NONLINEAR HILL'S EQUATIONS AND THE BEAM-BEAM INTERACTION†

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I. INTRODUCTION.

Nonlinear Hill's equations of the form

$$z''(\theta) + n(\theta)z(\theta) = F(\theta;z) \qquad (I.1)$$

frequently occur in the description of betatron oscillations in cyclic accelerators and in intersecting storage rings ([1] - [3]). In equation (I.1), θ stands for the azimuth around the machine (of radius 1), n denotes a periodic function with (minimal) period $T \leq 2\pi$, while F generally depends nonlinearly on z and also periodically on θ with, however, a minimal period T' in general different from T. In this paper, we present without (detailed) proofs new results regarding the stability properties of a class of equations of the form (I.1), relevant to the problem of the beam-beam interaction in the "weak-strong" approximation. Specifically, we discuss new inequalities for the corresponding action functional, valid in particular whenever the strong beam has an anisotropic (ribbon-like) Gaussian current density. We then solve the variational problem by direct methods, establish its connection with the existence problem of periodic orbits, and finally briefly indicate how to construct the minimizing sequences involved. A general theory, along with complete proofs, will appear in [4].

2. A CLASS OF NONLINEAR HILL'S EQUATIONS AND THE "WEAK-STRONG" BEAM-BEAM INTERACTION: A VARIATIONAL FORMULATION.

Consider a continuous periodic function Δ with minimal period 2π and let $W^1_{2,2\pi}$ be the space of all real square integrable functions z on $[0,2\pi]$ that have a square integrable (generalized) derivative z'; equip $W^1_{2,2\pi}$ with the norm

$$|||z|||^2_{1,2} = ||z||^2_2 + ||z'||^2_2 \tag{2.1}$$

where

$$||z||^2_2 = \int_0^{2\pi} z^2(\theta)d\theta \quad \text{and} \quad ||z'||^2_2 = \int_0^{2\pi} (z')^2(\theta)d\theta \tag{2.2}$$

Now consider a function G from $W^1_{2,2\pi}$ into itself which satisfies the following properties:

(1) $0 \le G(z) \le \frac{z^2}{2}$ for all $z \in W^1_{2,2\pi}$.

(2) G is concave in z^2; in other words, there exists a function H such that

$$H(x) = G(z) \tag{2.3}$$

where $x = z^2$, which satisfies the inequality

$$H(\lambda x + (1-\lambda)y) \ge \lambda H(x) + (1-\lambda)H(y) \tag{2.4}$$

for all nonnegative x and y in $W^1_{2,2\pi}$ and for each $\lambda \in (0;1)$.

(3) G is (Fréchet)-differentiable on $W^1_{2,2\pi}$ with bounded derivative

$$G'(z) = F(z) \tag{2.5}$$

In other words one has the relation

$$G(z+v) - G(z) = F(z)v + R(z;v) \tag{2.6}$$

for all $v \in W^1_{2,2\pi}$, where $R(z;v)$ is the remainder satisfying the relation

$$\lim_{||v||_{1,2}\to 0} \frac{R(z;v)}{||v||_{1,2}} = 0 \tag{2.7}$$

For z twice continuously differentiable on $[0,2\pi]$ we then consider the differential equation

$$z'' + nz = \beta\Delta F(z) \tag{2.8}$$

where n is a positive real number and β a real parameter. We are concerned with the stability properties of equation (2.8) in terms of n and β; in other words we would like to know for what values of n and β all the solutions of (2.8) are bounded (stability), and for which ones at least one of the solutions is unbounded (instability). Likewise, we would like to know how the solutions of (2.8) bifurcate away from those of the linear equation corresponding to $F(z) = z$ in (2.8). In this paper, we shall restrict our attention to the existence problem of periodic orbits and address ourselves to these more general questions in [4].

There are two elementary examples that have motivated this study in the first place, for which G satisfies the properties (1)-(3) above.

<u>EXAMPLE 1: The linear case</u>. We have $F(z) = z$ in (2.8); we may then choose $G(z) = \frac{z^2}{2}$ and thereby $H(x) = \frac{x}{2}$; properties (1)-(3) are here obvious.

<u>EXAMPLE 2: The anisotropic (ribbon-like) Gaussian beam</u>. In this case we have $F(z) = \mathrm{erf}(z)$ (error function), namely

$$F(z)(\theta) = 2\pi^{-1/2}\int_0^{z(\theta)} \exp[-t^2]dt \qquad (2.9)$$

for $z \geq 0$, and $F(-z) = -\mathrm{erf}(z)$ otherwise (see [1]-[3]). We then may choose

$$G(z) = z\,\mathrm{erf}(z) + \pi^{-1/2}(\exp[-z^2] - 1) \qquad (2.10)$$

and consequently

$$H(x) = \sqrt{x}\,\mathrm{erf}(\sqrt{x}) + \pi^{-1/2}(\exp[-x] - 1) \qquad (2.11)$$

An elementary calculation shows that H is concave in x if, and only if,

$$\int_0^{\sqrt{x}} \exp[-t^2]dt \geq \sqrt{x}\, \exp[-x] \tag{2.12}$$

for all nonnegative x's in $W^1_{2,2\pi}$. Relation (2.12) can then be proved using the power series expansions for exp[-x] and erf[$\sqrt{x}$], namely

$$\operatorname{erf}[\sqrt{x}] = 2\pi^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{x})^{2n+1}}{n!\,(2n+1)} \tag{2.13}$$

This shows that property (2) above is satisfied; property (1) can be proved by similar arguments. Property (3) is the result of a direct computation; in particular (2.7) follows from elementary estimates for $||R(z;v)||_{1,2}$. We refer the reader to [4] for details. Observe that, in this specific example, G itself is convex in z; this is, however, irrelevant. The crucial property is the concavity in z^2, as we shall see below.

Now consider the action functional

$$\begin{aligned} S[z] &= \frac{1}{2}\int_0^{2\pi} (z')^2(\theta)d\theta - \frac{n}{2}\int_0^{2\pi} z^2(\theta)d\theta + \beta\int_0^{2\pi} \Delta(\theta)G(z)(\theta)d\theta \\ &\equiv S_q[z] + \beta\int_0^{2\pi} \Delta(\theta)G(z)(\theta)d\theta \end{aligned} \tag{2.14}$$

where S_q stands for the quadratic, harmonic oscillator functional

$$S_q[z] = \frac{1}{2}\int_0^{2\pi} (z')^2(\theta)d\theta - \frac{n}{2}\int_0^{2\pi} z^2(\theta)d\theta \tag{2.15}$$

In terms of (2.2), one can then rewrite (2.14) as

$$2S[z] = ||z'||_2^2 - n||z||_2^2 + 2\beta\int_0^{2\pi} \Delta(\theta)G(z)(\theta)d\theta \quad (2.16)$$

In the next section, we shall present a set of inequalities for S and indicate how to determine its critical points using direct variational methods. This, in turn, will allow us to discuss the existence problem of periodic orbits of equation (2.8). Observe that S is not convex in z in general, so that the traditional convex minimization techniques (see for instance [5]) may not be applied.

3. INEQUALITIES FOR THE FUNCTIONAL S AND SOLUTION OF THE VARIATIONAL PROBLEM FOR PERIODIC ORBITS.

We shall denote by $W^1_{2,[2\pi]}$ the subspace of $W^1_{2,2\pi}$ containing all the real periodic functions of the form

$$z(\theta) = \sum_{k=-\infty}^{+\infty} a_k \exp[ik\theta] \quad (3.1)$$

which satisfy the conditions

$$z(0) = z(2\pi)$$

and

$$a_0 \equiv \int_0^{2\pi} z(\theta)d\theta = 0 \quad (3.2)$$

In order to detect the critical points of S on $W^1_{2,[2\pi]}$, we shall need an upper bound as well as a lower bound for $S[\frac{z-v}{2}]$, where both z and v belong to $W^1_{2,[2\pi]}$; we shall equip $W^1_{2,[2\pi]}$ with the kinetic energy norm

$$||z||^2_{1,2} = \int_0^{2\pi} (z')^2(\theta)d\theta \tag{3.3}$$

which, under the conditions (3.2), is equivalent to (2.1) since we have

$$||z||_2 \le ||z'||_2 \tag{3.4}$$

A typical situation is described in the following

PROPOSITION 3.1. Consider the functional (2.14) where G satisfies the properties (1)-(3) above. Assume moreover that $0 < n < 1$, $\beta\Delta \le 0$, $|\Delta(\theta)| \le K$ for some positive K independent of θ and that

$$0 \le |\beta| \le \frac{1-n}{K} \tag{3.5}$$

Then one has

$$0 \le \frac{1}{8}(1-n-|\beta|K)||z-v||^2_{1,2} \le S[\frac{z-v}{2}] \le \frac{1}{2}(S[z]+S[v]) - S[\frac{z+v}{2}] \tag{3.6}$$

for all $z,v \in W^1_{2,[2\pi]}$.

SKETCH OF THE PROOF (see [4] for details). From (2.15) one has

$$S_q[\frac{z+v}{2}] + S_q[\frac{z-v}{2}] = \frac{1}{2}(S_q[z] + S_q[v]) \qquad (3.7)$$

Moreover for G concave in z^2 and such that $G(0) \geq 0$, one has the estimate

$$G(\frac{z+v}{2}) + G(\frac{z-v}{2}) \geq \frac{1}{2}(G(z) + G(v)) \qquad (3.8)$$

Combination of (3.8), (3.7) and (2.14) with the fact that $\beta\Delta \leq 0$ then leads to the upper bound in (3.6). On the other hand one has

$$2\beta\int_0^{2\pi} \Delta(\theta)G(z)(\theta)d\theta \geq -K|\beta| \, ||z||_2^2 \qquad (3.9)$$

which follows from property (1) above and our assumptions on β and Δ; relation (3.9), along with (2.16), (3.4), (3.5) then implies the lower bounds in (3.6). This completes the proof.

REMARK. The concavity of G in z^2 is crucial to establish (3.8); concavity in z, along with the parity of G, would only lead to

$$G(\frac{z+v}{2}) + G(\frac{z-v}{2}) \geq G(z) + G(v) \qquad (3.10)$$

which is not sufficient to establish (3.6).

Proposition (3.1) now allows us to construct a critical point of S on $W^1_{2,[2\pi]}$; indeed, since $S[z] \geq 0$ for all z in $W^1_{2,[2\pi]}$, there exist a greatest lower bound

$$0 \leq \delta = \inf_{z \in W^1_{2,[2\pi]}} S[z] \tag{3.10}$$

and a minimizing sequence $z^{(N)}$ such that

$$\lim_{N\to\infty} S[z^{(N)}] = \delta \tag{3.11}$$

The fact that S actually takes on its minimal value δ in $W^1_{2,[2\pi]}$ is described in the following

PROPOSITION 3.2. Under the same conditions as in proposition (3.1), with the exception of (3.5) which is replaced by

$$0 \leq |\beta| < \frac{1-n}{K} \tag{3.12}$$

there exists a function z in $W^1_{2,[2\pi]}$ such that

$$S[z] = \delta \tag{3.13}$$

Moreover one has $\lim_{N\to\infty} z^{(N)} = z$ in the norm (3.3).

PROOF. Apply (3.6) to the minimizing sequence $z^{(N)}$; we get

$$0 \leq \frac{1}{8}(1-n-|\beta|K)\,||z^{(M)} - z^{(N)}||^2_{1,2} \leq \frac{1}{2}(S[z^{(M)}] + S[z^{(N)}]) - \delta \tag{3.14}$$

since $S\left[\frac{z^{(M)}+z^{(N)}}{2}\right] \geq \delta$; from (3.11) and (3.14) we then get

$$\lim_{M,N\to\infty} ||z^{(M)} - z^{(N)}||_{1,2} = 0 \qquad (3.15)$$

which proves that $\lim_{N\to\infty} z^{(N)} = z$ since $W^1_{2,[2\pi]}$ is complete. Relation (3.13) then follows from (3.11) and the continuity of S. This completes the proof.

<u>EXAMPLE: Solution to the variational problem for the Gaussian, ribbon-like beam-beam interaction</u>. We shall simply rephrase our results in physical terms, in the context of example 2. Consider the equation

$$z'' + nz = \beta\Delta \operatorname{erf}(z) \qquad (3.16)$$

which describes the vertical betatron oscillations of one particle in the weak beam, going through the strong Gaussian, ribbon-like, counterrotating beam at one of the interaction regions of an intersecting storage ring; one then has the following

<u>THEOREM 3.3</u>. Under the same conditions as in proposition 3.2, in particular with a magnetic field index n satisfying $0 < n < 1$ (weak focusing regime), there exists a periodic orbit z in $W^1_{2,[2\pi]}$ with period 2π which minimizes the action functional

$$S[z] = \frac{1}{2}\int_0^{2\pi}(z')^2(\theta)d\theta - \frac{n}{2}\int_0^{2\pi} z^2(\theta)d\theta$$

$$+ \beta\int_0^{2\pi}\Delta(\theta)(z\ \mathrm{erf}(z) + \pi^{-1/2}(\exp[-z^2] - 1))(\theta)d\theta$$

Moreover, the minimizing orbit z vanishes at least once in $[0,2\pi]$ (relation 3.2).

Similar results can be obtained for the strong focusing regime and for minimizing orbits which may vanish more than once in $[0,2\pi]$ (see [4]).

One important question now remains: is the minimizing orbit z in proposition (3.2) (respectively in theorem (3.3)) necessarily a (classical) solution of equation (2.8) (respectively of equation (3.16)) and is it possible to devise algorithms or iterative procedures to actually construct minimizing sequences $z^{(N)}$ converging to z?

We shall address ourselves to this question in the next section.

4. CONNECTION BETWEEN THE VARIATIONAL PROBLEM AND THE EXISTENCE OF NON TRIVIAL PERIODIC ORBITS.

We first have to mention that the solution to the variational problem of the preceding section may be chosen twice continuously differentiable if G(z) is regular enough in z; this follows from very general circumstances (see for instance [6]). In this case, we have the following

THEOREM 4.1. Let z be a twice continuously differentiable function in $W^1_{2,[2\pi]}$ which minimizes S; then z satisfies equation (2.8), namely

$$z'' + nz = \beta\Delta F(z) \qquad (4.1)$$

In this case one has the representation

$$s = \beta\int_0^{2\pi} \Delta(\theta)\{G(z) - \frac{1}{2} zF(z)\}(\theta)d\theta \qquad (4.2)$$

for the minimal value of S.

PROOF. Since z minimizes S on $W^1_{2,[2\pi]}$ one has

$$S[z+\lambda v] \geq S[z] \qquad (4.3)$$

for all v in $W^1_{2,[2\pi]}$ and for each real λ; thus the function $\lambda \longrightarrow S[z+\lambda v]$ has a minimum at $\lambda = 0$, which implies

$$\frac{d}{d\lambda} S[z+\lambda v](\lambda=0) = \int_0^{2\pi} \{-z'' - nz + \beta\Delta F(z)\}v(\theta)d\theta = 0 \qquad (4.4)$$

for all v in $W^1_{2,[2\pi]}$. An elementary density argument then shows that (4.4) actually holds for each v in $L^2_{[2\pi]}$ which, in turn, implies

$$z'' + nz - \beta\Delta F(z) = 0 \tag{4.5}$$

which is (4.1). Now from (4.5) (or (4.1)) one gets

$$z''z + nz^2 = \beta\Delta zF(z) \tag{4.6}$$

and consequently the relation

$$-\int_0^{2\pi} (z')^2(\theta)d\theta + n\int_0^{2\pi} z^2(\theta)d\theta = \beta\int_0^{2\pi} \Delta(\theta)zF(z)(\theta)d\theta \tag{4.7}$$

after an integration by parts of $z''z$. One can then express (4.7) in terms of $S[z]$ using (2.14), which leads to

$$s = S[z] = \beta\int_0^{2\pi} \Delta(\theta)\{G(z) - \frac{1}{2}\, zF(z)\}(\theta)d\theta \tag{4.8}$$

This completes the proof.

A few remarks are necessary at this point; we first observe that the relation

$$S[z] = \beta\int_0^{2\pi} \Delta(\theta)\{G(z) - \frac{1}{2}\, zF(z)\}(\theta)d\theta \tag{4.9}$$

is a necessary condition for any twice continuously differentiable function in $W^1_{2,[2\pi]}$ to be a periodic solution of equation (4.1) with period 2π. This fact, combined with the lower bound in (3.6), then leads to statements regarding the existence of periodic orbits which allow us to distinguish between the trivial solution $z \equiv 0$ and the non trivial ones $z \not\equiv 0$. A typical example is the following

THEOREM 4.2 (The linear case). Consider equation (2.8) with $F(z) = z$, namely

$$z'' + (n - \beta\Delta)z = 0 \tag{4.10}$$

Then, under the same conditions as in proposition (3.2), equation (4.10) has no non trivial periodic solution with period 2π.

PROOF. Choose any non zero z in $W^1_{2,[2\pi]}$. Since $1 - n - |\beta|K > 0$ from (3.12), the lower bound in (3.6) implies

$$S[z] > 0 \tag{4.11}$$

On the other hand one has

$$\beta\int_0^{2\pi} \Delta(\theta)\{G(z) - \frac{1}{2}\, zF(z)\} = 0$$

since $F(z) = z$ and $G(z) = \frac{z^2}{2}$; the necessary condition (4.9) can therefore not be satisfied. This proves the theorem.

REMARK. The preceding result has nothing surprising. Indeed, condition (3.12) can be rewritten as

$$0 \leq \frac{|\beta|}{n} < \frac{1 - n}{Kn} \tag{4.12}$$

and consequently represents the two-dimensional region in the $\frac{\beta}{n} - n$ plane bounded by the positive coordinate axes and the hyperbola

$$C(n) = \frac{1 - n}{Kn} \tag{4.13}$$

From Floquet's theory and the Liapounov-Haupt oscillation theorem however, it is known that the periodic orbits of equation (4.10) are not likely to exist in such two-dimensional domains, but only on well defined curves in the $\frac{\beta}{n} - n$ plane (see for instance [7] and [8]). In particular for $\beta = 0$, one has non trivial periodic orbits with period 2π only if $n = 1$, namely where the curve (4.13) intersects the horizontal axis; this is hardly a surprise since the fundamental period associated with the equation

$$z'' + nz = 0 \tag{4.14}$$

is $T = \frac{2\pi}{\sqrt{n}}$.

We now show that the above structure may persist in the nonlinear case: a typical example is the following

THEOREM 4.3 (The anisotropic (ribbon-like) Gaussian beam). Consider the equation

$$z'' + nz = \beta\Delta \operatorname{erf}(z) \tag{4.15}$$

in the weak focusing regime $0 < n < 1$, and under the same conditions as in proposition (3.2). Then equation (4.15) has no non trivial periodic orbit with period 2π.

PROOF. The same argument as in theorem (4.2) is applicable if one observes that one has

$$G(z) - \frac{1}{2} zF(z) = \frac{1}{2} z \operatorname{erf}(z) + \pi^{-1/2}(\exp[-z^2] - 1) \geq 0 \tag{4.16}$$

along with $\beta\Delta \leq 0$. Inequality (4.16) follows from the convexity of $G(z) - \frac{1}{2} zF(z)$ and $G(0) = 0$. One then has

$$\beta\int_0^{2\pi} \Delta(\theta)\{G(z) - \frac{1}{2} zF(z)\}(\theta)d\theta \leq 0 \tag{4.17}$$

so that the necessary condition (4.9) cannot be satisfied since $S[z] > 0$ for any non zero z. This completes the proof.

Similar results hold for the general case as long as $G(z) - \frac{1}{2}zF(z) \geq 0$.

The actual construction of approximation sequences $z^{(N)}$ converging to non trivial periodic orbits is a much less simple matter; we shall only give the main ideas here, and refer the reader to [4] for details. We first observe that the method of the variation of parameters applied to equation (4.1) leads to the solution

$$z(\theta) = z_0(\theta) + \frac{\beta}{\sqrt{n}} \int_0^\theta \sin(\sqrt{n}(\theta-\tau))\Delta(\tau)F(z(\tau))d\tau \qquad (4.18)$$

where z_0 satisfies (4.14). Define then the Volterra operator V by

$$V(f)(\theta) = \frac{\beta}{\sqrt{n}} \int_0^\theta \sin(\sqrt{n}(\theta-\tau)\Delta(\tau)f(\tau)d\tau \qquad (4.18)$$

on $W^1_{2,2\pi}$ and the function A from $W^1_{2,2\pi}$ into itself by

$$A(z) = z - z_0 - V(F(z)) \qquad (4.19)$$

Provided a sufficiently smooth F in (4.1) (typically once continuously differentiable), one can then apply the contraction mapping argument to show that there exists a $z \in W^1_{2,2\pi}$ satisfying (4.18) along with $z(0) = z(2\pi)$, in other words such that

$$A(z) = 0 \qquad (4.20)$$

One can then numerically implement the computation of the root in (4.20) using Newton's method. Indeed the derivative of A(z) is

$$A'(z) = 1 + V(F'(z)) \qquad (4.21)$$

where 1 denotes the identity function on $W^1_{2,2\pi}$; one can then show that $A'(z)$ is invertible, so that the sequence of approximations to the periodic orbit is recursively given by

$$z^{(N+1)} = z^{(N)} - (1+V(F'(z^{(N)}))^{-1}A(z^{(N)}) \qquad (4.22)$$

Quadratic convergence can be obtained. We hope to present our complete results at the next follow-up sessions.

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