rho}$. The connection ${\Gamma^{\mu}}_{\nu\rho}$ can be expressed in terms of the metric as

$$ {\Gamma^{\mu}}_{\nu\rho}[\boldsymbol{g}] = \frac{1}{2} g^{\mu\sigma} \left(\partial_{\nu} g_{\rho\sigma} + \partial_{\rho} g_{\nu\sigma} - \partial_{\sigma} g_{\nu\rho} \right). \tag{C.8} $$

Therefore ${\Gamma^{\mu}}_{\mu\rho}[\boldsymbol{g}] = \frac{1}{2} g^{\mu\nu} \partial_{\rho} g_{\mu\nu}$, and so

$$ \begin{aligned} \frac{d}{ds} {\Gamma^{\mu}}_{\mu\rho}[\boldsymbol{g} + s\boldsymbol{h}] \bigg|_{s=0} &= \frac{d}{ds} \left[\frac{1}{2} (g^{\mu\nu} - s h^{\mu\nu}) \partial_{\rho} (g_{\mu\nu} + s h_{\mu\nu}) + \mathcal{O}(s^2) \right]_{s=0} \\ &= \frac{1}{2} \left(g^{\mu\nu} \partial_{\rho} h_{\mu\nu} - h^{\mu\nu} \partial_{\rho} g_{\mu\nu} \right) \\ &= \frac{1}{2} \left(g^{\mu\nu} \partial_{\rho} h_{\mu\nu} + h_{\mu\nu} \partial_{\rho} g^{\mu\nu} \right) \\ &= \frac{1}{2} \partial_{\rho} {h^{\mu}}_{\mu}, \end{aligned} \tag{C.9} $$

where we have used the fact that the inverse of the perturbed metric $\boldsymbol{g} + s\boldsymbol{h}$ can be expanded as $g^{\mu\nu} - s h^{\mu\nu} + \mathcal{O}(s^2)$.

Now $\Box_{\boldsymbol{g}} \phi = \nabla_{\mu} g^{\mu\nu} \nabla_{\nu} = \partial_{\mu} (g^{\mu\nu} \partial_{\nu} \phi) + {\Gamma^{\mu}}_{\mu\rho} g^{\nu\rho} \partial_{\nu} \phi$. Consequently, we see that

$$ \frac{d}{ds} \Box_{\boldsymbol{g}+s\boldsymbol{h}} \phi \bigg|_{s=0} = -\partial_{\mu} (h^{\mu\nu} \partial_{\nu} \phi) + \frac{1}{2} (\partial_{\rho} {h^{\mu}}_{\mu}) g^{\nu\rho} \partial_{\nu} \phi - {\Gamma^{\mu}}_{\mu\rho} h^{\nu\rho} \partial_{\nu} \phi \tag{C.10} $$

$$ = \frac{1}{2} (\nabla_{\rho} {h^{\mu}}_{\mu}) \nabla^{\rho} \phi - \nabla_{\mu} (h^{\mu\nu} \nabla_{\nu} \phi). \tag{C.11} $$

We may also note that

$$ \frac{d}{ds} R_{\boldsymbol{g}+s\boldsymbol{h}} \bigg|_{s=0} = (g^{\mu\nu} \Box_{\boldsymbol{g}} - \nabla^{\mu} \nabla^{\nu} - R^{\mu\nu}) h_{\mu\nu} \tag{C.12} $$

(see e.g. [48]). Therefore, for any $f \in C_0^{\infty}(\boldsymbol{M})$, the limit $\lim_{s\to 0}(E_{\boldsymbol{M}}(P_{\boldsymbol{M}[s\boldsymbol{h}]} -$

$P_{\boldsymbol{M}})/s)E_{\boldsymbol{M}}f$ exists and is equal to

$$\left(\frac{1}{2}\nabla^\mu(h^\nu{}_\nu\nabla_\mu E_{\boldsymbol{M}}f)+\frac{1}{2}h^\mu{}_\mu m^2E_{\boldsymbol{M}}f+\frac{1}{2}h^\mu{}_\mu\xi RE_{\boldsymbol{M}}f\right.$$
$$\left.-\nabla^\mu(h_{\mu\nu}\nabla^\nu E_{\boldsymbol{M}}f)+\xi E_{\boldsymbol{M}}f\left(g^{\mu\nu}\Box_{\boldsymbol{g}}-\nabla^\mu\nabla^\nu-R^{\mu\nu}\right)h_{\mu\nu}\right), \tag{C.13}$$

where we have used the fact that $E_{\boldsymbol{M}}f$ solves the field equation.

By duality, the same limit holds for distributions in the weak topology. Moreover, the first term of (C.7) can now be seen to be of order $\mathcal{O}(s^2)$ as $s\to 0$, and therefore

$$\frac{d}{ds}E_{\boldsymbol{M}}\beta[s\boldsymbol{h}]t\Big|_{s=0}=-E_{\boldsymbol{M}}\left(\frac{1}{2}\nabla^\mu(h^\nu{}_\nu\nabla_\mu E_{\boldsymbol{M}}t)+\frac{1}{2}h^\mu{}_\mu m^2E_{\boldsymbol{M}}t\right.$$
$$+\frac{1}{2}h^\mu{}_\mu\xi RE_{\boldsymbol{M}}t-\nabla^\mu(h_{\mu\nu}\nabla^\nu E_{\boldsymbol{M}}t)$$
$$\left.+\xi E_{\boldsymbol{M}}t\left(g^{\mu\nu}\Box_{\boldsymbol{g}}-\nabla^\mu\nabla^\nu-R^{\mu\nu}\right)h_{\mu\nu}\right), \tag{C.14}$$

where the derivative is taken in the weak topology, and we have used the fact that $E_{\boldsymbol{M}}\zeta^+t=E_{\boldsymbol{M}}t$. For any $f\in C_0^\infty(\boldsymbol{M})$, the quantity $\frac{d}{ds}(\beta[s\boldsymbol{h}]t)[E_{\boldsymbol{M}}f]\Big|_{s=0}$ may then be seen to equal

$$\int_{\boldsymbol{M}}dvol_{\boldsymbol{M}}\,(E_{\boldsymbol{M}}f)\left(\frac{1}{2}\nabla^\mu(h^\nu{}_\nu\nabla_\mu E_{\boldsymbol{M}}t)+\frac{1}{2}h^\mu{}_\mu m^2E_{\boldsymbol{M}}t+\frac{1}{2}h^\mu{}_\mu\xi RE_{\boldsymbol{M}}t\right.$$
$$\left.-\nabla^\mu(h_{\mu\nu}\nabla^\nu E_{\boldsymbol{M}}t)+\xi E_{\boldsymbol{M}}t\left(g^{\mu\nu}\Box_{\boldsymbol{g}}-\nabla^\mu\nabla^\nu-R^{\mu\nu}\right)h_{\mu\nu}\right). \tag{C.15}$$

Integration by parts then yields

$$\frac{d}{ds}(\beta[s\boldsymbol{h}]t)[E_{\boldsymbol{M}}f]\Big|_{s=0}=\int_{\boldsymbol{M}}dvol_{\boldsymbol{M}}\,h_{\mu\nu}T^{\mu\nu}[E_{\boldsymbol{M}}t,E_{\boldsymbol{M}}f] \tag{C.16}$$

as required.

It remains to show that for any $u\in\mathcal{E}'(\boldsymbol{M})$ and $h\in H(\boldsymbol{M})$ we have $E^-_{\boldsymbol{M}[\boldsymbol{h}]}P_{\boldsymbol{M}[\boldsymbol{h}]}E^-_{\boldsymbol{M}}u=E^-_{\boldsymbol{M}}u$. We may see that this holds by considering an arbitrary $f \in C_0^\infty(\boldsymbol{M})$ and splitting $E^-_{\boldsymbol{M}}u = t + t'$ where $t \in \mathcal{E}'(\boldsymbol{M})$ and $t' \in \mathcal{D}'(\boldsymbol{M})$ with $J^-_{\boldsymbol{M}[\boldsymbol{h}]}(\mathrm{supp}(t')) \cap \mathrm{supp}(f) = \emptyset$. It follows that

$$\begin{aligned} E^-_{\boldsymbol{M}[\boldsymbol{h}]}P_{\boldsymbol{M}[\boldsymbol{h}]}E^-_{\boldsymbol{M}}u[f] &= E^-_{\boldsymbol{M}[\boldsymbol{h}]}P_{\boldsymbol{M}[\boldsymbol{h}]}t[f] + E^-_{\boldsymbol{M}[\boldsymbol{h}]}P_{\boldsymbol{M}[\boldsymbol{h}]}t'[f] \\ &= t[f]. \end{aligned} \tag{C.17}$$

But $t[f] = t[f] + t'[f] = E^-_{\boldsymbol{M}}u[f]$. Since f was arbitrary, we have

$$E^-_{\boldsymbol{M}[\boldsymbol{h}]}P_{\boldsymbol{M}[\boldsymbol{h}]}E^-_{\boldsymbol{M}}u = E^-_{\boldsymbol{M}}u. \tag{C.18}$$

□

We also include a proof for Lemma 4.2.1:

Lemma C.2. *Let* $\boldsymbol{M} \in \mathsf{Loc}$, *and* $t_n \in \mathscr{F}^n(\boldsymbol{M})$, $n \geq 1$. *If* $O \in \mathscr{O}(\boldsymbol{M})$, $K \in \mathscr{K}(\boldsymbol{M}; O)$ *and* $\left((\beta[s\boldsymbol{h}])^{\otimes n}t_n\right)[E_{\boldsymbol{M}}f] = t_n[E_{\boldsymbol{M}}f]$ *for all* $f \in C_0^\infty(\boldsymbol{M})$ *and for all* $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$, *then*

$$\mathrm{supp}(E_{\boldsymbol{M}}{}^{\otimes n}t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}. \tag{C.19}$$

Proof. For each $n \geq 1$ it is possible to differentiate (4.44) with respect to s and set $s = 0$; by corollary 4.1.7, this yields

$$\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{\mu\nu}T^{\mu\nu}\left[E_{\boldsymbol{M}}\tau^n_f, E_{\boldsymbol{M}}f\right] = 0 \tag{C.20}$$

for each $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$ and $f \in C_0^\infty(\boldsymbol{M})$, where τ^n_f is defined as in (4.43). It follows that for all $n \geq 1$, we have

$$T^{\mu\nu}[E_{\boldsymbol{M}}\tau^n_f, E_{\boldsymbol{M}}f](x) = 0 \tag{C.21}$$

for $x \in K^\perp$.

Now consider an arbitrary point $x \in K^\perp$, and a null geodesic $u : I \to K^\perp$, where $I \subset \mathbb{R}$ is an open interval containing 0 and $u(0) = x$. Since u is a null geodesic, it satisfies both $u^\mu u^\nu g_{\mu\nu} = 0$ and $u^\mu \nabla_\mu u^\nu = 0$, where u^μ is the tangent vector to u. For each point p on the geodesic we have $u_\mu(p)u_\nu(p)T^{\mu\nu}[E_{\boldsymbol{M}}\tau^n_f, E_{\boldsymbol{M}}f](p) = 0$, and consequently for our chosen $x \in K^\perp$

we have

$$(\nabla_u E_{\boldsymbol{M}} \tau_f^n(x))(\nabla_u E_{\boldsymbol{M}} f(x)) \\ + \xi \left(-\nabla_u^2 - R_{\mu\nu}(x) u^\mu u^\nu\right) \Big((E_{\boldsymbol{M}} \tau_f^n(x))(E_{\boldsymbol{M}} f(x))\Big) = 0. \quad \text{(C.22)}$$

Note that this is equivalent to

$$(1-2\xi)(\nabla_u E_{\boldsymbol{M}} \tau_f^n(x))(\nabla_u E_{\boldsymbol{M}} f(x)) - \xi R_{\mu\nu} u^\mu u^\nu (E_{\boldsymbol{M}} \tau_f^n(x))(E_{\boldsymbol{M}} f(x)) \\ + \xi (E_{\boldsymbol{M}} \tau_f^n(x) \nabla_u^2 E_{\boldsymbol{M}} f(x) + E_{\boldsymbol{M}} f(x) \nabla_u^2 E_{\boldsymbol{M}} \tau_f^n(x)) = 0. \quad \text{(C.23)}$$

It follows that for any $f \in C_0^\infty(\boldsymbol{M})$ for which $E_{\boldsymbol{M}} f(x) = 0 = \nabla_u E_{\boldsymbol{M}} f(x)$ and $\nabla_u^2 E_{\boldsymbol{M}} f(x) \neq 0$,[1] we have $E_{\boldsymbol{M}} \tau_f^n(x) = 0$, as $\xi \neq 0$.

We now split up the remainder of this proof into three cases, for $n = 1$, $n = 2$ and $n > 2$. The $n = 1$ case is simplest, since $E_{\boldsymbol{M}} \tau_f^1 = E_{\boldsymbol{M}} t_1$ for all f; we immediately see that $E_{\boldsymbol{M}} t_1(x) = 0$ for all $x \in K^\perp$.

Now, we look at $n = 2$ case. We have $E_{\boldsymbol{M}} \tau_f^2(x) = \int_{\boldsymbol{M}} dy\, t_2(x,y) E_{\boldsymbol{M}} f(y)$, which is linear in f. Let f again be chosen in such a way that $E_{\boldsymbol{M}} f(x) = 0 = \nabla_u E_{\boldsymbol{M}} f(x)$ and $\nabla_u^2 E_{\boldsymbol{M}} f(x) \neq 0$; additionally, we choose $f' \in C_0^\infty(\boldsymbol{M})$ such that $\mathrm{supp}(f') \subset \{x\}^\perp$. Then $E_{\boldsymbol{M}} f + E_{\boldsymbol{M}} f' = E_{\boldsymbol{M}} f$ in an open neighbourhood of x, so

$$E_{\boldsymbol{M}} \tau_{f'}^2(x) = E_{\boldsymbol{M}} \tau_{f+f'}^2(x) - E_{\boldsymbol{M}} \tau_f^2(x) = 0. \quad \text{(C.24)}$$

It follows that for any $f' \in C_0^\infty(\boldsymbol{M})$ supported outside $J_{\boldsymbol{M}}(x)$, we have

$$\int_{\boldsymbol{M}} dy\, ({E_{\boldsymbol{M}}}^{\otimes 2} t_2)(x,y) f'(y) = -E_{\boldsymbol{M}} \tau_{f'}^2(x) = 0. \quad \text{(C.25)}$$

Therefore ${E_{\boldsymbol{M}}}^{\otimes 2} t_2(x,y) = 0$ whenever $x \in K^\perp$ and $y \in \{x\}^\perp$.

However, we may note that ${E_{\boldsymbol{M}}}^{\otimes 2} t_2(x,\cdot) \in E_{\boldsymbol{M}} C_0^\infty(\boldsymbol{M})$ for any fixed

[1] Such a solution always exists; we may explicitly construct one as follows. We work in normal coordinates p^μ in a neighbourhood $S \ni x$ such that x is at the origin, and the $p^0 = 0$ hyperplane is a subset of a spacelike Cauchy surface $\Sigma \subset \boldsymbol{M}$, and we take our null geodesic u such that in coordinates, the tangent at x is $u^\mu(x) = (1, 1, 0, \ldots, 0)$. Then any solution ψ is uniquely determined by its data (φ, π) on Σ, where $\varphi(\underline{p}) = \psi|_\Sigma(\underline{p})$ and $\pi(\underline{p}) = (\nabla_0 \psi)|_\Sigma(\underline{p})$. It is then easy to check that defining $\varphi(\underline{p}) = (p^1)^2$, $\pi(\underline{p}) = 0$ for $\underline{p} \in \Sigma \cap S$ gives us a solution ψ satisfying the above conditions.

$x \in \boldsymbol{M}$ by the definition of $\mathscr{F}^2(\boldsymbol{M})$, and $E_{\boldsymbol{M}}{}^{\otimes 2}t_2(x,\cdot)$ is therefore a smooth classical Klein-Gordon solution. If Σ is a spacelike Cauchy surface containing x, then the data for $E_{\boldsymbol{M}}{}^{\otimes 2}t_2(x,\cdot)$ on Σ is supported in $\{x\}$ for any $x \in K^\perp$ by the above result. But the data for a smooth solution is itself smooth, and therefore cannot be both nonzero and supported at a point. Consequently $E_{\boldsymbol{M}}{}^{\otimes 2}t_2(x,y) = 0$ for any $(x,y) \in K^\perp \times \boldsymbol{M}$, and by symmetry we have $\mathrm{supp}(E_{\boldsymbol{M}}{}^{\otimes 2}t_2) \subset J_{\boldsymbol{M}}(K)^{\times 2}$.

We finally consider the case where $n > 2$. Suppose that we have $f, f_1 \in C_0^\infty(\boldsymbol{M})$ such that $E_{\boldsymbol{M}}f(x) = 0 = \nabla_u E_{\boldsymbol{M}}f(x)$, $E_{\boldsymbol{M}}f_1(x) = 0 = \nabla_u E_{\boldsymbol{M}}f_1(x)$, and $\nabla_u^2 E_{\boldsymbol{M}}f(x) \neq 0$. For sufficiently small κ we have $\nabla_u^2 E_{\boldsymbol{M}}(f + \kappa f_1)(x) \neq 0$, and so $E_{\boldsymbol{M}}\tau^n_{f+\kappa f_1}(x) = 0$. Therefore, by symmetry of t_n we have

$$E_{\boldsymbol{M}}\tau^n_f(x) + (-1)^{n-1}(n-1)\kappa \int_{\boldsymbol{M}^{\times(n-1)}} d^{n-1}y \left[(E_{\boldsymbol{M}}{}^{\otimes n}t_n)(x, y_1, \ldots, y_{n-1}) \right.$$
$$\left. f_1(y_1)f(y_2)\cdots f(y_{n-1})\right] + \mathcal{O}(\kappa^2) = 0. \quad \text{(C.26)}$$

Differentiating this expression with respect to κ and setting $\kappa = 0$ yields

$$\int_{\boldsymbol{M}^{\times(n-1)}} d^{n-1}y\, (E_{\boldsymbol{M}}{}^{\otimes n}t_n)(x, y_1, \ldots, y_{n-1}) f_1(y_1)f(y_2)\cdots f(y_{n-1}) = 0. \quad \text{(C.27)}$$

We may repeat this argument to see that

$$\int_{\boldsymbol{M}^{\times(n-1)}} d^{n-1}y\, (E_{\boldsymbol{M}}{}^{\otimes n}t_n)(x, y_1, \ldots, y_{n-1}) f_1(y_1)\cdots f_{n-1}(y_{n-1}) = 0 \quad \text{(C.28)}$$

for any $f_1, \ldots, f_{n-1}$ such that $E_{\boldsymbol{M}}f_i(x) = 0 = \nabla_u E_{\boldsymbol{M}}f_i(x)$, $i = 1, \ldots, n-1$. It follows that for any $x_1 \in K^\perp$, we have $E_{\boldsymbol{M}}{}^{\otimes n}t_n(x_1, \ldots, x_n) = 0$ whenever at least one of $x_2, \ldots, x_n$ lies in $x_1{}^\perp$. Fixing $x_1 \in K^\perp$, we note that $E_{\boldsymbol{M}}{}^{\otimes n}t_n(x_1, y_1, \ldots, y_{n-1})$ is a smooth Klein-Gordon $(n-1)$-solution; its data on a spacelike Cauchy surface $\Sigma \ni x$ is supported in $\{x\}^{\times(n-1)}$. Consequently we must have $E_{\boldsymbol{M}}{}^{\otimes n}t_n(x_1, y_1, \ldots, y_{n-1}) = 0$ for $x_1 \in K^\perp,\ y_1, \ldots, y_{n-1} \in \boldsymbol{M}$ by smoothness. Again, we conclude that $\mathrm{supp}(E_{\boldsymbol{M}}{}^{\otimes n}t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}$ by symmetry. □

Lemma C.3. *Let $\boldsymbol{M}$ be a spacetime, and consider $E_{\boldsymbol{M}}$ as a map from $\mathscr{T}^1(\boldsymbol{M})$*

to $\mathcal{D}'(\boldsymbol{M})$*. Then for all* $n \in \mathbb{N}$*,*

$$\ker {E_{\boldsymbol{M}}}^{\otimes n} = \left\{ \sum_{k=1}^{n} (P_{\boldsymbol{M}})_k u_k : u_k \in \mathscr{T}^n(\boldsymbol{M}) \right\}, \tag{C.29}$$

where $(P_{\boldsymbol{M}})_k = \mathbf{1}^{\otimes k-1} \otimes P_{\boldsymbol{M}} \otimes \mathbf{1}^{\otimes n-k}$*.*

Proof. Let S_n denote the set in the right hand side of the above equation. Clearly any distribution in S_n lies in $\ker {E_{\boldsymbol{M}}}^{\otimes n}$; therefore, we need only prove the inclusion $\ker {E_{\boldsymbol{M}}}^{\otimes n} \subseteq S_n$. Suppose that $t \in \mathscr{T}^1(\boldsymbol{M})$, with $E_{\boldsymbol{M}} t = 0$. We have $E^+_{\boldsymbol{M}} t = E^-_{\boldsymbol{M}} t$; t is compactly supported, and so by the support properties of $E^{\pm}_{\boldsymbol{M}} t$ we must have that $E^+_{\boldsymbol{M}} t$ is compactly supported. But $t = P_{\boldsymbol{M}} E^+_{\boldsymbol{M}} t$, and therefore $\ker E_{\boldsymbol{M}} \subseteq P_{\boldsymbol{M}} \mathscr{T}^1(\boldsymbol{M})$. This proves the case where $n = 1$. Now suppose that $t_n \in \mathscr{T}^n(\boldsymbol{M})$ with ${E_{\boldsymbol{M}}}^{\otimes n} t_n = 0$. Then pick a Cauchy partition function χ for $\boldsymbol{M}$. We know that $E_{\boldsymbol{M}} P_{\boldsymbol{M}} \chi E_{\boldsymbol{M}} t = E_{\boldsymbol{M}} t$ and that $P_{\boldsymbol{M}} \chi E_{\boldsymbol{M}} t$ is compactly supported for any $t \in \mathscr{T}^1(\boldsymbol{M})$; it follows that

$$(E^+_{\boldsymbol{M}} \otimes (P_{\boldsymbol{M}} \chi E_{\boldsymbol{M}})^{\otimes n-1}) t_n = (E^-_{\boldsymbol{M}} \otimes (P_{\boldsymbol{M}} \chi E_{\boldsymbol{M}})^{\otimes n-1}) t_n, \tag{C.30}$$

and that by the support properties given above the left hand side of the above equation must be compactly supported. Denoting this as u_1 we therefore have $t_n = (P_{\boldsymbol{M}})_1 u_1 + v_1$, where

$$v_1 = t_n - \mathbf{1} \otimes (P_{\boldsymbol{M}} \chi E_{\boldsymbol{M}})^{\otimes n-1} t_n. \tag{C.31}$$

As observed earlier, since u_1 is obtained from an element of $\mathscr{T}^n(\boldsymbol{M})$ by the application of $E^{\pm}_{\boldsymbol{M}}$, differential operators and multiplication by smooth functions, it follows that its wavefront set also has the desired properties for u_1 itself to be an element of $\mathscr{T}^n(\boldsymbol{M})$.

Now $t_n, (P_{\boldsymbol{M}})_1 u_1 \in \mathscr{T}^n(\boldsymbol{M})$, so $v_1 \in \mathscr{T}^n(\boldsymbol{M})$; but $\mathbf{1} \otimes {E_{\boldsymbol{M}}}^{\otimes n-1} v_1 = 0$, so $v_1 \in \ker(\mathbf{1} \otimes {E_{\boldsymbol{M}}}^{\otimes n-1})$: we may repeat the argument to see that $v_1 = (P_{\boldsymbol{M}})_2 u_2 + v_2$ for some $u_2, v_2 \in \mathscr{T}^n(\boldsymbol{M})$ with $v_2 \in \ker(\mathbf{1} \otimes \mathbf{1} \otimes {E_{\boldsymbol{M}}}^{\otimes n-2})$. Continuing the argument further, we may eventually see that $t_n = (P_{\boldsymbol{M}})_1 u_1 + \cdots + (P_{\boldsymbol{M}})_n u_n$ for some $u_1, \ldots, u_n \in \mathscr{T}^n(\boldsymbol{M})$, and consequently $t_n \in S_n$.

□

Here we prove Lemma 4.2.5:

Lemma C.4. *Let* $\boldsymbol{M} \in \mathsf{Loc}$*,* $O \in \mathscr{O}(\boldsymbol{M})$ *and* $K \in \mathscr{K}(\boldsymbol{M}; O)$*. Let* $t_n \in \mathscr{T}^n(\boldsymbol{M})$ *for some* $n \geq 1$*, and suppose that for all* $f \in C_0^\infty(\boldsymbol{M})$ *and* $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$ *we have*

$$\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{\mu\nu} T^{\mu\nu}[E_{\boldsymbol{M}}\tau_f^n, E_{\boldsymbol{M}} f] = 0, \tag{C.32}$$

where τ_f^n *is defined as in* (4.43)*. Then, in the massive minimally coupled case (*$m \neq 0$*,* $\xi = 0$*) and massive conformally coupled case (*$m \neq 0$*,* $\xi = \frac{d-2}{4(d-1)}$*, where* d *is the dimension of* $\boldsymbol{M}$*), we have* $\mathrm{supp}({E_{\boldsymbol{M}}}^{\otimes n} t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}$*.*

Proof. We will consider the massive minimally coupled case first, in which $m \neq 0$ and $\xi = 0$. Clearly (C.32) implies that $T^{\mu\nu}[E_{\boldsymbol{M}}\tau_f^n, E_{\boldsymbol{M}} f](x) = 0$ for all $f \in C_0^\infty(\boldsymbol{M})$ and $x \in K^\perp$; now, we fix $x \in K^\perp$ and pick some $f \in C_0^\infty(\boldsymbol{M})$ such that $(E_{\boldsymbol{M}} f)(x) \neq 0$. In the case where $\boldsymbol{M}$ has dimension 2, we note that

$$0 = g_{\mu\nu} T^{\mu\nu}[E_{\boldsymbol{M}}\tau_f^n, E_{\boldsymbol{M}} f](x) = m^2 E_{\boldsymbol{M}}\tau_f^n(x) E_{\boldsymbol{M}} f(x), \tag{C.33}$$

and consequently $E_{\boldsymbol{M}}\tau_f^n(x) = 0$ for any such f; in higher dimensions, we choose normal coordinates at x oriented such that $\nabla_2 E_{\boldsymbol{M}} f(x) = \cdots = \nabla_{d-1} E_{\boldsymbol{M}} f(x) = 0$, and define the tensor $v_{\mu\nu} \in T_x^*(\boldsymbol{M}) \otimes T_x^*(\boldsymbol{M})$ such that in these coordinates we have $v_{00} = 1$, $v_{11} = -1$, and all other entries zero. It follows that $v_{\mu\nu} g^{\mu\nu}(x) = 2$ and $v_{\mu\nu} \nabla^{(\mu} E_{\boldsymbol{M}}\tau_f^n(x) \nabla^{\nu)} E_{\boldsymbol{M}} f(x) = \nabla^\mu E_{\boldsymbol{M}}\tau_f^n(x) \nabla_\mu E_{\boldsymbol{M}} f(x)$, so that we have

$$0 = v_{\mu\nu} T^{\mu\nu}[E_{\boldsymbol{M}}\tau_f^n, E_{\boldsymbol{M}} f](x) = m^2 E_{\boldsymbol{M}}\tau_f^n(x) E_{\boldsymbol{M}} f(x). \tag{C.34}$$

Again, we may conclude that $E_{\boldsymbol{M}}\tau_f^n(x) = 0$ for any such f.

When $n = 1$ we deduce immediately that $E_{\boldsymbol{M}} t_1(x) = 0$ for all $x \in K^\perp$. For $n = 2$, we note that τ_f^2 is linear in f, and as any $f \in C_0^\infty(\boldsymbol{M})$ may be expressed as $f = f_1 - f_2$ where $E_{\boldsymbol{M}} f_1(x) \neq 0 \neq E_{\boldsymbol{M}} f_2(x)$ we have $E_{\boldsymbol{M}}\tau_f^2(x) = -\int_{\boldsymbol{M}} dy\, ({E_{\boldsymbol{M}}}^{\otimes 2} t_2)(x, y) f(y) = 0$ for all $f \in C_0^\infty(\boldsymbol{M})$. Therefore ${E_{\boldsymbol{M}}}^{\otimes 2} t_2(x, y) = 0$ for all $x \in K^\perp$, and so $\mathrm{supp}({E_{\boldsymbol{M}}}^{\otimes 2} t_2) \subset J_{\boldsymbol{M}}(K)^{\times 2}$ by symmetry. For $n > 2$, we pick $f \in C_0^\infty(\boldsymbol{M})$ with $E_{\boldsymbol{M}} f(x) \neq 0$ and let

$f_1 \in C_0^\infty(\boldsymbol{M})$ be arbitrary; for sufficiently small κ we have $E_{\boldsymbol{M}}(f+\kappa f_1)(x) \neq 0$, and so $E_{\boldsymbol{M}}\tau^n_{f+\kappa f_1}(x) = 0$. We differentiate this expression with respect to κ and set $\kappa = 0$, which yields

$$\int_{\boldsymbol{M}^{\times(n-1)}} d^{n-1}y\,(E_{\boldsymbol{M}}{}^{\otimes n}t_n)(x, y_1, \ldots, y_{n-1}) f_1(y_1) f(y_2) \cdots f(y_{n-1}) = 0; \quad \text{(C.35)}$$

we may then repeat this argument to see that

$$\int_{\boldsymbol{M}^{\times(n-1)}} d^{n-1}y\,(E_{\boldsymbol{M}}{}^{\otimes n}t_n)(x, y_1, \ldots, y_{n-1}) f_1(y_1) f_2(y_2) \cdots f_{n-1}(y_{n-1}) = 0 \quad \text{(C.36)}$$

for any $f_1, \ldots, f_{n-1} \in C_0^\infty(\boldsymbol{M})$. It follows that $E_{\boldsymbol{M}}{}^{\otimes n}t_n(x, y_1, \ldots, y_{n-1}) = 0$ for all $x \in K^\perp$, and by symmetry we have $\operatorname{supp}(E_{\boldsymbol{M}}{}^{\otimes n}t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}$. This concludes the proof for the massive minimally coupled case.

In the massive conformally coupled case, where $m \neq 0$ and $\xi = \frac{d-2}{4(d-1)}$, where d is the dimension of $\boldsymbol{M}$, we have $g_{\mu\nu}T^{\mu\nu}[\phi_1, \phi_2] = m^2\phi_1\phi_2$ for any $\phi_1, \phi_2 \in E_{\boldsymbol{M}}C_0^\infty(\boldsymbol{M})$. It follows that $E_{\boldsymbol{M}}\tau^n_f(x)E_{\boldsymbol{M}}f(x)$ for all $x \in K^\perp$ and $f \in C_0^\infty(\boldsymbol{M})$. We may then use the same argument as above to show that $\operatorname{supp}(E_{\boldsymbol{M}}{}^{\otimes n}t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}$. □

Finally, we give the proof of Lemma 4.2.7:

Lemma C.5. *Let $K \in \mathscr{K}(\boldsymbol{M})$ and $W \in \mathscr{W}^\bullet(\boldsymbol{M}; K)$, with W_H represented by $T_H = \sum_{n=0}^N t_n \in \mathscr{T}_H(\boldsymbol{M})$ for some fixed $H \in \mathscr{H}(\boldsymbol{M})$. Then, in the massive minimally coupled and massive conformally coupled cases,*

(a). *$\tilde{t}_{n;\boldsymbol{h}} \sim_{\boldsymbol{M}} t_n$ for all $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$ and $n \geq 0$,*

(b). *$\operatorname{supp}(E_{\boldsymbol{M}}{}^{\otimes n}t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}$ for each $n \geq 1$.*

Proof. We recall from Lemma 4.2.6 that $W \in \mathscr{W}^\bullet(\boldsymbol{M}; K)$ if and only if (4.53) and (4.54) are both satisfied. We may freely replace $\boldsymbol{h}$ in (4.53) with $s\boldsymbol{h}$ for sufficiently small $s \in \mathbb{R}$, and then differentate with respect to s to see that

$$\frac{d}{ds}\left((\beta[s\boldsymbol{h}])^{\otimes n}\tilde{t}_{n;s\boldsymbol{h}}\right)[E_{\boldsymbol{M}}f]\bigg|_{s=0} = 0 \quad \text{(C.37)}$$

for all $f \in C_0^\infty(\boldsymbol{M})$ and $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$. Since $\beta[\mathbf{0}] = \mathbf{1}$ and $\tilde{t}_{n;\mathbf{0}} = t_n$, this is equivalent to

$$\frac{d}{ds}\left((\beta[s\boldsymbol{h}])^{\otimes n} t_n\right)[E_{\boldsymbol{M}} f]\bigg|_{s=0} + \frac{d}{ds}\tilde{t}_{n;s\boldsymbol{h}}[E_{\boldsymbol{M}} f]\bigg|_{s=0} = 0, \tag{C.38}$$

and by corollary 4.1.7, we have

$$\frac{d}{ds}\left((\beta[s\boldsymbol{h}])^{\otimes n} t_n\right)[E_{\boldsymbol{M}} f]\bigg|_{s=0} = n \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{\mu\nu} T^{\mu\nu}[E_{\boldsymbol{M}} \tau_f^n, E_{\boldsymbol{M}} f], \tag{C.39}$$

where as before τ_f^n is defined according to (4.43).

Now, rather than attempting to prove both parts of this lemma separately, we will instead show that (a) implies (b) for any particular n, then prove (a) for $n = N$ and $n = N - 1$ and proceed by descent. We therefore assume that $\tilde{t}_{n;\boldsymbol{h}} \sim_{\boldsymbol{M}} t_n$ for some $n \geq 0$; this implies that the second term of (C.38) vanishes, and by (C.39) we have

$$\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{\mu\nu} T^{\mu\nu}[E_{\boldsymbol{M}} \tau_f^n, E_{\boldsymbol{M}} f] = 0. \tag{C.40}$$

It follows from Lemma 4.2.5 that in the massive minimally/conformally coupled theories, we have $\mathrm{supp}({E_{\boldsymbol{M}}}^{\otimes n} t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}$.

We may observe from (4.52) that $\tilde{t}_{N;\boldsymbol{h}} = t_N$ and $\tilde{t}_{N-1;\boldsymbol{h}} = t_{N-1}$ for any $\boldsymbol{h} \in H(\boldsymbol{M})$, and therefore (a) (and consequently (b)) certainly hold for $n = N$ and $n = N - 1$. Now, for a given $n < N - 1$ we assume that (a) and (b) are satisfied for all $n + 2k$ with $2 \leq 2k \leq N - n$. We may use Lemma 4.2.2 to see that for any open neighbourhood S of an arbitrary Cauchy surface, each distribution t_{n+2k} may be written

$$t_{n+2k} = s + \sum_{j=1}^{n+2k} (P_{\boldsymbol{M}})_j u_{jk} \tag{C.41}$$

where $s, u_{jk} \in \mathscr{T}^{n+2k}(\boldsymbol{M})$ and $\mathrm{supp}(s) \subset (J_{\boldsymbol{M}}(K) \cap S)^{\times(n+2k)}$. If we now fix some $\boldsymbol{h} \in H(\boldsymbol{M}, K^\perp)$ and choose S such that $J_{\boldsymbol{M}}(\mathrm{supp}(\boldsymbol{h})) \cap J_{\boldsymbol{M}}(K) \cap S = \emptyset$, it follows that

$$(\eta_H - \eta_{\check{H}_{\boldsymbol{h}}})^k(s) = 0, \tag{C.42}$$

recalling from Lemma 4.1.5 that $\operatorname{supp}(H - \check{H}_{\boldsymbol{h}}) \subset (J_{\boldsymbol{M}}(\operatorname{supp} \boldsymbol{h}))^{\times 2}$. But this means that for all $f \in C_0^\infty(\boldsymbol{M})$, we have

$$\Big((\eta_H - \eta_{\check{H}_{\boldsymbol{h}}})^k (t_{n+2k})\Big)[E_{\boldsymbol{M}} f] = \sum_{j=1}^{n+2k} \Big((\eta_H - \eta_{\check{H}_{\boldsymbol{h}}})^k ((P_{\boldsymbol{M}})_j u_{jk})\Big)[E_{\boldsymbol{M}} f] = 0 \tag{C.43}$$

for $2 \leq 2k \leq N - n$, where we have used the fact that $(P_{\boldsymbol{M}} \otimes \mathbf{1})H = 0 = (\mathbf{1} \otimes P_{\boldsymbol{M}})H$ for any $H \in \mathscr{H}(\boldsymbol{M})$. By (4.52), we therefore have $\tilde{t}_{n;\boldsymbol{h}} \sim_{\boldsymbol{M}} t_n$ for any $\boldsymbol{h} \in H(\boldsymbol{M}; K^\perp)$, and consequently $\operatorname{supp}(E_{\boldsymbol{M}}{}^{\otimes n} t_n) \subset J_{\boldsymbol{M}}(K)^{\times n}$.

Since (a) and (b) hold for $n = N$ and $n = N - 1$, it follows by descent that they are satisfied for all $n \geq 0$. □

Lemma C.6. *Let $\boldsymbol{M} \in \mathsf{Loc}$ and $O \in \mathscr{O}(\boldsymbol{M})$, and suppose that a distribution $t_n \in \mathscr{T}^n(\boldsymbol{M})$ is supported within $O^{\times n}$. Then there exists some $u \in \mathscr{T}^n(\boldsymbol{M})$ such that $t_n \sim_{\boldsymbol{M}} u$ and u may be written as a finite sum of distributions, $u = \sum_{r=1}^R u_r$, where each $u_r \in \mathscr{T}^n(\boldsymbol{M})$ is supported within $K_r^{\times n}$ for some $K_r \in \mathscr{K}(\boldsymbol{M}; O)$.*

Proof. O is globally hyperbolic, so $\boldsymbol{M}|_O$ admits a spacelike Cauchy surface Σ_O. This may or may not be extendible to a spacelike Cauchy surface of $\boldsymbol{M}$, as seen in Example 1.1.6. However, since t_n is compactly supported, by taking a compact exhaustion of Σ_O we will find some compact $X \subset \Sigma_O$ with $\operatorname{supp}(t_n) \subset D_{\boldsymbol{M}}(X)^{\times n}$; we may then extend X to a spacelike Cauchy surface Σ of $\boldsymbol{M}$, by [5, Theorem 1.1].

It is possible to take a cover of $\Sigma_X = X \cap \Sigma$ by Cauchy balls based on $\Sigma \cap O$ that are sufficiently 'small' that every union of $m \leq n$ of the balls has an open neighbourhood in $\Sigma \cap O$ with at most m connected components, each of which is diffeomorphic to a ball in $\mathbb{R}^{d-1}$. We take a finite subcover $B_1, \ldots, B_N$ of this cover (which must obey the same property). Now, we may find a globally hyperbolic neighbourhood S of Σ with the property that $S_X = J_{\boldsymbol{M}}(X) \cap S$ is contained within $\bigcup_{i=1}^N D_{\boldsymbol{M}}(B_i)$. We let χ be a Cauchy partition function for S and write $O_i = D_{\boldsymbol{M}}(B_i)$ for each i, so that each O_i is a diamond region in $\boldsymbol{M}$.

We take a smooth partition of unity $\sum_{i=1}^N \kappa_i = 1$ for S_X, satisfying $\operatorname{supp}(\kappa_i) \subset O_i$. We denote $K_i := \operatorname{supp}(\kappa_i) \cap S_X$, which is compact. Now

$t_n \sim_{\boldsymbol{M}} u = (P_{\boldsymbol{M}} \chi E_{\boldsymbol{M}})^{\otimes n} t_n$, and $\operatorname{supp}(u) \subset {S_X}^{\otimes n} \subset \left(\bigcup_{i=1}^N K_i\right)^{\otimes n}$, so

$$u = (\kappa_1 + \cdots + \kappa_N)^{\otimes n} u = \sum_{\boldsymbol{a} \in \{1,\dots,N\}^{\times n}} \kappa_{\boldsymbol{a}} u \tag{C.44}$$

where $\kappa_{\boldsymbol{a}} := \kappa_{a_1} \otimes \cdots \otimes \kappa_{a_n}$. Now $\kappa_{\boldsymbol{a}} u$ is supported within $K_{a_1} \times \cdots \times K_{a_n}$; by the defining property of $\{B_1, \dots, B_N\}$, there exists a collection $m \leq n$ disjoint Cauchy balls $\hat{B}_1, \dots, \hat{B}_m$ in Σ that are based in O (some of which may be elements of $\{B_1, \dots, B_N\}$) such that $K_{\boldsymbol{a}} := \bigcup_{i=1}^N K_{a_i}$ is contained within $\bigcup_{j=1}^m D_{\boldsymbol{M}}(\hat{B}_j)$. Since the $\hat{B}_j$ are disjoint and contained within a single Cauchy surface, it follows that they are spacelike separated, and therefore $K_{\boldsymbol{a}}$ has a multi-diamond neighbourhood based in O.

Since $\{1, \dots, N\}^{\times n}$ is finite, we may write all the $\kappa_{\boldsymbol{a}} u$ in a single list $u_1, \dots, u_R$, with u_r supported in $K_r^{\times n} \in \mathscr{K}(\boldsymbol{M}; O)$. We clearly have $t_n \sim_{\boldsymbol{M}} \sum_{r=1}^R u_r$ and $u_r \in \mathscr{T}^n(\boldsymbol{M})$. □

Appendix D

Proof of Lemma 7.2.7

We present a proof of the following result, which is essential for computing the dynamical solution spaces for the framed-spacetime classical Dirac field theory. We reiterate that a proof of a very similar result is contained in [55, §4.3.2], although the result in question refers directly to the quantized version of the Dirac theory contained therein. We present the following proof for the sake of completeness.

Lemma D.1. *Let* $\mathcal{M} = (\boldsymbol{M}, \boldsymbol{\varepsilon}) \in \mathsf{FLoc}_4$*, and consider* $\boldsymbol{h} \in H(\boldsymbol{M})$*. For any* $u \in \mathscr{L}^s(\mathcal{M})$ *and* $v \in \mathscr{L}^c(\mathcal{M})$*, we have*

$$\begin{aligned} \frac{d}{ds} S^s_{\mathcal{M}} P^s_{\mathcal{M}[s\boldsymbol{h}]} u \bigg|_{s=0} &= S^s_{\mathcal{M}} \left(\frac{i}{2} h^{ab} \boldsymbol{\gamma}_a \nabla_b u - \frac{i}{4} (\delta_{\boldsymbol{h}} \Gamma^\mu{}_{\nu\rho}) \varepsilon^b_\mu \varepsilon^\nu_a \varepsilon^\rho_c \, \boldsymbol{\gamma}^a \boldsymbol{\gamma}_b \boldsymbol{\gamma}^c u \right), \\ \frac{d}{ds} S^c_{\mathcal{M}} P^c_{\mathcal{M}[s\boldsymbol{h}]} v \bigg|_{s=0} &= S^c_{\mathcal{M}} \left(-\frac{i}{2} h^{ab} (\nabla_b v) \boldsymbol{\gamma}_a - \frac{i}{4} (\delta_{\boldsymbol{h}} \Gamma^\mu{}_{\nu\rho}) \varepsilon^b_\mu \varepsilon^\nu_a \varepsilon^\rho_c \, v \boldsymbol{\gamma}^a \boldsymbol{\gamma}_b \boldsymbol{\gamma}^c \right), \end{aligned} \tag{D.1}$$

where $\delta_{\boldsymbol{h}} \Gamma^\mu{}_{\nu\rho} := \frac{d}{ds} (\Gamma_{\boldsymbol{g}+s\boldsymbol{h}})^\mu{}_{\nu\rho} \Big|_{s=0}$*. Moreover, for* $u' \in \mathscr{L}^s(\mathcal{M})$ *and* $v' \in \mathscr{L}^c(\mathcal{M})$*, it also holds that*

$$\begin{aligned} s^s_{\mathcal{M}}(u', F^s_{\mathcal{M}}[\boldsymbol{h}] u) &= -\frac{1}{4} \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \, h_{ab} \left(\langle \overline{A_{\boldsymbol{M}} u'}, \boldsymbol{\gamma}^{(a} \nabla^{b)} u \rangle - \langle \overline{A_{\boldsymbol{M}} \nabla^{(a} u'}, \boldsymbol{\gamma}^{b)} u \rangle \right), \\ s^c_{\mathcal{M}}(v', F^c_{\mathcal{M}}[\boldsymbol{h}] v) &= -\frac{1}{4} \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \, h_{ab} \left(\langle \nabla^{(a} v \boldsymbol{\gamma}^{b)}, A^{-1}_{\boldsymbol{M}} \overline{v'} \rangle - \langle v \boldsymbol{\gamma}^{(a}, A^{-1}_{\boldsymbol{M}} \overline{\nabla^{b)} v'} \rangle \right). \end{aligned} \tag{D.2}$$

Proof. Throughout, the symbol $\delta_{\boldsymbol{h}}$ will denote a derivation defined on frame-dependent quantities as follows. We choose a smoothly varying family of frames $\boldsymbol{\varepsilon}(s) \in \Gamma^\infty(F_+^\uparrow \boldsymbol{M}[s\boldsymbol{h}])$ for $\{s \in \mathbb{R} : s\boldsymbol{h} \in H(\boldsymbol{M})\}$. These frames must satisfy $\boldsymbol{\varepsilon}(0) = \boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}(s) = \boldsymbol{\varepsilon}$ outside supp$(\boldsymbol{h})$. It follows that there is always a homotopy from $\boldsymbol{\varepsilon}(s)$ to $\boldsymbol{\varepsilon}$ as ordered bases of $T\boldsymbol{M}$ that is fixed and equal to $\boldsymbol{\varepsilon}$ outside supp$(\boldsymbol{h})$, so for each s the relative Cauchy evolution may be defined as in (7.96) with $\mathcal{M}[s\boldsymbol{h}] = (\boldsymbol{M}[s\boldsymbol{h}], \boldsymbol{\varepsilon}(s))$. Now, for every frame dependent quantity $\boldsymbol{Q}[\boldsymbol{\varepsilon}]$ we define

$$\delta_{\boldsymbol{h}}\boldsymbol{Q} := \frac{d}{ds}\boldsymbol{Q}[\boldsymbol{\varepsilon}(s)]\bigg|_{s=0}. \tag{D.3}$$

Since the perturbation $s\boldsymbol{h}$ may be derived from the frame $\boldsymbol{\varepsilon}(s)$ through $s\boldsymbol{h} = \eta_{ab}(\varepsilon(s)^a \otimes \varepsilon(s)^b) - \boldsymbol{g}$, it follows that $\delta_{\boldsymbol{h}}$ is also defined (in the usual way) for any quantity that depends explicitly only on the perturbation $\boldsymbol{h}$.

We have

$$\begin{aligned}\delta_{\boldsymbol{h}}(\nabla\!\!\!/^{s}_{\mathcal{M}}u) &= \boldsymbol{\gamma}^a\delta_{\boldsymbol{h}}(D_a u + \boldsymbol{\sigma}_a u)\\ &= (\delta_{\boldsymbol{h}}\varepsilon^\mu_a)\varepsilon^b_\mu\boldsymbol{\gamma}^a D_b u + \frac{1}{4}(\delta_{\boldsymbol{h}}\Gamma^b{}_{ac})\boldsymbol{\gamma}^a\boldsymbol{\gamma}_b\boldsymbol{\gamma}^c u,\end{aligned} \tag{D.4}$$

and through $\Gamma^\mu{}_{\nu\rho} = \Gamma^a{}_{bc}\varepsilon^\mu_a\varepsilon^b_\nu\varepsilon^c_\rho - \varepsilon^a_\rho\partial_\nu\varepsilon^\mu_a$, we may see that

$$\begin{aligned}\delta_{\boldsymbol{h}}\Gamma^b{}_{ac} &= \delta_{\boldsymbol{h}}\left(\varepsilon^b_\mu\varepsilon^\nu_a(\Gamma^\mu{}_{\nu\rho}\varepsilon^\rho_c + \partial_\nu\varepsilon^\mu_c)\right)\\ &= (\delta_{\boldsymbol{h}}\varepsilon^\mu_a)\varepsilon^d_\mu\Gamma^b{}_{dc} + (\delta_{\boldsymbol{h}}\varepsilon^\mu_c)\varepsilon^d_\mu\Gamma^b{}_{ad} - (\delta_{\boldsymbol{h}}\varepsilon^\mu_d)\varepsilon^b_\mu\Gamma^d{}_{ac}\\ &\qquad + D_a((\delta_{\boldsymbol{h}}\varepsilon^\mu_c)\varepsilon^b_\mu) + (\delta_{\boldsymbol{h}}\Gamma^\mu{}_{\nu\rho})\varepsilon^b_\mu\varepsilon^\nu_a\varepsilon^\rho_c,\end{aligned} \tag{D.5}$$

so

$$\begin{aligned}\delta_{\boldsymbol{h}}(\nabla\!\!\!/^{s}_{\mathcal{M}}u) = (\delta_{\boldsymbol{h}}\varepsilon^\mu_a)\varepsilon^b_\mu\boldsymbol{\gamma}^a\nabla_b u &+ \frac{1}{4}D_c((\delta_{\boldsymbol{h}}\varepsilon^\mu_a)\varepsilon^b_\mu)\boldsymbol{\gamma}^c\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a u\\ &+ \frac{1}{4}(\delta_{\boldsymbol{h}}\varepsilon^\mu_a)\varepsilon^b_\mu\left(\Gamma^d{}_{cb}\boldsymbol{\gamma}^c\boldsymbol{\gamma}_d\boldsymbol{\gamma}^a - \Gamma^a{}_{dc}\boldsymbol{\gamma}^d\boldsymbol{\gamma}_b\boldsymbol{\gamma}^c\right)u\\ &\qquad + \frac{1}{4}(\delta_{\boldsymbol{h}}\Gamma^\mu{}_{\nu\rho})\varepsilon^b_\mu\varepsilon^\nu_a\varepsilon^\rho_c\boldsymbol{\gamma}^a\boldsymbol{\gamma}_b\boldsymbol{\gamma}^c u.\end{aligned} \tag{D.6}$$

Now,

$$\begin{aligned} D_c((\delta_{\boldsymbol{h}}\varepsilon_a^\mu)\varepsilon_\mu^b)\boldsymbol{\gamma}^c\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a u &= \boldsymbol{\gamma}^c D_c((\delta_{\boldsymbol{h}}\varepsilon_a^\mu)\varepsilon_\mu^b\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a u) - (\delta_{\boldsymbol{h}}\varepsilon_a^\mu)\varepsilon_\mu^b\boldsymbol{\gamma}^c\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a D_c u \\ &= \nabla\!\!\!\!/^s_{\mathcal{M}}((\delta_{\boldsymbol{h}}\varepsilon_a^\mu)\varepsilon_\mu^b\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a u) - (\delta_{\boldsymbol{h}}\varepsilon_a^\mu)\varepsilon_\mu^b\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a\nabla\!\!\!\!/^s_{\mathcal{M}} u \\ &\quad + (\delta_{\boldsymbol{h}}\varepsilon_a^\mu)\varepsilon_\mu^b\left([\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a, \boldsymbol{\gamma}^c]D_c u + [\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a, \boldsymbol{\gamma}^c\boldsymbol{\sigma}_c]u\right). \end{aligned} \tag{D.7}$$

Since $P^s_{\mathcal{M}} u = 0$, and any variation along $\boldsymbol{h}$ must be supported within $\mathrm{supp}(\boldsymbol{h})$, we have

$$\begin{aligned} S^s_{\mathcal{M}} &\left(\nabla\!\!\!\!/^s_{\mathcal{M}}((\delta_{\boldsymbol{h}}\varepsilon_a^\mu)\varepsilon_\mu^b\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a u) - (\delta_{\boldsymbol{h}}\varepsilon_a^\mu)\varepsilon_\mu^b\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a\nabla\!\!\!\!/^s_{\mathcal{M}} u\right) \\ &= iS^s_{\mathcal{M}}\left(P^s_{\mathcal{M}}((\delta_{\boldsymbol{h}}\varepsilon_a^\mu)\varepsilon_\mu^b\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a u) - (\delta_{\boldsymbol{h}}\varepsilon_a^\mu)\varepsilon_\mu^b\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a P^s_{\mathcal{M}} u\right) = 0 \end{aligned} \tag{D.8}$$

Therefore $D_c((\delta_{\boldsymbol{h}}\varepsilon_a^\mu)\varepsilon_\mu^b)\boldsymbol{\gamma}^c\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a u$ is equivalent, modulo $S^s_{\mathcal{M}}$, to

$$(\delta_{\boldsymbol{h}}\varepsilon_a^\mu)\varepsilon_\mu^b\left([\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a, \boldsymbol{\gamma}^c]D_c u + [\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a, \boldsymbol{\gamma}^c\boldsymbol{\sigma}_c]u\right). \tag{D.9}$$

We have $[\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a, \boldsymbol{\gamma}^c] = \boldsymbol{\gamma}_b\{\boldsymbol{\gamma}^a, \boldsymbol{\gamma}^c\} - \{\boldsymbol{\gamma}_b, \boldsymbol{\gamma}^c\}\boldsymbol{\gamma}^a = 2\eta^{ac}\boldsymbol{\gamma}_b - 2\delta_b^c\boldsymbol{\gamma}^a$, so

$$[\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a, \boldsymbol{\gamma}^c]D_c u = 2(\eta^{ac}\boldsymbol{\gamma}_b D_c u - \boldsymbol{\gamma}^a D_b u). \tag{D.10}$$

We may also use the antisymmetry property (7.18) and

$$[\boldsymbol{\gamma}^a, \boldsymbol{\sigma}_b] = \Gamma^a{}_{bc}\boldsymbol{\gamma}^c = \frac{1}{2}\Gamma^a{}_{dc}\{\boldsymbol{\gamma}_b, \boldsymbol{\gamma}^d\}\boldsymbol{\gamma}^c \tag{D.11}$$

to show that

$$
\begin{aligned}
[\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a, \boldsymbol{\gamma}^c\boldsymbol{\sigma}_c] &= \frac{1}{4}\Gamma^d{}_{ce}[\boldsymbol{\gamma}_b\boldsymbol{\gamma}^a, \boldsymbol{\gamma}^c\boldsymbol{\gamma}_d\boldsymbol{\gamma}^e] \\
&= \frac{1}{2}\Gamma^d{}_{ce}(\eta^{ac}\boldsymbol{\gamma}_b\boldsymbol{\gamma}_d\boldsymbol{\gamma}^e - \delta^a_d\boldsymbol{\gamma}_b\boldsymbol{\gamma}^c\boldsymbol{\gamma}^e + \eta^{ac}\boldsymbol{\gamma}_b\boldsymbol{\gamma}^c\boldsymbol{\gamma}_d \\
&\qquad - \delta^c_b\boldsymbol{\gamma}_d\boldsymbol{\gamma}^e\boldsymbol{\gamma}^a + \eta_{bd}\boldsymbol{\gamma}^c\boldsymbol{\gamma}^e\boldsymbol{\gamma}^a - \delta^e_b\boldsymbol{\gamma}^c\boldsymbol{\gamma}_d\boldsymbol{\gamma}^a) \\
&= 2(\eta^{ac}\boldsymbol{\gamma}_b\boldsymbol{\sigma}_c - \boldsymbol{\sigma}_b\boldsymbol{\gamma}^a) - \Gamma^a{}_{ce}\boldsymbol{\gamma}_b\boldsymbol{\gamma}^c\boldsymbol{\gamma}^e - \Gamma^d{}_{cb}\boldsymbol{\gamma}^c\boldsymbol{\gamma}_d\boldsymbol{\gamma}^a \\
&= 2(\eta^{ac}\boldsymbol{\gamma}_b\boldsymbol{\sigma}_c - \boldsymbol{\gamma}^a\boldsymbol{\sigma}_b + [\boldsymbol{\gamma}^a, \boldsymbol{\sigma}_b]) - \Gamma^a{}_{dc}\boldsymbol{\gamma}_b\boldsymbol{\gamma}^d\boldsymbol{\gamma}^c - \Gamma^d{}_{cb}\boldsymbol{\gamma}^c\boldsymbol{\gamma}_d\boldsymbol{\gamma}^a \\
&= 2(\eta^{ac}\boldsymbol{\gamma}_b\boldsymbol{\sigma}_c - \boldsymbol{\gamma}^a\boldsymbol{\sigma}_b) + \Gamma^a{}_{dc}\boldsymbol{\gamma}^d\boldsymbol{\gamma}_b\boldsymbol{\gamma}^c - \Gamma^d{}_{cb}\boldsymbol{\gamma}^c\boldsymbol{\gamma}_d\boldsymbol{\gamma}^a. \qquad \text{(D.12)}
\end{aligned}
$$

This and (D.10) entail that (D.9) is equal to

$$
2(\delta_{\boldsymbol{h}}\varepsilon^\mu_a)\varepsilon^b_\mu(\eta^{ac}\boldsymbol{\gamma}_b\nabla_c u - \boldsymbol{\gamma}^a\nabla_b u) + (\delta_{\boldsymbol{h}}\varepsilon^\mu_a)\varepsilon^b_\mu\left(\Gamma^a{}_{dc}\boldsymbol{\gamma}^d\boldsymbol{\gamma}_b\boldsymbol{\gamma}^c - \Gamma^d{}_{cb}\boldsymbol{\gamma}^c\boldsymbol{\gamma}_d\boldsymbol{\gamma}^a\right)u. \qquad \text{(D.13)}
$$

Substituting into (D.6), it follows that $\delta_{\boldsymbol{h}}(\nabla\!\!\!/^s_{\mathcal{M}}u)$ is then equivalent modulo $S^s_{\mathcal{M}}$ to

$$
\frac{1}{2}(\delta_{\boldsymbol{h}}\varepsilon^\mu_a)\varepsilon^b_\mu(\eta^{ac}\boldsymbol{\gamma}_b\nabla_c u + \boldsymbol{\gamma}^a\nabla_b u) + \frac{1}{4}(\delta_{\boldsymbol{h}}\Gamma^\mu{}_{\nu\rho})\varepsilon^b_\mu\varepsilon^\nu_a\varepsilon^\rho_c\boldsymbol{\gamma}^a\boldsymbol{\gamma}_b\boldsymbol{\gamma}^c u. \qquad \text{(D.14)}
$$

Finally, we have

$$
\begin{aligned}
\frac{1}{2}(\delta_{\boldsymbol{h}}\varepsilon^\mu_a)\varepsilon^b_\mu\,(\eta^{ac}\boldsymbol{\gamma}_b\nabla_c u + \boldsymbol{\gamma}^a\nabla_b u) &= \frac{1}{2}\delta_{\boldsymbol{h}}(\varepsilon^\mu_c\varepsilon^\nu_d)\eta^{cd}\varepsilon^a_\mu\varepsilon^b_\nu\boldsymbol{\gamma}_a\nabla_b u \\
&= \frac{1}{2}(\delta_{\boldsymbol{h}}g^{\mu\nu})\varepsilon^a_\mu\varepsilon^b_\nu\boldsymbol{\gamma}_a\nabla_b u. \qquad \text{(D.15)}
\end{aligned}
$$

We use the fact that $(\delta_{\boldsymbol{h}}g^{\mu\nu})\varepsilon^a_\mu\varepsilon^b_\nu = -h^{\mu\nu}\varepsilon^a_\mu\varepsilon^b_\nu = -h^{ab}$ to see finally that $\delta_{\boldsymbol{h}}(P^s_{\mathcal{M}}u) = -i\delta_{\boldsymbol{h}}(\nabla\!\!\!/^s_{\mathcal{M}}u)$ is equivalent modulo $S^s_{\mathcal{M}}$ to

$$
\mathcal{P}^s_{\mathcal{M}}[\boldsymbol{h}]u := \frac{i}{2}h^{ab}\boldsymbol{\gamma}_a\nabla_b u - \frac{i}{4}(\delta_{\boldsymbol{h}}\Gamma^\mu{}_{\nu\rho})\varepsilon^b_\mu\varepsilon^\nu_a\varepsilon^\rho_c\boldsymbol{\gamma}^a\boldsymbol{\gamma}_b\boldsymbol{\gamma}^c u, \qquad \text{(D.16)}
$$

which concludes the first part of the proof for spinors. The proof for cospinors is similar.

Now, using (7.105) and writing $u' = S^s_{\mathcal{M}} f'$, we see that

$$\begin{aligned} s^s_{\mathcal{M}}(u', F_{\mathcal{M}}[\boldsymbol{h}]u) &= (f', \mathcal{P}^s_{\mathcal{M}}[\boldsymbol{h}]u)^s \\ &= -i \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \, \langle \overline{A_{\boldsymbol{M}} f'}, S^s_{\mathcal{M}} \mathcal{P}^s_{\mathcal{M}}[\boldsymbol{h}]u \rangle \\ &= i \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \, \langle \overline{A_{\boldsymbol{M}} u'}, \mathcal{P}^s_{\mathcal{M}}[\boldsymbol{h}]u \rangle. \end{aligned} \tag{D.17}$$

We use (C.8) to see that

$$\delta_{\boldsymbol{h}} \Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\nabla_\rho h_{\sigma\nu} + \nabla_\nu h_{\sigma\rho} - \nabla_\sigma h_{\nu\rho}), \tag{D.18}$$

and consequently

$$\begin{aligned} \mathcal{P}^s_{\mathcal{M}}[\boldsymbol{h}]u &= \frac{i}{2} h^{ab} \boldsymbol{\gamma}_a \nabla_b u + \frac{i}{8} \left((\nabla_b \boldsymbol{h})_{ac} - (\nabla_c \boldsymbol{h})_{ab} - (\nabla_a \boldsymbol{h})_{bc} \right) \boldsymbol{\gamma}^a \boldsymbol{\gamma}^b \boldsymbol{\gamma}^c u, \\ &= \frac{i}{2} h^{ab} \boldsymbol{\gamma}_a \nabla_b u + \frac{i}{8} (\nabla_c \boldsymbol{h})_{ab} \left(\boldsymbol{\gamma}^a \boldsymbol{\gamma}^c \boldsymbol{\gamma}^b - \boldsymbol{\gamma}^c \boldsymbol{\gamma}^b \boldsymbol{\gamma}^a - \boldsymbol{\gamma}^b \boldsymbol{\gamma}^a \boldsymbol{\gamma}^c \right) u \\ &= \frac{i}{2} h^{ab} \boldsymbol{\gamma}_a \nabla_b u + \frac{i}{8} (\nabla_c \boldsymbol{h})_{ab} \left(2\eta^{ac} \boldsymbol{\gamma}^b - \boldsymbol{\gamma}^c \boldsymbol{\gamma}^a \boldsymbol{\gamma}^b - \boldsymbol{\gamma}^c \boldsymbol{\gamma}^b \boldsymbol{\gamma}^a - \boldsymbol{\gamma}^b \boldsymbol{\gamma}^a \boldsymbol{\gamma}^c \right) u, \end{aligned} \tag{D.19}$$

where $(\nabla_c \boldsymbol{h})_{ab} = [\nabla_c \boldsymbol{h}](\varepsilon_a \otimes \varepsilon_b)$. But since $\boldsymbol{h}$ is symmetric, it holds that $(\nabla_c \boldsymbol{h})_{ab} \boldsymbol{\gamma}^a \boldsymbol{\gamma}^b = \frac{1}{2} (\nabla_c \boldsymbol{h})_{ab} \{\boldsymbol{\gamma}^a, \boldsymbol{\gamma}^b\} = (\nabla_c \boldsymbol{h})_{ab} \eta^{ab}$, therefore

$$\begin{aligned} &(\nabla_c \boldsymbol{h})_{ab} (2\eta^{ac} \boldsymbol{\gamma}^b - \boldsymbol{\gamma}^c \boldsymbol{\gamma}^a \boldsymbol{\gamma}^b - \boldsymbol{\gamma}^c \boldsymbol{\gamma}^b \boldsymbol{\gamma}^a - \boldsymbol{\gamma}^b \boldsymbol{\gamma}^a \boldsymbol{\gamma}^c) \\ &\qquad = 2\eta^{ac} (\nabla_c \boldsymbol{h})_{ab} \boldsymbol{\gamma}^b - 3\eta^{ab} (\nabla_c \boldsymbol{h})_{ab} \boldsymbol{\gamma}^c, \end{aligned} \tag{D.20}$$

and consequently

$$\begin{aligned} &i \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \, \langle \overline{A_{\boldsymbol{M}} u'}, \mathcal{P}^s_{\mathcal{M}}[\boldsymbol{h}]u \rangle \\ &\qquad = - \int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \bigg(\frac{1}{2} h^{ab} \langle \overline{A_{\boldsymbol{M}} u'}, \boldsymbol{\gamma}_a \nabla_b u \rangle + \frac{1}{4} \eta^{ac} (\nabla_c \boldsymbol{h})_{ab} \langle \overline{A_{\boldsymbol{M}} u'}, \boldsymbol{\gamma}^b u \rangle \\ &\qquad\qquad - \frac{3}{8} \eta^{ab} (\nabla_c \boldsymbol{h})_{ab} \langle \overline{A_{\boldsymbol{M}} u'}, \boldsymbol{\gamma}^c u \rangle \bigg). \end{aligned} \tag{D.21}$$

We may integrate by parts (using the fact that $\boldsymbol{h}$ is compactly supported to discard the integrals over $\boldsymbol{M}$ of total derivatives) to see that

$$\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}} \left(\frac{1}{2}h^{ab}\langle\overline{A_{\boldsymbol{M}}u'}, \boldsymbol{\gamma}_a\nabla_b u\rangle + \frac{1}{4}\eta^{ac}(\nabla_c\boldsymbol{h})_{ab}\langle\overline{A_{\boldsymbol{M}}u'}, \boldsymbol{\gamma}^b u\rangle\right) \\ = \frac{1}{4}\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{ab}\left(\langle\overline{A_{\boldsymbol{M}}u'}, \boldsymbol{\gamma}^a\nabla^b u\rangle - \langle\overline{A_{\boldsymbol{M}}\nabla^a u'}, \boldsymbol{\gamma}^b u\rangle\right), \quad \text{(D.22)}$$

whereas

$$\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, \eta^{ab}(\nabla_c\boldsymbol{h})_{ab}\langle\overline{A_{\boldsymbol{M}}u'}, \boldsymbol{\gamma}^c u\rangle \\ = -\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h^a{}_a\left(\langle\overline{A_{\boldsymbol{M}}u'}, \nabla\!\!\!/^s_{\mathcal{M}}u\rangle + \langle\overline{A_{\boldsymbol{M}}\nabla\!\!\!/^s_{\mathcal{M}}u'}, u\rangle\right) \\ = -\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h^a{}_a\left(\langle\overline{A_{\boldsymbol{M}}u'}, -imu\rangle + \langle\overline{A_{\boldsymbol{M}}(-imu')}, u\rangle\right) \\ = 0. \quad \text{(D.23)}$$

Putting this all together, and using the symmetry of $\boldsymbol{h}$, we see that

$$s^s_{\mathcal{M}}(u', F^s_{\mathcal{M}}[\boldsymbol{h}]u) = -\frac{1}{4}\int_{\boldsymbol{M}} dvol_{\boldsymbol{M}}\, h_{ab}\left(\langle\overline{A_{\boldsymbol{M}}u'}, \boldsymbol{\gamma}^{(a}\nabla^{b)}u\rangle - \langle\overline{A_{\boldsymbol{M}}\nabla^{(a}u'}, \boldsymbol{\gamma}^{b)}u\rangle\right) \quad \text{(D.24)}$$

as required. The result for cospinors may be obtained from this by noting that

$$s^c_{\mathcal{M}}(v', F^c_{\mathcal{M}}[\boldsymbol{h}]v) = s^s_{\mathcal{M}}(A^{-1}_{\boldsymbol{M}}\overline{F^c_{\mathcal{M}}[\boldsymbol{h}]v}, A^{-1}_{\boldsymbol{M}}\overline{v'}) \\ = s^s_{\mathcal{M}}(F^s_{\mathcal{M}}[\boldsymbol{h}]A^{-1}_{\boldsymbol{M}}\overline{v}, A^{-1}_{\boldsymbol{M}}\overline{v'}) \\ = s^s_{\mathcal{M}}(A^{-1}_{\boldsymbol{M}}\overline{v}, (F^s_{\mathcal{M}}[\boldsymbol{h}])^{-1}A^{-1}_{\boldsymbol{M}}\overline{v'}) \\ = -s^s_{\mathcal{M}}(A^{-1}_{\boldsymbol{M}}\overline{v}, F^s_{\mathcal{M}}[\boldsymbol{h}]A^{-1}_{\boldsymbol{M}}\overline{v'}), \quad \text{(D.25)}$$

where we have used the fact that $(F^s_{\mathcal{M}}[\boldsymbol{h}])^{-1} = -F^s_{\mathcal{M}}[\boldsymbol{h}]$. □

Appendix E

Table of categories

The following table contains an overview of the various categories defined within this thesis. It should only be used as a quick reference and not as an indication of their full definitions, since these are often fairly complicated and there is not enough space for all the relevant details here. In cases where significant details have been omitted, the entry is marked with †. Where the entry in the 'Arrow' column for a particular category is simply another category, this means that the former category is a full subcategory of the latter.

Table 1: Glossary of categories

Category	Object	Arrow
Alg	Unital $*$-algebra	Unit-preserving $*$-monomorphism
$\mathsf{Bund}_{(d)}$	Smooth fibre bundle over base in $\mathsf{Loc}_{(d)}$	Smooth bundle morphism
$\mathsf{Bund}^{c}_{(d)}$	Smooth fibre bundle over base in $\mathsf{Loc}^{c}_{(d)}$	$\mathsf{Bund}_{(d)}$
$\mathsf{Bund}^{sc}_{(d)}$	Smooth fibre bundle over base in $\mathsf{Loc}^{sc}_{(d)}$	$\mathsf{Bund}_{(d)}$
FLoc_4	Loc_4-spacetime with global frame	Frame-preserving Loc-arrow

[F]Loc_4	Loc_4-spacetime with equivalence class of frames	Equivalence class-preserving Loc-arrow
Herm	Hermitian space	Injective linear map†
HermAdj	Hermitian adjoint structure	Pair of Herm-arrows†
$\mathsf{HermAdj}_\mathsf{C}$	HermAdj object admitting a charge conjugation	HermAdj
$\mathsf{Loc}_{(d)}$	(d-dimensional) globally hyperbolic spacetime	Isometric orientation-preserving embedding
$\mathsf{Loc}^c_{(d)}$	Connected $\mathsf{Loc}_{(d)}$-spacetime	$\mathsf{Loc}_{(d)}$
$\mathsf{Loc}^{sc}_{(d)}$	Simply connected $\mathsf{Loc}_{(d)}$-spacetime	$\mathsf{Loc}_{(d)}$
Phys	*Arbitrary category of physical systems*	
$\mathsf{RPBund}_{(d)}$	Principal bundle over base in $\mathsf{Loc}_{(d)}$ with right action	Smooth bundle morphism with group homomorphism†
RTor	Right torsor	Function with group homomorphism†
SAdj	Squared adjoint structure	Injective linear map†
Sp	*Arbitrary category of spacetimes*	
Test	*Arbitrary category of test function spaces*	
$\mathsf{Vect}_\mathbb{C}$	Vector space over $\mathbb{C}$	Injective linear map

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