Aspects of integrability in string sigma-models

A thesis presented for the degree of Doctor of Philosophy

by

Andrej Stepanchuk



Department of Physics The Blackett Laboratory Imperial College London SW7 2AZ, U.K.

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Declaration and Copyright

I hereby certify that the research presented in this thesis is my own work except where otherwise indicated. The research presented in chapters 3, 5 and 7 is based on the following papers (referred to as [1], [2] and [3] in the bibliography)

- A. Stepanchuk and A. Tseytlin, "On (non)integrability of classical strings in p-brane backgrounds," J.Phys. A46 (2013) 125401, arXiv:1211.3727
- B. Hoare, A. Stepanchuk, and A. Tseytlin, "Giant magnon solution and dispersion relation in string theory in AdS₃ × S³ × T⁴ with mixed flux," *Nucl. Phys.* B879 (2014) 318–347, arXiv:1311.1794
- A. Stepanchuk, "String theory in AdS₃ × S³ × T⁴ with mixed flux: semiclassical and 1-loop phase in the S-matrix," J.Phys. A48 no. 19, (2015) 195401, arXiv:1412.4764

Additionally chapter 6 is based on unpublished work.

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> Andrej Stepanchuk London, UK 16th of June 2015

Abstract

The recent success in applying integrability-based methods to study examples of gauge/gravity dualities in highly (super)symmetric settings motivates the question of whether such methods can be carried over to more physical and less symmetric cases. In this thesis we consider two such examples of string σ -models, which interpolate between integrable or solvable limits.

First we consider classical string motion on curved *p*-brane backgrounds for which the σ -model interpolates between the integrable flat space and $\operatorname{AdS}_k \times \operatorname{S}^k$ coset or WZW σ -models. We find that while the equations for particle (i.e. geodesic) motion are integrable in these backgrounds, the equations for extended string motion are not.

The second example we consider is string theory on $AdS_3 \times S^3 \times T^4$ with mixed Ramond-Ramond (R-R) and Neveu-Schwarz-Neveu-Schwarz (NS-NS) 3-form fluxes, which interpolates between the integrable pure R-R and the pure NS-NS theory that can be solved using CFT methods. The dispersion relation and S-matrix for world-sheet excitations, which are the essential ingredients in solving for the string spectrum, are only partially fixed by integrability and symmetry arguments. By constructing the mixed flux generalisation of the dyonic giant magnon soliton, which we show can be interpreted as a bound-state of excitations, we determine the dispersion relation for massive excitations. We also construct the mixed flux generalisation of the folded string on $AdS_3 \times S^1$ and show that, at leading order in large angular momentum on AdS_3 , its energy is given by the pure R-R expression with the string tension rescaled by the R-R flux coefficient. Further, we derive the bound-state S-matrix and its 1-loop correction by considering the scattering of dyonic giant magnons and plane waves. From this we deduce the semiclassical and 1-loop dressing phases in the massive sector S-matrix, which we find to agree with recent proposals.

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Chapter 1

Introduction

One of the fundamental problems in physics is to understand strongly coupled quantum systems which arise in many different theories ranging from the strong nuclear force, described by QCD, over condensed-matter systems to cosmological models. More precisely we would like to be able to compute observables in such theories using analytical methods rather than purely numerical simulations. In QCD for example we can use standard perturbative Feynman diagram techniques at high energies where the quark interaction becomes weak due to asymptotic freedom. However, these methods break down at low energies where confinement takes over.

Over the past decade a new approach to this problem has emerged in terms of the gauge/string duality, which relates the strong-coupling regime of certain quantum field theories to the weak coupling regime of some associated string theories. In effect, string theory could potentially provide us with new insights and tools for studying strongly coupled quantum systems. An important step in this long-term programme is to test and explore this correspondence by identifying special examples which we can solve explicitly. In mathematical terms such models exhibit integrability, which encodes "hidden" symmetries associated with a conservation law for each degree of freedom. More recently this has allowed for significant progress in highly symmetric settings such as in the presence of supersymmetry [4, 5].

In particular the gauge/gravity duality or AdS/CFT correspondence relates string theories on Antide-Sitter spacetime to conformal field theories on the AdS boundary [6]. This is also a realisation of the holographic principle [7], by which the information about a volume of space can be encoded on its boundary. The most prominent example is type IIB superstring theory on AdS₅ × S⁵, which is dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in four dimensions with the gauge group SU(N). The parameters on the string side are the string tension h, which can be given in terms of the AdS₅ and S⁵ radius R and the inverse string tension α' or the string length l_s as $h = \frac{R^2}{2\pi \alpha'} = \frac{R^2}{2\pi l_s^2}$, and the string coupling g_s . On the gauge side we have the parameter N^1 of the gauge group and the coupling constant g_{YM} , which are related by the AdS/CFT correspondence to the string side by

$$q_s = \frac{\lambda}{4\pi N}, \qquad \mathbf{h} = \frac{\sqrt{\lambda}}{2\pi}, \tag{1.0.1}$$

where $\lambda \equiv g_{YM}^2 N$ is the 't Hooft coupling. In general the AdS/CFT dictionary relates the CFT parti-

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¹Essentially one can use the rank for this parameter since this is N-1 for SU(N) and we are interested in large N.

tion function with sources J for local operators to the string partition function with vertex operators of value J on the AdS boundary

$$Z_{\text{string}}[\phi|_{\partial \text{AdS}} = J] = Z_{\text{CFT}}[J].$$
(1.0.2)

However, the CFT side is fully characterised by the scaling dimensions of primary operators, operators of lowest scaling dimension in their multiplet, and three-point correlation functions. This information can essentially be used to determine any observable using operator product expansions (OPEs). The classical scaling dimensions also receive quantum corrections, the coupling-dependent anomalous dimensions. The scaling dimensions Δ are the eigenvalues of the dilatation operator \mathfrak{D} , which is one of the Casimir operators of the conformal group, SO(4, 2), in 3 + 1 dimensions

$$\mathfrak{D}\hat{\mathcal{O}}(x) = \Delta\left(\lambda, \frac{1}{N}\right)\hat{\mathcal{O}}(x). \tag{1.0.3}$$

The conformal group SO(4, 2) and the R-symmetry group SO(6) form the global bosonic symmetry group on the gauge side and they correspond to the global isometry groups for AdS₅ and S⁵ on the string side. Therefore operators on the gauge side and states on the string side both fall into representations of the global symmetry group and are labelled by the Casimir eigenvalues Δ , the two spins (S_1, S_2) for SO(4, 2) and the three angular momenta or R-charges (J_1, J_2, J_3) for SO(6).

The AdS/CFT dictionary then relates the spectrum of energy eigenstates (as measured in global AdS coordinates) on the string theory side,

$$\mathcal{H}_{\text{string}}|\mathcal{O}\rangle = E(\mathbf{h}, g_s)|\mathcal{O}\rangle,$$
 (1.0.4)

to the spectrum of scaling dimensions

$$\Delta\left(\lambda, \frac{1}{N}\right) = E\left(\mathbf{h}, g_s\right), \qquad g_s = \frac{\lambda}{4\pi N}, \qquad \mathbf{h} = \frac{\sqrt{\lambda}}{2\pi}.$$
(1.0.5)

The $\mathcal{N} = 4$ Super-Yang-Mills gauge theory with an SU(N) gauge group admits a 't Hooft expansion in 1/N with λ kept fixed [8]. In this expansion Feynman diagrams are reorganised according to their topology. For example the correlation function of n single-trace operators can be expanded as

$$\left\langle \prod_{i=1}^{n} \hat{\mathcal{O}}_i \right\rangle = N^{2-n} \qquad \qquad + N^{-n} \qquad \qquad + \frac{1}{N^{2+n}} \qquad + \dots, \quad (1.0.6)$$

which resembles a genus expansion of a correlation function in string theory. On the string side this is a weak coupling expansion since $g_s \sim \lambda/N$. Thus in the planar limit

$$N \to \infty, \qquad \lambda = \text{fixed}$$
(1.0.7)

the string theory reduces to a free theory and the spectrum of string states can be determined for $h \gg 1$ using perturbative and semiclassical methods in the string world-sheet theory, e.g. by looking at classical string solutions. However, on the gauge side the perturbative regime corresponds to the opposite limit $\lambda \ll 1$ making a direct comparison of the spectra beyond special cases, such as BPS operators whose scaling dimensions are protected from quantum corrections by supersymmetry,

extremely difficult.

Nevertheless it has been discovered that the problem of finding the spectrum of anomalous dimensions and free string states can be solved exactly on both sides of the duality due to the presence of integrability. This is quite a remarkable achievement as it puts the solution of an interacting higherdimensional CFT for the first time within our reach.

Spectrum of anomalous dimensions

The dilatation operator can be expanded in a perturbative series

$$\mathfrak{D} = \sum_{n} \lambda^n \mathfrak{D}^{(2n)}, \qquad (1.0.8)$$

where all the $\mathfrak{D}^{(2n)}$ operators commute with $\mathfrak{D}^{(0)}$ as well as all Casimirs. Thus the dilatation operator preserves the classical scaling dimension Δ_0 as well as $\Delta = \Delta_0 + \gamma(\lambda)$, the spins and R-charges. Labelling operators by their classical charges $(\Delta_0, S_1, S_2, J_1, J_2, J_3)$ let us consider two complex scalar fields X, Z with the charges (1, 0, 0, 1, 0, 0) and (1, 0, 0, 0, 1, 0) in the $\mathfrak{su}(2)$ subsector of the $\mathfrak{so}(6)$ Rsymmetry algebra. This forms a closed subsector, i.e. the dilatation operator cannot introduce mixing with operators of other scaling dimension, spins and R-charges.

We can construct gauge invariant primary local operators by taking a trace of scalar fields, all evaluated at the same point in spacetime

$$\mathcal{O}(x) = \operatorname{tr}[X(x)X(x)Z(x)...].$$
(1.0.9)

General gauge invariant operators can then be obtained from products of such single trace operators and since we are considering the large N limit the composite dimensions are sums of the single trace operator dimensions. Additionally contributions to correlation functions from multi-trace operators are suppressed by factors of 1/N. In the perturbative approach, i.e. $\lambda \ll 1$, the dilatation operator can then be diagonalised by computing Feynman diagrams for two-point correlation functions, which are partly fixed by conformal symmetry to have the form

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle = \frac{1}{|x-y|^{2\Delta}}.$$
(1.0.10)

In [9] it was realised that at the 1-loop level the spectral problem can be reformulated in terms of an integrable spin-chain. The local single trace operators are identified with states of the $XXX_{1/2}$ Heisenberg spin-chain for nearest neighbour interactions

$$\mathcal{O} = \operatorname{tr}(XZXX\cdots XZX) \qquad \Leftrightarrow \qquad |\Psi\rangle = |\uparrow\downarrow\uparrow\uparrow\cdots\uparrow\downarrow\uparrow\rangle \tag{1.0.11}$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ denote spin states at each site

$$S^{z} \left|\uparrow\right\rangle = +\frac{1}{2}\left|\uparrow\right\rangle, \qquad S^{z} \left|\downarrow\right\rangle = -\frac{1}{2}\left|\downarrow\right\rangle.$$
 (1.0.12)

The length L of the spin-chain is given by the classical (coupling independent) scaling dimension of



Figure 1.1: *Magnon* excitations or *spin waves* on a spin-chain correspond to quasi-particles which undergo scattering.

the operator, i.e. the number of scalar fields. The dilatation operator is identified with the spin-chain Hamiltonian

$$\mathfrak{D}_{\mathfrak{su}(2)}^{\mathrm{planar}} = L + \frac{\lambda}{4\pi^2} \mathcal{H}_{su(2)} + O(\lambda^2), \qquad \mathcal{H}_{su(2)} = \sum_i \left(\frac{1}{4} - \vec{S}_i \cdot \vec{S}_{i+1}\right), \tag{1.0.13}$$

where \vec{S}_i form the $\mathfrak{su}(2)$ algebra in terms of Pauli matrices σ_i acting on the *i*th site

$$[S_i^a, S_j^b] = \delta_{ij} \varepsilon^{abc} S_i^c, \qquad \vec{S}_i = \vec{S}_{i+L}, \qquad S_i^a = \frac{1}{2} \sigma_i^a.$$
(1.0.14)

The first term in (1.0.13) is the tree-level contribution, which is simply the classical scaling dimension. At 1-loop the spectral problem reduces to the problem of diagonalising the Hamiltonian $\mathcal{H}_{su(2)}$ of an integrable spin-chain, which can be readily solved using a Bethe ansatz, e.g. see the review [10]. The Hamiltonian $\mathcal{H}_{su(2)}$ has the ground state

$$|0\rangle = |\uparrow \cdots \uparrow\rangle \qquad \Leftrightarrow \qquad \operatorname{tr}(X^L).$$
 (1.0.15)

Its eigenstates for single "magnon" excitations of momentum p and energy $\varepsilon(p)$ are

$$|p\rangle = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} e^{ip\ell} |\uparrow \cdots \downarrow^{\ell} \cdots \uparrow\rangle, \qquad (1.0.16)$$

$$\mathcal{H}_{su(2)} |p\rangle = \varepsilon(p) |p\rangle, \qquad \varepsilon(p) = \frac{\lambda}{2\pi^2} \sin^2 \frac{p}{2}.$$
 (1.0.17)

Periodicity of the spin-chain implies that the magnon momentum is quantized

$$e^{ipL} = 1.$$
 (1.0.18)

Further, the cyclic property of the trace implies that the states must be invariant under the shifts $l \rightarrow l + 1$ which imposes p = 0. Hence non-trivial single magnon states do not exist and one should consider multi-particle scattering states. In order to define well separated incoming and outgoing magnon states, which can scatter, one can take the limit of a long spin-chain, $L \rightarrow \infty$, with periodic boundary conditions. In a scattering process magnons acquire only a phase shift such that their S-matrix is given by a phase factor $S_{12} = e^{i\phi}$. The two-particle state can be written in terms of the incoming and outgoing parts as

$$|p_1, p_2\rangle = \sum_{l_1 < l_2} e^{ip_1 l_1 + ip_2 l_2} |\cdots \downarrow^{l_1} \cdots \downarrow^{l_2} \cdots \rangle + e^{i\phi} \sum_{l_1 > l_2} e^{ip_1 l_1 + ip_2 l_2} |\cdots \downarrow^{l_2} \cdots \downarrow^{l_1} \cdots \rangle$$
(1.0.19)

and requiring this to be an eigenstate of $\mathcal{H}_{su(2)}$ one finds

$$S_{jk} = \frac{u_j - u_k - i}{u_j - u_k + i}, \qquad \varepsilon(u) = \frac{\lambda}{8\pi^2} \frac{1}{u^2 + 1/4}, \qquad e^{ip_j} = \frac{u_j + i/2}{u_j - i/2}, \tag{1.0.20}$$

where u_j is the rapidity variable. Considering a periodic spin-chain again, the periodicity requires that the state remains invariant when moving the magnon once around the chain. However, we now pick up a phase shift given by the S-matrix, hence momentum quantization gives the *Bethe equations*

$$e^{ip_1L}S_{12} = 1, \qquad e^{ip_2L}S_{21} = 1$$
 (1.0.21)

and the trace condition leads to momentum conservation

$$p_1 + p_2 = 0. \tag{1.0.22}$$

For multi-particle states integrability ensures that the S-matrix factorises into two-particle S-matrices. For M particle scattering states the Bethe equation and momentum conservation then become

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^M \frac{u_j - u_k + i}{u_j - u_k - i}, \qquad \prod_{j=1}^M \frac{u_j + i/2}{u_j - i/2} = 1, \tag{1.0.23}$$

with their energies giving the 1-loop spectrum of anomalous dimensions in the $\mathfrak{su}(2)$ sector

$$\gamma(\lambda) = \sum_{j_1}^M \varepsilon(u_j). \tag{1.0.24}$$

At higher loop orders the interaction length increases and length changing interactions appear [11]. Nevertheless in the asymptotic limit, i.e. when the length of the spin-chain is much greater than the interaction range and thus tends to infinity, this approach can be generalised to all loop orders. This amounts to considering operators for which one of the R-charges is infinite. In the case of the $\mathfrak{su}(2)$ sector one finds magnons [12] with the all-loop dispersion relation, fixed by supersymmetry [13],

$$\varepsilon(p) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}.$$
(1.0.25)

In [14] this was further generalised to all sectors by considering an asymptotic Bethe ansatz for the full superconformal algebra $\mathfrak{psu}(2,2|4)$ of $\mathcal{N} = 4$ SYM. Let us note that this approach does not include wrapping interactions, which must be treated separately [15].

Free string spectrum

On the string side the $g_s = 0$ string theory on $AdS_5 \times S^5$ is described by a 2d non-linear σ -model (NLSM). The bosonic part of the action is

$$S = -\frac{h}{2} \int d^2 \sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^M \partial_b X^N G_{MN}(X), \qquad (1.0.26)$$

where γ^{ab} is the 2d world sheet metric, X^M are the string embedding coordinates and G_{MN} is the target-space metric.

The $AdS_5 \times S^5$ background is also supported by a 5-form Ramond-Ramond (R-R) flux, to which the string world-sheet can be coupled using the Green-Schwarz formalism [16]. The Green-Schwarz (GS) action takes the form of a σ -model on the coset superspace [17]

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1) \times \mathrm{SO}(5)}.$$
(1.0.27)

Its bosonic subgroup is isomorphic to $AdS_5 \times S^5$

$$\frac{SO(4,2)}{SO(4,1)} \simeq AdS_5, \qquad \frac{SO(6)}{SO(5)} \simeq S^5.$$
 (1.0.28)

For example, restricting to the target space $\mathbb{R} \times S^3$ with $\mathbb{R} \subset AdS_5$ and $S^3 \subset S^5$ the bosonic string action is equivalent to the SU(2) principal chiral model (PCM)

$$S_{PCM} = -\frac{h}{4} \int d^2 \sigma \sqrt{-\gamma} \gamma^{ab} tr \Big[g^{-1} \partial_a g g^{-1} \partial_b g \Big], \qquad g \in SU(2).$$
(1.0.29)

Integrability for string σ -models requires an infinite number of conserved charges since we are dealing with a field theory, which has a degrees of freedom at every point in space. At the classical level integrability of a theory can be established by finding a Lax representation for its equations of motion. The Lax connection is then associated with a monodromy matrix, which generates an infinite tower of conserved quantities. Such a construction has been achieved for string σ -models on semi-symmetric coset spaces such as $AdS_5 \times S^5$ [18].

Using the Lax connection classical solutions and their energy spectrum can be characterised in terms of algebraic curves in the finite-gap picture [19, 20, 21]. The energy spectrum then corresponds to the spectrum on the gauge side in the strong-coupling regime $\lambda \gg 1$. For example in the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM the operators of large SO(6) R-charge J and Δ , but $\Delta - J$ being finite, can be identified with string solutions on $\mathbb{R} \times S^3$ of energy $E = \Delta$ and an angular momentum J. The ground state of the spin-chain corresponding to the operator $\operatorname{tr}(X^J)$ is identified with the BMN string, a point-particle moving along the great circle of S^3 at the speed of light with E - J = 0. Operators corresponding to a magnon excitation are identified with an open string solution, the giant magnon, on $\mathbb{R} \times S^2$ [22]. Its endpoints move at the speed of light around the equator with the separation angle staying constant. This angle is identified with the magnon momentum and the dispersion relation

$$E - J = \frac{\sqrt{\lambda}}{2\pi} \left| \sin \frac{p}{2} \right| \tag{1.0.30}$$

is found to match the first term in the large λ expansion of (1.0.25). In [23] these solutions were extended to $\mathbb{R} \times S^3$ with the dispersion relation

$$E - J = \sqrt{J_2^2 + 4h^2(\lambda)\sin^2\frac{p}{2}},$$
(1.0.31)

where J_2 is another angular momentum on S³, which takes integer values in the quantum theory. These *dyonic* giant magnon solutions can be interpreted as bound states of $J_2 = 1$ magnons with the dispersion relation (1.0.25). Integrability ensures that, like the magnon excitations on the spinchain, these soliton solutions undergo factorised scattering without particle production and one can use semiclassical methods to compute quantum corrections to their S-matrix and energy spectrum [24, 25]. Making use of the non-perturbative S-duality it was argued in [26] that the dispersion relation (1.0.31) in fact does not receive perturbative corrections in h and therefore that $h(\lambda)$ has the exact form

$$h(\lambda) = \frac{\sqrt{\lambda}}{4\pi}.$$
(1.0.32)

Quantising the Green-Schwarz superstring in general is not that well understood. However, the presence of integrability allows one to determine the full non-perturbative spectrum of the theory. This can be achieved by quantizing the GS superstring in light-cone gauge where one fixes the light-cone momentum $P_+ = E + J$. The world-sheet cylinder has essentially the circumference P_+ . Taking $P_+ \to \infty$ then effectively decompactifies the world-sheet to a plane and one can define asymptotic states for worldsheet excitations

$$|p_1, ..., p_m\rangle_{\alpha_1, ..., \alpha_n} = a^{\dagger}_{\alpha_1}(p_1) ... a^{\dagger}_{\alpha_n}(p_n) |0\rangle.$$
 (1.0.33)

These are eigenstates of the Hamiltonian with the energies

$$\mathcal{H} | p_1, .., p_m \rangle_{\alpha_1, .., \alpha_n} = E_{\alpha_1, .., \alpha_n}(p_1, .., p_n) | p_1, .., p_m \rangle_{\alpha_1, .., \alpha_n}$$
(1.0.34)

$$E_{\alpha_1,..,\alpha_n}(p_1,..,p_n) = \sum_i \varepsilon_{\alpha_i}(p_i), \qquad (1.0.35)$$

where $\varepsilon_{\alpha}(p)$ is the dispersion relation of fundamental excitations of different flavours α and masses m_{α} . Fixing light-cone gauge breaks the 2d Lorentz invariance and instead of a relativistic dispersion relation $\varepsilon_{\alpha}(p) = \sqrt{m_{\alpha}^2 + p^2}$ one finds a non-relativistic periodic relation

$$\varepsilon_{\alpha}(p) = \sqrt{m_{\alpha}^2 + 4h^2 \sin^2 \frac{p}{2}}.$$
 (1.0.36)

For physical states the total world-sheet momentum must vanish. However, in order to deal with scattering states of arbitrary momentum one must go off-shell by giving up this *level matching* condition. This enhances the global symmetry algebra of the light-cone theory

$$\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \subset \mathfrak{psu}(2,2|4) \tag{1.0.37}$$

by two central charges, which vanish on-shell [27].

Quantum integrability then ensures that scattering factorises into two-particle processes and that particle production is absent. Additionally the two-particle S-matrix must satisfy the Yang-Baxter equation (YBE) and some physical unitarity conditions. Together with the constraints from the centrally extended symmetry algebra this allows one to determine the two-particle world-sheet S-matrix up to an overall scalar factor, the *dressing phase* [13, 28, 29].

The dressing phase can be further constrained by Lorentz invariance and crossing symmetry, under which particles can be exchanged with anti-particles [30]. In the light-cone gauge this is somewhat subtle since manifest Lorentz invariance is not present. Nevertheless also in this case crossing equations for the S-matrix can be established [31]. In [32] an all-loop solution, the BES phase, has been proposed, which was further given in a nice integral (DHM) representation in [33].

Surprisingly quantum integrability in this context is formulated in a very similar fashion to the gauge side suggesting a deep connection of the algebraic structures beyond the equivalence of the spectra. If we consider a single excitation of momentum p on the world-sheet cylinder of finite circumference l, assuming l is sufficiently large to define asymptotic states [14], the periodicity of the wave function implies momentum quantization

$$e^{ipl} = 1.$$
 (1.0.38)

However, by the level-matching condition such single particle states are not physical. For an *M*-particle state $|p_1, ..., p_M\rangle_{\alpha_1, ..., \alpha_M}$ scattering of the individual particles of flavours α_i factorises into two-body processes. In the two-body scattering the particles scatter elastically without particle production and the outgoing state will consist of the same particle flavours and momenta. The individual particles only acquire a phase shift which is given by the S-matrix. The periodicity condition then implies the asymptotic Bethe ansatz or *Bethe-Yang equations* [34]

$$e^{ip_k l} \prod_{j \neq k}^M S_{kj}(p_k, p_j) = 1, \qquad k = 1, ..., M,$$
 (1.0.39)

in analogy to a periodic spin-chain.

The full asymptotic spectrum includes bound states of elementary excitations. Their S-matrix and dressing phase are closely related to those for fundamental world-sheet excitations. Bound-states can be understood in terms of poles in the S-matrix and their dressing phase can be obtained by fusing together S-matrices of fundamental excitations in the $\mathfrak{su}(2)$ sector [35, 29].

The asymptotic spectrum essentially describes a periodic but decompactified world-sheet. Instead we would like to determine the spectrum for a finite-size world-sheet cylinder, which additionally includes interactions that wrap the world-sheet cylinder. One possible approach is to compute perturbative corrections to the asymptotic spectrum in powers of $1/P_+$ from the Bethe-Yang equations and to use the Lüscher approach for wrapping interactions, which are exponentially suppressed [36]. The latter can also be used on the gauge side giving agreement with Feynman diagram calculations for the Konishi-operator for example [37].



Figure 1.2: The finite size world-sheet cylinder can be interpreted as a torus with the time-period $R \to \infty$. A double Wick rotation then relates this theory to an infinite volume theory at finite temperature T = 1/L.

In order to find the spectrum non-perturbatively one can consider an infinite volume but finite temperature integrable theory for which the fundamental excitations and their bound-states are in thermodynamic equilibrium. The spectrum of such theories is governed by finite density particle states and can be formulated implicitly in terms of *thermodynamic Bethe ansatz* (TBA) equations, see e.g. [38] for a review. The basic idea is to consider the world-sheet cylinder of circumference L as a torus with a periodic time-direction of size $R \to \infty$ and to perform a double Wick rotation to a *mirror theory* of infinite volume but finite temperature, T = 1/L, as shown in figure 1.2. The ground-state energy $E_0(L)$ in the original finite-volume theory is given by the leading contribution to the partition function

$$\lim_{R \to \infty} Z(R, L) = \lim_{R \to \infty} \operatorname{tr}[e^{-RH(L)}] = \lim_{R \to \infty} e^{-RE_0(L)} + \dots$$
(1.0.40)

The partition function can then be evaluated in the mirror theory

$$Z(R,L) = \tilde{Z}(R,L) = \lim_{R \to \infty} \sum_{n} e^{-L\tilde{E}_{n}(R)},$$
(1.0.41)

where the energies \tilde{E} are determined from the mirror TBA non-linear integral equations. This procedure can also be generalised to excited states by analytic continuation [39]. However, one subtlety is that the light-cone theory is not invariant under the mirror transformation in contrast to a relativistic theory, as one can see from the transformation of the dispersion relation

$$p \to i\tilde{E}, \qquad E \to i\tilde{p}$$
 (1.0.42)

$$\tilde{E} = 2\operatorname{ArcSinh}\left[\frac{\sqrt{m^2 + \tilde{p}^2}}{2\mathrm{h}}\right].$$
(1.0.43)

Thus the mirror theory is genuinely a different theory. It has been established for $AdS_5 \times S^5$ in [29]. Essentially the mirror model corresponds to an analytic continuation of the original theory and is therefore integrable [40]. Its TBA equations have been formulated in terms of a simpler set of equations, the *Y*-system [41, 42, 43], and more recently in terms of the quantum spectral curve [44].

Integrability beyond AdS_5/CFT_4



Figure 1.3: This quiver diagram for ABJM theory shows the representations in which the fields transform. The direction of arrows is from the anti-fundamental to the fundamental representation. The gauge field A_{μ} is in the adjoint representation of U(N). The scalars Y^A and fermions Ψ_A also carry an index A = 1, ..., 4 in the fundamental representation of the R-symmetry group SU(4).

Integrability methods have also been successfully applied to AdS_4/CFT_3 , the duality between type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ and ABJM theory, which is $\mathcal{N} = 6$ supersymmetric Chern-Simons

theory with the gauge group $U(N) \times U(N)$ and Chern-Simons levels k and -k. This gauge theory has matter fields in the bifundamental and adjoint representations of the gauge group, see figure 1.3. It is therefore somewhat less special compared to $\mathcal{N} = 4$ SYM, for which all fields are in the adjoint representation of SU(N).

The $AdS_4 \times \mathbb{CP}^3$ background is supported by two- and four-form R-R fluxes and it only preserves 24 supersymmetries in contrast to the 32 supersymmetries for $AdS_5 \times S^5$. Despite this reduction in manifest symmetries, integrability remains a feature in the planar limit

$$k, N \to \infty, \qquad \lambda = \frac{N}{k} = \text{fixed.}$$
 (1.0.44)

This allows one to apply the same techniques as in the case of for AdS_5/CFT_4 to solve the spectral problem. On the gauge side an integrable spin-chain Hamiltonian for the dilatation operator was identified at two loops in [45, 46]. On the string side the σ -model can be formulated on the coset space [47]

$$\frac{\text{OSp}(6|4)}{\text{SO}(3,1) \times \text{U}(3)}$$
(1.0.45)

with classical (dyonic) giant magnon solutions [46, 48, 49], an all-loop S-matrix [50], a Bethe ansatz [51] and a mirror-TBA description [42, 52]. For a review also see [53]. However, an important difference is that the *slope function* $h(\lambda)$ cannot be fixed by S-duality as in the case of AdS₅/CFT₄. Instead in [54] an integrability-based proposal was given by employing the quantum spectral curve formulation of the mirror-TBA and comparing with localisation based results.

These exciting developments motivate the question of whether we can carry over integrability-based methods and results to more physical and less symmetric settings - after all QCD for example does not possess supersymmetry. A possible approach to this question comes from the string side where we have an integrable string σ -model, which we can try to modify or deform. For example one can consider less symmetric backgrounds or introduce new physical parameters in such a way that integrability is preserved but some manifest symmetry is lost. In this thesis we investigate integrability of two such examples of string σ -models with a continuous parameter which interpolate between known integrable or solvable theories.

One examples that we consider comes from AdS_3/CFT_2 dualities associated with the maximally supersymmetric AdS_3 backgrounds $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$, which preserve only 16 supersymmetries. These backgrounds can be supplemented with a mixture of R-R and Neveu-Schwarz-Neveu-Schwarz (NS-NS) 3-form fluxes².

For $AdS_3 \times S^3 \times T^4$ the free string spectrum of the pure NS-NS theory can be found using a chiral decomposition of its formulation as a supersymmetric extension of an $SL(2, \mathbb{R}) \times SU(2)$ WZW model [55]. In the pure R-R case there is no equivalent of this technique. Instead the exact spectrum is believed to be described in terms of an integrability-based approach in analogy to the $AdS_5 \times S^5$ case [56, 57, 58, 59, 60, 61, 62, 63]. However, the mixed flux theory connects these seemingly distinct cases.

²All other fluxes are zero and the dilaton is constant.

Its R-R and NS-NS 3-form fluxes are given by (choosing unit curvature radii)

$$F = \hat{q} \Big(\operatorname{vol}(\operatorname{AdS}_3) + \operatorname{vol}(\operatorname{S}^3) \Big), \qquad H = q \Big(\operatorname{vol}(\operatorname{AdS}_3) + \operatorname{vol}(\operatorname{S}^3) \Big), \tag{1.0.46}$$

with their coefficients related by the supergravity equations as

$$q^2 + \hat{q}^2 = 1. \tag{1.0.47}$$

This provides us with an interpolating string σ -model between the pure R-R theory at q = 0 and the pure NS-NS theory at q = 1. Therefore, solving for the spectrum in the mixed flux case should improve our understanding of the connection between the world-sheet CFT methods and the integrability-based approach.

Physically AdS₃ backgrounds are of interest since they describe the local spacetime geometry of BTZ black holes, which are solutions of Einstein gravity in 2+1 dimensions with a negative cosmological constant and without matter [64]. Their global geometry is described by AdS₃ but with points identified under a discrete subgroup of the isometry group SO(2, 2). Interestingly there is no curvature singularity at the origin unless matter is present. Instead the origin becomes a singularity for the causal structure: analytically continuing beyond the origin introduces closed timelike curves. These solutions also preserve supersymmetry and form solutions of 2+1 dimensional AdS-supergravity [65]. Their supersymmetric properties can be obtained directly from their construction in terms of a quotient of the supergroup OSp(1|2) by a discrete isometry subgroup [66].

In the pure R-R case a Lax construction has been achieved for the above maximally supersymmetric AdS_3 backgrounds [56, 67] in terms of their supercosets

$$AdS_3 \times S^3 \simeq \frac{PSU(1,1|2) \times PSU(1,1|2)}{SU(1,1) \times SU(2)}, \quad AdS_3 \times S^3_+ \times S^3_- \simeq \frac{D(2,1;\alpha) \times D(2,1;\alpha)}{SU(1,1) \times SU(2) \times SU(2)}, \quad (1.0.48)$$

where α parametrizes the relative size of the two S³ spheres which, by the supergravity equations, are related to the AdS radius as

$$\frac{1}{R_{\mathrm{S}_{+}^{3}}^{2}} + \frac{1}{R_{\mathrm{S}_{-}^{3}}^{2}} = \frac{1}{R_{\mathrm{AdS}_{3}}^{2}}, \quad \alpha = \frac{R_{\mathrm{AdS}_{3}}^{2}}{R_{\mathrm{S}_{+}^{3}}^{2}}, \quad 0 < \alpha < 1.$$
(1.0.49)

Taking the limit $\alpha \to 0$ or $\alpha \to 1$ decompactifies one of the spheres to which, after recompactifying on a torus, gives $AdS_3 \times S^3 \times T^4$. In the mixed flux theory classical integrability and UV finiteness have been shown by supplementing the coset construction with an additional Wess-Zumino (WZ) term for the NS-NS flux in the action [68]. The integrability is also expected to extend to the quantum level, leading to the possibility of determining the exact string spectrum using a thermodynamic Bethe ansatz (TBA) and improving our understanding of the corresponding CFT dual. For a general review on integrability in AdS_3/CFT_2 also see [69].

The essential ingredients in solving for the string spectrum using integrability methods is the dispersion relation and the two-particle S-matrix for the scattering of fundamental world-sheet excitations. However, while in AdS_5/CFT_4 and AdS_4/CFT_3 all world-sheet excitations are massive, a novel feature of AdS_3/CFT_2 is the presence of additional massless excitations from the T^4 , S^1 and the mixed $S^3 \times S^3$ directions as well as their superpartners.

In the pure R-R case (q = 0) a first sign of quantum integrability was observed in the finitegap picture where classical solutions are represented in terms of meromorphic functions, the quasimomenta, with poles and branch cuts in the complex spectral parameter plane. The quasi-momenta are characterised by the densities of discontinuities at their branch cuts. These densities satisfy the finite-gap integral equations, which arise from the redundancies and symmetries of the Lax formulation. The finite-gap integral equations are the classical limit of the quantum Bethe equations: for large numbers of Bethe roots the configuration becomes macroscopic and is described by densities of roots. In [56] it was observed that the finite-gap equations have the same structure as in higher dimensional AdS/CFT examples leading to a conjecture for the quantum Bethe equations and dispersion relation in the massive sector. In terms of h the dispersion relation was found to have the same form as in the case of AdS₅ × S⁵. This was also found to be consistent with the spectrum of excitations near a BMN geodesic from the coset action and fixes the function $h(\lambda)$ at strong coupling as

$$h(\lambda) \approx \frac{\sqrt{\lambda}}{2\pi}, \qquad \lambda \gg 1.$$
 (1.0.50)

Rapid progress followed in determining the massive sector S-matrix and Bethe equations for $AdS_3 \times S^3 \times T^4$ [60, 61] and $AdS_3 \times S^3 \times S^3 \times S^1$ [58, 59].

However, incorporating massless modes into the well established integrability framework has posed some challenges. In a relativistic treatment massless excitations have the group velocity

$$v_g = \frac{\partial \varepsilon}{\partial p} = \pm 1, \tag{1.0.51}$$

where the two signs indicate left and right movers under the world-sheet chirality. Since all the particles move at the speed of light scattering for the same world-sheet chirality cannot take place and in general a relativistic treatment requires a more abstract notion of an S-matrix [70].

At the semiclassical level the dynamics of massless modes is captured by the finite-gap construction once one relaxes the way Virasoro constraints are imposed on the finite-gap equations [71, 72]. At the all-loop level there has been recent progress in resolving this issue in light-cone gauge where the dispersion relation is non-relativistic [63]. The group velocity for massless excitations then depends on the momentum,

$$v_g = \pm h \cos \frac{p}{2},\tag{1.0.52}$$

putting massless modes on the same footing as massive modes in terms of scattering processes. In general the form of the dispersion relation and S-matrix of a model follow directly from the off-shell symmetry algebra with the central charges properly identified as functions of the string tension and world-sheet momentum [13]. In this framework the complete all-loop world-sheet S-matrix has been obtained for string theory on $AdS_3 \times S^3 \times T^4$ [62, 63].

Dispersion relation and giant magnons

That the massive dispersion relation has the same form in terms of h as in the case of $AdS_5 \times S^5$ is not unexpected since dyonic giant magnon solutions, which live on $\mathbb{R} \times S^3$ [23], can be embedded in $AdS_3 \times S^3$. The Bohr-Sommerfeld quantization condition should still imply that the angular momentum J_2 is quantized and therefore one can expect that these solutions correspond to semiclassical bound states of elementary massive magnon excitations. One important difference is that these giant magnon solutions are now split into left- and right movers corresponding to opposite signs of J_2 . Solutions with $\pm J_2$ are not smoothly connected by rotations anymore since intermediate configurations would not lie on S^3 .

In the mixed flux $AdS_3 \times S^3 \times T^4$ case the first step towards finding the exact S-matrix was the construction of the dispersion relation and the tree-level light-cone gauge S-matrix for excitations around the BMN solution [73] which, as a geodesic, is not affected by the presence of the NS-NS flux [74]. These excitations have the following perturbative dispersion relation

$$\varepsilon_{\pm} = \sqrt{1 - q^2 + (p \pm q)^2} = \sqrt{(1 \pm qp)^2 + (1 - q^2)p^2} .$$
 (1.0.53)

Here $0 \le q \le 1$ is the coefficient of the NS-NS flux ($\hat{q} = \sqrt{1-q^2}$ is the coefficient of the R-R flux) and p is the spatial momentum of a 2d string fluctuation.³ In general, (1.0.53) is expected to receive corrections at higher orders in the inverse string tension ($h = \frac{\sqrt{\lambda}}{2\pi}$) expansion. To obtain the exact S-matrix the first step is to find the exact generalisation of the dispersion relation (1.0.53). In the case of $AdS_5 \times S^5$ the form of the dispersion relation can be obtained from the off-shell symmetry algebra with the central charges properly identified as functions of the string tension and world-sheet momentum [13, 4]. Using these symmetry considerations the exact mixed flux generalisation of the dispersion relation was suggested to be

$$\varepsilon_{\pm} = \sqrt{M_{\pm}^2 + 4(1-q^2)h^2 \sin^2 \frac{p}{2}}$$
, (1.0.54)

where the "central charge" M_{\pm} is not uniquely determined. The condition that (1.0.54) should reduce to (1.0.53) in the near-BMN limit $h \gg 1$, $p \ll 1$ with p = hp fixed implies that

$$M_{\pm} = 1 \pm q \,\mathrm{hp} + \ldots = 1 \pm q \,p + O(\mathrm{h}^{-1}) \;. \tag{1.0.55}$$

If one assumes that the dispersion relation should be manifestly periodic in p (i.e. with M_{\pm} being a smooth periodic function of p, which would apply if there were an underlying spin chain system) then the simplest consistent form of M would be [75]

$$M_{\pm} = 1 \pm 2qh\sin\frac{p}{2} . \qquad (1.0.56)$$

As was noted in [75], such a manifestly periodic dispersion relation (1.0.54),(1.0.56) suggestive of an underlying spin chain picture also naturally emerges upon formally discretizing the spatial direction in the string action (with step h^{-1}).

There is, however, no a priori reason to expect a spin chain interpretation to apply to the string integrable system for $q \neq 0$. It does not apparently apply for q = 1 when the world-sheet theory

³The quantized coefficient of the WZ term in the string action is $k = 2\pi qh$.

is related to a WZW model (which is solved in conformal gauge using, e.g., an effective free-field representation). For this reason it would be important to have an independent argument for or against the explicitly periodic choice (1.0.56) made in [75].

In this thesis we will determine the mixed flux dispersion relation by constructing a generalisation of the $AdS_5 \times S^5$ dyonic giant magnon solution to the presence of NS-NS flux. A priori it is not clear how a solitonic solution ansatz for $q \neq 0$ should look in terms of the world-sheet coordinates.

In the $AdS_5 \times S^5$ case the explicit form of the dyonic giant magnon solution in conformal gauge was found from a U(1) charged soliton of the complex sine-Gordon (CsG) model [23]. String motion on $\mathbb{R} \times S^2$ and $\mathbb{R} \times S^3$ is classically equivalent to the sine-Gordon and complex sine-Gordon (CsG) models through Pohlmeyer reduction, which is a procedure that solves the Virasoro constraints [76]. Thus the reduced theory only describes the physical degrees of freedom. It is also integrable and Lorentz invariant. The (dyonic) giant magnons then correspond to solitons of the (complex) sine-Gordon model. Their string embedding coordinates can be reconstructed from the solitons using the Pohlmeyer map. This procedure has also been used to obtain various other string solutions in $AdS_3 \times S^3$ including finite-size giant magnons, spiky and helical strings [77].⁴

Using Bäcklund transformations in the CsG model or equivalently the dressing transformations in the σ -model it is also possible to construct scattering solutions. During scattering the solitons experience a time-delay which matches the time-delay experienced by the string solutions. However, the energies in the string and reduced theory differ leading to different phase shifts and different S-matrices in general.

In the mixed flux case the reduced theory is again the CsG model, but with the mass rescaled by $\sqrt{1-q^2}$. It is not clear how the CsG soliton should be modified to incorporate the dependence on q and therefore we will instead consider string motion on $\mathbb{R} \times S^3$ as a principal chiral model with an additional WZ term for the NS-NS flux. In terms of the PCM currents the q = 0 and $q \neq 0$ models are related by a simple map of the world-sheet coordinates. Using this map will allow us to find the explicit $q \neq 0$ dyonic giant magnon solution in conformal gauge. Arguing that the world-sheet momentum is again identified by the opening angle between the string endpoints we find the exact dispersion relation (1.0.54) with

$$M_{\pm} = 1 \pm qhp. \tag{1.0.57}$$

Dressing phases

Symmetry and integrability alone only fix the form of the S-matrix up to the dressing phases, which must be constrained using additional physical requirements. A strong constraint is given by *crossing symmetry*, which relates S-matrix elements by an analytic continuation under the exchange of particles and anti-particles, see figure 1.4b. In a relativistic theory with energy and momentum parametrised

⁴Applying Pohlmeyer procedure to general symmetric space σ -models results in generalisations of the sine-Gordon model, the symmetric space sine-Gordon (SSSG) models [78, 49]. However, at the quantum level the original bosonic string σ -models have a conformal anomaly and are therefore not equivalent to the reduced theories. The Pohlmeyer reduction has also been formulated for coset σ -models, which has allowed for the inclusion of fermionic degrees of freedom. In particular, the reduced theory for superstrings in AdS₅ × S⁵ was found to correspond to a gauged WZW model with a potential [79, 80]. For a detailed study of the semiclassical and quantum properties of these reduced models see e.g. [81, 82, 83, 84, 85].

CHAPTER 1. INTRODUCTION



by a rapidity θ ,

$$E^{2} = m^{2} + p^{2}, \qquad E = m \cosh \theta, \qquad p = m \sinh \theta, \qquad (1.0.58)$$

the analytic continuation exchanges the branches of the dispersion relation

$$\theta \to \theta + i\pi, \qquad E \to -E, \qquad p \to -p,$$
 (1.0.59)

while the particle and antiparticle representations are mapped into each other. The concept of crossing symmetry can also be extended to the non-relativistic case and gives the *crossing equations*, a set of functional equations for the dressing phases $\sigma(p_1, p_2)$. The solutions to these equations are not unique since they can be multiplied by solutions of the homogeneous crossing equations - the *CDD factors* [86].

A further constraint on these factors comes from the analytic structure of the complete S-matrix. Its simple poles correspond to the formation and exchange of bound states. The poles can be understood in terms of propagators going on-shell in a Feynman diagram expansion of the S-matrix elements. However, this still allows for different CDD factors without poles.

Originally in the case of $AdS_5 \times S^5$ the dressing phases were established at the semiclassical (AFS) and 1-loop (HL) level using a Bethe ansatz in [87, 88] and later in [32] also the all-loop (BES) dressing phase was found. The AFS and HL phases were also reproduced from a first-principle semiclassical soliton quantization approach in [24]. The classical scattering of dyonic giant magnons gives rise to the semiclassical bound-state S-matrix. It can be found from the time-delay, experienced by the soliton, during scattering compared to free propagation [89]. For an integrable theory the bound-state S-matrix then corresponds to the fusion of S-matrices for the scattering of elementary constituents [30] matching the predicted dressing phases. In [25] this approach was further generalised to 1-loop corrections of the bound-state S-matrix by considering the scattering of plane waves off dyonic giant magnons. These plane waves represent small fluctuations around the dyonic giant magnon solutions and the 1-loop corrections to the bound-state S-matrix and energy can be determined from their scattering phase shifts. The time-delay and phase shifts are found directly from the multi soliton solutions, which in an integrable theory can be constructed using the *dressing method*. This method is based on reformulating the integrable model in terms of an auxiliary linear problem using the Lax connection and performing a gauge transformation on the auxiliary system. Starting with the BMN solution and successively applying the dressing transformation gives the dyonic giant magnon and its multi soliton solutions [90, 91].

Also in the case of $AdS_3 \times S^3 \times T^4$ without flux these dressing phase factors have been established at tree-level and 1-loop orders using semiclassical methods in [92, 93, 94] and an all-loop proposal was given in [61] by solving the crossing equations [60, 62, 63]. In [93] the 1-loop phase was further constrained in the pure R-R case by considering quantum corrections to spinning strings.

In the massive sector of $AdS_3 \times S^3 \times T^4$ with mixed flux results for the semiclassical and 1-loop phases have been obtained using the approach of finite-gap equations and algebraic curve quantization in [95] and matching conjectures for the 1-loop phases have also been proposed using unitarity cut based methods in [96, 97]. For the semiclassical phase the proposal in [95] relies on the assumption that the dressing phase for the scattering of same-type excitations (in terms of left- and right movers) remains unchanged when switching on the flux and therefore is given by the usual AFS expression [87]. In order to test this assumption and to provide an independent check of these results we exploit the additional information coming from the scattering of mixed-flux dyonic giant magnons and plane waves to derive the semiclassical and 1-loop phases, as was done in the cases of $AdS_5 \times S^5$ in [24, 25] and for $AdS_3 \times S^3$ in [92].

AdS/CFT in the brane picture

Another class of "interpolating" string σ -models, which we investigate, comes from string motion in brane backgrounds. In general AdS backgrounds naturally arise as near-horizon limits of brane systems. As such one might wonder if integrability is present for string motion on the full brane background, which interpolates between AdS and flat spacetime, both being integrable backgrounds.

Let us briefly see how branes and their near-horizon geometries appear in the context of AdS/CFT. We consider N coincident D3 branes. In the low energy limit of superstring theory the D-branes are described by supergravity solutions which source the geometry, in which closed strings propagate. Their geometry is given by the metric

$$ds^{2} = H(r)^{-1/2} \eta_{ij} dx^{i} dx^{j} + H(r)^{1/2} (dr^{2} + r^{2} d\Omega_{5}^{2})$$
(1.0.60)

$$H(r) = \left(1 + \frac{L^4}{r^4}\right), \qquad L^4 = 4\pi g_s N \alpha'^2, \qquad (1.0.61)$$

supplemented with an R-R 5-form flux. This supergravity description is only valid for a large curvature radius ~ L compared to the string scale $l_s = \sqrt{\alpha'}$ and thus $g_s N \gg 1$. In the near-horizon region $r \ll L$ the geometry reduces to $\operatorname{AdS}_5 \times \operatorname{S}^5$ whereas in the bulk $r \gg L$ the closed strings propagate in flat spacetime. Taking the low-energy or *Maldacena* limit $\alpha' \to 0, r/\alpha' = \text{fixed}$ [6] string propagation in the bulk decouples and one is left with string propagation in the near-horizon $\operatorname{AdS}_5 \times \operatorname{S}^5$ geometry.

In the opposite regime of string perturbation theory, $g_s N \ll 1$, closed strings propagate in a flat spacetime with the branes corresponding to hypersurface on which open strings end. In this picture closed strings describe massless excitations in flat spacetime and open strings describe massless excitations on the brane hypersurface. Taking again the Maldacena limit one finds that closed and open strings decouple, e.g. closed strings cannot form from open strings. Closed string excitations then induce free type IIB supergravity whereas open string excitations for a single brane induce a massless U(1) gauge theory with $\mathcal{N} = 4$ supersymmetry. For N coincident branes the open strings can end on different branes and the gauge theory is enhanced to U(N) giving rise to SU(N) $\mathcal{N} = 4$ SYM ⁵.

 $^{{}^{5}}$ The U(1) factor does not contribute to the brane dynamics since it is only related to the overall brane position.

In a similar way $\operatorname{AdS}_4/\operatorname{CFT}_3$ can be understood in terms of N coincident M2-branes on a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold. M-theory on the near-horizon geometry $\operatorname{AdS}_4 \times \operatorname{S}^7/\mathbb{Z}_k$ is then dual to the ABJM gauge theory, which is the world-volume theory of the M2 brane stack [98]. The $\operatorname{S}^7/\mathbb{Z}_k$ can be written as an S^1 fibration over \mathbb{CP}^3 with the S^1 radius ~ 1/k. For a small string coupling $g_s \sim (N/k^5)^{1/4}$ the radius vanishes and one obtains weakly coupled type IIA string theory on $\operatorname{AdS}_4 \times \mathbb{CP}^3$. Thus in this case the $\operatorname{AdS}_4/\operatorname{CFT}_3$ gauge/string duality is a special case of the more general ABJM/M-theory duality. Integrability then appears in the type IIA regime since it encompasses the planar limit, $N, k \to \infty$ with N/k = fixed, and one may also wonder if integrability in general can be present beyond the planar limit. For a more detailed review also see [99, 53, 100].

In the case of AdS_3/CFT_2 the $AdS_3 \times S^3 \times T^4$ background arises as a near-horizon limit of stacks of N_1 D1 branes and N_5 D5 branes. The D1 branes and D5 branes extend along a common direction and the system is compactified on T^4 along four D5 brane directions transverse to the D1 branes. The curvature radii of AdS_3 and S^3 and the T^4 volume are given by

$$R_{\text{AdS}_3}^2 = R_{\text{S}^3}^2 = \sqrt{N_1 N_5}, \qquad \text{vol}(\text{T}^4) = \frac{N_1}{N_5}.$$
 (1.0.62)

This construction gives rise to a 1+1 dimensional $U(N_1) \times U(N_5)$ supersymmetric gauge theory on the brane intersection with 16 real supercharges, which are chirally decomposed under the symmetry algebra of boosts along the intersection, $\mathfrak{so}(1,1)$, giving $\mathcal{N} = (4,4)$ supersymmetry. This two-dimensional UV gauge theory is not conformal but instead flows to a 2d CFT in the low energy limit. In contrast to $\mathcal{N} = 4$ SYM and ABJM this gauge theory also has matter content in both the fundamental and adjoint representations of the gauge group.

The UV theory has two branches: the Higgs branch, which describes the motion of D1 branes inside the D5 branes, and the Coulomb branch, which describes the separation of the D1 and D5 branes. In the low energy limit the Higgs branch CFT can be viewed in terms of the D1 branes being instantons, with instanton number N_1 , in the SU(N_5) gauge theory on the D5 branes [101]. In this picture the CFT is described by a σ -model on the instanton moduli space, which is given by a deformation of the symmetric product orbifold [6, 102]

$$(T^4)^{N_1N_5}/S_{N_1N_2},$$
 (1.0.63)

where S_N is the symmetric group.

This Higgs branch CFT is then conjectured to be dual to string theory on the near-horizon $AdS_3 \times S^3 \times T^4$ geometry of the D5-D1 brane system. In the planar limit the string theory is described by a 2d string σ -model, which is believed to be integrable in analogy to higher-dimensional AdS/CFT dualities. The superisometry algebra of this model consists of the global subalgebra of $\mathcal{N} = (4, 4)$

$$\mathfrak{psu}(1,1|2)_{\mathcal{L}} \oplus \mathfrak{psu}(1,1|2)_{\mathcal{R}}$$
(1.0.64)

and the additional $\mathfrak{u}(1)$ factors for T^4 . Here the labels L and R denote the left- and right sectors for the $\mathfrak{so}(1,1)$ chiral decomposition.

Recently it was shown in [103] that the Higgs branch CFT admits a 't Hooft expansion in powers of $1/N_1$ with non-planar corrections suppressed by powers of $1/N_1^2$ in the large N_1 limit. The parameters N_1 and N_5 correspond to the number of colour and flavours with N_5 playing the role of the Yang-Mills coupling $1/g_{YM}^2$. The planar limit then corresponds to

$$N_1 \to \infty, \qquad \lambda = \frac{N_1}{N_5} = \text{fixed.}$$
 (1.0.65)

In this limit integrability appears in terms of an integrable spin chain as one would expect from higher dimensional AdS/CFT examples. It was found that the 1-loop dilatation operator for single-trace operators consisting of scalars charged under $\mathfrak{so}(4) = \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, which forms a closed subsector at 1-loop, is given by an integrable homogeneous $\mathfrak{so}(4)$ spin-chain Hamiltonian.

An interesting feature of AdS_3 backgrounds is the presence of mixed flux. While the $AdS_3 \times S^3 \times T^4$ type IIB supergravity solution with pure R-R flux arises as the near-horizon limits of the D5-D1 system, the theories with pure NS-NS and mixed flux arise as the near-horizon limits of the NS5-NS1 and the D5-D1 + NS5-NS1 brane systems [104]. These R-R and NS-NS solutions are related by S-duality and, in particular, we can obtain the mixed flux background by applying S-duality to either the pure R-R or the pure NS-NS solution. Thus, string theory on the mixed flux background also provides an important model to study the non-perturbative S-duality.

For $\operatorname{AdS}_3 \times \operatorname{S}^3 \times \operatorname{S}^1 \times \operatorname{S}^1$ the situation is less clear: the presence of the two 3-spheres leads to two SU(2) R-symmetries. Therefore the superconformal algebra is enhanced to a *large* $\mathcal{N} = (4, 4)$ [105], which has two $\mathfrak{su}(2)$ subalgebras in contrast to the small $\mathcal{N} = (4, 4)$ algebra of $\operatorname{AdS}_3 \times \operatorname{S}^3 \times \operatorname{T}^4$. However, much less is known about the holographic dual in this case. Recently a brane system based on $\mathcal{N} = (0, 4)$ supersymmetry has been suggested, for which the corresponding gauge theory is conjectured to flow to an IR fixed-point with large $\mathcal{N} = (4, 4)$ superconformal symmetry and for which the CFT has a central charge matching the holographic dual of string theory on $\operatorname{AdS}_3 \times \operatorname{S}^3 \times \operatorname{S}^3 \times \operatorname{S}^1$. For details on progress in this direction see [106] and the references therein.

(Non-)integrability of string motion in brane backgrounds

Integrability is a powerful technique for studying the properties of certain σ -models and has lead to significant progress in finding classical solutions to the non-linear equations of motion as well as determining the spectrum of the quantum theory. However, given a 2d σ -model there is no general procedure to construct the Lax connection or to show integrability in another way. Moreover there is no classification of integrable 2d σ -models. In the case of σ -models for string motion on curved backgrounds this raises the question of which backgrounds lead to integrable string theories? A large class of such backgrounds consists of symmetric spaces, i.e. spaces that have an inversion symmetry, such as spheres [107] and related constant curvature spaces [108]. String motion on these spaces can often be described in terms of integrable group G or G/H coset σ -model [109, 110, 111]. For example string motion on $\mathbb{R} \times S^3$ is equivalent to the principal chiral model, a σ -model on the SU(2) group space [109]. There are also some integrable deformations of these models including the sausage model, which is a 2-sphere deformation [112], the squashed sphere as well as multi-parameter deformations of the 3-sphere [113, 114, 115]. In terms of the group spaces the SU(2) model has a 2-parameter deformation, the diagonal anisotropic principal chiral model [116]

$$S_C[g] = -\frac{1}{2} \int d^2 \sigma \operatorname{tr} \left[g^{-1} \partial_+ g K g^{-1} \partial_- g \right], \qquad K = \operatorname{diag}(K_1, K_2, K_3), \quad g \in \operatorname{SU}(2), \tag{1.0.66}$$

which includes the squashed 3-sphere as the special case $K_1 = K_2 \neq K_3$ with the squashing parameter $C = K_3/K_1$. In this 1-parameter case the deformation can be further extended to any compact Lie group G giving the Yang-Baxter σ -model [117]

$$S_K[g] = -\frac{1}{2} \int d^2 \sigma \left(g^{-1} \partial_+ g, \frac{(1+\eta^2)^2}{1-\eta R} g^{-1} \partial_- g \right), \qquad g \in G$$
(1.0.67)

where $\eta \geq 0$ is the deformation parameter, (,) is the Killing form for the Lie algebra of G and R is an operator given by the solution of some modified classical Yang-Baxter equation. For G = SU(2)this model reduces to the anisotropic squashed sphere case. There has also been some work towards generalising the principal chiral model to a non-diagonal form by studying the conditions for the existence of a Lax connection [118, 119, 120]. More recently in [121] a procedure has been given to construct 1-parameter deformations of string σ -models that preserve integrability and which recovers the Yang-Baxter σ -model when applied to a compact Lie group. It also provides a natural generalisation of the Yang-Baxter σ -model to the case of symmetric space cosets.

In addition to the target space metric G_{MN} string motion can also be coupled to an antisymmetric B_{MN} -field corresponding to an NS-NS flux. This leads to an additional Wess-Zumino term in the string σ -model action and in particular to integrable (gauged) WZW σ -models [122, 123, 68]. For backgrounds that can be supplemented with a mixture of NS-NS and R-R fluxes the coefficient of the Wess-Zumino term is unfixed. In the case of string motion on $\mathbb{R} \times S^3$ this gives a principal chiral model with a WZ term of arbitrary coefficient, which is an integrable model [124, 125, 126] and, as we show in this thesis, gives rise to the dyonic giant magnon in the presence of mixed flux. Other integrable examples include new 3d target space models [115], coset space σ -models with additional WZ terms [126] and pp-wave models that are related to (massive) light-cone gauge string actions [127, 128, 129, 130, 131].

Furthermore integrability is also preserved by classical transformations which do not affect the equations of motion. Such transformations include abelian and non-abelian T-dualities in combination with field redefinitions. For more details on T-duality and integrability also see [132, 120]. Among the resulting models are different gaugings and marginal deformations of WZW models [133, 134, 135, 136, 137] as well as models related to $AdS_5 \times S^5$ [138]. It is also possible to obtain new integrable theories from integrable group space models by taking a quotient by a discrete subgroup. This gives rise to integrable orbifold string models such as in the case of strings on the BTZ black hole background, which is an orbifold of AdS_3 . In this case the locally defined flat Lax connection is preserved when taking the quotient and string motion is integrable [139]. However, in general discrete subgroup quotients can also break integrability [140].

Even though there is no general Lax construction for string σ -models one might wonder what

necessary conditions are imposed on the backgrounds by integrability. For example, while integrability is a hidden symmetry enhancement, the presence of non-abelian symmetries is not necessary. There are integrable models which arise from non-abelian duality [141], e.g. by integrating out the gauge field in a gauged WZW model, and which have no non-abelian isometries. There is however one necessary condition for the integrability of 2d σ -models, namely that any consistent 1d truncation of the equations of motion to a mechanical system is also integrable. For example restricting classical string motion on a sphere or AdS space to rigid strings gives an integrable Neumann-Rosochatius system [142, 143]. Another example is stationary string motion on AdS-Kerr-NUT backgrounds, which is described by geodesic motion on an effective lower dimensional space and is known to be integrable [144].

Using necessary conditions one can also rule out integrability of a 2d σ -model. For this it is sufficient to find a consistent 1d truncation of the equations of motion that is not integrable. For integrable differential equations the existence of a constant of motion for each degree of freedom implies that solutions can be obtained by quadratures, i.e. in terms of some known functions. Therefore for non-integrability it suffices to show that a system does not admit such solutions. This can be done, for example, by further reducing the system to second order linear differential equations for small fluctuations around phase space solutions and showing that they cannot be solved by quadratures [145], which amounts to exhibiting chaotic motion of the system. Recently this method has been applied to string motion on AdS₅ × T^{p,q}, AdS₅ × Y^{p,q} [146, 147] and "confining" supergravity backgrounds [148]. While these are highly symmetric backgrounds, which have isometries, preserve supersymmetry and admit integrable geodesics, the corresponding string σ -models were found to be non-integrable.

In this thesis we will consider a class of less symmetric *p*-brane backgrounds, which interpolate between the integrable limits of flat space and $\operatorname{AdS}_n \times \operatorname{S}^m \times \operatorname{T}^k$. A priori one might hope that there exists an interpolation between the charges of these integrable limits such that integrability is preserved. As a first step we will consider geodesic, i.e. point particle, motion. Geodesic motion is known to be integrable in higher-dimensional rotating black hole spacetimes [149] and in some spacetimes such as $\operatorname{T}^{1,1}$ and $\operatorname{Y}^{p,q}$ geodesic motion is even super-integrable, i.e. with more constants of motion than degrees of freedom [150]. Also in our case we will see that geodesic motion is integrable. However, applying the non-integrability approach we find that the string σ -model is not integrable.

Thesis outline

This thesis is split into two parts. In the first part we review classical integrability and investigate integrability for geodesic and string motion on *p*-brane backgrounds. In the second part we apply semiclassical methods to investigate quantum integrability of string theory on $AdS_3 \times S^3 \times T^4$ with mixed flux.

The first part starts with a review of classical integrability for Hamiltonian systems and string σ -models. In chapter 2 we discuss the properties and consequences of integrability for classical and quantum systems and introduce the Lax pair formalism. Further, we review the Lax pair construction and the finite-gap equations for string motion on semi-symmetric coset spaces with an emphasis on $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times T^4$ with mixed flux. Finally, in preparation of the second part of the thesis, we present the explicit example of string solutions on $\mathbb{R} \times S^3$.

In chapter 3 we then consider the question of possible integrability of classical string motion on curved p-brane backgrounds. For example, the D3-brane metric interpolates between the flat and the $AdS_5 \times S^5$ regions in which string propagation is integrable. We find that while the point-like string (geodesic) equations are integrable, the equations describing an extended string on the complete D3brane geometry are not. The same conclusion is reached for similar brane intersection backgrounds interpolating between flat space and $AdS_k \times S^k$. We consider, in particular, the case of the NS 5-brane fundamental string background. To demonstrate non-integrability we make a special "pulsating string" ansatz for which the string equations reduce to an effective one-dimensional system. Expanding near this simple solution leads to a linear differential equation for small fluctuations that cannot be solved in quadratures, implying non-integrability of the original set of string equations.

In the first chapter of part II we then turn to the dyonic giant magnon, a string solution on $\mathbb{R} \times S^3$. We show how the explicit form of this solution in conformal gauge was originally obtained in the case of $AdS_5 \times S^5$. We also briefly review the formulation of the dyonic giant magnon as a finite-gap solution as this will allow us to include the fermionic world-sheet fluctuations when looking at the dressing phases.

In chapter 5 we fix the exact form of the dispersion relation for light-cone string excitations in string theory on $AdS_3 \times S^3 \times T^4$ with mixed R-R and NS-NS 3-form fluxes. For this we construct a generalisation of the known dyonic giant magnon soliton on S^3 to the presence of a non-zero NS-NS flux described by a WZ term in the string action (with coefficient q). We find that the angular momentum of this soliton gets shifted by a term linear in world-sheet momentum p. We also review and discuss the symmetry algebra of the string light-cone gauge S-matrix and show that the exact dispersion relation, which should have the correct perturbative BMN and semiclassical giant magnon limits, should also contain such a linear momentum term.

In chapter 6 we also construct the $q \neq 0$ generalisation of the folded string on S³ and on AdS₃×S¹. We show that the resulting solutions are closed strings if the angular momenta are taken to satisfy certain quantization conditions. For the solution on AdS₃×S¹ we find that, to leading order in large angular momentum on AdS₃, the energy takes the same form as in the pure R-R case but with the string tension rescaled by $\sqrt{1-q^2}$.

In chapter 7 we present a semiclassical derivation of the tree-level and 1-loop dressing phases in the massive sector of string theory on $AdS_3 \times S^3 \times T^4$ with mixed flux. In analogy with the $AdS_5 \times S^5$ case, we use the dressing method to obtain scattering solutions for dyonic giant magnons which allows us to determine the semiclassical bound-state S-matrix and its 1-loop correction. We also find that the 1-loop correction to the dyonic giant magnon energy vanishes. Looking at the relation between the bound-state picture and elementary magnons in terms of the fusion procedure we deduce the elementary dressing phases. In both the semiclassical and 1-loop cases we find agreement with recent proposals from finite-gap equations and unitarity-cut methods. Further, we find consistency with the finite-gap picture by determining the resolvent for the dyonic giant magnon from the semiclassical bosonic scattering data. In chapter 8 we conclude by summarising the results presented in this thesis and discussing them in the light of recent developments. We also outline some open questions and suggest possible future directions for investigating integrability of geodesic and string motion on curved backgrounds and applying integrability-based methods to string σ -models.

Part I

Classical integrability of string σ -models

Chapter 2

Review of classical integrability

In general systems described by non-linear differential equations cannot be solved exactly except for special cases with sufficient symmetry. The notion of such solvable systems is made precise by the concept of integrability. In classical mechanics integrable systems exhibit as many constants of motion as degrees of freedom and their analytic solutions can then be obtained by quadratures in terms of Liouvillian functions, which are algebraic expressions and integrals of some known functions. In the case of field theory this notion of integrability becomes less clear since exhibiting infinitely many conserved charges might not be sufficient for an infinite number of degrees of freedom. Instead integrability can be rephrased in terms of a Lax pair consisting of two matrices. The equations of motion then take the form of an isospectral evolution equation for one of these Lax matrices. This Lax equation can be also understood as the compatibility condition of an auxiliary linear problem. The conserved quantities are then encoded by an eigenvalue equation, the spectral curve. This concept is readily generalised to field theories with the Lax pair becoming a zero curvature connection.

Let us begin with a review of classical integrability for mechanical (1d) systems in section 2.1 and for 2d classical and quantum field theories in section 2.2. We then show how these concepts apply to string theories in AdS backgrounds in section 2.3. For more extensive reviews on these topics see e.g. [151, 5, 69].

2.1 Integrability of 1d Hamiltonian systems

Let us consider a classical Hamiltonian system with n degrees of freedom. Its phase space M is a 2n dimensional manifold with local canonical coordinates

$$(p_1, .., p_n, q_1, .., q_n) \tag{2.1.1}$$

and a Poisson bracket for differentiable functions on M, defined by

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i}\right).$$
(2.1.2)

The Hamiltonian equations of motion

$$\dot{\vec{x}} = J \cdot \nabla H, \quad \vec{x} = (p_1, .., q_1, ..), \quad \nabla H = \left(\frac{\partial H}{\partial p_1}, .., \frac{\partial H}{\partial q_1}, ..\right), \quad J = \left(\begin{array}{cc} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{array}\right)$$
(2.1.3)

imply that, for any function f(p, q, t), the evolution of f(p(t), q(t), t) is given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{H, f\} + \frac{\partial f}{\partial t}.$$
(2.1.4)

A constant of motion or first integral is then a function on phase space, F(p,q), that satisfies

$$F(p(t), q(t)) = \text{const} \qquad \Leftrightarrow \qquad \{H, F\} = 0. \tag{2.1.5}$$

We can now make the definition of integrability precise. A Hamiltonian system is *Liouville integrable* if it has n constants of motion F_i , i = 1, ..n, i.e. $\{H, F\} = 0$, which are in involution

$$\{F_i, F_j\} = 0 \tag{2.1.6}$$

and which are independent, i.e. the dF_i are linearly independent (the tangent space of the surface $F_i(p,q) = f_i$ has dimension *n* everywhere). For a non-degenerate Poisson bracket there cannot be more than *n* independent constants of motion. The Hamiltonian itself provides a conserved quantity and is therefore a function of the F_i .

The motion in an integrable system takes place on the level set manifold M_f of the phase space

$$M_f = \{(p,q) \in M : F_i(p,q) = f_i = \text{const}\}, \qquad i = 1, .., n.$$
(2.1.7)

By the Liouville-Arnold theorem a compact and connected level surface M_f is isomorphic to a torus T^n . Thus the motion takes place on these level set tori which foliate the phase space. The equations of motion can be solved by introducing a new set of canonical coordinates which include the constants of motion. In particular one can introduce action-angle coordinates (I_i, ϕ_i) through a canonical transformation (i.e. the Poisson bracket and the Hamiltonian equations are preserved)

$$(p_1, .., p_n, q_1, .., q_n) \to (I_1, .., I_n, \phi_1, .., \phi_n), \qquad 0 \le \phi_i \le 2\pi,$$

$$(2.1.8)$$

where the angles ϕ_i are coordinates on M_f and the actions $I_i = I_i(f_1, ..., f_n)$ are constants of motion. In these coordinates the equations of motion take the simple form

$$\dot{I}_i = -\frac{\partial H}{\partial \phi_i} = 0, \qquad \dot{\phi}_i = \frac{\partial H}{\partial I_i} = \omega_i(I_1, ..., I_n), \qquad i = 1, ..., n,$$
(2.1.9)

which corresponds to circular motion at constant velocity. The appropriate canonical transformation can be generated using the canonical 1-form $\alpha = \sum_i p_i \, dq_i$. The action variables are then defined as integrals of α over the fundamental cycles C_j of the torus for a level set $\{f_i\}$

$$I_j = \frac{1}{2\pi} \oint_{C_j} \alpha = \frac{1}{2\pi} \oint_{C_j} \sum_i p_i(q, f) \, \mathrm{d}q_i.$$
(2.1.10)

The angle coordinates on the other hand satisfy

$$\frac{1}{2\pi} \oint_{C_j} \mathrm{d}\phi_i = \delta_{ij}. \tag{2.1.11}$$

Using the generating function

$$S(I,q) = \int_{m_0}^m \alpha = \int_{m_0}^m \sum_i p_i(q',f) \, \mathrm{d}q'_i \tag{2.1.12}$$

one can define the transformation $(p,q) \to (I,\phi)$ through

$$p_i = \frac{\partial S}{\partial q_i}, \qquad \phi_i = \frac{\partial S}{\partial I_i}.$$
 (2.1.13)

This is indeed a canonical transformation since it preserves the symplectic form $\omega = d\alpha = \sum_i dp_i \wedge dq_i$ as follows from

$$0 = \mathrm{d}^2 S = \sum_i \left(\mathrm{d} p_i \wedge \mathrm{d} q_i + \mathrm{d} \phi_i \wedge \mathrm{d} I_i \right).$$
(2.1.14)

One can show that S always exists, i.e. it is path independent, and thus the system is solved in quadratures by calculating S and obtaining p_i as a function of q and F using algebraic manipulations.

As an example let us consider the simple harmonic oscillator

$$H = \frac{1}{2}(p^2 + \omega^2 q^2), \qquad p(q, E) = \pm \sqrt{2E - \omega^2 q^2}.$$
 (2.1.15)

The action variable and generating function are

$$I = \frac{1}{2\pi} \oint_E \mathrm{d}q \sqrt{2E - \omega^2 q^2} = \frac{E}{\omega}, \qquad S(I,q) = \omega \int^q \mathrm{d}x \sqrt{\frac{2I}{\omega} - x^2}$$
(2.1.16)

and the angle variable is then found from

$$\phi = \frac{\partial S}{\partial I} \Rightarrow q = \sqrt{\frac{2I}{\omega}} \sin \phi.$$
(2.1.17)

In these action angle variables the equations of motion are simply

$$H = \omega I, \qquad \dot{\phi} = \frac{\partial H}{\partial I}, \qquad \phi = \omega t + \phi_0.$$
 (2.1.18)

Lax pair formalism

The concept of integrability can be formulated as an auxiliary linear problem by recasting the Hamiltonian equations of motion into the form

$$\dot{L} = [L, M],$$
 (2.1.19)

where L and M are the Lax pair matrices. This equation is solved in terms of a matrix g(t)

$$L(t) = g(t)L(0)g^{-1}(t), \qquad M = \dot{g}(t)g^{-1}(t).$$
(2.1.20)

In this formalism the conserved quantities can be identified as functions of the time-independent eigenvalues of L and a common choice is $F_j = \text{tr}L^j$. In general the choice of the Lax pair is not unique:

for example we can obtain a new Lax pair from any invertible matrix g using the transformation

$$L \to gLg^{-1}, \qquad M \to gMg^{-1} + \dot{g}g^{-1}.$$
 (2.1.21)

For a set of k independent simple harmonic oscillator a possible Lax pair would consist of block diagonal matrices with entries

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & -\omega/2 \\ \omega/2 & 0 \end{pmatrix}, \qquad (2.1.22)$$

with the conserved quantities

$$\operatorname{tr} L^{2p+1} = 0, \qquad \operatorname{tr} L^{2p} = 2 \sum_{i}^{k} (2F_i)^p, \quad F_i = \frac{1}{2} (p_i^2 + \omega^2 q_i^2).$$
 (2.1.23)

In general one still needs to ensure that the conserved quantities obtained from the Lax pair are in involution. Defining the matrix of Poisson brackets between the elements of L

$$\{L_1, L_2\} = \sum_{ij,kl} \{L_{ij}, L_{kl}\} E_{ij} \otimes E_{kl}, \quad (E_{ij})_{kl} = \delta_{ik} \delta_{jl}, \quad L_1 \equiv L \otimes \mathbb{I}, \quad L_2 \equiv \mathbb{I} \otimes L$$
(2.1.24)

the involution of the conserved charges is then equivalent to the existence of an r-matrix, which is a function on phase space satisfying

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2].$$
(2.1.25)

The Jacobi identity for the Poisson bracket leads to a constraint on the r-matrix. For a constant r-matrix this constraint is solved by a matrix that satisfies

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0.$$
(2.1.26)

For an antisymmetric r-matrix, i.e. $r_{12} = -r_{21}$, this is the classical Yang-Baxter equation. In the case of the simple harmonic oscillator the r-matrix is

$$r_{12} = \frac{\omega}{4E} \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \otimes L.$$
(2.1.27)

The Lax pair formalism can be further extended by introducing a spectral parameter z such that L and M become analytic functions in z satisfying

$$\dot{L}(z) = [L(z), M(z)].$$
 (2.1.28)

The advantage of introducing the spectral parameter is that the analytic structure in z essentially encodes the model and its properties. A particular system can be specified by the number and order of poles in $L(\lambda)$ and the analytic structure of $M(\lambda)$ is associated with the dynamical Hamiltonian flows. The eigenvalues μ of L(z) are now encoded by the characteristic equation

$$\Gamma(z;\mu) \equiv \det(L(z) - \mu \mathbb{I}) = 0, \qquad (2.1.29)$$

which defines the spectral curve $\Gamma = \{(z, \mu) : \Gamma(z; \mu) = 0\}$ on \mathbb{C}^2 . As we shall see next, these ideas are particularly useful in generalising integrability to field theories.

2.2 Integrability of 2d field theories

The Lax construction is generalised to 2d field theories by rephrasing the equations of motion in terms of the zero-curvature condition

$$\partial_{\tau} L_{\sigma} - \partial_{\sigma} L_{\tau} - [L_{\tau}, L_{\sigma}] = 0, \qquad (2.2.1)$$

where $L_a = L_a(\tau, \sigma; z)$ is a function of the world-sheet coordinates τ, σ and the spectral parameter z. This flatness condition arises as the compatibility condition $\partial_{\tau}\partial_{\sigma}\Psi = \partial_{\sigma}\partial_{\tau}\Psi$ of an auxiliary linear problem

$$(\partial_a - L_a)\Psi(\tau,\sigma;z) = 0. \tag{2.2.2}$$

The flatness of the Lax connection allows one to define the monodromy matrix

$$T(\tau,\sigma;z) = \mathcal{P}\exp\oint_{[\gamma(\tau,\sigma)]} \mathrm{d}\sigma'^a L_a(\tau',\sigma';z), \qquad (2.2.3)$$

which is a path-ordered exponential, i.e.

$$\mathcal{P}\exp\int_{a}^{b}\mathrm{d}x\,f(x) = \lim_{\epsilon \to 0}\int_{a}^{a+\epsilon}\mathrm{d}x\,f(x)\int_{a+\epsilon}^{a+2\epsilon}\mathrm{d}x\,f(x)\dots,$$
(2.2.4)

along a closed path $\gamma(\tau, \sigma)$ on the world-sheet cylinder in the homotopy class $[\gamma(\tau, \sigma)]$ of paths through a base point (τ, σ) with single winding. The monodromy matrices for two base points (τ, σ) and (τ', σ') are related by

$$T(\tilde{\tau}, \tilde{\sigma}; z) = UT(\tau, \sigma; z)U^{-1}, \qquad U(\tau, \sigma, \tilde{\tau}, \tilde{\sigma}; z) = \mathcal{P} \exp \int_{\tilde{\gamma}} \mathrm{d}\sigma'^a L_a(\tau', \sigma'; z), \tag{2.2.5}$$

where $\bar{\gamma}$ is a path that connects (τ, σ) and $(\tilde{\tau}, \tilde{\sigma})$ and has no winding. This evolution property can be seen by applying the non-abelian version of Stokes' theorem to the path in figure 2.1. It also implies that $\operatorname{tr} T(\tau, \sigma; z)^p$ is independent of the base point and therefore the eigenvalues of the monodromy matrix are conserved. One can also show that the traces of powers of the monodromy matrix are in involution with respect to the Dirac bracket, which generalises the Poisson bracket to include second class constraints in the Hamiltonian formalism (see [152]).¹

A common choice for the path on the world-sheet is a circle of constant τ such that the monodromy matrix becomes

$$T(\tau,\sigma;z) = \mathcal{P}\exp\int_{\sigma}^{\sigma+2\pi} \mathrm{d}\sigma' L_{\sigma}(\tau,\sigma';z).$$
(2.2.6)

¹For string motion the Virasoro constraints are first class constraints. However, for example the additional static gauge constraint $X_0 = \kappa \tau$ is a second class constraint since the Poisson bracket $\{X_0(\sigma'), \pi_0(\sigma)\} = \delta(\sigma' - \sigma)$ does not vanish.



Figure 2.1: Path on the world-sheet cylinder for obtaining the evolution of the monodtromy matrix using Stokes' theorem. Changing the direction of a path corresponds to inverting the associated monodromy matrix.

In terms of the linear problem (2.2.2) this matrix corresponds to the *monodromy* of a solution

$$\Psi(\tau, \sigma + 2\pi; z) = T(\tau, \sigma; z)\Psi(\tau, \sigma; z).$$
(2.2.7)

The time evolution of $T(\tau, \sigma; z)$ can be obtained from (2.2.5) for an infinitesimal path $\tilde{\tau} = \tau + \delta \tau$, $\tilde{\sigma} = \sigma + \delta \sigma$

$$\partial_{\tau}T = [L_{\tau}, T], \qquad \partial_{\sigma}T = [L_{\sigma}, T],$$
(2.2.8)

which resembles the Lax equation. The time independent eigenvalues $\mu(z)$ of the Lax connection are described by the spectral curve

$$\Gamma = \{(z,\mu) \in \mathbb{C}^2 : \Gamma(z,\mu) = 0\}, \qquad \Gamma(z,\mu) \equiv \det(T(\tau;z) - \mu\mathbb{I}).$$
(2.2.9)

For every solution of the equations of motion this curve encodes an infinite tower of conserved quantities, which can be determined by expanding in z. In general the spectral curve can have essential singularities and infinite genus. However, it is possible to consider a subclass of configurations corresponding to a curve of finite genus, the *algebraic curve*, which describes classical solutions. Also since the eigenvalues $\mu(z)$ can have essential singularities these *finite-gap* solutions are instead encoded in terms of the analytic structure of the *quasi-momenta* [153] p_i defined by

$$\mu_i(z) = e^{ip_i(z)}.$$
(2.2.10)

Finding the explicit form of a solution to the equations of motion (2.2.1) for a given set of quasimomenta is a difficult problem in general. Another way to obtain classical solutions comes from directly solving the auxiliary linear problem instead of the non-linear equations (2.2.1). The Lax connection is a function of the fields of the model and specifying initial conditions at $\tau = 0$ for the fields (and thus the Lax connection) one can solve for the time evolution of $\Psi(\tau, \sigma; z)$ in (2.2.2). The time evolution of the fields is then found from $\Psi(\tau, \sigma; z)$ by solving a linear integral equation, the Gel'fand-Levitan-Marchenko equation [154]. This is essentially a non-linear analogue of the Fourier transform for partial differential equations, i.e. the non-linear problem is rephrased as a linear problem
in the spectral space. This technique is known as the *inverse scattering method* [155, 109].

In general there is no constructive method known to find the Lax connection for a given model and to show integrability. One necessary condition for integrability of a 2d field theory is that all its consistent 1d mechanical subsystems are also integrable. For integrable string σ -models on curved backgrounds this implies that, for example, geodesic motion on the background must be integrable. Using this condition it is also possible to show explicitly that a model is not integrable. The idea is to consider an integrable 1d subsystem and expand in small fluctuations orthogonal to this system. The resulting linearised equations take a standard form, for which differential Galois theory can be used to determine whether Liouvillian solutions exist. If no such solutions exist the model is not integrable. In chapter 3 we review and apply this method to study integrability for string motion on brane background.

2.2.1 Quantum integrability

At the quantum level integrability places strong constraints on the dynamics of the theory and in particular on the S-matrix. Let us consider an infinite tower of commuting conserved charges

$$\hat{F}_n, \qquad [\hat{F}_i, \hat{F}_j] = 0, \quad \forall i, j,$$
(2.2.11)

which are diagonalised in a basis of Hilbert states of momentum p and flavour α

$$\ddot{F}_n \left| p \right\rangle_{\alpha} = F_n \left| p \right\rangle_{\alpha}.$$
 (2.2.12)

For example in the sine-Gordon model an excitation of mass m has the charges

$$F_{2n+1} = p^{2n+1}, \quad F_{2n} = p^{2n}\sqrt{p^2 + m^2}.$$
 (2.2.13)

Acting with the symmetry in the non-relativistic case, i.e. $F_n \approx p^n$ (setting m = 1), on a localised wave packet

$$\psi(x) = \int_{-\infty}^{\infty} \mathrm{d}p \, e^{-a(p-p_0)^2} e^{ip(x-x_0) + i\frac{p^2}{2}(t-t_0)} \tag{2.2.14}$$

one finds

$$\tilde{\psi}(x) = e^{i\epsilon F_n}\psi(x) = \int_{-\infty}^{\infty} \mathrm{d}p \, e^{-a(p-p_0)^2} e^{ip(x-x_0) + i\frac{p^2}{2}(t-t_0)} e^{i\epsilon p^n}.$$
(2.2.15)

The symmetry translated wave packet is localised near the stationary point of the phase

$$0 = \left. \frac{\mathrm{d(phase)}}{\mathrm{d}p} \right|_{p=p_0} = x - x_0 + p_0(t - t_0) + n\alpha p_0^{n-1}.$$
(2.2.16)

The cases n = 1 and n = 2 correspond to shifts in x_0 and t_0 . However, for n > 2 the symmetry operator shifts the wave packet in space by a value depending on its momentum. Since the symmetry operator commutes with the S-matrix this implies that in two dimensions any multi particle scattering factorises into a sequence of two particle scattering processes. In higher dimension all the wave packets become well separated and scattering becomes trivial. This is the Coleman-Mandula theorem, which gives trivial S-matrices for theories with higher spin conserved currents (see for example [156]).

Let us consider a multi particle scattering process

$$|\tilde{p}_{1},..,\tilde{p}_{\tilde{M}}\rangle_{\tilde{\alpha}_{1},..,\tilde{\alpha}_{M}}^{(\text{out})} = S |p_{1},..,p_{M}\rangle_{\alpha_{1},..,\alpha_{M}}^{(\text{in})}.$$
 (2.2.17)

The *in* and *out* states are eigenstates of the symmetry operators

$$\hat{F}_{k} |p_{1},..,p_{M}\rangle_{\alpha_{1},..,\alpha_{M}} = \sum_{j=1}^{M} F_{k}(p_{j},\alpha_{j}) |p_{1},..,p_{M}\rangle_{\alpha_{1},..,\alpha_{M}}$$
(2.2.18)

and all the charges must be conserved during scattering

$$\sum_{j=1}^{\tilde{M}} F_n(\tilde{p}_j, \tilde{\alpha}_j) = \sum_{j=1}^{M} F_n(p_j, \alpha_j), \quad \forall n.$$
(2.2.19)

In general this condition can only be satisfied when the sets of incoming and outgoing momenta $\{p_j\}$ and $\{\tilde{p}_j\}$ are the same and thus $\tilde{M} = M$. In other words there is no particle production in an integrable theory. Furthermore if we consider 3-particle scattering there are two different ways to resolve the Smatrix into two particle processes. Consistency requires that both ways are equivalent and this imposes the Yang-Baxter equation on the two particle S-matrix, see figure 1.4a.

2.3 Integrability of string motion on AdS backgrounds

In this section we briefly look at how the Lax construction can be achieved for string σ -models on semi-symmetric coset spaces such as in the case of $AdS_5 \times S^5$. In our discussion we closely follow [157]. The AdS_{d+1} space can be defined as a surface embedded in $\mathbb{R}^{2,d}$ given by the hyperboloid

$$Y^{2} = \eta_{PQ}Y^{P}Y^{Q} = -Y^{2}_{-1} - Y^{2}_{0} + \dots + Y^{2}_{d} = -1, \qquad (2.3.1)$$

where $\eta_{PQ} = \text{diag}(-1, 1, ..., -1)$ and P, Q = 0, ..., d. The corresponding AdS_{d+1} metric is then

$$\mathrm{d}s^2_{\mathrm{AdS}_{d+1}} = \eta_{PQ} \,\mathrm{d}Y^P \mathrm{d}Y^Q. \tag{2.3.2}$$

Alternatively AdS_{d+1} can be formulated as a coset space. Any point on AdS_{d+1} can e reached using the isometry group $\operatorname{SO}(d, 2)$. However, once a point is fixed the remaining rotations under the little group $\operatorname{SO}(d, 1)$ leave it invariant. Therefore the AdS space can be defined by the equivalence classes of $\operatorname{SO}(d, 2)$ transformations under the action of the little group

$$AdS_{d+1} = \{g : g \sim gh | g \in SO(d, 2), h \in SO(d, 1)\}.$$
 (2.3.3)

In this language string configurations correspond to a map from the world-sheet Σ into SO(d, 2) which is gauged under the left action of SO(d, 1)

$$g: \Sigma \to \mathrm{SO}(d,2), \qquad g(\tau,\sigma) \to g(\tau,\sigma)H(\tau,\sigma), \quad H \in \mathrm{SO}(d,1).$$
 (2.3.4)

A gauge invariant string σ -model can then obtained from the left-invariant current

$$J = g^{-1} \mathrm{d}g \in \mathfrak{so}(d, 2). \tag{2.3.5}$$

This current is flat by construction, i.e. it satisfies the Maurer-Cartan equation

$$\partial_a J_b - \partial_b J_a + [J_a, J_b] = 0. \tag{2.3.6}$$

Since J transforms as a gauge connection

$$J \to H^{-1}JH + H^{-1}\mathrm{d}H \tag{2.3.7}$$

one can split $\mathfrak{so}(d,2)$ into $\mathfrak{so}(d,1)$ and an orthogonal part $\mathfrak{f}, \mathfrak{so}(d,2) = \mathfrak{so}(d,1) \oplus \mathfrak{f}$,

$$J = J^{(0)} + J^{(2)}, \qquad J^{(0)} \in \mathfrak{so}(d, 1), \quad J^{(2)} \in \mathfrak{f}.$$
(2.3.8)

This decomposition arises from the fact that AdS is a symmetric space with the \mathbb{Z}_2 inversion symmetry $Y \to -Y$. This induces a \mathbb{Z}_2 automorphism on $\mathfrak{so}(d, 2)$

 $\Omega:\mathfrak{so}(d,2)\to\mathfrak{so}(d,2),\qquad [\Omega(\omega_1),\Omega(\omega_2)]=\Omega([\omega_1,\omega_2]),\qquad\forall\,\omega_1,\omega_2\in\mathfrak{so}(d,2),\tag{2.3.9}$

which acts on the split currents as

$$J^{(0)} \to J^{(0)}, \qquad J^{(2)} \to -J^{(2)}.$$
 (2.3.10)

The full decomposition of $\mathfrak{so}(d,2)$ under this symmetry is then

 $[\mathfrak{so}(d,1),\mathfrak{so}(d,1)] \subset \mathfrak{so}(d,1), \quad [\mathfrak{so}(d,1),\mathfrak{f}] \subset \mathfrak{f}, \quad [\mathfrak{f},\mathfrak{f}] \subset \mathfrak{so}(d,1). \tag{2.3.11}$

The current $J^{(2)}$ now transforms as

$$J^{(2)} \to H^{-1} J^{(2)} H$$
 (2.3.12)

allowing one to define a gauge invariant action

$$S = \frac{h}{2} \int tr \left[J^{(2)} \wedge *J^{(2)} \right] = \frac{h}{2} \int d^2 \sigma \sqrt{-\gamma} \gamma^{ab} tr \left[J_a^{(2)} J_b^{(2)} \right].$$
(2.3.13)

Substituting a particular coset parametrisation g(Y) in terms of some embedding coordinates Y one recovers the bosonic string action (1.0.26). The advantage of this coset formulation is that it can be easily extended to the full supersymmetric string σ -model. Also the flatness of the current J leads to a natural construction for the Lax connection.

The equations of motion take the form

$$D_a(\sqrt{-\gamma}\gamma^{ab}J_b^{(2)}) = 0, \qquad D_a = \partial_a + [J_a^{(0)}, \cdot].$$
(2.3.14)

Taking $J^{(0)}$ and $J^{(2)}$ to be independent the flatness condition (2.3.6) becomes another equation of

motion and under (2.3.11) it is decomposed into

$$D_a J_b^{(2)} - D_b J_a^{(2)} = 0, \qquad F_{ab} + [J_a^{(2)}, J_b^{(2)}] = 0, \qquad (2.3.15)$$

$$F_{ab} = \partial_a J_b^{(0)} - \partial_b J_a^{(0)} + [J_a^{(0)}, J_b^{(0)}].$$
(2.3.16)

These equations are supplemented by the Virasoro constraints, which arise from varying the action with respect to the world-sheet metric γ_{ab}

$$\operatorname{tr}\left[J_{a}^{(2)}J_{b}^{(2)} - \frac{1}{2}\gamma_{ab}\gamma^{cd}J_{c}^{(2)}J_{d}^{(2)}\right] = 0.$$
(2.3.17)

The equations of motion (2.3.14) and (2.3.15) can now be rephrased as the flatness condition for a 1-parameter family of Lax connections [18]

$$dL - L \wedge L = 0, \quad L_a = J_a^{(0)} + \frac{z^2 + 1}{z^2 - 1} J_a^{(2)} - \frac{2z}{z^2 - 1} \frac{1}{\sqrt{-\gamma}} \gamma_{ab} \epsilon^{bc} J_c^{(2)}, \quad (2.3.18)$$

where z is the spectral parameter. For consistent superstring theory backgrounds of the form $AdS_{d+1} \times M_{9-d}$ the fermionic degrees of freedom are naturally incorporated into a supergroup coset construction. In general such backgrounds can also be supported by NS-NS and R-R fluxes which are coupled as bosonic and fermionic Wess-Zumino terms in the Green-Schwarz action. For example the Green-Schwarz action for the superstring on $AdS_5 \times S^5$, which is supported by an R-R flux, takes the form of a σ -model on the semi-symmetric coset space

$$\frac{G}{H_0} = \frac{\text{PSU}(2,2|4)}{\text{SO}(4,1) \times \text{SO}(5)}.$$
(2.3.19)

The \mathbb{Z}_2 symmetry of the bosonic case is now extended to a \mathbb{Z}_4 automorphism Ω on the Lie algebra \mathfrak{g} of the group G. This automorphism gives a \mathbb{Z}_4 grading of the algebra \mathfrak{g}

$$\mathfrak{psu}(2,2|4) = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)}, \qquad \Omega(\mathfrak{g}^{(n)}) = i^n \mathfrak{g}^{(n)}$$
(2.3.20)

with the compatible (anti)commutation relations

$$[\mathfrak{g}^{(n)},\mathfrak{g}^{(m)}] \subset \mathfrak{g}^{((m+n) \mod 4)}. \tag{2.3.21}$$

Since Ω should also be compatible with the Grassmann parity, i.e. $\Omega^2 = (-1)^F$, the algebra is split into the bosonic subalgebra $\mathfrak{g}^{(0)} + \mathfrak{g}^{(2)}$, which is Grassmann even, and the fermionic subalgebra $\mathfrak{g}^{(1)} + \mathfrak{g}^{(3)}$, which is Grassmann odd. Decomposing the left current under the \mathbb{Z}_4 grading as

$$J = g^{-1} dg = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)}, \qquad g \in \text{PSU}(2, 2|4), \quad J \in \mathfrak{psu}(2, 2|4)$$
(2.3.22)

the current $J^{(0)}$ transforms as a gauge field and $J^{(m)}$, $m \neq 0$ transform by conjugation under the gauged left action of H_0

$$g(\tau,\sigma) \to g(\tau,\sigma)H(\tau,\sigma), \qquad H(\tau,\sigma) \in \mathrm{SO}(4,1) \times \mathrm{SO}(5),$$

$$(2.3.23)$$

$$J^{(0)} \to H^{-1}J^{(0)}H + H^{-1}dH, \qquad J^{(m)} \to H^{-1}J^{(m)}H, \ m \neq 0.$$
 (2.3.24)

Coupling bosonic degrees of freedom with the world-sheet metric γ^{ab} and the fermionic degrees of

freedom with ϵ^{ab} , as expected for a Green-Schwarz action, one obtains the gauge invariant action for the superstring on $AdS_5 \times S^5$ [158]

$$S = \frac{h}{2} \int \operatorname{str} \left[J^{(2)} \wedge *J^{(2)} - J^{(1)} \wedge J^{(3)} + \Lambda \wedge J^{(2)} \right].$$
 (2.3.25)

Here the supertrace str (a \mathbb{Z}_2 graded trace) is the unique *G*-invariant bilinear form on \mathfrak{g} and the Lagrange multiplier Λ ensures that str $J^{(2)} = 0$ as required for PSU(2, 2|4). The equations of motion for this system are equivalent to the zero-curvature condition for the Lax connection

$$L = J^{(0)} + \frac{z^2 + 1}{z^2 - 1}J^{(2)} - \frac{2z}{z^2 - 1}(*J^{(2)} - \Lambda) + \sqrt{\frac{z + 1}{z - 1}}J^{(1)} + \sqrt{\frac{z - 1}{z + 1}}J^{(3)},$$
(2.3.26)

which shows classical integrability [18].

2.3.1 Finite-gap equations

In an integrable system classical solutions are characterised by the eigenvalues of the monodromy matrix

$$T(\tau,\sigma;z) = \mathcal{P}\exp\oint_{[C(\tau,\sigma)]} \mathrm{d}\sigma'^a L_a(\tau',\sigma';z).$$
(2.3.27)

In the case of a coset construction this is a group element of G which transforms by conjugation under gauge transformations

$$T(\tau,\sigma;z) \to H^{-1}(\tau,\sigma)T(\tau,\sigma;z)H(\tau,\sigma), \qquad h(\tau,\sigma) \in H_0$$
(2.3.28)

and under shifts of the base point

$$T(\tau,\sigma;z) \to U^{-1}T(\tau',\sigma';z)U, \qquad U \in G.$$
(2.3.29)

Here U is the monodromy along a zero-winding path connecting the base points (τ, σ) and (τ', σ') . These transformations do not change the conjugacy class of a monodromy matrix and as such the conjugacy classes are gauge invariant and do not depend on a base point and are therefore timeindependent. The set of these conjugacy classes is isomorphic to the maximal torus of G modulo the Weyl group. Locally we can diagonalise the monodromy matrix in a Cartan basis $\{H_l\}, l = 1, ..., R$

$$T = U^{-1} \exp\left(\sum_{l=1}^{R} p_l(z) H_l\right) U, \qquad R = \operatorname{rank}(\mathfrak{g}).$$
(2.3.30)

The conserved charges are encoded in the asymptotics of this monodromy matrix. For example the Noether current associated with the global symmetry of left multiplication by $g \in G$ can be found from the first coefficient in the Laurent expansion around $z = \infty$

$$L_a = g^{-1} \left(\partial_a + \frac{1}{z} \frac{2}{\gamma} \epsilon_{ab} k^b \right) g + \dots, \qquad z \to \infty$$
(2.3.31)

$$\partial_a k^a = 0, \qquad k^a = g \left(\sqrt{-\gamma} \gamma^{ab} J_b^{(2)} - \frac{1}{2} \epsilon^{ab} J_b^{(1)} + \frac{1}{2} \epsilon^{ab} J_b^{(3)} \right) g^{-1}.$$
(2.3.32)

The associated Noether charges are encoded as the first coefficient in the corresponding expansion of the quasi-momenta

$$p_l(z) = -\frac{2}{z}Q_l + ..., \qquad z \to \infty.$$
 (2.3.33)

Together with the remaining coefficients this gives the infinite set of conserved charges. Expanding around the poles at $z = \pm 1$ one can also generate an infinite set of charges in terms of local densities.

The quasi-momenta p_l are defined up to shifts by 2π and Weyl group transformations. They are multivalued functions in z with poles at $z = \pm 1$ coming from the Lax connection. Additionally they can have branch points arising from the diagonalisation in a particular Cartan basis. The associated monodromies, i.e. the change of p_l when encircling a branch point, correspond to Weyl group transformations. In this picture elementary string excitations are characterised by Weyl reflections which can be used to construct any element of the Weyl group. Composite monodromies then correspond to composite string states. The monodromy for a Weyl reflection with respect to the *l*th root of \mathfrak{g} is

$$p_l(z) \to p_l(z) - A_{lm} p_m(z) + 2\pi n_{l,i}, \qquad n_{l,i} \in \mathbb{Z},$$
(2.3.34)

where A_{lm} is the Cartan matrix of \mathfrak{g} . Depending on whether the root generators are bosonic or fermionic the diagonal entries of the Cartan matrix have different values giving the two cases

fermionic:
$$A_{ll} = 0, \qquad p_l \to p_l + \dots$$
 (2.3.35)

bosonic:
$$A_{ll} = 2, \qquad p_l \to -p_l + ...$$
 (2.3.36)

In the fermionic case the quasi-momentum is shifted by some locally analytic function in z giving a logarithmic branch point. This fully specifies the discontinuity across the fermionic cut except at the endpoints. In the bosonic case there is an additional sign change which leads to a square root branch point with some freedom for the discontinuity along the cut. Thus the quasi-momenta satisfy the discontinuity relations

$$A_{lm} p_m(z) = 2\pi n_{l,i}, \qquad z \in C_{l,i}, \quad n_{l,i} \in \mathbb{Z}$$

$$(2.3.37)$$

$$p_l(z) \equiv \frac{1}{2} \lim_{\epsilon \to 0} (p_l(z+i\epsilon)) + p_l(z-i\epsilon)), \quad \epsilon \in \mathbb{C}$$
(2.3.38)

along a set of bosonic branch cuts $\{C_{l,i}\}$ with square root singularities at their endpoints and at the endpoints of fermionic branch cuts corresponding to logarithmic singularities. Let us also parametrise the residues around the poles at $z = \pm 1$ by the constants κ_l and m_l

$$p_l(z) = \frac{1}{2} \frac{\kappa_l z \mp 2\pi m_l}{z \pm 1} + \dots$$
 (2.3.39)

Additionally the system has a \mathbb{Z}_4 symmetry, which corresponds to an inversion symmetry in z at the level of the quasi-momenta. The inversion symmetry arises from the action of the automorphism Ω on the Lax connection and monodromy matrix (by lifting Ω to the group action using the exponential map)

$$\Omega(L_a(z)) = L_a\left(\frac{1}{z}\right), \qquad \Omega(T(z)) = T\left(\frac{1}{z}\right). \tag{2.3.40}$$

The action of Ω on the Cartan generators can be parametrised by a matrix S_{lm} such that

$$\Omega(H_l) = \sum_{m=1}^{R} H_m S_{ml}.$$
(2.3.41)

This matrix satisfies $S^2 = 1$ and therefore has eigenvalues ± 1 , since $\Omega^2 = (-1)^F$ and the H_l are bosonic generators. The quasi-momenta are then found to satisfy

$$p_l\left(\frac{1}{z}\right) = \sum_{m=1}^R S_{lm} p_m(z).$$
 (2.3.42)

Thus it is sufficient to specify $p_l(z)$ in the *physical* region |z| > 1. The quasi-momenta can be further parametrised by the density $\rho_l(z)$ for the discontinuities along the bosonic branch cuts and fermionic singularities, denoted by C_l ,

$$\rho_l(z) = \lim_{\epsilon \to 0} \left(p_l(z+\epsilon) - p_l(z-\epsilon) \right), \qquad z \in C_l.$$
(2.3.43)

This gives the spectral representation

$$p_l(z) = -\frac{\kappa_l z + 2\pi m_l}{z^2 - 1} + \int_{C_l} \mathrm{d}y \frac{\rho_l(y)}{z - y} + \int_{1/C_l} \mathrm{d}y \frac{\tilde{\rho}_l(y)}{z - y}, \tag{2.3.44}$$

where the integral is split into cuts and poles C_l in the physical region |z| > 1 and those in the unit circle |z| < 1, denoted by $1/C_l$. The inversion symmetry (2.3.42) gives the additional conditions

$$\kappa_l = -S_{lk}\kappa_k, \qquad m_l = -S_{lk}m_k. \tag{2.3.45}$$

The inversion symmetry also determines the density $\tilde{\rho}(w)$ in terms of $\rho(1/w)$ giving

$$p_l(z) = -\frac{\kappa_l z + 2\pi m_l}{z^2 - 1} + \int_{C_l} \mathrm{d}y \frac{\rho_l(y)}{z - y} - S_{lm} \int_{C_m} \frac{\mathrm{d}y}{y^2} \frac{\rho_m(y)}{z - \frac{1}{y}},\tag{2.3.46}$$

$$2\pi m_l = (\delta_{lk} - S_{lk}) \int_{C_k} \frac{\mathrm{d}y}{y} \rho_k(y), \qquad (2.3.47)$$

with all integration contours in the physical region. Further imposing the discontinuity relations (2.3.37) on (2.3.46) by using the Sochocki-Plemelj theorem one obtains the *finite-gap equations*, a set of Cauchy principal value integral equations for the densities $\rho_l(z)$ on the collection of bosonic cuts and fermionic singularities $\{C_{l,i}\}$

$$A_{lk} \oint_{C_k} \mathrm{d}y \frac{\rho_k(y)}{z - y} - A_{lk} S_{kn} \int_{C_n} \frac{\mathrm{d}y}{y^2} \frac{\rho_n(y)}{z - \frac{1}{y}} = A_{lk} \frac{\kappa_k z + 2\pi m_k}{z^2 - 1} + 2\pi n_{l,i}, \quad z \in C_{l,i}.$$
(2.3.48)

Additionally one needs to impose the Virasoro constraints, which lead to conditions on the residues of the poles at $z = \pm 1$ of the quasi-momenta. Let us review these conditions following [72]. At the leading order in $\eta = z \mp 1$ the auxiliary linear problem

$$\partial_{\sigma}\Psi_i(\sigma;z) = (L_{\sigma})_{ij}\Psi_j(\sigma;z) \tag{2.3.49}$$

reduces to

$$\partial_{\sigma}\Psi_i(\sigma;z) = -\frac{1}{i\eta}V_{ij}\Psi_j(\sigma;z), \quad V = -i\eta L_{\sigma} \sim -i(J_{\tau}^{(2)} \pm J_{\sigma}^{(2)}) + \mathcal{O}(\eta).$$
(2.3.50)

Here $V(\eta)$ is an analytic function since L_{σ} has only simple poles at $z = \pm 1$. This equation can be solved by the WKB ansatz

$$\Psi_i(\sigma; z) = \exp\left(\frac{i}{\eta} S_l(\sigma; \eta)\right) \xi_i(\eta).$$
(2.3.51)

This gives the eigenvalue problem for the $R = \operatorname{rank}(\mathfrak{g})$ eigenvalues of $V \in \mathfrak{g}$

$$V_{ij}\Psi_j(\sigma;z) = \partial_\sigma S_l \Psi_i(\sigma;z), \qquad l = 1, .., R.$$
(2.3.52)

The quasi-momenta are related to the linear auxiliary problem through the monodromy of a solution which, in a diagonal basis, takes the form

$$\Psi_i(\tau, \sigma + 2\pi; z) = \exp(ip_l(z))\Psi_i(\tau, \sigma; z).$$
(2.3.53)

Combining this with the WKB ansatz (2.3.51) the quasi-momenta can be written as

$$p_l(z) = \frac{1}{\eta} (S_l(\sigma + 2\pi; \eta) - S_l(\sigma; \eta)) = \frac{1}{\eta} \int_0^{2\pi} \mathrm{d}\sigma \,\partial_\sigma S_l(\sigma; \eta).$$
(2.3.54)

In conformal gauge $\gamma_{ab} = \text{diag}(+1, -1)$ the Virasoro constraints (2.3.17) take the form

$$\operatorname{tr}\left[(J_{\tau}^{(2)} \pm J_{\sigma}^{(2)})^2 \right] = 0, \qquad (2.3.55)$$

which implies that

$$trV^2 = \mathcal{O}(\eta). \tag{2.3.56}$$

Writing V in the Cartan-Weyl basis with its eigenvalues $\partial_{\sigma} S_l(\sigma; \eta)$ and using $A_{ij} = \operatorname{str} H_i H_j$ the condition becomes

$$A_{lm}\partial_{\sigma}S_{l}\partial_{\sigma}S_{m} = \mathcal{O}(\eta), \qquad (2.3.57)$$

where A_{ij} is the Cartan matrix. Defining the functions

$$f_l^{\pm}(\sigma) = \lim_{\eta \to 0} \partial_{\sigma} S(\sigma; \eta)$$
(2.3.58)

the limit $\eta \to 0$ of (2.3.54) then determines the residues in terms of f_l^{\pm} with the equation (2.3.57) giving the Virasoro constraints. Thus the Virasoro constraints are equivalent to the existence of the functions $f_l^{\pm}(\sigma)$ such that

$$\frac{1}{2}(\kappa_l \pm 2\pi m_l) = \int_0^{2\pi} \mathrm{d}\sigma \, f_l^{\pm}(\sigma), \qquad A_{lm} f_l^{\pm} f_m^{\pm} = 0.$$
(2.3.59)

In the case that there are not more than two product factors in the target spacetime, as for $AdS_5 \times S^5$

and $AdS_4 \times \mathbb{CP}^3$ for example, the functions $f_l^{\pm}(\sigma)$ can be chosen to be constant giving the simpler relations [157]

$$f_l^{\pm}(\sigma) = \frac{1}{2} (\kappa_l \pm 2\pi m_l), \qquad (\kappa_l \pm 2\pi m_l) A_{lk} (\kappa_k \pm 2\pi m_k) = 0.$$
(2.3.60)

However, in cases such as $AdS_3 \times S^3 \times T^4$, for which one needs to consider massless modes, the Virasoro constraints take the more general form (2.3.59).

2.3.2 Quasi-momenta for string solutions on $AdS_5 \times S^5$

Classical solutions can be classified in terms of quasi-momenta by using the constraints on their asymptotics from the global charges of the solution, the constraints on the residues of the poles from the Virasoro constraints and the \mathbb{Z}_4 symmetry from the automorphism of the algebra. As an example let us see how this characterises quasi-momenta of classical solutions for strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$. The Lax connection is super-traceless and therefore the monodromy matrix T(z) is unimodular

$$sdet T(z) = 1.$$
 (2.3.61)

It is convenient to define the quasi-momenta in the fundamental representation, i.e. one can consider the set of quasi-momenta given by the diagonal form of the monodromy matrix

$$T(z) = \operatorname{diag}\left(e^{ip_1^A(z)}, e^{ip_2^A(z)}, e^{ip_3^A(z)}, e^{ip_4^A(z)} \middle| e^{ip_1^S(z)}, e^{ip_2^S(z)}, e^{ip_3^S(z)}, e^{ip_4^S(z)}\right),$$
(2.3.62)

where the label A and S distinguishes between the AdS_5 and S^5 matrix blocks. These quasi-momenta are not independent since unimodularity gives the additional condition

$$\sum_{i=1}^{4} \left(p_i^A(z) - p_i^S(z) \right) = 2\pi k, \qquad k \in \mathbb{Z}.$$
(2.3.63)

In this representation the quasi-momenta are only unique up to shifts by some constants corresponding to a central charge.² The quasi-momenta are defined on the cover of the complex plane and the number of sheets of the Riemann surface is given by the degree of the characteristic polynomial for T(z)

$$\operatorname{sdet}(T(z) - \mu \mathbb{I}) = 0, \qquad (2.3.64)$$

which in this case is eight. Each branch cut C_n^{ij} connects a pair of these eight sheets with the quasimomenta having the discontinuity

$$p_i(z+i\epsilon) - p_j(z-i\epsilon) = 2\pi n_{ij}, \qquad z \in C_n^{ij}.$$

$$(2.3.65)$$

In this picture classical solutions are described by quasi-momenta with macroscopic cuts. The global charges (E, S_1, S_2) and (J_1, J_2, J_3) for the SO(4, 2) and SO(6) isometry groups are related to the

²We will elaborate on this point in the next section when considering $AdS_3 \times S^3$.

asymptotics of the quasi-momenta as

$$\begin{pmatrix} p_{1}^{A} \\ p_{2}^{A} \\ p_{3}^{A} \\ p_{3}^{A} \\ \frac{p_{4}^{A}}{p_{1}^{S}} \\ p_{2}^{S} \\ p_{3}^{S} \\ p_{4}^{S} \end{pmatrix} = \frac{2\pi}{z} \begin{pmatrix} +\mathcal{E} - \mathcal{S}_{1} + \mathcal{S}_{2} \\ +\mathcal{E} + \mathcal{S}_{1} - \mathcal{S}_{2} \\ -\mathcal{E} - \mathcal{S}_{1} - \mathcal{S}_{2} \\ -\mathcal{E} - \mathcal{S}_{1} - \mathcal{S}_{2} \\ -\mathcal{E} - \mathcal{S}_{1} - \mathcal{S}_{2} \\ -\mathcal{I} - \mathcal{I} - \mathcal{I}_{2} - \mathcal{I}_{3} \\ +\mathcal{I}_{1} - \mathcal{I}_{2} + \mathcal{I}_{3} \\ -\mathcal{I}_{1} - \mathcal{I}_{2} - \mathcal{I}_{3} \end{pmatrix} + \mathcal{O}\left(\frac{1}{z^{2}}\right),$$
(2.3.66)

where the charges are rescaled by the 't Hooft coupling $Q = \sqrt{\lambda}Q$. The inversion symmetry of the quasi-momenta, which arises from the automorphism of the $\mathfrak{psu}(2,2|4)$ algebra and relates the |z| < 1 and |z| > 1 regions, takes the explicit form

$$\tilde{p}_{1,2}(z) = -\tilde{p}_{2,1}\left(\frac{1}{z}\right) - 2\pi m \tag{2.3.67}$$

$$\tilde{p}_{3,4}(z) = -\tilde{p}_{4,3}\left(\frac{1}{z}\right) + 2\pi m \tag{2.3.68}$$

$$\hat{p}_{1,2,3,4}(z) = -\hat{p}_{2,1,4,3}\left(\frac{1}{z}\right).$$
(2.3.69)

Finally the Virasoro constraints (2.3.60) relate the residues of the poles at $z = \pm 1$ for different quasimomenta

$$\{p_1^A, p_2^A, p_3^A, p_4^A | p_1^S, p_2^S, p_3^S, p_4^S\} = \frac{\{\alpha_{\pm}, \alpha_{\pm}, \beta_{\pm}, \beta_{\pm} | \alpha_{\pm}, \alpha_{\pm}, \beta_{\pm}, \beta_{\pm}\}}{z \pm 1} + \mathcal{O}(1),$$
(2.3.70)

where α_{\pm} and β_{\pm} parametrise the residues. Having specified the quasi-momenta one can define the *filling fraction* for each cut

$$S_{ij} = \pm \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{C_{ij}} dz \left(1 - \frac{1}{z^2}\right) p_i(z), \qquad (2.3.71)$$

which correspond to the action variables of the system. At the semiclassical level these are quantized by the Bohr-Sommerfeld condition [159, 152]. Hence S_{ij} corresponds to the number of quanta for a macroscopic excitation.

Small fluctuations around a classical solution can be described by adding perturbations $\delta p_i(z)$ to the quasi-momenta of the classical solution, i.e.

$$p_i(z) \to p_i(z) + \delta p_i(z). \tag{2.3.72}$$

The perturbations take the form of microscopic cuts (i.e poles), which backreact with the macroscopic cuts and shift their position. The discontinuity condition (2.3.65) now applies to all the cuts. At the microscopic cuts one finds

$$p_i(z_n^{ij}) - p_j(z_n^{ij}) = 2\pi n, \qquad |z_n^{ij}| > 1,$$
(2.3.73)

which determines the position z_n^{ij} of the poles to leading order. Along the macroscopic cuts on the

other hand one finds that the perturbations satisfy

$$\delta p_i(z+i\epsilon) - \delta p_j(z-i\epsilon) = 0, \qquad z \in C_n^{ij}. \tag{2.3.74}$$

The 16 different polarisations of physical world-sheet excitations for the superstring on $AdS_5 \times S^5$ are given by microscopic cuts which connect the following sheets following [160]

$$\begin{split} \mathbf{S}^5 : & (p_1^S, p_3^S), (p_1^S, p_4^S), (p_2^S, p_3^S), (p_2^S, p_4^S), \\ \mathbf{AdS}_5 : & (p_1^A, p_3^A), (p_1^A, p_4^A), (p_2^A, p_3^A), (p_2^A, p_4^A), \\ \mathbf{fermionic} : & (p_1^S, p_3^A), (p_1^A, p_4^S), (p_2^S, p_3^A), (p_2^A, p_4^S), \\ & (p_1^A, p_3^S), (p_1^S, p_4^A), (p_2^A, p_3^S), (p_2^S, p_4^A). \end{split}$$

Together with the asymptotic behaviour and the conditions on the simple poles from the Lax connection this can be used to find the spectrum of small fluctuations around classical solutions.

2.3.3 Finite-gap equations for strings on $AdS_3 \times S^3 \times T^4$ with mixed flux

In the case of string theory in $AdS_3 \times S^3 \times T^4$ with mixed flux the coset construction is supplemented with an additional Wess-Zumino term in the action for the NS-NS flux. In the massive sector of this mixed flux theory, i.e. in the case of coset fields only, the finite-gap equations were determined in [95]. An important new feature of the $AdS_3 \times S^3$ backgrounds is that there are two sets of quasi-momenta, one for the left moving and one for the right moving sector. The coset for $AdS_3 \times S^3$ is

$$\frac{G_L \times G_R}{H_0} = \frac{\text{PSU}(1,1|2) \times \text{PSU}(1,1|2)}{\text{SU}(1,1) \times \text{SU}(2)}$$
(2.3.75)

The algebra of this permutation supercoset is split into the left and right moving sectors

$$\mathfrak{g} = \mathfrak{g}_L \oplus \mathfrak{g}_R, \qquad \mathfrak{g}_L = \mathfrak{g}_R. \tag{2.3.76}$$

The algebra elements can be written in the matrix form

$$J = g^{-1} dg \in \mathfrak{g}, \quad g \in G_L \times G_R, \quad J = \begin{pmatrix} J_L & 0\\ 0 & J_R \end{pmatrix}, \quad J_{L,R} = g_{L,R}^{-1} dg_{L,R}, \quad g_{L,R} \in G_{L,R}. \quad (2.3.77)$$

The \mathbb{Z}_4 automorphism acts as

$$\Omega(J) = \begin{pmatrix} J_R & 0\\ 0 & (-1)^F J_L \end{pmatrix}, \qquad (2.3.78)$$

where $(-1)^F$ is 1 for Grassmann even and -1 for Grassmann odd elements. Explicitly one can take

$$\Omega(J) = MJM^{-1}, \qquad M = \begin{pmatrix} 0 & \frac{+\mathbb{I} & 0}{0 & +\mathbb{I}} \\ \frac{+\mathbb{I} & 0}{0 & -\mathbb{I}} & 0 \end{pmatrix}, \qquad \Omega(AB) = \Omega(A)\Omega(B), \qquad (2.3.79)$$

where each entry in M acts on either even or odd subblocks of a supermatrix. This automorphism decomposes the Lie algebra \mathfrak{g} in the usual way into the four parts

$$J \in \mathfrak{g}, \qquad J = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)}, \qquad \Omega(J^{(n)}) = i^n J^{(n)}. \tag{2.3.80}$$

The gauge transformations $H \in H_0$ act simultaneously on the left and right group elements by

$$g_{L,R} \to g_{L,R} H \tag{2.3.81}$$

and the currents transform as

$$J_{L,R} = J_{L,R}^B + J_{L,R}^F, \qquad J_{L,R}^B \to H^{-1} J_{L,R}^B H + H^{-1} dH, \qquad J_{L,R}^F \to H^{-1} J_{L,R}^F H.$$
(2.3.82)

Here *B* and *F* label the even (bosonic) and odd (fermionic) components. The NS-NS flux is represented by a Wess-Zumino term in the action, which is a three-dimensional integral over a ball \mathcal{B} with its boundary $\mathcal{M} = \partial \mathcal{B}$ corresponding to the world-sheet. The integrand is a total derivative ensuring that after integration one is left with a term that only depends on the world-sheet coordinates. In the bosonic case, i.e. for the bosonic subgroup H_0 with the only non-vanishing current $J^{(2)}$ in the action, the Wess-Zumino term has the standard form [122]

$$S_{WZ}^{\text{bosonic}} = q \frac{2}{3} \int_{\mathcal{B}} \text{str} \, J^{(2)} \wedge J^{(2)} \wedge J^{(2)}.$$
(2.3.83)

In the case of the full supergroup this expression by itself is not a total derivative. Instead in [68] it was shown that a total derivative can be obtained by supplementing the bosonic WZ term with two additional terms giving the unique fermionic completion

$$S_{WZ} = q \int_{\mathcal{B}} \operatorname{str} \left[\frac{2}{3} J^{(2)} \wedge J^{(2)} \wedge J^{(2)} + J^{(1)} \wedge J^{(3)} \wedge J^{(2)} + J^{(3)} \wedge J^{(1)} \wedge J^{(2)} \right].$$
(2.3.84)

The action therefore takes the form

$$S = \frac{1}{2} \int_{\mathcal{M}} \operatorname{str} \left[J^{(2)} \wedge *J^{(2)} + \kappa J^{(1)} \wedge J^{(3)} \right] + S_{WZ}, \qquad (2.3.85)$$

where κ and q are now free relative parameters for the different terms. For $\kappa^2 + q^2 = 1$ the equations of motion admit a flat Lax connection in terms of the currents $J^{(i)}$ showing classical integrability. The above form of the action is not invariant under the \mathbb{Z}_4 automorphism due to the presence of the WZ term S_{WZ} , which has a grading of two, i.e. $\Omega(S_{WZ}) = -S_{WZ}$. Instead, applying Ω is equivalent to sending $q \to -q$. For convenience it is possible to write a \mathbb{Z}_4 invariant action by introducing a matrix of grading two into the WZ term in the action. Defining such a matrix by

$$W = \begin{pmatrix} +\mathbb{I} & 0\\ 0 & -\mathbb{I} \end{pmatrix}, \qquad W^2 = \mathbb{I}, \qquad \Omega(W) = -W, \qquad (2.3.86)$$

the \mathbb{Z}_4 invariant action takes the form [95]

$$S_{WZ} = q \int_{\mathcal{B}} \operatorname{str} W \Big[\frac{2}{3} J^{(2)} \wedge J^{(2)} \wedge J^{(2)} + J^{(1)} \wedge J^{(3)} \wedge J^{(2)} + J^{(3)} \wedge J^{(1)} \wedge J^{(2)} \Big].$$
(2.3.87)

One should note that the automorphism is not a physical symmetry since it acts non-trivially on W which is not a dynamical field. The equations of motion are given by

$$d * K + *K \wedge J + J \wedge *K = 0, \qquad K \equiv K_{GS} + K_{WZ}$$
(2.3.88)

$$*K_{GS} = -(*2J^{(2)} + \kappa(J^{(3)} - J^{(1)})), \qquad (2.3.89)$$

$$*K_{WZ} = -qW(2J^{(2)} + J^{(1)} + J^{(3)}), (2.3.90)$$

where $k \equiv gKg^{-1}$ is a Noether current

$$d * k = 0. (2.3.91)$$

Making the same ansatz for the Lax connection as in the pure R-R case (i.e. q = 0)

$$L = J^{(0)} + \gamma_2 J^{(2)} + \gamma_* \star J^{(2)} + \gamma_1 J^{(1)} + \gamma_3 J^{(3)}$$
(2.3.92)

the flatness condition $dL + L \wedge L = 0$ gives a set of equations for the coefficients γ_i , $i = 0, ..., 3, \star$ which involve W. Hence in this parametrisation the coefficients are matrices of the form

$$\gamma_i = \alpha_i \mathbb{I} + \beta_i W. \tag{2.3.93}$$

The flatness condition is invariant under transformations of the form

$$L \to hLh^{-1} - dhh^{-1}, \qquad h \in G,$$
 (2.3.94)

which can be used to gauge away the $J^{(0)}$ term by choosing h = g where g is obtained from $J = g^{-1} dg$ and thus

$$L \to \tilde{L} = g(L - J)g^{-1}.$$
 (2.3.95)

Finally the coefficients α_i and β_i can depend on q and should also be functions of the spectral parameter z. In particular one can choose a parametrisation in z such that the Lax connection inherits the following properties from the pure R-R case:

• When the NS-NS flux vanishes the Lax connection should reduce to the standard expression in the pure R-R case, which in particular implies

$$\alpha_2(z) \to \frac{z^2 + 1}{z^2 - 1}, \qquad \beta_2(z) \to 0, \qquad q \to 0.$$
(2.3.96)

• The asymptotics of the Lax connection should give the global Noether current k as in the pure R-R case, i.e.

$$\tilde{L} \sim \frac{k}{z} + \dots, \qquad z \to \infty.$$
 (2.3.97)

• The \mathbb{Z}_4 automorphism should induce the inversion symmetry

$$\Omega(L(z)) = L\left(\frac{1}{z}\right). \tag{2.3.98}$$

• The coefficients $\delta_2 = \alpha_2 + \beta_2$ and $\delta_* = \alpha_* + \beta_*$ for the bosonic current $J^{(2)}$ contributions should be rational functions of the spectral parameter. This is particularly useful for the construction of the algebraic curve for classical solutions [20].

Imposing these conditions a possible parametrisation for the Lax connection is given by

$$L(z) = \frac{1}{2} \sum_{i=1,2,3,*} \left[(\delta_i(q) + \delta_i(-q)) J^{(i)} + (\delta_i(q) - \delta_i(-q)) W J^{(i)} \right], \qquad J^{(*)} \equiv *J^{(2)}$$
(2.3.99)

$$\delta_2 = \frac{(z^2 + 1)\kappa}{(z^2 - 1)\kappa - 2qz}, \qquad \delta_1 = (z + 1)\sqrt{\frac{\kappa}{(z^2 - 1)\kappa - 2qz}}$$
(2.3.100)

$$\delta_* = -\frac{-2\kappa z}{(z\kappa - q)^2 - 1}, \qquad \delta_3 = (z - 1)\sqrt{\frac{\kappa}{(z^2 - 1)\kappa - 2qz}}.$$
 (2.3.101)

The original pure R-R Lax connection poles at $z = \pm 1$ now get shifted by the NS-NS flux coefficient q to z = s, -1/s, where

$$s = \sqrt{\frac{1+q}{1-q}}.$$
 (2.3.102)

The \mathbb{Z}_4 symmetry also gives the two additional poles at z = 1/s, -s. The monodromy matrix can now be diagonalised in a Cartan basis

$$T(z,q) = U^{-1}(z,q) \exp\left(\sum_{k=1}^{R} \left[p_{+k}(z,q)H_{k}^{L} + p_{-k}(z,q)H_{k}^{R}\right]\right) U(z,q),$$
(2.3.103)

where the labels L and R denote non-vanishing elements in the upper left and lower right corners of the matrix decomposition for $\mathfrak{g} = \mathfrak{g}_L \oplus \mathfrak{g}_R$. The \mathbb{Z}_4 automorphism exchanges the left and right currents and therefore the left quasi-momenta $p_{+k}(z)$ inherit the poles at z = s, -1/s while the right quasi-momenta $p_{-k}(z)$ inherit the poles at z = 1/s, -s. The Weyl reflections lead to the usual logarithmic and square root branch points with the monodromy condition along the cuts $C_{l,i}$

where A_{lm}^{\pm} are the Cartan matrices for \mathfrak{g}_L and \mathfrak{g}_R . Expanding the quasi-momenta near the poles one can parametrise the residues as

$$p_{+l}(z) = \pm \frac{s_{\pm}}{2} \frac{\kappa_l^+ \pm 2\pi m_l^+}{z - s_{\pm}} + \dots, \qquad z \sim s_{\pm}, \qquad s_+ = s, \quad s_- = -\frac{1}{s}$$
(2.3.105)

$$p_{-l}(z) = \mp \frac{s_{\pm}}{2} \frac{\kappa_l^- \mp 2\pi m_l^-}{z + s_{\pm}} + \dots, \qquad z \sim -s_{\pm} .$$
(2.3.106)

At large z the Lax connection and quasi-momenta expand as

$$L = g^{-1} \left(d + \frac{1}{\kappa z} * k \right) g + \mathcal{O}\left(\frac{1}{z^2}\right), \qquad p_{\pm l}(z) = \frac{Q_l}{\kappa z} + \mathcal{O}\left(\frac{1}{z^2}\right), \tag{2.3.107}$$

which gives the charges Q_l associated with the Noether current k for the global symmetry. In the spectral representation the quasi-momenta take the form

$$p_{\pm l}(z) = \frac{\frac{z}{\kappa}(2\pi q m_l^{\pm} \pm \kappa_l^{\pm}) \pm 2\pi m_l^{\pm}}{(z \mp s)(z \pm s^{-1})} + \int_{C_l} \mathrm{d}y \, \frac{\rho_l^{\pm}(y)}{z - y} + \int_{1/C_l} \mathrm{d}y \, \frac{\tilde{\rho}_l^{\pm}(y)}{z - y}.$$
(2.3.108)

The action of the \mathbb{Z}_4 automorphism on the Cartan generators can be parametrised as

$$\Omega(H_l^L) = H_m^R S_{ml}^+, \qquad \Omega(H_l^R) = H_m^L S_{ml}^-.$$
(2.3.109)

Together with $\Omega(T(z)) = T(1/z)$ this gives the inversion symmetry for the quasi-momenta

$$p_{\pm l}\left(\frac{1}{z}\right) = S_{lm}^{\mp} p_{\mp m}(z).$$
 (2.3.110)

For the spectral representation of the quasi-momenta these relations imply

$$\kappa_l^{\pm} = S_{lk}^{\mp} \kappa_k^{\mp}, \qquad m_l^{\pm} = S_{lk}^{\mp} \kappa_k^{\mp}, \qquad 2\pi m_l^{\pm} = \mp \int_{C_l} \mathrm{d}y \, \frac{\rho^{\pm}(y)}{y} \pm S_{lk}^{\mp} \int_{C_k} \mathrm{d}y \, \frac{\rho_k^{\mp}(y)}{y} \tag{2.3.111}$$

$$p_{\pm l}(z) = \frac{\frac{z}{\kappa}(2\pi q m_l^{\pm} \pm \kappa_l^{\pm}) \pm 2\pi m_l^{\pm}}{(z \mp s)(z \pm s^{-1})} + \int_{C_l} \mathrm{d}y \, \frac{\rho_l^{\pm}(y)}{z - y} - S_{lk}^{\mp} \int_{C_k} \frac{\mathrm{d}y}{y^2} \, \frac{\rho_k^{\mp}(y)}{z - y}.$$
 (2.3.112)

The monodromy condition (2.3.104) then gives the finite-gap equations

$$A_{lk}^{\pm} \oint_{C_k} \mathrm{d}y \frac{\rho_k^{\pm}(y)}{z - y} - A_{lk}^{\pm} S_{kn}^{\mp} \int_{C_n} \frac{\mathrm{d}y}{y^2} \frac{\rho_n^{\mp}(y)}{z - 1/y} = -A_{lk}^{\pm} \frac{(2\pi q m_k^{\pm} \pm \kappa_k^{\pm}) \frac{z}{\kappa} \pm 2\pi m_k^{\pm}}{(z \mp s)(z \pm 1/s)} \pm 2\pi n_{l,i}^{\pm}.$$
 (2.3.113)

The Virasoro constraints do not change in the presence of the Wess-Zumino term and for coset fields are given by

$$(\kappa_l^+ \pm 2\pi m_l^+) A_{lk}^+ (\kappa_k^+ \pm 2\pi m_k^+) = 0, \qquad (\kappa_l^- \pm 2\pi m_l^-) A_{lk}^- (\kappa_k^- \pm 2\pi m_k^-) = 0.$$
(2.3.114)

For superalgebras the choice of simple roots is not unique, which results in different possible inequivalent Dynkin diagrams and Cartan matrices. At the level of the finite-gap equations all these grading choices are equivalent. However, matching the finite-gap equations with the semiclassical limit of Bethe ansatz equations leads the preferred choice of opposite gradings for the two copies of $\mathfrak{psu}(1,1|2)$

$$A^{\pm} = \begin{pmatrix} 0 & \mp 1 & 0 \\ \mp 1 & \pm 2 & \mp 1 \\ 0 & \mp 1 & 0 \end{pmatrix}, \qquad S^{\pm} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$
 (2.3.115)

In the fundamental representation, i.e. with the quasi-momenta defined as the eigenvalues of the monodromy matrix, the finite-gap equations reduce to the six relations

$$p_{+1}^A - p_{+1}^S = 2\pi n_1^+, \qquad p_{-2}^S - p_{-2}^A = 2\pi n_3^-,$$
 (2.3.116)

$$p_{+1}^{S} - p_{+2}^{S} = 2\pi n_{2}^{+}, \qquad p_{-2}^{A} - p_{-1}^{A} = 2\pi n_{2}^{-}, \qquad (2.3.117)$$

$$p_{+2}^S - p_{+2}^A = 2\pi n_3^+, \qquad p_{-1}^A - p_{-1}^S = 2\pi n_1^-.$$
 (2.3.118)

For fluctuations along directions on $AdS_3 \times S^3$ there are now 8 physical world-sheet excitations, which correspond to microscopic cuts connecting the following pairs of sheets (as shown in figure 2.2)

$$S^{3}: (p_{-1}^{S}, p_{-2}^{S}), (p_{+2}^{S}, p_{+1}^{S}),$$
(2.3.119)

$$AdS_3: (p_{-1}^A, p_{-2}^A), (p_{+2}^A, p_{+1}^A),$$
(2.3.120)

fermionic:
$$(p_{-1}^A, p_{-2}^S), (p_{-1}^S, p_{-2}^A), (p_{+2}^A, p_{+1}^S), (p_{+2}^S, p_{+1}^A).$$
 (2.3.121)



Figure 2.2: The left and right figures show the sheets of the quasi-momenta in the gradings for the two copies of $\mathfrak{psu}(1,1|2)$. The ± 1 gradings are characterised by the middle nodes, which in the left and right figures correspond to $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$. The blue and red wiggly lines show which sheets are connected by the poles for the eight bosonic and fermionic excitations. Polarisations that cross the dashed lines next to the fermionic nodes an odd number of times are fermionic.

2.3.4 String motion on $\mathbb{R} \times S^3$

In this section we briefly review classical bosonic string motion on the $\mathbb{R} \times S^3$ subspace of the $AdS_3 \times S^3 \times M^4$ backgrounds with mixed flux. In particular we make the connection between the group space σ -model and the bosonic string action in terms of embedding coordinates.

In general bosonic string motion on a curved background is described by the Polyakov action

$$S = -\frac{h}{2} \int d^2\sigma \left[\sqrt{-\gamma} \gamma^{ab} G_{MN}(Y) + \epsilon^{ab} B_{MN}(Y) \right] \partial_a Y^M \partial_b Y^N, \qquad (2.3.122)$$

where Y^M , M = 0, ..., 9 are the string embedding coordinates, G_{MN} is the target space metric, B_{MN} is the antisymmetric tensor for the NS-NS flux and γ_{ab} is the 2d world-sheet metric. The target-space metric G_{MN} and world-sheet metric γ_{ab} are assumed to have signatures (-, +, ..., +) and (-, +) respectively.

For string motion on the $\mathbb{R} \times S^3 \subset AdS_3 \times S^3 \times T^4$ subspace with embedding coordinates $(t, X_1, ..., X_4)$ in $\mathbb{R}^{1,4}$ the action takes the form

$$S = -\frac{h}{2} \int d^2 \sigma \left[\sqrt{-\gamma} \gamma^{ab} (\partial_a X \cdot \partial_b X - \partial_a t \partial_b t) + \Lambda (X^2 - 1) \right] + S_{WZ}, \qquad (2.3.123)$$

$$S_{WZ} = -\frac{h}{2} \int d^2 \sigma \, \epsilon^{ab} B_{ij} \partial_a X^i \partial_b X^j, \qquad i, j = 1, .., 4, \qquad (2.3.124)$$

where Λ is a Lagrange multiplier restricting string motion to S³ and we use the shorthand notation

$$X^2 \equiv X^i X^i, \qquad X \cdot Y \equiv X^i Y^i, \qquad i = 1, .., 4.$$
 (2.3.125)

The Kalb-Ramond 2-form B_{MN} is a potential for the 3-form field strength H = dB. The 3-form H must be proportional to the volume form since it is the only non-trivial 3-form on S³

$$\mathrm{d}B = q \operatorname{vol}(\mathrm{S}^3). \tag{2.3.126}$$

Using Stokes theorem the B-field contribution S_{WZ} can be written in terms of the embedding coordi-

nates as a Wess-Zumino term, which in this case is given by the 3-dimensional integral

$$S_{WZ} = -hq \int d^3\sigma \, \frac{1}{3} \epsilon^{abc} \epsilon_{ijkl} X_i \partial_a X_j \partial_b X_k \partial_c X_l.$$
(2.3.127)

Varying the action with respect to the world-sheet metric gives the Virasoro constraints

$$T^{ab} = 0, \qquad T^{ab} = \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{ab}} = \frac{h}{2} \left(g^{ab} - \frac{1}{2} \gamma^{ab} \gamma_{cd} g^{cd} \right), \tag{2.3.128}$$

$$g_{ab} \equiv G_{MN} \partial_a Y^M \partial_b Y^N = \partial_a X \cdot \partial_b X - \partial_a t \partial_b t, \qquad (2.3.129)$$

where T^{ab} is the 2d energy-momentum tensor. These constraints are independent of q since the worldsheet metric does not appear in the Wess-Zumino term. The equations of motion for X_i and t are given by

$$\partial_a (\sqrt{-\gamma}\gamma^{ab}\partial_b X_i) + \sqrt{-\gamma}\gamma^{ab}(\partial_a X \cdot \partial_b X)X_i + qK_i = 0, \qquad X^2 = 1$$

$$\partial_a (\sqrt{-\gamma}\gamma^{ab}\partial_b t) = 0.$$
(2.3.130)
$$(2.3.131)$$

$$\partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b t) = 0. \tag{2.3.131}$$

Here $K_i = \epsilon^{ab} \epsilon_{ijkl} X_j \partial_a X_k \partial_b X_l$ and we have used that $\epsilon_{ijkl} \delta X_i \partial_a X_j \partial_b X_k \partial_c X_l = 0$ since δX and ∂X are orthogonal to X, which follows from $X^2 = 1$, and thus only span a 3-dimensional space. Alternatively since $S^3 \simeq SU(2)$ this string model can be written as a principal chiral model with a Wess-Zumino term, which is a group space σ -model on SU(2)

$$S = \frac{h}{2} \Big[\int d^2 \sigma \sqrt{-\gamma} \gamma^{ab} \Big(\frac{1}{2} tr[J_a J_b] + \partial_a t \partial_b t \Big) + q \int d^3 \sigma \frac{1}{3} \epsilon^{abc} tr[J_a J_b J_c] \Big], \qquad J_a = g^{-1} \partial_a g, \quad (2.3.132)$$

where $g \in SU(2)$ and $J \in \mathfrak{su}(2)$. This action can also be written in a more compact notation

$$S = \frac{h}{2} \left[\int d^2 \sigma \left(\frac{1}{2} \operatorname{tr}[J \wedge *J] + dt \wedge \star dt \right) + \frac{q}{3} \int d^3 \sigma \operatorname{tr}[J \wedge J \wedge J] \right], \qquad J = g^{-1} dg, \qquad (2.3.133)$$

where $J = J_{\tau} d\tau + J_{\sigma} d\sigma$. We can explicitly parametrise SU(2) in terms of the embedding coordinates as, for example,

$$g = \begin{pmatrix} Z_1 & Z_2 \\ -Z_2^* & Z_1^* \end{pmatrix}, \quad Z_1 = X_1 + iX_2, \quad Z_2 = X_3 + iX_4, \quad |Z_i|^2 = 1.$$
(2.3.134)

Substituting this parametrisation into the PCM action (2.3.132) one recovers the Polyakov action (2.3.123). The equations of motion and Virasoro constraints now take the form

$$d \star dt = 0,$$
 (2.3.135)

$$d \star J - qJ \wedge J = 0, \qquad dJ - J \wedge J = 0, \qquad (2.3.136)$$

$$T^{ab} = 0, \qquad g_{ab} = -\frac{1}{2} \operatorname{tr}[J_a J_b] - \partial_a t \partial_b t. \qquad (2.3.137)$$

In order to determine the Lax connection we can make the ansatz

$$L = \alpha(z)J + \beta(z) \star J. \tag{2.3.138}$$

Requiring that this connection is flat gives

$$0 = (\alpha - \alpha^2 + q\beta - \beta^2)J \wedge J, \qquad (2.3.139)$$

where we used the equations of motion (2.3.136) as well as the following identities for 1-forms in two dimensions

$$\star \star = 1, \qquad a \wedge \star b = - \star a \wedge b. \tag{2.3.140}$$

The equation (2.3.139) can be solved for $\beta = (-q \pm \sqrt{q^2 - 4\alpha + 4\alpha^2})/2$. However, the Lax connection should only have simple poles and we would like it to be a rational function in z. Therefore we need to choose a parametrisation such that $q^2 - 4\alpha + 4\alpha^2$ gives a square. One possible solution is $\alpha(x) = (1 - qf(z))/(1 - f(z)^2)$ where f(z) is an arbitrary rational function³. For example, a convenient choice is

$$\alpha(z) = \frac{1 - qz}{1 - z^2}, \qquad \beta(z) = \frac{z - q}{1 - z^2}, \qquad L = \frac{1 - qz}{1 - z^2}J + \frac{z - q}{1 - z^2} \star J. \tag{2.3.141}$$

Global symmetries

The action for string motion on $\mathbb{R} \times S^3$, in the form of (2.3.123) or (2.3.132), is invariant under constant shifts of the target space time coordinate t. This gives the Noether charge for spacetime energy

$$E = h \int_0^{2\pi} d\sigma \,\partial_\tau t. \tag{2.3.142}$$

The principal chiral model (2.3.132) also has a global $SU(2)_L \times SU(2)_R$ symmetry

$$g \to U_L g U_R. \tag{2.3.143}$$

The associated Noether charges are given by

SU(2)_L:
$$Q_L = \frac{h}{2} \int \star j_L, \quad j_L = K + q \star K, \quad K = dgg^{-1} = gJg^{-1}$$
 (2.3.144)

SU(2)_R:
$$Q_R = \frac{h}{2} \int \star j_R$$
, $j_R = J - q \star J$, $J = g^{-1} dg$. (2.3.145)

Expanding the Lax parametrisation (2.3.141) around $z \to \infty$ we obtain the Noether currents

$$L = -\frac{1}{z} \star j_R + ..., \qquad j_R = J - q \star J, \qquad z \to \infty.$$
 (2.3.146)

Let us note that the action (2.3.132) is written in terms of the SU(2)_R current J, which is invariant under SU(2)_L but under SU(2)_R transforms non-trivially as

$$J \to U_R^{-1} J U_R.$$
 (2.3.147)

Equivalently the action can also be written in terms of the $SU(2)_L$ current K.

³This is not a unique solution. Another possible solution is $\alpha = \frac{1}{2} \left(1 - \sqrt{1 - q^2} \frac{1 + f(z)^2}{1 - f(z)^2}\right)$.

Gauge fixing

String σ -model actions are invariant under world-sheet reparametrisations

$$(\tau, \sigma) \to (\tilde{\tau}, \tilde{\sigma}),$$
 (2.3.148)

which is a world-sheet gauge symmetry. The target space fields transform as world-sheet scalars and the world-sheet metric transforms as a world-sheet tensor

$$Y^{M}(\tau,\sigma) \to \tilde{Y}^{M}(\tilde{\tau},\tilde{\sigma}) = Y^{M}(\tau,\sigma), \qquad (2.3.149)$$

$$\gamma^{ab}(\tau,\sigma) \to \tilde{\gamma}_{ab}(\tilde{\tau},\tilde{\sigma}) = \frac{\partial \sigma^c}{\partial \tilde{\sigma}^a} \frac{\partial \sigma^a}{\partial \tilde{\sigma}^b} \gamma_{cd}(\tau,\sigma).$$
(2.3.150)

For infinitesimal diffeomorphisms $\sigma^a \to \sigma^a - \epsilon^a(\tau, \sigma)$ with

$$\delta Y^M = \epsilon^a \partial_a Y^M, \qquad \delta \gamma_{ab} = \nabla_a \epsilon_b + \nabla_b \epsilon_a \tag{2.3.151}$$

this world-sheet symmetry gives the Noether currents

$$j^a = 2\epsilon_b T^{ab}, \qquad \nabla_a j^a = 0. \tag{2.3.152}$$

For global diffeomorphisms, i.e. constant ϵ_b , this reduces to the conservation of the energy-momentum tensor with world-sheet energy and momentum as the corresponding charges

$$\nabla_a T^{ab} = 0, \qquad P_a = \int \mathrm{d}\sigma \, T^0{}_a. \tag{2.3.153}$$

Since ϵ_b is an arbitrary function of the world-sheet coordinates the remaining condition from (2.3.152) implies that the energy-momentum tensor vanishes, $T^{ab} = 0$, which is consistent with the Virasoro constraints. Therefore the world-sheet energy and momentum vanish on shell

$$P_a = 0. (2.3.154)$$

Another gauge symmetry of string σ -model actions is the invariance under Weyl transformations

$$\gamma_{ab} \to e^{2\phi(\tau,\sigma)}\gamma_{ab}.\tag{2.3.155}$$

The topology of the string world-sheet is that of a sphere. Therefore all metrics are conformally equivalent. In particular the reparametrisation invariance can be used to fix conformal gauge, i.e. to bring the world-sheet metric into a conformally flat form

$$\sqrt{-\gamma}\gamma^{ab} = \eta^{ab} \equiv \operatorname{diag}(-1, +1). \tag{2.3.156}$$

Any remaining overall scale factor of the metric can be further eliminated using Weyl invariance. This still leaves a residual invariance of the action under diffeomorphisms that only affect the scale factor of the metric, i.e. *conformal transformations* of the form

$$(\tau, \sigma) \to (\tilde{\tau}, \tilde{\sigma}), \qquad \gamma_{ab} \to \tilde{\gamma}_{ab} = \Omega \gamma_{ab}.$$
 (2.3.157)

In two dimensions they form an infinite dimensional subgroup of diffeomorphisms with an infinite number of Noether currents

$$j^a = 2\epsilon_b T^{ab}, \tag{2.3.158}$$

where ϵ_b is a solution of the conformal Killing equation

$$\mathcal{L}_{\epsilon}\gamma = (\Omega - 1)\gamma, \quad \Leftrightarrow \quad \nabla_{a}\epsilon_{b} + \nabla_{b}\epsilon_{a} = \nabla_{c}\epsilon^{c}\gamma_{ab}, \qquad \Omega - 1 = \nabla_{c}\epsilon^{c}. \tag{2.3.159}$$

This residual gauge freedom can be used to further fix the gauge. For this it is convenient to introduce world-sheet light-cone coordinates

$$\sigma^{\pm} = \frac{\tau \pm \sigma}{2}, \qquad \partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}, \qquad (2.3.160)$$

such that the world-sheet metric takes the form

$$\mathrm{d}s^2 = -4\mathrm{d}\sigma^+\mathrm{d}\sigma^-.\tag{2.3.161}$$

Conformal transformations preserve this form and are therefore given by

$$\sigma^+ \to \tilde{\sigma}^+ = f^+(\sigma^+), \qquad \sigma^- \to \tilde{\sigma}^- = f^-(\sigma^-), \qquad (2.3.162)$$

where $f^{\pm}(\sigma^{\pm})$ are arbitrary invertible functions. In these coordinates the equation of motion for t takes the form

$$\partial_+\partial_- t = 0, \tag{2.3.163}$$

which has the general solution

$$t = t^{+}(\sigma^{+}) + t^{-}(\sigma^{-}).$$
(2.3.164)

Applying a conformal transformation allows one to fix static gauge by bringing t into the form

$$t = \kappa \tau. \tag{2.3.165}$$

Here the constant κ is related to the target-space energy of a string solution

$$E = h \int d\sigma \, \dot{t} = \sqrt{\lambda} \kappa. \tag{2.3.166}$$

The bosonic string action simplifies in conformal gauge to

$$S = \frac{\mathrm{h}}{2} \int \mathrm{d}\sigma^{+} \mathrm{d}\sigma^{-} \left(G_{MN} + B_{MN}\right) \partial_{+} Y^{M} \partial_{-} Y^{N}.$$
(2.3.167)

In the case of string motion on $\mathbb{R}\times \mathrm{S}^3$ this gives the action

$$S = -\frac{h}{2} \Big[\int d^2 \sigma \left[\partial_+ X \cdot \partial_- X + \Lambda (X^2 - 1) \right] - q \int d^3 \sigma \, \frac{1}{3} \epsilon^{abc} \epsilon_{ijkl} X_i \partial_a X_j \partial_b X_k \partial_c X_l \Big], \qquad (2.3.168)$$

with the equations of motion and Virasoro constraints

$$\partial_{+}\partial_{-}X_{i} + (\partial_{+}X \cdot \partial_{-}X)X_{i} + qK_{i} = 0, \qquad K_{i} = \epsilon_{ijkl}X_{j}\partial_{+}X_{k}\partial_{-}X_{l}, \qquad X^{2} = 1, \qquad (2.3.169)$$

$$(\partial_{\pm}X)^2 = \kappa^2.$$
 (2.3.170)

In the case of the principal chiral model with a Wess-Zumino term the action becomes

$$S = -\frac{\mathrm{h}}{2} \left[\int \mathrm{d}^2 \sigma \, \frac{1}{2} \mathrm{tr}(J_+ J_-) - q \int \mathrm{d}^3 \sigma \, \frac{1}{3} \epsilon^{abc} \mathrm{tr}(J_a J_b J_c) \right],\tag{2.3.171}$$

with the equations of motion and Virasoro constraints

$$\partial_{\pm} J_{\mp} \pm \frac{1}{2} (1 \pm q) [J_{+}, J_{-}] = 0, \quad \operatorname{tr}(J_{\pm}^{2}) = -2\kappa^{2}.$$
 (2.3.172)

The Lax connection (2.3.141) now takes the simple form

$$L_{\pm} = \frac{1 \pm q}{1 \pm z} J_{\pm}.$$
 (2.3.173)

Solving the Virasoro constraints by defining the unit vectors \vec{S}_{\pm} and parametrising J_{\pm} as

$$J_{\pm} = i\kappa \vec{S}_{\pm} \cdot \vec{\sigma}, \qquad (2.3.174)$$

where $\vec{\sigma}$ is the vector of Pauli matrices, the equations of motion become

$$\partial_{+}S_{-}^{i} - \kappa(1+q)\varepsilon^{ijk}S_{+}^{j}S_{-}^{k} = 0, \quad \partial_{-}S_{+}^{i} + \kappa(1-q)\varepsilon^{ijk}S_{+}^{j}S_{-}^{k} = 0.$$
(2.3.175)

These equations follow from the Faddeev-Reshetikhin action [161]

$$S = \int d^2 \sigma [(1-q)C_+(\vec{S}_-) + (1+q)C_-(\vec{S}_+) - \frac{1}{2}(1-q^2)\kappa \vec{S}_+ \cdot \vec{S}_-], \qquad (2.3.176)$$

where C_\pm is a Wess-Zumino-type term of the form

$$C_{\pm} \equiv -\frac{1}{2} \int_{0}^{1} \mathrm{d}\xi \,\varepsilon^{ijk} S_i \partial_{\xi} S_j \partial_{\pm} S_k, \quad \delta C_{\pm} = \frac{1}{2} \varepsilon^{ijk} \delta S_i S_j \partial_{\pm} S_k + \partial_{\pm} \chi \tag{2.3.177}$$

$$\vec{S}_{\pm}(\xi = 1) = \vec{S}_{\pm}, \qquad \vec{S}_{\pm}(\xi = 0) = \vec{S}_{0\pm} = \text{const.}$$
 (2.3.178)

Now transforming the world-sheet coordinates as

$$\sigma^{\pm} \to \tilde{\sigma}^{\pm} = (1 \pm q)\sigma^{\pm} \tag{2.3.179}$$

maps the $q \neq 0$ FR action to the q = 0 action

$$S = \int d^2 \tilde{\sigma} [\tilde{C}_+(\vec{S}_-) + \tilde{C}_-(\vec{S}_+) - \frac{1}{2}\kappa \vec{S}_+ \cdot \vec{S}_-].$$
(2.3.180)

In part II of this thesis we will use the map (2.3.179) to construct $q \neq 0$ generalisations of q = 0 string solutions.

2.3.5 Finite-gap equations for strings on $\mathbb{R} \times S^3$

The monodromy matrix

$$T(z) = \mathcal{P} \exp \int_0^{2\pi} \mathrm{d}\sigma \, \frac{1}{2} \Big(\frac{1+q}{1+z} J_+ - \frac{1-q}{1-z} J_- \Big)$$
(2.3.181)

is a unimodular 2×2 matrix with the eigenvalues $e^{\pm ip(z)}$ and the quasi-momentum p(z) given by

$$\operatorname{tr} T(z) = 2\cos p(z).$$
 (2.3.182)

The Dynkin diagram for SU(2) consists of a single node with the Cartan matrix and inversion symmetry matrix given by

$$A = 2, \qquad S = -1. \tag{2.3.183}$$

The Lax connection (2.3.173) can be diagonalised near the poles $z = \pm 1$ by a gauge transformation which is regular in z and periodic in σ (see e.g. chapter 3.8 in [151])

$$L(z) \to L^{\text{diag}}(z) = g^{\pm}(z)L(z)g^{\pm}(z)^{-1} - \mathrm{d}g^{\pm}(z)g^{\pm}(z)^{-1}, \qquad g^{\pm}(z) \in \mathrm{SU}(2), \tag{2.3.184}$$

$$L^{\text{diag}}(z) = \sum_{n=-1}^{\infty} L_n^{\pm} (1 \pm z)^n, \qquad g^{\pm} = \sum_{n=0}^{\infty} g_n^{\pm} (\tau, \sigma) (1 \pm z)^n.$$
(2.3.185)

In this abelian gauge the flatness of L^{\pm} reduces to $dL_n^{\pm} = 0$ and the coefficients L_n^{\pm} give an infinite set of local conserved quantities

$$Q_n^{\pm} = \int_{\gamma} L_n^{\pm}, \qquad n = -1, 0, \dots.$$
 (2.3.186)

Their conservation follows from Stokes theorem

$$\int_{\gamma_1} L_n^{\pm} - \int_{\gamma_2} L_n^{\pm} = \int_D \mathrm{d}L_n^{\pm} = 0, \qquad (2.3.187)$$

where γ_1 and γ_2 are closed paths corresponding to the boundary of D. The Q_n^{\pm} are diagonal elements of $\mathfrak{su}(2)$. They are therefore proportional to the Pauli matrix $\sigma_3 = \operatorname{diag}(+1, -1)$, which gives the conserved charges

$$Q_n^{\pm} = \frac{1}{2i} \operatorname{tr}(\mathcal{Q}_n^{\pm} \sigma_3).$$
(2.3.188)

The mondoromy matrix then takes the diagonal form

$$T(z) \to T^{\text{diag}}(z) = g^{\pm}(z)T(z)g^{\pm}(z)^{-1} = \exp\left[\sum_{n=-1}^{\infty} \mathcal{Q}_n^{\pm}(1\pm z)^n\right]$$
 (2.3.189)

with the quasi-momentum

$$p(z) = \sum_{n=-1}^{\infty} Q_n^{\pm} (1 \pm z)^n.$$
(2.3.190)

In particular the Virasoro constraints determine the residues at $z = \pm 1$ of the quasi-momentum, i.e. Q_{-1}^{\pm} . Expanding the σ -component of the Lax connection (2.3.173) in the abelian gauge we have

$$L_{\sigma} = \pm \frac{1}{2} \frac{1 \pm q}{1 \pm z} J_{\pm}^{\text{diag}} + \dots, \qquad z \to \pm 1.$$
 (2.3.191)

Since $J^{\text{diag}}_{\pm} \in \mathfrak{su}(2)$ it is proportional to σ_3 and the Virasoro constraints $\text{tr} J^2_{\pm} = -2\kappa^2$ then give

$$J_{\pm}^{\text{diag}} = i\kappa\sigma_3. \tag{2.3.192}$$

The residues at $z = \pm 1$ of the quasi-momentum are then found from the diagonal monodromy matrix

$$T^{\text{diag}} = \exp\left[\pm \frac{1\pm q}{1\pm z}i\pi\kappa\sigma_3 + \ldots\right], \qquad z \to \mp 1.$$
(2.3.193)

We can therefore write the quasi-momentum in the spectral representation as

$$p(z) = G(z) + \pi \frac{E}{\sqrt{\lambda}} \Big[\frac{1+q}{1+z} - \frac{1-q}{1-z} \Big], \qquad G(z) = \int_C \mathrm{d}y \, \frac{\rho(y)}{z-y} + \int_C \frac{\mathrm{d}y}{y^2} \frac{\rho(y)}{z-\frac{1}{y}}, \tag{2.3.194}$$

where G(z) is the *resolvent*. The discontinuity relation (2.3.37) for the quasi-momentum reads

$$p(z+i\epsilon) + p(z-i\epsilon) = 2\pi n_i.$$
(2.3.195)

This gives the finite-gap integral equation

$$G(z+i\epsilon) + G(z-i\epsilon) = 2 \oint dy \left(\frac{\rho(y)}{z-y} + \frac{1}{y^2} \frac{\rho(y)}{z-\frac{1}{y}}\right) = 2\pi n_i - \frac{2\pi E}{\sqrt{\lambda}} \left[\frac{1+q}{1+z} - \frac{1-q}{1-z}\right], \quad z \in C_i.$$
(2.3.196)

2.3.6 BMN string on $\mathbb{R} \times S^3$ and $AdS_3 \times S^3 \times T^4$ with mixed flux

As a simple example let us consider the point-like BMN string. It has the same form on pure R-R and mixed flux backgrounds since its coordinates do not depend on σ

$$Z_1 = e^{i\kappa\tau}, \qquad Z_2 = 0, \qquad \mathcal{E} = \frac{E}{\sqrt{\lambda}} = \kappa, \qquad J_{\pm} = J_{\tau} \pm J_{\sigma} = i\kappa \begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix}. \tag{2.3.197}$$

For this solution the monodromy matrix (2.3.181) is found to have the eigenvalues

$$e^{\pm ip(z)}, \qquad p(z) = \pi \mathcal{E}\Big[\frac{1+q}{1+z} - \frac{1-q}{1-z}\Big].$$
 (2.3.198)

In other words the BMN string corresponds to a quasi-momentum with the two simple poles and no additional cuts. In the case of $AdS_3 \times S^3 \times T^4$ with mixed flux the BMN string gives the flat connection

$$L_{\sigma} = \frac{i}{\hat{q}} \frac{\mathcal{E}z}{(z-s)(z+\frac{1}{s})} \operatorname{diag}(1,-1,1,-1) \oplus \frac{i}{\hat{q}} \frac{\mathcal{E}z}{(z+s)(z-\frac{1}{s})} \operatorname{diag}(-1,1,-1,1), \quad (2.3.199)$$

$$\hat{q} = \sqrt{1 - q^2}.$$
 (2.3.200)

This gives the diagonalised monodromy matrix in terms of the quasi-momenta

$$T = \operatorname{diag}\left(e^{ip_{+1}^{A}}, e^{ip_{+2}^{A}} \left| e^{ip_{+1}^{S}}, e^{ip_{+2}^{S}} \right| \left| e^{ip_{-1}^{A}}, e^{ip_{-2}^{A}} \right| e^{ip_{-1}^{S}}, e^{ip_{-2}^{S}}\right)$$
(2.3.201)

$$= \operatorname{diag}\left(e^{ip(z)}, e^{-ip(z)} | e^{ip(z)}, e^{-ip(z)} | | e^{ip(1/z)}, e^{-ip(1/z)} | e^{ip(1/z)}, e^{-ip(1/z)}\right)$$
(2.3.202)

$$p(z) = \frac{2\pi \mathcal{E}z}{\sqrt{1 - q^2}(z - s)(z + \frac{1}{s})}.$$
(2.3.203)

2.3.7 Circular strings on $AdS_5 \times S^5$

Another simple example is the circular string solution moving on the $\mathbb{R} \times S^3$ subspace with $\mathbb{R} \subset AdS_5$, $S^3 \subset S^5$, equal angular momenta and equal winding numbers

$$E = \sqrt{\lambda}\mathcal{E}, \quad J_1 = \sqrt{\lambda}\mathcal{J}_1, \quad J_2 = \sqrt{\lambda}\mathcal{J}_2, \qquad \mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}/2, \qquad m_1 = -m_2 = m \qquad (2.3.204)$$

$$Z_{1} = \frac{1}{\sqrt{2}} e^{i(\omega\tau + m\sigma)}, \qquad Z_{2} = \frac{1}{\sqrt{2}} e^{i(\omega\tau - m\sigma)}, \quad m^{2} + \omega^{2} = \kappa^{2}, \quad \mathcal{J} = \omega, \quad \mathcal{E} = \kappa.$$
(2.3.205)

The quasi-momenta on the AdS subspace have only the two simple poles (in analogy to the BMN string solution)

$$\hat{p}_1 = \hat{p}_2 = -\hat{p}_3 = -\hat{p}_4 = \frac{2\pi\kappa z}{z^2 - 1}, \qquad \kappa = \mathcal{E} = \sqrt{\mathcal{J}^2 + m^2}.$$
 (2.3.206)

Choosing a coset parametrisation in terms of Z_1 and Z_2 and evaluating the Lax connection one finds that the quasi-momenta on the S⁵ subspace have a single cut connecting the sheets of \tilde{p}_2 and \tilde{p}_3 [153, 160]

$$\tilde{p}_1 = -\tilde{p}_4 = \frac{z}{z^2 - 1} K(1/z), \quad \tilde{p}_2 = -\tilde{p}_3 = \frac{z}{z^2 - 1} K(z) - m, \quad K(x) \equiv \sqrt{\mathcal{J}^2 + m^2 x^2}.$$
(2.3.207)

Chapter 3

(Non-)integrability of classical strings on p-brane backgrounds

In this chapter, based on [1], we investigate integrability of classical string motion on brane backgrounds. The equations of motion for strings on curved backgrounds are non-linear as they are described by σ -models. Ideally we would like to have a detailed description of their dynamics at the classical and quantum level to a degree similar to flat spacetime, for which the equations of motion are linear. This is indeed possible for certain backgrounds that lead to integrable string σ -models. The most prominent example is given by superstring theory on the maximally symmetric AdS₅ × S⁵ spacetime, for which much progress towards a quantitative description of the quantum spectrum has been achieved due to its integrability [162].

One may wonder if integrability can also apply to string motion on closely related but less symmetric *p*-brane backgrounds [163, 164, 165], such as the D3-brane background [165, 166], which is a 1-parameter (D3-brane charge Q) interpolation between flat space (Q = 0) and the AdS₅ × S⁵ space ($Q = \infty$). While string theory is integrable in these two limiting cases, what happens at finite Q is a priori an open question. This is the question we are going to address in this chapter.

A first step towards establishing quantum integrability of a bosonic model or integrability of its superstring counterpart is the study of classical integrability of the corresponding bosonic σ -model. Following the same method as used previously in [146, 167, 147] we will demonstrate non-integrability of classical extended string motion on the D3-brane background (particle, i.e. geodesic, motion is still integrable). We will also reach similar conclusions for other *p*-brane backgrounds that interpolate between flat space and integrable $\operatorname{AdS}_n \times \operatorname{S}^k$ backgrounds.

The main cases include the D3-brane, the D5-D1 background and the four D3-brane intersection that interpolate between flat space and $AdS_5 \times S^5$, $AdS_3 \times S^3 \times T^4$ and $AdS_2 \times S^2 \times T^6$ respectively. While geodesic motion on these backgrounds is integrable, making a special ansatz for string motion we will find that the resulting 1d system of equations is not integrable by applying the variational non-integrability technique for Hamiltonian systems as used in [147] (section 3.3).

In section 3.1 we first give a brief summary of the ideas from differential Galois theory on which the non-integrability techniques for Hamiltonian systems are based on. In section 3.2 we then illustrate these techniques on a simple example of the non-integrable two-particle system with the potential

 $V(x_1, x_2) = \frac{1}{2}x_1^2 x_2^2.$

In section 3.4 we study the NS5-F1 background and show that integrability is absent for generic values of the two charges Q_1 and Q_5 , but it is restored in the limit when $Q_5 \to \infty$. In section 3.5 we also demonstrate non-integrability of string propagation on the pp-wave geometry T-dual to the fundamental string background [168, 130].

3.1 Integrability and differential Galois theory

Integrability in the sense of Liouville implies that the equations of motion are solvable in quadratures, i.e. the solutions are combinations of integrals of rational functions, exponentials, logarithms and algebraic expressions. Solutions of this type are referred to as Liouvillian functions. Thus the question of integrability is related to the question of when differential equations admit only Liouvillian solutions. This is answered by differential Galois theory and the main integrability theorem, which our analysis relies on, can be stated as follows [169]:

Theorem Let H be a Hamiltonian defined on a phase space manifold M with an associated Hamiltonian vector field X. Let P be a submanifold of M which is invariant under the flow of X with the invariant curves Γ and let $X|_P$ be completely integrable. Also let G be the differential Galois group of the variational equation of the flow of $X|_P$ along Γ normal to P. If X is completely integrable on M then the largest connected algebraic subgroup of G which contains the identity is abelian.

Since differential Galois theory is not frequently used in physics we shall give a brief introduction and illustrate some basic concepts on a simple example below. We shall follow ref.[170]. First, let us give some basic definitions:

- 1. A differential field is a field F with a derivation D_F , which is an additive map $D_F: F \to F$ thus satisfying $D_F(ab) = D_F(a)b + aD_F(b)$, $\forall a, b \in F$.
- 2. A differential homomorphism/isomorphism between two differential fields F_1 and F_2 is a homomorphism/isomorphism $f: F_1 \to F_2$ which satisfies $D_{F_2}(f(a)) = f(D_{F_1}(a)), \forall a \in F_1$.
- 3. A linear differential operator L is defined by

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_0y$$
(3.1.1)

where $y^{(n)} = D_F^n(y)$ and $a_i \in F$.

- 4. A differential field E with derivation D_E is called a differential field extension of a differential field F with derivation D_F iff $E \supseteq F$ and the restriction of D_E to F coincides with D_F .
- 5. Elements $a \in F$ whose derivative vanishes are called constants of F. The subfield of constants of F is denoted by C_F .

In order to investigate the types of solutions a differential equation admits one has to encode the relations and symmetries of the independent solutions. Relations can be encoded in the Picard-Vessiot extension of a differential field which includes the solutions of L(y) = 0:

Definition An extension $E \supseteq F$ is called a Picard-Vessiot extension of F for L(y) = 0 iff:

E is generated over F as a differential field by the set of solutions of L(y) = 0 in E;

E and F have the same constants;

L(y) = 0 has n solutions in E linearly independent over the constants.

One can show that the Picard-Vessiot extension always exists for differential fields with an algebraically closed field of constants and this extension is unique up to isomorphisms.

Symmetries of the solution space are linear transformations which, applied to a solution, give a new solution. The simplest formulation is in terms of automorphisms of the Picard-Vessiot extension and leads to the notion of the differential Galois group:

Definition For a differential field F with the Picard-Vessiot extension $E \supseteq F$ the differential Galois group G(E/F) is the group of differential automorphisms $\sigma : E \to E$ whose restriction to F is the identity map, i.e.

$$G(E/F) = \{ \sigma : E \to E \, | \, \sigma(a) = a \quad \forall a \in F \}.$$

A solution y(x) of a linear differential equation L(y) = 0 is called

algebraic if it is a root of a polynomial over F; primitive if $y'(x) \in F$, i.e. $y(x) = \int^x f(z) dz$ for $f(x) \in F$; exponential if $\frac{y'(x)}{y(x)} \in F$, i.e. $y(x) = \exp\left(\int^x f(z) dz\right)$ with $f(z) \in F$.

Definition The solution y(x) is called *Liouvillian* if there exist differential extensions E_i , i = 0, ..., m

$$F = E_0 \subset E_1 \subset \ldots \subset E_m = E$$

such that $y(x) = y_m(x) \in E$, $E_i = E_{i-1}(y_i(x))$ and $y_i(x)$ is algebraic, primitive or exponential over E_{i-1} .

We will be interested in second order homogeneous linear ordinary differential equations

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0, (3.1.2)$$

where $a(x), b(x) \in F$ and $F = \mathbb{C}(x)$ is the differential field of rational functions over complex numbers. The equation (3.1.2) can be cast into the normal form

$$y''(x) + U(x) y(x) = 0 , \qquad (3.1.3)$$

$$U(x) = -\frac{1}{4}a^{2}(x) - \frac{1}{2}a'(x) + b(x) , \qquad U(x) \in F$$
(3.1.4)

by the transformation $y(x) \to y(x) \exp\left(-\frac{1}{2}\int^x a(z)dz\right)$. Let us identify the corresponding Galois group. Let *E* be a Picard-Vessiot extension of $F = \mathbb{C}(x)$ and $z_1(x), z_2(x) \in E$ be two independent solutions of (3.1.3). The Galois group G(E/F) consists of differential automorphisms σ which act on the space of solutions of (3.1.3), i.e. map solutions to solutions. Thus

$$\sigma(z_i(x)) = C_{ij} z_j(x) , \qquad i, j = 1, 2 .$$
(3.1.5)

where det C is the determinant of the C_{ij} matrix in (3.1.5). The Wronskian is a constant since $w'(x) = z_1(x)z_2''(x) - z_2(x)z_1''(x) = 0$ by virtue of (3.1.3), i.e. $w(x) \in F$. Therefore, $\sigma(w(x)) = w(x)$, giving det C = 1. Thus the Galois group for (3.1.3) is a subgroup of SL(2, \mathbb{C}). For more general equations the Galois group is a subgroup of GL(n, \mathbb{C}).

Let us now address the main question: whether a differential equation can be solved in terms of Liouvillian functions. Since the minimal differential extension generated by all solutions is the Picard-Vessiot extension we therefore require that the Picard-Vessiot extension is generated by a tower of Liouvillian extensions. By the fundamental theorem of differential Galois theory this tower of extensions is related to subgroups of the Galois group through a bijective correspondence. One can show that for the Picard-Vessiot extension to be Liouvillian, the *identity-connected component* of the Galois group G^0 has to be *solvable* (see theorem 25 in [171]).¹

A simple example is provided by the equation y''(x) + 2xy'(x) = 0 which has the normal form

$$y''(x) - (x^2 + 1)y(x) = 0. (3.1.7)$$

For this equation $F = \mathbb{C}(x)$ is the field of rational functions over complex numbers. The solution space may be represented as $\mathbb{C} e^{\frac{1}{2}x^2} \oplus \mathbb{C} e^{\frac{1}{2}x^2} \int e^{-x^2}$. The Picard-Vessiot extension is $E = \mathbb{C}(x, e^{\frac{1}{2}x^2}, e^{-x^2}, \int e^{-x^2})$ and the tower of extensions of E is

$$\mathbb{C}(x) \subset \mathbb{C}\left(x, e^{\frac{1}{2}x^2}\right) \subset \mathbb{C}\left(x, e^{\frac{1}{2}x^2}, e^{-x^2}\right) \subset E = \mathbb{C}\left(x, e^{\frac{1}{2}x^2}, e^{-x^2}, \int e^{-x^2}\right).$$

Then the Galois group G(E/F) and its identity component G^0 are

$$G = G^{0} = \left\{ \left(\begin{array}{cc} a & 0 \\ c & 1/a \end{array} \right), \quad a \neq 0, c \in \mathbb{C} \right\}$$
(3.1.8)

This is a solvable group since it is a subgroup of $SL(2, \mathbb{C})$ whose algebraic subgroups are all solvable except for $SL(2, \mathbb{C})$ itself.

3.2 On the non-integrability of classical Hamiltonian systems

Integrability of classical Hamiltonian systems is closely related to the behaviour of variations around phase space curves. Based on this observation, ref.[172] obtained necessary conditions for the existence of additional functionally independent integrals of motion. These conditions are given in terms of the monodromy group properties of the equations for small variations around phase space curves. Using differential Galois theory, refs.[173, 174, 145] improved these results by showing that integrability implies that the identity component of the differential Galois group of variational equations normal

¹A group is solvable if it has a subnormal series whose factor groups are all abelian, that is, if there are subgroups $\{1\} \leq G_1 \leq ... \leq G_n = G$ such that G_{k-1} is normal in G_k (i.e. invariant under conjugation) and G_k/G_{k-1} is an abelian group for all k = 1, ..., n.

to an integrable plane of solutions must be abelian (see the theorem in section 3.1). This necessary integrability condition allows one to show non-integrability from the properties of the Galois group. However, determining the Galois group can be difficult and therefore one usually takes a slightly different route in analysing the normal variational equation (NVE).

One considers a special class of solutions of the NVE which are functions of exponentials, logarithms, algebraic expressions of the independent variables and their integrals and which are known as Liouvillian solutions [175]. The existence of such solutions is equivalent to the condition that the identity component G^0 of the Galois group is solvable (see, for example, theorem 25 in [171]). This in turn implies that if the NVE does not admit Liouvillian solutions then G^0 is non-solvable (and thus non-abelian) leading to the conclusion of non-integrability.

For Hamiltonian systems the NVE is a second order linear homogeneous differential equation and for such equations with rational coefficients Liouvillian solutions can be determined by the Kovacic algorithm [176] which fails if and only if no such solutions exist. Therefore, whenever no Liouvillian solutions of the NVE are found by the Kovacic algorithm one can deduce non-integrability of a Hamiltonian system.

An important subtlety here is that we first need to algebrize the NVE, i.e. rewrite it as a differential equation with rational coefficients. This can be done by means of a Hamiltonian change of the independent variable and it was shown in [177] that the identity component of the Galois group is preserved under this procedure.

In summary, if the algebrized NVE is not solvable in terms of Liouvillian functions or, equivalently, if the component of the quadratic fluctuation operator normal to an integrable subsystem does not admit Liouvillian zero modes, the original system is non-integrable. This is consistent with the usual definition of integrability in the sense of Liouville which, for Hamiltonian systems, implies that the equations of motion should be solvable in quadratures, i.e. in terms of Liouvillian functions.

The main steps to prove non-integrability of a Hamiltonian system are thus:

- Choose an invariant plane of solutions.
- Obtain the NVEs, i.e. the variational equations normal to the invariant plane.
- Check that the algebrized NVEs have no Liouvillian solutions using the Kovacic algorithm.

This approach was recently used in [147, 148] to show the non-integrability of string motion in several curved backgrounds and we shall also follow it here.

Let us first explain what the algebraization procedure of the last step means. A generic NVE is a second order differential equation of the form

$$\ddot{\eta} + q(t)\dot{\eta} + r(t)\eta = 0$$
. (3.2.1)

A change of the variable $t \to x(t)$ is called Hamiltonian if x(t) is a solution of the Hamiltonian system $H(p, x) = \frac{p^2}{2} + V(x)$ where V(x) is some potential. Then x satisfies the first integral equation $\frac{1}{2}\dot{x}^2 + V(x) = h$, implying that \dot{x} (and also \ddot{x}) is a function of x only. Changing the variable $t \to x(t)$

in (3.2.1) we obtain (here $' \equiv \frac{d}{dx}$)

$$\eta'' + \left(\frac{\ddot{x}}{\dot{x}^2} + \frac{q(t(x))}{\dot{x}}\right)\eta' + \frac{r(t(x))}{\dot{x}^2}\eta = 0.$$
(3.2.2)

We should now require that the coefficients in this equation are rational functions of x.

Let us consider the following simple example of a Hamiltonian system with the canonical variables (x_i, p_i) (i = 1, 2) [178]

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}x_1^2x_2^2.$$
(3.2.3)

A choice of an invariant plane, referred to above, is $P = \{(x_1, x_2, p_1, p_2) : x_2 = p_2 = 0\}$ along which the solution curves are $x_1(\tau) = \kappa \tau + \text{const}, p_1 = \kappa, \kappa = \sqrt{2H}$. The system along the plane P with the coordinates (x_1, p_1) is integrable since the only required constant of motion is provided by the Hamiltonian.

To determine the direction normal to the solution curves of this integrable subsystem we can use the Hamiltonian vector field along the curves $X^{x_1} = \kappa$, $X^{p_1} = 0$, $X^{x_2} = 0$, $X^{p_2} = 0$. Thus the normal direction is along x_2 and p_2 and expanding around $x_2 = 0$ with $\delta x_2 \equiv \eta$ we obtain the following normal variational equation (NVE)

$$\eta''(\tau) = -\bar{x}_1^2(\tau) \,\eta(\tau), \qquad \bar{x}_1(\tau) \equiv \kappa \tau \;, \tag{3.2.4}$$

which is solved by the parabolic cylinder functions $D_{-1/2}((i-1)\kappa\tau)$ and $D_{-1/2}((i+1)\kappa\tau)$. These are not Liouvillian functions and this implies the *non-integrability* of the original Hamiltonian (3.2.3).

To illustrate the previously discussed algebraization procedure let us consider the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{8}(x_1^2 - x_2^2)^2$$
(3.2.5)

which is equivalent to (3.2.3) by the trivial canonical transformation

$$x_1 \to \frac{x_1 - x_2}{\sqrt{2}}, \quad x_2 \to \frac{x_1 + x_2}{\sqrt{2}}, \quad p_1 \to \frac{p_1 - p_2}{\sqrt{2}}, \quad p_2 \to \frac{p_1 + p_2}{\sqrt{2}}.$$
 (3.2.6)

Restricting to the invariant plane $P = \{(x_1, x_2, p_1, p_2) : x_2 = p_2 = 0\}$ we obtain an integrable subsystem

$$\dot{x}_1 = p_1, \qquad \dot{p}_1 = -\frac{1}{2}x_1^3.$$
 (3.2.7)

The Hamiltonian vector field on this solution plane is $X^{p_1} = -\frac{1}{2}\bar{x}_1^3$, $X^{p_2} = 0$, $X^{x_1} = \dot{x}_1$, $X^{x_2} = 0$. The NVE is now obtained by expanding the Hamiltonian equations for the coordinates normal to the invariant plane, i.e. x_2 and p_2 , along (3.2.7). Denoting $\delta x_2 = \eta$ we thus obtain

$$\ddot{\eta} = \frac{1}{2}\bar{x}_1^2(t)\eta$$
, $\bar{x}_1(t) = 2^{3/4}h^{1/4}\operatorname{sn}\left(\left(\frac{h}{2}\right)^{1/4}t, -1\right)$ (3.2.8)

where \bar{x}_1 is a solution of (3.2.7) representing the curves in P parametrised by the Hamiltonian h > 0.

We can algebrize this NVE by changing the variable $t \to x = \bar{x}_1(t)$. This leads to $(' = \frac{d}{dx})$

$$\eta'' - \frac{2x^3}{8h - x^4}\eta' - \frac{2x^2}{8h - x^4}\eta = 0.$$
(3.2.9)

Further changing the variable $\eta \rightarrow \xi = (8h - x^4)^{1/4}\eta$ one obtains the NVE in the normal form

$$\xi'' + \frac{8hx^2 + 2x^6}{(8h - x^4)^2}\,\xi = 0. \tag{3.2.10}$$

This equation is solved in terms of hypergeometric functions, i.e. it has no Liouvillian solutions. This implies non-integrability of (3.2.5), in agreement with the previous discussion of the equivalent system (3.2.3).

To summarise, proving non-integrability of a Hamiltonian system can be achieved by demonstrating that given an integrable subsystem defined by some invariant plane in the phase space, the corresponding normal variational equations are not integrable in quadratures and thus do not admit sufficiently many conserved quantities.

3.3 String motion on *p*-brane backgrounds

In what follows we shall consider classical bosonic string motion on a curved background. In the conformal gauge the action is

$$I = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \left[G_{MN}(Y)\partial_a Y^M \partial^a Y^N + \epsilon^{ab} B_{MN}(Y)\partial_a Y^M \partial_b Y^N \right], \qquad (3.3.1)$$

and the corresponding equations of motion should be supplemented by the Virasoro constraints

$$G_{MN} \dot{Y}^M \dot{Y}^N = 0 , \qquad (3.3.2)$$

$$G_{MN}(\dot{Y}^M \dot{Y}^N + \dot{Y}^M \dot{Y}^N) = 0. ag{3.3.3}$$

In this section we shall consider the case of $B_{MN} = 0$ and the target space metric given by the following *p*-brane ansatz ($Q \ge 0$)

$$ds^{2} = f^{-2}(r) dx^{\mu} dx^{\nu} \eta_{\mu\nu} + f^{2}(r) (dr^{2} + r^{2} d\Omega_{k}^{2}) , \qquad (3.3.4)$$

$$f(r) = \left(1 + \frac{Q}{r^n}\right)^m, \qquad n, m \neq 0$$
 (3.3.5)

$$\eta_{\mu\nu} = \text{diag}(-1, +1, ..., +1), \qquad \mu, \nu = 0, ..., p$$

where $d\Omega_k^2$ is the metric on a k-sphere.

For k = 8 - p, n = 7 - p > 0 and $m = \frac{1}{4}$ this metric describes the standard single *p*-brane geometry in D = 10 supergravity. It may be supported by a R-R field strength and dilation backgrounds, but since they do not couple to the classical bosonic string we shall ignore them here.

Keeping k, n, m, p generic also allows one to describe lower-dimensional backgrounds representing (up to a flat factor) some special brane intersection geometries. In particular, the metric (3.3.4) with nm = 1 describes a one-parameter interpolation [179, 180] between flat space (for Q = 0) and the $AdS_{p+2} \times S^k$ space (for $Q \to \infty$).²

²More precisely, the metric (3.3.4) interpolates between the flat-space region $(r^n \gg Q)$ and the $AdS_{p+2} \times S^k$ region

Special cases include:

- (i) $n = 4, m = \frac{1}{4}, k = 5, p = 3$: D3-brane interpolating between flat space and $AdS_5 \times S^5$;
- (ii) $n = 2, m = \frac{1}{2}, k = 3, p = 1$: D5-D1 (or NS5-F1 with non-zero B_{MN} , see next section) background [181, 182] with $Q_5 = Q_1 = Q$ interpolating between flat 10d space and $AdS_3 \times S^3 \times T^4$;
- (iii) n = 1, m = 1, k = 2, p = 0: four equal-charge D3-brane intersection [183] (or U-duality related backgrounds, see, e.g. [184]) interpolating between flat 10d space and $AdS_2 \times S^2 \times T^6$; this may be viewed as a generalised Bertotti-Robinson geometry.

Below we shall show that the point-like string (i.e. geodesic) motion on this background is integrable. We shall then demonstrate that extended string motion on the *p*-brane geometry with p = 0, ..., 6and on the backgrounds (ii), (iii) is not integrable for generic values of Q, despite integrability being present in the limits Q = 0 and $Q = \infty$.

3.3.1 Complete integrability of geodesic motion

The symmetries of the metric (3.3.4) are shifts in the x^{μ} coordinates giving p+1 conserved quantities and spherical symmetry which gives k/2 conserved commuting angular momenta for even k and (k + 1)/2 for odd k (generators of the Cartan subgroup of SO(k+1)). Thus for $k \ge 3$ the spherical symmetry does not, a priori, provide us with sufficiently many conserved quantities.

Parametrising the k-sphere by k+1 embedding coordinates y^i the effective Lagrangian for point-like string motion is

$$\mathcal{L} = f^{-2}(r)\dot{x}^{\mu}\dot{x}^{\nu}\eta_{\mu\nu} + f^{2}(r)(\dot{r}^{2} + r^{2}\dot{\bar{y}}^{2}) + \Lambda(\bar{y}^{2} - 1).$$
(3.3.6)

The first integrals for x^{μ} and r are $E^{\mu} = f^{-2}(r)\dot{x}^{\mu}$ and the Hamiltonian respectively. The equation of motion for y^{i} reads

$$\partial_{\tau} (f^2(r) r^2 \dot{y}^i) + \Lambda y^i = 0 . ag{3.3.7}$$

Taking the scalar product of this equation with \vec{y} and with $\dot{\vec{y}}$ and using $y^i y^i = 1$, $y^i \dot{y}^i = 0$ we conclude that

$$\Lambda = f^2(r)r^2 \dot{\vec{y}}^2 = f^{-2}(r)r^{-2}m^2 , \qquad m^2 = f^4(r)r^4 \dot{\vec{y}}^2 = \text{const} .$$
(3.3.8)

Substituting this expression into (3.3.7) allows one to eliminate Λ giving

$$\partial_{\tau}(f^2(r)r^2\dot{y}^i) + m^2 f^{-2}(r)r^{-2}y^i = 0.$$
(3.3.9)

Multiplying this equation by $f^2(r)r^2y^i$ for fixed *i* and integrating once we obtain the constants of motion C^i (no summation over *i*)

$$f^4(r)r^4(\dot{y}^i)^2 + m^2(y^i)^2 = C^i , \qquad \sum_i C^i = 2m^2 .$$
 (3.3.10)

 $[\]overline{(r^n \ll Q)}$. Moreover, since $f(r) = (c + \frac{Q}{r^n})^m$ gives a solution for any constant c including c = 0, $\operatorname{AdS}_{p+2} \times \operatorname{S}^k$ is also an exact solution.

We can rewrite the C^i in terms of phase space coordinates $(y^i, p^i = 2f^2(r)r^2\dot{y}^i)$ as

$$C^{i} = \frac{1}{4} (p^{i})^{2} + m^{2} (y^{i})^{2} . \qquad (3.3.11)$$

The integrals C^i are easily checked to be in involution. Only k of the k+1 integrals C^i are functionally independent and correspond to the angles on the Ω_k sphere. Altogether with the p+1 constants for the x^{μ} coordinates and the Hamiltonian this gives in total k+p+2 constants of motion which is the same as the number of degrees of freedom.

3.3.2 Non-integrability of string motion

Let us now study extended string motion on the geometry (3.3.4), (3.3.5). We shall choose a particular "pulsating string" ansatz for the dependence of the string coordinates on the world-sheet directions (τ, σ) : we shall assume that (i) only x^0 , r and two angles ϕ , θ of $S^2 \subset S^k$ (with $d\Omega_2^2 = d\theta^2 + \sin^2 \theta \, d\phi^2$) are non-constant, and (ii) x^0, r, θ depend only on τ while ϕ depends only on σ , i.e. ³

$$x^0 = t(\tau), \qquad r = r(\tau), \qquad \phi = \phi(\sigma), \qquad \theta = \theta(\tau) .$$
 (3.3.12)

This ansatz is consistent with the conformal-gauge string equations of motion and the Virasoro constraint (3.3.2). The remaining string equations of motion and the Virasoro constraint give

$$\dot{t} = Ef^2 , \qquad \qquad E = \text{const} , \qquad (3.3.13)$$

$$\dot{\phi} = \nu = \text{const}, \qquad \phi = \nu\sigma, \qquad (3.3.14)$$

$$2\partial_{\tau}(f^{2}\dot{r}) = \partial_{r}(f^{2})E^{2} + \partial_{r}(f^{2})\dot{r}^{2} + \partial_{r}(r^{2}f^{2})(\dot{\theta}^{2} - \nu^{2}\sin^{2}\theta) , \qquad (3.3.15)$$

$$\partial_{\tau}(f^2 r^2 \dot{\theta}) = -f^2 r^2 \nu^2 \sin \theta \cos \theta , \qquad (3.3.16)$$

$$E^{2} = \dot{r}^{2} + r^{2}\dot{\theta}^{2} + \nu^{2}r^{2}\sin^{2}\theta . \qquad (3.3.17)$$

Thus the string is wrapped (with winding number ν) on a circle of S² whose position in r and θ changes with time. The equations for r and θ can be derived from the following effective Lagrangian

$$\mathcal{L} = f^2 \Big(\dot{r}^2 + r^2 \dot{\theta}^2 - \nu^2 r^2 \sin^2 \theta + E^2 \Big) , \qquad (3.3.18)$$

with the corresponding Hamiltonian restricted to be zero by (3.3.17). We shall show that this 1d Hamiltonian system is *not* integrable, implying non-integrability of string motion on the *p*-brane background (3.3.4).

Let us choose as an invariant plane $\{(r, \theta; p_r, p_\theta) : \theta = \frac{\pi}{2}, p_\theta = 0\}$, corresponding to the a string wrapped on the equator of S² and moving only in r. Then (3.3.17) gives $\dot{r}^2 + \nu^2 r^2 = E^2$ which is readily solved by (assuming e.g. that r(0) = 0)

$$r = \bar{r}(\tau) = \frac{E}{\nu}\sin(\nu\tau) . \qquad (3.3.19)$$

According to the general method described in section 2, we have to show that small fluctuations near this special solution are not integrable, or, more precisely, that the variational equation normal to this surface of solutions parametrized by E and ν has no Liouvillian solutions along the curves inside the

³Note that when only one of the angles of S^k is non-constant, 1-d truncations are not sufficient to establish non-integrability, see below.

surface.

For the invariant subspace $\{\theta = \frac{\pi}{2}, p_{\theta} = 0, r = \bar{r}, p_r = 2f^2(\bar{r})\dot{\bar{r}}\}$ the Hamiltonian vector field is

$$X^{r} = \dot{\bar{r}}, \qquad X^{p_{r}} = \dot{p}_{r} = 4ff'\dot{\bar{r}}^{2} + 2f^{2}\ddot{\bar{r}}, \qquad X^{\theta} = 0, \qquad X^{p_{\theta}} = 0$$
(3.3.20)

and thus the normal direction to this plane is along θ and p_{θ} . Expanding the Hamiltonian equation for θ around $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$, $r = \bar{r}$ one obtains the NVE ($\delta \theta \equiv \eta$)

$$\ddot{\eta} + 2\nu \cot(\nu\tau) \Big(1 + \frac{E}{\nu} \sin(\nu\tau) \frac{f'(\frac{E}{\nu} \sin(\nu\tau))}{f(\frac{E}{\nu} \sin(\nu\tau))} \Big) \dot{\eta} - \nu^2 \eta = 0 .$$
(3.3.21)

If f'/f is a rational function (as is the case for (3.3.5)) this equation can be algebrized through the change of variable $\tau \to x = \sin(\nu \tau)$ giving

$$\eta'' + \left(\frac{2}{x} - \frac{x}{1 - x^2} + 2\frac{E}{\nu} \frac{f'(\frac{E}{\nu}x)}{f(\frac{E}{\nu}x)}\right)\eta' - \frac{1}{1 - x^2}\eta = 0.$$
(3.3.22)

Using the explicit form of f(r) in (3.3.5) this NVE reduces to

$$\eta'' + \left(\frac{2}{x} - \frac{x}{1 - x^2} - 2\frac{mn\nu^n Q}{x(\nu^n Q + E^n x^n)}\right)\eta' - \frac{1}{1 - x^2}\eta = 0.$$
(3.3.23)

We shall assume that $\nu \neq 0$ as otherwise the string is point-like and we go back to the case of geodesic motion.

Bringing (3.3.23) to the normal form by changing the variable to $\xi(x) = g(x)\eta(x)$, where g(x) is a suitably chosen function, one obtains

$$\xi'' + \left[\frac{2+x^2}{4(x^2-1)^2} - \frac{(m-1)mn^2}{x^2(1+Bx^n)^2} + \frac{mn(n-1-(n-2)x^2)}{x^2(x^2-1)(1+Bx^n)}\right]\xi = 0 , \qquad B \equiv \frac{E^n}{Q\nu^n} .$$
(3.3.24)

There are two special limits of (3.3.23) and (3.3.24)

$$Q \to 0: \quad \xi'' + \frac{2+x^2}{4(x^2-1)^2}\xi = 0 ,$$
 (3.3.25)

$$Q \to \infty: \quad \xi'' + \frac{2x^2 + x^4 - 4(mn)^2(x^2 - 1)^2 + 4mn(1 - 3x^2 + 2x^4)}{4x^2(x^2 - 1)^2}\xi = 0 , \qquad (3.3.26)$$

corresponding to the flat spacetime and $AdS_2 \times S^2$ for nm = 1.

We can now apply the Kovacic algorithm⁴ to determine whether these NVEs do not admit Liouvillian solutions and thus the identity component of their Galois group is not solvable which would imply non-integrability.

For the special cases $Q \to 0$ and for $Q \to \infty$ with nm = 1 Liouvillian solutions are found which is consistent with the fact that string motion on flat spacetime and on $\operatorname{AdS}_2 \times \operatorname{S}^2$ is integrable. However, for a finite value of Q Liouvillian solutions do not exist for generic values of E, ν for the cases of a p-brane background with p = 0, ..., 6 and the intersecting brane backgrounds of (ii) and (iii) mentioned above. This implies non-integrability of string motion in those special cases of the general background (3.3.4), (3.3.5).

Finally, let us comment on the special case of a 7-brane in 10 dimensions when the transverse space

⁴We use Maple's function kovacicsols.

is 2-dimensional, i.e. the string motion is described by

$$\mathcal{L} = f^{-2}(r)\partial_a x^{\mu}\partial^a x^{\nu}\eta_{\mu\nu} + f^2(r)\left(\partial_a r\partial^a r + r^2\partial_a\theta\partial^a\theta\right) \,. \tag{3.3.27}$$

It is easy to see that if we truncate this model to a 1-parameter system by taking each of the string coordinates to be a function of τ or σ only then the corresponding equations always admit constants of motion for x^{μ} and θ , and the Virasoro condition then provides the solution for r. To test integrability in this case one is to consider some more non-trivial truncations. Assuming the spatial coordinates x^{i} are constant the corresponding effective metric is 3-dimensional: $ds^{2} = -f^{-2}(r)dt^{2} + f^{2}(r)(dr^{2} + r^{2}d\theta^{2})$. Integrability of string motion on such a background deserves further study.

3.4 String motion on NS5-F1 background

Let us now include the possibility of a non-zero B_{MN} coupling and consider the case of string motion on a background produced by fundamental strings delocalised inside NS5 branes [185, 164, 181, 186] (here we use the notation $x^0 = t$, $x^1 = z$ and i, j = 2, ..., 5)

$$ds^{2} = H_{1}^{-1}(r)(-dt^{2} + dz^{2}) + dx^{i}dx_{i} + H_{5}(r)(dr^{2} + r^{2}d\Omega_{3}^{2}) , \qquad (3.4.1)$$

$$d\Omega_{3}^{2} = d\theta^{2} + \sin^{2}\theta \,d\varphi^{2} + \cos^{2}\theta \,d\phi^{2} ,$$

$$B = -H_1^{-1}(r) \,\mathrm{d}t \wedge \mathrm{d}z + Q_5 \sin^2 \theta \,\mathrm{d}\varphi \wedge \mathrm{d}\phi \,\,, \qquad (3.4.2)$$

$$H_5 = 1 + \frac{Q_5}{r^2}, \qquad H_1 = 1 + \frac{Q_1}{r^2}, \qquad (3.4.3)$$

with the dilation given by $e^{-2\Phi} = \frac{H_1(r)}{H_5(r)}$. The corresponding string Lagrangian in (3.3.1) $\mathcal{L} = (G_{MN} + B_{MN})\partial_+ Y^M \partial_- Y^N$ (where $\partial_{\pm} = \partial_0 \pm \partial_1$) is

$$\mathcal{L} = \frac{r^2}{r^2 + Q_1} \partial_+ u \,\partial_- v + \partial_+ x^i \partial_- x_i + \frac{r^2 + Q_5}{r^2} \partial_+ r \partial_- r + (r^2 + Q_5) (\partial_+ \theta \partial_- \theta + \sin^2 \theta \,\partial_+ \varphi \partial_- \varphi + \cos^2 \theta \,\partial_+ \phi \partial_- \phi) + Q_5 \sin^2 \theta \left(\partial_+ \varphi \partial_- \phi - \partial_- \varphi \partial_+ \phi\right), \quad (3.4.4)$$
$$u \equiv -t + z , \qquad v \equiv t + z .$$

It interpolates between the flat space model for $Q_1, Q_5 = 0$ and the SL(2) × SU(2) WZW model (plus 4 free directions) for $Q_1, Q_5 \to \infty$. Both of these limits are obviously integrable.

Another integrable special case is $Q_1 = 0, Q_5 \to \infty$ when we get the SU(2) WZW model plus flat directions.

The opposite limit of $Q_5 = 0$, $Q_1 \to \infty$ is described (after a rescaling of coordinates) by $\mathcal{L} = r^2 \partial_+ u \partial_- v + \partial_+ r \partial_- r + r^2 (\partial_+ \theta \partial_- \theta + \sin^2 \theta \partial_+ \varphi \partial_- \varphi + \cos^2 \theta \partial_+ \phi \partial_- \phi) +$ free directions. Solving the equations for u, v as $(\sigma^{\pm} = \tau \pm \sigma)$

$$r^2 \partial_+ u = \tilde{f}(\sigma^+) , \qquad r^2 \partial_- v = f(\sigma^-) , \qquad (3.4.5)$$

one finds the effective Lagrangian for r being

$$\mathcal{L} = \partial_{+}r\partial_{-}r - \frac{f(\sigma^{-})\tilde{f}(\sigma^{+})}{r^{2}} + r^{2}(\partial_{+}\theta\partial_{-}\theta + \sin^{2}\theta\,\partial_{+}\varphi\partial_{-}\varphi + \cos^{2}\theta\,\partial_{+}\phi\partial_{-}\phi) \,. \tag{3.4.6}$$

As we shall find below, this model is not integrable.⁵

Let us note that in the limit $Q_5 \to \infty$ the action (3.4.4) reduces to a combination of the SU(2) WZW model, flat directions and the following 3-dimensional sigma model

$$\mathcal{L} = \frac{1}{1 + Q_1 e^{-2\rho}} \partial_+ u \,\partial_- v + Q_5 \partial_+ \rho \partial_- \rho , \qquad \rho \equiv \ln r , \qquad (3.4.7)$$

This model is T-dual to a pp-wave model with an exponential potential function [168, 135] and is thus related to the SL(2) WZW model by a combination of T-dualities and a coordinate transformation (it can be interpreted, upon a rescaling of u, v, as an exactly marginal deformation of the SL(2, \mathbb{R}) WZW model [136]). It should thus be integrable. Indeed, solving the equations for $z \pm t$ as in (3.4.5) we get the following effective Lagrangian for ρ (cf. (3.4.6))

$$\mathcal{L} = Q_5 \partial_+ \rho \partial_- \rho - f(\sigma^-) \tilde{f}(\sigma^+) (1 + Q_1 e^{-2\rho}) .$$
(3.4.8)

Since $f(\sigma^{-})\tilde{f}(\sigma^{+})$ can be made constant by conformal redefinitions of σ^{\pm} (reflecting residual gauge freedom in the conformal gauge) this model is equivalent to the Liouville theory, i.e. it is integrable. Indeed, the results of our analysis below are consistent with this conclusion.

A point-like string does not couple to the *B*-field, so geodesic motion on the background (3.4.1) can be shown to be integrable in the same way as in the previous section. To study extended string motion let us consider the following ansatz describing the probe string being stretched along the fundamental string direction z (that may be assumed to be compactified to a circle) and the S³ angle φ , i.e.

$$t = t(\tau), \quad z = z(\sigma), \quad x^i = 0, \quad r = r(\tau), \quad \theta = \theta(\tau), \quad \varphi = \varphi(\sigma), \quad \phi = \phi(\tau)$$
 (3.4.9)

Then the string equations are solved by

$$\dot{t} = \kappa_1 H_1(r(\tau)) + \kappa_2 , \qquad z = \kappa_2 \sigma , \qquad \qquad \phi = \nu \sigma, \qquad \dot{\phi} = \frac{\nu Q_5}{r^2(\tau) H_5(r(\tau))} .$$
(3.4.10)

The resulting 1d subsystem of equations for r and θ is described by the following effective Lagrangian

$$\mathcal{L} = H_5(r) \Big[\dot{r}^2 + r^2 \dot{\theta}^2 - \nu^2 r^2 \sin^2 \theta - Q_5^2 \nu^2 r^{-2} H_5^{-2}(r) \cos^2 \theta \Big] + \kappa_1^2 H_1(r) , \qquad (3.4.11)$$

which should be supplemented by the consequence of the Virasoro constraint

$$H_5(r) \Big[\dot{r}^2 + r^2 \dot{\theta}^2 + \nu^2 r^2 \sin^2 \theta + Q_5^2 \nu^2 r^{-2} H_5^{-2}(r) \cos^2 \theta \Big] - \kappa_1^2 H_1(r) = 2\kappa_1 \kappa_2 , \qquad (3.4.12)$$

implying that the Hamiltonian corresponding to (3.4.11) is equal to $2\kappa_1\kappa_2$. This system admits the special solution

$$\theta = \frac{\pi}{2} , \qquad r = \bar{r}(\tau) , \qquad H_5(\bar{r})(\dot{\bar{r}}^2 + \nu^2 \bar{r}^2) = 2\kappa_1 \kappa_2 + \kappa_1^2 H_1(\bar{r}) , \qquad (3.4.13)$$

⁵Let us note that the constant term in the harmonic function of the fundamental string background (1 in H_1) can be changed by a combination of T-duality, coordinate shift and another T-duality [168, 180], so that the fundamental string background is not integrable for any value of Q_1 if it is not integrable for large Q_1 . A similar shift in the harmonic function can also be achieved for D-brane backgrounds but this involves S-duality (mapping e.g. F1 to D1) which is not a symmetry of the world-sheet string action; thus there is no contradiction with integrability of the D3-brane background at large Q.
which may be chosen as an invariant plane in the phase space. The corresponding Hamiltonian vector field is

$$X^r = \dot{\bar{r}}, \qquad X^{p_r} = 2\partial_\tau (\dot{\bar{r}} H_5(\bar{r})), \qquad X^\theta = 0, \qquad X^{p_\theta} = 0.$$
 (3.4.14)

Expanding along the normal direction to this solution plane we find for the NVE $(\theta = \frac{\pi}{2} + \eta)$

$$\ddot{\eta} + \left(\frac{H_5'(\bar{r})}{H_5(\bar{r})} + \frac{2}{\bar{r}}\right) \left(\kappa_1^2 \frac{H_1(\bar{r})}{H_5(\bar{r})} + \frac{2\kappa_1 \kappa_2}{H_5(\bar{r})} - \nu^2 \bar{r}^2\right)^{1/2} \dot{\eta} - \nu^2 \left(1 - \frac{Q_5^2}{\bar{r}^4 H_5^2(\bar{r})}\right) \eta = 0.$$
(3.4.15)

Changing the variable τ to $x = \bar{r}(\tau)$ we obtain the NVE in the normal form $(' \equiv \frac{d}{dx})$

$$\xi''(x) + U(x)\xi(x) = 0 , \qquad (3.4.16)$$

where

$$U(x) = \frac{1}{4x^2} \left[\frac{Q_5(Q_5 - 2x^2)}{(Q_5 + x^2)^2} + \frac{4Q_5x^4 - 2x^6 + 2Q_1\beta^2(2Q_5 + x^2)}{(Q_5 + x^2)(-Q_1\beta^2 + x^2[Q_5 + x^2 - \beta^2(1 + \alpha)])} + \frac{3(Q_1\beta^2 + x^4)^2}{(Q_1\beta^2 + x^2[-Q_5 - x^2 + \beta^2(1 + \alpha)])^2} \right], \qquad \beta \equiv \frac{\kappa_1}{\nu}, \quad \alpha \equiv 2\frac{\kappa_2}{\kappa_1}.$$
(3.4.17)

One finds that already for $\alpha = 0$, i.e. for $\kappa_2 = 0$, this NVE has no Liouvillian solutions for general values of Q_1 and Q_5 , implying non-integrability of string motion on the NS5-F1 background with generic values of the charges. Our results are summarised in the table 3.1.

Let us discuss some special cases. Taking $Q_1 = 0$ or $Q_5 = 0$ one can absorb the remaining brane charge by rescaling x, β and κ_1 . The resulting equations do not admit Liouvillian solutions for arbitrary values of β which implies non-integrability of string motion on the NS5-brane or on the fundamental string backgrounds.

In the limit $Q_1 \to \infty$, $Q_5 \to \infty$, in which the non-trivial part of the string action becomes that of the SL(2, \mathbb{R}) × SU(2) WZW model, one finds Liouvillian solutions in agreement with the expected integrability. The same applies to the case of $Q_1 \to 0$, $Q_5 \to \infty$ described by the SU(2) WZW model plus free fields and, in fact, to the model with $Q_5 \to \infty$ and arbitrary Q_1 described by (3.4.7)(3.4.8).

In the opposite case of $Q_5 \to 0$, $Q_1 \to \infty$ described by (3.4.6) integrability appears to be absent as we did not find Liouvillian solutions (see table).⁶

Since 2d duality transformations of string coordinates preserve (non)integrability, similar conclusions can be reached for string backgrounds related to this NS5-F1 background or to its limits via T-dualities (and coordinate transformations). In particular, this applies to the pp-wave background related to the fundamental string by T-duality [168] which we study in the next section.

3.5 String motion on pp-wave background

Here we shall study string motion on a pp-wave metric (i = 1, .., d)

$$ds^{2} = du \, dv + H(u, x) \, du^{2} + dx^{i} dx_{i} . \qquad (3.5.1)$$

⁶Let us note that the results about the existence of Liouvillian solutions turn out to be independent of whether the truncated solutions contains any contribution from the B-flux of the fundamental string solution ($\alpha \neq 0$ or $\alpha = 0$), reflecting a special choice of our ansatz (3.4.9).

limit		$U(x)$ in NVE $\xi''(x) + U(x)\xi(x) = 0$, $\gamma \equiv \frac{Q_5}{Q_1}$				rescaling
$Q_5 \rightarrow 0$	$Q_1 \rightarrow 0$	$rac{2eta^2(1+lpha)+x^2}{4(eta^2(1+lpha)-x^2)^2}$				-
$Q_5 \to \infty$	$Q_1 \to \infty$	$\frac{(\gamma-\beta^2\alpha)(\gamma x^2+\beta^2(2-x^2\alpha)}{4(\gamma x^2-\beta^2(1+x^2\alpha))^2}$			$\alpha \to Q_1 \alpha$	
$Q_5 \to \infty$	$Q_1 \rightarrow 0$	$\frac{1}{4x^2}$				-
$Q_5 \rightarrow 0$	$Q_1 \to \infty$	$\frac{Q_1^2\beta^4 + 10Q_1\beta^2x^4 + x^8 - 2Q_1\beta^4x^2\alpha + 2\beta^2x^6\alpha}{4(Q_1\beta^2x - x^5 + \beta^2x^3\alpha)^2}$				-
$Q_5 \to \infty$	$Q_1 \neq 0$	$\frac{(\gamma - \beta^2 (1 + \alpha))(\gamma x^2 - \beta^2 (-2 + x^2 (1 + \alpha)))}{4(\gamma x^2 - \beta^2 (1 + x^2 (1 + \alpha)))^2}$				$\begin{aligned} x \to \sqrt{Q_1} x, \\ \kappa_{1,2} \to \sqrt{Q_1} \kappa_{1,2} \end{aligned}$
$Q_5 \neq 0$	$Q_1 \to \infty$	$ \left[\gamma^{-2}\beta^4 (x^2 - 2) + 2\gamma^{-1}\beta^2 (1 + x^2)^2 (1 + 5x^2) - 2\gamma^{-1}\beta^4 (1 + 5x^2 + x^4)\alpha \right. \\ \left. + x^2 ((1 + x^2)^2 - \beta^2 \alpha) ((1 + x^2)^2 + \beta^2 (2x^2 - 1)\alpha) \right] \\ \left. \times \left[4(1 + x^2)^2 (x^2 + x^4 - \beta^2 (\gamma^{-1} + x^2\alpha))^2 \right]^{-1} \right] $				$x \to \sqrt{Q_5} x,$ $\kappa_{1,2} \to \sqrt{Q_5} \kappa_{1,2}$
$Q_5 \rightarrow 0$	$Q_1 \neq 0$	$\frac{x^8 + \beta^4 (1 - 2x^2(1 + \alpha)) + 2\beta^2 x^4(5 + x^2(1 + \alpha))}{4x^2 (x^4 - \beta^2 (1 + x^2(1 + \alpha)))^2}$				$\begin{aligned} x &\to \sqrt{Q_1} x, \\ \kappa_{1,2} &\to \sqrt{Q_1} \kappa_{1,2} \end{aligned}$
$Q_5 \neq 0$	$Q_1 \rightarrow 0$	$\frac{(1+x^2)^4+2\beta^2(x^2-1)(1+x^2)^2(1+\alpha)-\beta^4(2x^2-1)(1+\alpha)^2}{4x^2(1+x^2)^2(1+x^2-\beta^2(1+\alpha))^2}$				$\begin{aligned} x &\to \sqrt{Q_5} x, \\ \kappa_{1,2} &\to \sqrt{Q_5} \kappa_{1,2} \end{aligned}$
			2-form contributes to the NVE			s Liouvillian
limit		spacetime	etime truncated system		SO	lutions
			$\alpha = 0$	$\alpha \neq 0$	$\alpha = 0$	$\alpha \neq 0$
$Q_5 \to 0$	$Q_1 \to 0$	$\mathbb{R}^{1,9}$	×	×		
$Q_5 \to \infty$	$Q_1 \to \infty$	AdS ₃ × S ³ × \mathbb{R}^4				
$Q_5 \to \infty$	$Q_1 \to 0$	$\mathbb{R}^{1,6} \times \mathrm{S}^{3}$				
$Q_5 \to 0$	$Q_1 \to \infty$	-	×		×	X
$Q_5 \to \infty$	$Q_1 \neq 0$	-			1	
$Q_5 \neq 0$	$Q_1 \to \infty$	-	√		×	×
$Q_5 \to 0$	$Q_1 \neq 0$	F1	×	1	×	X
$Q_5 \neq 0$	$Q_1 \to 0$	NS5	1		×	×

Figure 3.1: This table summarises the NVEs and the results for the existence of Liouvillian solutions for string motion on the various limits of the NS5-F1 background.

In conformal gauge the equation for v reads $\partial_+\partial_- u = 0$ and is solved by $u = f(\sigma^+) + \tilde{f}(\sigma^-)$. We can fix the residual conformal symmetry by choosing $u = p\tau$. Then v is determined from the Virasoro constraints and we get the following effective Lagrangian for x_i

$$\mathcal{L} = \dot{x}^{i} \dot{x}^{i} - \dot{x}^{i} \dot{x}^{i} + p^{2} H(u, x), \qquad (3.5.2)$$

which describes light-cone gauge string motion in a potential.

One familiar example is the Ricci flat space with $H(x) = \mu_{ij}x^ix^j$, $\mu_i^i = 0$. Another is the pp-wave limit of the AdS₅ × S⁵ background for which $H(x) = x^ix_i$ [187]. Here the string motion takes place in a quadratic potential and is obviously integrable. Penrose limits of the brane backgrounds we consider in this paper also take the pp-wave metric form with $H(u, x) = h_{ij}(u)x^ix^j$ (see e.g. [188]). In the light-cone gauge string motion takes place in a quadratic potential with a τ -dependent coefficient and it is solvable [189] but formally the resulting model is not integrable.⁷

There are also other integrable examples with H(x) corresponding, for example, to the Liouville

⁷For the relevant case of $h \sim \frac{1}{\tau^2}$ the Kovacic algorithm gives no Liouvillian solutions for string modes that depend on σ .

or Toda potential.

The case that we shall study below is the pp-wave solution T-dual to the fundamental string [185, 168] for which H is a harmonic function

$$H(x) = 1 + \frac{Q}{r^{d-2}}$$
, $r^2 = x^i x_i$. (3.5.3)

Not surprisingly, the conclusions will be the same as for the fundamental string in section 3.4: the geodesic motion is integrable but an extended string motion is not.

In the point-like string limit (3.5.2) takes the form (we set $x^i = ry^i$ with $\vec{y}^2 = 1$)

$$\mathcal{L} = \dot{r}^2 + r^2 \dot{\vec{y}}^2 + p^2 H(r) + \Lambda(\vec{y}^2 - 1) .$$
(3.5.4)

The corresponding geodesic motion is completely integrable since we have d-1 constants of motion from the coordinates y^i (see section 3.3.1) plus the Hamiltonian.

Let us now consider an extended string moving only on a 3-space $dx^i dx_i = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ and choose the following ansatz (similar to the one in (3.3.12))

$$r = r(\tau), \qquad \phi = \phi(\sigma) = \nu\sigma, \qquad \theta = \theta(\tau)$$
 (3.5.5)

Then the effective Lagrangian for r and θ takes the form

$$\mathcal{L} = \dot{r}^2 + r^2 \dot{\theta}^2 - \nu^2 r^2 \sin^2 \theta + p^2 H(r) . \qquad (3.5.6)$$

If we also assume that v = 0 (i.e. the string also moves in the light-cone direction) then the Virasoro condition for (3.5.1) is equivalent to the vanishing of the Hamiltonian corresponding to (3.5.6)

$$\dot{r}^2 + r^2 \dot{\theta}^2 + \nu^2 r^2 \sin^2 \theta - p^2 H(r) = 0 . \qquad (3.5.7)$$

Restricting motion to the invariant plane $\{(r, p_r, \theta, p_\theta) : \theta = \frac{\pi}{2}, p_\theta = 0\}$ gives a one-dimensional integrable system parametrised by ν and p with the Hamiltonian as the constant of motion. Imposing the zero Hamiltonian constraint (3.5.7) gives the solution for r

$$\dot{r}^2 + \nu^2 r^2 = p^2 H(r) , \qquad r = \bar{r}(\tau) .$$
 (3.5.8)

The Hamiltonian vector field on the plane of these solutions is

$$X^{r} = \dot{\bar{r}}, \qquad X^{p_{r}} = 2\ddot{\bar{r}} = -2\nu^{2}\bar{r} + p^{2}H'(\bar{r}), \qquad X^{\theta} = 0, \qquad X^{p_{\theta}} = 0$$
(3.5.9)

and the direction normal to this plane is along θ and p_{θ} . Expanding the equation of motion for θ gives the NVE $(\eta = \delta \theta = \theta - \frac{\pi}{2})$

$$\ddot{\eta} + 2\frac{\dot{\bar{r}}}{\bar{r}}\dot{\eta} - \nu^2\eta = 0.$$
(3.5.10)

Changing the independent variable $\tau \to r = \bar{r}(\tau)$ one obtains $(\prime \equiv \frac{d}{dr})$

$$\eta'' + \left(\frac{2}{r} + \frac{\frac{1}{2}p^2H'(r) - \nu^2r}{p^2H(r) - \nu^2r^2}\right)\eta' - \frac{\nu^2}{p^2H(r) - \nu^2r^2}\eta = 0$$
(3.5.11)

Bringing this equation to the normal form via the change of variable $\eta(r) = g(r)\xi(r)$ we get

$$\xi'' + U(r)\,\xi = 0 , \qquad (3.5.12)$$

$$U(r) = \frac{-(d-2)\beta^4 Q \left[(d-6)Qr^2 + 4(d-3)r^d \right] + 4\beta^2 r^d \left[(d^2 - 2d + 2)Qr^2 + 2r^d \right] + 4r^{2+2d}}{16 \left[\beta^2 (Qr^2 + r^d) - r^{2+d} \right]^2} ,$$

where $\beta \equiv \frac{p}{\nu}$. For 2 < d < 13 the Kovacic algorithm does not yield Liouvillian solutions for general finite non-zero values of Q and β which implies non-integrability.

Note that in the limit $Q \to 0$, when the metric becomes flat, we get

$$\xi'' + \frac{2\beta^2 + r^2}{4(\beta^2 - r^2)^2} \xi = 0, \qquad (3.5.13)$$

which has Liouvillian solutions in agreement with flat space integrability.

3.6 Summary

In summary we have investigated classical integrability of string motion on p-brane type backgrounds. The corresponding string σ -models interpolate between integrable flat space and coset or WZW σ -models. We have seen that particle, i.e. geodesic, motion is integrable by constructing as many conserved quantities as degrees of freedom which are in involution. For extended string motion we considered a particular string ansatz which reduces the model to an effective one-dimensional Hamiltonian system. Perturbing around a subclass of integrable solutions of this system we found that the resulting differential equation for small fluctuations cannot be solved in quadratures. Therefore integrability is not present for extended string motion on these p-brane backgrounds.

Let us note that the lack of integrability does not necessarily imply the absence of a quantitative description of these string σ -models. For example, we have seen that the string model on the pp-wave background that gives rise to motion in a time-dependent potential is not integrable. Nevertheless this model is solved in terms of the well-understood Bessel functions, which are not Liouvillian functions.

Part II

String theory on $AdS_3 \times S^3 \times T^4$ with mixed flux

Chapter 4

Dyonic giant magnons on $AdS_5 \times S^5$

For AdS_5/CFT_4 the spectral problem takes a particularly simple form in the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM. The dilatation operator is identified with an integrable $\mathfrak{su}(2)$ spin-chain Hamiltonian. In this sector operators of a large U(1) R-charge J_1 (and thus large scaling dimension $\Delta \geq J_1$) and

$$\Delta - J_1 = \text{fixed}, \qquad J_1, \Delta \to \infty, \tag{4.0.1}$$

can be built from long chains of scalars X. The spin-chain ground state corresponds to the operator

$$\operatorname{tr} X^{J_1}, \qquad \Delta - J_1 = 0$$
 (4.0.2)

and excited states are obtained from operators with a finite number of Z "impurities"

$$\mathcal{O} \sim \sum_{\ell} e^{ip\ell} (\dots XXX \stackrel{\ell}{Z} XXX\dots). \tag{4.0.3}$$

These operators correspond to long spin chains with magnon excitations propagating along the chain. In [22] the dual string state of a magnon excitation was identified for large coupling λ as a classical string solution, the giant magnon, on an $\mathbb{R} \times S^2$ subspace of $AdS_5 \times S^5$ with energy $E = \Delta$ and angular momentum J_1 . Its endpoints move at the speed of light around the equator and identifying their angular separation with the momentum p of the magnon excitation one recovers the magnon dispersion relation

$$\Delta - J_1 = \frac{\sqrt{\lambda}}{\pi} \Big| \sin \frac{p}{2} \Big|. \tag{4.0.4}$$

In [23] these giant magnon string configurations were extended to string solutions on $\mathbb{R} \times S^3$, the dyonic giant magnons, which carry an additional angular momentum J_2 . This was done by exploiting the relation of string motion on $(\mathbb{R} \times S^3) \mathbb{R} \times S^2$ to the (complex) sine-Gordon equation through the Pohlmeyer reduction. The giant magnon solution corresponds to a kink soliton of the sine-Gordon equation. Similarly the dyonic giant magnon solution is related to a U(1) charged soliton of the CsG theory, which in the limit $J_2 \to 0$ reduces to the kink soliton of the giant magnon. Additionally imposing the boundary condition that the string ends move at the speed of light around a great circle

of S^3 then leads to the unique dyonic magnon solution. Its charges satisfy

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}},$$
(4.0.5)

where p is again the angular separation of the string endpoints along a great circle of S³. For $J_2 = 1$ this is the all-loop dispersion relation for a single magnon excitation of momentum p. For integer $J_2 > 1$ this dispersion relation describes bound states of J_2 elementary magnon excitations. Therefore the dyonic giant magnon string configurations are dual to bound states of elementary magnon excitations.

The dyonic giant magnons are also related to the folded string configurations. For example taking two dyonic giant magnons of momenta $p = \pm \pi$ one obtains the folded spinning string on $\mathbb{R} \times S^3$ [35]. Also various configurations of multi spin giant magnons on $AdS_5 \times S^5$, their scattering and bound-state solutions have been studied in [190, 90, 191].

In this chapter we review the construction of the (dyonic) giant magnon solutions in some detail. In section 4.1 we give the Hofman-Maldacena (HM) giant magnon solution on $\mathbb{R} \times S^2$ and show that it reproduces the dispersion relation (4.0.4). We also review its explicit form in conformal gauge and we obtain the associated sine-Gordon kink soliton. In section 4.2 we review the generalisation of the giant magnon to $\mathbb{R} \times S^3$ using the Pohlmeyer reduction. We also show that J_2 should be quantized.

In section 4.3 we briefly consider string motion on $\mathbb{R} \times S^3$ with an additional Wess-Zumino term for the NS-NS flux and show that the Pohlmexer reduction gives again the CsG model, but with a rescaled mass parameter. Finally in section 4.4 we review the dyonic giant magnon solution in the context of the finite-gap construction.

4.1 Giant magnons in $AdS_5 \times S^5$

Let us write the $\mathbb{R} \times S^5$ metric as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\theta^2 + \sin^2\theta \,\mathrm{d}\phi^2 + \cos^2\theta \,\mathrm{d}\Omega_3^2,\tag{4.1.1}$$

where t is the global AdS time coordinate and we denote by J the angular momentum associated with shifts in ϕ . Following [22] we are interested in string solutions of energy E and angular momentum J in the limit

$$E, J \to \infty, \quad E - J = \text{fixed.}$$
 (4.1.2)

The ground state E - J = 0 corresponds to the BMN solution, which is point-like string located at the spatial origin of AdS and which moves around the great circle of an $S^2 \in S^5$. In the above coordinates it takes the form

$$\phi - t = \text{const}, \qquad \theta = \pi/2. \tag{4.1.3}$$

The giant magnon is a string solution which has the lowest non-vanishing energy $\epsilon = E - J$ for a given world-sheet momentum p. Therefore it suffices to find a solution that gives the expected dispersion relation and verify that it is indeed the minimum energy solution. This can be done by restricting motion to $\mathbb{R} \times S^2$ described by the coordinates t, θ and ϕ and considering excitations in the θ direction along the motion of ϕ such that θ is rigid, i.e. $\theta = \theta(\sigma)$. Choosing the gauge

$$t = \tau, \quad \phi = \tau + \sigma, \tag{4.1.4}$$

the Nambu-Goto action takes the form

$$S = \frac{\sqrt{\lambda}}{2\pi} \int d\tau \, d\sigma \, \sqrt{\cos^2 \theta \, \theta'^2 + \sin^2 \theta}. \tag{4.1.5}$$

and the equation of motion for θ is solved by

$$\sin\theta(\sigma) = \frac{\sin\theta_0}{\cos\sigma}, \quad -\left(\frac{\pi}{2} - \theta_0\right) \le \sigma \le \frac{\pi}{2} - \theta_0, \quad 0 \le \theta_0 \le \frac{\pi}{2}.$$
(4.1.6)

The endpoints of this giant magnon solution are moving at the speed of light around the equator and they are separated at a given time t by $\Delta \varphi = 2(\frac{\pi}{2} - \theta_0)$ which is identified with the momentum, i.e. $p = \Delta \varphi$, of the excitation. This identification leads to the expected dispersion relation

$$E - J = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma \left(\frac{\partial \mathcal{L}}{\partial \dot{t}} - \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma \sqrt{\cos^2 \theta \, \theta'^2 + \sin^2 \theta} \tag{4.1.7}$$

$$= \frac{\sqrt{\lambda}}{2\pi} \int_{-(\pi/2-\theta_0)}^{\pi/2-\theta_0} \mathrm{d}\sigma \frac{\sin\theta_0}{\cos^2\sigma} = \frac{\sqrt{\lambda}}{\pi} \Big| \sin\frac{p}{2} \Big|. \tag{4.1.8}$$

This is indeed the minimum energy solution for a given momentum p since one can show that it is BPS [22].

4.1.1 Conformal gauge

Let us now see how this solution looks in conformal gauge with the choice $t = \kappa \tau$. Since in this gauge $E = \sqrt{\lambda} \kappa/(2\pi) \to \infty$ and λ is held fixed let us rescale the world-sheet coordinates $(\tau, \sigma) \to (\kappa \tau, \kappa \sigma) \equiv (t, x)$, which decompactifies the world-sheet σ -direction into a straight line and thus turns the closed string into an open and infinitely long string. This also allows one to actually consider a single magnon excitation as otherwise momentum conservation would require the presence of additional magnons. In these rescaled coordinates one can place these additional magnons at the boundary $x = \pm \infty$ such that they are well separated from any interaction with the magnon excitation in the "bulk". Such solutions can then be glued together to form closed string solutions corresponding to asymptotic states long before or after any interaction of the magnons.

In this gauge the giant magnon solution can be found by making an ansatz for a soliton profile with the embedding coordinates X_i parametrised in terms of a boosted world-sheet coordinate $\xi = x - vt$

$$t = \kappa\tau, \quad Z_i = z_i(\xi)e^{i\omega_i t}, \quad z_i(\xi) = r_i(\xi)e^{i\varphi_i(\xi)}, \quad \xi = x - vt, \quad r_1^2 + r_2^2 = 1$$
(4.1.9)

where $Z_1 = X_1 + iX_2$ and $Z_2 = X_3 + iX_4$ are complex embedding coordinates for S³. Writing the metric on S³ in Hopf coordinates

$$ds^{2} = d\theta^{2} + \sin^{2}\theta \, d\phi_{1}^{2} + \cos^{2}\theta \, d\phi_{2}^{2}, \qquad (4.1.10)$$

the action and the ansatz take the form

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int dt \, dx \, \left(-\dot{t}^2 + \dot{t}^2 + \dot{\theta}^2 - \dot{\theta}^2 + \sin^2\theta (\dot{\phi}_1^2 - \dot{\phi}_1^2) + \cos^2\theta (\dot{\phi}_2^2 - \dot{\phi}_2^2) \right), \tag{4.1.11}$$

$$t = \kappa \tau, \quad \theta = \theta(\xi), \quad \phi_1 = \omega_1 t + \varphi_1(\xi), \quad \phi_2 = \omega_2 t + \varphi_2(\xi), \quad \xi = x - vt,$$
 (4.1.12)

$$r_1 = \sin \theta, \quad r_2 = \cos \theta, \tag{4.1.13}$$

where $\dot{=} \partial_t$ and $\dot{=} \partial_x$. In order to consider the giant magnon solution on S² we shall take $X_4 = 0$ or equivalently $\omega_2 = 0$ and $\varphi_2 = 0$.

Furthermore the endpoints of the giant magnon must be moving at the speed of light around the equator with a constant angular separation $\Delta \phi_1$. This gives the boundary conditions

$$\theta \to \frac{\pi}{2}, \quad \phi_1 \to t \pm \frac{\Delta \phi_1}{2} \quad \text{as } x \to \pm \infty,$$
(4.1.14)

which implies $\omega_1 = 1$. This ansatz gives the effective Lagrangian density

$$\mathcal{L} = -(1 - v^2)\theta'^2 + \sin^2\theta (1 - 2v\varphi'_1 - (1 - v^2)\varphi'_1^2), \qquad (4.1.15)$$

which is supplemented by the Virasoro constraints

$$\theta'(\xi)^2 + \sin^2\theta \left(-\frac{1}{v} + \varphi_1'\right)\varphi_1' = 0, \quad \theta'^2(1+v^2) + \sin^2\theta \left[(1-v\varphi_1')^2 + \varphi_1'^2\right] = 1.$$
(4.1.16)

The equation of motion for φ_1

$$\varphi_1' = \frac{A_1}{1 - v^2} \sin^{-2} \theta - \frac{v}{1 - v^2} \tag{4.1.17}$$

can be used to eliminate φ'_1 from the Virasoro constraints obtaining an equation of motion for θ and a condition on the integration constant A_1

$$\theta^{\prime 2} = \frac{\cos^2 \theta (1 - v^2 - \cos^2 \theta)}{(1 - v^2)^2 \sin^2 \theta}, \quad A_1 = v.$$
(4.1.18)

This can be used to solve for θ and subsequently ϕ_1 giving

$$\theta(x,t) = \arccos(\gamma \operatorname{sech}(\gamma^{-1}\xi(x,t))), \quad \gamma = \sqrt{1-v^2}$$
(4.1.19)

$$\phi_1(x,t) = t + \arctan\left(\frac{\gamma}{v}\tanh(\gamma^{-1}\xi(x,t))\right).$$
(4.1.20)

As before the magnon momentum is identified with the angular separation of the string endpoints, which correspond to $x \to \pm \infty$. In terms of the soliton velocity v the separation angle is given by ¹

$$\Delta \varphi = \lim_{x \to \infty} \phi_1(x, t) - \lim_{x \to -\infty} \phi_1(x, t) = 2 \arctan\left(\frac{\sqrt{1 - v^2}}{v}\right) \qquad \Rightarrow \qquad \cos\frac{\Delta \varphi}{2} = v. \tag{4.1.22}$$

¹Note that same result can also be obtained by performing the integral

$$\Delta \varphi = 2 \int_{\arccos\sqrt{1-v^2}}^{\pi/2} \mathrm{d}\theta \; \frac{\varphi_1'}{\theta'} = 2 \arcsin\sqrt{1-v^2} \tag{4.1.21}$$

which is useful when the explicit solution cannot be given in closed form.

In this gauge the identification of the magnon momentum with the separation angle is also supported by comparing the group velocity of the dispersion relation with the string soliton velocity

$$v_{\text{gauge}} = \frac{\mathrm{d}\epsilon(p)}{\mathrm{d}p} = \frac{\sqrt{\lambda}}{2\pi} \cos\frac{p}{2}, \qquad v_{\text{string}} = \frac{\mathrm{d}x}{\mathrm{d}t} = v. \tag{4.1.23}$$

Figure 4.1: The giant magnon solution is a string solution moving on the $\mathbb{R} \times S^2$ subspace of $AdS_5 \times S^5$. Its endpoints move at the speed of light around the equator and their separation angle is identified with the magnon momentum p.

4.1.2 Giant magnons and the Pohlmeyer reduction

In conformal gauge the action for string motion on a sphere is

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d^2 \sigma \left[\partial_+ X \cdot \partial_- X + \Lambda (X^2 - 1) \right], \tag{4.1.24}$$

where $X(\tau, \sigma)$ are the string embedding coordinates, $\sigma_{\pm} = (\tau \pm \sigma)/2$ are world-sheet light-cone coordinates and Λ is a Lagrange multiplier. The equations of motion and Virasoro constrains are

$$\partial_{+}\partial_{-}X + (\partial_{+}X \cdot \partial_{-}X)X = 0, \quad (\partial_{\pm}X)^{2} = \kappa^{2}.$$

$$(4.1.25)$$

In the case of $\mathbb{R} \times S^2$ the Pohlmeyer reduction is performed by introducing a new basis for \mathbb{R}^3 consisting of the three vectors

$$X, \qquad \partial_+ X, \qquad \partial_- X. \tag{4.1.26}$$

Their inner products are given by

$$X^{2} = 1 X \cdot \partial_{+} X = 0, X \cdot \partial_{-} X = 0, (\partial_{+} X)^{2} = \kappa^{2}, (\partial_{-} X)^{2} = \kappa^{2}, (4.1.27)$$

$$\partial_+ X \cdot \partial_- X = \kappa^2 \cos 2\phi, \qquad (4.1.28)$$

where ϕ is the remaining degree of freedom of the system after the Virasoro constraints are taken into account. The system can be reduced to an equation of motion for this degree of freedom by expressing

higher derivatives in the above basis

$$\partial_{+}^{2}X = 2\cot 2\phi \,\partial_{+}\phi \,\partial_{+}X - 2\csc 2\phi \,\partial_{+}\phi \,\partial_{-}X - \kappa^{2}X, \qquad (4.1.29)$$

$$\partial_{-}^{2}X = 2\cot 2\phi \,\partial_{-}\phi \,\partial_{-}X - 2\csc 2\phi \,\partial_{-}\phi \,\partial_{+}X - \kappa^{2}X. \tag{4.1.30}$$

Taking derivatives of (4.1.28) and using (4.1.29), (4.1.30) one finds that the O(3) σ -model describing string motion on $\mathbb{R} \times S^2$ reduces to the sine-Gordon equation

$$\partial_+\partial_-\phi - \frac{\kappa^2}{2}\sin 2\phi = 0, \qquad (4.1.31)$$

which arises from the sine-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_+\phi\partial_-\phi + \frac{\kappa^2}{4}(\cos 2\phi - 1).$$
(4.1.32)

In the sine-Gordon model the giant magnon takes the form of the kink soliton (after rescaling τ , σ by κ)

$$\phi(t,x) = \arcsin\left[\operatorname{sech}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\right].$$
(4.1.33)

4.2 Dyonic giant magnons and Pohlmeyer reduction

The giant magnon solution can be extended to a solution on $\mathbb{R} \times S^3$ carrying a second angular momentum J_2 by exploiting the relation of the string σ -model to the complex sine-Gordon model. Following [23] we review the construction of this dyonic giant magnon here. Let us consider string motion on $\mathbb{R} \times S^3$ in the limit

$$E, J_1 \to \infty, \quad E - J_1 = \text{fixed}, \quad J_2, p = \text{fixed}.$$
 (4.2.1)

In order to perform the Pohlmeyer reduction one needs to identify four basis vectors which span \mathbb{R}^4 . This can be done by supplementing the vectors X, $\partial_+ X$ and $\partial_- X$, which span \mathbb{R}^3 , by

$$K_i = \varepsilon_{ijkl} X_j \partial_+ X_k \partial_- X_l. \tag{4.2.2}$$

The three degrees of freedom of the system ϕ , u and v can now be written in terms of SO(4) invariants as

$$\cos\phi = \partial_+ X \cdot \partial_- X, \quad u \sin\phi = \partial_+^2 X \cdot K, \quad v \sin\phi = \partial_-^2 X \cdot K.$$
(4.2.3)

where we have set $\kappa = 1$ since we consider a decompactified world-sheet with the light-cone coordinates rescaled by κ , i.e. $\sigma^{\pm} = (t \pm x)/2$. Expressing derivatives in the above basis allows one to cast the equations of motion and Virasoro constraints into the form

$$u = \partial_+ \chi \tan \frac{\phi}{2}, \quad v = -\partial_- \chi \tan \frac{\phi}{2}$$
 (4.2.4)

$$\partial_{+}\partial_{-}\phi + \sin\phi - \frac{\tan^{2}\frac{\phi}{2}}{\sin\phi}\partial_{+}\chi\partial_{-}\chi = 0, \quad \partial_{+}\partial_{-}\chi + \frac{1}{\sin\phi}(\partial_{+}\phi\partial_{-}\chi + \partial_{-}\phi\partial_{+}\chi) = 0, \quad (4.2.5)$$

where χ is a new field replacing u and v, which are not independent. Defining the complex field $\psi = \sin(\phi/2) \exp(i\chi/2)$ one obtains the complex sine-Gordon equation

$$\partial_{+}\partial_{-}\psi + \psi^{*}\frac{\partial_{+}\partial_{-}\psi}{1 - |\psi|^{2}} + \psi(1 - |\psi|^{2}) = 0.$$
(4.2.6)

This equation is integrable and therefore its soliton solutions undergo factorised scattering. The most general single soliton solution is

$$\psi_{1-\text{soliton}}(x,t) = e^{i\mu} \frac{\cos\alpha \exp(i\sin\alpha\mathcal{T})}{\cosh((\mathcal{X} - \mathcal{X}_0)\cos\alpha)}$$
(4.2.7)

$$\mathcal{X} = x \cosh \vartheta - t \sinh \vartheta, \quad \mathcal{T} = t \cosh \vartheta - x \sinh \vartheta.$$
 (4.2.8)

where \mathcal{X}, \mathcal{T} are boosted world-sheet coordinates, ϑ is the soliton rapidity and α corresponds to the soliton U(1) charge. The parameter \mathcal{X}_0 can be absorbed by a shift in x and the phase $e^{i\mu}$ does not affect the string embedding coordinates since it only contributes a constant shift to χ . Hence without loss of generality we set $\mathcal{X}_0 = \mu = 0$.

In order to relate this solution solution to the string embedding coordinates one can solve the defining equation of ψ for $\cos \phi = \partial_+ X \cdot \partial_- X$ giving a linear differential equation for the embedding coordinates through (4.1.25)

$$\partial_+\partial_- X + \cos\phi X = 0, \quad \cos\phi = 1 - \frac{2\cos^2\alpha}{\cosh^2(\mathcal{X}\cos\alpha)}.$$
 (4.2.9)

The soliton $\psi_{1-\text{soliton}}$ is also the only known solution for which the field ϕ reduces to the giant magnon kink soliton (4.1.33) when taking $\alpha \to 0$.

In order to solve (4.2.9) it is convenient to introduce complex embedding coordinates $Z_1 = X_1 + iX_2$, $Z_2 = X_3 + iX_4$, which carry charges (1,0) and (0,1) under the U(1)_L × U(1)_R Cartan subgroup of the full SU(2)_L × SU(2)_R isometry group of this system. In terms of the boosted world-sheet coordinates the equation (4.2.9) takes the form

$$\frac{\partial^2 Z}{\partial \mathcal{T}^2} - \frac{\partial^2 Z}{\partial \mathcal{X}^2} + \left[1 - \frac{2\cos^2\alpha}{\cosh^2(\mathcal{X}\cos\alpha)}\right]Z = 0.$$
(4.2.10)

Additionally the solution should satisfy the giant magnon boundary conditions

$$Z_1 \to \exp\left(it \pm i\frac{p}{2}\right), \qquad Z_2 \to 0 \qquad \text{as } x \to \pm \infty.$$
 (4.2.11)

The equation (4.2.10) describes a relativistic particle incident on a potential well and is solved in terms of stationary states

$$Z_{\omega} = F_{\omega}(\mathcal{X}) \exp(i\omega\mathcal{T}). \tag{4.2.12}$$

Defining the new variables $y = \mathcal{X} \cos \alpha$, $f_{\omega}(y) = F_{\omega}(\mathcal{X})$ and $\varepsilon = \sqrt{\omega^2 - 1} / \cos \alpha$ one obtains the

time-independent Schrödinger equation in a Rosen-Morse potential V [192] from (4.2.10)

$$\frac{\mathrm{d}^2 f}{\mathrm{d}^2 y} - V(y)f = \varepsilon^2 f, \quad V(y) = -\frac{2}{\cosh^2 y}.$$
(4.2.13)

The solutions to this equation consist of one normalisable bound state of energy $\varepsilon^2 = -1$ and scattering states with a continuous dispersion relation $\varepsilon^2 = k^2$, k > 0. Their wavefunctions are

$$f_{-1}(y) = \operatorname{sech}(y)$$
 (4.2.14)

$$f_{k^2}(y) = \exp(iky)(\tanh y - ik),$$
 (4.2.15)

which for $y \to \pm \infty$ behave as

$$f_0 \to 0 \tag{4.2.16}$$

$$f_{\sqrt{1+k^2}} \to \exp(iky \pm i\frac{\delta}{2}), \quad \delta = 2\arctan(k^{-1}).$$

$$(4.2.17)$$

The full solution is a linear combination of these modes which should satisfy the boundary conditions (4.2.11). This is only possible for a single mode since k is fixed by the condition $p = \delta$. Therefore the string solution is

$$Z_1 = c_1 f_{k_p^2}(\mathcal{X} \cos \alpha) \exp(i\omega_{k_p^2} \mathcal{T}), \quad k_p = \frac{\sinh \vartheta}{\cos \alpha}, \quad \omega_{k_p^2} = \sqrt{1 + k_p^2 \cos^2 \alpha}$$
(4.2.18)

$$Z_2 = c_2 f_{-1}(\mathcal{X} \cos \alpha) \exp(i\omega_{-1}\mathcal{T}), \quad \omega_{-1} = \sin \alpha, \quad p = 2 \arctan(k_p^{-1}).$$
(4.2.19)

Finally taking c_1 and c_2 to be real and imposing $|Z_1|^2 + |Z_2|^2 = 1$ gives $c_1 = c_2 = (1 + k_p^2)^{-1/2}$ and the full solution takes the form

$$Z_1 = \frac{(\tanh(\mathcal{X}\cos\alpha) - ik_p)\exp(it)}{(1+k_p^2)^{1/2}} \quad Z_2 = \frac{\operatorname{sech}(\mathcal{X}\cos\alpha)\exp(i\mathcal{T}\sin\alpha)}{(1+k_p^2)^{1/2}},$$
(4.2.20)

which reduces to the HM giant magnon as $\alpha \to 0$.

The conserved charges for this solution can be obtained from

$$E - J_1 = \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \left(1 - \mathrm{Im}(\bar{Z}_1 \partial_t Z_1)\right), \quad J_2 = \frac{\sqrt{\lambda}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{Im}(\bar{Z}_2 \partial_t Z_2). \tag{4.2.21}$$

Note that these charges are not necessarily the same as the charges in the original world-sheet coordinates (τ, σ) since they can carry additional contributions from $x = \pm \infty$. Substituting the explicit solution (4.2.20) into these expressions one finds

$$E - J_1 = \frac{\sqrt{\lambda}}{\pi} \frac{1}{1 + k_p^2} \frac{(1 + k_p^2 \cos^2 \alpha)^{1/2}}{\cos \alpha}, \quad J_2 = \frac{\sqrt{\lambda}}{\pi} \frac{1}{1 + k_p^2} \tan \alpha.$$
(4.2.22)

Eliminating α and replacing k_p by the momentum as $p = 2 \arctan(k^{-1})$ gives the dispersion relation

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}.$$
(4.2.23)

4.2.1 Semiclassical quantization of J₂

In an integrable Hamiltonian system motion takes place on a phase space torus. The action variables are associated with the Liouville torus cycles and by the Bohr-Sommerfeld condition for semiclassical quantization they should be integer-valued. The dyonic giant magnon solution (4.2.20) is time-periodic in the ϕ_2 direction up to world-sheet translations in x and its period is given by

$$T = 2\pi \frac{\cosh \vartheta}{\sin \alpha} = \frac{2\pi}{J_2} \sqrt{J_2^2 + 4h^2 \sin^2 \frac{p}{2}}.$$
 (4.2.24)

The periodicity of the full solution can also be explicitly seen in light-cone gauge as we will show in the next chapter for the mixed flux case. Time periodicity can be used to define the action variable Iover the associated Liouville torus cycle γ

$$2\pi I = \oint_{\gamma} p_i \,\mathrm{d}q^i = \int_0^T \mathrm{d}t \int_{-\infty}^\infty \mathrm{d}x \, p_i \dot{q}^i = S + TH = S - T \frac{\partial S}{\partial T},\tag{4.2.25}$$

where we used the Hamilton-Jacobi equation

$$H = -\frac{\partial S(t)}{\partial t}.$$
(4.2.26)

In (4.2.25) the action S = S(T, p) is evaluated over one period of the dyonic giant magnon solution. The value of the action does not depend on the gauge due to reparametrisation invariance and one finds in conformal gauge

$$S(T,p) = -4\pi h \sin \frac{p}{2} \sqrt{\left(\frac{T}{2\pi}\right)^2 - 1}.$$
(4.2.27)

The action variable is then obtained from (4.2.25) and (4.2.27), giving

$$I = 4\pi h \sin \frac{p}{2} \left[\left(\frac{T}{2\pi} \right)^2 - 1 \right]^{-1/2} = J_2.$$
(4.2.28)

Therefore the Bohr-Sommerfeld condition implies that J_2 should be quantized, which is consistent with the expectation that J_2 is integer-valued in the quantum theory.

4.3 Pohlmeyer reduction for $\mathbb{R} \times S^3$ with a Wess-Zumino term

In preparation of the mixed flux case let us also briefly review the Pohlmeyer reduction for string motion on $\mathbb{R} \times S^3$ with an additional Wess-Zumino term in the action for the NS-NS flux (for a more detailed review see [74]). The action in conformal gauge is

$$S = \frac{\sqrt{\lambda}}{2\pi} \Big(\frac{1}{2} \int d^2 \sigma [\partial_+ X \cdot \partial_- X + \Lambda (X^2 - 1)] + \frac{1}{3} q \int d^3 \sigma \, \varepsilon^{abc} \varepsilon_{mnpq} X_m \partial_a X_n \partial_b X_p \partial_c X_q \Big).$$
(4.3.1)

The the equations of motion and Virasoro constraints are give by

$$\partial_+ \partial_- X - \Lambda X + qK = 0, \tag{4.3.2}$$

$$K_m = \varepsilon_{mnpl} X_n \partial_+ X_p \partial_- X_l, \quad (\partial_\pm X)^2 = \kappa^2, \tag{4.3.3}$$

where $X^2 = 1$ and the Lagrange multiplier Λ is given by $\Lambda = -\partial_+ X \cdot \partial_- X$. As in the q = 0 case the set of basis vectors $\{X, \partial_+ X, \partial_- X, K\}$ spans \mathbb{R}^4 and they satisfy the inner product relations (4.1.27). In terms of the SO(4) invariants the two degrees of freedom take the form

$$\partial_+ X \cdot \partial_- X = \kappa^2 \cos 2\varphi, \quad K \cdot \partial_{\pm}^2 X = f_{\pm}(\varphi) \partial_{\pm} \chi,$$
(4.3.4)

where φ and χ are the fields of the reduced model. Expressing $\partial_{\pm}^2 X$ in the above basis as

$$\partial_{\pm}^{2}X = -\kappa^{2}X + 2\partial \pm \varphi \cot 2\varphi \partial_{\pm}X - 2\partial_{\pm}\varphi \csc(2\varphi)\partial_{\mp}X + \frac{f_{\pm}\partial_{\pm}\chi}{\kappa^{4}\sin^{2}2\varphi}K$$
(4.3.5)

and taking the inner product of $\partial_{\pm}^2 X$ with itself gives the equation of motion for φ

$$\partial_+\partial_-\varphi + \frac{f_+f_-\partial_+\chi\partial_-\chi}{2\kappa^6\sin^3 2\varphi} + \frac{\kappa^2(1-q^2)}{2}\sin 2\varphi = 0.$$
(4.3.6)

The functions f_{\pm} must be chosen such that they satisfy the compatibility condition

$$\partial_{-}\left(\frac{\partial_{+}^{2}X\cdot K}{f_{+}}\right) = \partial_{+}\left(\frac{\partial_{-}^{2}X\cdot K}{f_{-}}\right),\tag{4.3.7}$$

where we used $K^2 = \kappa^4 \sin^2 2\varphi$. This condition has the solutions

$$f_{+} = -f_{-} = A\sin^{2}\varphi \tag{4.3.8}$$

$$f_{+} = f_{-} = B\cos^{2}\varphi. \tag{4.3.9}$$

Both solutions give the same reduced model. Therefore choosing the former one finds for χ

$$\partial_{-}(\tan^{2}\varphi\partial_{+}\chi) + \partial_{+}(\tan^{2}\varphi\partial_{-}\chi) = 0.$$
(4.3.10)

Choosing $A = 4\kappa^2$, for consistency with the q = 0 case, one obtains the equation of motion for φ

$$\partial_{+}\partial_{-}\varphi - \sec^{2}\varphi \tan\varphi \partial_{+}\chi \partial_{-}\chi + \frac{1}{2}(1-q^{2})\kappa^{2}\sin 2\varphi = 0.$$
(4.3.11)

The equations (4.3.10) and (4.3.11) follow from the complex sine-Gordon model

$$\mathcal{L} = \partial_{+}\varphi \partial_{-}\varphi + \tan^{2}\varphi \partial_{+}\chi \partial_{-}\chi + \frac{1}{2}(1-q^{2})\kappa^{2}\cos 2\varphi, \qquad (4.3.12)$$

which has the mass parameter $\frac{1}{2}(1-q^2)\kappa^2$. As we see the Pohlmeyer reduced theory is the same as for q = 0 but with a rescaled mass parameter. Also exchanging the solutions (4.3.8) and (4.3.9) interchanges $\tan^2 \varphi$ and $\cot^2 \varphi$ in the CsG Lagrangian. It is also interesting to note that for the special points $q = \pm 1$, where the σ -model becomes a WZW model, the reduced model is massless.

4.4 Dyonic giant magnon as a finite-gap solution

As a finite-gap solution of the principal chiral model the dyonic giant magnon takes the form of a *condensate cut* [193, 194], which is a constant density cut with logarithmic singularities at the

endpoints. The resolvent

$$G(z) = p(z) - \frac{\pi E}{\sqrt{\lambda}} \Big[\frac{1}{1+z} - \frac{1}{1-z} \Big], \qquad z \in C$$
(4.4.1)

admits a spectral representation in terms of a density $\rho(y)$ along the condensate cut C defined by

$$2\pi i\rho(z) = G(z+i\epsilon) - G(z-i\epsilon), \qquad z \in C$$
(4.4.2)

$$G(z) = \int dy \frac{\rho(y)}{z - y}.$$
(4.4.3)

The discontinuity condition (2.3.37) reads

$$2 \int dy \frac{\rho(y)}{z-y} = 2\pi n_i - \frac{2\pi E}{\sqrt{\lambda}} \Big[\frac{1}{1+z} - \frac{1}{1-z} \Big], \qquad z \in C.$$
(4.4.4)

A single dyonic giant magnon is described by the constant density along a contour C,

$$\rho(z) = -i, \qquad z \in C, \tag{4.4.5}$$

for which the endpoints X^{\pm} are complex conjugates of each other (see figure 4.2). The resolvent for this density is given by

$$G_{\rm DGM}(z) = -i \int_{X^-}^{X^+} \frac{\mathrm{d}y}{z - y} = -i \ln \frac{z - X^+}{z - X^-}.$$
(4.4.6)

The position of the endpoints X^{\pm} is fixed in terms of J_2 and p, which can be seen from the asymptotics

Figure 4.2: The dyonic giant magnon is described by a condensate cut, which is a constant density cut with logarithmic singularities at its endpoints. The endpoints are the dyonic spectral parameters X^{\pm} .

of the Lax connection

$$L = -\frac{1}{z} \star J + \mathcal{O}\left(\frac{1}{z^2}\right), \qquad z \to \infty$$
(4.4.7)

$$L = J + z \star J + \mathcal{O}(z^2), \qquad z \to 0.$$
(4.4.8)

Expanding the monodromy matrix to leading order gives for $z \to \infty$

$$T = \mathbb{I} - \frac{1}{z} \int_{\gamma} \star J + \dots = \mathbb{I} - \frac{1}{z} \frac{4\pi}{\sqrt{\lambda}} Q_R + \dots, \qquad z \to \infty$$
(4.4.9)



and similarly for $z \to 0$

$$T = g(2\pi)^{-1} \mathcal{P} \exp\left(z \int_{\gamma} \star K + \dots\right) g(0), \qquad z \to 0$$
(4.4.10)

$$= \mathbb{I} + z \int_{\gamma} \star K + \dots = \mathbb{I} + z \frac{4\pi}{\sqrt{\lambda}} Q_L + \dots , \qquad (4.4.11)$$

where we used $g(\sigma + 2\pi) = g(\sigma)$, $d + J + z \star J = g^{-1}(d + z \star K)g$ and γ is a closed path of single winding around the world-sheet cylinder. Without loss of generality one can take the SU(2)_L and SU(2)_R charges to lie in the σ_3 direction of $\mathfrak{su}(2)$, i.e. one considers highest weight solutions. This leaves the Cartan subgroup U(1)_L × U(1)_R with the two angular momenta J_1 and J_2 as Noether charges and one finds

$$Q_L = \frac{\sqrt{\lambda}}{4\pi} \int_{\gamma} \star K = (J_2 + J_1) \frac{1}{2i} \sigma_3, \qquad Q_R = \frac{\sqrt{\lambda}}{4\pi} \int_{\gamma} \star J = (J_2 - J_1) \frac{1}{2i} \sigma_3. \tag{4.4.12}$$

The asymptotics of the quasi-momentum p(z) around $z \to \infty$ and $z \to 0$ are given by

$$p(z) = -\frac{2\pi}{\sqrt{\lambda}} \frac{J_2 - J_1}{2z} + ..., \qquad z \to \infty,$$
 (4.4.13)

$$p(z) = 2\pi m + \frac{2\pi}{\sqrt{\lambda}} \frac{J_2 + J_1}{2} z + \dots, \qquad z \to 0,$$
(4.4.14)

where $2\pi m$ arises from $T(0) = g(2\pi)^{-1}g(0) = \mathbb{I}$ as a consequence of the closed string boundary condition $g(\sigma + 2\pi) = g(\sigma)$. This can be interpreted as the level matching condition with the string momentum given by $2\pi m$. Combined with the finite-gap equation (4.4.4) these give the relations

$$-\oint \frac{\mathrm{d}y}{2\pi i} G(y) = \int \mathrm{d}y \,\rho(y) = \frac{4\pi}{\sqrt{\lambda}} \Big[J_2 + \frac{E - J_2 - J_1}{2} \Big],\tag{4.4.15}$$

$$-\oint \frac{\mathrm{d}y}{2\pi i} \frac{G(y)}{y} = \int \mathrm{d}y \,\frac{\rho(y)}{y} = 2\pi m,\tag{4.4.16}$$

$$-\oint \frac{\mathrm{d}y}{2\pi i} \frac{G(y)}{y^2} = \int \mathrm{d}y \, \frac{\rho(y)}{y^2} = \frac{2\pi}{\sqrt{\lambda}} (E - J_1 - J_2). \tag{4.4.17}$$

The dyonic giant magnon is an open string solution and a physical string configuration is obtained by combining dyonic giant magnons such that the total momentum vanishes. However, in order to consider a single dyonic giant magnon we can relax the level matching condition to allow for a momentum which is not a multiple of 2π . The above relations then give

$$-i\int_{X^{-}}^{X^{+}} \mathrm{d}y = -i(X^{+} - X^{-}) = \frac{2\pi}{\sqrt{\lambda}} \Big[E - J_1 + J_2 \Big], \qquad (4.4.18)$$

$$-i\int_{X^{-}}^{X^{+}} \frac{\mathrm{d}y}{y} = -i\ln\frac{X^{+}}{X^{-}} = p, \qquad (4.4.19)$$

$$-i\int_{X^{-}}^{X^{+}} \frac{\mathrm{d}y}{y^{2}} = i\left(\frac{1}{X^{+}} - \frac{1}{X^{-}}\right) = \frac{2\pi}{\sqrt{\lambda}} \Big[E - J_{1} - J_{2}\Big],\tag{4.4.20}$$

which reproduces the dispersion relation (4.0.5) after eliminating X^{\pm} . Therefore the position of the condensate cut is determined by the charges $E - J_1$, J_2 and p as

$$X^{\pm} = re^{\pm ip/2}, \qquad r = \frac{E - J_1 + J_2}{2h\sin\frac{p}{2}} = \frac{2h\sin\frac{p}{2}}{E - J_1 - J_2}.$$
 (4.4.21)

For fundamental magnon excitations, i.e. $J_2 = 1$, the dyonic spectral parameters X^{\pm} are denoted by lower case letters x^{\pm} . These Zhukovsky variables also appear naturally in the Bethe ansatz and the S-matrix. The dispersion relation written in these variables takes the form

$$x^{+} + \frac{1}{x^{+}} - \left(x^{-} + \frac{1}{x^{-}}\right) = \frac{2i}{h}.$$
(4.4.22)

This dispersion relation can be solved by introducing the Zhukovsky map to the *rapidity* plane $u \in \mathbb{C}$ through

$$u = x + \frac{1}{x}, \qquad x^{\pm} = x\left(u \pm \frac{i}{h}\right).$$
 (4.4.23)

The rapidity for a single magnon is given by

$$u = \frac{1}{h} \cot \frac{p}{2} \sqrt{1 + 4h^2 \sin^2 \frac{p}{2}}.$$
(4.4.24)

These rapidity parameters correspond to the Bethe roots in the Bethe ansatz describing the scattering of magnons. For example, in the $\mathfrak{su}(2)$ sector the exact S-matrix for the scattering of two magnons with the spectral parameters x^{\pm} and y^{\pm} takes the form

$$s(x,y) = s_{\text{BDS}}(x,y)\sigma(x,y), \qquad s_{\text{BDS}}(x,y) = \frac{x^+ - y^-}{x^- - y^+} \frac{1 - \frac{1}{x^+ y^-}}{1 - \frac{1}{x^- y^+}} = \frac{u_x - u_y + \frac{2i}{h}}{u_x - u_y - \frac{2i}{h}}, \tag{4.4.25}$$

where $\sigma(x, y)$ is the dressing factor and $s_{BDS}(x, y)$ is the factor from the BDS all-loop asymptotic Bethe ansatz [12]

$$e^{ip_k} = \frac{x(u_k + \frac{i}{h})}{x(u_k - \frac{i}{h})}, \qquad \frac{x(u_k + \frac{i}{h})^L}{x(u_k - \frac{i}{h})^L} = \prod_{j=1, j \neq k}^M \frac{u_k - u_j + \frac{2i}{h}}{u_k - u_j - \frac{2i}{h}}, \qquad x(u) = \frac{1}{2} \left(u + \sqrt{u^2 - 4} \right). \quad (4.4.26)$$

The BDS factor has a pole at $x^- = y^+$, which corresponds to the formation of a bound state of two magnons [35]. In terms of the spectral parameters these Bethe equations take the form

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j=1, j \neq k}^M \frac{x_k^+ + \frac{1}{x_k^+} - \left(x_j^- + \frac{1}{x_j^-}\right)}{x_k^- + \frac{1}{x_k^-} - \left(x_j^+ + \frac{1}{x_j^+}\right)}.$$
(4.4.27)

In the semiclassical limit of a large number of excitations the Bethe roots condense to form cuts [153] and the Bethe equations reduce to the finite-gap integral equations describing the semiclassical dyonic giant magnon spectrum.

Chapter 5

Giant magnon solution and dispersion relation

In the case of string theory on $\operatorname{AdS}_5 \times \operatorname{S}^5$, in addition to the light-cone symmetry algebra considerations and the perturbative near-BMN expansion, there is a third string-theory-based source of information for the dispersion relation – the semiclassical giant magnon solution. In this chapter, based on [2], we use this third approach to complement previous work on the first two approaches [74, 75] and shed further light on the exact form of the mixed-flux dispersion relation for massive excitations. Following [22, 23, 35], we consider a giant magnon solution on S^3 with two angular momenta (J_1, J_2) and find that its energy is given by $(E, J_1 \to \infty)$

$$E - J_1 = \sqrt{M_{\pm}^2 + 4(1 - q^2)h^2 \sin^2 \frac{p}{2}} , \qquad (5.0.1)$$

$$M_{\pm} = J_2 \pm q h p$$
 . (5.0.2)

For q = 0 this reduces to the standard dispersion relation for a dyonic giant magnon [23, 35]. In the giant magnon construction the momentum p is related to the angle $\Delta \phi_1$ between the end-points of an open rigid string moving along a circle of S³ so that $p \in (-\pi, \pi)$. One may formally consider the energy as periodic in p by periodically extending (5.0.2) to the whole interval $p \in (-\infty, \infty)$.¹

The giant magnon solution is interpreted as a bound state of J_2 elementary "magnons" (string excitations) so that for $J_2 = 1$ this relation corresponds to (1.0.54) with an exact linear expression for M_{\pm} (i.e. without any higher order corrections in (1.0.55))

$$M_{\pm} = 1 \pm q h p$$
 . (5.0.3)

The resulting dispersion relation (5.0.1), (5.0.3) has the nice feature that for q = 1, i.e. in WZW model limit, it directly reduces to the expected massless dispersion relation

$$q = 1:$$
 $\varepsilon_{\pm} = 1 \pm hp$. (5.0.4)

To derive (5.0.1), (5.0.2) we shall start with the bosonic string moving in $\mathbb{R} \times S^3$ in the presence of an NS-NS flux, i.e. described by an action with a WZ term proportional to q, and consider its classical solutions (see also [74] and references therein). Some previous discussions of similar classical solutions

¹The periodicity in p becomes irrelevant in the perturbative string theory limit of $h \gg 1$ when we set $p = h^{-1}p$ for fixed p so that p goes to zero.

in this model and their charges, based on various ansatzes for the string coordinates, appeared in [195] but they will not be used here. Since the string model on $\mathbb{R} \times S^3$ in the conformal gauge reduces to a principal chiral model with a WZ term proportional to q, to find solutions for $q \neq 0$ from solutions in the q = 0 case one may use the fact that the $q \neq 0$ equations of motion written in terms of SU(2) currents are related to the q = 0 equations of motion through a world-sheet coordinate transformation.

In section 5.1 we review the classical string equations on $\mathbb{R} \times S^3$ in conformal gauge described by the SU(2) principal chiral model with a WZ term proportional to q. We then discuss the corresponding conserved charges, pointing out an ambiguity in the action related to boundary terms, and describe a procedure for constructing classical solutions for $q \neq 0$ from their q = 0 counterparts, illustrating it on the example of the rigid circular string solution on S^3 .

In section 5.2 we construct the dyonic giant magnon solution generalising the solution of [22, 23] to the $q \neq 0$ case. We find the corresponding relation between the energy, the finite angular momentum component J_2 , and the effective kink charge, equal to the separation angle $\Delta \phi_1$ between the rigid open string endpoints. Claiming that the latter should be interpreted as in [22, 23] as the magnon world-sheet momentum p, we obtain the dispersion relation (5.0.1),(5.0.2).

In section 5.3 we further justify this momentum identification by considering the limit of large angular momentum which isolates and effectively decouples fast string motion of extended slowly varying string configurations such as the giant magnon. In this limit the string motion is described by a $q \neq 0$ generalisation of the familiar Landau-Lifshitz model [196, 197]. The Landau-Lifshitz equations are known to admit a "spin wave" soliton [198, 199, 200] which may be interpreted as the large J_2 limit of the dyonic giant magnon solution. The world-sheet momentum p of this Landau-Lifshitz soliton has a straightforward definition that confirms its identification with $\Delta \phi_1$ of the giant magnon. The resulting dispersion relation represents the large J_2 limit of (5.0.1), i.e.

$$E_{\rm LL} = E - J_1 - J_2 = -qhp + \frac{2(1-q^2)h^2}{J_2}\sin^2\frac{p}{2} + O(J_2^{-2}) .$$
 (5.0.5)

In section 5.4 we revisit the discussion of the world-sheet S-matrix of the mixed-flux $AdS_3 \times S^3$ theory in [74, 75]. We first review the light-cone symmetry algebra and then suggest a modification to the conjecture for the central charge function M_{\pm} in [75], switching from (5.0.3) to (1.0.56). Doing so, we recover the semiclassical $q \neq 0$ dyonic giant magnon dispersion relation (5.0.1),(5.0.2) by considering the bound states of elementary excitations (with J_2 being the number of constituents) and taking an appropriate strong-coupling limit. The simplicity of the bound-state picture provides a strong argument in favour of the linear momentum function (5.0.3).

In section 5.5 we also comment on the relation between the dyonic giant magnon solution and the soliton of the corresponding Pohlmeyer reduced theory.

5.1 Classical string solutions on $\mathbb{R} \times S^3$ for $q \neq 0$

In this section we shall discuss the relation between the q = 0 and $q \neq 0$ classical string equations on $\mathbb{R} \times S^3$ that we will use in the following section to find the unique generalisation of the standard q = 0 dyonic giant magnon solution of [23] to $q \neq 0$. We will see that the q = 0 and $q \neq 0$ equations written in terms of the current $\mathfrak{J} = g^{-1}dg^2$ are related by a world-sheet coordinate transformation. Our strategy will be (i) to perform this world-sheet coordinate transformation on the q = 0 current of a given solution to obtain its $q \neq 0$ counterpart and (ii) starting with this new current to solve for the coordinates of the $q \neq 0$ solution.

The string action in the conformal gauge is equivalent to that of the principal chiral model with a Wess-Zumino term with the coefficient $q \in (0, 1)$

$$S = -\frac{h}{2} \left[\int d^2 \sigma \, \frac{1}{2} \operatorname{tr}(\mathfrak{J}_+ \mathfrak{J}_-) - q \int d^3 \sigma \, \frac{1}{3} \varepsilon^{abc} \operatorname{tr}(\mathfrak{J}_a \mathfrak{J}_b \mathfrak{J}_c) \right] \,, \qquad \mathfrak{J}_a = g^{-1} \partial_a g \,, \tag{5.1.1}$$

where h is the string tension, $g \in SU(2)$ and $\sigma^{\pm} = \frac{1}{2}(\tau \pm \sigma), \ \partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}.$

5.1.1 Classical equations

The equation of motion for the above action is

$$(1+q)\partial_{-}\mathfrak{J}_{+} + (1-q)\partial_{+}\mathfrak{J}_{-} = 0 , \qquad \mathfrak{J} = g^{-1}dg , \qquad (5.1.2)$$

or, equivalently

$$(1-q)\partial_{-}\mathfrak{K}_{+} + (1+q)\partial_{+}\mathfrak{K}_{-} = 0 , \qquad \mathfrak{K} = dgg^{-1} .$$
(5.1.3)

Supplemented with the flatness condition (5.1.2) can be rewritten as

$$\partial_{+}\mathfrak{J}_{-} + \frac{1}{2}(1+q)[\mathfrak{J}_{+},\mathfrak{J}_{-}] = 0 , \qquad \partial_{-}\mathfrak{J}_{+} - \frac{1}{2}(1-q)[\mathfrak{J}_{+},\mathfrak{J}_{-}] = 0 .$$
 (5.1.4)

The formal transformation of the world-sheet coordinates

$$\sigma^{\pm} \to \tilde{\sigma}^{\pm} = (1 \pm q)\sigma^{\pm} \tag{5.1.5}$$

then maps the $q \neq 0$ current equations to the q = 0 equations, provided the currents are left unaltered (i.e. this is not a conformal transformation that leaves the classical equations invariant). Furthermore, the Virasoro conditions (assuming that the target space time coordinate is $t = \kappa \tau$)

$$\operatorname{tr}(\mathfrak{J}_{\pm}^2) = -2\kappa^2 \tag{5.1.6}$$

are invariant under this transformation. Given a solution for q = 0 the map (5.1.5) allows us to construct the $q \neq 0$ counterpart, \mathfrak{J} , of the q = 0 current. It then remains to solve the defining equations of \mathfrak{J} for the function g, or, e.g., for the 4 real (2 complex) S³ embedding coordinates X_m , m = 1, ..., 4 ($Z_i, i = 1, 2$)

$$\mathfrak{J}_{\pm} = g^{-1} \partial_{\pm} g , \qquad g = \begin{pmatrix} Z_1 & Z_2 \\ -Z_2^* & Z_1^* \end{pmatrix} \in \mathrm{SU}(2) ,$$
(5.1.7)

$$Z_1 = X_1 + iX_2$$
, $Z_2 = X_3 + iX_4$, $X_m^2 = 1$, $|Z_i|^2 = 1$. (5.1.8)

²Here we denote algebra elements by \mathfrak{J} instead of using plain J notation as in part I. This is to avoid confusion with the various angular momenta denoted by J.

$$\partial_{\pm} \vec{Z} = \mathfrak{J}_{\pm}^T \vec{Z} . \tag{5.1.9}$$

Taking an additional derivative these imply

$$\partial_{+}\partial_{-}\vec{Z} = (\partial_{-}\mathfrak{J}_{+})^{T}\vec{Z} + \mathfrak{J}_{+}^{T}\mathfrak{J}_{-}^{T}\vec{Z} , \qquad \partial_{-}\partial_{+}\vec{Z} = (\partial_{+}\mathfrak{J}_{-})^{T}\vec{Z} + \mathfrak{J}_{-}^{T}\mathfrak{J}_{+}^{T}\vec{Z} .$$
(5.1.10)

Subtracting these equations gives the compatibility condition $\partial_+\partial_-\vec{Z} = \partial_-\partial_+\vec{Z}$, which corresponds to the flatness condition for \mathfrak{J} . Adding the two equations one obtains

$$\partial_{+}\partial_{-}\vec{Z} + \Omega\vec{Z} + \frac{1}{2}q[\mathfrak{J}_{+},\mathfrak{J}_{-}]^{T}\vec{Z} = 0 , \qquad \Omega = -\frac{1}{2}\mathrm{tr}(\mathfrak{J}_{+}\mathfrak{J}_{-}) , \qquad (5.1.11)$$

where have we used the fact that since \mathfrak{J}_{\pm} are traceless and anti-hermitian the anti-commutator $\{\mathfrak{J}_+,\mathfrak{J}_-\}$ is proportional to the identity.

In order to solve for \vec{Z} we decouple (5.1.9) into two first order equations

$$\mathfrak{J}_{-}^{21}\partial_{+}Z_{1} - \mathfrak{J}_{+}^{21}\partial_{-}Z_{1} + (\mathfrak{J}_{+}^{21}\mathfrak{J}_{-}^{11} - \mathfrak{J}_{-}^{21}\mathfrak{J}_{+}^{11})Z_{1} = 0 , \qquad (5.1.12)$$

$$\mathfrak{J}_{-}^{12}\partial_{+}Z_{2} - \mathfrak{J}_{+}^{12}\partial_{-}Z_{2} + (\mathfrak{J}_{+}^{12}\mathfrak{J}_{-}^{22} - \mathfrak{J}_{-}^{12}\mathfrak{J}_{+}^{22})Z_{2} = 0.$$
(5.1.13)

These linear first order partial differential equations can be solved using the method of characteristics and their solution will involve an undetermined function. At the same time, the original equations (5.1.9) are four first order equations for two variables, which uniquely determine the solution up to integration constants. Therefore, we still need to impose some additional conditions. This we can do by decoupling the second order equations (5.1.11) as

$$\partial_{+}\partial_{-}Z_{1} + q\frac{C_{12}}{\mathfrak{J}_{+}^{21}}\partial_{+}Z_{1} + \left(\Omega + qC_{11} - q\frac{C_{12}}{\mathfrak{J}_{+}^{21}}\mathfrak{J}_{+}^{11}\right)Z_{1} = 0 , \qquad (5.1.14)$$

$$\partial_{+}\partial_{-}Z_{2} + q\frac{C_{21}}{\mathfrak{J}_{+}^{12}}\partial_{+}Z_{2} + \left(\Omega + qC_{22} - q\frac{C_{21}}{\mathfrak{J}_{+}^{12}}\mathfrak{J}_{+}^{22}\right)Z_{2} = 0 , \qquad (5.1.15)$$

where $C = \frac{1}{2} [\mathfrak{J}_+, \mathfrak{J}_-]^T$. The undetermined function can then be fixed by substituting the solution of the first order equations into the above second order equations.³

5.1.2 Conserved charges

The equations of motion (5.1.2), (5.1.3) imply that we have two conserved SU(2) currents, the left-invariant and the right-invariant one

$$L_a = \mathfrak{J}_a - q\epsilon_{ab}\mathfrak{J}^b , \qquad R_a = \mathfrak{K}_a + q\epsilon_{ab}\mathfrak{K}^b , \qquad \partial_a L^a = \partial_a R^a = 0 . \tag{5.1.16}$$

³Notice that the equations (5.1.14) and (5.1.15) are related through complex conjugation. This does not imply that Z_1 and Z_2 are complex conjugates of each other as we shall see in more detail later on. The above second order differential equations admit two separate solutions corresponding to roots of a quadratic equation. Requiring that the final solution is consistent with the original q = 0 solution then uniquely fixes the choice of these roots giving rise to solutions for Z_1 and Z_2 that in general are not related by complex conjugation.

Using these we can construct conserved charges in the standard way

$$Q_L = h \int d\sigma \left(\mathfrak{J}_0 + q\mathfrak{J}_1\right), \qquad Q_R = h \int d\sigma \left(\mathfrak{K}_0 - q\mathfrak{K}_1\right). \tag{5.1.17}$$

For the dyonic giant magnon solution that is the main subject of this paper, there are a particular pair of charges that we will be interested in

$$J = -\frac{i}{4}(tr[Q_L \cdot \sigma_3] + tr[Q_R \cdot \sigma_3]) , \qquad M = -\frac{i}{4}(-tr[Q_L \cdot \sigma_3] + tr[Q_R \cdot \sigma_3]) .$$
(5.1.18)

We will parametrise the 3-sphere as

$$Z_1 = X_1 + iX_2 = \sin\theta \ e^{i\phi_1} , \qquad Z_2 = X_3 + iX_4 = \cos\theta \ e^{i\phi_2} , \qquad (5.1.19)$$

or, equivalently, in terms of the Euler angles

$$g = e^{\frac{i}{2}\theta_L \sigma_3} e^{\frac{i}{2}\psi\sigma_2} e^{\frac{i}{2}\theta_R \sigma_3} , \qquad \psi = \pi - 2\theta , \quad \theta_L = \phi_1 + \phi_2 , \quad \theta_R = \phi_1 - \phi_2 .$$
(5.1.20)

The bosonic string action (5.1.1) then takes the form

$$S = -\frac{h}{2} \int d^2\sigma \left[\partial_a \theta \partial^a \theta + \sin^2 \theta \partial_a \phi_1 \partial^a \phi_1 + \cos^2 \theta \partial_a \phi_2 \partial^a \phi_2 + q(\cos 2\theta + c)(\dot{\phi}_1 \dot{\phi}_2 - \dot{\phi}_2 \dot{\phi}_1) \right], \quad (5.1.21)$$

where a, b = 0, 1 stand for the world-sheet coordinates τ , σ with the metric $\eta = \text{diag}(-1, 1)$ and $\dot{=} \partial_{\tau}, \dot{=} \partial_{\sigma}$. The last q-dependent term comes from the Wess-Zumino term, in which the parameter c corresponds to an ambiguity in defining a local 2-d action.⁴ The c-term is a total derivative and does not, of course, affect the equations of motion. However, if we consider string solutions with non-trivial boundary conditions (which includes the case of interest – the dyonic giant magnon) then it will affect the corresponding Noether global charges as we shall discuss below.

As we are using the conformal gauge with the residual conformal symmetry fixed by choosing $t = \kappa \tau$ the Virasoro constraints take the following explicit form

$$\dot{\theta}^2 + \dot{\theta}^2 + \sin^2\theta(\dot{\phi}_1^2 + \dot{\phi}_1^2) + \cos^2\theta(\dot{\phi}_2^2 + \dot{\phi}_2^2) = \kappa^2 , \qquad \dot{\theta}\dot{\theta} + \sin^2\theta\dot{\phi}_1\dot{\phi}_1 + \cos^2\theta\dot{\phi}_2\dot{\phi}_2 = 0.$$
(5.1.22)

The translational invariance of the full string action under shifts of t, ϕ_1 and ϕ_2 leads to the following conserved Noether charges: the energy and the angular momenta (here $\sigma \in (-\pi, \pi)$)

$$E = 2\pi h \kappa , \qquad (5.1.23)$$

$$J_1 = h \int d\sigma \left[\sin^2 \theta \dot{\phi}_1 - \frac{q}{2} (\cos 2\theta + c) \dot{\phi}_2 \right] , \qquad (5.1.24)$$

$$J_2 = h \int d\sigma \left[\cos^2 \theta \dot{\phi}_2 + \frac{q}{2} (\cos 2\theta + c) \dot{\phi}_1 \right] .$$
 (5.1.25)

Here $J_{1,2}$ follow directly from the action (5.1.21). The c-terms are of course total derivatives and thus contribute only if ϕ_1 or ϕ_2 have non-trivial boundary values or if periodicity in σ is not imposed (as

⁴In general, the string couples locally to the antisymmetric *B*-field, while the defining equations – the conformal invariance conditions or, to leading order, the supergravity equations of motion – depend on the three-form field strength H. Therefore, there is a gauge freedom in choice of the *B*-field and in the presence of the boundary this necessitates a boundary term parametrising this ambiguity. At the moment we will leave it arbitrary and fix it later via natural physical requirements appropriate for the giant magnon solution.

will be the case for the giant magnon solution).

Let us compare (5.1.24), (5.1.25) with the charges J and M (5.1.18) that were derived from the SU(2)-invariant currents (5.1.17). Substituting the parametrisation (5.1.19) into (5.1.17) we find

$$\mathbf{J} = \mathbf{h} \int \mathrm{d}\sigma \left[\sin^2 \theta \dot{\phi}_1 - \frac{q}{2} (\cos 2\theta + 1) \dot{\phi}_2 \right] \,, \tag{5.1.26}$$

$$M = h \int d\sigma \left[\cos^2 \theta \dot{\phi}_2 + \frac{q}{2} (\cos 2\theta - 1) \dot{\phi}_1 \right], \qquad (5.1.27)$$

and hence

$$J_1 = J - \frac{1}{2}hq(c-1)\Delta\phi_2 , \qquad J_2 = M - \frac{1}{2}hq(c+1)\Delta\phi_1 , \qquad (5.1.28)$$

$$\Delta \phi_i = \phi_i(\pi) - \phi_i(-\pi) .$$
 (5.1.29)

Thus to match J and M with J_1 and J_2 we need different choices of c $(=\pm 1)$, i.e. J and M cannot be obtained as Noether charges from a local action (5.1.21) with equations of motion equivalent to (5.1.2). This, of course, is not a contradiction as the difference appears only due to the boundary "twist" terms $\Delta \phi_i$, but if non-zero such terms break manifest SU(2) symmetry.⁵

The dyonic giant magnon solution we will be interested in is a classical soliton representing a "bound state" of string excitations above the BMN vacuum. The latter corresponds to a point-like string moving along a great circle of S^3

$$\theta = \frac{\pi}{2}$$
, $\phi_1 = \kappa \tau$, $\phi_2 = 0$. (5.1.30)

For the point-like BMN solution

$$E - J_1 = 0$$
, $J_1 = J$. (5.1.31)

In the q = 0 case, the giant magnon limit [22] involves taking both E and J_1 to infinity (i.e. $\kappa \to \infty$) with their difference held fixed

$$E, J_1 \to \infty, \qquad \epsilon \equiv E - J_1, J_2 = \text{fixed} .$$
 (5.1.32)

Also, as in [22] the string is assumed to be open so that rescaling τ and σ by $\kappa \to \infty$ the spatial interval may be decompactified

$$x = \kappa \sigma$$
, $\kappa \to \infty$, $x \in (-\infty, +\infty)$, (5.1.33)

and the non-zero angle between the end points of the string

$$\Delta \phi_1 = \phi_1(x = \infty) - \phi_1(x = -\infty) , \qquad (5.1.34)$$

may be related to the 2-d momentum p. Then ϵ , which plays the role of the energy of the state relative to the BMN vacuum, can be expressed as a function of p and J_2 .

As we shall see below, for $q \neq 0$ the requirement that $E - J_1$ remains finite in the $\kappa \to \infty$ limit (and

⁵Let us note also that in general, the currents conserved $(\partial_a j_i^a = 0)$ on the equations of motion are defined modulo a trivial term $\epsilon^{ab}\partial_b f_i$ where the functions f_i (that may, in principle, break some manifest global symmetries) contribute to the corresponding charges only if they have non-trivial boundary twists.

also the classical action is finite when evaluated on one period of the dyonic giant magnon solution) implies that

$$c = 1$$
. (5.1.35)

In this case the action (5.1.21) becomes explicitly

$$S = -\frac{h}{2} \int d^2 \sigma \left(\partial_a \theta \partial^a \theta + \sin^2 \theta \, \partial_a \phi_1 \partial^a \phi_1 + \cos^2 \theta \, \partial_a \phi_2 \partial^a \phi_2 + 2q \epsilon^{ab} \cos^2 \theta \, \partial_a \phi_1 \partial_b \phi_2 \right) \,. \tag{5.1.36}$$

The physical reason for this particular choice of B_{mn} -term in the string action is that it vanishes at $\theta = \frac{\pi}{2}$. This implies the vanishing of force on the ends of the open string (representing the giant magnon solution) moving along the great circle corresponding to ϕ_1 . As usual, the boundary term in the variation of the string action specifies the boundary conditions for the open string end-points. The variation of (5.1.36) under the variation of ϕ_1 gives the condition $\int d\tau \ \delta \phi_1 (\sin^2 \theta \partial_\sigma \phi_1 - q \cos^2 \theta \partial_\tau \phi_2) \Big|_{\sigma=0,\pi} = 0$. Since the end-points of the giant magnon move along the great circle $\theta|_{\sigma=0,\pi} = \frac{\pi}{2}$ the q-dependent term vanishes and we just have the standard free-ends condition $\partial_\sigma \phi_1|_{\sigma=0,\pi} = 0$.

From (5.1.29) we then get

$$J = J_1$$
, $M = J_2 - qh\Delta\phi_1$. (5.1.37)

Here $\Delta \phi_1$ plays the role of kink charge, which, as for q = 0, can be identified with the 2-d spatial momentum p of the soliton. Recalling that the quantized WZ level k is related to q as $k = 2\pi hq$, we may write M as

$$M = J_2 - qhp = J_2 - k\frac{p}{2\pi}$$
, $p = \Delta\phi_1$. (5.1.38)

Here $\Delta \phi_1 \in (0, \pi)$ but being an angular coordinate it may defined modulo 2π , and then the same may be assumed about p, i.e. M may be considered as a periodic function of p.⁶

Also, for a physical closed string $\Delta \phi_1$ should be equal to $2\pi n$ where n is an integer winding number, or, equivalently, the total momentum of a bound state of magnons representing a physical state should be quantized

$$\sum_{i} \mathbf{p}_i = 2\pi n \ . \tag{5.1.39}$$

This is consistent with both M and J_2 in (5.1.38) taking integer values for such states.

This relation between M and J_2 is suggestive of how the dyonic giant magnon dispersion relation is to be modified in the presence of the NS-NS flux (cf. (5.0.1), (5.0.2)).

5.1.3 An example of a solution: rigid circular string

Before turning to the construction of the giant magnon solution for $q \neq 0$ let us illustrate the general procedure of finding $q \neq 0$ solutions from their q = 0 counterparts on the example of a rigid circular string on S³ [201, 143]. The standard q = 0 solution written in the embedding coordinates reads

$$Z_{1} = \frac{1}{\sqrt{2}} \exp[i(\omega + m)\sigma^{+} + i(\omega - m)\sigma^{-}],$$

$$Z_{2} = \frac{1}{\sqrt{2}} \exp[i(\omega - m)\sigma^{+} + i(\omega + m)\sigma^{-}], \qquad m^{2} + \omega^{2} = \kappa^{2}. \qquad (5.1.40)$$

⁶The issue of periodicity is a subtle one and we will return to it later in section 5.2.3 after we have derived the relevant expressions for the energy and angular momenta of the dyonic giant magnon as functions of the solution parameters (which include $\Delta \phi_1$).

For this solution the SU(2) currents are (see (5.1.7))

$$\mathfrak{J}_{+} = i \begin{pmatrix} m & \omega \exp[-2im(\sigma^{+} - \sigma^{-})] \\ \omega \exp[2im(\sigma^{+} - \sigma^{-})] & -m \end{pmatrix}, \\
\mathfrak{J}_{-} = i \begin{pmatrix} -m & \omega \exp[-2im(\sigma^{+} - \sigma^{-})] \\ \omega \exp[2im(\sigma^{+} - \sigma^{-})] & m \end{pmatrix}, \\
\Omega = -\frac{1}{2} \operatorname{tr}(\mathfrak{J}_{+}\mathfrak{J}_{-}) = \omega^{2} - m^{2}.$$
(5.1.41)

Performing the world-sheet coordinate transformation (5.1.5) gives the $q \neq 0$ currents

$$\begin{aligned} \mathfrak{J}_{+} &= i \begin{pmatrix} m & \omega \exp[-2im[\sigma^{+} - \sigma^{-} + q(\sigma^{+} + \sigma^{-})]] \\ \omega \exp[2im[\sigma^{+} - \sigma^{-} + q(\sigma^{+} + \sigma^{-})]] & -m \end{pmatrix}, \\ \mathfrak{J}_{-} &= i \begin{pmatrix} -m & \omega \exp[-2im[\sigma^{+} - \sigma^{-} + q(\sigma^{+} + \sigma^{-})]] \\ \omega \exp[2im[\sigma^{+} - \sigma^{-} + q(\sigma^{+} + \sigma^{-})]] & m \end{pmatrix}, \\ C &= 2m\omega \begin{pmatrix} 0 & \exp[2im(\sigma^{+} - \sigma^{-} + q(\sigma^{+} + \sigma^{-}))] \\ -\exp[-2im(\sigma^{+} - \sigma^{-} + q(\sigma^{+} + \sigma^{-}))] & 0 \end{pmatrix}, \quad (5.1.42) \end{aligned}$$

with Ω unchanged, while the decoupled equations for the embedding coordinates (5.1.14)-(5.1.15) become

$$\partial_{+}\partial_{-}Z_{1} - 2qmi\,\partial_{+}Z_{1} + \left(\omega^{2} - m^{2} - 2qm^{2}\right)Z_{1} = 0$$
, (5.1.43)

$$\partial_{+}\partial_{-}Z_{2} + 2qmi\,\partial_{+}Z_{2} + \left(\omega^{2} - m^{2} - 2qm^{2}\right)Z_{2} = 0$$
 (5.1.44)

Fourier decomposing the solution as

$$Z_1 = a_n \exp[i\mu_n \tau + in\sigma] , \qquad Z_2 = b_n \exp[i\nu_n \tau - in\sigma] , \qquad (5.1.45)$$

and requiring that the modes reduce to those of the q = 0 circular string solution one obtains

$$\mu_m = qm + \sqrt{q^2 m^2 + \omega^2} , \qquad \nu_m = -qm + \sqrt{q^2 m^2 + \omega^2} . \qquad (5.1.46)$$

The normalisation condition $|Z_1|^2 + |Z_2|^2 = 1$ together with the Virasoro constraints $\partial_{\pm} Z_1 \partial_{\pm} Z_1^* + \partial_{\pm} Z_2 \partial_{\pm} Z_2^* = \kappa^2 = \omega^2 + m^2$ then determine a_n and b_n up to a phase giving

$$Z_1 = \sqrt{\frac{W - qm}{2W}} \exp(i[(W + qm)\tau + m\sigma]) , \qquad (5.1.47)$$

$$Z_2 = \sqrt{\frac{W+qm}{2W}} \exp(i[(W-qm)\tau - m\sigma]) , \qquad W = \sqrt{\omega^2 + q^2m^2} .$$
 (5.1.48)

In the parametrisation (5.1.19) this solution takes the form

$$\sin \theta = \sqrt{\frac{W - qm}{2W}}, \qquad \phi_1 = (W + qm)\tau + m\sigma, \qquad \phi_2 = (W - qm)\tau - m\sigma.$$
 (5.1.49)

The two angular momenta associated to shifts in ϕ_1 and ϕ_2 computed from (5.1.24),(5.1.25) are

$$J_1 = J_2 = \pi h(W + cqm) , \qquad (5.1.50)$$

where c parametrises the ambiguity in the choice of the total derivative term in the action (5.1.21). The expression for energy then takes the form

$$E = 2\pi h\kappa = \sqrt{(J - 2\pi hcqm)^2 + 4\pi^2 h^2 m^2 (1 - q^2)}, \qquad J \equiv J_1 + J_2 = 2J_1.$$
(5.1.51)

Expanding in large J we get

$$E = J - 2\pi hcqm + \frac{2\pi^2 h^2 (1 - q^2)m^2}{J} + O(J^{-2}) .$$
 (5.1.52)

The choice c = 0 here gives the standard BMN limit E = J when $J \to \infty$.

5.2 Dyonic giant magnon on $\mathbb{R} \times S^3$ in the presence of NS-NS flux

5.2.1 Review of q = 0 case

Let us start with a review of the standard dyonic giant magnon solution on S³ in the absence of an NS-NS flux, i.e. for q = 0 in the action (5.1.21). In the notation of section 5.1 the dyonic giant magnon solution, labelled by the two independent parameters b (or v) and ρ , takes the form [23]

$$Z_1 = \frac{\left[b + i \tanh(\mathcal{X}\cos\rho)\right] \exp(it)}{(1+b^2)^{1/2}} , \qquad Z_2 = \frac{\operatorname{sech}(\mathcal{X}\cos\rho) \exp(i\mathcal{T}\sin\rho)}{(1+b^2)^{1/2}} , \qquad (5.2.1)$$

$$b = \frac{v \sec \rho}{\sqrt{1 - v^2}} , \qquad v \in (0, 1) , \qquad \rho \in (0, \frac{\pi}{2}) , \qquad b \in (0, \infty) , \qquad (5.2.2)$$

where \mathcal{X} and \mathcal{T} are related to the world-sheet coordinates τ , σ through a boost of velocity v and a rescaling by κ

$$\mathcal{X} = \frac{x - vt}{\sqrt{1 - v^2}}, \qquad \qquad \mathcal{T} = \frac{t - vx}{\sqrt{1 - v^2}}, \qquad (5.2.3)$$

$$t = \kappa \tau, \quad x = \kappa \sigma, \qquad \tau \in (-\infty, \infty), \qquad \sigma \in (-\pi, \pi), \qquad x \in (-\infty, \infty).$$
 (5.2.4)

Here we have already taken the limit $\kappa \to \infty$ and thus "decompactified" the spatial direction x. $x \to \pm \infty$ correspond to the endpoints of the string moving in the ϕ_1 direction, while $\rho \in (0, \frac{\pi}{2})$ is the parameter associated with the angular momentum in the ϕ_2 direction. We may of course extend the parameter ranges so that $v \in (-1, 1)$, $\rho \in (-\pi, \pi) \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ to cover also the soliton moving in the opposite direction. These ranges correspond to $b \in (-\infty, \infty)$ and hence $\Delta \phi_1 \in (-\pi, \pi)$.

This solution satisfies the boundary conditions

$$x \to \pm \infty$$
: $Z_1 \to \exp\left(it \pm i\frac{\Delta\phi_1}{2}\right)$, $Z_2 \to 0$, (5.2.5)

where

$$\Delta \phi_1 = 2 \arctan b^{-1} \quad \in (0,\pi) \tag{5.2.6}$$

corresponds to the angle between the rigid open string endpoints which move in the ϕ_1 direction on the great circle $\theta = \frac{\pi}{2}$.

The finite combination of energy E with J_1 and the angular momentum J_2 for this solution are

given by

$$E - J_1 = \frac{2h}{1+b^2} \frac{(1+b^2\cos^2\rho)^{1/2}}{\cos\rho} , \qquad J_2 = \frac{2h}{1+b^2} \tan\rho . \qquad (5.2.7)$$

The case of $\rho = 0$ thus corresponds to the S² giant magnon of [22] (then Z_2 in (5.2.1) becomes real) with $J_2 = 0$. In addition to J_2 another "charge" parameter of this solution is the "kink charge" $\Delta \phi_1$. Expressing the energy in terms of these charges we get

$$E - J_1 = \sqrt{J_2^2 + \frac{4h^2}{1+b^2}} = \sqrt{J_2^2 + 4h^2 \sin^2 \frac{\Delta\phi_1}{2}} .$$
 (5.2.8)

This becomes the usual dyonic giant magnon dispersion relation upon the identification [22, 23] of the magnon momentum with the separation angle: $p = \Delta \phi_1$.

Let us mention that if one considers a more general solution where the string moves also along an S^1 in the torus part of $AdS_3 \times S^3 \times T^4$ then the dispersion relation (5.2.8) is modified as follows:

$$\sqrt{E^2 - P^2} - J_1 = \sqrt{J_2^2 + 4h^2 \sin^2 \frac{p}{2}} , \qquad (5.2.9)$$

where P is the (large) momentum in S¹ with E, P and J_1 scaling as $\kappa \to \infty$. This follows simply from the formal Lorentz invariance in the $\mathbb{R}_t \times S^1_{\psi}$ subspace in the decompactification limit (equivalently, the contribution of the \mathbb{R}_t and S^1_{ψ} to the Virasoro condition can be absorbed into a rescaling of τ and σ).

5.2.2 Dyonic giant magnon for $q \neq 0$

Let us now generalise the above solution to the $q \neq 0$ case using the procedure outlined in section 5.1. First we are to re-express the current, constructed from the q = 0 solution (5.2.1) via (5.1.7), in terms of $\tilde{\sigma}^{\pm}$, defined in (5.1.5), giving us the current \mathfrak{J} of the $q \neq 0$ solution. Anticipating that the $q \neq 0$ solution is again most conveniently written in terms of boosted world-sheet coordinates we introduce the boosted world-sheet coordinates $\tilde{\mathcal{X}}, \tilde{\mathcal{T}}$ which are related to the $q \neq 0$ world-sheet coordinates \tilde{t}, \tilde{x} by a boost of velocity v, i.e.

$$\tilde{\sigma}^{\pm} = (1 \pm q)\sigma^{\pm} = \frac{1}{2}(\tilde{\tau} \pm \tilde{\sigma}) , \qquad \tilde{t} = \kappa \tilde{\tau} , \quad \tilde{x} = \kappa \tilde{\sigma} , \qquad \tilde{\mathcal{X}} = \frac{\tilde{x} - v\tilde{t}}{\sqrt{1 - v^2}} , \quad \tilde{\mathcal{T}} = \frac{\tilde{t} - v\tilde{x}}{\sqrt{1 - v^2}} .$$
(5.2.10)

Note that the boosted world-sheet coordinates of the q = 0 and $q \neq 0$ cases are related via

$$\tilde{\mathcal{X}} = \mathcal{X} + q\mathcal{T}$$
, $\tilde{\mathcal{T}} = \mathcal{T} + q\mathcal{X}$. (5.2.11)

It is useful also to introduce rescaled coordinates $\tilde{\xi}$ and $\tilde{\eta}$, defined as

$$\tilde{\xi} = \tilde{\mathcal{X}} \cos \rho , \qquad \qquad \tilde{\eta} = \tilde{\mathcal{T}} \sin \rho . \qquad (5.2.12)$$

The coordinate transformation from the original light-cone coordinates σ^{\pm} to $\tilde{\xi}$, $\tilde{\eta}$ is then

$$\sigma^{+} = \frac{1}{2\kappa(1+q)}\sqrt{\frac{1+v}{1-v}}\left(\frac{\tilde{\eta}}{\sin\rho} + \frac{\tilde{\xi}}{\cos\rho}\right), \qquad \sigma^{-} = \frac{1}{2\kappa(1-q)}\sqrt{\frac{1-v}{1+v}}\left(\frac{\tilde{\eta}}{\sin\rho} - \frac{\tilde{\xi}}{\cos\rho}\right).$$
(5.2.13)

Written in $\tilde{\xi}$, $\tilde{\eta}$ coordinates the equations (5.1.12)-(5.1.13) become

$$\partial_{\tilde{\xi}} Z_1 + A \,\partial_{\tilde{\eta}} Z_1 + B \,Z_1 = 0 , \qquad \partial_{\tilde{\xi}} Z_2 + A^* \,\partial_{\tilde{\eta}} Z_2 + B^* \,Z_2 = 0 , \qquad (5.2.14)$$

where

$$A = \tan \rho \,\frac{(1+q)\sqrt{\frac{1-v}{1+v}}\mathfrak{J}_{-}^{21} - (1-q)\sqrt{\frac{1+v}{1-v}}\mathfrak{J}_{+}^{21}}{(1+q)\sqrt{\frac{1-v}{1+v}}\mathfrak{J}_{-}^{21} + (1-q)\sqrt{\frac{1+v}{1-v}}\mathfrak{J}_{+}^{21}} = \tan \rho \,\frac{m+qb+iq\tanh\tilde{\xi}}{qm+b+i\tanh\tilde{\xi}} \,, \tag{5.2.15}$$

$$B = \frac{\kappa^{-1} \cos^{-1} \rho(\mathfrak{J}_{+}^{21} \mathfrak{J}_{-}^{11} - \mathfrak{J}_{-}^{21} \mathfrak{J}_{+}^{11})}{(1+q)\sqrt{\frac{1-v}{1+v}} \mathfrak{J}_{-}^{21} + (1-q)\sqrt{\frac{1+v}{1-v}} \mathfrak{J}_{+}^{21}} = -i \frac{\operatorname{sech}^{2} \xi + (k+i \tanh{\tilde{\xi}})(b+iu \tanh{\tilde{\xi}})}{(u+1)(b+qm+i \tanh{\tilde{\xi}})} , \qquad (5.2.16)$$
$$u = \frac{\sqrt{1+b^{2} \cos^{2} \rho}}{\sin \rho} , \qquad m = (1-u) \tan \rho . \qquad (5.2.17)$$

We can then write (5.2.14) as an ordinary differential equation

$$\frac{\mathrm{d}Z_1}{\mathrm{d}\tilde{\xi}} + BZ_1 = 0 \tag{5.2.18}$$

valid along the characteristic curve

$$\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\tilde{\xi}} = A(\tilde{\xi}) \quad \Rightarrow \quad \tilde{\eta} = \int \mathrm{d}\tilde{\xi} A(\tilde{\xi}) + C_0 \;. \tag{5.2.19}$$

Evaluating the integrals of A and B we obtain

$$I_{1} = \int d\tilde{\xi} B(\tilde{\xi}) = -\frac{2i[k+k^{2}s_{1}+(k-s_{1})u]\tilde{\xi}}{2(1+s_{1}^{2})(1+u)} + \ln\cosh\tilde{\xi} \\ -\frac{[1+k^{2}+s_{1}(s_{1}-k)(1+u)]\ln[2(s_{1}\cosh\tilde{\xi}+i\sinh\tilde{\xi})^{2}]}{2(1+s_{1}^{2})(1+u)} ,$$

$$I_{2} = \int d\tilde{\xi} A(\tilde{\xi}) = \tan\rho \frac{2(1+s_{1}s_{2})\tilde{\xi}-im(1-q^{2})\ln[2(s_{1}\cosh\tilde{\xi}+i\sinh\tilde{\xi})^{2}]}{2(1+s_{1}^{2})} ,$$

$$s_{1} = k + qm , \qquad s_{2} = m + qk . \qquad (5.2.20)$$

The solution for Z_1 is then obtained by integrating (5.2.18)

$$Z_1 = f(C_0(\tilde{\xi}, \tilde{\eta})) \exp\left[-\int \mathrm{d}\tilde{\xi} B(\tilde{\xi})\right] = f(\tilde{\eta} - I_2(\tilde{\xi})) \exp\left[-I_1(\tilde{\xi})\right].$$
(5.2.21)

We can determine f by substituting this solution into (5.1.14). This gives

$$f''(x) - 2rf'(x) + r^2 - \delta^2 = 0 , \qquad (5.2.22)$$

$$r \equiv \frac{i}{2} \left(\frac{1}{\sin \rho \sqrt{1 - v^2}} - 1 \right) , \qquad \delta \equiv \frac{i}{2} \left(1 + \frac{1 + q(q - 2v)}{\sin \rho (1 - q^2) \sqrt{1 - v^2}} \right) , \qquad (5.2.23)$$

which has the solutions

$$f(x) = e^{a_{\pm}x}$$
, $a_{\pm} = r \pm \delta$. (5.2.24)

Requiring that in the limit $q \to 0$ we recover the dyonic giant magnon solution (5.2.1) leads to

$$f(z) = e^{a_{+}z} = \exp\left(i\frac{z}{\sin\rho\sqrt{1-v^2}}\frac{1-qv}{1-q^2}\right).$$
(5.2.25)

We can now determine the Z_2 solution by taking the complex conjugate of (5.2.21), but to ensure the correct q = 0 limit in this case we should take

$$f(z) = e^{a_{-}z} = \exp\left(iz\left[\frac{q(q-v)}{\sin\rho(1-q^2)\sqrt{1-v^2}} + 1\right]\right).$$
(5.2.26)

After fixing the normalisation constants using the Virasoro condition and $|Z_1|^2 + |Z_2|^2 = 1$ we obtain the solution written in terms of the original \mathcal{X} , \mathcal{T} coordinates (5.2.3)

$$Z_{1} = \frac{\left(\tilde{b} + i \tanh[\cos\rho(\mathcal{X} + q\mathcal{T})]\right) \exp(it)}{(1 + \tilde{b}^{2})^{1/2}} , \qquad (5.2.27)$$

$$Z_2 = \frac{\operatorname{sech}[\cos\rho(\mathcal{X}+q\mathcal{T})] \exp\left(i\left[\sin\rho(\mathcal{T}+q\mathcal{X})-qx\right]\right)}{(1+\tilde{b}^2)^{1/2}} , \qquad (5.2.28)$$

$$\tilde{b} = \sec \rho \left(\frac{v - q}{\sqrt{1 - v^2}} + q \sin \rho \right) .$$
(5.2.29)

This generalises (5.2.1), (5.2.2) to the $q \neq 0$ case. It is straightforward to verify that the solution (5.2.27)-(5.2.29) satisfies the defining equations (5.1.9). Written in the parametrisation (5.1.19) it takes the form⁷

$$\cos \theta = \frac{\operatorname{sech}[\cos \rho \ (\mathcal{X} + q\mathcal{T})]}{(1 + \tilde{b}^2)^{1/2}} , \qquad (5.2.30)$$

$$\phi_1 = t + \arctan\left(\tilde{b}^{-1} \tanh\left[\cos\rho \left(\mathcal{X} + q\mathcal{T}\right)\right]\right), \qquad \phi_2 = \sin\rho \left(\mathcal{T} + q\mathcal{X}\right) - qx , \qquad (5.2.31)$$

where as in (5.2.3) here $\mathcal{X} = \frac{x - vt}{\sqrt{1 - v^2}}$, $\mathcal{T} = \frac{t - vx}{\sqrt{1 - v^2}}$.

The asymptotics of this $q \neq 0$ dyonic giant magnon solution (5.2.27),(5.2.28) have the same form as in the q = 0 case (5.2.5)

$$x \to \pm \infty$$
: $Z_1 \to \exp\left(it \pm i\frac{\Delta\phi_1}{2}\right)$, $Z_2 \to 0$, (5.2.32)

$$\Delta \phi_1 = 2 \arctan \tilde{b}^{-1} \in (0, \pi) .$$
 (5.2.33)

Here we have restricted so that $\Delta \phi_1 \in (0, \pi)$, corresponding to $\tilde{b} \in (0, \infty)$. As in the q = 0 case these ranges can be extended to $(-\pi, \pi)$ and $(-\infty, \infty)$ respectively.

5.2.3 Conserved charges and dispersion relation

For the q = 0 dyonic giant magnon the energy E and the angular momentum J_1 diverge with their difference staying finite. This is no longer true in general for $q \neq 0$: the behaviour of $E - J_1$ happens to depend on the definition of J_1 in (5.1.24) which is sensitive to the total derivative ambiguity (~ c)

⁷One can check that this solution remains valid also for q = 1 (it satisfies the Virasoro constraints and equations of motion for (5.1.21)) even though the world-sheet coordinate transformation (5.1.5) which we used to derive it becomes degenerate. Furthermore, written in terms of the group element (5.1.20) the solution factorises as expected: $g = \exp\left(\frac{i}{2}(t-x)\sigma_3\right) \cdot g_R(t+x)$. It is interesting to note that the right-invariant current is particularly simple in this limit: $\partial_-gg^{-1} = i\sigma_3$.

in the Wess-Zumino term in (5.1.21).⁸ We find from (5.1.23)-(5.1.25)

$$E - J_1 = h \int_{-\infty}^{\infty} dx \left(1 - \left[\sin^2 \theta \partial_t \phi_1 - \frac{q}{2} (\cos 2\theta + c) \partial_x \phi_2 \right] \right), \qquad (5.2.34)$$

$$J_2 = h \int_{-\infty}^{\infty} dx \left[\cos^2 \theta \partial_t \phi_2 + \frac{q}{2} (\cos 2\theta + c) \partial_x \phi_1 \right], \qquad (5.2.35)$$

where we used the rescaled world-sheet coordinates $(t, x) = (\kappa \tau, \kappa \sigma)$ with $t, x \in (-\infty, \infty)$. Computing these integrals for the solution (5.2.30), (5.2.31) we find

$$E - J_1 = 2h \frac{\sqrt{1 - q^2 + (\tilde{b}\cos\rho - q\sin\rho)^2}}{(1 + \tilde{b}^2)\cos\rho} + \frac{1}{2}hq(c-1)\Delta\phi_2 , \qquad (5.2.36)$$

$$\Delta\phi_2 = -\cos\rho \,\left(q\cos\rho + \tilde{b}\sin\rho\right) \, x \Big|_{-\infty}^{\infty} \,, \tag{5.2.37}$$

$$J_2 = M + \frac{1}{2}(c+1)hq\Delta\phi_1 , \qquad M = 2h\frac{\tan\rho - qb}{1 + \tilde{b}^2} , \qquad (5.2.38)$$

where $\Delta \phi_1$ is given in (5.2.33), the divergent expression for $\Delta \phi_2 = \phi_2(x = \infty) - \phi_2(x = -\infty)$ follows from (5.2.31) and M was defined in (5.1.18),(5.1.27). We conclude that $E - J_1$ is finite only if $c = 1.^9$ Remarkably, this is exactly the case (cf. (5.1.28)) when the charge J_1 (5.1.24) coincides with J in (5.1.18),(5.1.26) which corresponds to manifestly SU(2) invariant current.

Eliminating ρ and expressing \tilde{b} in terms of $\Delta \phi_1$ in (5.2.33) gives

c = 1:
$$E - J_1 = \sqrt{M^2 + 4h^2(1 - q^2)\sin^2\frac{\Delta\phi_1}{2}}, \qquad M = J_2 - qh\Delta\phi_1.$$
 (5.2.39)

Let us comment on the values of parameters here (with $q \in (0,1)$). As in the q = 0 case, when constructing the solution we restrict to $\Delta \phi_1 \in (0,\pi)$, or equivalently $\tilde{b} \in (0,\infty)$. Taking also $\rho \in (0,\frac{\pi}{2})$, this implies the restriction $v > v_*(q,\rho) > v_*(q,0) = q$, where v_* is a function of q and ρ whose explicit form follows from (5.2.29). As before, we may extend the parameter ranges so that $v \in (-1,1)$ and $\rho \in (-\pi,\pi) \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ and thus $\tilde{b} \in (-\infty,\infty)$ and $\Delta \phi_1 \in (-\pi,\pi)$.

Note also that M in (5.2.38) is single-valued; $J_2 = M + qh\Delta\phi_1$ formally shifts if we shift $\Delta\phi_1$ by its period. As was already mentioned in section 5.1.2, the shift is integer as the WZ level $k = 2\pi hq$ should be quantized.

It remains to relate the "kink charge" $\Delta \phi_1$ (5.2.33) to the world-sheet momentum p. In general, there is no universal definition of world-sheet momentum (the total momentum vanishes as we are dealing with a reparametrisation-invariant theory). In the present case the preferred gauge used to define the near-BMN S-matrix is the uniform light-cone gauge (see [5] for a review). In the q = 0 case the momentum was identified in [22, 23] with the angular separation

$$\mathbf{p} = \Delta \phi_1 \;, \tag{5.2.40}$$

and this relation was indeed demonstrated to apply in the uniform light-cone gauge for the original

⁸The infinite contribution of the total derivative term comes from the infinite (in the $\kappa \to \infty$ limit) number of "windings" of the $q \neq 0$ giant magnon around the circle of ϕ_2 , i.e. $\Delta \phi_2 = \infty$ in (5.2.31).

⁹Let us note again that this choice is not related to the parameters of the solution itself but only to the total derivative term in the action (5.1.21) or to the definition of the corresponding Noether charge J_1 . Let us mention also that the importance of similar WZ-term related boundary terms in the presence of non-trivial kinks was emphasised in a similar context in [82].

 $(J_2 = 0)$ [22] giant magnon [202, 5]. Heuristically, the relation (5.2.40) is not expected to change upon switching on non-zero values of J_2 and q. First, there should be no momentum flow in ϕ_2 direction in (5.2.31) as it is linear in t and x, i.e. the relevant momentum should be associated with ϕ_1 . Expressing ϕ_1 and ϕ_2 in terms of two other string coordinates – t and θ in (5.2.30),(5.2.31) and treating θ as a spatial coordinate along the string ($\cos \theta$ changes from its maximal value to zero and then back) we get

$$\phi_1(t,\theta) = t + \arctan\left[\tilde{b}^{-1}\sqrt{1 - (1 + \tilde{b}^2)\cos^2\theta}\right] , \qquad (5.2.41)$$

$$\phi_2(t,\theta) = wt + r \operatorname{arccosh}\left[\left(\sqrt{1 + \tilde{b}^2 \cos\theta}\right)^{-1}\right], \qquad (5.2.42)$$

$$w = \frac{(1-q^2)\sin\rho\sqrt{1-v^2} - q(v-q)}{1-qv} = \frac{\sqrt{1-q^2 + (\tilde{b}\cos\rho - q\sin\rho)^2}}{\sin\rho - q\tilde{b}\cos\rho} , \qquad \mathbf{T} \equiv \frac{2\pi}{|w|} , \qquad (5.2.43)$$

$$r = w \frac{q + \tilde{b} \tan \rho}{q\tilde{b} - \tan \rho} .$$
(5.2.44)

In this form we can also see the qualitative features of the solution: the string motion is controlled by the three parameters \tilde{b} , w and r. In particular,

- (i) $\tilde{b} = \cot \frac{\Delta \phi_1}{2} \in (-\infty, \infty)$ controls the separation between the string end points,
- (ii) $w \in (-1, 1)$ controls the angular velocity in the ϕ_2 direction,
- (iii) $r \in (-\infty, \infty)$ controls the winding in the ϕ_2 direction.

From the expressions (5.2.29), (5.2.43) and (5.2.44) for \tilde{b} , w and r we see that the NS-NS flux does not introduce any new features for the motion, i.e. all of the above three behaviours are also present for q = 0. Instead for a given value of q there are only two independent parameters for the solution, e.g. (v, ρ) or (\tilde{b}, ρ) , and only certain combinations of the three features are allowed. Therefore the presence of the NS-NS flux allows for additional new $q \neq 0$ combinations of these features.

In (5.2.41)-(5.2.44) the independent parameters are ρ and \tilde{b} associated, respectively, with two conserved charges – J_2 and p (see (5.2.33),(5.2.38)). The expression for ϕ_1 has indeed the same form as for the $J_2 = 0$, q = 0 case, i.e. it depends on ρ (or J_2) and q only via \tilde{b} in (5.2.29). Then a natural definition of the world-sheet momentum corresponding to ϕ_1 is

$$\mathbf{p} = \int d\theta \ \partial_t \phi_1 \ \partial_\theta \phi_1 = \int d\theta \ \partial_\theta \phi_1 = \Delta \phi_1 \ , \tag{5.2.45}$$

where we have taken into account that $\partial_t \phi_1(t, \theta) = 1$. The same conclusion is indeed reached in the uniform light-cone gauge where one has [5]

$$x_{-} = \varphi - t, \qquad x_{+} = (1 - a)t + a\varphi = \tau, \qquad p_{+} = (1 - a)p_{\varphi} - ap_{t} = 1, \qquad p_{-} = p_{\varphi} + p_{t}.$$
 (5.2.46)

Here a is a gauge parameter (we ignore winding in the φ direction as we are interested in the decompactification limit $J_1 \to \infty$). The Virasoro condition, which is unchanged by the presence of the WZ term $\sim q$, then implies

$$\dot{x}_{-} + p_i \dot{x}^i = 0. \tag{5.2.47}$$

In the present case φ is to be identified with ϕ_1 (see [74]) and x^i stand for all other "transverse" coordinates. Thus the world-sheet momentum is

$$p_{\rm ws} \equiv -\int d\sigma \ p_i \acute{x}^i = \int d\sigma \ \acute{x}_- = \Delta \varphi = \Delta \phi_1 = {\rm p.}$$
(5.2.48)

Therefore the relation between world-sheet momentum and $\Delta \phi_1$ does not depend on the gauge parameter *a*, which was observed for q = 0 in [202] and agrees also with the near-BMN expansion for $q \neq 0$ in [74].

Let us note that the momentum for a rigid moving wave-type soliton, given by some profile function $\varphi = \varphi(x - vt)$, can be defined as

$$\mathbf{p} = \int \mathrm{d}x \, p_{\varphi} \, \varphi', \tag{5.2.49}$$

where p_{φ} is the conjugate momentum density for φ . For the S² giant magnon, which is described in light-cone gauge by a constant J_1 -density, this gives $p = 2 \arccos v$ [202]. The relation $\cos \frac{p}{2} = v$ between p and the soliton center of mass velocity v also generalises to $J_2 \neq 0$ and $q \neq 0$ (v = v in (5.2.1) when $\rho = 0$). From (5.2.27),(5.2.28) the string centre of mass coordinates are

$$z_i = \lim_{\kappa \to \infty} \int_{-\pi}^{\pi} \frac{\mathrm{d}\sigma}{2\pi} Z_i(t, \kappa\sigma), \qquad z_1 = \frac{\tilde{b}}{\sqrt{1+\tilde{b}^2}} e^{it}, \qquad z_2 = 0, \tag{5.2.50}$$

i.e. they describe a motion along a circle in the (X_1, X_2) plane with linear (tangent) velocity given by (cf. (5.2.33))

$$\mathbf{v} = \frac{\tilde{b}}{\sqrt{1+\tilde{b}^2}} = \cos\frac{\Delta\phi_1}{2}.$$
(5.2.51)

Using (5.2.40) in (5.2.39) we arrive at the following $q \neq 0$ generalisation of the dyonic magnon dispersion relation¹⁰

$$E - J_1 = \sqrt{(J_2 - qhp)^2 + 4h^2(1 - q^2)\sin^2\frac{p}{2}} .$$
 (5.2.52)

It is worth noting that, as in the q = 0 case (see equation (5.2.9)), the generalisation of this dispersion relation to the case when the string also moves along an S¹ in the torus part of the background is simply given by replacing $E \to \sqrt{E^2 - P^2}$ where P is the (large) momentum in S¹. Again the reason for this is the formal Lorentz invariance in the $\mathbb{R}_t \times S^1_{\psi}$ subspace in the decompactification limit.

Finally, let us derive the quantization condition for J_2 (a similar argument for the q = 0 case appeared in [22, 23]). As one can see from (5.2.28) or (5.2.42) the giant magnon motion is timeperiodic in the ϕ_2 direction with period T (5.2.43) assuming that the shift of t is compensated by a shift of x so that $\mathcal{X} + q\mathcal{T}$ and thus θ stays unchanged. In fact, the solution is explicitly periodic in

¹⁰For completeness, one should also check that the "off-diagonal" components of the SU(2) charges (5.1.17) vanish on this solution and indeed this is the case. We can also construct these "off-diagonal" charges by considering the corresponding Noether currents following from the local action (5.1.21). For these charges to be well-defined, i.e. for the spatial component of the current to go to zero as $x \to \pm \infty$ we again find that we should fix c = 1. Furthermore, these charges also vanish on the solution, i.e. there is no additional contribution from the non-trivial boundary conditions.

 $x_{+} = \tau$ in the light-cone gauge discussed above, where we have (see (5.2.41), (5.2.42))

$$x_{+} = (1-a)t + a\phi_{1} = t + ax_{-} = \tau, \qquad x_{-} = f(\theta), \qquad \phi_{2} = w\tau + g(\theta). \tag{5.2.53}$$

The changes over the period $\delta t = T$ are

$$\delta\theta = 0, \qquad \delta\phi_2 = 2\pi, \qquad \delta\phi_1 = \delta t = \mathbf{T},$$
(5.2.54)

so that $\delta x_{-} = \delta \phi_{1} - \delta t = 0$, $\delta x_{+} = T$.

This periodicity implies that there is an associated action variable, which should take integer values upon semiclassical quantization. Indeed, in general, for an integrable Hamiltonian system one can define action variables $I_s = \frac{1}{2\pi} \int_{\gamma_s} p_i dq^i$ where the γ_s form a basis of Liouville torus cycles. The Bohr-Sommerfeld condition then implies that I_s should take integer values in the quantum theory. In the present case we can obtain the action variable I associated to the above cycle in phase space from

$$2\pi I = S - T \frac{\partial S}{\partial T}\Big|_{p} . \qquad (5.2.55)$$

Here S = S(T, p) is the light-cone gauge string action computed over one period T on the giant magnon solution (we assume that the parameters ρ and \tilde{b} are expressed in terms of T in (5.2.43) and p in (5.2.40),(5.2.33)). Since the string action is reparametrisation-invariant, its value is gaugeindependent and so it can be evaluated, e.g., in the conformal gauge (even though the periodicity of the solution is not manifest in this gauge – ϕ_1 gets an additional shift ~ T). Considering w > 0we compute the action (5.1.21) (keeping c arbitrary and including the $-\partial^a t \partial_a t$ term) on the solution (5.2.30),(5.2.31) to find

$$S = 2\pi h \left[-\frac{2(1-q^2)}{\tan \rho - q\tilde{b}} + \frac{1}{2}q(c+1)\Delta\phi_1 - \frac{1}{4\pi}Tq(c-1)\Delta\phi_2 \right], \qquad (5.2.56)$$

where T is given in (5.2.43), $\Delta \phi_1$ in (5.2.33),(5.2.40), and $\Delta \phi_2$ in (5.2.37) is divergent. Thus the action, like $E - J_1$ in (5.2.36), is finite only if c = 1, once again supporting the choice of the boundary term made above in (5.1.35),(5.1.36). Eliminating ρ in favour of T or $w = \frac{2\pi}{T}$ (recall we consider w > 0) using (5.2.43), i.e.

$$\tan^2 \rho = \frac{1 - q^2 + \tilde{b}^2 - \left[\sqrt{(1 - q^2)(1 + \tilde{b}^2)(1 - w^2)} - qw\tilde{b}\right]^2}{1 - w^2} , \qquad (5.2.57)$$

gives (here $\tilde{b} = \cot \frac{p}{2}$ and $w = \frac{2\pi}{T}$)

c = 1 :
$$\frac{S}{2\pi h} = \frac{2(1-q^2)\sqrt{1-w^2}}{q\tilde{b}\sqrt{1-w^2} - \left(1-q^2+\tilde{b}^2 - \left[\sqrt{(1-q^2)(1+\tilde{b}^2)(1-w^2)} - qw\tilde{b}\right]^2\right)^{1/2}} + qp.$$
(5.2.58)

Substituting into (5.2.55) we find that the action variable associated to the periodic motion in ϕ_2 is nothing but J_2 given in (5.2.38), i.e.

$$I = J_2$$
 . (5.2.59)

Thus J_2 should be quantized, which is consistent with the near-BMN perturbation theory [74] where

the dispersion relation is a limit of (5.2.52) with $J_2 = 1$, and with the bound-state analysis in section 5.4.

5.3 Giant magnon in the Landau-Lifshitz limit

Let us now check (5.2.40), (5.2.52) by considering a particular large angular momentum limit (when both J_1 and J_2 are large) in which the string action reduces to a Landau-Lifshitz (LL) model in which there is a natural definition for the world-sheet momentum.

In the q = 0 case the LL model admits a well-known "spin wave" soliton solution which, in fact, may be interpreted as a limit of the giant magnon of the original string sigma model, and we shall find its generalisation to $q \neq 0$. In the LL model one can give a natural definition to the spatial 2-d momentum of the soliton and as we shall see it is consistent with (5.2.40), (5.2.45) and the resulting energy-momentum relation agrees with large J_2 expansion of (5.2.52).

5.3.1 Landau-Lifshitz model for $q \neq 0$

To derive the LL model from the string action on $\mathbb{R} \times S^3$ one introduces a collective coordinate to isolate the "fast" string motion associated to the large total angular momentum and obtains the effective action describing the remaining "slow" degrees of freedom [196, 197]. Let us parametrise S^3 as

$$Z_{1} = X_{1} + iX_{2} = \sin\theta \, e^{i\phi_{1}} = U_{1}e^{i\alpha} , \qquad U_{1} = \sin\theta \, e^{i\beta} ,$$

$$Z_{2} = X_{3} + iX_{4} = \cos\theta \, e^{i\phi_{2}} = U_{2}e^{i\alpha} , \qquad U_{2} = \cos\theta \, e^{-i\beta} ,$$

$$\alpha = \frac{1}{2}(\phi_{1} + \phi_{2}) , \qquad \beta = \frac{1}{2}(\phi_{1} - \phi_{2}) , \qquad |U_{1}|^{2} + |U_{2}|^{2} = 1 .$$
(5.3.1)

The angle α and the $\mathbb{C}\mathbf{P}^1$ coordinates U_1, U_2 correspond to the S¹ Hopf fibration of S³. The conformalgauge string Lagrangian is $\mathcal{L} = -\frac{1}{2}\partial_+ t \partial_- t + \frac{1}{2}\mathcal{L}_S$ where the S³ part in (5.1.21), written in the above coordinates, takes the form

$$\mathcal{L}_{S} = \partial_{+}\theta\partial_{-}\theta + \partial_{+}\alpha\partial_{-}\alpha + \partial_{+}\beta\partial_{-}\beta - (1+q)\partial_{+}\alpha C_{-} - (1-q)\partial_{-}\alpha C_{+} - qc(\partial_{+}\alpha\partial_{-}\beta - \partial_{+}\beta\partial_{-}\alpha) ,$$

$$C_{\pm} = \cos 2\theta \,\partial_{\pm}\beta . \qquad (5.3.2)$$

With $t = \kappa \tau$ the Virasoro constraints are

$$(\partial_{\pm}\alpha)^2 - 2\partial_{\pm}\alpha C_{\pm} + (\partial_{\pm}\theta)^2 + (\partial_{\pm}\beta)^2 = \kappa^2 .$$
(5.3.3)

Introducing $n_i = U^{\dagger} \sigma_i U$ or explicitly

$$\vec{n} = (\sin 2\theta \, \cos 2\beta, \, \sin 2\theta \, \sin 2\beta, \, \cos 2\theta) \,, \qquad \vec{n}^2 = 1 \,, \tag{5.3.4}$$

$$\partial_{+}C_{-} - \partial_{-}C_{+} = -\frac{1}{2}\varepsilon_{ijk}n_{i}\partial_{+}n_{j}\partial_{-}n_{k} , \qquad \frac{1}{4}\partial_{+}\vec{n}\cdot\partial_{-}\vec{n} = \partial_{+}\theta\partial_{-}\theta + \partial_{+}\beta\partial_{-}\beta - C_{+}C_{-} , \qquad (5.3.5)$$

we may rewrite the Lagrangian in (5.1.21) and the Virasoro constraints as

$$\mathcal{L}_{S} = \frac{1}{4}\partial_{+}n \cdot \partial_{-}n + (\partial_{+}\alpha - C_{+})(\partial_{-}\alpha - C_{-}) - q(\partial_{+}\alpha C_{-} - \partial_{-}\alpha C_{+}) - qc(\partial_{+}\alpha \partial_{-}\beta - \partial_{+}\beta \partial_{-}\alpha) , \qquad (5.3.6)$$

$$\partial_{\pm}\alpha - C_{\pm} = \kappa \sqrt{1 - \frac{(\partial_{\pm}n)^2}{4\kappa^2}} . \tag{5.3.7}$$

Let us now take the large total angular momentum limit directly in the action (as in [197]) using the Virasoro constraints to eliminate α . We take the limit directly in the action rather than the equations of motion in order to determine the contribution to the LL action from the total derivative in the WZ term. Introducing $u = \alpha - t$ and expanding in large κ (which corresponds to large angular momentum limit with both J_1 and J_2 being large¹¹) we find, after solving for u using the Virasoro constraints $\partial_{\pm} u = C_{\pm} + O(\kappa^{-1}),^{12}$

$$\mathcal{L}_S = \frac{1}{4} \partial_+ n \cdot \partial_- n - 2\kappa C_\tau + 2q\kappa C_\sigma + 2q\kappa \partial_\sigma \beta + O(\kappa^{-1}) .$$
(5.3.8)

Finally, using the equation of motion $\partial_{\tau} n_i = q \partial_{\sigma} n_i + O(\kappa^{-1})$ we arrive at the following $q \neq 0$ generalisation of the Landau-Lifshitz action

$$S_{\rm LL} = -h \int dt dx \left[C_t - qC_x + \frac{1}{8} (1 - q^2) (\partial_x n_i)^2 - qc \partial_x \beta \right] , \qquad (5.3.9)$$

where $t = \kappa \tau$, $x = \kappa \sigma$ and $C_a = \cos 2\theta \ \partial_a \beta$.

In this procedure we use the Virasoro constraints in the action which in general may not necessarily lead to a correct result but in the present case indeed gives the same expression for the LL action as the systematic procedure based on uniform gauge fixing and large κ expansion developed in [197]. The same conclusion is also easily reached for $q \neq 0$ by taking the same limit directly at the level of string equations of motion:

$$\partial_+ n_i (\partial_- \alpha - C_-) + \partial_- n_i (\partial_+ \alpha - C_+) - \varepsilon_{ijk} n_j \partial_+ \partial_- n_k + q (\partial_+ \alpha \partial_- n_i - \partial_- \alpha \partial_+ n_i) + O(\kappa^{-1}) = 0.$$

Using the expansion of the Virasoro constraints (5.3.7) to eliminate α gives

$$2\kappa(\partial_{\tau} - q\partial_{\sigma})n_i + \varepsilon_{ijk}n_j(\partial_{\sigma}^2 - \partial_{\tau}^2)n_k + q(C_+\partial_-n_i - C_-\partial_+n_i) + O(\kappa^{-1}) = 0.$$

For $q \neq 0$ the time derivatives of n_i are not suppressed but go as $\partial_{\tau} n_i = q \partial_{\sigma} n_i + O(\kappa^{-1})$. Eliminating them recursively from the above equation gives $(\partial_{\tau} - q\partial_{\sigma})n_i = -\frac{1}{2\kappa}(1-q^2)\varepsilon_{ijk}n_j\partial_{\sigma}^2n_k + O(\kappa^{-2})$ which follows from the action (5.3.9).

The LL model action (5.3.9) is invariant under translations of (t, x) and SO(3) rotations of n_i . The former give two conserved charges – 2-d energy and 2-d momentum of the "slow" variables (which are

¹¹This one can see from the large κ expansion of $J_{\alpha} = 2(J_1 + J_2) = \kappa + \int \frac{d\sigma}{2\pi} q[c + \cos(2\theta)]\dot{\beta} + O(\kappa^{-1})$. ¹²We dropped the total derivative term $2\kappa\partial_{\tau}u$ and the constant term κ^2 which do not depend on q.
no longer fixed by the Virasoro constraints)

$$E_{\rm LL} = \hbar \int \mathrm{d}x \Big(-q(\cos 2\theta + c)\partial_x\beta + \frac{1}{2}(1-q^2) \big[(\partial_x\theta)^2 + \sin^2 2\theta (\partial_x\beta)^2 \big] \Big) , \qquad (5.3.10)$$

$$P_{\rm LL} = -\frac{h}{2} \int dx \, \frac{\partial \mathcal{L}_S}{\partial (\partial_t \beta)} \partial_x \beta = -h \int dx \, \cos 2\theta \, \partial_x \beta \,. \tag{5.3.11}$$

Then

$$E_{\rm LL} + qP_{\rm LL} = \hbar \int \mathrm{d}x \left(-qc\partial_x\beta + \frac{1}{2}(1-q^2) \left[(\partial_x\theta)^2 + \sin^2 2\theta \left(\partial_x\beta \right)^2 \right] \right) \,. \tag{5.3.12}$$

Before discussing the LL model counterpart of the giant magnon solution let us consider the corresponding LL limit of the rigid circular string solution of section 5.1.3. The solution in (5.1.47), (5.1.48) may be written as

$$\cos\theta = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{qm}{\sqrt{\kappa^2 - (1 - q^2)m^2}}} , \qquad \alpha = \sqrt{\kappa^2 - (1 - q^2)m^2} \tau , \qquad \beta = m(\sigma + q\tau) .$$
(5.3.13)

Taking the large κ limit we get, to leading order,

$$\cos \theta = \frac{1}{\sqrt{2}} , \qquad \alpha = \left[\kappa - \frac{(1-q^2)m^2}{2\kappa}\right]\tau , \qquad \beta = m(\sigma + q\tau) , \qquad (5.3.14)$$

which solve the LL equations of motion. The corresponding conserved charges (5.3.10), (5.3.11) are

$$E_{\rm LL} = 2\pi h \left[-q c m + \frac{1}{2} (1 - q^2) \frac{m^2}{\kappa}\right], \qquad P_{\rm LL} = 0, \qquad (5.3.15)$$

which is consistent with a large angular momentum expansion of the full string energy (5.1.52) (here $E_{\rm LL} = E - J_1$ and $J_1 = 2\pi h\kappa$).

5.3.2 Landau-Lifshitz limit of the dyonic giant magnon solution

When taking the Landau-Lifshitz or large κ limit we required that the derivative $\partial_{\sigma} n_i$ stays finite. However, since the giant magnon solution is itself a large κ solution depending only on $(t, x) = (\kappa \tau, \kappa \sigma)$, in this case $\partial_{\sigma} n_i \sim O(\kappa)$. Therefore we also need to take an appropriate limit of the parameters in (5.2.27)-(5.2.29) to obtain the corresponding solution of the LL model. We have from (5.2.30), (5.2.31)

$$\partial_{\sigma} \cos \theta = -\frac{\kappa \cos \rho}{\sqrt{1+\tilde{b}^2}} \frac{1-qv}{\sqrt{1-v^2}} \frac{\tanh(\cos \rho(\mathcal{X}+q\mathcal{T}))}{\cosh(\cos \rho(\mathcal{X}+q\mathcal{T}))} ,$$

$$\partial_{\sigma}\beta = \kappa \cos \rho \frac{1-qv}{\tilde{b}\sqrt{1-v^2}} \frac{\cos^2(\arctan(\tilde{b}^{-1}\tanh[\cos \rho(\mathcal{X}+q\mathcal{T})]))}{\cosh(\cos \rho(\mathcal{X}+q\mathcal{T}))} + \frac{\kappa}{2} \left(q-\sin \rho \frac{q-v}{\sqrt{1-v^2}}\right) , \quad (5.3.16)$$

so that to take the LL limit we need to assume that in the large κ limit

$$\sin \rho \sim 1 + O(\kappa^{-2})$$
, $v \sim O(\kappa^{-1})$. (5.3.17)

In this limit we have (cf. (5.2.2), (5.2.29))

$$\tan \rho \sim \kappa \gg 1 , \qquad b = \frac{v \sec \rho}{\sqrt{1 - v^2}} = \text{fixed} , \qquad \tilde{b} = b + O(\kappa^{-1}) . \qquad (5.3.18)$$

Under this assumption the conserved charges (5.2.34), (5.2.35) of the $q \neq 0$ dyonic giant magnon take the form (c = 1)

$$E - J_1 = \frac{2h\kappa}{1+b^2} + O(\kappa^0) , \qquad J_2 = \frac{2h\kappa}{1+b^2} + O(\kappa^0) , \qquad \Delta\phi_1 = 2 \arctan b^{-1} + O(\kappa^{-1}) ,$$

$$\frac{E - J_1}{J_2} = 1 - q \frac{(1+b^2) \arctan b^{-1}}{\kappa} + O(\kappa^{-2}) . \qquad (5.3.19)$$

Thus in the Landau-Lifshitz limit $E - J_1$ and J_2 diverge with their ratio staying finite. Eliminating b and κ from the above expressions we reproduce the large J_2 expansion of (5.2.52)

$$E - J_1 = J_2 - qh\Delta\phi_1 + O(J_2^{-1}) .$$
(5.3.20)

To construct the corresponding LL solution let us first consider the q = 0 case. Expanding the q = 0 giant magnon (5.2.1) at large κ we get¹³

$$2\alpha = -b\sigma + 2\kappa\tau + \frac{b^2 - 1}{2\kappa}\tau + \arctan\left[b^{-1}\tanh(\sigma - \frac{b}{\kappa}\tau)\right],$$

$$2\beta = b\sigma - \frac{b^2 - 1}{2\kappa}\tau + \arctan\left[b^{-1}\tanh(\sigma - \frac{b}{\kappa}\tau)\right],$$

$$\cos\theta = \frac{\operatorname{sech}(\sigma - \frac{b}{\kappa}\tau)}{\sqrt{1 + b^2}}.$$
(5.3.21)

These β and θ indeed solve the q = 0 LL equations: they describe the known "pulse" or "spin wave" LL soliton found in [198, 199].¹⁴ The corresponding conserved charges are

$$E_{\rm LL} = \frac{\mathrm{h}}{2} \int \mathrm{d}x \left(\theta^{\prime 2} + \sin^2 2\theta \,\beta^{\prime 2}\right) = \frac{\mathrm{h}}{\kappa} \,, \qquad (5.3.22)$$

$$P_{\rm LL} = h \int \mathrm{d}x \left(1 + \cos 2\theta\right) \beta' = 2h \arctan b^{-1} , \qquad (5.3.23)$$

$$J_{\beta} = h \int dx \, (1 + \cos 2\theta) = \frac{4h\kappa}{1 + b^2} \,, \qquad (5.3.24)$$

where we have subtracted the values for the ground state solution $\theta = \pi/2$, $\phi_1 = \kappa \tau$, $\phi_2 = 0$ to obtain finite expressions. Here J_β is the angular momentum corresponding to translations in β .

Comparing with (5.3.19) we see that in the Landau-Lifshitz limit we have

$$P_{\rm LL} = h \Delta \phi_1 , \qquad J_\beta = 2J_2 .$$
 (5.3.25)

This then supports the identification of $\Delta \phi_1$ with the spatial momentum p and leads to the following familiar dispersion relation for the LL soliton

$$E_{\rm LL} = \frac{2h^2}{J_2}\sin^2\frac{p}{2}$$
, $p = h^{-1}P_{\rm LL}$, (5.3.26)

which is also the leading term in the large J_2 expansion of the dyonic giant magnon energy in (5.2.8), $E - J_1 \rightarrow J_2 + E_{\rm LL}$.¹⁵

¹³Note also that the Virasoro constraints take the expected form $\partial_{\pm}\alpha - C_{\pm} = \kappa - \frac{1}{2\kappa} \operatorname{sech}^2(\sigma - \frac{b}{\kappa}\tau)$.

¹⁴This soliton is non-topological (i.e. it can be continuously deformed into the vacuum $\theta = \frac{\pi}{2}$). Upon semiclassical quantization [203, 199] its U(1) charge J_2 is quantized and the quantum soliton $J_2 = 1$ state may be identified with the elementary magnon state.

 $^{^{15}}$ In the AdS₅×S⁵ case the leading term of the expansion is protected and thus it also agrees with the small h expansion

The generalisation of the relevant large κ expansion of the giant magnon solution to $q \neq 0$ is

$$2\alpha = -b(\sigma + q\tau) + 2\kappa\tau + \frac{(1 - q^2)(b^2 - 1)}{2\kappa}\tau + \arctan\left[b^{-1}\tanh(\sigma + q\tau - \frac{b}{\kappa}(1 - q^2)\tau)\right],$$

$$2\beta = b(\sigma + q\tau) - \frac{(1 - q^2)(b^2 - 1)}{2\kappa}\tau + \arctan\left[b^{-1}\tanh(\sigma + q\tau - \frac{b}{\kappa}(1 - q^2)\tau)\right],$$

$$\cos\theta = \frac{\operatorname{sech}\left[\sigma + q\tau - \frac{b}{\kappa}(1 - q^2)\tau\right]}{\sqrt{1 + b^2}}.$$
(5.3.27)

These β and θ satisfy the $q \neq 0$ LL equations of motion for (5.3.9) while α solves the Virasoro constrains

$$\partial_{\pm}\alpha - C_{\pm} = \kappa - \frac{1}{2\kappa} (1 - q^2) \operatorname{sech}^2 \left[\sigma + q\tau - \frac{b}{\kappa} (1 - q^2)\tau \right]$$
 (5.3.28)

Note that one can also obtain the $q \neq 0$ solution for β and θ by applying the world-sheet coordinate transformation $\tilde{\tau} = \tau$, $\sigma \to \tilde{\sigma} = \sigma - q\tau$, $\partial_{\sigma} = \tilde{\partial}_{\sigma}$, $\partial_{\tau} = \tilde{\partial}_{\tau} - q\tilde{\partial}_{\sigma}$ after which the LL equations following from (5.3.9) take the standard form $\tilde{\partial}_{\tau} n_i = \frac{1}{2\kappa} (1 - q^2) \varepsilon_{ijk} n_j \tilde{\partial}_{\sigma}^2 n_k$.

The corresponding energy (5.3.10) that generalises (5.3.22) to the $q \neq 0$ case is found to be (taking c = 1)

$$\begin{split} E_{\rm LL} &= h \int dx \left[-q(1+\cos 2\theta)\beta' + \frac{1}{2}(1-q^2)(\theta'^2 + \sin^2 2\theta \,\beta'^2) \right] \\ &= -2hq \arctan b^{-1} + (1-q^2)\frac{h}{\kappa} \,, \end{split} \tag{5.3.29}$$

while the expressions for $P_{\rm LL}$ and J_{β} remain the same as in (5.3.23) and (5.3.24). As a result, eq. (5.3.25) is unchanged while we find the following generalisation of the LL soliton dispersion relation (5.3.26)

$$E_{\rm LL} = -qhp + \frac{2h^2(1-q^2)}{J_2}\sin^2\frac{p}{2}, \qquad p = h^{-1}P_{\rm LL}. \qquad (5.3.30)$$

This agrees with the large J_2 expansion of the giant magnon energy (5.2.52) found in section 5.2 thus supporting the identification of the magnon momentum (5.2.40) made there.

5.4 Symmetry algebra of light-cone gauge S-matrix and exact dispersion relation

As in the pure R-R case (q = 0) the world-sheet light-cone gauge S-matrix of the mixed-flux theory is determined, up to overall dressing phases, by the centrally extended off-shell symmetry algebra. In [74, 75] this was used to show that the massive sector S-matrix takes the same form as in the pure R-R case when written in terms of Zhukovsky variables x_{\pm}^{\pm} [60]. The form of Zhukovsky variables depends on the exact dispersion relation, which, by the symmetry algebra, is only fixed up to an identification of the "central charge" M_{\pm} in terms of the string tension and world-sheet momentum. In [75] a conjecture for M_{\pm} was given based on the assumption of an underlying spin-chain picture for the world-sheet theory. In this section we modify this conjecture such that in the strong coupling limit the bound-state dispersion relation reproduces the semiclassical dispersion relation (5.2.52).

of the dyonic giant magnon energy, matching the expression following from the coherent state expectation value of the one-loop ferromagnetic spin chain Hamiltonian.

5.4.1 Symmetry algebra

Let us first review the symmetry algebra of the light-cone gauge world-sheet S-matrix following [75]. The mixed flux $AdS_3 \times S^3 \times T^4$ background is described by the same supercoset as in the pure R-R case [204]

$$\frac{\text{PSU}(1,1|2) \times \text{PSU}(1,1|2)}{\text{SU}(1,1) \times \text{SU}(2)}.$$
(5.4.1)

As such the superisometries of the mixed flux $AdS_3 \times S^3$ backgrounds do not depend on q, which only enters as a parameter in front of the Wess-Zumino term in the action [68]. In light-cone gauge the worldsheet S-matrix for the scattering of excitations above the BMN vacuum has a residual symmetry. This symmetry corresponds to a centrally extended subalgebra of the coset superisometries which leaves the BMN geodesic invariant. Therefore it should not depend on q and instead q enters the representation of this algebra on states.

Explicitly the symmetry algebra is given by two copies of the centrally extended algebra

$$[\mathfrak{u}(1) \in \mathfrak{psu}(1|1)^2] \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3 , \qquad (5.4.2)$$

with their central extensions $(\ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3)$ identified. Thus the full symmetry algebra of the S-matrix is given by

$$[\mathfrak{u}(1) \in \mathfrak{psu}(1|1)^2]^2 \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3 .$$
(5.4.3)

The generators for a single copy of the algebra (5.4.2) are:

- (i) two U(1) generators \mathfrak{R} and \mathfrak{L} ;
- (ii) four supercharges $\mathfrak{Q}_{\pm\mp}$ and $\mathfrak{S}_{\pm\mp}$, where the subscript \pm denotes the charges under the $U(1) \times U(1)$ bosonic subalgebra;
- (iii) three generators \mathfrak{C} , \mathfrak{P} and \mathfrak{K} for the central extension.

Defining

$$\mathfrak{M} = \frac{1}{2}(\mathfrak{R} + \mathfrak{L}) , \qquad \mathfrak{B} = \frac{1}{2}(\mathfrak{R} - \mathfrak{L}) , \qquad (5.4.4)$$

they satisfy the following non-vanishing (anti-)commutation relations

$$[\mathfrak{B}, \mathfrak{Q}_{\pm\mp}] = \pm i \mathfrak{Q}_{\pm\mp} , \qquad [\mathfrak{B}, \mathfrak{S}_{\pm\mp}] = \pm i \mathfrak{S}_{\pm\mp} , \{\mathfrak{Q}_{\pm\mp}, \mathfrak{Q}_{\mp\pm}\} = \mathfrak{P} , \qquad \{\mathfrak{S}_{\pm\mp}, \mathfrak{S}_{\mp\pm}\} = \mathfrak{K} , \qquad \{\mathfrak{Q}_{\pm\mp}, \mathfrak{S}_{\mp\pm}\} = \pm i \mathfrak{M} + \mathfrak{C} .$$
(5.4.5)

Additionally the generators satisfy the reality conditions

$$\mathfrak{B}^{\dagger} = -\mathfrak{B}$$
, $\mathfrak{Q}_{\pm\mp}^{\dagger} = \mathfrak{S}_{\mp\pm}$, $\mathfrak{M}^{\dagger} = -\mathfrak{M}$, $\mathfrak{P}^{\dagger} = \mathfrak{K}$, $\mathfrak{C}^{\dagger} = \mathfrak{C}$. (5.4.6)

The dispersion relation arises from the closure or shortening condition for the algebra with the central charges identified as functions of the string tension, world-sheet momentum and energy. The central extension generators are \mathfrak{C} , \mathfrak{P} , \mathfrak{K} for $\mathfrak{psu}(1|1)^2$ and \mathfrak{M} for $\mathfrak{u}(1)$. Their action on one-particle states $|\Psi\rangle$ for a representation consisting of a complex boson ϕ and a complex fermion ψ is given by

$$\{\mathfrak{M}, \mathfrak{C}, \mathfrak{P}, \mathfrak{K}\} | \Phi_{\pm} \rangle = \{ \pm \frac{i}{2} M_{\pm}, C_{\pm}, P_{\pm}, K_{\pm} \} | \Phi_{\pm} \rangle , \qquad \Phi_{\pm} \in \{ \phi_{\pm}, \psi_{\pm} \},$$
(5.4.7)

where $\phi_{+} = \phi$, $\phi_{-} = \phi^{*}$, $\phi_{+} = \psi$ and $\psi_{-} = \psi^{*}$. Here $\{\phi_{+}, \psi_{+}\}$ and $\{\phi_{-}, \psi_{-}\}$ form two irreducible representations which are related by charge conjugation and with the subscript \pm denoting left- and right movers. Once the representation parameters C_{\pm} , M_{\pm}^{2} , P_{\pm} and K_{\pm} are determined as real functions of the energy and momentum the dispersion relation follows from the closure condition of the algebra

$$C_{\pm}^2 = \frac{M_{\pm}^2}{4} + P_{\pm}K_{\pm} . \qquad (5.4.8)$$

In [74] the tree-level light-cone gauge S-matrix for excitations around the BMN solution was computed and in [75] this was used to obtain the representation parameters at leading order in the near-BMN limit

$$h \to \infty, \qquad p = \frac{p}{h} = \text{fixed} , \qquad (5.4.9)$$

where p is the momentum of a near-BMN excitation and p denotes the usual magnon momentum used in sections 5.2 and 5.3. The representation parameters take the form

$$M_{\pm} = 1 \pm q \, p \, , \qquad C_{\pm} = \frac{\varepsilon_{\pm}}{2} \, , \qquad P_{\pm} = -\frac{i}{2} \sqrt{1 - q^2} \, p \, , \qquad K_{\pm} = \frac{i}{2} \sqrt{1 - q^2} \, p \, , \qquad (5.4.10)$$

and substituting these expressions into the closure condition (5.4.8) gives the near-BMN dispersion relation

$$\varepsilon_{\pm} = \sqrt{(1 \pm q \, p)^2 + (1 - q^2)p^2} = \sqrt{1 - q^2 + (p \pm q)^2} \,. \tag{5.4.11}$$

In [75] also exact expressions for the representation parameters were conjectured based on the analogy with the pure R-R q = 0 case and based on various algebraic requirements giving

$$C_{\pm} = \frac{\varepsilon_{\pm}}{2} , \qquad P_{\pm} = \frac{h}{2}\sqrt{1-q^2}(1-e^{ip}) , \qquad K_{\pm} = \frac{h}{2}\sqrt{1-q^2}(1-e^{-ip}) . \tag{5.4.12}$$

Substituting these expressions into (5.4.8) gives the form of the dispersion relation up to an undetermined central charge M_{\pm}

$$\varepsilon_{\pm} = \sqrt{M_{\pm}^2 + 4h^2(1-q^2)\sin^2\frac{p}{2}}$$
 (5.4.13)

Here M_{\pm} was assumed to be a smooth periodic function in the momentum, as one would expect if there was an underlying spin-chain picture, and it was conjectured to take the form

$$M_{\pm} = 1 \pm 2qh\sin\frac{p}{2} . \qquad (5.4.14)$$

However, this is not consistent with our semiclassical result (5.2.52). Instead we propose

$$M_{\pm} = 1 \pm q h p$$
, (5.4.15)

which is consistent with both the near-BMN and our semiclassical result. As we shall see in the next section this expression also leads to a more natural construction of magnon bound-states.

5.4.2 Bound states

Bound states are characterised by poles in the S-matrix. Therefore we first need to review the effect of the above choice (5.4.15) on the S-matrix. In the pure R-R case the S-matrix can be written as

a function of the Zhukovsky variables x^{\pm} , which are defined in terms of the energy and momentum. Imposing invariance of the S-matrix under the four supercharges then fixes its form up to overall dressing phases. This S-matrix also satisfies the Yang-Baxter equation, QFT unitarity and braiding unitarity if the dressing phases satisfy certain crossing equations. The dispersion relation then enters the S-matrix only through the map from the Zhukovsky variables to the energy and momentum. In [75] the world-sheet S-matrix in the massive sector of the mixed flux case was found to take the same form as in the pure R-R case when written as a function of the new Zhukovsky variables x_{\pm}^{\pm} . Here the \pm subscript distinguishes the left and right moving sectors. For an arbitrary M_{\pm} the Zhukovsky map between x_{\pm}^{\pm} and the energy and momentum is given by ¹⁶

$$e^{ip} = \frac{x_{\pm}^{+}}{x_{\pm}^{-}}, \qquad \varepsilon_{\pm} = \frac{h\sqrt{1-q^{2}}}{2i} \left(x_{\pm}^{+} - \frac{1}{x_{\pm}^{+}} - x_{\pm}^{-} + \frac{1}{x_{\pm}^{-}}\right),$$

$$x_{\pm}^{\pm} = r_{\pm}e^{\pm i\frac{p}{2}}, \qquad r_{\pm} = \frac{\varepsilon_{\pm} + M_{\pm}}{2h\sqrt{1-q^{2}}\sin\frac{p}{2}} = \frac{2h\sqrt{1-q^{2}}\sin\frac{p}{2}}{\varepsilon_{\pm} - M_{\pm}}.$$
(5.4.16)

In these variables the dispersion relation (5.4.13) takes the form

$$x_{\pm}^{+} + \frac{1}{x_{\pm}^{+}} - x_{\pm}^{-} - \frac{1}{x_{\pm}^{-}} = \frac{2iM_{\pm}}{h\sqrt{1-q^2}} .$$
 (5.4.17)

The choice (5.4.15) for M_{\pm} then gives

$$M_{\pm} = 1 \pm q \ln p = 1 \mp i q \ln \ln \frac{x_{\pm}^{\pm}}{x_{\pm}^{-}} . \qquad (5.4.18)$$

Substituting this expression into (5.4.17) one finds

$$\left[\sqrt{1-q^2}\left(x_{\pm}^+ + \frac{1}{x_{\pm}^+}\right) \mp 2q\ln x_{\pm}^+\right] - \left[\sqrt{1-q^2}\left(x_{\pm}^- + \frac{1}{x_{\pm}^-}\right) \mp 2q\ln x_{\pm}^-\right] = \frac{2i}{h} .$$
 (5.4.19)

We can use this to define a q-modified Zhukovsky map, which solves this dispersion relation

$$u_{\pm}(x_{\pm}) = \sqrt{1 - q^2} \left(x_{\pm} + \frac{1}{x_{\pm}} \right) \mp 2q \ln x_{\pm} , \qquad x_{\pm}^{\pm} = x_{\pm} \left(u \pm \frac{i}{h} \right) . \qquad (5.4.20)$$

However, in contrast to the q = 0 pure R-R and q = 1 pure NS-NS cases the inverse of this map, i.e. $x_{\pm}(u)$, cannot be given in closed form.

Let us now determine the bound-state dispersion relation. The bound states are characterised by poles in the S-matrix, which requires the knowledge of the dressing factors. However, bound states that preserve some supersymmetry should transform in a short representation, which in our case consists of the boson ϕ and the fermion ψ . In particular the residues of the S-matrix at the poles give the bound-state S-matrix and should therefore project onto a short representation. For the S-matrix given

¹⁶Here the definitions of x_{\pm}^{\pm} have a natural periodic extension of the region $p \in (0, \pi)$ to the whole line, which is consistent with the semiclassical identification of p with the angular separation of the dyonic giant magnon string end-points.

in [75] this leads to the same possible poles as in the q = 0 case

(i):
$$x_{\pm}^{+} = x_{\pm}^{\prime -}$$
,
(ii): $x_{\pm}^{-} = x_{\pm}^{\prime +}$, (5.4.21)

where x and x' denote the spectral parameters for the two constituents. The case (i) corresponds to a bound state in the $\mathfrak{su}(2)$ sector associated with the boson and thus S³ whereas the case (ii) would give an $\mathfrak{sl}(2)$ bound state. In the pure R-R case the bound state forms in the $\mathfrak{su}(2)$ sector [61]. This can be seen by looking at the bound state wave function and imposing that the outgoing wave should decay. The wave function for a scattering state in a region where the excitations are well separated can be written as a sum of an incoming and outgoing wave

$$\Psi(\sigma,\sigma') \sim \frac{1}{S(p,p')} e^{i(p\sigma+p'\sigma')} + e^{i(p\sigma'+p'\sigma)}, \qquad \sigma \ll \sigma'.$$
(5.4.22)

The bound state wave function can be obtained from (5.4.22) by analytically continuing the momenta to complex values

$$p = \frac{P}{2} + iv, \qquad p' = \frac{P}{2} - iv.$$
 (5.4.23)

This gives

$$\Psi(\sigma,\sigma') \sim \frac{1}{S(p,p')} e^{v(\sigma'-\sigma)} + e^{-v(\sigma'-\sigma)}, \qquad \sigma \ll \sigma'.$$
(5.4.24)

For a decaying outgoing wave the momentum p should have a positive imaginary part. From (5.4.16) one finds that the imaginary part is positive in the case (i), but negative in the case (ii). Therefore also in the mixed flux case the bound-state pole is given by the condition (i) in (5.4.21).

The bound-state energy $E_{\pm}^{(2)}$ and momentum $p^{(2)}$ are given by the sum of the constituents

$$E_{\pm}^{(2)} = \varepsilon_{\pm} + \varepsilon'_{\pm} , \qquad p^{(2)} = p + p' . \qquad (5.4.25)$$

Using (5.4.16) one finds

$$e^{ip^{(2)}} = \frac{x_{\pm}^{\prime +}}{x_{\pm}^{-}}, \qquad E_{\pm}^{(2)} = \frac{h\sqrt{1-q^2}}{2i} \left(x_{\pm}^{\prime +} - \frac{1}{x_{\pm}^{\prime +}} - x_{\pm}^{-} + \frac{1}{x_{\pm}^{-}} \right).$$
(5.4.26)

From this we can identify the bound-state spectral parameters as

$$x_{\pm}^{(2)+} = x_{\pm}^{\prime+}, \qquad x_{\pm}^{(2)-} = x_{\pm}^{-}.$$
 (5.4.27)

Defining $x_{\pm} \equiv x_{\pm}^{+} = x_{\pm}^{\prime-}$ the dispersion relations of the constituents take the form

$$\mathbf{x}_{\pm} + \frac{1}{\mathbf{x}_{\pm}} - x_{\pm}^{(2)-} - \frac{1}{x_{\pm}^{(2)-}} = \frac{2iM_{\pm}}{h\sqrt{1-q^2}} , \qquad x_{\pm}^{(2)+} + \frac{1}{x_{\pm}^{(2)+}} - \mathbf{x}_{\pm} - \frac{1}{\mathbf{x}_{\pm}} = \frac{2iM_{\pm}'}{h\sqrt{1-q^2}} . \tag{5.4.28}$$

Their sum gives the bound-state dispersion relation

$$x_{\pm}^{(2)+} + \frac{1}{x_{\pm}^{(2)+}} - x_{\pm}^{(2)-} - \frac{1}{x_{\pm}^{(2)-}} = \frac{2iM_{\pm}^{(2)}}{h\sqrt{1-q^2}} , \qquad (5.4.29)$$

where

$$M_{\pm}^{(2)} = M_{\pm} + M_{\pm}' . \qquad (5.4.30)$$

Therefore the central charge M_{\pm} is additive when acting with the central generator \mathfrak{M} on multi-particle states ¹⁷

For the bound state the value of M_{\pm} is found from (5.4.18)

$$M_{\pm}^{(2)} = 2 \pm q \ln^{(2)} = 2 \mp i q \ln \ln \frac{x_{\pm}^{(2)+}}{x_{\pm}^{(2)-}} .$$
 (5.4.31)

This simple bound-state picture with the x^{\pm} dropping out follows directly from the linearity of M_{\pm} as a function of p. Iterating the above procedure gives general N bound states with

$$M_{\pm}^{(N)} = N \pm q \ln p^{(N)} = N \mp i q \ln \ln \frac{x_{\pm}^{(N)+}}{x_{\pm}^{(N)-}} .$$
 (5.4.32)

Their dispersion relation takes the form

$$E_{\pm}^{(N)} = \sqrt{\left(N \pm q \ln p^{(N)}\right)^2 + 4\hbar^2 (1 - q^2) \sin^2 \frac{p^{(N)}}{2}} , \qquad (5.4.33)$$

which agrees with the semiclassical dispersion relation (5.2.52) with a quantized angular momentum $J_2 = N$.

It is worth noting that this agreement requires the bound-state momentum to satisfy the bound $|\mathbf{p}^{(N)}| \leq \pi$ (at least in the semiclassical $\mathbf{h} \to \infty$ limit). This is implied by the identification of this momentum with that of the semiclassical dyonic giant magnon, which in turn is given by the separation angle of the string end-points. This suggests that any momentum p should be understood as defined modulo 2π (and thus may be taken to lie in the range $|\mathbf{p}| \leq \pi$). Then the momentum conservation should also be considered modulo 2π .¹⁸ The consequences of this rather unusual dispersion relation and its interpretation are important topics for further study.

5.5 Relation to soliton of the Pohlmeyer reduced theory

Let us briefly describe the relation between the $q \neq 0$ generalisation of the giant magnon solution, found in section 5.2.2, and the corresponding soliton of the Pohlmeyer reduction of $\mathbb{R} \times S^3$ string theory with $q \neq 0$, which is the complex sine-Gordon model with the mass parameter rescaled by $\sqrt{1-q^2}$ [74]. This generalises the relation between the soliton of the complex sine-Gordon model and the q = 0

 $^{^{17}}$ This also follows from the fact that its coproduct is the standard one [75, 60, 74, 58] (i.e. it is just given by the usual Leibniz action).

¹⁸Note that in the $q \to 1$ limit the $\sin^2 \frac{p}{2}$ term drops out of the dispersion relation and it might appear somewhat unnatural to take this definition. However, in this limit the classical string solution remains well-defined. Furthermore, $e^{\frac{ip}{2}}$ still appears in the definition of x_{\pm}^{\pm} and thus the S-matrix. These both suggest that the momentum should continue to be defined modulo 2π .

dyonic giant magnon used in [23].

The Lagrangian density of the Pohlmeyer reduced theory and the relation of the string embedding coordinates X_m to the reduced variables are given by [74]

$$\mathcal{L} = \partial_{+}\varphi\partial_{-}\varphi + \tan^{2}\varphi\,\partial_{+}\chi\partial_{-}\chi + \frac{1}{2}\kappa^{2}(1-q^{2})\cos 2\varphi ,$$

$$\kappa^{2}\cos 2\varphi = \partial_{+}X \cdot \partial_{-}X , \qquad \kappa^{3}\sin^{2}\varphi\,\partial_{\pm}\chi = \mp \frac{1}{2}\varepsilon_{mnkl}X^{m}\partial_{+}X^{n}\partial_{-}X^{k}\partial_{\pm}^{2}X^{l} .$$
(5.5.1)

These can be written in terms of the SU(2) current \mathfrak{J} in (5.1.7) as follows:

$$\kappa^2 \cos 2\varphi = -\frac{1}{2} \operatorname{tr}(\mathfrak{J}_+ \mathfrak{J}_-) , \qquad \kappa^3 \sin^2 \varphi \,\partial_\pm \chi = \pm \frac{1}{8} \operatorname{tr}([\mathfrak{J}_+, \mathfrak{J}_-] \partial_\pm \mathfrak{J}_\pm) . \tag{5.5.2}$$

Substituting the expressions (5.2.27)–(5.2.31) for the $q \neq 0$ giant magnon solution into \mathfrak{J}_{\pm} the corresponding reduced theory solution is found to be

$$\sin \varphi = \frac{\cos \rho}{\cosh \left[\cos \rho (\mathcal{X} + q\mathcal{T})\right]} , \qquad \chi = 2 \sin \rho (\mathcal{T} + q\mathcal{X}) , \qquad (5.5.3)$$

where \mathcal{T} and \mathcal{X} were defined in (5.2.3). Then

$$\psi \equiv \sin \varphi \, e^{\frac{i}{2}\chi} = \frac{\cos \rho \, \exp\left[i \sin \rho \left(\mathcal{T} + q\mathcal{X}\right)\right]}{\cosh\left[\cos \rho \left(\mathcal{X} + q\mathcal{T}\right)\right]} \tag{5.5.4}$$

is recognised as the familiar complex sine-Gordon soliton solution.

5.6 Summary

In this chapter we have supplemented the information provided by the perturbative near-BMN expansion [74] and the light-cone symmetry algebra [75] with the construction of the semiclassical dyonic giant magnon solution in $AdS_3 \times S^3 \times T^4$ string theory with mixed flux to propose the exact form of the corresponding dispersion relation. We have seen that the presence of the WZ term representing the NS-NS flux in the bosonic string action leads to subtleties associated to the proper choice of boundary terms and the definition of angular momenta, which become important for non-trivial open-string solutions like the giant magnon.

We reviewed the symmetry algebra for the string light-cone gauge S-matrix and introduced a new set of Zhukovsky variables corresponding to the proposed dispersion relation. Analyzing the resulting bound-state dispersion relation, we found that it has a simple structure (5.4.33) and agrees with the giant magnon dispersion relation (5.2.52).

Another check of (5.0.1), (5.0.3), that we perform in the next chapter, is to confirm that the first semiclassical (one-loop) correction to the giant magnon energy (5.2.52) vanishes¹⁹ as was shown in the case of the AdS₅ × S⁵ giant magnon in [193, 205, 25]. This should indeed be the case since (i) the one-loop corrections in the string and the corresponding Pohlmeyer reduced theory should match [83] (since the classical equations and thus the leading fluctuations near a classical solution are directly related) and (ii) the solution of the reduced theory corresponding to the giant magnon is essentially the same as in the q = 0 case up to a simple rescaling of the mass scale by $\sqrt{1-q^2}$ (see [75] and the section 5.5).

¹⁹In semiclassical limit $J_2 \sim h \mathcal{J}_2$ and \mathcal{J}_2 and p are fixed while one expands in large h.

Chapter 6

Folded strings

In AdS₅/CFT₄ the spectral problem has also been studied in the $\mathfrak{sl}(2)$ sector of $\mathcal{N} = 4$ SYM. Operators in this sector are constructed from a single scalar field, Z, and a covariant derivative \mathcal{D} . A general single trace operator takes the form

$$\mathcal{O} = \operatorname{tr}[\mathcal{D}^S Z^L] + \dots, \qquad L = \Delta - S, \tag{6.0.1}$$

where the dots indicates all possible ways to distribute the covariant derivative on the scalar field Z, L is the length or *twist* of the operator and S is the Lorentz spin. Such operators are also of particular interest since they appear in QCD [206]. In the context of integrability the anomalous dimension of large-spin twist operators has lead to a proposal for the dressing phase in the S-matrix [207, 32].

For twist -two operators the dispersion relation takes the form

$$\Delta - (\mathcal{S} + 2) = f(\lambda) \ln \mathcal{S} + \mathcal{O}\left(\frac{1}{\mathcal{S}}\right).$$
(6.0.2)

The interpolating function $f(\lambda)$ can be found perturbatively at weak coupling

$$f(\lambda) \sim \lambda, \qquad \lambda \ll 1.$$
 (6.0.3)

The function $f(\lambda)$ also appears in the expectation value of a straight Wilson loop with a cusp and is therefore often referred to as the *cusp anomalous dimension*. On the string side the twist-two operators are dual to a folded spinning string on the AdS₃ subspace of AdS₅ [208, 209]. Its energy in the large spin $S = S/\sqrt{\lambda}$ limit is given by

$$E = S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} + \dots, \qquad \lambda \gg 1, \quad S = \frac{S}{\sqrt{\lambda}} \gg 1, \tag{6.0.4}$$

which gives the strong coupling behaviour of $f(\lambda)$

$$f(\lambda) \sim \frac{\sqrt{\lambda}}{\pi}, \qquad \lambda \gg 1.$$
 (6.0.5)

The folded string is a special case of a spinning spiky string, which is related to twist operators with

derivatives on several fields. The corresponding dispersion relation is [210]

$$E = S + \frac{n}{2} \frac{\sqrt{\lambda}}{\pi} \ln\left(\frac{4\pi}{n} \frac{S}{\sqrt{\lambda}}\right) + \dots, \qquad \lambda \gg 1, \quad \frac{S}{\sqrt{\lambda}} \gg 1, \tag{6.0.6}$$

where n is the number of spikes. For n = 2 this reduces to (6.0.4).

In the case of $AdS_3 \times S^3$ backgrounds with mixed flux the folded spinning string has been recently studied as a perturbative series around the pure R-R case q = 0 in [211]. This has lead to the following proposal for a perturbative generalisation of (6.0.4) (in q)

$$E = S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} - \frac{\sqrt{\lambda}}{2\pi} q^2 \ln^2 \frac{S}{\sqrt{\lambda}} + \dots , \qquad (6.0.7)$$

where $q \ln S \ll 1$ and with the leading term in $\ln S$ given at each order in q.

In order to check this relation we use the same procedure as in the case of the mixed flux dyonic giant magnon to construct the $q \neq 0$ generalisation of the folded spinning string solution on $\operatorname{AdS}_3 \times \operatorname{S}^1$. The new solution is a string with spikes and carries an angular momentum $\mathcal{J} = J/\sqrt{\lambda}$ on S^1 in additional to the angular momentum \mathcal{S} on AdS_3 . For small \mathcal{J} its energy, as a large \mathcal{S} expansion, is given by

$$E = S + \frac{\sqrt{\lambda}}{\pi} \sqrt{1 - q^2} \ln \frac{S}{\sqrt{\lambda}} + \dots, \qquad \lambda \gg 1, \quad \mathcal{S} = \frac{S}{\sqrt{\lambda}} \gg 1, \quad \mathcal{J} \ll \mathcal{S}.$$
(6.0.8)

In contrast to (6.0.7) we find the usual $\ln S$ behaviour at leading order, as in the pure R-R case q = 0, but with the string tension rescaled by $\sqrt{1-q^2}$.

This chapter is structured as follows. In sections 6.1-6.2 we review the folded string solution on $R \times S^3$, its relation to dyonic giant magnons and its $AdS_3 \times S^1$ equivalent. In section 6.3 we then solve for its $q \neq 0$ counterpart. The resulting string solution is closed once we generalise the periodicity condition to $q \neq 0$ and impose quantization conditions on the angular momenta. In section 6.4 we then construct its $AdS_3 \times S^1$ equivalent by performing a Wick rotation. For this AdS solution we additionally require that the AdS time coordinate is single valued. As we shall see, this constraint can be satisfied by performing a world-sheet Lorentz transformation on the solution. Finally we compute the relation between the energy and angular momenta of the string for large S in the two cases of small and large \mathcal{J} .

6.1 Folded string and the dyonic giant magnon

In conformal gauge the spinning folded string on $\mathbb{R} \times S^3$ has the form [212, 213, 193]

$$Z_1 = \sin\theta \, e^{i\omega_1\tau}, \qquad Z_2 = \cos\theta \, e^{i\omega_2\tau} \tag{6.1.1}$$

$$t = \kappa \tau, \qquad \theta = \theta(\sigma), \qquad \phi_1 = \omega_1 \tau, \qquad \phi_2 = \omega_2 \tau.$$
 (6.1.2)

This ansatz satisfies the equations of motion for ϕ_1 and ϕ_2 . Taking $\omega_1 > \omega_2$ the remaining equation for θ and the Virasoro constraint can be written as

$$\dot{\theta}^2 = \omega_{21}^2 (\cos^2 \theta - \cos^2 \theta_{\min}), \qquad \cos \theta_{\min} \equiv \nu, \tag{6.1.3}$$

where $-\theta_{\min} \le \theta(\sigma) \le \theta_{\min}$ and¹

$$\cos \theta = \operatorname{dn}(\omega_{21}\sigma, \nu), \qquad \omega_{21} = \sqrt{\omega_1^2 - \omega_2^2}, \quad \omega_1 > \omega_2,$$

$$\sin \theta = \sqrt{\nu} \operatorname{sn}(\omega_{21}\sigma, \nu), \qquad \nu = \frac{\kappa^2 - \omega_2^2}{\omega_1^2 - \omega_2^2}, \quad 0 \le \nu \le 1.$$
(6.1.4)

For a closed string solution we also have the periodicity condition

$$2\pi = \int_0^{2\pi} d\sigma = 2n \int_0^{\theta_{\min}} \frac{d\theta}{\omega_{21}\sqrt{\cos^2\theta - \nu^2}}, \qquad \omega_{21} = \frac{n}{\pi}K(\nu).$$
(6.1.5)

This string solution folds along an arc through the north pole as shown in figure 6.1. The energy and



Figure 6.1: A folded spinning string with $\omega_2 = 0$ rotating at an angular velocity ω_1 . The right figure shows a diagrammatic representation of the string profile from the top.

the two angular momenta are given by

$$E = \sqrt{\lambda}\kappa, \qquad J_1 = \omega_1 \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} \mathrm{d}\sigma \sin^2\theta = \sqrt{\lambda}\omega_1 \Big[1 - \frac{E(\nu)}{K(\nu)}\Big] \tag{6.1.6}$$

$$J_2 = \omega_2 \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} \mathrm{d}\sigma \cos^2 \theta = \sqrt{\lambda} \omega_2 \frac{E(\nu)}{K(\nu)}.$$
(6.1.7)

For $\nu \to 1$, i.e. when the string profile teaches $\theta = \pi/2$, this solution is related to the dyonic giant magnon. From (6.1.4) we see that $\omega_1 = \kappa$. The periodicity condition (6.1.5) then implies $\kappa \to \infty$ (since $\omega_1 = \kappa > \omega_2$). Therefore in decompactified world-sheet coordinates the folded string reduces to

$$Z_1 = \tanh(\sqrt{1-\omega^2}x)e^{it}, \qquad Z_2 = \operatorname{sech}(\sqrt{1-\omega^2}x)e^{i\omega t}, \tag{6.1.8}$$

¹Here dn, sn and cn are the usual Jacobi elliptic functions defined as $\operatorname{sn}(u,m) = \sin \phi$, $\operatorname{cn}(u,m) = \cos \phi$, $\operatorname{dn}(u,m) = \sqrt{1 - m \sin^2 \phi}$ where $\phi = \operatorname{am}(u,m)$ is the inverse of the elliptic integral of the first kind $u = F(\phi,m) = \int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1 - m \sin^2 \theta}}$.

which is a dyonic giant magnon with the momentum $p = \pm \pi$ and $J_2^{\text{DGM}} = \frac{\sqrt{\lambda}}{2\pi} \frac{\omega}{\sqrt{1-\omega^2}}$. For the folded string the energy and angular momenta reduce to

$$E = \sqrt{\lambda} \frac{n}{\pi} \frac{K(\nu)}{\sqrt{1-\omega^2}} \to \infty, \qquad J_1 = \sqrt{\lambda} \frac{n}{\pi} \frac{1}{\sqrt{1-\omega^2}} (K(\nu) - E(\nu)) \to \infty, \qquad \nu \to 1$$
(6.1.9)

$$J_2 = \sqrt{\lambda} \frac{n}{\pi} \frac{\omega}{\sqrt{1 - \omega^2}} E(\nu), \qquad (6.1.10)$$

where we used $\kappa = \frac{n}{\pi} \frac{K(\nu)}{\sqrt{1-\omega^2}}$. Eliminating ω one finds [35]

$$E - J_1 = n\sqrt{\frac{J_2^2}{n^2} + \frac{\lambda}{\pi^2}} = n\sqrt{(J_2^{\text{DGM}})^2 + \frac{\lambda}{\pi^2}}, \qquad \nu \to 1.$$
(6.1.11)

Therefore in the $\nu \to 1$ limit the string with n/2 folds consists of pairs of dyonic giant magnons of opposite momenta $p = \pm$ with each carrying J_2/n units of angular momentum in the ϕ_2 direction.

6.2 Folded string on $AdS_3 \times S^1$

A string solution on $\mathbb{R} \times S^3$ can be Wick rotated to give a solution on $AdS_3 \times S^1$. Starting with the metric on $\mathbb{R} \times S^3$

$$ds^{2} = -dt^{2} + d\theta^{2} + \sin^{2}\theta d\phi_{1}^{2} + \cos^{2}\theta d\phi_{2}^{2}$$
(6.2.1)

we perform the Wick rotation

$$t \to \varphi, \qquad \theta \to i\rho, \qquad \phi_1 \to \phi, \qquad \phi_2 \to t.$$
 (6.2.2)

This gives, up to an overall sign, the $AdS_3 \times S^1$ metric in global coordinates

$$ds^{2} = -(d\varphi^{2} + d\rho^{2} - \cosh^{2}\rho \,dt^{2} + \sinh^{2}\rho \,d\phi^{2}), \qquad (6.2.3)$$

and the bosonic string action takes the form

$$S = \frac{h}{2} \int d^2 \sigma \left[\partial_a \rho \partial^a \rho + \sinh^2 \rho \partial_a \phi \partial^a \phi - \cosh^2 \rho \partial_a t \partial^a t + \partial_a \varphi \partial^a \varphi \right], \tag{6.2.4}$$

where $\eta = \text{diag}(-1, 1)$. Hence given a solution on $\mathbb{R} \times S^3$ we can analytically continue it to $\text{AdS}_3 \times S^1$ using (6.2.2). However, for a physical solution we also require that ρ should be real. For the folded string this can be achieved by changing the range of the ν parameter to

$$\nu \le 0. \tag{6.2.5}$$

The string profile is now given by

$$\hat{\rho}^2 = \omega_{21}^2 (\cosh^2 \rho_{\max} - \cosh^2 \rho), \qquad \cosh^2 \rho_{\max} = 1 - \nu \ge 1, \quad \cosh \rho = \operatorname{dn}(\omega_{21}\sigma, \nu), \tag{6.2.6}$$

with ρ moving between $\rho = 0$ and $\rho = \rho_{\text{max}}$.

6.2.1 Dispersion relation in the large spin limit

The energy and the angular momenta of the AdS folded string are given by

$$E \equiv \sqrt{\lambda} \mathcal{E} = \omega_2 \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} \mathrm{d}\sigma \,\cosh^2\rho = \sqrt{\lambda}\omega_2 \frac{E(\nu)}{K(\nu)},\tag{6.2.7}$$

$$S \equiv \sqrt{\lambda}S = \omega_1 \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} \mathrm{d}\sigma \,\sinh^2\rho = \sqrt{\lambda}\omega_1 \frac{E(\nu) - K(\nu)}{K(\nu)},\tag{6.2.8}$$

$$J \equiv \sqrt{\lambda} \mathcal{J} = \sqrt{\lambda} \kappa. \tag{6.2.9}$$

Since we are considering the semiclassical limit $\lambda \gg 1$ the charges of the quantum theory become large. Therefore we use instead the semiclassical equivalents \mathcal{E} , \mathcal{S} and \mathcal{J} , which are assumed to remain finite as $\lambda \to \infty$.

The large spin limit $S \gg 1$ corresponds to a long string, i.e. $\rho_{\max} \to \infty$ [209]. It is convenient to introduce the parameter η

$$\eta = -\frac{1}{\nu}, \qquad \coth^2 \rho_{\max} = 1 + \eta,$$
(6.2.10)

such that the long string limit corresponds to $\eta \to 0$. The periodicity condition and the defining relation for η

$$\sqrt{\omega_2^2 - \kappa^2} = \frac{1}{\sqrt{\eta}} \frac{n}{\pi} K(\nu), \qquad \eta = \frac{\omega_1^2 - \omega_2^2}{\omega_2^2 - \kappa^2}$$
(6.2.11)

can be used to express ω_1 and ω_2 in terms of the angular momentum \mathcal{J} and η

$$\omega_1^2 = \kappa^2 + \left(1 + \frac{1}{\eta}\right) \frac{n^2}{\pi^2} K \left(-\frac{1}{\eta}\right)^2, \qquad \omega_2^2 = \kappa^2 + \frac{1}{\eta} \frac{n^2}{\pi^2} K \left(-\frac{1}{\eta}\right)^2.$$
(6.2.12)

For small $\eta = 16\tilde{\eta}$ these give

$$\omega_1^2 = \kappa^2 + \frac{n^2}{4\pi^2} (1 + 8\tilde{\eta}) \ln^2 \frac{1}{\tilde{\eta}} + \mathcal{O}\Big(\tilde{\eta} \ln \frac{1}{\tilde{\eta}}\Big), \tag{6.2.13}$$

$$\omega_2^2 = \kappa^2 + \frac{n^2}{4\pi^2} (1 - 8\tilde{\eta}) \ln^2 \frac{1}{\tilde{\eta}} + \mathcal{O}\Big(\tilde{\eta} \ln \frac{1}{\tilde{\eta}}\Big).$$
(6.2.14)

The angular momentum \mathcal{S} goes as

$$\mathcal{S} \sim \omega_1 \Big[\frac{1}{8} \frac{1}{\tilde{\eta} \ln \frac{1}{\tilde{\eta}}} + \mathcal{O}(1) \Big], \tag{6.2.15}$$

and therefore, in the long string limit, we have that $S \gg 1$ for any value of $\mathcal{J} \sim \kappa$. Further expanding \mathcal{E}, \mathcal{S} in \mathcal{J} one finds two cases

1. In the case $\kappa \gg \ln \frac{1}{\tilde{\eta}}$, which corresponds to large \mathcal{S} and large \mathcal{J} with $\ln \frac{\mathcal{S}}{\mathcal{J}} \ll \mathcal{J} \ll \mathcal{S}$, one finds

$$\mathcal{E} = \mathcal{S} + \mathcal{J} + \frac{n^2}{8\pi^2 \mathcal{J}} \ln^2 \frac{\mathcal{S}}{\mathcal{J}} + \dots$$
 (6.2.16)

On the gauge side this predicts that operators of large spin and R-charge have a contribution of $\frac{\lambda}{2\pi^2 J} \ln^2 \frac{S}{J}$ in their anomalous dimension.

2. In the opposite case $\kappa \ll \ln \frac{1}{\tilde{n}}$, which corresponds to large S and small \mathcal{J} (i.e. $\mathcal{J} \ll S$), one has

$$\mathcal{E} = \mathcal{S} + \frac{n}{2\pi} \ln \mathcal{S} + \frac{\pi \mathcal{J}^2}{n \ln \mathcal{S}} + \dots$$
 (6.2.17)

For $\mathcal{J} \to 0$ and n = 2 this is simply (6.0.4). Thus on the gauge side operators of large R-charge but small angular momentum have a contribution of $\frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}}$ in their anomalous dimension.

6.3 $q \neq 0$ solution on $\mathbb{R} \times \mathbf{S}^3$

In this section we derive the $q \neq 0$ generalisation of the folded string by following the same approach as for the mixed flux dyonic giant magnon. Let us rewrite equation (5.1.12) in terms of the coordinates $\tilde{\sigma}^{\pm} = \frac{1}{2}(\tilde{\tau} \pm \tilde{\sigma}) = (1 \pm q)\sigma^{\pm}$ giving

$$0 = \partial_{\tilde{\sigma}} Z_{1} + A \partial_{\tilde{\tau}} Z_{1} + B Z_{1} ,$$

$$A \equiv \frac{\mathfrak{J}_{-}^{21} - \mathfrak{J}_{+}^{21} + q(\mathfrak{J}_{-}^{21} + \mathfrak{J}_{+}^{21})}{\mathfrak{J}_{-}^{21} + \mathfrak{J}_{+}^{21} + q(\mathfrak{J}_{-}^{21} - \mathfrak{J}_{+}^{21})} = \frac{w_{-} \operatorname{cn}(\omega_{21}\tilde{\sigma}, \nu) + iqw_{+} \operatorname{dn}(\omega_{21}\tilde{\sigma}, \nu) \operatorname{sn}(\omega_{21}\tilde{\sigma}, \nu)}{qw_{-} \operatorname{cn}(\omega_{21}\tilde{\sigma}, \nu) + iw_{+} \operatorname{dn}(\omega_{21}\tilde{\sigma}, \nu) \operatorname{sn}(\omega_{21}\tilde{\sigma}, \nu)} ,$$

$$B \equiv \frac{\mathfrak{J}_{+}^{21}\mathfrak{J}_{-}^{11} - \mathfrak{J}_{-}^{21}\mathfrak{J}_{+}^{11}}{\mathfrak{J}_{-}^{21} + \mathfrak{J}_{+}^{21} + q(\mathfrak{J}_{-}^{21} - \mathfrak{J}_{+}^{21})} = \frac{1}{2}\frac{iw_{-} \operatorname{cn}(\omega_{21}\tilde{\sigma}, \nu)(w_{-}^{2} - w_{+}^{2} + 2\nu w_{+}^{2} \operatorname{sn}^{2}(\omega_{21}\tilde{\sigma}, \nu))}{qw_{-} \operatorname{cn}(\omega_{21}\tilde{\sigma}, \nu) + iw_{+} \operatorname{dn}(\omega_{21}\tilde{\sigma}, \nu) \operatorname{sn}(\omega_{21}\tilde{\sigma}, \nu)} , \qquad (6.3.1)$$

where $\mathfrak{J} = \mathfrak{J}((1+q)\sigma^+, (1-q)\sigma^-) = \mathfrak{J}(\tilde{\sigma}^+, \tilde{\sigma}^-)$ and $w_{\pm} = \sqrt{\omega_1 \pm \omega_2}$. It is also convenient to further rescale $\tilde{\sigma}$ to the new coordinate $y = \omega_{21}\tilde{\sigma}$ and reparametrise the constants $w_+ = \sqrt{\omega_{21}/m}$, $w_- = \sqrt{\omega_{21}m}$ (which corresponds to $\omega_1 = \frac{1}{2}\omega_{21}(m^{-1}+m)$, $\omega_2 = \frac{1}{2}\omega_{21}(m^{-1}-m)$, i.e. $m = \sqrt{\frac{\omega_1-\omega_2}{\omega_1+\omega_2}}$) giving

$$0 = \partial_y Z_1 + A \partial_{\tilde{\tau}} Z_1 + B Z_1 ,$$

$$A = \frac{1}{\omega_{21}} \frac{m \operatorname{cn}(y, \nu) + iq \operatorname{dn}(y, \nu) \operatorname{sn}(y, \nu)}{qm \operatorname{cn}(y, \nu) + i \operatorname{dn}(y, \nu) \operatorname{sn}(y, \nu)} , \qquad B = \frac{1}{2} \frac{i \operatorname{cn}(y, \nu) (m^2 - 1 + 2\nu \operatorname{sn}^2(y, \nu))}{qm \operatorname{cn}(y, \nu) + i \operatorname{dn}(y, \nu) \operatorname{sn}(y, \nu)} .$$
(6.3.2)

Equation (6.3.2) becomes the ODE $\frac{dZ_1}{dy} + BZ_1 = 0$ along the characteristic curves

$$\frac{\mathrm{d}\tilde{\tau}}{\mathrm{d}y} = A(y) \Rightarrow \tilde{\tau}(y) = C_0 + \int \mathrm{d}y \, A(y) \;. \tag{6.3.3}$$

The full solution is then given by

v =

$$Z_1 = f(C_0(\tilde{\tau}, \tilde{\sigma})) \exp\left[-\int \mathrm{d}y \, B(y)\right] \,. \tag{6.3.4}$$

The integral in the above expression can be evaluated to give

$$I_{1}(y) \equiv \int dy B(y) = imqy + \frac{imq}{2} \sum_{z \in \{z_{+}, z_{-}\}} l_{1}(z)\Pi(z^{-1}, \operatorname{am}(y, \nu), \nu) - \frac{1}{4} \ln\left[(1-\nu)m^{2}q^{2} - (1+m^{2}q^{2})\operatorname{dn}^{2}(y, \nu) + \operatorname{dn}^{4}(y, \nu)\right] + (1-q^{2})\frac{m^{2}}{4\nu} \ln\left[\frac{\nu+1+m^{2}q^{2}-2\operatorname{dn}^{2}(y, \nu)}{\nu-1-m^{2}q^{2}+2\operatorname{dn}^{2}(y, \nu)}\right], \sqrt{4m^{2}q^{2}\nu + (1-m^{2}q^{2})^{2}}, \quad l_{1}(z) = \frac{(2m^{2}q^{2}+2\nu-m^{2}-1)z+m^{2}-2m^{2}q^{2}-1}{z(2\nu z+m^{2}q^{2}-1)}, \quad (6.3.5)$$

where z_{\pm} are the roots of $z^2 + \frac{m^2 q^2 - 1}{\nu} z - \frac{m^2 q^2}{\nu} = 0$ and Π is the incomplete elliptic integral of the third kind

$$\Pi(n,\phi,m) = \int_0^\phi \frac{\mathrm{d}\theta}{(1-n\sin^2\theta)\sqrt{1-m\sin^2\theta}} \,. \tag{6.3.6}$$

Evaluating the integral in (6.3.3) gives

$$I_{2}(y) \equiv \omega_{21} \int dy A(y) = qy + qm^{2} \sum_{z \in \{z_{+}, z_{-}\}} l_{2}(z) \Pi(z^{-1}, \operatorname{am}(y, \nu), \nu) - i(1 - q^{2}) \frac{m}{2v} \ln\left[\frac{v + 1 + m^{2}q^{2} - 2\operatorname{dn}^{2}(y, \nu)}{v - 1 - m^{2}q^{2} + 2\operatorname{dn}^{2}(y, \nu)}\right], l_{2}(z) = \frac{(1 - q^{2})(1 - z)}{z(2\nu z + m^{2}q^{2} - 1)}.$$
(6.3.7)

Substituting the solution

$$Z_1 = f\left(\tilde{\tau} - \frac{1}{\omega_{21}}I_2(y)\right)e^{-I_1(y)}$$
(6.3.8)

into the second order differential equation for Z_1 (5.1.14) gives the differential equation for f

$$f''(x) + im\omega_{21}f'(x) + \omega_{21}^2 \frac{1 - m^4 + 2m^2(-1 + 2\nu + m^2)q^2}{4m^2(1 - q^2)^2} f(x) = 0.$$
 (6.3.9)

Its solutions are

$$f(x) = a_{\pm}e^{l_{\pm}x}$$
, $l_{\pm} = i\omega_{21}\left(-\frac{m}{2} \pm \frac{v}{2m(1-q^2)}\right)$, (6.3.10)

where a_{\pm} are integration constants. Taking the limit $q \to 0$ only the choice $f = a_{+} \exp(l_{+}x)$ recovers the q = 0 folded string solution (6.1.4).

Thus the full solution becomes

$$Z_{1} = a_{1}e^{l_{+}\tilde{\tau}} \Big[\frac{v - 1 - m^{2}q^{2} + 2\operatorname{dn}^{2}(y,\nu)}{v + 1 + m^{2}q^{2} - 2\operatorname{dn}^{2}(y,\nu)} ((1 + m^{2}q^{2})\operatorname{dn}^{2}(y,\nu) - \operatorname{dn}^{4}(y,\nu) - (1 - \nu)m^{2}q^{2}) \Big]^{1/4} \\ \times \exp\Big\{ -\frac{qi}{2} \Big[\Big(m + \frac{v}{m(1 - q^{2})}\Big)y + m\sum_{z \in \{z_{+}, z_{-}\}} \Big(l_{1}(z) + \Big(\frac{v}{1 - q^{2}} - m^{2}\Big)l_{2}(z)\Big)\Pi(z^{-1}, \operatorname{am}(y,\nu),\nu) \Big] \Big\} .$$

We can immediately write down the Z_2 solution by noticing that the equations for Z_1 and Z_2 are related via complex conjugation. Therefore taking the complex conjugate of the general Z_1 solution, but now choosing $f = a_- \exp(l_- x)$, such that we recover the q = 0 solution, we obtain

$$Z_{2} = a_{2}e^{l_{-}^{*}\tilde{\tau}} \Big[\frac{v+1+m^{2}q^{2}-2\operatorname{dn}^{2}(y,\nu)}{v-1-m^{2}q^{2}+2\operatorname{dn}^{2}(y,\nu)} ((1+m^{2}q^{2})\operatorname{dn}^{2}(y,\nu)-\operatorname{dn}^{4}(y,\nu)-(1-\nu)m^{2}q^{2}) \Big]^{1/4} \times \exp\Big\{ \frac{qi}{2} \Big[\Big(m-\frac{v}{m(1-q^{2})}\Big)y + m\sum_{z\in\{z_{+},z_{-}\}} \Big(l_{1}(z) - \Big(\frac{v}{1-q^{2}}+m^{2}\Big)l_{2}(z)\Big)\Pi(z^{-1},\operatorname{am}(y,\nu),\nu) \Big] \Big\}.$$

From the normalisation condition $|Z_1|^2 + |Z_2|^2 = 1$ we then find

$$a_1 = a_2 = v^{-1/2} . (6.3.11)$$

After some algebra this solution can be further simplified to

$$Z_1 = \sin \theta e^{i\phi_1}, \qquad Z_2 = \cos \theta e^{i\phi_2}, \qquad \cos^2 \theta = \frac{v - 1 - q^2 m^2}{2v} + \frac{1}{v} \operatorname{dn}^2(\omega_{21}(\sigma + q\tau), \nu), \qquad (6.3.12)$$

$$\phi_1 = \frac{v + (1+q^2)m^2}{2m}\omega_{21}\tau + qm\omega_{21}\sigma - \frac{1+q^2m^2 + v}{2mq}\pi\Big(\frac{2\nu}{1-q^2m^2 - v}, \omega_{21}(\sigma + q\tau)\Big), \tag{6.3.13}$$

$$\phi_2 = \frac{v - (1 + q^2)m^2}{2m}\omega_{21}\tau - qm\omega_{21}\sigma + \frac{1 + q^2m^2 - v}{2mq}\pi\Big(\frac{2\nu}{1 - q^2m^2 + v}, \omega_{21}(\sigma + q\tau)\Big), \tag{6.3.14}$$

$$v = \sqrt{4m^2q^2\nu + (1 - m^2q^2)^2}, \qquad \pi(x, y) = \Pi(x, \operatorname{am}(y, \nu), \nu).$$
(6.3.15)

The winding numbers for ϕ_1 and ϕ_2 are given by

$$h_{\phi_2} = \omega_{21} \Big(-qm + \frac{1 + q^2 m^2 - v}{2mq} \frac{\Pi(\frac{2\nu}{1 - q^2 m^2 + v}, \nu)}{K(\nu)} \Big), \tag{6.3.16}$$

$$h_{\phi_1} = \omega_{21} \left(qm - \frac{1 + q^2 m^2 + v}{2mq} \frac{\Pi(\frac{2\nu}{1 - q^2 m^2 - v}, \nu)}{K(\nu)} \right).$$
(6.3.17)

For a closed string solution these must be integer-valued, which gives two "quantization" conditions for κ , ω_1 and ω_2 . This solution corresponds to a string with spikes rotating in the ϕ_1 and ϕ_2 directions as shown in figure 6.2.

In order to avoid these quantization conditions it should also be possible to start with a more general ansatz with winding, which is essentially a spiky string [210, 214]. For the resulting $q \neq 0$ string solution the original winding parameters then become auxiliary parameters and should be eliminated in favour of the physical windings of the $q \neq 0$ solution, on which the closed string conditions are imposed. This is similar to the dyonic giant magnon case where we expressed the q = 0 and $q \neq 0$ solutions in terms of their respective separation angles between the endpoints since these play the role of the physical momentum. We leave this more general case to a future study.

In the $q \to 0$ limit the coordinates θ and ϕ_2 reduce to those of the folded string as expected. However, for ϕ_1 the limit is subtle: the σ derivative

$$\partial_{\sigma}\phi_1 = mq\omega_{21} \Big(1 - \frac{v - 1 + 2\nu + q^2m^2}{v + 1 + q^2m^2 - 2\operatorname{dn}^2(\omega_{21}(\sigma + q\tau), \nu)} \Big).$$
(6.3.18)

is not well defined at the points $\omega_{21}(\sigma + q\tau) = 2K(\nu)n, n \in \mathbb{Z}$ when q = 0. Taking $q \to 0$ the derivative $\partial_{\sigma}\phi_1$ approaches a set of Dirac δ -functions centred at these points such that ϕ_1 reduces to a step function in σ and h_{ϕ_1} approaches

$$h_{\phi_1} \to -\frac{\pi}{2} \frac{\omega_{21}}{K(\nu)}.$$
 (6.3.19)

Thus ϕ_1 jumps by a value

$$\Delta\phi_1 = h_{\phi_1}\Delta\sigma = -\pi,\tag{6.3.20}$$

which is consistent with a folded string as shown in figure 6.1.



Figure 6.2: The $q \neq 0$ generalisation of the folded string shown as a parametric plot from the top of the sphere for the values $\omega_2 = 0.3, q = 0.836152, \omega_1 = 2.69191, \kappa = 2.28482, h_{\phi_1} = -1, h_{\phi_2} = -1, n = 4$. The string rotates in the ϕ_1 and ϕ_2 directions as indicated by the arrows.

6.3.1 Relation to the dyonic giant magnon

Taking the $\nu \to 1$ limit in (6.3.12)-(6.3.15) gives

$$\cos \theta = \frac{\operatorname{sech}(y)}{\sqrt{1+q^2m^2}}, \qquad y = \sqrt{1-\omega^2}(x+qt)$$
 (6.3.21)

$$\phi_1 = t - \arctan\left(\frac{1}{qm}\tanh(y)\right), \qquad \phi_2 = \omega(t+qx) - qx, \tag{6.3.22}$$

where we changed to the decompactified world-sheet coordinates $(t, x) = (\kappa \tau, \kappa \sigma), \kappa \to \infty$. The spikes of the string now reach $\theta = \pi/2$ with each segment corresponding to a dyonic giant magnon with

$$\tan\frac{p}{2} = \pm\frac{1}{qm}, \qquad v = 0, \qquad \sin\rho = \omega, \qquad m = \sqrt{\frac{1-\omega}{1+\omega}}, \tag{6.3.23}$$

$$J_2^{\text{DGM}} = h \left[2q \arctan\left(\frac{1}{qm}\right) + \frac{1}{m} \right].$$
 (6.3.24)

Here ω is fixed by the closed string condition

$$p = \pm \frac{2\pi}{n}.\tag{6.3.25}$$

For the energy and angular momenta of the $q \neq 0$ solution this gives

$$E - J_1 = n(E^{\text{DGM}} - J_1^{\text{DGM}}), \qquad J_2 = nJ_2^{\text{DGM}} = \pm qh\left(2\pi + n\tan\frac{\pi}{n}\right).$$
 (6.3.26)

$$E - J_1 = n \sqrt{\left(\frac{J_2}{n} - qhp\right)^2 + 4h^2(1 - q^2)\sin^2\frac{p}{2}}.$$
 (6.3.27)

Therefore in the limit $\nu \to 1$ the $q \neq 0$ solution consists of n/2 pairs of dyonic giant magnons of opposite momenta $p = \pm \frac{2\pi}{n}$.

6.3.2 Relation to soliton of the Pohlmeyer reduced theory

The soliton for the $q \neq 0$ solution can be found from the expressions (5.5.2), which give

$$\sin\varphi = \frac{\sqrt{\kappa^2 - \omega_2^2}}{\kappa} \operatorname{cn}(\omega_{21}(\sigma + q\tau), \nu), \qquad \chi = 2\frac{\omega_1\omega_2}{\kappa}(\tau + q\sigma).$$
(6.3.28)

The CsG soliton soliton $\psi = \sin \varphi e^{\frac{i}{2}\chi}$ then takes the form

$$\psi = \frac{\sqrt{\kappa^2 - \omega_2^2}}{\kappa} \operatorname{cn}(\omega_{21}(\sigma + q\tau), \nu) e^{i\frac{\omega_1\omega_2}{\kappa}(\tau + q\sigma)}.$$
(6.3.29)

As in the case of the mixed flux dyonic giant magnon, this is the q = 0 soliton with the coordinates replaced by

$$\sigma \to \sigma + q\tau, \qquad \tau \to \tau + q\sigma,$$
 (6.3.30)

which is a Lorentz boost once we rescale κ , ω_1 and ω_2 by $\sqrt{1-q^2}$.

6.4 $q \neq 0$ solution on $AdS_3 \times S^1$

Let us now construct the $q \neq 0$ folded string solution in $AdS_3 \times S^1$. In global AdS coordinates the metric and NS-NS flux take the form

$$ds^{2} = -\cosh^{2}\rho \,dt^{2} + d\rho^{2} + \sinh^{2}\rho \,d\phi^{2} + d\varphi^{2}, \qquad B_{t\phi} = q \sinh^{2}\rho, \qquad (6.4.1)$$

where φ is the angle on S¹. The bosonic string action is given by

$$S = -\frac{h}{2} \int d^2\sigma \left[\partial_a \rho \partial^a \rho + \sinh^2 \rho \partial_a \phi \partial^a \phi - \cosh^2 \rho \partial_a t \partial^a t + \partial_a \varphi \partial^a \varphi -q(2\cosh^2 \rho + c - 1)(\dot{\phi}\dot{t} - \dot{t}\dot{\phi}) \right].$$
(6.4.2)

We can obtain a real solution to the equations of motion and Virasoro constraints by Wick rotating the $R \times S^3$ coordinates $(t, \theta, \phi_1, \phi_2)$ of the solution (6.3.12)-(6.3.15) to

$$t \to \varphi, \quad \theta \to i\rho, \quad \phi_1 \to \phi, \quad \phi_2 \to t$$
 (6.4.3)

and changing the range of ν to

$$\nu \le 0. \tag{6.4.4}$$

We still need to ensure that the new global time coordinate t, which originates from ϕ_1 , is single valued. We shall do this by performing the world-sheet transformation

$$\tau \to \tau - w\sigma, \qquad \sigma \to \sigma - w\tau \tag{6.4.5}$$

and fixing w to ensure that the winding of the new t coordinate vanishes. This is possible since the map between the q = 0 and $q \neq 0$ equations of motion, which we used to obtain the $q \neq 0$ solution, is invariant under such world-sheet Lorentz transformations and coordinate rescalings. However, for

solutions on $R \times S^3$ this symmetry is broken by requiring that $t = \kappa \tau$ is preserved for $q \neq 0$.

Imposing that the winding for t vanishes we obtain

$$0 = h_t = \omega_{21} \Big(-w \frac{v - (1 + q^2)m^2}{2m} - qm + \frac{1 + q^2m^2 - v}{2mq} \frac{\Pi(\frac{2\nu}{1 - q^2m^2 + v}, \nu)}{K(\nu)} (1 - qw) \Big), \tag{6.4.6}$$

$$w = \frac{\frac{1+q^2m^2-\nu}{q}\Pi(\frac{2\nu}{1-q^2m^2+\nu},\nu) - 2qm^2K(\nu)}{(1+q^2m^2-\nu)\Pi(\frac{2\nu}{1-q^2m^2+\nu},\nu) + (\nu-(1+q^2)m^2)K(\nu)}.$$
(6.4.7)

This solution is characterised by the variables κ , ω_1 and ω_2 subject to the periodicity condition

$$2\pi = 2n \int_{\rho_{min}}^{\rho_{max}} \frac{\mathrm{d}\rho}{\partial_{\sigma}\rho}.$$
 (6.4.8)

Writing the derivative $\partial_{\sigma}\rho$ as

$$(\partial_y \rho)^2 = v \frac{\sinh^2 \rho + \gamma}{\sinh^2 \rho \cosh^2 \rho} (\sinh^2 \rho_{max} - \sinh^2 \rho) (\sinh^2 \rho - \sinh^2 \rho_{min}), \qquad (6.4.9)$$

$$\sinh^2 \rho_{max} = \frac{-\nu}{v} + \frac{1 - v - q^2 m^2}{2v}, \qquad \sinh^2 \rho_{min} = \frac{1 - v - q^2 m^2}{2v}, \quad \gamma = \frac{1 + v + q^2 m^2}{2v}, \quad (6.4.10)$$

we see that along the ρ direction the string stretches between $\rho = \rho_{\min}$ and $\rho = \rho_{\max}$. Using this the periodicity condition (6.4.8) evaluates to

$$\pi\omega_{21}(1-qw) = nK(\nu), \quad n \in \mathbb{N}^*,$$
(6.4.11)

where n is the number of spikes. Thus, apart from the Lorentz transformation associated with w, the periodicity condition does not acquire any q-dependence.

The Noether charges are given by

$$E = h \int d\sigma \left[\cosh^2 \rho \, \dot{t} + \frac{q}{2} (2 \cosh^2 \rho + c - 1) \dot{\phi} \right] \tag{6.4.12}$$

$$S = h \int d\sigma \left[\sinh^2 \rho \, \dot{\phi} + \frac{q}{2} (2 \cosh^2 \rho + c - 1) \dot{t} \right]$$
(6.4.13)

$$J = h \int d\sigma \,\dot{\varphi}. \tag{6.4.14}$$

For a closed string solution these evaluate to (choosing c = -1)

$$J = 2\pi h\kappa$$
(6.4.15)
$$S = \frac{h}{8mq^2v} \Big\{ 4\pi\omega_{21} \Big[(1-q^2)(1-q^4m^4) - (1+q^2)v^2 \Big]$$

$$+2q^{2}n\Big[m^{2}(1-q^{2})+v\Big]E(\nu)-n(1-q^{2}+v)(1+m^{2}q^{2}-v)K(\nu)\Big\}$$
(6.4.16)

$$E = -\frac{\pi}{8mq^2v} \Big(4\pi\omega_{21} \Big[(1-q^2)(1-q^4m^4) - (1+q^2)v^2 \Big] + 2q^2n \Big[m^2(1-q^2) - v \Big] E(\nu) - n(1-q^2-v)(1+m^2q^2+v)K(\nu) \Big).$$
(6.4.17)

For q = 0 it is useful to note that from (6.4.12),(6.4.13) the energy E and spin S satisfy the relation

$$\frac{E}{\omega_2} - \frac{S}{\omega_1} = 1. (6.4.18)$$

Similarly we can write the equivalent relation for $q \neq 0$ as

$$E = \left[\frac{2v}{v - m^2(1 - q^2) + 2m^2r} - 1\right]S$$

$$-\frac{\pi\omega_{21}}{mq^2(v - m^2(1 - q^2) + 2m^2r)} \left[2m(1 + m^2)r^2 + [q^2(m^2((m^2 + 2)(q^2 - 1) - 4s) + 1) + v^2 - 1]r + q^2(m^2(q^2 - 1)(m^2q^2 - 4s) - v^2 - 1) - v^2 + 1]$$

$$= \frac{nK(\nu)}{nE(\nu)} = \frac{nE(\nu)}{nE(\nu)}$$

(6.4.19)

$$r = \frac{nK(\nu)}{\pi\omega_{21}}, \qquad s = \frac{nE(\nu)}{\pi\omega_{21}}.$$
 (6.4.20)

6.4.1 Long string limit

Having established the $q \neq 0$ string solution we can now find the dispersion relation in the long string limit $\rho_{max} \rightarrow \infty$. Defining the variable η as

$$\eta = -\frac{1}{\nu} = \frac{\omega_1^2 - \omega_2^2}{\omega_2^2 - \kappa^2},\tag{6.4.21}$$

this limit corresponds to $\eta \to 0$ as can be seen from (6.4.10),

$$\operatorname{coth}^2 \rho_{max} \approx 1 + \sqrt{1 - q^2 \left(1 - \frac{\kappa^2}{\omega_2^2}\right)} \Big|_{\eta=0} \eta + \mathcal{O}(\eta^2).$$
 (6.4.22)

In this limit $\omega_1^2 - \omega_2^2 \to 0$ and we can determine their asymptotic behaviour for small η from the periodicity relation (6.4.11) by solving (6.4.21) for ω_1

$$\omega_1 = \sqrt{\omega_2^2 (1+\eta) - \eta \kappa^2}$$
(6.4.23)

and expanding

$$\omega_2^2 = \kappa^2 + \left[\frac{n}{\pi(1 - qw)\sqrt{\eta}}K\left(-\frac{1}{\eta}\right)\right]^2.$$
 (6.4.24)

For (6.4.24) there are two different cases in κ to consider:

1. For $\kappa \ll \ln \frac{1}{\eta}$ (as we shall see this corresponds to $\mathcal{J} \ll \mathcal{S}$) we first expand in small κ . Then we notice that $\omega_2 \to \infty$ as $\eta \to 0$. This can be seen by expanding the RHS of (6.4.24) in small ω_2 . However, the resulting expression for ω_2 becomes large for small η and we should instead expand

(6.4.24) in large ω_2 . This gives the recursive relation

$$\begin{split} \omega_{2}^{2} &\approx \left[\frac{n^{2}}{4\pi^{2}}\ln^{2}\frac{1}{\tilde{\eta}} - \frac{2n^{2}}{\pi^{2}}e^{\frac{q\operatorname{ArcSin}q}{\sqrt{1-q^{2}}}}\left(1 + \frac{q^{2}}{\sqrt{1-q^{2}}}\right)\tilde{\eta}\ln^{2}\frac{1}{\tilde{\eta}} \\ &+ \frac{4n^{2}}{\pi^{2}}e^{\frac{q\operatorname{ArcSin}q}{\sqrt{1-q^{2}}}}\left(1 + \frac{q\operatorname{ArcSin}q}{2(1-q^{2})}\right)\tilde{\eta}\ln\frac{1}{\tilde{\eta}} + \mathcal{O}\left(\tilde{\eta}^{2}\ln^{2}\frac{1}{\tilde{\eta}}\right)\right] \\ &+ \kappa^{2}\left[1 + \frac{1}{\omega_{2}^{2}}\mathcal{O}\left(\ln\frac{1}{\tilde{\eta}}\right)\right] + \mathcal{O}(\kappa^{4}), \end{split}$$
(6.4.25)

where we rescaled $\eta \to 16e^{\frac{q\operatorname{ArcSin} q}{\sqrt{1-q^2}}}\tilde{\eta}$. Iterating this relation we find

$$\omega_2^2 \approx \left[\frac{n^2}{4\pi^2}\ln^2\frac{1}{\tilde{\eta}} - \frac{2n^2}{\pi^2}e^{\frac{q\operatorname{ArcSin}q}{\sqrt{1-q^2}}}\left(1 + \frac{q^2}{\sqrt{1-q^2}}\right)\tilde{\eta}\ln^2\frac{1}{\tilde{\eta}} + \mathcal{O}\left(\tilde{\eta}\ln\frac{1}{\tilde{\eta}}\right)\right] + \kappa^2\left[1 + \mathcal{O}\left(\frac{1}{\ln\frac{1}{\tilde{\eta}}}\right)\right] + \mathcal{O}(\kappa^4).$$
(6.4.26)

Alternatively (6.4.26) can also be obtained by solving (6.4.25) for ω_2^2 and expanding in small κ and η . It is worth noting that for q = 0 the coefficient in front of κ^2 in (6.4.25) and (6.4.26) is identically one.

From (6.4.21) we then have

$$\omega_{1}^{2} \approx \left[\frac{n^{2}}{4\pi^{2}}\ln^{2}\frac{1}{\tilde{\eta}} - \frac{2n^{2}}{\pi^{2}}e^{\frac{q\operatorname{ArcSin}q}{\sqrt{1-q^{2}}}}\left(-1 + \frac{q^{2}}{\sqrt{1-q^{2}}}\right)\tilde{\eta}\ln^{2}\frac{1}{\tilde{\eta}} + \mathcal{O}\left(\tilde{\eta}\ln\frac{1}{\tilde{\eta}}\right)\right] + \kappa^{2}\left[1 + \mathcal{O}\left(\frac{1}{\ln\frac{1}{\tilde{\eta}}}\right)\right] + \mathcal{O}(\kappa^{4}), \quad (6.4.27)$$

and from (6.4.16), (6.4.19) we find for the conserved charges

$$\mathcal{S} \approx \frac{n}{8} e^{-\frac{q\operatorname{ArcSin} q}{\sqrt{1-q^2}}} \frac{1}{\tilde{\eta}} - \frac{n\sqrt{1-q^2}}{2} \ln \frac{1}{\tilde{\eta}} + O(1) + \kappa^2 \mathcal{O}\left(\frac{1}{\tilde{\eta}\ln^2 \frac{1}{\tilde{\eta}}}\right)$$
(6.4.28)

$$\mathcal{E} \approx \left[1 + \mathcal{O}(\tilde{\eta}) + \kappa^2 \mathcal{O}\left(\frac{\eta}{\ln^2 \tilde{\eta}}\right) + \mathcal{O}(\kappa^4)\right] \mathcal{S} + n\sqrt{1 - q^2} \ln \frac{1}{\tilde{\eta}} + \mathcal{O}(1) + \kappa^2 \left[\frac{2\pi^2}{n\sqrt{1 - q^2} \ln \frac{1}{\tilde{\eta}}} + \mathcal{O}\left(\frac{1}{\ln^2 \frac{1}{\tilde{\eta}}}\right)\right] + \mathcal{O}(\kappa^4).$$
(6.4.29)

We can now give the energy as an expansion in large spin

$$\mathcal{E} \approx \mathcal{S} + \frac{n}{2\pi}\sqrt{1-q^2}\ln\mathcal{S} + \mathcal{O}(1) + \mathcal{J}^2\Big[\frac{\pi}{n\sqrt{1-q^2}\ln\mathcal{S}} + \mathcal{O}\Big(\frac{1}{\ln^2\mathcal{S}}\Big)\Big] + \mathcal{O}(\mathcal{J}^4).$$
(6.4.30)

In deriving the conserved charges we made a choice of the total derivative by setting c = -1. Explicitly expanding the contribution from this total derivative term in \mathcal{E} we find

$$\Delta \mathcal{E} \approx n\pi q + \mathcal{O}(\tilde{\eta}) + q\kappa^2 \Big[2\pi^2 \frac{q\sqrt{1-q^2} + \operatorname{ArcSin} q}{n(1-q^2)\ln^2 \frac{1}{\tilde{\eta}}} + \mathcal{O}(\tilde{\eta}) \Big] + \mathcal{O}(\kappa^4).$$
(6.4.31)

Hence to leading order the result (6.4.30) is independent of the choice for the total derivative term in the action and the effect of q is a rescaling of the string tension by $\sqrt{1-q^2}$ with the

same $\ln S$ behaviour as in the pure R-R case. Also an interesting feature is that the limits $q \to 1$ and $\mathcal{J} \to 0$ do not commute.

2. For $\kappa \gg \ln \frac{1}{\eta}$ we need to determine the asymptotic behaviour of ω_2 as $\kappa \to \infty$. Assuming that ω_2 is small and expanding in large κ we find again that in contradiction $\omega_2 \sim \kappa$ is large. Hence first expanding in large ω_2 and then small η at each order we find the recursion relation

$$\omega_2^2 \approx \kappa^2 + \frac{n^2}{4\pi^2} \ln^2 \frac{1}{\tilde{\eta}} - \frac{2n^2}{\pi^2} e^{\frac{q \operatorname{ArcSin} q}{\sqrt{1-q^2}}} \left(1 + \frac{q^2}{\sqrt{1-q^2}}\right) \tilde{\eta} \ln^2 \frac{1}{\tilde{\eta}} + \mathcal{O}\left(\tilde{\eta} \ln \frac{1}{\tilde{\eta}}\right) + \mathcal{O}\left(\frac{1}{\omega_2^2}\right). \quad (6.4.32)$$

Thus we have

$$\omega_{2}^{2} \approx \kappa^{2} + \frac{n^{2}}{4\pi^{2}} \ln^{2} \frac{1}{\tilde{\eta}} - \frac{2n^{2}}{\pi^{2}} e^{\frac{q \operatorname{ArcSin} q}{\sqrt{1-q^{2}}}} \left(\frac{q^{2}}{\sqrt{1-q^{2}}} + 1\right) \tilde{\eta} \ln^{2} \frac{1}{\tilde{\eta}} + \mathcal{O}\left(\tilde{\eta} \ln \frac{1}{\tilde{\eta}}\right) + \frac{1}{\kappa^{2}} + \mathcal{O}\left(\frac{1}{\kappa^{4}}\right),$$

$$(6.4.33)$$

$$\omega_{1}^{2} \approx \kappa^{2} + \frac{n^{2}}{4\pi^{2}} \ln^{2} \frac{1}{\tilde{\eta}} - \frac{2n^{2}}{\pi^{2}} e^{\frac{q \operatorname{ArcSin} q}{\sqrt{1-q^{2}}}} \left(\frac{q^{2}}{\sqrt{1-q^{2}}} - 1\right) \tilde{\eta} \ln^{2} \frac{1}{\tilde{\eta}} + \mathcal{O}\left(\tilde{\eta} \ln \frac{1}{\tilde{\eta}}\right) + \frac{1}{\kappa^{2}} + \mathcal{O}\left(\frac{1}{\kappa^{4}}\right).$$

$$(6.4.34)$$

This gives the expansion for the angular momentum

$$\mathcal{S} \approx \left[\frac{1}{4}e^{-\frac{q\operatorname{ArcSin}q}{\sqrt{1-q^2}}}\frac{1}{\tilde{\eta}\ln\frac{1}{\tilde{\eta}}} + \mathcal{O}(1)\right]\kappa + \mathcal{O}(\kappa^{-1}).$$
(6.4.35)

Thus asymptotically we have $\ln \frac{1}{\bar{\eta}} \approx \ln \frac{S}{\mathcal{J}}$ and $S \gg \mathcal{J} = \kappa$. Hence for $S \gg \mathcal{J} \gg \ln \frac{S}{\mathcal{J}}$ we obtain the energy from (6.4.19)

$$\mathcal{E} \approx [1 + \mathcal{O}(\kappa^{-2})]\mathcal{S} + 2\pi\kappa \Big[1 + \frac{q \operatorname{ArcSin} q}{\sqrt{1 - q^2 \ln \frac{1}{\tilde{\eta}}}} + \mathcal{O}(\tilde{\eta}) \Big] \\ + \frac{n^2}{4\pi\kappa} \Big[(1 - q^2) \ln^2 \frac{1}{\tilde{\eta}} + \mathcal{O}(\ln \frac{1}{\tilde{\eta}}) \Big] + \mathcal{O}(\kappa^{-3}).$$
(6.4.36)

The contribution coming from the total derivative in this expression is

$$\Delta \mathcal{E} \approx 2\pi \kappa \Big[\frac{q \operatorname{ArcSin} q}{\sqrt{1 - q^2} \ln \frac{1}{\tilde{\eta}}} + \mathcal{O}(\tilde{\eta}) \Big] + n\pi q + \frac{1}{\kappa} \mathcal{O}\Big(\ln \frac{1}{\tilde{\eta}} \Big) + \mathcal{O}(\kappa^{-3}).$$
(6.4.37)

Eliminating κ and $\tilde{\eta}$ in favour of \mathcal{J} and \mathcal{S} we find

$$\mathcal{E} \approx \left[1 + \mathcal{O}\left(\frac{1}{\mathcal{J}^2}\right)\right] \mathcal{S} + \mathcal{J}\left[1 + \frac{q \operatorname{ArcSin} q}{\sqrt{1 - q^2 \ln \frac{\mathcal{S}}{\mathcal{J}}}} + \mathcal{O}\left(\frac{1}{\frac{\mathcal{S}}{\mathcal{J}} \ln \frac{\mathcal{S}}{\mathcal{J}}}\right)\right] \\ + \frac{n^2}{8\pi^2 \mathcal{J}} \left[(1 - q^2) \ln^2 \frac{\mathcal{S}}{\mathcal{J}} + \mathcal{O}\left(\ln \frac{\mathcal{S}}{\mathcal{J}}\right)\right] + \mathcal{O}\left(\frac{1}{\mathcal{J}^3}\right).$$
(6.4.38)

Again the total derivative only contributes at subleading order in this expression and at leading order the effect of q is the rescaling of the string tension by $\sqrt{1-q^2}$ with the same $\frac{1}{\mathcal{J}} \ln^2 \frac{S}{\mathcal{J}}$ behaviour as in the pure R-R case.

6.5 Summary

In this chapter we have constructed the mixed flux generalisation of the folded string on the $\mathbb{R} \times S^3$ subspace of $AdS_3 \times S^3 \times T^4$. We have seen that the winding numbers of the mixed flux solution depend on q. In other words when the flux is switched on the string opens up. Imposing closed string boundary conditions, i.e. that the $q \neq 0$ winding numbers are integer-valued, we have found that the two angular momenta S and \mathcal{J} become quantized. We have also seen that in the limit of the spikes reaching the great circle of S^3 this solution reduces to a set of pairs of dyonic giant magnons, which supports the interpretation of the $q \neq 0$ solution as a generalisation of the folded string.

Further, we analytically continued this $q \neq 0$ solution to $\operatorname{AdS}_3 \times \operatorname{S}^1$ and used a Lorentz transformation to ensure that the global time coordinate is single valued. Considering the long string limit we found its energy at leading order in large S for small and large \mathcal{J} . In both cases the leading order terms are given by the pure R-R terms with the string tension rescaled by $\sqrt{1-q^2}$. We have also seen that this is independent of the choice of the total derivative contribution in the local form of the Wess-Zumino term in the action.

Chapter 7

Semiclassical and 1-loop phase in the S-matrix

In this chapter, based on [3], we consider the dressing phases in the massive sector world-sheet S-matrix for string theory on $AdS_3 \times S^3 \times T^4$ with mixed flux. As we have seen in chapter 5, the mixed flux theory admits dyonic giant magnon soliton solutions moving on the $\mathbb{R} \times S^3$ subspace with the dispersion relation

$$\varepsilon \equiv E - J_1 = \sqrt{(J \pm qhp)^2 + 4h^2 \hat{q}^2 \sin^2 \frac{p}{2}}, \qquad \hat{q} = \sqrt{1 - q^2}, \qquad E, J_1 \to \infty,$$
 (7.0.1)

where $q \in (0, 1)$ is the coefficient of the NS-NS flux, (J_1, J) are two angular momenta and the worldsheet momentum p is related to the effective kink-charge corresponding to the opening angle between the end-points of the string along a circle of S³.

The parameter h denotes the string tension in the semiclassical limit, i.e. the 't Hooft coupling λ is large with h given by

$$h = \frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi\alpha'},\tag{7.0.2}$$

where R is the curvature radius of AdS_3 and S^3 . This relation could receive corrections in $1/\sqrt{\lambda}$ as in the case of $AdS_4 \times \mathbb{CP}^3$ [215, 46, 54]. In the action h appears in the coefficient hq/2 of the WZ term for the NS-NS flux and as such the combination

$$2\pi hq = q\sqrt{\lambda} \tag{7.0.3}$$

is the quantized WZ level.

In the quantum theory soliton solutions are associated to asymptotic states. Their dispersion relation and S-matrix $S(p_1, p_2) = \exp(i\Theta(p_1, p_2))$ have the semiclassical expansion, i.e. $h \gg 1$,

$$E(p) = hE_{cl}(p) + \Delta E(p) + \mathcal{O}\left(\frac{1}{h}\right), \qquad (7.0.4)$$

$$\Theta(p_1, p_2) = \mathbf{h}\Theta_{\mathrm{cl}}(p_1, p_2) + \Delta\Theta(p_1, p_2) + \mathcal{O}\left(\frac{1}{\mathbf{h}}\right).$$
(7.0.5)

For classical scattering, integrability implies that such soliton solutions experience only an overall timedelay, ΔT , relative to free propagation. This time-delay is related to the leading order S-matrix by the WKB approximation [89]

$$\Theta_{\rm cl}(p_1, p_2) = \frac{1}{\rm h} \int_{E_{\rm th}}^{E(p_1)} \mathrm{d}E_1 \,\Delta T(E_1, E_2; J_1, J_2), \tag{7.0.6}$$

where $E_{\rm th} = E_1|_{p_1=0}$. For dyonic giant magnon solitons the semiclassical limit corresponds to

$$h \to \infty, \qquad \frac{J}{h} \text{ fixed}, \qquad p \text{ fixed},$$
 (7.0.7)

with associated bound states in the quantum theory such that their dispersion relation has the same form (7.0.1) as the classical solution

$$E_{\rm cl} = \frac{1}{\rm h}\varepsilon = \sqrt{\left(\frac{J}{\rm h} \pm qp\right)^2 + 4\hat{q}^2\sin^2\frac{p}{2}}.$$
(7.0.8)

Further the 1-loop corrections were obtained for an integrable field theory in [25]

$$\Delta E(p) = \frac{1}{2\pi} \sum_{I=1}^{N_F} (-1)^{F_I} \int_{-\infty}^{\infty} \mathrm{d}k \, \frac{\partial \delta_I(k;p)}{\partial k} \omega(k) \tag{7.0.9}$$

$$\Delta\Theta(p_1, p_2) = \frac{1}{2\pi} \sum_{I=1}^{N_F} (-1)^{F_I} \int_{-\infty}^{\infty} \mathrm{d}k \, \frac{\partial \delta_I(k; p_1)}{\partial k} \delta_I(k; p_2).$$
(7.0.10)

where the $\delta_I(k; p)$ are the phase shifts for plane waves of momentum k scattering off classical soliton solutions of momentum p. These plane waves have the dispersion relation $\omega(k)$ and they represent small fluctuations around the soliton solution labelled by the index $I = 1, ..., N_F$, with $(-1)^{F_I} = 1$ for bosononic and $(-1)^{F_I} = -1$ for fermionic fields.

For an integrable theory the resulting bound-state S-matrix corresponds to the fusion of S-matrices for the scattering of elementary constituents [30]. This will allow us to obtain the elementary semiclassical and 1-loop phases from the above dyonic giant magnon bound-state picture. To evaluate the above expressions we first need to find the time-delay for dyonic giant magnon scattering and the phase shifts for plane wave scattering off dyonic giant magnons. Both can be obtained by constructing multi-soliton scattering solutions using the dressing method [90, 91].

In section 7.1 we extend this method to the case of mixed flux and as a consitency check we reproduce the mixed-flux dyonic giant magnon by dressing the BMN solution. Applying the dressing method again we find the scattering solution for two dyonic giant magnons from which we extract the time-delay. In general such classical solutions on $\mathbb{R} \times S^3$ are related to soliton solutions of the Complex sine-Gordon (CsG) model via the Pohlmeyer reduction [76] (for $q \neq 0$ also see [74]). Scattering solutions in the CsG model can then be obtained using Bäcklund transformations and for q = 0 the soliton for the scattering of two dyonic giant magnons is known explicitly. In section 7.4 we generalise this soliton to $q \neq 0$ and we find the time-delay from the CsG picture as an independent check for our result.

In section 7.2 we determine the bound-state S-matrix using the WKB formula (7.0.6). The bound-

state S-matrix includes contributions from the fusion of the elementary dressing (AFS) phase as well as the BDS factors and its equivalent for mixed excitation scattering. Eliminating the latter and using arguments relating to the functional form of the resulting AFS contribution we arrive at the dressing phase for elementary excitations.

In section 7.3 we turn our attention to the 1-loop corrections (7.0.9) and (7.0.10). Using the generalised dressing method we determine the bosonic phase shifts experienced by small fluctuations when scattering off dyonic giant magnons on $AdS_3 \times S^3$. We do not consider directions along T⁴ as we are interested in the massive sector only.

We then establish the fermionic phase shifts as well as the explicit form of the dyonic giant magnon as a finite-gap solution by considering the connection between bosonic phase shifts and finite-gap equations. This also serves as an additional check for the finite-gap picture obtained in [95]. Finally in section 7.3.3 we use this scattering data to evaluate the 1-loop correction to the bound-state S-matrix. Following similar arguments as in the leading order case we find the 1-loop elementary dressing phases, which agree with [95, 96, 97] in the semiclassical limit ($h \rightarrow \infty, hp = fixed$). We also find that the 1-loop energy shift (7.0.9) vanishes.

7.1 Dressing method for solutions on $\mathbb{R} \times S^3$ with NS-NS flux

In this section we extend the dressing method [109, 216] for solutions on $\mathbb{R} \times S^3$ to the $q \neq 0$ case and applying this method to the BMN solution we reproduce the mixed-flux dyonic giant magnon. In our analysis we closely follow [90, 91]. We then apply the dressing method to the dyonic giant magnon to find the explicit scattering solution for two dyonic giant magnons from which we read off the time-delay.

In conformal gauge the string action is equivalent to a principal chiral model with a Wess-Zumino term with a coefficient $q \in (0, 1)$

$$S = -\frac{h}{2} \left[\int d^2 \sigma \, \frac{1}{2} \operatorname{tr}(\mathfrak{J}_+ \mathfrak{J}_-) - q \int d^3 \sigma \, \frac{1}{3} \varepsilon^{abc} \operatorname{tr}(\mathfrak{J}_a \mathfrak{J}_b \mathfrak{J}_c) \right] \,, \qquad \mathfrak{J}_a = g^{-1} \partial_a g \,, \tag{7.1.1}$$

where $h = \frac{\sqrt{\lambda}}{2\pi}$ is the string tension, $g \in SU(2)$ and $\sigma^{\pm} = \frac{1}{2}(\tau \pm \sigma)$, $\partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}$. The equations of motion

$$\partial_{\pm}\tilde{\mathfrak{J}}_{\mp} \pm \frac{1}{2}[\tilde{\mathfrak{J}}_{+},\tilde{\mathfrak{J}}_{-}] = 0, \qquad \tilde{\mathfrak{J}}_{\pm} = (1\pm q)\mathfrak{J}_{\pm}$$
(7.1.2)

can be rephrased as the compatibility condition of the Lax pair equations

$$\partial_{\pm}\Psi(\sigma^{+},\sigma^{-};\lambda) = \Psi(\sigma^{+},\sigma^{-};\lambda)A_{\pm}^{(\lambda)}$$
(7.1.3)

where λ is the spectral parameter and

$$A_{\pm}^{(\lambda)} = \frac{1}{1 \pm \lambda} \tilde{\mathfrak{J}}_{\pm} = \frac{1 \pm q}{1 \pm \lambda} \mathfrak{J}_{\pm}.$$
(7.1.4)

We shall also impose unitarity

$$\Psi^{\dagger}(\bar{\lambda})\Psi(\lambda) = 1. \tag{7.1.5}$$

Given a solution $\Psi(\lambda)$ to the linear system (7.1.3) we can obtain a solution to the equations of motion (7.1.2) by taking $g = \Psi(q)$. Conversely given a solution g to the equations of motion we can solve the linear system for $\Psi(\lambda)$ such that $\Psi(q) = g$.

Starting with a given solution Ψ we can perform a λ -dependent gauge transformation to obtain a new solution

$$\Psi \to \Psi' = \Psi \chi \tag{7.1.6}$$

$$A_{\pm} \to A'_{\pm} = \chi^{-1} A_{\pm} \chi + \chi^{-1} \partial_{\pm} \chi.$$
 (7.1.7)

Here we need to choose $\chi(\lambda)$ such that the transformed SU(2) current

$$\mathfrak{J}_{\pm}' = \chi^{-1}\mathfrak{J}_{\pm}\chi + \frac{1\pm\lambda}{1\pm q}\chi^{-1}\partial_{\pm}\chi \tag{7.1.8}$$

is independent of λ . This can be done by requiring that χ is a meromorphic function with $\chi \to 1$ as $\lambda \to \infty$. The simplest choice is then a single pole at $\lambda = \lambda_1$. The unitarity condition (7.1.5) then requires

$$\chi^{\dagger}(\bar{\lambda})\chi(\lambda) = 1 \tag{7.1.9}$$

which fixes χ to be of the form

$$\chi = 1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P \tag{7.1.10}$$

where P is an idempotent hermitian operator, i.e. $P = P^2 = P^{\dagger}$. Finally it remains to choose P such that \mathfrak{J}'_{\pm} has no poles at $\lambda = \lambda_1$. This is achieved by choosing P such that its image is spanned by $\{\Psi^{-1}(\lambda_1)e_1, \Psi^{-1}(\lambda_1)e_2, ...\}$ where e_i are arbitrary constant vectors. For our purposes P shall have rank 1 and is explicitly given by

$$P = \frac{\Psi^{-1}(\lambda_1)ee^{\dagger}\Psi(\bar{\lambda}_1)}{e^{\dagger}\Psi(\bar{\lambda}_1)\Psi^{-1}(\lambda_1)e}.$$
(7.1.11)

The resulting dressing factor χ has the determinant

$$\det \chi = \frac{\lambda - \bar{\lambda}_1}{\lambda - \lambda_1} \tag{7.1.12}$$

and the dressed solution is given by

$$g_{\text{dressed}} = \sqrt{\frac{\lambda_1 - q}{\bar{\lambda}_1 - q}} \Psi(q) \chi(q)$$
(7.1.13)

where the normalisation factor ensures that $g_{\text{dressed}} \in \text{SU}(2)$.

7.1.1 Scattering solution and time-delay

Using this method let us now derive the $q \neq 0$ dyonic giant magnon solution by dressing up the BMN solution

$$Z_1 = e^{it}, \qquad Z_2 = 0. \tag{7.1.14}$$

Parametrising SU(2) in terms of the embedding coordinates as

$$g = \begin{pmatrix} Z_1 & Z_2 \\ -Z_2^* & Z_1^* \end{pmatrix}, \qquad |Z_1|^2 + |Z_2|^2 = 1$$
(7.1.15)

we have for the BMN solution

$$g = \begin{pmatrix} e^{i(\sigma^+ + \sigma^-)} & 0\\ 0 & e^{-i(\sigma^+ + \sigma^-)} \end{pmatrix}, \quad \Psi(\lambda) = \begin{pmatrix} e^{iZ(\lambda)} & 0\\ 0 & e^{-iZ(\lambda)} \end{pmatrix}, \quad (7.1.16)$$

$$Z(\lambda) = \frac{1+q}{1+\lambda}\sigma^{+} + \frac{1-q}{1-\lambda}\sigma^{-}.$$
 (7.1.17)

Since the projection operator P does not depend on the scale of e we can parametrise e as

$$e = (c, 1/c), \quad c \in \mathbb{C}^*.$$
 (7.1.18)

Furthermore c only enters P in

$$\Psi^{-1}(\lambda_1)e = \begin{pmatrix} c e^{iZ(\lambda_1)} \\ c^{-1}e^{-iZ(\lambda_1)} \end{pmatrix}$$
(7.1.19)

allowing us to absorb c by shifting $Z(\lambda_1) \to Z(\lambda_1) + i \ln c$ which corresponds to a shift in (σ^+, σ^-) . Therefore we can set c = 1 without loss of generality. The projector P is then given by

$$P = \frac{1}{1 + e^{2i(Z(\lambda_1) - Z(\bar{\lambda}_1))}} \begin{pmatrix} 1 & e^{-2iZ(\bar{\lambda}_1)} \\ e^{2iZ(\lambda_1)} & e^{2i(Z(\lambda_1) - Z(\bar{\lambda}_1))} \end{pmatrix}$$
(7.1.20)

and the dressed solution takes the form

$$Z_{1} = \frac{e^{it}}{|\tilde{\lambda}_{1}|} \frac{\tilde{\lambda}_{1} e^{2iZ(\lambda_{1})} + \tilde{\bar{\lambda}}_{1} e^{2iZ(\bar{\lambda}_{1})}}{e^{2iZ(\lambda_{1})} + e^{2iZ(\bar{\lambda}_{1})}}, \qquad Z_{2} = \frac{e^{-it}}{|\tilde{\lambda}_{1}|} \frac{i(\tilde{\lambda}_{1} - \tilde{\bar{\lambda}}_{1})}{e^{2iZ(\lambda_{1})} + e^{2iZ(\bar{\lambda}_{1})}}.$$
(7.1.21)

where $\tilde{\lambda}_1 = \lambda_1 - q$, $\tilde{\bar{\lambda}}_1 = \bar{\lambda}_1 - q$. This is the $q \neq 0$ dyonic giant magnon solution which we can see by parametrising λ_1 as

$$\lambda_1 = q + r e^{ip/2} \tag{7.1.22}$$

and introducing the world-sheet coordinates

$$u = i(Z(\lambda_1) - Z(\bar{\lambda}_1)), \qquad v = t + qx - (Z(\lambda_1) + Z(\bar{\lambda}_1)).$$
(7.1.23)

Equivalently we can parametrise these coordinates in terms of a rapidity θ and a parameter ρ as

$$u = \cos \rho(\mathcal{X} + q\mathcal{T}), \qquad v = \sin \rho(\mathcal{T} + q\mathcal{X}),$$
(7.1.24)

$$\mathcal{X} = x \cosh \theta - t \sinh \theta, \qquad \mathcal{T} = t \cosh \theta - x \sinh \theta.$$
 (7.1.25)

These dyonic giant magnon soliton parameters θ and ρ are related to q, r and p as

$$\tanh \theta = 2 \frac{q + r \cos \frac{p}{2}}{1 + r^2 + q^2 + 2qr \cos \frac{p}{2}}, \qquad \cot \rho = \frac{2r \sin \frac{p}{2}}{r^2 - 1 + q^2 + 2qr \cos \frac{p}{2}}$$
(7.1.26)

$$\tan\frac{p}{2} = \frac{\cos\rho}{\sinh\theta - q\cosh\theta + q\sin\rho}$$
(7.1.27)

and we can solve these relations explicitly for r giving

$$r^{2} = \frac{\cosh\theta + \sin\rho - 2q\sinh\theta}{\cosh\theta - \sin\rho} + q^{2}.$$
(7.1.28)

In terms of the energy E, angular momentum J and the world-sheet momentum p this becomes

$$r = \frac{E \pm M_{\mp}}{2h \sin \frac{p}{2}}, \qquad M_{\pm} = J \pm qhp$$
 (7.1.29)

where the sign distinguishes left and right movers. The solution (7.1.21) then takes the standard $q \neq 0$ dyonic giant magnon form

$$Z_1 = e^{it} (\cos\frac{p}{2} + i\sin\frac{p}{2}\tanh u), \qquad Z_2 = \sin\frac{p}{2}e^{i(v-qx)}\operatorname{sech} u.$$
(7.1.30)

Applying the dressing method a second time gives the scattering solution of two magnons

$$Z_{1} = \frac{e^{it}}{2|\tilde{\lambda}_{1}||\tilde{\lambda}_{2}|} \frac{R + |\tilde{\lambda}_{1}|^{2}\lambda_{1\bar{1}}\lambda_{2\bar{2}}e^{-i(v_{1}-v_{2})} + |\tilde{\lambda}_{2}|^{2}\lambda_{1\bar{1}}\lambda_{2\bar{2}}e^{i(v_{1}-v_{2})}}{\lambda_{12}\lambda_{\bar{1}\bar{2}}\cosh(u_{1}+u_{2}) + \lambda_{1\bar{2}}\lambda_{\bar{1}2}\cosh(u_{1}-u_{2}) + \lambda_{1\bar{1}}\lambda_{2\bar{2}}\cos(v_{1}-v_{2})}$$
(7.1.31)

$$Z_{2} = \frac{e^{-iqx}}{2|\tilde{\lambda}_{1}||\tilde{\lambda}_{2}|} \frac{\lambda_{\bar{1}1}e^{iv_{1}}(\lambda_{\bar{1}\bar{2}}\lambda_{1\bar{2}}\tilde{\lambda}_{2}e^{u_{2}} + \lambda_{12}\lambda_{\bar{1}2}\tilde{\lambda}_{2}e^{-u_{2}}) + \lambda_{\bar{2}2}e^{iv_{2}}(\lambda_{\bar{2}\bar{1}}\lambda_{2\bar{1}}\tilde{\lambda}_{1}e^{u_{1}} + \lambda_{21}\lambda_{\bar{2}1}\tilde{\lambda}_{1}e^{-u_{1}})}{\lambda_{\bar{1}\bar{2}}\lambda_{12}\cosh(u_{1} + u_{2}) + \lambda_{\bar{1}2}\lambda_{1\bar{2}}\cosh(u_{1} - u_{2}) + \lambda_{\bar{1}1}\lambda_{\bar{2}2}\cos(v_{1} - v_{2})}$$
(7.1.32)

$$R = \lambda_{12}\lambda_{\bar{1}\bar{2}}(\tilde{\lambda}_{1}\tilde{\lambda}_{2}e^{u_{1}+u_{2}} + \bar{\tilde{\lambda}}_{1}\bar{\tilde{\lambda}}_{2}e^{-u_{1}-u_{2}}) + \lambda_{\bar{1}2}\lambda_{1\bar{2}}(\tilde{\lambda}_{1}\bar{\tilde{\lambda}}_{2}e^{u_{1}-u_{2}} + \bar{\tilde{\lambda}}_{1}\tilde{\lambda}_{2}e^{-u_{1}+u_{2}})$$
(7.1.33)

where

$$\lambda_{ij} = \lambda_i - \lambda_j, \qquad \lambda_{\bar{i}} = \bar{\lambda}_i. \tag{7.1.34}$$

In terms of the familiar coordinates (7.1.24) this solution takes the form

$$Z_{1} = e^{it} \frac{\left[\cos(v_{1} - v_{2})\cosh\beta + i\sin(v_{1} - v_{2})\sinh\beta\right]\sin\frac{p_{1}}{2}\sin\frac{p_{2}}{2}\operatorname{sech} u_{1}\operatorname{sech} u_{2} + R + iI}{\sin\frac{p_{1}}{2}\sin\frac{p_{2}}{2}\left[\cos(v_{1} - v_{2})\operatorname{sech} u_{1}\operatorname{sech} u_{2} + \tanh u_{1}\tanh u_{2}\right] + \cos\frac{p_{1}}{2}\cos\frac{p_{2}}{2} - \cosh\beta}$$

$$Z_{2} = e^{-iqx} \frac{V_{1} + V_{2}}{\sin\frac{p_{1}}{2}\sin\frac{p_{2}}{2}\left[\cos(v_{1} - v_{2})\operatorname{sech} u_{1}\operatorname{sech} u_{2} + \tanh u_{1}\tanh u_{2}\right] + \cos\frac{p_{1}}{2}\cos\frac{p_{2}}{2} - \cosh\beta}$$

$$(7.1.35)$$

$$R = \cos^2 \frac{p_1}{2} + \cos^2 \frac{p_2}{2} - \cosh \beta \cos \frac{p_1}{2} \cos \frac{p_2}{2} - 1$$
(7.1.37)

$$I = \tanh u_1 \sin \frac{p_1}{2} \left(\cos \frac{p_1}{2} - \cosh \beta \cos \frac{p_2}{2} \right) + \tanh u_2 \sin \frac{p_2}{2} \left(\cos \frac{p_2}{2} - \cosh \beta \cos \frac{p_1}{2} \right)$$
(7.1.38)

$$V_{1} = e^{iv_{1}} \sin \frac{p_{1}}{2} \operatorname{sech} u_{1} \left[i \left(\cosh \beta \cos \frac{p_{2}}{2} - \cos \frac{p_{1}}{2} \right) + \sinh \beta \sin \frac{p_{2}}{2} \tanh u_{2} \right]$$
(7.1.39)

$$V_{2} = e^{iv_{2}} \sin \frac{p_{2}}{2} \operatorname{sech} u_{2} \left[i \left(\cosh \beta \cos \frac{p_{1}}{2} - \cos \frac{p_{2}}{2} \right) - \sinh \beta \sin \frac{p_{1}}{2} \tanh u_{1} \right]$$
(7.1.40)

$$\beta = \ln \frac{r_2}{r_1}.$$
(7.1.41)

We can extract the time delay by comparing the scattering solution to a freely propagating giant magnon as $t \to \pm \infty$. The velocity of a single free propagating giant magnon is given by

$$v_s = \frac{1}{h} \frac{\mathrm{d}E}{\mathrm{d}p} = \frac{v - q}{1 - qv} = \frac{\sinh\theta - q\cosh\theta}{\cosh\theta - q\sinh\theta}.$$
(7.1.42)

Taking $x = v_s t$ and $t \to +\infty$ the freely propagating soliton phase shifted by δt_+ has the asymptotic form

$$Z_{1}^{(1)} = e^{i(t-\delta t_{+})} \Big(\cos \frac{p_{1}}{2} + i \sin \frac{p_{1}}{2} \tanh \Big[\cos \rho_{1} (\sinh \theta_{1} - q \cosh \theta_{1}) \delta t_{+} \Big] \Big), \quad Z_{1}^{(2)} = e^{i(t-\delta t_{+}+p_{2}/2)}$$
(7.1.43)

$$Z_2^{(1)} = e^{i(v_1 - qv_{s1}t)} \sin\frac{p_1}{2} \operatorname{sech} \left[\cos\rho_1(\sinh\theta_1 - q\cosh\theta_1)\delta t_+ \right], \qquad Z_2^{(2)} = 0 \qquad (7.1.44)$$

whereas the scattering solution asymptotes to

$$Z_1^{(s)} = e^{it} \frac{e^{i\frac{p_2}{2}} (\cos\frac{p_2}{2} - \cos\frac{p_1}{2}\cosh\beta) - \sin^2\frac{p_1}{2}}{\cos\frac{p_1}{2}\cos\frac{p_2}{2} - \cosh\beta} \quad \Rightarrow |Z_1^{(s)}|^2 = \cos^2\frac{p_1}{2} + \frac{\sin^4\frac{p_1}{2}\cos^2\frac{p_2}{2}}{(\cosh\beta - \cos\frac{p_1}{2}\cos\frac{p_2}{2})^2}.$$
(7.1.45)

In the COM frame $(\sum \cos \rho_i(\sinh \theta_i - q \cosh \theta_i) = 0)$ both solitons experience the same time delay giving in total

$$\Delta T = 2\delta t_{+} = \frac{1}{(\sinh\theta_{1} - q\cosh\theta_{1})\cos\rho_{1}} \Big| \ln \frac{\cosh(\theta_{1} - \theta_{2}) - \cos(\rho_{1} - \rho_{2})}{\cosh(\theta_{1} - \theta_{2}) + \cos(\rho_{1} + \rho_{2})} \Big|.$$
(7.1.46)

For q = 0 this reduces to the standard expression for the time-delay due to the collison of two solitons in the CsG model [217].

7.2 Semiclassical bound-state S-matrix

In order to evaluate the semiclassical bound-state S-matrix

$$\Theta(p_1, p_2) = \int_0^{p_1} \mathrm{d}p'_1 \frac{\mathrm{d}E}{\mathrm{d}p}(p'_1) \,\Delta T(p'_1, p_2) \tag{7.2.1}$$

let us rewrite the above time-delay expression (7.1.46) in terms of the momentum p_i , angular momentum J_i and energy E_i of the associated individual dyonic giant magnons. In terms of the soliton parameters

 θ_i and ρ_i these charges are given by (see chapter 5)

$$E_l = \sqrt{M^2 + 4h^2(1 - q^2)\sin^2\frac{p}{2}}, \quad M_l \equiv J + qlhp, \quad M_l = -2lh\sin\frac{p}{2}(\tan\rho\sin\frac{p}{2} - q\cos\frac{p}{2}) \quad (7.2.2)$$

$$\sinh \theta = \frac{m + q\sqrt{m^2 + 1}}{\sqrt{1 - q^2}}, \quad \cosh \theta = \frac{qm + \sqrt{m^2 + 1}}{\sqrt{1 - q^2}}, \quad m = \frac{1}{\sqrt{1 - q^2}} \Big(\frac{\cos \rho}{\tan \frac{p}{2}} - q\sin \rho\Big) \quad (7.2.3)$$

$$\cosh(\theta_1 - \theta_2) = \sqrt{m_1^2 + 1}\sqrt{m_2^2 + 1} - m_1m_2$$
 (7.2.4)

where $l = \pm 1$ represents left and right movers which are related by sending $J \rightarrow -J$. The time delay becomes

$$\Delta T_{l_1 l_2} = C_{l_1} L_{l_1 l_2} \qquad C_{l_1} = \frac{4h^2 \sin^4 \frac{p_1}{2} + (M_1 - qhl_1 \sin p_1)^2}{2 \sin^2 \frac{p_1}{2} (2h^2 \hat{q}^2 \cos \frac{p_1}{2} \sin \frac{p_1}{2} + qhl_1 M_1)}$$
(7.2.5)

$$L_{l_1 l_2} = \left| \ln \frac{E_1 E_2 - h^2 \hat{q}^2 \sin p_1 \sin p_2 - 4h^2 \hat{q}^2 \sin^2 \frac{p_1}{2} \sin^2 \frac{p_2}{2} - l_1 l_2 M_1 M_2}{E_1 E_2 - h^2 \hat{q}^2 \sin p_1 \sin p_2 + 4h^2 \hat{q}^2 \sin^2 \frac{p_1}{2} \sin^2 \frac{p_2}{2} - l_1 l_2 M_1 M_2} \right|.$$
(7.2.6)

We can further simplify (7.2.1) by transforming the integral over energy into a contour integral over the dyonic spectral parameters

$$X^{\pm}(p) = \frac{e^{\pm ip/2}}{2h\hat{q}\sin\frac{p}{2}} \left[E(p) + M(p) \right], \qquad \hat{q} = \sqrt{1 - q^2}$$
(7.2.7)

which for J = 1 reduce to the usual spectral parameters x^{\pm} corresponding to fundamental magnon excitations and which satisfy

$$X^{+} + \frac{1}{X^{+}} - \left(X^{-} + \frac{1}{X^{-}}\right) = \frac{2i}{h\hat{q}}M, \quad \frac{X^{+}}{X^{-}} = e^{ip},$$
(7.2.8)

$$X^{+} - \frac{1}{X^{+}} - \left(X^{-} - \frac{1}{X^{-}}\right) = \frac{2i}{h\hat{q}}E.$$
(7.2.9)

In terms of these variables the time delay can be written as

$$C_{l_1} = i \frac{(1 + X^+ X^-)^2 - (\hat{q}(X^+ + X^-) - l_1 q(1 - X^+ X^-))^2}{\hat{q}(X^+ - X^-)(\hat{q}(X^+ + X^-) - l_1 q(1 - X^+ X^-))},$$
(7.2.10)

$$L_{l_1 l_2} = -\begin{cases} \ln \frac{X^- - Y^-}{X^+ - Y^+}, & l_1 l_2 = 1\\ \ln \frac{\left(1 - \frac{1}{X^+ Y^-}\right) \left(1 - \frac{1}{X^- Y^+}\right)}{\left(1 - \frac{1}{X^- Y^-}\right)}, & l_1 l_2 = -1. \end{cases}$$
(7.2.11)

Noting the identity

$$\frac{\mathrm{d}E_1}{\mathrm{d}p_1}C_{l_1} = \frac{\mathrm{d}X^{\pm}}{\mathrm{d}p_1}\mathrm{h}\hat{q}\frac{1-(X^{\pm})^2+2l_1\frac{q}{\hat{q}}X^{\pm}}{(X^{\pm})^2}$$
(7.2.12)

we can split the integral in (7.2.1) into two separate integrals over X^+ and X^- respectively giving

$$\frac{1}{h\hat{q}}\Theta(X,Y) = I(X^+,Y^+) - I(X^+,Y^-) + I(X^-,Y^-) - I(X^-,Y^+)$$
(7.2.13)

$$I(X^{\pm}, Y) = \int_{X^{\pm}(0)}^{X^{\pm}(p_1)} \mathrm{d}z \, \frac{1 - z^2 + 2\frac{ql_1}{\hat{q}}z}{z^2} \begin{cases} -\ln(z - Y), & l_1l_2 = 1\\ \ln\left(1 - \frac{1}{zY}\right), & l_1l_2 = -1 \end{cases}$$
(7.2.14)

where for $l_1 l_2 = 1$ the integral evaluates to

$$I(X,Y) = -X + \frac{1}{Y}\ln X - \left[Y + \frac{1}{Y} - \left(X + \frac{1}{X}\right) + 2l_1\frac{q}{\hat{q}}\ln\frac{X}{Y}\right]\ln(X-Y)$$
(7.2.15)

$$-2l_1\frac{q}{\hat{q}}\mathrm{Li}_2\left(1-\frac{X}{Y}\right)-I\Big|_{p_1=0}$$

$$(7.2.16)$$

and for $l_1 l_2 = -1$ we have

$$\bar{I}(X,Y) = \frac{1}{X} - \frac{\ln Y}{Y} + \frac{\ln X}{Y} + \left[Y + \frac{1}{Y} - (X + \frac{1}{X})\right] \ln\left(1 - \frac{1}{XY}\right)$$
(7.2.17)

$$+ 2l_1 \frac{q}{\hat{q}} \operatorname{Li}_2\left(\frac{1}{XY}\right) - I\Big|_{p_1=0}.$$
(7.2.18)

The last term $I|_{p_1=0}$ in these integrals corresponds to an infinite contribution from the integral at $p_1 = 0$. These infinite contributions arise from the way we chose to split the integral and they cancel in the sum $I(X^+, Y) - I(X^-, Y)$.

We can therefore write the bound-state scattering matrix for $l_1 l_2 = 1$ as

$$\Theta(X,Y) = h\hat{q} \Big[K(X^+,Y^+) + K(X^-,Y^-) - K(X^+,Y^-) - K(X^-,Y^+) \Big] + p_1(E_2 - J_2)$$
(7.2.19)

$$K(X,Y) = -\Big[Y + \frac{1}{Y} - \Big(X + \frac{1}{X} \Big) + 2l_1 \frac{q}{\hat{q}} \ln \frac{X}{Y} \Big] \ln(X-Y) - 2l_1 \frac{q}{\hat{q}} \text{Li}_2 \Big(1 - \frac{X}{Y} \Big) + l_1 \frac{q}{\hat{q}} \ln X \ln Y$$
(7.2.20)

and for $l_1 l_2 = -1$ as

$$\bar{\Theta}(X,Y) = h\hat{q}\Big[\bar{K}(X^+,Y^+) + \bar{K}(X^-,Y^-) - \bar{K}(X^+,Y^-) - \bar{K}(X^-,Y^+)\Big] + p_1(E_2 - J_2)$$
(7.2.21)

$$\bar{K}(X,Y) = \left[Y + \frac{1}{Y} - (X + \frac{1}{X})\right] \ln\left(1 - \frac{1}{XY}\right) + 2l_1 \frac{q}{\hat{q}} \operatorname{Li}_2\left(\frac{1}{XY}\right) - l_1 \frac{q}{\hat{q}} \ln X \ln Y.$$
(7.2.22)

Here the terms proportional to p_1 are gauge terms originating from the choice of conformal gauge (for details see [22]). The remaining part of the S-matrix contains another such contribution from the terms of $\ln X \ln Y$. However, the gauge term is uniquely fixed by imposing that the S-matrix without the gauge term is antisymmetric under the exchange $X \leftrightarrow Y$ and $l_1 \leftrightarrow l_2$ as required by unitarity.

7.2.1 Dressing phases for elementary excitations

In order to obtain the elementary dressing phases we need to eliminate the bound-state contributions in the S-matrix coming from the BDS factor for same-type excitations and the equivalent factor for mixed-type excitations [218]

$$s_{BDS}(x,y) = \frac{x^+ - y^-}{x^- - y^+} \frac{1 - \frac{1}{x^+ y^-}}{1 - \frac{1}{x^- y^+}}, \qquad s_{mix}(x,y) = \frac{1 - \frac{1}{x^- y^+}}{1 - \frac{1}{x^- y^-}} \frac{1 - \frac{1}{x^+ y^-}}{1 - \frac{1}{x^+ y^+}}.$$
 (7.2.23)

Integrability allows us to fuse together these elementary factors into their bound-state contributions

$$S = \prod_{j_1=1}^{J_1} \prod_{j_2=1}^{J_2} s(x_{j_1}^+, x_{j_1}^-; y_{j_2}^+, y_{j_2}^-)$$
(7.2.24)

where x_{j_1}, y_{j_2} are the spectral parameters of the constituents satisfying the shortening conditions

$$x_{j_1}^+ + \frac{1}{x_{j_1}^+} - 2l_X \frac{q}{\hat{q}} \ln x_{j_1}^+ - \left(x_{j_1}^- + \frac{1}{x_{j_1}^-} - 2l_X \frac{q}{\hat{q}} \ln x_{j_1}^-\right) = \frac{2i}{h\hat{q}}, \quad l_X = \pm 1, \quad j_1 = 1, .., J_1 - 1 \quad (7.2.25)$$

$$y_{j_{2}}^{+} + \frac{1}{y_{j_{2}}^{+}} - 2l_{Y}\frac{q}{\hat{q}}\ln y_{j_{2}}^{+} - \left(y_{j_{2}}^{-} + \frac{1}{y_{j_{2}}^{-}} - 2l_{Y}\frac{q}{\hat{q}}\ln y_{j_{2}}^{-}\right) = \frac{2i}{h\hat{q}}, \quad l_{Y} = \pm 1, \quad j_{2} = 1, .., J_{2} - 1 \quad (7.2.26)$$

and the pole conditions for the formation of the \mathcal{J}_1 and \mathcal{J}_2 bound states

$$x_{j_1}^- = x_{j_1+1}^+, \qquad j_1 = 1, ..., J_1 - 1$$
 (7.2.27)

$$y_{j_2}^- = y_{j_2+1}^+, \qquad j_2 = 1, .., J_2 - 1.$$
 (7.2.28)

The bound-state spectral parameters are then identified as

$$X^+ = x_1^+, \qquad X^- = x_{J_1}^-, \qquad Y^+ = y_1^+, \qquad Y^- = y_{J_2}^-.$$
 (7.2.29)

Let us first compute the semiclassical bound-state contribution for the BDS factor. Splitting off the $j_1 = 1, j_2 = 1, j_1 = J_1$ and $j_2 = J_2$ terms in the product (7.2.24) we get

$$S_{BDS} = \exp(i\Theta_{BDS})$$

$$i\Theta_{BDS} = \ln\frac{X^{+} - Y^{-}}{X^{-} - Y^{+}} \frac{1 - \frac{1}{X^{+}Y^{-}}}{1 - \frac{1}{X^{-}Y^{+}}} + \sum_{j_{2}=1}^{J_{2}-1} \ln\frac{X^{+} - y_{j_{2}}^{-}}{X^{-} - y_{j_{2}}^{+}} \frac{1 - \frac{1}{X^{+}y_{j_{2}}^{-}}}{1 - \frac{1}{X^{-}y_{j_{2}}^{+}}} + \sum_{j_{1}=2}^{J_{1}} \ln\frac{x_{j_{1}}^{+} - Y^{-}}{x_{j_{1}}^{-} - Y^{+}} \frac{1 - \frac{1}{x_{j_{1}}^{+}Y^{-}}}{1 - \frac{1}{x_{j_{1}}^{-}Y^{+}}}.$$

$$(7.2.30)$$

$$(7.2.31)$$

Taking the semiclassical dyonic giant magnon limit $h \to \infty, J \sim h$ the sums transform into integrals giving

$$i\Theta_{BDS} = h \int_0^{J_2/h} dj \ln \frac{X^+ - y^-(j)}{X^- - y^+(j)} \frac{1 - \frac{1}{X^+ y^-(j)}}{1 - \frac{1}{X^- y^+(j)}} + h \int_0^{J_1/h} dj \ln \frac{x^+(j) - Y^-}{x^-(j) - Y^+} \frac{1 - \frac{1}{x^+(j)Y^-}}{1 - \frac{1}{x^-(j)Y^+}} + \mathcal{O}(1).$$

In the semiclassical limit the shortening condition of the constituents and the pole condition combined give

$$x^{-}(j) + \frac{1}{x^{-}(j)} - 2l_1 \frac{q}{\hat{q}} \ln x^{-}(j) = X^{+} + \frac{1}{X^{+}} - 2l_1 \frac{q}{\hat{q}} \ln X^{+} - \frac{2i}{\hat{q}} j$$
(7.2.32)

$$y^{-}(j) + \frac{1}{y^{-}(j)} - 2l_2\frac{q}{\hat{q}}\ln y^{-}(j) = Y^{+} + \frac{1}{Y^{+}} - 2l_2\frac{q}{\hat{q}}\ln Y^{+} - \frac{2i}{\hat{q}}j$$
(7.2.33)

where $j \sim h$ has been rescaled by h. This allows us to rewrite the BDS contribution in terms of contour integrals over the spectral parameters as

$$\Theta_{BDS} = \frac{\mathbf{h}\hat{q}}{2} \int_{Y^+}^{Y^-} \mathrm{d}z \left(1 - \frac{1}{z^2} - 2l_2 \frac{q}{\hat{q}} \frac{1}{z} \right) \ln \frac{X^+ - z}{X^- - z} \frac{1 - \frac{1}{X^+ z}}{1 - \frac{1}{X^- z}}$$
(7.2.34)

$$+ \frac{\mathrm{h}\hat{q}}{2} \int_{X^{+}}^{X^{-}} \mathrm{d}z \left(1 - \frac{1}{z^{2}} - 2l_{1}\frac{q}{\hat{q}}\frac{1}{z}\right) \ln \frac{z - Y^{-}}{z - Y^{+}} \frac{1 - \frac{1}{zY^{-}}}{1 - \frac{1}{zY^{+}}} + \mathcal{O}(1).$$
(7.2.35)

Performing these integrals we obtain the bound-state BDS contribution

$$\Theta_{BDS} = h\hat{q} [\hat{k}(X^{-}, Y^{-}) - \hat{k}(X^{+}, Y^{-}) - \hat{k}(X^{-}, Y^{+}) + \hat{k}(X^{+}, Y^{+})] + \mathcal{O}(1)$$
(7.2.36)

$$\hat{k}(X,Y) = \left[X + \frac{1}{X} - \left(Y + \frac{1}{Y}\right)\right] \ln\left[(X - Y)\left(1 - \frac{1}{XY}\right)\right]$$
(7.2.37)

$$-2l_1\frac{q}{\hat{q}}\ln\frac{X}{Y}\ln(X-Y) - l_1\frac{q}{\hat{q}}\left(2\operatorname{Li}_2\left(1-\frac{X}{Y}\right) - \ln X\ln Y\right).$$
(7.2.38)

For q = 0 this agrees with the BDS contribution in the case of $AdS_5 \times S^5$ given in [24] (see equation (34)). Performing this fusion procedure on the factor (7.2.23) for mixed-type magnon scattering one finds that the integrals cancel at the linear order in h, i.e.

$$\Theta_{\rm mix} \sim \mathcal{O}(1). \tag{7.2.39}$$

Subtracting these contributions from the bound-state S-matrix (7.2.22) we are left with

$$\Theta_{AFS}(X,Y) = h\hat{q} \Big[\chi(X^+,Y^+) - \chi(X^+,Y^-) - \chi(X^-,Y^+) + \chi(X^-,Y^-) \Big]$$
(7.2.40)

$$\bar{\Theta}(X,Y) = h\hat{q} \left[\bar{\chi}(X^+,Y^+) - \bar{\chi}(X^+,Y^-) - \bar{\chi}(X^-,Y^+) + \bar{\chi}(X^-,Y^-) \right]$$
(7.2.41)

$$\chi(x,y) = \left(y + \frac{1}{y} - x - \frac{1}{x}\right) \ln\left(1 - \frac{1}{xy}\right)$$
(7.2.42)

$$\bar{\chi}(x,y) = \left(y + \frac{1}{y} - x - \frac{1}{x}\right) \ln\left(1 - \frac{1}{xy}\right) + l_1 \frac{q}{\sqrt{1 - q^2}} \left(2\mathrm{Li}_2\left(\frac{1}{xy}\right) - \ln x \ln y\right)$$
(7.2.43)

where Θ_{AFS} is the contribution for same-type magnons $(X, Y \text{ with } l_X = l_Y)$ and $\overline{\Theta}$ is the contribution for mixed-type magnons $(X, Y \text{ with } l_X = -l_Y)$.

These bound-state AFS and mixed scattering contributions are related to the tree-level elementary dressing phases by the fusion procedure. However, since both the elementary dressing phases and the associated bound-state contributions are of the same leading order in h, the fusion procedure results in the same functional form for the bound-state result as for the elementary phases (see figure 7.1)¹. Therefore we can deduce that the elementary dressing phase must have the functional form of the associated bound-state contribution (7.2.40)-(7.2.43) in agreement with the prediction in [95].

7.3 One-loop corrections

In this part of the paper we will determine the 1-loop corrections to the dispersion relation and the soliton S-matrix (7.0.9)-(7.0.10). The involved phase shifts vanish for fluctuations on the AdS₃ and T⁴ parts since we are only considering the dyonic giant magnon which moves in the $\mathbb{R} \times S^3$ subspace of AdS₃ × S³ × T⁴. In order to find the phase shifts for fluctuations on S³ we will use the dressing method

$$\theta(x,y) = f(x^+, y^+) - f(x^+, y^-) - f(x^-, y^+) + f(x^-, y^-),$$
(7.2.44)

which fused into the bound-state contribution Θ has the same functional form

$$\Theta(X,Y) = f(X^+,Y^+) - f(X^+,Y^-) - f(X^-,Y^+) + f(X^-,Y^-).$$
(7.2.45)

¹This is because any sum of elementary factors contributes an order of $\mathcal{O}(h)$. However, all such contributions must cancel for the bound-state and elementary phase to be of the same leading order $\mathcal{O}(h)$. For q = 0 this is apparent from the particular form of the elementary dressing phase

$$\begin{split} \Theta &= \Theta_{\text{BDS}} + \Theta_{\text{AFS}} & \bar{\Theta} &= \Theta_{\text{mix}} + \bar{\Theta} \\ &\sim \mathcal{O}(h) &\sim \mathcal{O}(h) & & & & & \\ && & \uparrow^{\text{fusion}} & & \uparrow^{\text{fusion}} & & & & & \\ \theta &= & \theta_{\text{BDS}} + & \theta_{\text{AFS}} & & & & & & \\ &\sim \mathcal{O}(1) &\sim \mathcal{O}(h) & & & & & & \\ \theta &= & \theta_{\text{mix}} + & \theta_{\text{mix}} + & & & \\ &\sim \mathcal{O}(1) &\sim \mathcal{O}(h) & & & & & & \\ (a) \text{ same type excitations } (l_1 l_2 = 1) & & (b) \text{ mixed type excitations } (l_1 l_2 = -1) \end{split}$$

Figure 7.1: Fusion of the AFS and BDS contributions: θ denotes elementary S-matrix contributions and Θ denotes bound-state contributions.

which allows us to obtain multi-soliton scattering solutions. Identifying the limit in which one of these dyonic giant magnon solitons reduces to a plane wave we obtain a solution for a plane wave scattering off multiple dyonic giant magnons. From the asymptotic behaviour of this solution at $x \to \pm \infty$ we then find the associated bosonic phase shifts.

The dressing method only covers bosonic phase shifts, however in the formulation of classical solutions in terms of the finite-gap picture fermionic and bosonic fluctuations are closely related. Exploiting this relation we will see in section 7.3.2 how the information coming from the bosonic phase shifts is already sufficient to determine the fermionic phase shifts as well as the exact form of the dyonic giant magnon solution in the finite-gap picture. In section 7.3.3 we use this semiclassical scattering data to evaluate the 1-loop bound-state corrections (7.0.9)-(7.0.10) and we deduce the 1-loop corrections for the scattering of elementary excitations.

7.3.1 Phase shifts for bosonic fluctuations

Let us first relate the plane wave solutions to the dyonic giant magnon. In order to obtain a plane wave solution to the linearised equations of motion we take the semiclassical limit

$$h \to \infty, \qquad k \equiv hp \text{ fixed}, \qquad J \text{ fixed.}$$
 (7.3.1)

In this plane wave limit the spectral parameters have the expansion

$$X^{\pm} \sim w + \mathcal{O}\left(\frac{1}{h}\right), \qquad w = \frac{r}{\hat{q}}, \quad r = \frac{1}{k} \left[J + qlk + \varepsilon(k)\right]$$

$$(7.3.2)$$

with the dispersion relation

$$\varepsilon(k) = \sqrt{(k+qlJ)^2 + \hat{q}^2 J^2}.$$
(7.3.3)

In the large momentum limit the spectral parameter reduces to

$$w_l \to s^{\pm l}, \qquad k \to \pm \infty, \qquad s = \frac{\sqrt{1+q}}{\sqrt{1-q}}.$$
 (7.3.4)
Expanding the dyonic giant magnon solution in the limit (7.3.1) we obtain a plane wave

$$\delta Z_1 = Z_1 - Z_1^{BMN} \sim 0 \tag{7.3.5}$$

$$\delta Z_2 = Z_2 - Z_2^{BMN} \sim \sin\frac{p}{2} e^{i(v-qx)} = \sin\frac{p}{2} \exp\left(-i\frac{\varepsilon(k)t - kx}{Jl}\right)$$
(7.3.6)

Since we are interested in the phase shifts for an elementary plane wave we take from now on J = 1. Let us further parametrise its frequency ω and momentum k in terms of the spectral parameter w as

$$\omega = \sqrt{(k+ql)^2 + \hat{q}^2} = \frac{1-q^2+r^2}{(r+1-lq)(r-(1+lq))} = \frac{\hat{q}(1+w_l^2)}{(w_l+s^{-l})(w_l-s^l)},$$
(7.3.7)

$$k = \frac{2r}{(r+1-lq)(r-(1+lq))} = \frac{2w_l}{\hat{q}(w_l+s^{-l})(w_l-s^l)}.$$
(7.3.8)

This dispersion relation corresponds to two separate plane waves with $l = \pm 1$ representing left or right movers. In the following derivation of the phase shifts we will only consider l = -1 plane waves and dyonic giant magnons for simplicity. In order to generalise these to arbitrary combinations of left and right moving plane waves and dyonic giant magnons we can simply send $J \rightarrow -J$. In terms of the dyonic giant magnon spectral parameters X_l^{\pm} this corresponds to the transformation

$$X_{-}^{\pm} \to \frac{1}{X_{+}^{\mp}}$$
 (7.3.9)

For plane waves we have $x^+ \sim x^- \sim w$ and this transformation reduces to

$$w_{-} \to \frac{1}{w_{+}}.\tag{7.3.10}$$

Using the dressing method we can now calculate the phase shift of a plane wave scattering off an N-soliton solution. For this we construct the asymptotic form of the N-soliton solution and take the plane wave limit in the N + 1 dressing step. The recursive dressing relation is

$$g_N = \sqrt{\frac{\lambda_N - q}{\bar{\lambda}_N - q}} \Psi_{N-1}(q) \left(1 + \frac{\lambda_N - \bar{\lambda}_N}{q - \lambda_N} P \right).$$
(7.3.11)

Identifying

$$\lambda_N = q + r_N e^{ip_N/2}, \qquad \bar{\lambda}_N = q + r_N e^{-ip_N/2}, \qquad \Psi_N(q) = g_N$$
 (7.3.12)

we obtain

$$g_N = g_{N-1} \left(e^{ip_N/2} - 2i \sin \frac{p_N}{2} P_{N-1} \right).$$
(7.3.13)

Expanding this expression in the plane wave limit (in small $\eta = x^+ - x^- = 2ir \sin \frac{p}{2}$) gives the phase shifts

$$\delta g = g_{N+1} - g_N = g_N \left(e^{ip/2} - 2i \sin \frac{p}{2} P_N - 1 \right) \sim i \sin \frac{p}{2} \left(g_N (1 - 2P_{N+1})|_{\eta = 0, x \to \pm \infty} \right).$$
(7.3.14)

We now need to determine the N-soliton solution g_N and the projector P_N in the plane wave limit using the dressing transformation

$$\Psi_N(\lambda) = \Psi_{N-1}(\lambda)\chi_N(\lambda), \qquad \chi_N(\lambda) = 1 + \frac{\lambda_N - \lambda_N}{\lambda - \lambda_N}P_N.$$
(7.3.15)

First let us show that the asymptotic form of the projector (without taking the plane wave limit) is

$$P_i|_{x \to +\infty} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_i|_{x \to -\infty} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$
 (7.3.16)

We can establish this by induction. Starting with the projector P_1 for the giant magnon

$$P_{1} = \frac{1}{2} \operatorname{sech} u \begin{pmatrix} e^{-u} & e^{i(v-t-qx)} \\ e^{-i(v-t-qx)} & e^{u} \end{pmatrix}$$
(7.3.17)

we obtain the above asymptotic behaviour as $u \to \pm \infty$. Assuming this also holds for all the projectors up to P_N we can construct the next projector P_{N+1} by applying the dressing transformation using the above asymptotic form. The dressing transformation (7.3.15) becomes

$$\Psi_N = e^{iP/2} \Psi_0 \prod_{k=1}^N \chi_k(\lambda), \qquad P = \sum_{k=1}^N p_k, \qquad A(\lambda) = \prod_{k=1}^N \frac{\lambda - \bar{\lambda}_k}{\lambda - \lambda_k}, \tag{7.3.18}$$

$$\chi_k(\lambda)|_{x \to +\infty} = \begin{pmatrix} 1 & 0\\ 0 & \frac{\lambda - \bar{\lambda}_k}{\lambda - \lambda_k} \end{pmatrix}, \qquad \Psi_N|_{x \to +\infty} = e^{iP/2} \begin{pmatrix} e^{iZ(\lambda)} & 0\\ 0 & e^{-iZ(\lambda)}A(\lambda) \end{pmatrix}, \qquad (7.3.19)$$

$$\chi_k(\lambda)|_{x \to -\infty} = \begin{pmatrix} \frac{\lambda - \bar{\lambda}_k}{\lambda - \lambda_k} & 0\\ 0 & 1 \end{pmatrix}, \qquad \Psi_N|_{x \to -\infty} = e^{iP/2} \begin{pmatrix} e^{iZ(\lambda)}A(\lambda) & 0\\ 0 & e^{-iZ(\lambda)} \end{pmatrix}$$
(7.3.20)

and using

$$P_N = \frac{[\Psi_{N-1}(\bar{\lambda}_N)]^{\dagger} e e^{\dagger} \Psi_{N-1}(\bar{\lambda}_N)}{e^{\dagger} [\Psi_{N-1}(\bar{\lambda}_N)]^{\dagger} \Psi_{N-1}(\bar{\lambda}_N) e}, \qquad e = (1,1)$$
(7.3.21)

we find the projector

$$P_{N+1}|_{x \to +\infty} = \frac{1}{e^{-u} + e^u |\Pi|^2} \begin{pmatrix} e^{-u} & e^{i(v-t-qx)}\Pi \\ e^{-i(v-t-qx)}\bar{\Pi} & e^u |\Pi|^2 \end{pmatrix}|_{x \to +\infty}, \qquad \Pi \equiv \prod_{k=1}^N \frac{\lambda_{N+1} - \bar{\lambda}_k}{\lambda_{N+1} - \lambda_k}$$
(7.3.22)

$$P_{N+1}|_{x \to -\infty} = \frac{1}{e^{-u}|\Pi|^2 + e^u} \begin{pmatrix} e^{-u}|\Pi|^2 & e^{i(v-t-qx)}\bar{\Pi} \\ e^{-i(v-t-qx)}\Pi & e^u \end{pmatrix}|_{x \to -\infty}.$$
(7.3.23)

Taking $u \to \pm \infty$ we arrive at the asymptotic behaviour (7.3.16) as required. However, in the plane wave limit we have $\bar{\lambda}_{N+1} = \lambda_{N+1} = q + r + \mathcal{O}(1/h)$. Therefore by definition $u \equiv 0$ and $\bar{\Pi} = 1/\Pi$ giving the projector

$$P_{N+1}|_{x \to +\infty} \sim \frac{1}{2} \begin{pmatrix} 1 & e^{i(v-t-qx)} \Pi \\ e^{-i(v-t-qx)} \bar{\Pi} & 1 \end{pmatrix},$$
(7.3.24)

$$P_{N+1}|_{x \to -\infty} \sim \frac{1}{2} \left(\begin{array}{cc} 1 & e^{i(v-t-qx)}\bar{\Pi} \\ e^{-i(v-t-qx)}\Pi & 1 \end{array} \right).$$
(7.3.25)

Finally the N-soliton solution as obtained from (7.3.20) takes the form

$$g_N|_{x \to +\infty} = \begin{pmatrix} e^{i(t+P/2)} & 0\\ 0 & e^{-i(t+P/2)} \end{pmatrix}, \qquad g_N|_{x \to -\infty} = \begin{pmatrix} e^{i(t-P/2)} & 0\\ 0 & e^{-i(t-P/2)} \end{pmatrix}$$
(7.3.26)

giving

$$\delta g|_{x \to \pm \infty} \sim -i \sin \frac{p}{2} \left(\begin{array}{cc} 0 & e^{\pm iP/2} e^{i(v-qx)} \Pi^{\pm 1} \\ e^{\mp iP/2} e^{-i(v-qx)} \Pi^{\mp 1} & 0 \end{array} \right)$$
(7.3.27)

or in terms of the coordinates

$$\delta Z_1|_{x \to \pm \infty} = 0, \qquad \delta Z_2|_{x \to \pm \infty} = -i \sin \frac{p}{2} e^{\pm i P/2} e^{i(v-qx)} \Pi^{\pm 1}$$
(7.3.28)

$$\delta \bar{Z}_1|_{x \to \pm \infty} = 0, \qquad \delta \bar{Z}_2|_{x \to \pm \infty} = i \sin \frac{p}{2} e^{\pm i P/2} e^{-i(v-qx)} \Pi^{\pm 1}.$$
 (7.3.29)

Note that $v - qx = \omega t - kx$ such that these are plane wave solutions of frequency ω and momentum k given by (7.3.8). For the conjugate fields we have $\omega \to -\omega$ and $k \to -k$ corresponding to $w_l \to 1/w_{-l}$. We read off the phase shifts

$$\delta_{Z_1} = 0 \tag{7.3.30}$$

$$\delta_{Z_2} \equiv i \ln \delta Z_2|_{x \to +\infty} - i \ln \delta Z_2|_{x \to -\infty} = -P + 2i \ln \Pi = -P - 2i \sum_{k=1}^N \ln \frac{w_- - X_{-k}^+}{w_- - X_{-k}^-}$$
(7.3.31)

$$\delta_{\bar{Z}_1} = 0 \tag{7.3.32}$$

$$\delta_{\bar{Z}_2} = P - 2i \ln \Pi = P + 2i \sum_{k=1}^{N} \ln \frac{\frac{1}{w_+} - X_{-k}^+}{\frac{1}{w_+} - X_{-k}^-}$$
(7.3.33)

where we used (7.1.22) to write these expressions in terms of the spectral parameters $X_{\pm k}^{\pm}$ with the lower sign denoting left or right movers. Finally we can obtain phase shifts for any combination of left and right moving plane waves and dyonic giant magnons using the transformations (7.3.9), (7.3.10).

7.3.2 Phase shifts for fermionic fluctuations

The fermionic phase shifts can be directly extracted from the fermionic solution of a single giant magnon given in [219, 205]. However, instead we will follow here the same route as in [25] and consider the dyonic giant magnon in terms of finite-gap solutions. Using the fact that the dyonic giant magnon only lives on the sphere part of $AdS_3 \times S^3$ will be sufficient to determine the fermionic phase shifts without knowing the precise form of the finite-gap equations. The consistency of the resulting finite-gap picture also serves as an additional check for the mixed-flux dyonic giant magnon results in [95].

The equations of motion for strings on $\operatorname{AdS}_3 \times \operatorname{S}^3$ correspond to the flatness condition of a current j with an associated monodromy matrix $\Omega \sim P \exp(\oint j)$. Its analytic properties give rise to the finite-gap equations for classical solutions with periodic boundary conditions.

In our case the monodromy matrix is an element of the supergroup SU'(1,1|2) and its eigenvalues

have the form

$$(e^{ip_1^A(X)}, e^{ip_2^A(X)}|e^{ip_1^S(X)}, e^{ip_2^S(X)}), \qquad p_1(X) = -p_2(X)$$
(7.3.34)

where the quasi-momenta $p_i(X)$ are complex functions of the spectral parameter X and the label A, S distinguishes between the AdS₃ and S³ matrix blocks. These quasi-momenta have poles and branch cuts and thus can be written as

$$p_i(X) = G_i(X) + f_i(X) \tag{7.3.35}$$

where f(X) contains the poles and the resolvent G(X) contains the branch cuts with the discontinuity relation across each cut

$$p_i(X+i\epsilon) + p_j(X-i\epsilon) = 2\pi n_{ij}, \quad n_{ij} \in \mathbb{Z}.$$
(7.3.36)

The resolvents characterise the different classical solutions and since we are only interested in solutions with non-trivial motion on the sphere the quasi-momenta take the form

$$p_1^A(X) = -p_2^A(X) = f(X), \qquad p_i^S(X) = -p_2^S(X) = f(X) + G(X)$$
(7.3.37)

In this formalism the phase shifts are encoded by introducing a micoscopic probe cut corresponding to small fluctuations. The associated discontinuity condition then becomes the quantization condition of the associated plane wave momentum k for a string solution of finite length L

$$2\pi n_{ij} = p_i(X(k) + i\epsilon) + p_j(X(k) - i\epsilon) = -\delta(k) - kL.$$
(7.3.38)

Each fluctuation corresponds to a connection of two particular sheets $(p_i^{\{S,A\}}, p_j^{\{S,A\}})$ and their exact relation was determined in [160].

In the plane wave limit the spectral parameter expands as $X \sim w \equiv \frac{r}{\hat{q}}$ and thus we have for l = -1

$$p_{-1}^{A}(w_{-}) - p_{-2}^{S}(w_{-}) = 2G_{-}(w_{-}) + 2f_{-}(w_{-}) = -\delta_{Z_{2}}(w_{-}) - k(w_{-})L$$
(7.3.39)

$$p_{-1}^{A}(w_{-}) - p_{-2}^{A}(w_{-}) = 2f_{-}(w_{-}) = -\delta_{Y_{2}}(w_{-}) - k(w_{-})L$$
(7.3.40)

Since the phase shifts δ_{Y_2} vanish in the AdS part we can identify the form of $f_{-}(x)$ as

$$f_{-}(x) = -\frac{L}{2\hat{q}} \frac{2x}{(x+s)(x-s^{-1})}.$$
(7.3.41)

Furthermore we can read off the resolvent for the dyonic giant magnon from the bosonic phase shifts as

$$G_{-}(x) = G_{mag}(x) - \frac{1}{2}G_{mag}(0), \quad G_{mag}(x) = i\ln\frac{x - X^{+}}{x - X^{-}}, \quad G_{mag}(0) = p.$$
(7.3.42)

where p is the magnon world-sheet momentum. For l = 1 excitations we use (7.3.10) to get

$$p_{+1}^{S}(w_{+}) - p_{+2}^{S}(w_{+}) = 2G_{+}(w_{+}) + 2f_{+}(w_{+}) = -\delta_{Z_{2}}\left(\frac{1}{w_{+}}\right) - k\left(\frac{1}{w_{+}}\right)L$$
(7.3.43)

$$p_{+1}^{A}(w_{+}) - p_{+2}^{A}(w_{+}) = 2f_{+}(w_{+}) = -\delta_{Y_{2}}\left(\frac{1}{w_{+}}\right) - k\left(\frac{1}{w_{+}}\right)L$$
(7.3.44)

and therefore

$$f_{+}(x) = \frac{L}{2\hat{q}} \frac{2x}{(x-s)(x+s^{-1})}, \qquad G_{+}(x) = G_{mag}\left(\frac{1}{x}\right) - \frac{1}{2}G_{mag}(0). \tag{7.3.45}$$

We see that the resolvent has the same form as in the q = 0 case. Also these results for $f_{\pm}(x)$ and $G_{\pm}(x)$ match (C.1) in [95] up to factors of $G_{mag}(0)$ which depend on the choice of boundary conditions for the dyonic giant magnon. In our case we have $Z_1 \sim e^{it \pm ip/2}$ as $x \to \pm \infty$ corresponding to $-\frac{1}{2}G_{mag}(0)$ in both $G_{+}(x)$ and $G_{-}(x)$.

Finally for the fermionic fluctuations we have for plane waves of l = -1 type

$$p_{-1}^{S}(w_{-}) - p_{-2}^{A}(w_{-}) = G_{-}(w_{-}) + 2f_{-}(w_{-}) = -\delta_{\eta}(w_{-}) - k(w_{-})L$$
(7.3.46)

$$p_{-2}^{S}(w_{-}) - p_{-1}^{A}(w_{-}) = G_{-}(w_{-}) + 2f_{-}(w_{-}) = -\delta_{\theta}(w_{-}) - k(w_{-})L.$$
(7.3.47)

The equations for plane waves of type l = 1 are then obtained using (7.3.10) and using the same arguments for conjugate fields we find altogether

$$\delta_{\eta}(k) = \delta_{\theta}(k) = \frac{1}{2}\delta_{Z_2}(k), \qquad \delta_{\bar{\eta}}(k) = \delta_{\bar{\theta}}(k) = \frac{1}{2}\delta_{\bar{Z}_2}(k).$$
 (7.3.48)

7.3.3 Corrections to the dispersion relation and the dressing phase

We can now evaluate the 1-loop corrections (7.0.9) and (7.0.10) using the phase shifts δ_I for a plane wave scattering off a single dyonic giant magnon explicitly given by

$$AdS_3: \delta_{Y_k} = \delta_{\bar{Y}_k} = 0 \tag{7.3.49}$$

$$S^{3}: \delta_{Z_{1}} = \delta_{\bar{Z}_{1}} = 0, \quad \delta_{Z_{2},l}(k,X) = -2G(w_{l}(k)^{-l},X), \quad \delta_{\bar{Z}_{2},l}(k,X) = 2G(w_{-l}(k)^{l},X) \quad (7.3.50)$$

fermionic:
$$\delta_{\theta,l}(k,X) = \delta_{\eta,l}(k,X) = -G(w_l(k)^{-l},X)$$
(7.3.51)

$$: \delta_{\bar{\theta},l}(k,X) = \delta_{\bar{\eta},l}(k,X) = G(w_{-l}(k)^l,X)$$
(7.3.52)

$$G(w,X) = i\left(-\frac{1}{2}\ln\frac{X^{+}}{X^{-}} + \ln\frac{w-X^{+}}{w-X^{-}}\right).$$
(7.3.53)

where the lower label l specifies whether the phase shift is for a left or right moving plane wave of spectral parameter $w_l(k)$ given by (7.3.8) (for example $\delta_{\bar{Z}_2,-}(k,X) = 2G(1/w_+(k),X)$). Here X is the spectral parameter a dyonic giant magnon with $l_X = -1$ and we obtain the phase shift for $l_X = 1$ using (7.3.9).

We immediately see that the 1-loop energy shift vanishes

$$\Delta E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \,\omega(k) \frac{\partial}{\partial k} \Big([\delta_{Z_2} - \delta_{\theta} - \delta_{\eta}] + [\delta_{\bar{Z}_2} - \delta_{\bar{\theta}} - \delta_{\bar{\eta}}] \Big) = 0.$$
(7.3.54)

For the 1-loop phase correction (7.0.10)

$$\Delta\Theta(p_1, p_2) = \frac{1}{2\pi} \sum_{I=1}^{N_F} (-1)^{F_I} \int_{-\infty}^{\infty} \mathrm{d}k \, \frac{\partial\delta_I(k; p_1)}{\partial k} \delta_I(k; p_2) \tag{7.3.55}$$

we have the choice of evaluating the integral for a left or right moving plane wave. Both choices are equivalent since

$$\int_{-s^{-1}}^{s} \mathrm{d}w_{+} \frac{\partial G(\frac{1}{w_{+}};X)}{\partial w_{+}} G\left(\frac{1}{w_{+}};Y\right) = \int_{-s}^{s^{-1}} \mathrm{d}w_{-} \frac{\partial G(w_{-};X)}{\partial w_{-}} G\left(w_{-};Y\right).$$
(7.3.56)

This also implies that the integrals for conjugate and non-conjugate fields are in fact the same. Let us also rewrite (7.0.10) in a manifestly antisymmetric form as we expect from unitarity. Dropping a total derivative and noticing that under (7.3.9)

$$G(w_l, X) \to -G(1/w_l, X)$$
 (7.3.57)

we can write the 1-loop phase correction for same-type excitations $l_1 = l_2 = l$ as

$$\Delta\Theta_{ll}(p_1, p_2) = \frac{1}{\pi} \int_{-s^{-l}}^{s^l} \mathrm{d}w \,\left(\frac{\partial G(w, X)}{\partial w} G(w, Y) - (X \leftrightarrow Y)\right),\tag{7.3.58}$$

$$\frac{\partial G(w,X)}{\partial w} = \frac{i}{w - X^+} - \frac{i}{w - X^-}, \quad s = \frac{\sqrt{1+q}}{\sqrt{1-q}}.$$
(7.3.59)

We can further split this integral and perform the integration explicitly to give

$$\Delta\Theta_{ll}(p_1, p_2) = \chi_l(X^+, Y^+) - \chi_l(X^+, Y^-) - \chi_l(X^-, Y^+) + \chi_l(X^-, Y^-)$$
(7.3.60)

where

$$\chi_l(X,Y) = -\frac{1}{\pi} \int_{-s^{-l}}^{s^l} \frac{\mathrm{d}u}{u-X} \left(-\frac{1}{2} \ln Y + \ln(u-Y) \right) - (X \leftrightarrow Y), \tag{7.3.61}$$

$$= -\frac{1}{\pi} [I_l(X,Y) - I_l(Y,X)]$$
(7.3.62)

$$I_l(X,Y) = \ln\left(\frac{X-s^l}{X+s^{-l}}\right) \left[\ln(X-Y) - \frac{1}{2}\ln Y\right] - \text{Li}_2\left(\frac{X-s^l}{X-Y}\right) + \text{Li}_2\left(\frac{X+s^{-l}}{X-Y}\right).$$
(7.3.63)

For the case of mixed excitations, i.e. $l_1 = -l_2 = l$, we can use (7.3.9) directly on the integral (7.3.63). However, in order to make manifest its antisymmetry under $X \leftrightarrow Y$, $l_1 \leftrightarrow l_2$ let us write the integral (7.3.55) as

$$\Delta\bar{\Theta}_{l_1l_2}(p_1, p_2) = -\frac{1}{\pi} \int_{-s^{-l_1}}^{s^{l_1}} \mathrm{d}w \,\left(\frac{\partial G(w, X)}{\partial w} G\left(\frac{1}{w}, Y\right) - (X \leftrightarrow Y, \, l_1 \leftrightarrow l_2)\right) \tag{7.3.64}$$

giving

$$\Delta\bar{\Theta}_{l_1l_2}(p_1, p_2) = \bar{\chi}_l(X^+, Y^+) - \bar{\chi}_l(X^+, Y^-) - \bar{\chi}_l(X^-, Y^+) + \bar{\chi}_l(X^-, Y^-)$$
(7.3.65)

$$\bar{\chi}_l(X,Y) = -\frac{1}{\pi}(\bar{I}_l(X,Y) - \bar{I}_l(Y,X))$$
(7.3.66)

$$\bar{I}_{l}(X,Y) = -\frac{1}{2}\ln Y \ln \frac{X-s^{l}}{X+s^{-l}} + \ln(X-s^{l})\ln(Y-s^{-l}) - \ln(X+s^{-l})\ln(Y+s^{l}) + \ln\left(s^{-2l}\frac{X-s^{l}}{X+s^{-l}}\right)\ln(1-XY) - \operatorname{Li}_{2}\left(s^{-l}\frac{Y+s^{l}}{1-XY}\right) + \operatorname{Li}_{2}\left(-s^{l}\frac{Y-s^{-l}}{1-XY}\right).$$
(7.3.67)

These 1-loop corrections to the bound-state S-matrix come directly from the elementary dressing phases. This is because the bound-state S-matrix does not receive contributions from the BDS factor (or its mixed excitation scattering equivalent) as was pointed out in [25]. In order to find the elementary dressing phases we notice that we can use the same arguments as in the tree-level case in section 7.2.1 since the bound-state 1-loop corrections are of the form (7.2.44). Thus the elementary 1-loop dressing phases are given by the above expressions but with the dyonic giant magnon spectral parameters X, Y replaced by the elementary magnon spectral parameters x, y.

As a consistency check let us take the semiclassical near-BMN limit $h \to \infty$ with hp fixed such that the shortening conditions (7.2.8) are solved by

$$x_l^{\pm} = x_l \pm \frac{i}{2} \alpha_l(x_l) + \mathcal{O}\left(\frac{1}{h}\right), \quad \alpha_l(x) = \frac{2x^2}{h\hat{q}(x-s^l)(x+s^{-l})}, \tag{7.3.68}$$

$$m_l = 1 + qlk, \quad x_l \equiv \frac{m_l + \sqrt{m_l^2 + \hat{q}^2 k^2}}{\hat{q}k}.$$
 (7.3.69)

We obtain

$$\theta_{ll}(x,y) = \frac{1}{2}\Delta\Theta_{ll}, \qquad \bar{\theta}_{l_1l_2}(x,y) = \frac{1}{2}\Delta\bar{\Theta}_{l_1l_2}(p_1,p_2)$$
(7.3.70)

$$\Delta \Theta_{ll} = -\frac{\alpha_l(x)\alpha_l(y)}{2\pi} \Big[\frac{1}{\hat{q}} \frac{(x+y)(1-\frac{1}{xy}) - \frac{4ql}{\hat{q}}}{(x-s^l)(x+s^{-l})(y-s^l)(y+s^{-l})} \frac{x+y}{x-y} + \frac{2}{(x-y)^2} \ln\left(\frac{y-s^l}{x-s^l} \frac{x+s^{-l}}{y+s^{-l}}\right) \Big],$$
(7.3.71)

$$\Delta\bar{\Theta}_{l_{1}l_{2}}(p_{1},p_{2}) = -\frac{\alpha_{l_{1}}(x)\alpha_{l_{2}}(y)}{2\pi} \Big[\frac{1}{\hat{q}} \frac{(x-y)(1+\frac{1}{xy}) - \frac{4ql_{1}}{\hat{q}}}{(x-s^{l_{1}})(x+s^{-l_{1}})(y+s^{-l_{2}})(y-s^{l_{2}})} \frac{1+xy}{1-xy} + \frac{2}{(1-xy)^{2}} \ln\left(\frac{y-s^{l_{2}}}{x-s^{l_{1}}}\frac{x+s^{-l_{1}}}{y+s^{-l_{2}}}s^{l_{1}-l_{2}}\right) \Big],$$
(7.3.72)

which matches the 1-loop dressing phase predictions in [95, 96, 97].

7.4 Time-delay from the complex sine-Gordon model

As an additional check of our results let us also find the time-delay from the CsG model. The CsG soliton anti-soliton scattering solution for q = 0 can be obtained from the single soliton solution by

applying Bäcklund transformations and is given by (see e.g. [220])

$$\psi_{2s} = \frac{e^{i\gamma}(\delta_2 v_1^* - \delta_1 v_2^*)(\delta_1 u_1 - \delta_2 u_2) - e^{-i\gamma}(\delta_2 v_2 - \delta_1 v_1)(\delta_1 u_2 - \delta_2 u_1)}{\delta_1^2 - (u_2 u_1^* + u_2^* u_1 + v_2 v_1^* + v_2^* v_2)\delta_1 \delta_2 + \delta_2^2}$$
(7.4.1)

where

$$u_k = \frac{N_k \cos \rho_k \exp(i \sin(\rho_k) \mathcal{T}_k)}{\cosh(\cos(\rho_k) \mathcal{X}_k)}, \qquad v_k = -e^{i\gamma} \Big(\cos(\rho_k) \tanh(\cos(\rho_k) \mathcal{X}_k) + i \sin \rho_k \Big)$$
(7.4.2)

$$\delta_k = \exp(-\theta_k), \quad N_k = e^{i\phi_k} = \text{const}$$
(7.4.3)

$$\mathcal{X}_k = \cosh(\theta_k)x - \sinh(\theta_k)t, \qquad \mathcal{T}_k = \cosh(\theta_k)t - \sinh(\theta_k)x.$$
 (7.4.4)

Note that γ drops out of the full scattering solution ψ_{2s} . Considering the form of a single soliton solution for $q \neq 0$ in 5 we can generalise ψ_{2s} to $q \neq 0$ by performing the replacement

$$\mathcal{X}_k \to \tilde{\mathcal{X}}_k = \mathcal{X}_k + q\mathcal{T}_k,$$
(7.4.5)

$$\mathcal{T}_k \to \tilde{\mathcal{T}}_k = \mathcal{T}_k + q\mathcal{X}_k.$$
 (7.4.6)

One can easily check that this new solution satisfies the $q \neq 0$ CsG equation of motion

$$\ddot{\psi} - \psi'' + \psi^* \frac{\dot{\psi}^2 - \dot{\psi}^2}{1 - |\psi|^2} + (1 - q^2)\psi(1 - |\psi|^2) = 0.$$
(7.4.7)

Even though the $q \neq 0$ CsG model differs from the q = 0 model only by a mass rescaling, the $q \neq 0$ soliton solution, which corresponds to the giant magnon, is not obtained through the simple mass rescaling $(\mathcal{X}, \mathcal{T}) \rightarrow \sqrt{1-q^2}(\mathcal{X}, \mathcal{T})$ but rather through a mixing of \mathcal{X} and \mathcal{T} .²

The energy of this soliton is

$$E = \int \mathrm{d}x \,\mathcal{H} = \int_{-\infty}^{+\infty} \mathrm{d}x \left(\frac{|\dot{\psi}|^2 + |\dot{\psi}|^2}{1 - |\psi|^2} + (1 - q^2)|\psi|^2 \right) = \sum_i 4(\cosh\theta_i - q\sinh\theta_i)\cos\rho_i \tag{7.4.8}$$

and we shall consider the COM frame where

$$\sum_{i} E_{i} v_{si} = \sum_{i} (\sinh \theta_{i} - q \cosh \theta_{i}) \cos \rho_{i} = 0.$$
(7.4.9)

Taking the limit $t \to \infty$ with $x = v_{s1}t$ the free solitons shifted by $t \to t - \delta t_+$ become

$$u_{1} = e^{i(v_{1} + \delta t_{+} \sin \rho_{1}(\cosh \theta_{1} - q \sinh \theta_{1}))} \cos \rho_{1} \operatorname{sech} \left[\delta t_{+} \cos \rho_{1}(\sinh \theta_{1} - q \cosh \theta_{1}) \right], \ u_{2} = 0.$$
(7.4.10)

whereas the scattering soliton solution asymptotes to

$$|\psi_{2s}|^2 = \cos^2 \rho_1 \frac{1 - \cos(2\rho_1) - \cos(2\rho_2) + \cosh(2(\theta_1 - \theta_2)) - 4\cosh(\theta_1 - \theta_2)\sin\rho_1\sin\rho_2}{2(\cosh(\theta_1 - \theta_2) - \sin\rho_1\sin\rho_2)^2}.$$
 (7.4.11)

²A simple mass rescaling in the CsG soliton solution would lead to embedding coordinates with x, t rescaled breaking the Virasoro constraints.

Comparing these expression we obtain the expected result for the time-delay

$$\Delta T = \frac{1}{(\sinh \theta_1 - q \cosh \theta_1) \cos \rho_1} \Big| \ln \frac{\cosh(\theta_1 - \theta_2) - \cos(\rho_1 - \rho_2)}{\cosh(\theta_1 - \theta_2) + \cos(\rho_1 + \rho_2)} \Big|.$$
(7.4.12)

7.5 Summary

In summary we have presented a semiclassical derivation of the tree-level and 1-loop dressing phases in the massive sector of string theory on $AdS_3 \times S^3 \times T^4$ with R-R and NS-NS 3-form fluxes. In both cases we have found agreement with the proposals from finite-gap equations and algebraic curve quantization in [95] and unitarity cut methods in [96, 97].

For the 1-loop phase we have seen that the semiclassical scattering data for bosonic fluctuations, given in terms of phase shifts experienced by plane waves scattering off dyonic giant magnons, is sufficient to determine the fermionic scattering data and the resolvent for the mixed-flux dyonic giant magnon which agrees with the suggested resolvent in [95].

Chapter 8

Conclusions and outlook

The problem of understanding the physics of strongly coupled quantum systems has motivated much interest in gauge/gravity dualities, which relate the strong coupling regime of gauge theories to the weak coupling regime of string theories. In particular this strong/weak nature provides a promising setting to study strongly coupled gauge theories from the perspective of a weakly coupled theory.

In highly symmetric settings, such as in the case of the duality between $\mathcal{N} = 4$ SYM and type IIB superstring theory in the maximally supersymmetric 10-dimensional $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, significant progress has been made in understanding these dualities in the planar limit due to the presence of integrability. This has motivated the search for new integrable string σ -models in the hope of uncovering dualities in less symmetric and more physical settings. The most promising starting point are σ -models which interpolate between known integrable or solvable limits. In this thesis we have investigated integrability in two examples of such string σ -models.

Classical integrability of string σ -models

In part I we have looked at classical integrability of string motion on the following curved brane backgrounds

- (i) p-brane of 10-dimensional supergravity for p = 0, ..., 6, which includes the D3-brane;
- (ii) the equal-charge D5-D1 brane intersection;
- (iii) the four equal-charge D3-brane intersection;
- (iv) the NS 5-brane-fundamental string system.

The backgrounds (i)-(iii) interpolate between the integrable limits of flat space and their respective near-horizon geometries, which are $AdS_5 \times S^5$, $AdS_3 \times S^3 \times T^4$ and $AdS_2 \times S^2 \times T^6$. The background (iv) is supported by an NS-NS flux, to which the classical bosonic string couples. Therefore the corresponding string σ -model interpolates between the integrable flat and SU(2) or $SL(2) \times SU(2)$ WZW models. We have shown that, while geodesic motion is integrable in these backgrounds, extended classical string motion in general is not integrable.

This is not surprising in the light of previous non-integrability results for string motion on various curved backgrounds [147, 146, 167, 148]. In particular integrability of string motion is a much stronger

constraint on the σ -model than integrability of geodesic motion. Nevertheless without a general classification of integrable 2d σ -models it is not clear what other interesting, e.g. less symmetric and more physical, integrable string σ -models exist which cannot be obtained through classical equivalence, such as T-dualities and coordinate transformations (cf. [130, 129]), from known integrable models.

Recently in [221] progress has been made in classifying D-brane backgrounds that admit integrable geodesic motion, which is equivalent to the existence of variables in which the Hamilton-Jacobi equation separates. In particular it was shown that for backgrounds associated with a single type of D-branes the Hamilton-Jacobi equation separates only in the cases of SO(m) or $SO(m) \times SO(n)$ isometries in terms of ellipsoidal coordinates. The former case of SO(m) corresponds to a single stack of D*p*-branes. In the latter case the branes were found to localise in certain configurations. However, classical string motion on these backgrounds was shown not to be integrable except in the case of $AdS_5 \times S^5$.

Interestingly for more general supersymmetric D-brane intersections, such as the D*p*-D(*p*+4) system, the separation in ellipsoidal coordinates was found to only occur in cases with the usual pp-wave or $AdS_p \times S^q$ asymptotics.

Similar non-integrability results for classical string motion have also been obtained in the dual background of the marginal deformation of $\mathcal{N} = 4$ SYM [222] as well as in certain classes of non-relativistic theories [223]. It should also be possible to further extend these studies to other highly symmetric backgrounds like the Pilch-Warner [224] and Klebanov-Tseytlin [225] geometries to identify possible candidates for new integrable models.

More recently classically integrable deformations of the $AdS_5 \times S^5$ coset σ -model and its lower dimensional equivalents, [226], have been constructed based on a generalisation of the bosonic Yang-Baxter σ -model [227, 121, 228, 117, 229] and based on a gauged WZW model which is related to the non-abelian T-dual of the $AdS_5 \times S^5 \sigma$ -model [230, 231, 141]. These deformed models are of interest since they have significantly fewer manifest symmetries compared to the undeformed case. For example the background of the first model has no spacetime supersymmetry and its isometries correspond to $SO(2)^3 \times SO(2)^3$, the Cartan subgroup of $SO(4, 2) \times SO(6)$.

It would be interesting to extend the study of geodesic motion to these deformed models and identify the coordinates in which the Hamilton-Jacobi equation separates. This might shed some light on more general candidates for integrable string σ -models beyond backgrounds with the usual pp-wave or $\operatorname{AdS}_p \times \operatorname{S}^q$ asymptotics.

In general it would also be interesting to understand in more detail why moving away from the near-horizon region of a brane, such as the $AdS_5 \times S^5$ region of a D3 brane, by turning on the brane charge breaks the hidden symmetries associated with string integrability.

String theory in $AdS_3 \times S^3 \times T^4$ with mixed flux

In part II we have looked at superstring theory on $AdS_3 \times S^3 \times T^4$ with mixed R-R and NS-NS 3form fluxes, which interpolates between the Green-Schwarz coset σ -model in the pure R-R case and a supersymmetric extension of the $SL(2) \times SU(2)$ WZW model in the pure NS-NS case. The quantum spectrum of the pure R-R and mixed flux cases is believed to be described by an integrability-based approach, as in the case of $AdS_5 \times S^5$, whereas in the pure NS-NS case the spectrum is found using CFT methods [104, 55, 232].

The starting point for finding the quantum spectrum of the mixed flux theory is the dispersion relation and S-matrix for elementary world-sheet excitations. While integrability and symmetry put strong constraints on both they do not fully determine the dispersion relation and S-matrix. In this thesis we have addressed this problem by supplementing the integrability and symmetry based results with information coming from semiclassical string solutions. In particular we have considered bosonic string motion on the $\mathbb{R} \times S^3$ subspace of $AdS_3 \times S^3 \times T^4$ in the presence of mixed flux, which is described by a principal chiral model with an additional Wess-Zumino term, with coefficient q, for the NS-NS flux. We have shown how to obtain classical solutions of the mixed flux theory ($q \neq 0$) from solutions of the pure R-R theory (q = 0) by using the fact that the q = 0 and $q \neq 0$ equations of motion in terms of the SU(2) currents are related by the world-sheet coordinate transformation

$$\sigma^{\pm} \to (1 \pm q)\sigma^{\pm}. \tag{8.0.1}$$

We used this method to construct the $q \neq 0$ generalisations of rigid circular strings and the dyonic giant magnon moving on $\mathbb{R} \times S^3$ and the folded string moving on $AdS_3 \times S^1$. We have seen that, since a local 2d action in the presence of a Wess-Zumino is only defined up a total derivative, subtleties arise in the definition of the angular momenta for solutions with non-trivial boundary conditions.

Dyonic giant magnon and dispersion relation

In chapter 5 we used the fact that the dyonic giant magnon solution describes bound states of elementary world-sheet string excitations above the BMN vacuum to determine their exact dispersion relation. For this we found the relation between the energy, the finite angular momentum J_2 and a kink charge corresponding to the separation angle between the endpoints of the dyonic giant magnon. As in the q = 0 case we identified this angle with the world-sheet momentum. In support of this identification we considered the large J_2 limit, in which the theory reduces to the Landau-Lifshitz model. In this model the dyonic giant magnon takes the form of a spin wave soliton and its momentum is well defined and coincides with the separation angle.

The dispersion relation we obtained,

$$\varepsilon = \sqrt{M_{\pm}^2 + 4(1 - q^2)h^2 \sin^2 \frac{p}{2}}, \qquad M_{\pm} = J_2 \pm qhp, \qquad (8.0.2)$$

contains a shift of J_2 by a linear term in the world-sheet momentum p. This is a somewhat surprising feature since one would expect a smooth and periodic dispersion relation in p in the case of an underlying spin-chain interpretation.

We further found evidence for this dispersion relation from the consistency of the associated bound state picture and the quantisation condition for the angular momentum suggesting that this mixed flux dyonic giant magnon indeed represents a bound state of elementary magnon excitations. In chapter 7 we also checked that the semiclassical 1-loop correction to the energy of the giant magnon vanishes as in the case of $AdS_5 \times S^5$ [24, 25].

CHAPTER 8. CONCLUSIONS AND OUTLOOK

More recently the off-shell symmetry algebra has been used to determine the full all-loop S-matrix, i.e. including the massless and mixed-mass sectors, of the mixed flux theory for both $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$ [62, 63, 218, 233]. The ratios of the S-matrix elements take the same form as in the pure R-R case but with q-modified Zhukovsky variables. In the $AdS_3 \times S^3 \times T^4$ case the central charges were found from the Poisson brackets of the supercharges and the dispersion relation for world-sheet excitations, including massless modes, has been suggested to take the form

$$\varepsilon = \sqrt{(m+qhp)^2 + 4(1-q^2)h^2 \sin^2 \frac{p}{2}}, \qquad m = 0, \pm 1.$$
 (8.0.3)

A direct check of the dispersion relation has been performed in [234] where the two-loop dispersion relation was obtained in the near flat-space limit [235] (cf. $AdS_5 \times S^5$ case [236] and the $AdS_2 \times S^2 \times T^6$ cases [237]) and more recently in [238] where the full BMN two-loop dispersion relation was computed for the $AdS_5 \times S^5$, $AdS_3 \times S^3$ and $AdS_2 \times S^2$ cases. For $AdS_5 \times S^5$ and the massive modes in the mixed flux $AdS_3 \times S^3$ case the results match the BMN expansion, i.e. $P \equiv ph = \text{fixed}, h \rightarrow \infty, P \ll 1$, of (8.0.3). However, for massless modes in the mixed flux $AdS_3 \times S^3$ case disagreement arises from diagrams with one massless and two massive modes in the loops and is speculated to be due to some unknown quantum corrections to the central charges or a mismatch of the asymptotic states in the two calculations. This still remains to be resolved.

Another interesting question is the exact dispersion relation for strings on $AdS_2 \times S^2$ backgrounds. It seems that in this case the off-shell symmetry algebra is not sufficient to determine the form of the dispersion relation [239]. Also there is no dyonic giant magnon in this case. However, the BMN two-loop dispersion relation in this case was found to be the same as in the $AdS_3 \times S^3$ case but with the string tension replaced with $h \rightarrow 2h$ [240, 238].

Finally, in order to compute the string spectrum in the $AdS_3 \times S^3$ case with mixed flux using the dispersion relation and S-matrix one needs to further understand the analytic structure of the curve (5.4.19) and if possible identify the uniformising variables, as was done in the case of $AdS_5 \times S^5$ in [31, 29].

In the $\operatorname{AdS}_5 \times \operatorname{S}^5$ case the dyonic giant magnon solution can also be used to study finite-size effects or wrapping interactions in the semiclassical, i.e. $\lambda \gg 1$, limit. These effects arise when considering a finite but large world-sheet cylinder instead of a plane in the decompactification limit and the interaction range of the magnons becomes of the order of the size of the system. In this case it is possible to obtain a finite-size dyonic giant magnon solution which has large but finite global charges (E, J_1) . The energy then receives finite-size corrections, which are exponentially suppressed in the large charges [202, 241].

Recently also in the $AdS_3 \times S^3$ mixed flux case the finite-size corrections to the dispersion relation have been obtained in [242] and they were further confirmed in [95] from the finite-gap construction by using the same resolvent for the $q \neq 0$ finite-size dyonic giant magnon as in the $AdS_5 \times S^5$ case [200, 243]. As in the $AdS_5 \times S^5$ case the finite-size corrections are exponentially suppressed in the large charges.

Folded strings

In chapter 6 we constructed the $q \neq 0$ generalisation of a folded string on $AdS_3 \times S^1$ with angular momenta S and \mathcal{J} on AdS_3 and S^1 . We found that the closed string boundary conditions can be satisfied for the $q \neq 0$ solution once we impose that the angular momenta are quantized. Considering the long string limit we obtained the energy as an expansion in large S in the case of large and small \mathcal{J} . We found that at leading order the effect of q amounts to a rescaling of the string tension

$$\mathbf{h} \to \sqrt{1 - q^2} \,\mathbf{h}.\tag{8.0.4}$$

This is somewhat surprising as it disagrees with the results in [211] where the energy of a spinning folded string was given perturbatively in q and a new leading order term of the form $\ln^2 S$ was found in addition to the standard $\ln S$ behaviour. In order to resolve this problem it would be interesting to better understand the relation between the solution ansatz in [211] and our $q \neq 0$ solution.

In particular our $q \neq 0$ generalisation of the folded string resembles a spiky or helical string. This suggests that one should in fact start with more general pure R-R solutions. For example, in order to obtain a genuine $q \neq 0$ folded string one should start with a q = 0 solution with enough parameters to impose that the $q \neq 0$ winding numbers vanish while the angular momenta stay unfixed (i.e. are not restricted by quantisation conditions).

Recently the class of spinning string solutions in the mixed flux theory was shown to be described by a modified Neumann-Rosochatius system [244, 245] in analogy to the $AdS_5 \times S^5$ case [142, 143]. This class of solutions is essentially related to our $q \neq 0$ generalisation of the folded string through a Lorentz transformation. However, the Lorentz symmetry is fixed in both cases to ensure that the global time coordinate is single valued. Therefore it would be interesting to further investigate the relation between these two types of solutions and to see if our solution ansatz can be related to some type of a Neumann-Rosochatius system.

Dressing phase

The two particle S-matrix for elementary world-sheet excitations is only fixed by the off-shell symmetry algebra up to overall dressing phase factors. In the case of $AdS_3 \times S^3 \times T^4$ there are five such factors¹: two for the scattering of massive excitations (massive-massive sector) and one for the massive-massless, massless-massive and massless-massless sectors respectively.

In [95] proposals for the semiclassical and 1-loop phases in the massive sector of the mixed flux theory have been obtained. For the semiclassical phases it was assumed that the phase for the scattering of same-type excitations (i.e. scattering of left-left or right-right movers) should be given by the $AdS_5 \times S^5$ AFS expression with the Zhukovsky variables replaced by their $q \neq 0$ counterparts. The crossing equations were then solved for the remaining phase describing the scattering of mixed-type (left-right or right-left) excitations. Using the resulting dressing phases in the all-loop Bethe equations it was found that the finite-gap equations are recovered in the semiclassical limit as expected. Also the 1-loop phases were obtained by quantising the algebraic curve around a general solution and matching results for the 1-loop phases were further found from unitarity cut techniques in [96, 97].

¹Once the left \leftrightarrow right symmetry is taken into account.

In chapter 7 we have checked these proposals and, in particular, confirmed the assumption for the semiclassical phase by independently deriving the semiclassical and 1-loop phases from the bound-state S-matrix and its 1-loop correction, as was done in the case of $AdS_5 \times S^5$ in [24, 25]. The bound-state S-matrix describes the scattering of dyonic giant magnons and its 1-loop correction can be understood as describing the scattering of plane waves off dyoinic giant magnons. Both are found from multi soliton dyonic giant magnon solutions, which can be constructed by applying the dressing method to the BMN solution.

By generalising the dressing method to the mixed flux case we were able to obtain multi soliton dyonic giant magnon solutions allowing us to extract the bound-state S-matrix and its 1-loop correction. Taking into account the normalisation factors for the dressing phases, given by the contributions of the BDS factor and its equivalent for mixed-type excitation scattering, we reproduced the proposals for the semiclassical and 1-loop phase. Additionally we determined the form of the mixed flux dyonic giant magnon resolvent in the finite-gap picture from the plane wave scattering data. In terms of the Zhukovsky variables the resolvent was found to be given by the same expression as in the pure R-R case.

In general, already at the semiclassical and 1-loop level, the analytic structure of the dressing phases is significantly more complicated than in the pure R-R case. In particular, in contrast to the S-matrix, the pure R-R phases cannot be generalised to the mixed flux case by replacing the Zhukovsky variables with their mixed flux equivalents.

With the recent progress in incorporating massless modes into the integrability-based framework it would also be interesting to extend the semiclassical methods to the massless and mixed-mass sectors if possible. One important question is whether one can provide a further check for the exact massless dispersion relation, for which the near-BMN 2-loop check gives disagreement, by identifying the analogue of the giant magnon for massless modes. In the pure R-R case of $AdS_3 \times S^3 \times S^3 \times S^1$ there has been some recent progress in identifying classical spinning string solutions for the massless modes [246]. In the mixed flux case there are two approaches one can take:

- (i) One should be able to choose an ansatz living on the appropriate subspace of $AdS_3 \times S^3 \times T^4$ which gives the dispersion relation we expect. However, this might be the giant magnon itself which does not move on T^4 where the massless modes arise. This leads to the question why the giant magnon on $\mathbb{R} \times S^2$ cannot be interpreted as the giant magnon in the massless sector.
- (ii) One might extend the mixed flux dressing method used in this thesis to include motion on T⁴. In order to interpret the resulting solution as the giant magnon for massless modes we would expect non-trivial motion on the torus. This would allow one to construct scattering solutions and address the question of the semiclassical and 1-loop dressing phases in the massless and mixed mass sectors.

Summary

In conclusion we have investigated integrability for strings on two types of backgrounds, which interpolate between integrable limits: curved *p*-brane backgrounds and $AdS_3 \times S^3 \times T^4$ with mixed flux. In the first case we found that already at the classical level string integrability can be ruled out by considering an appropriate truncation of the 2d σ -model to a mechanical system.

In the second case classical integrability is expected to extend to the quantum level and we have, indeed, seen evidence in support of this at the semiclassical level from the dyonic giant magnons and their scattering properties. While there are still many open questions the recent progress in the mixed flux case could lead to the exciting possibility of relating the integrability-based approach to the CFT methods in the pure NS-NS theory. For example, in the sine-Gordon model with an imaginary potential, which is a massless theory, the infinite tower of conserved quantities can be related to the Virasoro algebra [247].

In general it would also be interesting to extend the integrability-based methods to the various classically integrable, but less symmetric, deformations of the $AdS_5 \times S^5$ coset σ -model.

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