



## Leading relativistic corrections to the Kompaneets equation

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### ABSTRACT

We calculate the first relativistic corrections to the Kompaneets equation for the evolution of the photon frequency distribution brought about by Compton scattering. The Lorentz invariant Boltzmann equation for electron–photon scattering is first specialized to isotropic electron and photon distributions, the squared scattering amplitude and the energy–momentum conserving delta function are each expanded to order  $v^4/c^4$ , averages over the directions of the electron and photon momenta are then carried out, and finally an integration over the photon energy yields our Fokker–Planck equation. The Kompaneets equation, which involves only first- and second-order derivatives with respect to the photon energy, results from the order  $v^2/c^2$  terms, while the first relativistic corrections of order  $v^4/c^4$  introduce third- and fourth-order derivatives. We emphasize that our result holds when neither the electrons nor the photons are in thermal equilibrium; two effective temperatures characterize a general, non-thermal electron distribution. When the electrons are in thermal equilibrium our relativistic Fokker–Planck equation is in complete agreement with the most recent published results, but we both disagree with older work.

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### 1. Introduction

The Kompaneets [1] equation,

$$\frac{\partial}{\partial t} f(t, \omega) = \frac{\sigma_T n_e}{m_e c} \frac{1}{\omega^2} \frac{\partial}{\partial \omega} \omega^4 \left\{ T \frac{\partial f(t, \omega)}{\partial \omega} + \hbar [1 + f(t, \omega)] f(t, \omega) \right\}, \quad (1.1)$$

describes the scattering of unpolarized, low energy photons of frequency  $\omega$  on a dilute distribution of non-relativistic electrons when all the particles – photons and electrons – are distributed isotropically in their momenta. The non-relativistic total photon–electron cross section is the Thomson cross section  $\sigma_T$ . The electron number density and mass are denoted by  $n_e$  and  $m_e$ . The photon phase space distribution  $f(t, \omega)$  is normalized such that the number  $n_\gamma$  of photons per unit volume is given by

$$n_\gamma(t) = 2 \int \frac{(d^3 \mathbf{k})}{(2\pi)^3} f(t, \omega), \quad (1.2)$$

in which the prefactor 2 counts the number of photon polarization states and  $\mathbf{k}$  is the photon wave-number vector with  $|\mathbf{k}|c = \omega$ . If the electrons are in thermal equilibrium described by a Maxwell–Boltzmann distribution, then  $T$  is the temperature (in energy units)  $T_e$  of this thermal distribution. However, the Kompaneets equation (1.1) holds for any isotropic distribution of electron momenta with

$T$  defined to be 2/3 of the average energy in this distribution [2]. For photons with a Planck distribution,

$$f(t, \omega) \rightarrow f^{(0)}(\omega) = \frac{1}{\exp\{\hbar\omega/T_\gamma\} - 1}. \quad (1.3)$$

The terms in the curly braces in the Kompaneets equation (1.1) vanish when  $T_\gamma = T$ . In particular, if  $T = T_\gamma = T_e$ , there is a time-independent photon distribution in thermal equilibrium with the electrons.<sup>1</sup>

Our purpose here is to examine the first relativistic corrections to the Kompaneets equation. These corrections have been previously computed by Challinor and Lasenby (C&L) [3] for the case in which the electrons are in a thermal distribution. Using the method of C&L, Itoh et al. [4] carried out the expansion to a much higher order in  $v/c$ . Subsequently, Sazonov and Sunyaev [5] confirmed the previous work of Challinor and Lasenby. Here we use a method that is quite different from that employed by C&L, a method that does not require that the electrons be in thermal equilibrium. Moreover, this method explicitly exhibits the order of  $v/c$  in every term and thus provides a straightforward evaluation of the correct  $v^2/c^2$  corrections to the Kompaneets equation. Although the structure of our result is quite different from that found by C&L, we agree with C&L in the number of higher-order derivatives with respect to the photon frequency  $\omega$  which must supplement the Kompaneets equation to correctly account for the relativistic

<sup>1</sup> Since Compton scattering preserves the photon number, the collision term on the right-hand side of Eq. (1.1) also vanishes for a general Bose–Einstein distribution of massless particles at temperature  $T$ ,  $f(t, \omega) \rightarrow f^{(2)}(\omega) = [\exp\{\hbar\omega/T\} - \alpha]^{-1}$ .

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corrections. Such higher-order derivative terms are missing from the ad hoc treatments of Cooper [6] and of Prasad et al. [7]. These authors assume (incorrectly) that the relativistic corrections may be accounted for by simply replacing the factor  $\omega^4$  that stands just before the curly braces in Eq. (1.1) by a function  $\alpha(\omega, T)$  which is determined so as to give the rate of change of the photon energy density including the first relativistic corrections. We compute both the rate of energy exchange between the photons and electrons and the Sunyaev–Zel’dovich effect [9–11] which follow from the relativistically corrected Kompaneets equation. Including the first relativistic corrections, our results entail two effective temperatures  $T_{\text{eff1}}$  and  $T_{\text{eff2}}$  which are defined by energy moments of the electron phase-space distribution. When the electron distribution is restricted to a thermal, relativistic Maxwell–Boltzmann distribution at temperature  $T$ ,  $T_{\text{eff1}} = T_{\text{eff2}} = T$  and we find, after some algebra, that our results that have a completely different structure are, in fact, in complete agreement with those of C&L. Moreover, the rate of energy exchange that we compute (also written down by C&L) agrees with that found earlier by Woodward [8].

Our presentation is organized as follows: After describing the general method we use in Section 2, we then outline the calculation in Section 3 using the results of several Appendices. Finally, our results are shown in Section 4: Section 4.1 presents our general result, Section 4.2 gives its restriction to the case in which the photons are in thermal equilibrium at temperature  $T_\gamma$ , Section 4.3 derives the rate of energy transport between photons at temperature  $T_\gamma$  and the electrons in a general distribution, and finally, in Section 4.4 the Sunyaev–Zel’dovich effect for non-thermal electrons with the first relativistic correction is briefly described.

## 2. Relativistic Boltzmann equation for isotropic scattering

We start from the Lorentz invariant form of the Boltzmann equation for electron–photon scattering:

$$k\partial f(x, k) = \int \frac{(d^3\mathbf{p}')}{(2\pi)^3} \frac{1}{2E'} \frac{(d^3\mathbf{k}')}{(2\pi)^3} \frac{1}{2\omega'} \frac{(d^3\mathbf{p})}{(2\pi)^3} \frac{1}{2E} (2\pi)^4 \\ \times \delta^4(\mathbf{p}' + \mathbf{k}' - \mathbf{p} - \mathbf{k}) |T(\mathbf{p}', \mathbf{k}'; \mathbf{p}, \mathbf{k})|^2 \{ [1 + f(x, k)] \\ \times 2g(x, \mathbf{p}')f(x, \mathbf{k}') - [1 + f(x, \mathbf{k}')]2g(x, \mathbf{p})f(x, \mathbf{k}) \}. \quad (2.1)$$

Here we revert to units in which  $\hbar = 1 = c$ , but we shall return to conventional units when we write the final result. The left-hand side of the equation involves the relativistic scalar  $k\partial = \omega(\partial/\partial t) + \mathbf{k} \cdot \nabla$ . We are assuming that the electrons and photons are not polarized. Hence  $|T|^2$  denotes the square of the Lorentz invariant scattering amplitude that is summed over the initial and final electron and photon spins. It is divided by the initial electron spin weight  $g_e = 2$  so as to describe the average scattering from an initially unpolarized ensemble of electrons. It is divided by the square of the photon spin weight  $g_\gamma^2 = 4$  because initially there is an unpolarized mixture and finally the scattering is into the scalar density  $f(x, k)$  that describes a typical photon (with the factor  $g_\gamma = 2$  needed to provide the photon number count in Eq. (1.2)). The function  $g(x, p)$  is the electron phase space density. We choose our Lorentz metric to have signature  $(-+++)$  so that  $t = x^0 = -x_0$  while for the spatial coordinates  $x^k = x_k$ .

We now specialize to the isotropic case of interest where  $f(x, k) \rightarrow f(t, \omega)$  and  $g(x, p) \rightarrow g(t, E)$ , with the electron number density given by

$$n_e = 2 \int \frac{(d^3\mathbf{p})}{(2\pi)^3} g(t, E). \quad (2.2)$$

The integration variables  $p$  and  $p'$  in Eq. (2.1) are dummy variables. We shall make the interchange  $p \leftrightarrow p'$  in the first set of terms in Eq. (2.1) so as to have a common factor of  $g(t, E)$  for the two ‘scattering

in to’ and ‘scattering out of’ terms. To keep a convenient form, we shall also use the detailed balance relation

$$|T(\mathbf{p}', \mathbf{k}'; \mathbf{p}, \mathbf{k})|^2 = |T(\mathbf{p}, \mathbf{k}; \mathbf{p}', \mathbf{k}')|^2 \quad (2.3)$$

for this first term in Eq. (2.1). Finally, we note that the  $\mathbf{p}'$  integration is best performed using

$$\frac{(d^3\mathbf{p}')}{(2\pi)^3} \frac{1}{2E'} = \frac{(d^3\mathbf{p}')}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{p}'^2 + m_e^2}} = \frac{(d^4\mathbf{p}')}{(2\pi)^3} \delta(\mathbf{p}'^2 + m_e^2) \quad (2.4)$$

against the four-dimensional delta function which now replaces

$$\mathbf{p}' = \mathbf{p} + \mathbf{k} - \mathbf{k}' \quad (2.5)$$

giving

$$\mathbf{p}'^2 + m_e^2 = 2p(k - k') - 2kk'. \quad (2.6)$$

In this fashion, we obtain

$$\omega \frac{\partial}{\partial t} f(t, \omega) = \int \frac{(d^3\mathbf{k}')}{(2\pi)^3} \frac{1}{2\omega'} \frac{(d^3\mathbf{p})}{(2\pi)^3} \frac{1}{2E} 2g(t, E) 2\pi \delta(2p(k - k') \\ - 2kk') |T(\mathbf{p}', \mathbf{k}'; \mathbf{p}, \mathbf{k})|^2 \{ [1 + f(t, \omega)]f(t, \omega') \\ - |T(\mathbf{p}', \mathbf{k}'; \mathbf{p}, \mathbf{k})|^2 [1 + f(t, \omega')]f(t, \omega) \}, \quad (2.7)$$

in which the four-momentum  $p'$  in  $|T(\mathbf{p}', \mathbf{k}'; \mathbf{p}, \mathbf{k})|^2$  is determined by Eq. (2.5).

The angular part of the integrations over  $\mathbf{p}$  and  $\mathbf{k}'$  pick out the completely rotationally invariant part of the integrand. Thus, with angular brackets denoting the average over all the orientations of the vectors within it, we may make the replacement

$$\delta(2p(k - k') - 2kk') |T(\mathbf{p}', \mathbf{k}'; \mathbf{p}, \mathbf{k})|^2 \\ \rightarrow \langle \delta(2p(k - k') - 2kk') |T(\mathbf{p}', \mathbf{k}'; \mathbf{p}, \mathbf{k})|^2 \rangle \equiv s(p; \omega', \omega). \quad (2.8)$$

In view of these remarks, we may write Eq. (2.7) as

$$\frac{\partial}{\partial t} f(t, \omega) = \frac{1}{\omega^2} \int \frac{(d^3\mathbf{p})}{(2\pi)^3} \frac{1}{2E} 2g(p) F(t, \omega; p) \quad (2.9)$$

with

$$F(t, \omega; p) = \frac{\omega}{2\pi} \int_0^\infty \omega' d\omega' \{ s(p; \omega, \omega') [1 + f(t, \omega)] f(t, \omega') \\ - s(p; \omega', \omega) [1 + f(t, \omega')] f(t, \omega) \}. \quad (2.10)$$

For the evaluation of Eq. (2.10) it is convenient to separate the weight that appears there into symmetric and antisymmetric parts:

$$s(p; \omega', \omega) = s^S(p; \omega', \omega) + s^A(p; \omega', \omega) \quad (2.11)$$

with

$$s^S(p; \omega, \omega') = +s^S(p; \omega', \omega) \quad (2.12)$$

and

$$s^A(p; \omega, \omega') = -s^A(p; \omega', \omega). \quad (2.13)$$

With this decomposition, Eq. (2.10) becomes

$$F(t, \omega; p) = \frac{\omega}{2\pi} \int_0^\infty \omega' d\omega' [s^S(p; \omega', \omega) - s^A(p; \omega', \omega)] f(t, \omega') \\ - \frac{\omega}{2\pi} f(t, \omega) \int_0^\infty \omega' d\omega' [s^S(p; \omega', \omega) + s^A(p; \omega', \omega)] \\ - \frac{\omega}{2\pi} 2f(t, \omega) \int_0^\infty \omega' d\omega' s^A(p; \omega', \omega) f(t, \omega'). \quad (2.14)$$

### 3. Expansions and angular averages

It proves convenient to use the angles  $\alpha, \alpha'$  between  $\mathbf{p}$  and  $\mathbf{k}, \mathbf{k}'$ , and the angle  $\theta$  between  $\mathbf{k}$  and  $\mathbf{k}'$ . We also use the velocity

$$v = \frac{|\mathbf{p}|}{E} = \frac{|\mathbf{p}|}{E(|\mathbf{p}|)} < 1. \quad (3.1)$$

The delta function in Eq. (2.8) now becomes

$$\delta(2p(k-k') - 2kk') = \frac{1}{2E} \delta(\omega(1-v\cos\alpha) - \omega'(1-v\cos\alpha')) - (\omega\omega'/E)(1-\cos\theta) \quad (3.2)$$

and the squared scattering amplitude (A8) now appears as

$$|T(p', k'; p, k)|^2 = 6\pi m_e^2 \sigma_T \left\{ 2 + \frac{1-\cos\theta}{(1-v\cos\alpha)(1-v\cos\alpha')} \times \left[ \frac{\omega\omega'}{E^2} (1-\cos\theta) - 2(1-v^2) \right] + (1-v^2)^2 \frac{(1-\cos\theta)^2}{(1-v\cos\alpha)^2(1-v\cos\alpha')^2} \right\}. \quad (3.3)$$

As we shall see, the Kompaneets equation results from expanding Eqs. (3.2) and (3.3) in powers of  $v$ , with this equation resulting from the order  $v^2$  terms. The leading relativistic corrections that concern us require that the expansion be carried out to order  $v^4$ . We note that since in the applications that we envisage,  $\omega, \omega' \sim T \sim p^2/m_e \sim v^2 E$ ,  $\omega/E$  or  $\omega'/E$  should be counted as being of order  $v^2$ .

The needed expansion of the delta function (3.2) in powers of  $v$  reads

$$\begin{aligned} 2E\delta(2p(k-k') - 2kk') &= \delta(\omega - \omega') - [v(\omega\cos\alpha - \omega'\cos\alpha')\delta'(\omega - \omega')] \\ &+ \left[ \frac{1}{2}v^2(\omega\cos\alpha - \omega'\cos\alpha')^2\delta''(\omega - \omega') \right. \\ &\quad \left. - \frac{\omega\omega'}{E}(1-\cos\theta)\delta'(\omega - \omega') \right] \\ &- \left[ \frac{1}{3!}v^3(\omega\cos\alpha - \omega'\cos\alpha')^3\delta'''(\omega - \omega') \right. \\ &\quad \left. - v(\omega\cos\alpha - \omega'\cos\alpha') \right. \\ &\quad \left. \times \frac{\omega\omega'}{E}(1-\cos\theta)\delta''(\omega - \omega') \right] \\ &+ \left[ \frac{1}{4!}v^4(\omega\cos\alpha - \omega'\cos\alpha')^4\delta''''(\omega - \omega') \right. \\ &\quad \left. - \frac{1}{2}v^2(\omega\cos\alpha - \omega'\cos\alpha')^2 \right. \\ &\quad \left. \times \frac{\omega\omega'}{E}(1-\cos\theta)\delta''''(\omega - \omega') \right. \\ &\quad \left. + \frac{1}{2}\left(\frac{\omega\omega'}{E}\right)^2(1-\cos\theta)^2\delta''(\omega - \omega') \right] + \mathcal{O}(v^5). \end{aligned} \quad (3.4)$$

The term of order  $v^0$ ,  $\delta(\omega - \omega')$ , makes no contribution, since for it the two parts of the collision integral, the ‘scattering into’ and the ‘scattering out of’, cancel. Hence we need expand the squared amplitude (3.3) only to order  $v^3$  to obtain results good to order  $v^4$ :

$$\begin{aligned} |T(p', k'; p, k)|^2 &\simeq 6\pi m_e^2 \sigma_T \{ v^0[(1+\cos^2\theta)] - 2v(1-\cos\theta) \\ &\times [(\cos\alpha + \cos\alpha')\cos\theta] + v^2(1-\cos\theta) \\ &\times [(\cos^2\alpha + \cos^2\alpha' + 2\cos\alpha\cos\alpha')] \\ &+ (2-3\cos^2\alpha - 3\cos^2\alpha' - 4\cos\alpha\cos\alpha')\cos\theta \\ &+ v^3(1-\cos\theta)[(-2\cos\alpha - 2\cos\alpha' + 2\cos^3\alpha + 2\cos^3\alpha' \\ &+ 4\cos^2\alpha\cos\alpha' + 4\cos\alpha\cos^2\alpha') - (-4\cos\alpha - 4\cos\alpha' + 4\cos^3\alpha \\ &+ 4\cos^3\alpha' + 6\cos^2\alpha\cos\alpha' + 6\cos\alpha\cos^2\alpha')\cos\theta] \}. \end{aligned} \quad (3.5)$$

We now multiply the expressions (3.4) and (3.5) together and retain the resulting terms up to those of order  $v^4$ . We then use the result of angular averaging over the directions of  $\mathbf{p}$  detailed in Appendix B and subsequently average over the direction between  $\mathbf{k}$  and  $\mathbf{k}'$ , the average over  $\cos\theta$ . To present the results in a compact form, we separately record the order  $v^2$  result, the one that gives the Kompaneets equation,

$$\begin{aligned} s_2(p; \omega', \omega) &= \langle |T(p', k'; p, k)|^2 \delta(2p(k-k') - 2kk') \rangle_2 \\ &= 4\pi \frac{m_e^2}{E} \sigma_T \left\{ v^2 \frac{\omega\omega'}{3} \delta''(\omega - \omega') - \frac{\omega\omega'}{E} \delta'(\omega - \omega') \right\} \end{aligned} \quad (3.6)$$

and the order  $v^4$  result which gives the leading relativistic correction to the Kompaneets equation

$$\begin{aligned} s_4(p; \omega', \omega) &= \left\langle |T(p', k'; p, k)|^2 \delta(2p(k-k') - 2kk') \right\rangle_4 \\ &= \frac{\pi}{15} \frac{m_e^2}{E} \sigma_T \left\{ v^4 \left[ 2(\omega - \omega')^2 \omega\omega' + \frac{14}{5} \omega^2 \omega'^2 \right] \delta''''(\omega - \omega') \right. \\ &\quad \left. - 10v^2 \left[ \frac{2}{25} v^2 (\omega - \omega') \omega\omega' + ((\omega - \omega')^2 + \frac{14}{5} \omega\omega') \frac{\omega\omega'}{E} \right] \right. \\ &\quad \left. \times \delta''''(\omega - \omega') + \left[ -\frac{16}{5} v^4 \omega\omega' + 4v^2 (\omega - \omega') \frac{\omega\omega'}{E} \right. \right. \\ &\quad \left. \left. + 42 \left( \frac{\omega\omega'}{E} \right)^2 \right] \delta''(\omega - \omega') + \left[ 12v^2 \frac{\omega\omega'}{E} \right] \delta'(\omega - \omega') \right\}. \end{aligned} \quad (3.7)$$

There are no terms of odd order in  $v$  in the somewhat lengthy algebra required to obtain these formulae. They simplify with the aid of the delta function identities presented in Appendix C:

$$\frac{\omega\omega'}{2\pi m_e^2} s_2^{\text{prime}}(p; \omega', \omega) = 2\sigma_T v^2 \frac{(\omega\omega')^2}{3E} \delta''(\omega - \omega'), \quad (3.8)$$

$$\frac{\omega\omega'}{2\pi m_e^2} s_2^A(p; \omega', \omega) = -2\sigma_T \frac{(\omega\omega')^2}{E^2} \delta'(\omega - \omega') \quad (3.9)$$

and

$$\begin{aligned} \frac{\omega\omega'}{2\pi m_e^2} s_4^S(p; \omega', \omega) &= \sigma_T \frac{(\omega\omega')^2}{15E} \left\{ \frac{7}{5} v^4 \omega\omega' \delta''''(\omega - \omega') \right. \\ &\quad \left. + \left[ \frac{58}{5} v^4 + 21 \frac{\omega\omega'}{E^2} \right] \delta''(\omega - \omega') \right\}, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \frac{\omega\omega'}{2\pi m_e^2} s_4^A(p; \omega', \omega) &= -\sigma_T v^2 \\ &\times \frac{(\omega\omega')^2}{15E^2} \{ 14\omega\omega' \delta''''(\omega - \omega') + 28\delta'(\omega - \omega') \}. \end{aligned} \quad (3.11)$$

### 4. Results

It is now a straightforward although tedious matter to insert the forms above into Eq. (2.14), perform the  $\omega'$  integrals, and place the results into Eq. (2.9) to secure the Kompaneets equation and its leading relativistic corrections.

#### 4.1. General result

$$\begin{aligned}
\omega^2 \frac{\partial}{\partial t} f(t, \omega) = & \sigma_T \frac{n_e}{m_e c} \frac{d}{d\omega} \omega^4 \left\{ T_{\text{eff1}} \frac{df(t, \omega)}{d\omega} + \hbar [1 + f(t, \omega)] f(t, \omega) \right. \\
& + \frac{1}{m_e c^2} \left( \frac{5}{2} T_{\text{eff2}}^2 + 21 T_{\text{eff1}}^2 \right) \frac{df(t, \omega)}{d\omega} \\
& + \frac{47}{2 m_e c^2} T_{\text{eff1}} \hbar [1 + f(t, \omega)] f(t, \omega) \\
& - \left. \frac{7 \hbar \omega^2}{10 m_e c^2} \left[ 6 T_{\text{eff1}} \left( \frac{df(t, \omega)}{d\omega} \right)^2 - \hbar \frac{df(t, \omega)}{d\omega} + T_{\text{eff1}}^2 \frac{1}{\hbar} \frac{d^3 f(t, \omega)}{d\omega^3} \right] \right\} \\
& - \frac{21 n_e \sigma_T}{5 m_e^2 c^3} \frac{d^2}{d\omega^2} \omega^5 \left\{ \left( T_{\text{eff2}}^2 + 2 T_{\text{eff1}}^2 \right) \frac{df(t, \omega)}{d\omega} \right. \\
& + 3 T_{\text{eff1}} \hbar [1 + f(t, \omega)] f(t, \omega) \left. \right\} \\
& + \frac{7 n_e \sigma_T}{10 m_e^2 c^3} \frac{d^3}{d\omega^3} \omega^6 \left\{ \left( T_{\text{eff2}}^2 + T_{\text{eff1}}^2 \right) \frac{df(t, \omega)}{d\omega} \right. \\
& + 2 T_{\text{eff1}} \hbar [1 + f(t, \omega)] f(t, \omega) \left. \right\}. \tag{4.1}
\end{aligned}$$

Here we have reverted to conventional units and used the effective temperature definitions (D4) and (D7) of Appendix D which, for convenience, we repeat here:

$$T_{\text{eff1}} = \frac{1}{n_e} \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{p^2 c^2}{3E(p)} 2g(t, E) \tag{4.2}$$

and

$$T_{\text{eff2}}^2 = \frac{4}{15 n_e} \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \left( \frac{p^2}{2m_e} \right)^2 2g(t, E). \tag{4.3}$$

As explained in detail in Appendix D, when the electrons are in thermal equilibrium with a relativistic Maxwell-Boltzmann distribution at temperature  $T$ ,  $T_{\text{eff1}} \rightarrow T$  holds exactly. On the other hand, in this limit  $T_{\text{eff2}} \rightarrow T$  only to leading order in  $v^2/c^2$ ; but this is sufficiently accurate because  $T_{\text{eff2}}^2$  appears only in the relativistic corrections.

We have arranged the terms in the result (4.1) so as to exhibit certain features. We have arranged terms into groups that separately vanish in thermal equilibrium,  $T_{\text{eff1}} = T_{\text{eff2}} = T$ . These groups in Eq. (4.1) are: line 1; lines 2 and 3; line 4; lines 5 and 6; lines 7 and 8. To achieve this, it was necessary to introduce terms of order  $T_{\text{eff1}}^2$  in all of the lines representing relativistic corrections—all of the lines of Eq. (4.1) save for the first which has the form given by Kompaneets. However, the sum of all of these order  $T_{\text{eff1}}^2$  terms vanishes, as a simple calculation shows.

We have also chosen to order the terms in our result (4.1) so as to have successive parts involve overall higher derivatives. Since each part starts out with at least one overall derivative, the result conserves photon number as it must. Only the part with the single overall derivative  $d/d\omega$  contributes to the rate of energy exchange between the photons and the electrons. Similarly, the rate at which the second moment  $(\hbar\omega)^2$  changes with time is affected only by the parts involving  $d/d\omega$  and  $d^2/d\omega^2$  while all the parts contribute to the time rate of change of the  $(\hbar\omega)^3$  moment. We have kept some photon frequency derivatives within the sequence of increasingly higher overall derivative so that, as we have just mentioned, the sum of the terms in each of these groups vanishes in thermal equilibrium. Each of these groupings in the result (4.1) vanishes, in fact, in the more general situation in which the photon distribution is of the Planck form (1.3) but with the electron  $g(E)$  constrained only to have  $T_{\text{eff1}} = T_{\text{eff2}} = T_\gamma$ .

Our expression (4.1) is in complete agreement with the work of Challinor and Lasenby [3] in the limit in which the electrons are in thermal equilibrium at temperature  $T_e$ . However, as mentioned previously, the structure of our expression (4.1) for electrons in

thermal equilibrium—which is equivalent to that of C&L—differs completely from the previous (incorrect) results of Cooper [6] and of Prasad et al. [7].

#### 4.2. Photons in thermal equilibrium

For photons in thermal equilibrium at a temperature  $T_\gamma$  (the Planck distribution (1.2) or, more generally, a Bose-Einstein distribution of photons) the result reduces to

$$\begin{aligned}
\omega^2 \frac{\partial}{\partial t} f(t, \omega) = & -4 \frac{d}{d\omega} \omega^3 \sigma_T \frac{n_e}{m_e c} \left\{ \left( 1 - \frac{21}{20} \frac{\hbar^2 \omega^2}{m_e c^2 T_\gamma} \right) (T_{\text{eff1}} - T_\gamma) \right. \\
& + \frac{5}{2 m_e c^2} \left( T_{\text{eff2}}^2 - T_{\text{eff1}} T_\gamma \right) \left. \right\} f_0(\omega) \\
& + \frac{d^2}{d\omega^2} \omega^4 \sigma_T \frac{n_e}{m_e c} \left\{ \left( 1 - \frac{7}{10} \frac{\hbar^2 \omega^2}{m_e c^2 T_\gamma} \right) (T_{\text{eff1}} - T_\gamma) \right. \\
& + \frac{47}{2 m_e c^2} \left( T_{\text{eff2}}^2 - T_{\text{eff1}} T_\gamma \right) \left. \right\} f_0(\omega) - \frac{d^3}{d\omega^3} \omega^5 \sigma_T \frac{42 n_e}{5 m_e^2 c^3} \\
& \times \left( T_{\text{eff2}}^2 - T_{\text{eff1}} T_\gamma \right) f_0(\omega) + \frac{d^4}{d\omega^4} \omega^6 \sigma_T \frac{7 n_e}{10 m_e^2 c^3} \\
& \times \left( T_{\text{eff2}}^2 - T_{\text{eff1}} T_\gamma \right) f_0(\omega). \tag{4.4}
\end{aligned}$$

#### 4.3. Energy transport

The energy transfer per unit volume to the photons is given by

$$\dot{u}_\gamma = 2 \int \frac{(d^3 \mathbf{k})}{(2\pi)^3} \omega \frac{\partial}{\partial t} f(t, \omega). \tag{4.5}$$

From Eq. (4.4), we see that, including the first relativistic corrections, this energy transfer between a photon distribution in equilibrium at temperature  $T_\gamma$  and an arbitrary isotropic distribution of electron energies involves

$$\begin{aligned}
2 \int \frac{(d^3 \mathbf{k})}{(2\pi)^3} \omega f_0(\omega) &= \sum_{n=1}^{\infty} \frac{1}{\pi^2} \int_0^{\infty} d\omega \omega^3 \exp \left\{ -n \frac{\omega}{T_\gamma} \right\} \\
&= T_\gamma^4 \frac{3!}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^4} = T_\gamma^4 \frac{3!}{\pi^2} \zeta(4) = \frac{\pi^2}{15} T_\gamma^4 = u_\gamma \tag{4.6}
\end{aligned}$$

and

$$\begin{aligned}
2 \int \frac{(d^3 \mathbf{k})}{(2\pi)^3} \omega^3 f_0(\omega) &= \sum_{n=1}^{\infty} \frac{1}{\pi^2} \int_0^{\infty} d\omega \omega^5 \exp \left\{ -n \frac{\omega}{T_\gamma} \right\} \\
&= T_\gamma^6 \frac{5!}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^6} = T_\gamma^6 \frac{5!}{\pi^2} \zeta(6) = \frac{8\pi^4}{63} T_\gamma^6 \\
&= \frac{120\pi^2}{63} T_\gamma^2 u_\gamma. \tag{4.7}
\end{aligned}$$

Hence, again reverting to conventional units,

$$\begin{aligned}
\dot{u}_\gamma = & 4 \sigma_T \frac{n_e}{m_e c} \left\{ \left( 1 - 2\pi^2 \frac{T_\gamma}{m_e c^2} \right) (T_{\text{eff1}} - T_\gamma) \right. \\
& + \left. \frac{5}{2 m_e c^2} \left( T_{\text{eff2}}^2 - T_{\text{eff1}} T_\gamma \right) \right\} u_\gamma. \tag{4.8}
\end{aligned}$$

In the electron thermal equilibrium limit in which  $T_{\text{eff1}} = T_{\text{eff2}} = T_e$ , the energy transfer rate (4.8) agrees with the rate given by Woodward [8] which was later confirmed by Challinor and Lasenby [3]. The result (4.8) holds, of course, for a general isotropic distribution of electrons with the two effective temperatures  $T_{\text{eff1}}$  and  $T_{\text{eff2}}$  defined by the integrals (4.2) and (4.3).

#### 4.4. Sunyaev–Zel’dovich effect

Eq. (4.4) can be used to generalize published results on relativistic corrections to the Sunyaev–Zel’dovich effect, the distortion of the cosmic microwave background, a Planck distribution at a very low temperature, by high energy electrons in hot plasmas in galactic clusters. We sketch this generalization here both as an illustration of our result and to record a formula that can be used to ascertain if a more detailed treatment is warranted as more data comes along.

The distortion involves changing the time derivative in the Boltzmann equation to the proper coordinate distance  $\ell$  along the line of sight through the plasma cloud,  $t \rightarrow \ell/c$ . In doing so in the relativistic corrected Kompaneets formula (4.4), we encounter two dimensionless variables

$$y_1 = \sigma_T \int d\ell n_e(\ell) \frac{T_{\text{eff}1}(\ell)}{m_e c^2} \quad (4.9)$$

and

$$y_2 = \sigma_T \int d\ell n_e(\ell) \frac{T_{\text{eff}2}^2(\ell)}{m_e^2 c^4}, \quad (4.10)$$

that replace the conventional  $y$  parameter.<sup>2</sup> It is now convenient to define

$$x = \frac{\hbar\omega}{T_\gamma} \quad (4.11)$$

and write the Planck distribution as  $f_0(x) = [\exp\{x\} - 1]^{-1}$ . Since the microwave background temperature  $T_\gamma$  is so low, while the electrons have average energies that are several keV, we may neglect the very small ratios  $T_\gamma/T_{\text{eff}1}$  and  $T_\gamma/T_{\text{eff}2}$  as well as  $T_\gamma/m_e c^2$ . With the omission of these terms, carrying out the derivatives in Eq. (4.4) yields the spectral distortion

$$\begin{aligned} \frac{\Delta f(x)}{f_0(x)} &= y_1 x [1 + f_0(x)] \{x - 4 + 2x f_0(x)\} \\ &+ y_2 x [1 + f_0(x)] \left\{ \frac{x}{10} (235 - 84x + 7x^2) \right. \\ &- 10 + \frac{1}{5} (235 - 252x + 49x^2) x f_0(x) \\ &\left. + \frac{126}{5} (x - 2) x^2 f_0^2(x) + \frac{84}{5} x^3 f_0^3(x) \right\}. \end{aligned} \quad (4.12)$$

In the limit of small  $x$ , the Rayleigh–Jeans region, Eq. (4.12) simplifies to

$$\frac{\Delta f(x)}{f_0(x)} = -2y_1 + \frac{17}{5} y_2. \quad (4.13)$$

Eqs. (4.12) and (4.13) hold when the electrons are not in thermal equilibrium; when they are in equilibrium Eqs. (4.12) and (4.13) agree with the expressions obtained by Challinor and Lasenby<sup>3</sup> [3]. Elaborate discussions of the Sunyaev–Zel’dovich effect have previously been presented for non-thermal electrons [12,13] for specific choices of the electron distribution function  $g(E, t)$ .

#### Appendix A. Squared amplitude details

We first express the fully relativistic squared amplitude [14] as

$$|T(p', k'; p, k)|^2 = 6\pi m_e^2 \sigma_T \left\{ \left( \frac{\kappa'}{\kappa} + \frac{\kappa}{\kappa'} \right) + 2 \left( \frac{m_e^2}{\kappa} - \frac{m_e^2}{\kappa'} \right) + \left( \frac{m_e^2}{\kappa} - \frac{m_e^2}{\kappa'} \right)^2 \right\}, \quad (A1)$$

<sup>2</sup> Note that even in the case of a plasma in local thermodynamic equilibrium with an electron temperature  $T_e(\ell)$ , the relativistic corrections involve a different  $y$  parameter ( $y = y_2$ ) defined by the electron number weighted average of  $T_e^2(\ell)$  rather than the first power  $T_e(\ell)$  that appears in  $y = y_1$ .

<sup>3</sup> These authors, however, do not define the proper parameters  $y_1$  and  $y_2$  that, as noted in the previous footnote, are needed for the relativistic treatment, but rather use a  $y = y_1$ , and then multiply this by an undefined electron temperature to obtain a  $y_2$  parameter for the relativistic corrections.

in which  $\sigma_T = 8\pi r_0^2/3$  is the Thomson cross section with  $r_0$  the classical electron radius and, with our space-like metric

$$\kappa = -pk = p^0 k^0 - \mathbf{p} \cdot \mathbf{k}, \quad \kappa' = -pk' = p^0 k'^0 - \mathbf{p} \cdot \mathbf{k}'. \quad (A2)$$

Note that in terms of these variables the delta function constraint (3.2) reads

$$\kappa' - \kappa = kk'. \quad (A3)$$

It is convenient to use the variable

$$\bar{\kappa} = \sqrt{\kappa\kappa'}, \quad (A4)$$

so that the relation (A3) may be written as

$$\frac{\kappa'}{\bar{\kappa}} = 1 + \frac{kk'}{\bar{\kappa}} = 1 + \sqrt{\frac{\kappa'}{\bar{\kappa}}} \frac{kk'}{\bar{\kappa}}. \quad (A5)$$

The proper solution of this quadratic equation, written in terms of  $\kappa'/\bar{\kappa}$ , is

$$\frac{\kappa'}{\bar{\kappa}} = \frac{1}{2} \left[ \left( \frac{kk'}{\bar{\kappa}} \right)^2 + 2 + \frac{kk'}{\bar{\kappa}} \sqrt{\left( \frac{kk'}{\bar{\kappa}} \right)^2 + 4} \right]. \quad (A6)$$

Similarly,

$$\frac{\kappa}{\bar{\kappa}'} = \frac{1}{2} \left[ \left( \frac{kk'}{\bar{\kappa}'} \right)^2 + 2 - \frac{kk'}{\bar{\kappa}'} \sqrt{\left( \frac{kk'}{\bar{\kappa}'} \right)^2 + 4} \right]. \quad (A7)$$

Making use of Eq. (A3) and some algebra now presents

$$|T(p', k'; p, k)|^2 = 6\pi m_e^2 \sigma_T \left\{ \left( \frac{kk'}{\bar{\kappa}} \right)^2 + 2 + 2m_e^2 \frac{kk'}{\bar{\kappa}^2} + m_e^4 \left( \frac{kk'}{\bar{\kappa}^2} \right)^2 \right\}. \quad (A8)$$

#### Appendix B. Angular averages

The calculation outlined in the text involves the integration of

$$\cos \alpha = \hat{\mathbf{p}} \cdot \hat{\mathbf{k}} = \hat{p}^l \hat{k}^l, \quad \cos \alpha' = \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}' = \hat{p}^l \hat{k}'^l \quad (B1)$$

and of the products of the powers  $\cos^m \alpha \cos^n \alpha'$ , over the solid angle of the momentum  $\mathbf{p}$  associated with the electron distribution  $g(t, E)$ . This is equivalent to averaging over all orientations of the unit vector  $\hat{\mathbf{p}}$ , and this averaging can be performed at any stage of the computation—it may be performed before the actual integral over  $\mathbf{p}$  is carried out. The averages may be expressed as contractions of outer products  $\hat{k}^l \dots$  and  $\hat{k}'^m \dots$  with the rotationally invariant tensors that result from the angular averages  $\langle \hat{p}^k \hat{p}^l \dots \rangle$ . For example,

$$\langle \cos \alpha \cos \alpha' \rangle = \hat{k}^l \hat{k}'^m \langle \hat{p}^k \hat{p}^m \rangle. \quad (B2)$$

This is the method that we shall employ.

Under the angular average

$$\langle \hat{p}^l \rangle = 0 \quad (B3)$$

and since  $\mathbf{p}$  is a vector, not a pseudo-vector,

$$\langle \hat{p}^l \hat{p}^m \hat{p}^n \rangle = 0, \quad (B4)$$

because the only rotationally invariant, third rank tensor, is the pseudo-tensor  $\epsilon^{lmn}$ . The lowest-order correlation is

$$\langle \hat{p}^l \hat{p}^m \rangle = \frac{1}{3} \delta^{lm}, \quad (B5)$$

where  $\delta^{kl}$  is the matrix element of the invariant unit matrix. The overall coefficient is determined by the trace  $\langle \hat{p}^l \hat{p}^l \rangle = \langle 1 \rangle = 1$ . The final average that we shall need is

$$\langle \hat{p}^k \hat{p}^l \hat{p}^m \hat{p}^n \rangle = \frac{1}{15} [\delta^{kl} \delta^{mn} + \delta^{km} \delta^{ln} + \delta^{kn} \delta^{lm}]. \quad (\text{B6})$$

Here the particular combination of the delta symbols on the right-hand side is required to reproduce the complete symmetry of the left-hand side under any permutation of the indices  $k, l, m, n$ . The overall coefficient again follows from taking the trace over any index pair and comparing the result with Eq. (B5).

Therefore,

$$\langle \cos \alpha \rangle = 0 = \langle \cos \alpha' \rangle \quad (\text{B7})$$

and

$$\langle \cos \alpha \cos \alpha' \rangle = \frac{1}{3} \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' = \frac{1}{3} \cos \theta, \quad (\text{B8})$$

from which follow

$$\langle \cos^2 \alpha \rangle = \frac{1}{3} = \langle \cos^2 \alpha' \rangle. \quad (\text{B9})$$

Since the angular average of three momentum vectors vanishes,

$$0 = \langle \cos^3 \alpha \rangle = \langle \cos^2 \alpha \cos \alpha' \rangle = \langle \cos \alpha \cos^2 \alpha' \rangle = \langle \cos^3 \alpha' \rangle. \quad (\text{B10})$$

Next,

$$\langle \cos^3 \alpha \cos \alpha' \rangle = \frac{1}{15} [\hat{\mathbf{k}}^2 \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' + \hat{\mathbf{k}}^2 \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' + \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} \hat{\mathbf{k}}'^2] = \frac{1}{5} \cos \theta \quad (\text{B11})$$

and

$$\langle \cos \alpha \cos^3 \alpha' \rangle = \frac{1}{5} \cos \theta. \quad (\text{B12})$$

Finally

$$\begin{aligned} \langle \cos^2 \alpha \cos^2 \alpha' \rangle &= \frac{1}{15} [\hat{\mathbf{k}}^2 \hat{\mathbf{k}}'^2 + (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')^2 + (\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')^2] \\ &= \frac{1}{15} [1 + 2 \cos^2 \theta], \end{aligned} \quad (\text{B13})$$

from which follow

$$\langle \cos^4 \alpha \rangle = \frac{1}{5} = \langle \cos^4 \alpha' \rangle. \quad (\text{B14})$$

### Appendix C. Delta function identities

The work in the text involves various derivatives of  $\delta(\omega - \omega') = \delta(x)$  multiplied by various powers of  $\omega - \omega' = x$ . Simple manipulations can be performed to place a derivative  $d^n/dx^n$  to the left of a power  $x^m$ , leaving lower derivatives and lower powers of  $x$ . Again, the lower derivatives can be ordered to the left so that all the derivatives appear as total derivatives. According to the rules of generalized functions, any resulting term of the form  $x^l \delta(x)$  gives a vanishing contribution. The simplest example of this procedure is

$$x \delta'(x) = \frac{d}{dx} [x \delta(x)] - \delta(x). \quad (\text{C1})$$

As we have just noted, the first term may be discarded. Moreover, the delta function  $\delta(x) = \delta(\omega - \omega')$  with no derivative can also be omitted because it gives rise to equal contributions from the ‘scattering into’ and ‘scattering out of’ terms in the Boltzmann equation which cancel. We use the symbol ‘=’ to denote the only effective parts that remain after the manipulations described above have been made. Thus, we write Eq. (C1) as

$$x \delta'(x) = 0. \quad (\text{C2})$$

More involved computations lead to the effective results

$$x \delta''(x) = -2 \delta'(x), \quad (\text{C3})$$

$$x^2 \delta''(x) = 0, \quad (\text{C4})$$

$$x \delta'''(x) = -3 \delta''(x), \quad (\text{C5})$$

$$x^2 \delta'''(x) = 6 \delta'(x), \quad (\text{C6})$$

$$x^2 \delta''''(x) = 12 \delta''(x) \quad (\text{C7})$$

and

$$x^4 \delta''''(x) = 0. \quad (\text{C8})$$

### Appendix D. Effective temperatures defined by electron distribution integrals

Here we shall explain the definitions of the two effective temperatures  $T_{\text{eff1}}$  and  $T_{\text{eff2}}$  that reduce to the electron temperature  $T_e$  when the electron relativistic phase-space distribution  $g(t, E)$  is restricted to be a Maxwell–Boltzmann distribution.

For our system of free, relativistic electrons, the number density is given for an arbitrary phase space distribution  $g(t, E)$  by

$$n_e = \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} 2g(t, E). \quad (\text{D1})$$

For the case of thermal equilibrium,

$$g(p) = \text{const. exp} \left\{ -\frac{E(p)}{T} \right\}, \quad (\text{D2})$$

in which

$$E(p) = [p^2 + m_e^2]^{1/2} \quad (\text{D3})$$

is the total relativistic energy of an electron with momentum  $p$ .

We define

$$T_{\text{eff1}} = \frac{1}{n_e} \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{p^2}{3E(p)} 2g(t, E) \quad (\text{D4})$$

because, with this definition for an arbitrary electron distribution  $g(t, E)$ , we have the exact (to all orders of  $v^2$ ) limit  $T_{\text{eff1}} \rightarrow T$  when  $g(t, E)$  is replaced by a relativistic Maxwell–Boltzmann distribution with temperature  $T$ . This assertion is proved by noting that partial integration gives<sup>4</sup>

<sup>4</sup> The result that we have obtained for an effective temperature definition that reduces without approximation to the temperature of a relativistic Maxwell–Boltzmann distribution is motivated and explained by the following discussion of relativistic statistical mechanics. In general, even for an interacting system of particles, the grand canonical ensemble partition function  $Z$  defines the grand potential  $\Omega$  by  $\ln Z = \Omega/T$ . A fundamental theorem of the theory relates  $\Omega$  to the pressure  $p$  and the volume  $V$  of the system by  $\Omega = pV$ . For the case of a dilute free gas with number density  $n$ , it is a standard textbook result that  $\ln Z = nV$ , even for a relativistic ensemble of particles. Thus, for a dilute, non-interacting gas, the familiar equation of state  $p = nT$  holds even for relativistic particles. On the other hand, with an arbitrary phase space density  $f(t, \mathbf{p})$  for particles with spin weight  $g_s$ , the spatial stress is given by

$$T^{kl} = \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{p^k p^l}{E(p)} g_s f(t, \mathbf{p}).$$

The pressure is defined by the trace of the spatial stress,  $p = T^k_k/3$ . Thus the definition (D4) for electrons is just  $T_{\text{eff1}} = p_e/n_e$ . The general statistical mechanics discussion above now shows that  $T_{\text{eff1}} \rightarrow T$  for the case of an equilibrium distribution. This is a proof of this equilibrium limit, independent of the explicit computation given in Eq. (D5).

$$\begin{aligned}
n_e &= \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} 2g(t, E) \\
&= \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} 2 \text{const.} \exp \left\{ -\frac{E(p)}{T} \right\} \frac{1}{3} \frac{\partial}{\partial \mathbf{p}} \cdot \mathbf{p} \\
&= - \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{1}{3} \mathbf{p} \cdot \frac{\partial}{\partial \mathbf{p}} 2 \text{const.} \exp \left\{ -\frac{E(p)}{T} \right\} \\
&= \frac{1}{T} \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{p^2}{3E(p)} 2g(t, E). \tag{D5}
\end{aligned}$$

To obtain the definition of  $T_{\text{eff}2}$ , we note it will appear only in first relativistic corrections to the Kompaneets equation. Hence, for an equilibrium distribution, we may approximate a relativistic correction integral by

$$\begin{aligned}
\int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \left( \frac{p^2}{2m} \right)^2 2g(t, E) &\simeq 2 \text{const.} \exp \{ -m_e/T \} \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \left( \frac{p^2}{2m_e} \right)^2 \\
&\times \exp \left\{ -\frac{p^2}{2m_e T} \right\} \simeq \frac{15}{4} T^2 n_e. \tag{D6}
\end{aligned}$$

Therefore, for an arbitrary distribution, we shall define

$$T_{\text{eff}2}^2 = \frac{4}{15n_e} \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \left( \frac{p^2}{2m_e} \right)^2 2g(t, E). \tag{D7}$$

To leading order, the only order needed,  $T_{\text{eff}2} \rightarrow T$  when  $g(t, E)$  becomes a thermal distribution.

With these results in hand, we can now evaluate the the integrals needed in the text. The following two integrals appear with the lowest-order functions  $s_2^A$  and  $s_2^B$ , and thus they must be evaluated to both lowest and first non-leading orders:

$$\begin{aligned}
\int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{v^2}{E^2} g(t, E) &= \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{p^2}{E^4} g(t, E) \\
&\simeq \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{1}{m_e^3} \left[ \frac{p^2}{E} - \frac{6}{m_e} \left( \frac{p^2}{2m_e} \right)^2 \right] g(t, E) \\
&= \frac{3}{2} \frac{n_e}{m_e^3} \left[ T_{\text{eff}1} - \frac{15}{2} \frac{T_{\text{eff}2}^2}{m_e} \right], \tag{D8}
\end{aligned}$$

$$\begin{aligned}
\int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{1}{E^3} g(t, E) &\simeq \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{1}{m_e^3} \left[ 1 - \frac{3}{2} \frac{p^2}{m_e^2} \right] g(t, E) \\
&\simeq \frac{n_e}{2m_e^3} \left[ 1 - \frac{9}{2} \frac{T_{\text{eff}1}}{m_e} \right]. \tag{D9}
\end{aligned}$$

On the other hand, for the higher order  $s_4^A$  and  $s_4^B$  terms, we need only leading evaluations:

$$\int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{v^4}{E^2} g(t, E) \simeq \frac{15}{2} \frac{n_e}{m_e^2} \frac{T_{\text{eff}2}^2}{m_e^2}, \tag{D10}$$

$$\int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{1}{E^4} g(t, E) \simeq \frac{n_e}{2m_e^4}, \tag{D11}$$

$$\int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{v^2}{E^3} g(t, E) \simeq \frac{3}{2} \frac{n_e}{m_e^4} T_{\text{eff}1}. \tag{D12}$$

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