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Exact correlators from conformal Ward identities in momentum space and the perturbative TJJ vertex

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Abstract

We present a general study of 3-point functions of conformal field theory in momentum space, following a reconstruction method for tensor correlators, based on the solution of the conformal Ward identities (CWI's), introduced in recent works by Bzowski, McFadden and Skenderis (BMS). We investigate and detail the structure of the CWI's, their non-perturbative solutions and the transition to momentum space, comparing them to perturbation theory by taking QED as an example. We then proceed with an analysis of the TJJ correlator, presenting independent and detailed re-derivations of the conformal equations in the reconstruction method of BMS, originally formulated using a minimal tensor basis in the transverse traceless sector. A careful comparison with a second basis introduced in previous studies shows that this correlator is affected by one anomaly pole in the graviton (T) line, induced by renormalization. The result shows that the origin of the anomaly, in this correlator, should be necessarily attributed to the exchange of a massless effective degree of freedom. Our results are then exemplified in massless QED at one-loop in d-dimensions, expressed in terms of perturbative master integrals. An independent analysis of the Fuchsian character of the solutions, which bypasses the 3K integrals, is also presented. We show that the combination of field theories at one-loop – with a specific field content of degenerate massless scalar and fermions – is sufficient to generate the complete non-perturbative solution, in agreement with a previous study in coordinate space. The result shows that free conformal field theories, in specific dimensions, arrested at one-loop, reproduce the general result for the TJJ. Analytical checks of this correspondence are presented in $d = 3, 4$ and 5 spacetime dimensions. This implies that the generalized 3K integrals of the BMS solution can be expressed in terms of the two single master integrals B_0 and C_0 of 2- and 3-point functions, with significant simplifications.

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1. Introduction

The analysis of multi-point correlation functions in conformal field theory (CFT) is of outmost importance in high energy physics and in string theory, where exact results for lower (2- and 3-)point functions are combined with the operator product expansion (OPE) in order to characterize the structure of correlators of higher orders. This is the key motivation for a bootstrap program in $d = 4$ spacetime dimensions.

The enlarged $SO(2,4)$ symmetry of CFT's – respect to Poincaré invariance – has been essential for establishing the form of some of their correlation functions. For 3-point functions, the solution of the conformal constraints in coordinate space allows to determine such correlators only up to few constants [1,2], which can then be fixed within a specific realization of a theory. In the case of a Lagrangian realization of a given CFT, such constants are expressed in terms of its (massless) field content (number of scalars, vectors, fermions), according to rather simple algebraic relations.

Except for perturbative studies performed at Lagrangian level, such as in the case of the $N = 4$ super Yang–Mills theory, which reach considerably high orders in the gauge coupling expansion, most of these analyses are performed in coordinate space, with no reference to any specific Lagrangian.

There are obvious reasons for this. The first is that the inclusion of the conformal constraints is more straightforward to obtain in coordinate space, compared to momentum space. The second is that the operator product expansion (OPE) in momentum space is difficult to perform, especially for correlators of higher orders (≥ 3), in the Minkowski region. However, there are also some advantages which are typical of a momentum space analysis, and these are related to the availability of dimensional regularization (DR), at least at perturbative level, and to the technology of master integrals, which has allowed to compute large classes of multiloop amplitudes.

Another advantage has to do with the identification of the conformal anomaly [3], which can be automatically extracted in DR (in d spacetime dimensions), being proportional to the $1/(d-4)$ singularity of the corresponding correlators. In coordinate space, instead, the anomaly contributions has to be added by hand by the inclusion of an inhomogeneous local term (i.e. by pinching all of its external coordinates), whose structure has to be inferred indirectly [1].

Finally, a crucial issue concerns the physical character of the anomaly, which does not find any simple particle interpretation in position space, while it is clearly associated to the appearance of an anomaly pole in momentum space [4–6] in an uncontracted anomaly vertex. One finds, by a perturbative one-loop analysis of any anomalous correlator, that the anomaly is always associated with such massless exchanges in the corresponding diagram. It is therefore possible to identify them as effective degrees of freedom induced by the anomaly, present in the 1PI (one-particle irreducible) effective action. The physical significance of such contributions has been stressed in several previous works [4,6,7] along the years. They have recently discussed in condensed matter theory in the context of topological insulators and of Weyl semimetals [8,9].

One of the goals of our work is to compare and extend previous perturbative analysis of the TJJ correlator with more recent ones based on the solution of the conformal Ward identities (CWI's) in momentum space [11–14]. This correlator is the simplest one describing the coupling

of gravity to ordinary matter in QED and it has been investigated in perturbation theory from several directions [15–19].

1.1. Direct Fourier transform and the reconstruction program

In principle, one can move from coordinate space to position space in a CFT by a Fourier transform. This was the approach of [20] for 3-point functions, which can be explicitly worked out by introducing a regulator (ω) for the transform very much alike DR. The regulator serves as an intermediate step since some of the components of the correlators in position space are apparently non-transformable. It has been shown that $1/\omega$ poles generated by the transform *cancel* in all the correlators analyzed, giving a complete expression for these in momentum space. The result is expressed in terms of ordinary and logarithmic master integrals of Feynman type, for which, in the latter case, it is possible to derive recursion relations as for ordinary ones [20]. The advantage of such approach is of being straightforward and algorithmic. It may be essential and probably the only manageable way to re-express the bootstrap program of CFT's in momentum space beyond 3- and 4-point functions, from the original coordinate space analysis. Consistency with the analysis presented in [20] implied rather directly that such logarithmic integrals had to be re-expressed in terms of ordinary Feynman integrals. In fact, it was shown in the same study that the TJJ correlator was entirely reproduced by a free field theory in coordinate space. Our analysis in momentum space is in complete agreement with this former result.

1.2. Reconstruction

An alternative method has been developed more recently, based on the direct solution of the conformal Ward identities in momentum space. The method has been proposed in [14] and [24] for scalar 3-point functions and extensively generalized to tensor correlators in [14].

Several issues related to the renormalization of the solutions of the conformal Ward identities have been investigated in [12,13], adopting the formalism of the 3K integrals (i.e. parametric integrals of 3 Bessel functions). Several analyses in momentum space, for specific applications, have been worked out [10,23], but the generality of the approach is clearly a significant feature of [14], which reconstructs a tensor correlator starting from its transverse/traceless components and using the conservation/trace Ward identities (local terms). The latter are reconstructed from lower point functions.

The result is expressed in terms of two sets of primary and secondary conformal Ward identities (CWI's), the first involving the form factors of the transverse/traceless contributions, which are parametrized on a symmetry basis, the second emerging from CWI's of lower point functions. For 3-point functions, the secondary CWI's involve conservation, trace and special WI's. In all the cases, the reconstructed solutions for 3-point functions can be given in terms of generalized hypergeometrics of type F_4, [20], also known as Appell's hypergeometric function of two variables (F_4), related to 3-K integrals [14].

1.2.1. The anomaly pole of the TJJ

One of the results of our analysis will be to show how such contributions originate from the process of renormalization, taking as an example the case of the TJJ, filling in the intermediate steps of the discussion presented in [22]. We follow the general (BMS) approach introduced in [11] for the solution of the conformal constraints, which we detail in several of its parts, not offered in [11]. It has been compelling to proceed with an independent re-derivation of all the

lengthy equations. The method can be directly generalized to higher point functions and implemented algorithmically, as we are going to show in a separate work. When coming to discuss the momentum space approach in CFT, there are several gaps in the literature, which are of methodological nature and need to be addressed. These concerns the correct form of the differential equations, the treatment of the derivatives of the Dirac δ's induced by momentum conservation, violations of the Leibnitz rule for the special conformal transformations, or the choice of the Lorentz (spin) singlet operator in the action of the conformal group on a specific correlator. These are points that we will address systematically. We will illustrate how to merge the results of the BMS approach on the structure of the minimal set of (4) form factors (the A-basis), solutions of the CWI's for the TJJ correlator, with a basis of 13 ones (the F-basis) defined in previous perturbative studies. We will show how to extract from the F-basis 4 combinations of the 13 and we will verify that they respect the scalar equations identified within the BMS approach.

The use of this second basis is essential in order to prove that the WI's and the renormalization procedure for this correlator, imply that the anomaly can be attributed to the appearance of an anomaly pole in a single tensor structure of nonzero trace.

1.3. Our work

As we have just mentioned, one of the goals of this work, in a first part, is to present a systematic approach to the analysis of the CWI's in momentum space, closing a gap in the literature. The transition to momentum space raises the issue of how to include momentum conservation (i.e. translational invariance in coordinate space) in the presence of the dilatation and the special conformal generators. We will be dealing, in particular, with a rigorous treatment of such contributions which show up after a Fourier transform of the conformal generators to momentum space.

We are going to investigate in detail the role of these contributions relying on the theory of tempered distributions. In particular, the discussion of these terms will be performed using a Gaussian basis which converges – in a distributional sense – to a covariant δ function in $D = 4$ and allows to define a formal calculus for such distributions.

Such contributions do not cancel, but lead to specific forms of the conformal generators in momentum space which are, however, in agreement with those presented in [14,21,24]. We define operational methods for the treatment of the covariant derivatives of δ functions in a consistent way, which may find application also beyond the scope of the current treatment, being quite general.

In a second part we move to discuss scalar and tensor correlators and the solutions of the CWI's. We elaborate, in particular, one the apparent violation of the Leibnitz rule for the special conformal (SC) generator (K^κ), which emerges whenever we impose momentum conservation and eliminate one of the momenta, and the symmetric action of this operator, at an intermediate stage, is not evident. In position space this corresponds to choosing one coordinate to be zero, and treating the corresponding operator in a given correlation function, as spin singlet. The derivation of the constraints on the form factors is performed, in our case, by using Lorentz Ward identities, on which we elaborate in detail, confirming the results of [14]. We show how different choices for the singlet operator leads to an equivalent set of conformal equations. We then illustrate how to derive the solutions of the various form factors using some properties of the hypergeometric equations, bypassing the 3K integrals, showing that the Fuchsian indices of all the equations remain the same for all the solutions.

Such analysis is followed by a perturbative study of TJJ correlator in the transverse traceless basis both in QED and in scalar QED, deriving the associated anomalous conformal Ward identities from this perspective.

In the final part of our work we show how the perturbative solutions for the A_i, which are given in an appendix, reproduce the exact BMS result in a simplified way. We use the cases of $d = 3$ and $d = 5$ to show the exact correspondence between the two. This correspondence is studied by fixing an appropriate normalization of the photon two-point functions, on which we elaborate. This shows that the choice of different perturbative sectors (scalar, fermion) in both cases are sufficient to reproduce the entire nonperturbative result. This implies that only arbitrary constant in the nonperturbative solution, expressed in terms of the 3K integrals, has to simplify and be expressible in terms of simple integrals B_0 and C_0, the scalar 2- and 3-point functions. In our conclusions we briefly comment on the possible origin of such simplifications.

2. Special conformal Ward identities in the operatorial approach

In this section, to make our treatment self-contained, we briefly illustrate the operatorial derivation of the CWI's for correlators involving 3-point functions of stress energy tensors.

An infinitesimal transformation

$$x^\mu(x) \to x'^\mu(x) = x^\mu + v^\mu(x) \tag{2.1}$$

is classified as an isometry if it leaves the metric $g_{\mu\nu}(x)$ invariant in form. If we denote with $g'_{\mu\nu}(x')$ the new metric in the coordinate system x', then an isometry is such that

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x'). \tag{2.2}$$

This condition can be inserted into the ordinary covariant transformation rule for $g_{\mu\nu}(x)$ to give

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) = g_{\mu\nu}(x') \tag{2.3}$$

from which one derives the Killing equation for the metric

$$v^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\sigma}\partial_\nu v^\sigma + g_{\sigma\nu}\partial_\mu v^\sigma = 0. \tag{2.4}$$

For a conformal transformation the metric condition (2.2) is replaced by the condition

$$g'_{\mu\nu}(x') = \Omega^{-2} g_{\mu\nu}(x') \tag{2.5}$$

generating the conformal Killing equation (with $\Omega(x) = 1 - \sigma(x)$)

$$v^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\sigma}\partial_\nu v^\sigma + g_{\sigma\nu}\partial_\mu v^\sigma = 2\sigma g_{\mu\nu}. \tag{2.6}$$

In the flat spacetime limit this becomes

$$\partial_\mu v_\nu + \partial_\nu v_\mu = 2\sigma\, \eta_{\mu\nu}, \qquad \sigma = \frac{1}{d}\partial \cdot v. \tag{2.7}$$

From now on we switch to the Euclidean case, neglecting the index positions. Using the fact that every conformal transformation can be written as a local rotation matrix of the form

$$R^\mu_\alpha = \Omega \frac{\partial x'^\mu}{\partial x^\alpha} \tag{2.8}$$

we can first expand generically R around the identity as

$$R = \mathbf{1} + [\epsilon] + \dots \tag{2.9}$$

with an antisymmetric matrix $[\epsilon]$, which we can re-express in terms of antisymmetric parameters $(\tau_{\rho\sigma})$ and $1/2\,d\,(d-1)$ generators $\Sigma_{\rho\sigma}$ of $SO(d)$ as

$$\begin{aligned} [\epsilon]_{\mu\alpha} &= \frac{1}{2}\tau_{\rho\sigma}\left(\Sigma_{\rho\sigma}\right)_{\mu\alpha} \\ \left(\Sigma_{\rho\sigma}\right)_{\mu\alpha} &= \delta_{\rho\mu}\delta_{\sigma\alpha} - \delta_{\rho\alpha}\delta_{\sigma\mu} \end{aligned} \tag{2.10}$$

from which, using also (2.8) we derive a constraint between the parameters of the conformal transformation (v) and the parameters $\tau_{\mu\alpha}$ of R

$$R_{\mu\alpha} = \delta_{\mu\alpha} + \tau_{\mu\alpha} = \delta_{\mu\alpha} + \frac{1}{2}\partial_{[\alpha}v_{\mu]} \tag{2.11}$$

with $\partial_{[\alpha}v_{\mu]} \equiv \partial_\alpha v_\mu - \partial_\mu v_\alpha$.

Denoting with Δ_A the scaling dimensions of a vector field $A_\mu(x)'$, its variation under a conformal transformation can be expressed via R in the form

$$\begin{aligned} A'^{\,\mu}(x') &= \Omega^{\Delta_A} R_{\mu\alpha} A^\alpha(x) \\ &= (1 - \sigma + \dots)^{\Delta_A}(\delta_{\mu\alpha} + \frac{1}{2}\partial_{[\alpha}v_{\mu]} + \dots)A^\alpha(x) \end{aligned} \tag{2.12}$$

from which one can easily deduce that

$$\delta A^\mu(x) \equiv A'^{\,\mu}(x) - A^\mu(x) = -(v\cdot\partial + \Delta_A\sigma)A^\mu(x) + \frac{1}{2}\partial_{[\alpha}v_{\mu]}A^\alpha(x), \tag{2.13}$$

which is defined to be the Lie derivative of A^μ in the v direction, modulo a sign

$$L_v A^\mu(x) \equiv -\delta A^\mu(x). \tag{2.14}$$

As an example, in the case of a generic rank-2 tensor field $(\phi^{I\,K})$ of scaling dimension Δ_ϕ, transforming according to a representation $D^I_J(R)$ of the rotation group $SO(d)$, (2.12) takes the form

$$\phi'^{\,I\,K}(x') = \Omega^{\Delta_\phi} D^I_{I'}(R) D^K_{K'}(R)\phi^{I'\,K'}(x). \tag{2.15}$$

In the case of the stress energy tensor ($D(R) = R$), with scaling (mass) dimension Δ_T ($\Delta_T = d$) the analogue of (2.12) is

$$\begin{aligned} T'^{\,\mu\nu}(x') &= \Omega^{\Delta_T} R^\mu_\alpha R^\nu_\beta T^{\alpha\beta}(x) \\ &= (1 - \Delta_T\sigma + \dots)(\delta_{\mu\alpha} + \frac{1}{2}\partial_{[\alpha}v_{\mu]} + \dots)(\delta_{\mu\alpha} + \frac{1}{2}\partial_{[\alpha}v_{\mu]} + \dots)\,T^{\alpha\beta}(x) \end{aligned} \tag{2.16}$$

where $\partial_{[\alpha}v_{\mu]} \equiv \partial_\alpha v_\mu - \partial_\mu v_\alpha$. One gets

$$\delta T^{\mu\nu}(x) = -\Delta_T\,\sigma\,T^{\mu\nu} - v\cdot\partial\,T^{\mu\nu}(x) + \frac{1}{2}\partial_{[\alpha}v_{\mu]}\,T^{\alpha\nu} + \frac{1}{2}\partial_{[\nu}v_{\alpha]}T^{\mu\alpha}. \tag{2.17}$$

For a special conformal transformation (SCT) one chooses

$$v_\mu(x) = b_\mu x^2 - 2x_\mu b\cdot x \tag{2.18}$$

with a generic parameter b_μ and $\sigma = -2b\cdot x$ (from 2.7) to obtain

$$\delta T^{\mu\nu}(x) = -(b^\alpha x^2 - 2x^\alpha b\cdot x)\,\partial_\alpha T^{\mu\nu}(x) - \Delta_T \sigma T^{\mu\nu}(x) + 2(b_\mu x_\alpha - b_\alpha x_\mu)T^{\alpha\nu} + 2(b_\nu x_\alpha - b_\alpha x_\nu)\,T^{\mu\alpha}(x). \tag{2.19}$$

It is sufficient to differentiate this expression respect to b_κ in order to derive the form of the SCT K^κ on T in its finite form

$$\begin{aligned}\mathcal{K}^\kappa T^{\mu\nu}(x) &\equiv \delta_\kappa T^{\mu\nu}(x) = \frac{\partial}{\partial b^\kappa}(\delta T^{\mu\nu}) \\ &= -(x^2\partial_\kappa - 2x_\kappa x\cdot\partial)T^{\mu\nu}(x) + 2\Delta_T x_\kappa T^{\mu\nu}(x) + 2(\delta_{\mu\kappa}x_\alpha - \delta_{\alpha\kappa}x_\mu)T^{\alpha\nu}(x) \\ &\quad +2(\delta_{\kappa\nu}x_\alpha - \delta_{\alpha\kappa}x_\nu)T^{\mu\alpha}.\end{aligned} \tag{2.20}$$

The approach can be generalized to correlators built out of several operators. In the case of a TJJ correlator,

$$\Gamma^{\mu\nu\alpha\beta}(x_1,x_2,x_3) = \langle T^{\mu\nu}(x_1)J^\alpha(x_2)J^\beta(x_3)\rangle \tag{2.21}$$

with a vector current of dimension Δ_J, the CWI's take the explicit form

$$\begin{aligned}\mathcal{K}^\kappa\Gamma^{\mu\nu\alpha\beta}(x_1,x_2,x_3) &= \sum_{i=1}^{3} K_{i\,scalar}^{\kappa}(x_i)\Gamma^{\mu\nu\alpha\beta}(x_1,x_2,x_3) \\ &\quad +2\left(\delta^{\mu\kappa}x_{1\rho} - \delta^\kappa_\rho x_1^\mu\right)\Gamma^{\rho\nu\alpha\beta} + 2\left(\delta^{\nu\kappa}x_{1\rho} - \delta^\kappa_\rho x_1^\nu\right)\Gamma^{\mu\rho\alpha\beta} \\ &\quad 2\left(\delta^{\alpha\kappa}x_{2\rho} - \delta^\kappa_\rho x_2^\alpha\right)\Gamma^{\mu\nu\rho\beta} + 2\left(\delta^{\beta\kappa}x_{3\rho} - \delta^\kappa_\rho x_3^\beta\right)\Gamma^{\mu\nu\alpha\rho} = 0,\end{aligned} \tag{2.22}$$

where

$$\mathcal{K}_{i\,scalar}^{\kappa} = -x_i^2\frac{\partial}{\partial x_\kappa} + 2x_i^\kappa x_i^\tau\frac{\partial}{\partial x_i^\tau} + 2\Delta_i x_i^\kappa \tag{2.23}$$

is the scalar part of the special conformal operator acting on the i_{th} coordinate and $\Delta_i \equiv (\Delta_T, \Delta_J, \Delta_J)$ are the scaling dimensions of the operators in the correlation function.

2.1. Constraints from translational symmetry and the Leibniz rule

One of the main issues, when moving to momentum space, is to include the constraint from translational symmetry on a tensor correlator. The inclusion of this constraint at the beginning, for a tensor 3-point function of the form

$$\langle H_1^{\mu_1\nu_1}(x_1)H_2^{\mu_2\nu_2}(x_2)H_3^{\mu_3\nu_3}(x_3)\rangle, \tag{2.24}$$

with each of the H_i's of scaling dimensions Δ_i^H, reduces it to the form

$$\langle H_1^{\mu_1\nu_1}(x_{13})H_2^{\mu_2\nu_2}(x_{23})H_3^{\mu_3\nu_3}(0)\rangle, \tag{2.25}$$

and the action of K^κ, the special conformal generator, on (2.24) and (2.25) will obviously change. The transition to momentum space in the two cases above takes to two different forms of the special CWI. The first form will be symmetric in momentum space, but at the cost of generating derivatives of the delta-function, which enforce conservation of the total momentum, while the second one will be asymmetric, treating one of the momenta as dependent from the other two. The final result for the scalar equations of the corresponding form factors will obviously be symmetric respect the three momenta.

In particular, the Lorentz (spin) generator, in the case of (2.25), will act only the indices of H_1 and H_2, but not on those of H_3, although the differentiation respect to the 4-momentum p_3 will be performed implicitly by a chain rule in this second case, once we move to momentum space.

We are going to illustrate this point in detail.

Identifying $K_i^\kappa H_i^{\mu\nu}$ with the expression (2.20) (with $\Delta_T \to \Delta_i^H$), then the action of the special conformal transformation on (2.24) will take the forms

$$\begin{aligned} K^\kappa \langle H_1^{\mu_1\nu_1}(x_1) H_2^{\mu_2\nu_2}(x_2) H_3^{\mu_3\nu_3}(x_3)\rangle &= \sum_{i=1}^{3} K_i^\kappa \langle H_1^{\mu_1\nu_1}(x_1) H_2^{\mu_2\nu_2}(x_2) H_3^{\mu_3\nu_3}(x_3)\rangle \\ &= e^{-iPx_3} K^{\kappa\prime} \langle H_1^{\mu_1\nu_1}(x_{13}) H_2^{\mu_2\nu_2}(x_{23}) H_3^{\mu_3\nu_3}(0)\rangle \end{aligned} \tag{2.26}$$

with P the total translation operator ($P = P_1 + P_2 + P_3$) and

$$K^{\kappa\prime} = e^{iPx_3} K^\kappa e^{-iPx_3}. \tag{2.27}$$

Using the relations of the conformal algebra

$$[K^\kappa, P^\nu] = 2i(\eta^{\kappa\nu} D + M^{\kappa\nu}), \qquad [D, P^\mu] = -iP^\mu$$

$$[M^{\kappa\nu}, P^\mu] = -i(\eta^{\kappa\mu} P^\nu - \eta^{\mu\nu} P^\kappa),$$

and expanding (2.27) we obtain the relation

$$\begin{aligned} K^{\kappa\prime} &= K^\kappa + i x_{3\mu}[P^\mu, K^\kappa] + \frac{i^2}{2} x_{3\mu} x_{3\nu} [P^\mu, [P^\nu, K^\kappa]] + \dots \\ &= K^\kappa + 2x_3^\kappa D + 2x_{3\mu} M^{\kappa\mu} - 2x_3^\kappa x_{3\mu} P^\mu + x_3^2 P^\kappa, \end{aligned} \tag{2.28}$$

since the commutator of higher order vanish.

The explicit form of the operators (dilatation, Lorentz and special conformal) D, $M^{\mu\nu}$, K^κ is $K^\kappa = K_1^\kappa + K_2^\kappa + K_3^\kappa$, $D = D_1 + D_2 + D_3$, $M^{\mu\nu} = M_1^{\mu\nu} + M_2^{\mu\nu} + M_3^{\mu\nu}$ and

$$M_i^{\mu\nu} = L_i^{\mu\nu} + \Sigma_i^{\mu\nu},\ L_l = i\left(x_l^\mu \frac{\partial}{\partial x_l^\nu} - x_l^\nu \frac{\partial}{\partial x_l^\mu}\right), \tag{2.29}$$

split into angular momentum (L) and spin (Σ). We illustrate this crucial point in some detail, since it shows how the action of the Lorenz generators on the field at x_3 vanishes. We get (using $\hat{p}_l^\kappa \equiv i\partial/\partial x_l^\kappa$)

$$\begin{aligned} K^{\kappa\prime} = {} & x_1^2 \hat{p}_1^\kappa - 2x_1^\kappa x_{1\nu} \hat{p}_1^\nu - 2i x_1^\kappa \Delta_1 - 2x_{1\nu} \Sigma_1^{\kappa\nu} \\ & + x_2^2 \hat{p}_2^\kappa - 2x_2^\kappa x_{2\nu} \hat{p}_2^\nu - 2i x_2^\kappa \Delta_2 - 2x_{2\nu} \Sigma_2^{\kappa\nu} \\ & + x_3^2 \hat{p}_3^\kappa - 2x_3^\kappa x_{3\nu} \hat{p}_3^\nu - 2i x_3^\kappa \Delta_3 - 2x_{3\nu} \Sigma_3^{\kappa\nu} \\ & + 2i x_3^\kappa (\Delta_1 + \Delta_2 + \Delta_3) + 2x_3^\kappa (x_{1\nu} \hat{p}_1^\nu + x_{2\nu} \hat{p}_2^\nu + x_{3\nu} \hat{p}_3^\nu) \\ & + 2x_{3\nu} (\Sigma_1^{\kappa\nu} + \Sigma_2^{\kappa\nu} + \Sigma_3^{\kappa\nu}) + 2x_{3\nu} (x_1^\kappa \hat{p}_1^\nu - x_1^\nu \hat{p}_1^\kappa + x_2^\kappa \hat{p}_2^\nu - x_2^\nu \hat{p}_2^\kappa + x_3^\kappa \hat{p}_3^\nu - x_3^\nu \hat{p}_3^\kappa) \\ & - 2x_3^\kappa x_{3\mu} (\hat{p}_1^\mu + \hat{p}_2^\mu + \hat{p}_3^\mu) + x_3^2 (\hat{p}_1^\kappa + \hat{p}_2^\kappa + \hat{p}_3^\kappa) \end{aligned} \tag{2.30}$$

which shows the cancellation of the contribution from the generator M_3 since

$$K^{\kappa\prime} = (x_1 - x_3)^2 \hat{p}_1^{\kappa} - 2(x_1^{\kappa} - x_3^{\kappa})(x_{1\nu} - x_{3\nu})\hat{p}_1{}^{\nu} - 2i(x_1^{\kappa} - x_3^{\kappa})\Delta_1 - 2(x_{1\nu} - x_{3\nu})\Sigma_1^{\kappa\nu} + (x_2 - x_3)^2 \hat{p}_2^{\kappa} - 2(x_2^{\kappa} - x_3^{\kappa})(x_{2\nu} - x_{3\nu})\hat{p}_2^{\nu} - 2i(x_2^{\kappa} - x_3^{\kappa})\Delta_2 - 2(x_{2\nu} - x_{3\nu})\Sigma_2^{\kappa\nu}. \tag{2.31}$$

Notice that both Σ_1 and Σ_2 denote the two spin matrices, which act only on H_1 and H_2. In other words, in coordinate space the choice $x_3 = 0$ implies that H_3 behaves as a Lorentz singlet respect to the spin part. The result can be rewritten in the compact form

$$K^{\kappa\prime} = K_{13}^{\kappa} + K_{23}^{\kappa}, \tag{2.32}$$

where the action of K_{13}^{κ} on a rank-2 tensor $H^{\mu\nu}$, for instance, is given by

$$K^{\kappa} H^{\mu\nu}(x_{13}) = K^{\kappa}_{scalar}(x_{13}) H^{\mu\nu} + 2\left(\delta^{\mu\kappa} x_{13\rho} - \delta^{\kappa}_{\rho} x_{13}^{\mu}\right) H^{\rho\nu} + 2\left(\delta^{\nu\kappa} x_{13\rho} - \delta^{\kappa}_{\rho} x_{13}^{\nu}\right) H^{\mu\rho}, \tag{2.33}$$

with $K_{scalar}(x_{13})$ being given as in (2.23) with $x_i \to x_{13}$ and $\Delta_i \to \Delta_H$, and we obtain the equation

$$\begin{aligned} 0 &= K^{\kappa} \langle H_1^{\mu_1\nu_1}(x_1) H_2^{\mu_2\nu_2}(x_2) H_3^{\mu_3\nu_3}(x_3)\rangle \\ &= e^{-iPx_3}\left(K_{13}^{\kappa} + K_{23}^{\kappa}\right) \langle H_1^{\mu_1\nu_1}(x_{13}) H_2^{\mu_2\nu_2}(x_{23}) H_3^{\mu_3\nu_3}(0)\rangle. \end{aligned} \tag{2.34}$$

Notice that the solution of this equation can be obtained by solving the reduced equation

$$(K_{13}^{\kappa} + K_{23}^{\kappa}) \langle H_1^{\mu_1\nu_1}(x_{13}) H_2^{\mu_2\nu_2}(x_{23}) H_3^{\mu_3\nu_3}(0)\rangle = 0 \tag{2.35}$$

which is equivalent to finding the solution of (2.34) with $x_3 = 0$ and acting afterwards with the translation operator e^{-iPx_3} to restore the full dependence on the third coordinate. Setting

$$\chi^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(x_{12}, x_{13}) \equiv \langle H_1^{\mu_1\nu_1}(x_{13}) H_2^{\mu_2\nu_2}(x_{23}) H_3^{\mu_3\nu_3}(0)\rangle, \tag{2.36}$$

anticipating the discussion that will be presented for these equations in momentum space, the special CWI then can be cast into the form

$$\int d^4p_1\, d^4p_2 e^{-i(p_1 x_{13} + p_2 x_{23})} (K_{p_1}^{\kappa} + K_{p_2}^{\kappa}) \chi(p_1, p_2) = 0 \tag{2.37}$$

giving

$$(K_{p_1}^{\kappa} + K_{p_2}^{\kappa}) \chi(p_1, p_2) = 0. \tag{2.38}$$

Notice that the previous form of $\chi(p_1, p_2)$, which is a function of the independent momenta p_1 and p_2, conjugate to x_{12} and x_{13}, is the final form of the function, having re-expressed p_3 in terms of p_1 and p_2. In a direct explicit computation, one has to act with the transforms of $K^{\kappa}(x_{12})$ and $K^{\kappa}(x_{13})$, that we denote as $K^{\kappa}(p_1)$ and $K^{\kappa}(p_2)$, on the transform of the initial correlator

$$\begin{aligned} &(K^{\kappa}(p_1) + K^{\kappa}(p_2)) \langle H_1^{\mu_1\nu_1}(p_1) H_2^{\mu_2\nu_2}(p_2) H_3(\bar{p}_3)\rangle = \\ &= \int d^d x_{13} d^d x_{23}\, e^{-ip_1\cdot x_{12} - ip_2\cdot x_{23}} (K^{\kappa}(x_{12}) + K^{\kappa}(x_{13})) \\ &\quad \times \langle H_1^{\mu_1\nu_1}(x_{13}) H_2^{\mu_2\nu_2}(x_{23}) H_3^{\mu_3\nu_3}(0)\rangle \end{aligned} \tag{2.39}$$

with $p_3 \to \bar{p}_3 = -p_1 - p_2$ and the Leibniz rule is violated. The symmetry respect to the external invariants (p_1^2, p_2^2, p_3^2) of the conformal generator is only reobtained at the end, after applying the chain rule for the differentiation of p_3 respect to the two independent momenta.

The final result is that in momentum space we can treat $H_3(\bar{p}_3)$ as a single particle operator, in the sense that the differentials in p_1 and p_2 will act separately on H_1 and H_2, but also on H_3 implicitly, via a chain rule.

At the same time, as clear from (2.31), the spin rotation matrix $\Sigma^{\mu\nu}$ contained in the Lorentz generator $M^{\mu\nu}$ will act only on H_1 and H_2, treating H_3 as a Lorentz (spin) singlet. We will present on the sections below complete worked out examples of this action. Since we are free to set any of the 3 coordinates to zero, the intermediate steps of the computations of the CWI's will be completely different, and the choice of the Lorentz singlet operator can be dictated by convenience.

The choice of the point x which will be set to zero (e.g. $x_3 = 0$) is obviously arbitrary, but preferably should be suggested by the symmetry of the correlator. For instance, for correlators such as $\langle T(x_1)T(x_2)T(x_3)\rangle$ and $\langle T(x_1)T(x_2)O(x_3)\rangle$ setting $x_3 = 0$ and removing momentum p_3 in terms of p_1 and p_2 is the natural choice. In the $\langle T(x_1)J(x_2)J(x_3)\rangle$ case it is convenient to set $x_1 = 0$ and re-express the momentum p_1 in terms of p_2 and p_3.

3. The conformal generators in momentum space

In this section we discuss two formulations of the dilatation and SCT's, with the goal of clarifying the treatment of the constraints coming from the conservation of the total momentum in a generic correlator. We will be using some condensed notations in order to shorten the expressions of the transforms in momentum space. We will try to avoid the proliferation of indices, whenever necessary, with the conventions

$$\Phi(\underline{x}) \equiv \langle \phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)\rangle \qquad e^{i\underline{px}} \equiv e^{i(p_1x_1+p_2x_2+\dots p_nx_n)}$$
$$\underline{dp} \equiv dp_1dp_2\dots dp_n \qquad \Phi(\underline{p}) \equiv \Phi(p_1, p_2, \dots, p_n). \tag{3.1}$$

It will also be useful to introduce the total momentum $P = \sum_{j=1}^{n} p_j$.

The momentum constraint is enforced via a delta function $\delta(P)$ under integration. For instance, translational invariance of $\Phi(\underline{x})$ gives

$$\Phi(\underline{x}) = \int \underline{dp}\,\delta(P)\, e^{i\underline{px}}\,\Phi(p_1, p_2, p_3). \tag{3.2}$$

In general, for an n-point function $\Phi(x_1, x_2, \dots, x_n) = \langle \phi_1(x_1)\phi_2(x_2)...\phi_n(x_n)\rangle$, the condition of translation invariance

$$\langle \phi_1(x_1)\phi_2(x_2), \dots, \phi_n(x_n)\rangle = \langle \phi_1(x_1 + a)\phi_2(x_2 + a)\dots\phi_n(x_n + a)\rangle \tag{3.3}$$

generates the expression in momentum space of the form (3.2), from which we can remove one of the momenta, conventionally the last one, p_n, which is replaced by its "on shell" version $\overline{p}_n = -(p_1 + p_2 + \dots p_{n-1})$

$$\Phi(x_1, x_2, \dots, x_n) = \int dp_1dp_2...dp_{n-1}e^{i(p_1x_1+p_2x_2+\dots p_{n-1}x_{n-1}+\overline{p}_nx_n)}\Phi(p_1, p_2, \dots, \overline{p}_n). \tag{3.4}$$

We start by considering the dilatation WI.

The condition of scale covariance for the fields ϕ_i of scale dimensions Δ_i (in mass units)

$$\Phi(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^{-\Delta}\Phi(x_1, x_2, \dots, x_n), \qquad \Delta = \Delta_1 + \Delta_2 + \dots \Delta_n \tag{3.5}$$

after setting $\lambda = 1 + \epsilon$ and Taylor expanding up to $O(\epsilon)$ gives the scaling relation

$$(D_n + \Delta)\Phi \equiv \sum_{j=1}^{n}\left(x_j^\alpha \frac{\partial}{\partial x_j^\alpha} + \Delta_j\right)\Phi(x_1, x_2, \dots, x_n) = 0, \tag{3.6}$$

with

$$D_n = \sum_{j=1}^{n} x_j^\alpha \frac{\partial}{\partial x_j^\alpha}. \tag{3.7}$$

The corresponding equation in momentum space can be obtained either by a Fourier transform of (3.6), which can give either symmetric or asymmetric expressions of the equations in the respective momenta p_i or, more simply, exploiting directly (3.5). In the latter case, using the translational invariance of the correlator under the integral, by removing the δ-function constraint, one obtains

$$\begin{aligned}
&\Phi(\lambda x_1, \lambda x_2, \dots, \lambda x_n)\\
&\quad = \int d^d p_1 d^d p_2 \dots d^d p_{n-1} e^{i\lambda(p_1x_1 + p_2x_2 + \dots p_{n-1}x_{n-1} + \overline{p}_n x_n)}\Phi(p_1, p_2, \dots, \overline{p}_n)\\
&\quad = \lambda^{-\Delta}\int d^d p_1 d^d p_2 \dots d^d p_{n-1} e^{i(p_1x_1 + p_2x_2 + \dots p_{n-1}x_{n-1} + \overline{p}_n x_n)}\Phi(p_1, p_2, \dots, \overline{p}_n).
\end{aligned} \tag{3.8}$$

It is simply a matter of performing the change of variables $p_i = p_i'/\lambda$ on the rhs of the equation above (first line) with $dp_1 \dots dp_{n-1} = (1/\lambda)^{d(n-1)} d^d p_1' \dots d^d p_{n-1}'$ to derive the relation

$$\frac{1}{\lambda^{d(n-1)}}\Phi\left(\frac{p_1}{\lambda}, \frac{p_2}{\lambda}, \dots, \frac{\overline{p}_n}{\lambda}\right) = \lambda^{-\Delta}\Phi(p_1, p_2, \dots, \overline{p}_n). \tag{3.9}$$

Setting $\lambda = 1/s$ this generates the condition

$$s^{(n-1)d-\Delta}\Phi(sp_1, sp_2, \dots, s\overline{p}_n) = \Phi(p_1, p_2, \dots, \overline{p}_n) \tag{3.10}$$

and with $s \sim 1 + \epsilon$, expanding at $O(\epsilon)$ we generate the equation

$$\left[\sum_{j=1}^{n}\Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\right]\Phi(p_1, p_2, \dots, \overline{p}_n) = 0. \tag{3.11}$$

It is straightforward to reobtain the same equation from the direct Fourier transform of (3.6) if we use translational invariance since

$$\begin{aligned}
&(D_n + \Delta)\Phi(x_1, \dots, x_n) =\\
&\quad = (D_n + \Delta)\int d^d p_1 \dots d^d p_n \delta(P) e^{ip_1x_1 + \dots p_n x_n}\Phi(p_1, \dots, p_n)\\
&\quad = \left(\sum_{j=1}^{n}\Delta_j + \sum_{i=1}^{n-1} x_{i\,n}\frac{\partial}{\partial x_{in}}\right)\int d^d p_1 \dots d^d p_{n-1} e^{ip_1x_{1n} + \dots p_{n-1}x_{n-1\,n}}\\
&\qquad \times \Phi(p_1, \dots, p_{n-1}, \bar{p}_n)
\end{aligned}$$

$$= \int d^d p_1 \dots d^d p_{n-1} \left(\sum_{j=1}^{n} \Delta_j + \sum_{j=1}^{n-1} p_j \frac{\partial}{\partial p_j} \right) e^{i p_1 x_{1n} + \dots p_{n-1} x_{n-1\,n}} \times \Phi(p_1, \dots, p_{n-1}, \bar{p}_n). \tag{3.12}$$

At this point we perform a partial integration $n-1$ times, moving the derivatives from the exponential to the correlator Φ to reobtain (3.11)

$$\begin{aligned} 0 &= (D_n + \Delta)\Phi(x_1, \dots, x_n) \\ &= \int d^d p_1 \dots d^d p_{n-1} \left(\sum_{j=1}^{n} \Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \right) \\ &\quad \times \Phi(p_1, \dots, p_{n-1}, \bar{p}_n)\, e^{i p_1 x_{1n} + \dots i p_{n-1} x_{n-1\,n}}. \end{aligned} \tag{3.13}$$

A rigorous way to reobtain this result is to consider directly the conformal algebra for the dilatation operator $D = (i D_n + \Delta)$ and use the commutation relations. For our purpose we can consider a realization of $\Phi(x_1, \dots x_n)$ via some operators $O_j(x_j)$

$$\begin{aligned} D \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle &\equiv \sum_{j=1}^{n} D(x_j) \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle \\ &= \left[\sum_{j=1}^{n} \Delta_j + \sum_{j=1}^{n} x_j^\alpha \frac{\partial}{\partial x_j^\alpha} \right] \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle . \end{aligned} \tag{3.14}$$

This n-point function is translationally invariant, so that we can shift the fields using the translation operator $\exp(i P \cdot x_n)$, with P the total translation operator $P = \sum_{j=1}^{n} P_j$

$$D \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = e^{-i x_n \cdot P} D' \langle \mathcal{O}_1(x_1 - x_n) \dots \mathcal{O}_n(x_{n-1} - x_n) \mathcal{O}_n(0) \rangle = 0 \tag{3.15}$$

where

$$D' = e^{i x_n^\mu \cdot P_\mu} D\, e^{-i x_n^\mu \cdot P_\mu} = \sum_{k=0}^{\infty} \frac{i^k}{k!} x_n^{\nu_1} \dots x_n^{\nu_k} [P_{\nu_1}, [\dots [P_{\nu_k}, D] \dots]]. \tag{3.16}$$

Using the commutation relations of the conformal algebra, there are at most two non-vanishing terms in this sum. Evaluating the finite multiple commutators we get

$$D' = D + i x_n^\nu [P_\nu, D] = D - x_n^\nu P_\nu \tag{3.17}$$

and explicitly

$$\begin{aligned} D' &= \sum_{j=1}^{n} \Big[\Delta_j + x_j^\nu (P_j)_\nu \Big] - x_n^\nu \sum_{j=1}^{n} (P_j)_\nu \\ &= \sum_{j=1}^{n-1} (x_j - x_n)^\nu (P_j)_\nu + \sum_{j=1}^{n} \Delta_j \;=\; \sum_{j=1}^{n-1} (x_j - x_n)^\nu \frac{\partial}{\partial x_{j\,\nu}} + \sum_{j=1}^{n} \Delta_j . \end{aligned} \tag{3.18}$$

Notice that the solution of (3.15) can be obtained by solving the reduced equation

$$D' \langle \mathcal{O}_1(x_1 - x_n) \dots \mathcal{O}_n(x_{n-1} - x_n) \mathcal{O}_n(0) \rangle = 0 \tag{3.19}$$

and then acting with a total translation. In momentum space this relation generates the Ward identity

$$\int \prod_{j=1}^{n} d^d x_j \; e^{ix_1\cdot p_1+\ldots ip_n\cdot x_n} \; D' \langle \mathcal{O}_1(x_1-x_n)\ldots\mathcal{O}_n(x_{n-1}-x_n)\,\mathcal{O}_n(0)\rangle = 0, \tag{3.20}$$

that is

$$\int d^d p_1 \ldots d^d p_n \delta^{(d)} \left(\sum_{j=1}^{n} p_j\right) \left[\sum_{j=1}^{n} \Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\right] \times \langle \mathcal{O}_1(p_1)\ldots\mathcal{O}_n(p_{n-1})\,\mathcal{O}_n(\bar{p}_n)\rangle = 0, \tag{3.21}$$

where $\bar{p}_n = -\sum_{j=1}^{n-1} p_j$.

If we decided to work with symmetric expressions of the transform, the approach would be more cumbersome since it would involve δ' (derivative) terms in the integrand. We are going to discuss this second approach both for the dilatation and for the special conformal transformations. It requires a brief digression on the use of some relations for the covariant δ-functions which we are going to formulate below and that will be essential in order to clarify the correct way to treat such contributions.

3.1. Delta calculus and symmetric Gaussians

It is possible to derive a formal calculus for the derivatives of $\delta^d(P)$ using as defining conditions that, for a generic function $f(p)$ which is regular at $P^\mu = 0$ the rules

$$\int d^d P\, \partial_\alpha \delta^d(P) f(P) = -\partial_\alpha f(0) \tag{3.22}$$

and

$$\int d^d P\, \partial_\beta \partial_\alpha \delta^d(P) f(P) = \partial_\beta \partial_\alpha f(0) \tag{3.23}$$

hold. For instance one easily obtains formally

$$\partial_\alpha \delta^d(P) \equiv \frac{\partial}{\partial P^\alpha}\delta^d(P) = -\frac{d}{P^2}\delta^d(P) P^\alpha, \tag{3.24}$$

which can be checked using the following rule for symmetric integration in (3.22)

$$\int d^d P \frac{\delta^d(P)}{P^2} P^\alpha P^\beta f(0) = \frac{1}{d}\eta^{\alpha\beta}, \tag{3.25}$$

where $f(0)$ is a constant. Similarly, one can show that

$$\int d^d P\, P^\alpha \frac{\delta^d(P)}{P^2} f(0) = 0, \tag{3.26}$$

which is consistent with the fact that the integral

$$\int d^d P\, \partial_\alpha \delta^d(P) = 0, \tag{3.27}$$

has a zero boundary value. Notice that (3.26) can be extended to the more general form

$$\int d^d P\, P^\alpha \frac{\delta^d(P)}{P^2} f(P) = 0, \tag{3.28}$$

if the function $f(P)$ is regular at $P = 0$.

To derive such relations on a rigorous basis, we need to introduce a suitable family of functions converging to the $\delta^d(P)$ in the distributional limit.

We will be needing the relations

$$\delta^d(P) = \frac{\delta(P)\delta(\Omega)}{P^{d-1} C(\theta_1, ..., \theta_{d-1})}, \qquad \delta(\Omega) \equiv \prod_{l=1}^{D-1} \delta(\theta_l) \qquad C(\theta_1, ...\theta_{d-1}) = \prod_{l=1}^{d-2} \sin\theta_l^{d-l-1} \tag{3.29}$$

between the cartesian and the polar coordinates versions of the delta function. Then clearly

$$\int d^d P \delta^d(P) = \int_0^\infty dP \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \ldots \int_0^{2\pi} d\theta_{d-1} \delta(P)\delta(\Omega_d) = 1. \tag{3.30}$$

It is easily shown that the integral of a vector n of unit norm, expressed in the same variables

$$n^\alpha(\theta_1, \ldots, \theta_{d-1}) = (\cos\theta_1, \cos\theta_2 \sin\theta_1, \ldots, \sin\theta_{d-1} \ldots \sin\theta_1),$$
$$0 \le \theta_i \le \pi, i = 1, 2, \ldots d-2, \qquad 0 \le \theta_{d-1} \le 2\pi, \tag{3.31}$$

vanishes

$$\int d\Omega\, n^\alpha(\theta_1, \ldots, \theta_{d-1}) = 0, \qquad d\Omega \equiv d\theta_1 d\theta_2 \ldots d\theta_{d-1}. \tag{3.32}$$

These relations will be used to parametrize the tensor integrals over the (total) momentum P^μ of the correlators as $n^\mu |P|$, with $|P|$ being the magnitude of P. Notice that respect to the d-dimensional angular integration measure $d\Omega_d$, $d\Omega$ is stripped of the angular factors

$$d\Omega_d \equiv d\Omega \prod_{l=1}^{d-2} \sin\theta_l^{d-l-1}. \tag{3.33}$$

The vanishing of (3.32) is simply due to the symmetry of the angular integrations. This may not be obvious if we separate the angular from the radial parts of the δ function and performs the angular integration first, since the d-dimensional rotational symmetry is broken

$$\int d\Omega\, n^\alpha(\theta_1, \ldots, \theta_{d-1})\delta(\Omega) = \delta^{\alpha 0}, \tag{3.34}$$

where the nonzero component surviving in (3.34) depends on the directions chosen for the polar axis, and the integration measure has been stripped off of the angular factors. Notice that only one component (n^0), proportional to $\cos\theta_{d-1}$, is nonvanishing after the integration with $\delta(\Omega)$, the remaining ones being zero. Therefore, the vanishing of (3.26) has to be shown by using a *rotationally symmetric* sequence of functions which avoid the formal manipulation in (3.32). For this purpose we use a sequence of normalized Gaussians

$$G_k(P) = \frac{1}{(2\pi)^{d/2} k^d} e^{-\frac{P^2}{2k^2}}, \qquad \int d^d P G_k(P) = 1 \tag{3.35}$$

converging to $\delta^d(P)$ as $k \to 0$. We need to consider the distributional limit of

$$\int d^d P \frac{P^\alpha}{P^2} G_k(P) = \int_0^\infty P^{d-2} dP\, G_k(P)) \int n^\alpha(\theta_1, \ldots, \theta_{d-1}) d\Omega_d$$
$$= \frac{1}{2\sqrt{2}\pi^{d/2} k^{d/2+1/2}} \Gamma(\frac{d-1}{2}) \int n^\alpha(\theta_1, \ldots, \theta_{d-1}) d\Omega_d \tag{3.36}$$

which vanishes after angular integration since

$$Y^\alpha(n) \equiv \int n^\alpha(\theta_1, \ldots, \theta_{d-1}) d\Omega_d = 0 \tag{3.37}$$

with the boundaries given in (3.31). Therefore, the correct angular average should be taken before the distributional limit of $k \to 0$, giving a vanishing result, thereby proving (3.26). The result for the rank-2 integral in (3.25) which has been justified above by covariance and symmetric integration, can also be obtained by a similar method. In this case we get

$$\int d^d P \frac{P^\alpha P^\beta}{P^2} G_k(P) = \int_0^\infty P^{d-1} G_k(P) dP\, Y^{\alpha\beta}(n), \tag{3.38}$$

where we have defined the angular part

$$Y^{\alpha\beta}(n) \equiv \int n^\alpha n^\beta(\theta_1, \ldots, \theta_{d-1}) d\Omega_d = \frac{1}{d} V_d \delta^{\alpha\beta}, \qquad V_d = \frac{2\pi^{d/2}}{\Gamma[d/2]}. \tag{3.39}$$

Also in this case the integral (3.38) is factorized with

$$\int_0^\infty dP\, P^{d-1} G_k(P) = \frac{1}{V_d}, \tag{3.40}$$

which is independent of the k parameter of the distributional limit. Therefore

$$\int d^d P \frac{P^\alpha P^\beta}{P^2} G_k(P) = \int_0^\infty P^{d-1} G_k(P) dP\, Y^{\alpha\beta}(n) = \frac{1}{d} \delta^{\alpha\beta} \tag{3.41}$$

for any value of the parameter k. This takes directly to (3.25) if the parameter of the Gaussian family approaches $k = 0$ in order to extract the value of a test function $f(P)$ at $P = 0$.

One can expand on this result using the rules of the ordinary calculus formally, by taking multiple derivatives of $\delta(P)$ using (3.22), which imply that

$$\partial_\beta \partial_\alpha \delta^d(P) = \partial_\beta \left(-\frac{d}{P^2} \delta^d(P) P^\alpha \right)$$
$$= \frac{d(d+2)}{(P^2)^2} \delta^d(P) P^\alpha P^\beta - \frac{d}{P^2} \delta^d(P) \delta^{\alpha\beta}. \tag{3.42}$$

These correctly generate (3.23) using symmetric integration

$$\int d^d P \frac{\delta^d(P)}{(P^2)^2} P^\alpha P^\beta P^\rho P^\sigma f(0) = \frac{1}{d(d+2)} \left(\delta^{\alpha\beta} \delta^{\rho\sigma} + \delta^{\alpha\rho} \delta^{\beta\sigma} + \delta^{\alpha\sigma} \delta^{\beta\rho} \right). \tag{3.43}$$

Another useful relation is

$$\sum_{j=1}^{n} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\delta^d(P) = \sum_{j=1}^{n} p_j^\alpha \frac{\partial}{\partial P^\alpha}\delta^d(P)$$
$$= P^\alpha \partial_\alpha \delta^d(P) = -d\,\delta^d(P), \tag{3.44}$$

using (3.24), which is of immediate derivation. We will be using the relations above to illustrate the elimination of one of the momenta from the differential equations which characterize the CWI's in momentum space.

3.2. The dilatation Ward identity with one less momentum

We can reobtain the results of the previous sections for the dilatation WI by using the calculus derived above. The dilatation Ward identity of (3.6) can indeed be written in a $(p_1, p_2, \dots p_n)$ symmetric form using

$$\sum_{j=1}^{n}\left(x_j^\alpha \frac{\partial}{\partial x_j^\alpha} + \Delta_j\right)\Phi(x_1, x_2, \dots, x_n)$$
$$= \sum_{j=1}^{3}\left(x_j^\alpha \frac{\partial}{\partial x_j^\alpha} + \Delta_j\right)\int \underline{d^d p}\, e^{i\underline{px}}\,\delta(P)\,\Phi(\underline{p})$$
$$= \int \underline{d^d p}\,\delta^d(P)\, e^{i\underline{px}}\left(\sum_{j=1}^{n}\Delta_j - nd - \sum_{j=1}^{n} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\right)\Phi(\underline{p}) + \delta'_{\text{term}}, \tag{3.45}$$

with

$$\delta'_{\text{term}} = -\int \underline{d^d p}\, P^\alpha \partial_\alpha \delta^d(P) e^{i\underline{px}}\Phi(\underline{p}) = d\int \underline{d^d p}\,\delta^d(P) e^{i\underline{px}}\Phi(\underline{p}), \tag{3.46}$$

where we have used (3.44). Inserting (3.46) into (3.45) we obtain the *symmetric* expression of the scaling relation in momentum space

$$\sum_{j=1}^{n}\left(x_j^\alpha \frac{\partial}{\partial x_j^\alpha} + \Delta_j\right)\Phi(x_1, x_2, \dots, x_n)$$
$$= \int \underline{d^d p}\,\delta^d(P) e^{i\underline{px}}\left(\sum_{j=1}^{n}\Delta_j - (n-1)d - \sum_{j=1}^{n} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\right)\Phi(\underline{p}). \tag{3.47}$$

The expression given above depends only on $n-1$ momenta, since one of them can be eliminated. If we choose as independent ones $p_1, \dots p_{n-1}$ with $(p_1, \dots, p_n) \to (p_1, \dots, p_{n-1}, P)$, and define $q = p_1 + p_2 \dots + p_{n-1}$, then the dependence on p_n in (3.47) can be re-expressed in terms of the total momentum P and of the sum of the independent momenta q as

$$\Phi(\underline{p}) = \Phi(p_1, \dots, p_{n-1}, P) \qquad p_n^\alpha = P^\alpha - q^\alpha$$
$$\sum_{j=1}^{n} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\Phi(\underline{p}) = \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\Phi(p_1, \dots p_{n-1}, P)$$
$$+ (P^\alpha - q^\alpha)\frac{\partial}{\partial P^\alpha}\Phi(p_1, \dots p_{n-1}, P). \tag{3.48}$$

The last (n_{th}) term in (3.47) is given by

$$\sigma_n \equiv \int \underline{d^d p}\, \delta^d(P) e^{i\underline{p x}} p_n^\alpha \frac{\partial}{p_n^\alpha}\Phi(\underline{p}). \tag{3.49}$$

Rewriting the exponential as $e^{i\underline{p\cdot x}} \to e^{i(p_1\cdot x_{1n}+\dots p_{n-1}\cdot x_{n-1\,n}+iP\cdot x_n)}$ and using the $\delta^d(P)$ to remove the $P\cdot x_n$ term, σ_n takes the form

$$\begin{aligned}\sigma_n &= \int d^d p_1 d^d p_2 \dots d^d p_{n-1} d^d P\, \delta^d(P) e^{ip_1x_{1n}+ip_2x_{2n}+\dots ip_{n-1}x_{n-1n}} \times \\ &\qquad \times (P-q)^\alpha \frac{\partial}{P^\alpha}\Phi(p_1,\dots p_{n-1},P) \\ &= -\int d^d p_1 d^d p_2 \dots d^d p_{n-1} e^{ip_1x_{1n}+ip_2x_{2n}+\dots ip_{n-1}x_{n-1n}} q^\alpha \\ &\qquad \times \int d^d P \frac{\partial}{\partial P^\alpha}\Phi(p_1,\dots p_{n-1},P)\delta^d(P).\end{aligned} \tag{3.50}$$

Notice that in the expression above we have removed the $\delta^d(P)P^\alpha \frac{\partial}{\partial P^\alpha}\Phi(\underline{p})$ term, which after a partial integration becomes

$$\begin{aligned}&\int d^d P \delta^d(P) P^\alpha \frac{\partial}{\partial P^\alpha}\Phi(p_1,\dots p_{n-1},P) e^{ip_1x_{1n}+ip_2x_{2n}+\dots ip_{n-1}x_{n-1n}} \\ &= \int d^d P e^{ip_1x_{1n}+ip_2x_{2n}+\dots ip_{n-1}x_{n-1n}} \left(-d + P^\alpha \frac{\partial}{\partial P^\alpha}\delta^d(P)\right)\Phi(p_1,\dots p_{n-1},P)\end{aligned} \tag{3.51}$$

and vanishes by (3.24). Similarly, we derive the vanishing relation

$$\begin{aligned}\int d^d P\, P^\alpha \frac{\partial}{\partial P^\alpha}\Phi(p_1,\dots p_{n-1},P)\delta^d(P) &= d\int d^d P \delta^d(P)\frac{P^\alpha}{P^2}\Phi(p_1,\dots p_{n-1},P) \\ &= d\int d^d P \delta^d(P)\frac{P^\alpha}{P^2}\Phi(p_1,\dots p_{n-1},0) \quad (3.52) \\ &= 0 \quad (3.53)\end{aligned}$$

which has been obtained as a result of (3.26) and (3.28). Therefore we find that $\sigma_n = 0$, reobtaining the expected scaling equation in momentum space

$$\left(\sum_{j=1}^n \Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\right)\Phi(p_1,\dots p_{n-1},\bar{p}_n) = 0. \tag{3.54}$$

3.3. Special conformal WI's for scalar correlators

We now turn to the analysis of the special conformal transformations in momentum space. In this second case the δ' terms cancel identically. Also in this case we discuss both the symmetric and the asymmetric forms of the equations, focusing our attention first on the scalar case. The Ward identity in the scalar case is given by

$$\sum_{j=1}^n \left(-x_j^2\frac{\partial}{\partial x_j^\kappa} + 2x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} + 2\Delta_j x_j^\kappa\right)\Phi(x_1,x_2,\dots,x_n) = 0 \tag{3.55}$$

which in momentum space, using

$$x_j^\alpha \to -i\frac{\partial}{\partial p_j^\alpha} \qquad \frac{\partial}{\partial x_j^\kappa} \to i p_j^\kappa \tag{3.56}$$

becomes

$$\sum_{j=1}^{n}\int \underline{d^d p}\left(p_j^\kappa \frac{\partial^2}{\partial p_j^\alpha \partial p_j^\kappa} - 2p_j^\alpha \frac{\partial^2}{\partial p_j^\alpha \partial p_j^\kappa} - 2\Delta_j \frac{\partial}{\partial p_j^\kappa}\right) e^{i\underline{p\cdot x}}\delta^d(P)\phi(\underline{p}) = 0, \tag{3.57}$$

where the action of the operator is only on the exponential. At this stage we integrate by parts, bringing the derivatives from the exponential to the correlator and on the Dirac δ function obtaining

$$\int \underline{d^d p} e^{i\underline{p x}} K_s^k \Phi(\underline{p})\delta^d(P) + \delta'_{\rm term} = 0 \tag{3.58}$$

in the notations of Eq. (3.1), where we have introduced the differential operator acting on a scalar correlator in a symmetric form

$$K_s^k = \sum_{j=1}^{n}\left(p_j^\kappa \frac{\partial^2}{\partial p_j^\alpha \partial p_j^\alpha} + 2(\Delta_j - d)\frac{\partial}{\partial p_j^\kappa} - 2p_j^\alpha \frac{\partial^2}{\partial p_j^\kappa \partial p_j^\alpha}\right). \tag{3.59}$$

Some of the terms containing first and second derivatives of the Dirac delta function can be rearranged using also the intermediate relation

$$\begin{aligned} K_s^\kappa \delta^d(P) &= \left(P^k \frac{\partial^2}{\partial P^\alpha \partial P_\alpha} - 2P^\alpha \frac{\partial^2}{\partial P^\alpha \partial P^k} + 2(\Delta - n\,d)\frac{\partial}{\partial P^k}\right)\delta^d(P) \\ &= 2d(d\,n - d - \Delta)P^k \frac{\delta^d(P)}{P^2} \\ &= -2(d\,n - d - \Delta)\frac{\partial}{\partial P_k}\delta^d(P), \qquad\qquad \Delta = \sum_{j=1}^{n}\Delta_j, \end{aligned} \tag{3.60}$$

where we have repeatedly used (3.24) together with (3.42) and (3.44). Combining all the derivative terms, on the other hand, we obtain

$$\begin{aligned} \delta'_{\rm term} = \int \underline{d^d p}\, e^{i\underline{p\cdot x}} \Bigg[&\frac{\partial}{\partial P^\alpha}\delta^d(P)\sum_{j=1}^{n}\left(p_j^\alpha \frac{\partial}{\partial p_j^\kappa} - p_j^\kappa \frac{\partial}{\partial p_j^\alpha}\right)\Phi(\underline{p}) \\ &+2\frac{\partial}{\partial P^\kappa}\delta^d(P)\left(\sum_{j=1}^{n}(\Delta_j - p_j^\alpha \frac{\partial}{\partial p_j^\alpha}) - (n-1)d\right)\Phi(\underline{p})\Bigg]. \end{aligned} \tag{3.61}$$

Notice that such terms vanish by using rotational invariance of the scalar correlator

$$\sum_{j=1}^{3}\left(p_j^\alpha \frac{\partial}{\partial p_j^\kappa} - p_j^\kappa \frac{\partial}{\partial p_j^\alpha}\right)\Phi(\underline{p}) = 0, \tag{3.62}$$

as a consequence of the SO(4) symmetry

$$\sum_{j=1}^{3} L_{\mu\nu}(x_j)\langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle = 0, \tag{3.63}$$

with

$$L_{\mu\nu}(x) = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right), \tag{3.64}$$

and the symmetric scaling relation,

$$\left(\sum_{j=1}^{n}\Delta_j - \sum_{j}^{n-1} p_j^\alpha\frac{\partial}{\partial p_j^\alpha} - (n-1)d\right)\Phi(\underline{p}) = 0, \tag{3.65}$$

where we have used (3.24). Using (3.60) and the vanishing of the $\delta'_{\rm term}$ terms, the structure of the CWI on the correlator $\Phi(p)$ then takes the symmetric form

$$\sum_{j=1}^{n}\int \underline{d^d p} e^{i\underline{p\cdot x}}\left(p_j^\kappa\frac{\partial^2}{\partial p_{j\alpha}\partial p_j^\alpha} - 2p_j^\alpha\frac{\partial^2}{\partial p_j^\alpha\partial p_j^\kappa} + 2(\Delta_j - d)\frac{\partial}{\partial p_j^\kappa}\right)\phi(\underline{p})\delta^d(P) = 0. \tag{3.66}$$

This *symmetric* expression is the starting point in order to proceed with the elimination of one of the momenta, say p_n. Also in this case, one can proceed by following the same procedure used in the derivation of the dilatation identity, dropping the contribution coming from the dependent momentum p_n, thereby obtaining the final form of the equation

$$\sum_{j=1}^{n-1}\left(p_j^\kappa\frac{\partial^2}{\partial p_j^\alpha\partial p_j^\alpha} + 2(\Delta_j - d)\frac{\partial}{\partial p_j^\kappa} - 2p_j^\alpha\frac{\partial^2}{\partial p_j^\kappa\partial p_j^\alpha}\right)\Phi(p_1,\dots p_{n-1},\bar{p}_n) = 0. \tag{3.67}$$

Also in this case the differentiation respect to p_n requires the chain rule. For a certain sequence of scalar single particle operators

$$\Phi(p_1,\dots p_{n-1},\bar{p}_n) = \langle\phi(p_1)\dots\phi(\bar{p}_n)\rangle \tag{3.68}$$

the Leibnitz rule is therefore violated. As we have already mentioned, the complete symmetry of the solution respect to the three momenta is however respected. We will now move to a discussion of the general structure of the method, focusing first on scalars and then on tensor correlators.

4. Reduction of the action of K^κ_{scalar}

In the case of a scalar correlator all the anomalous conformal WI's can be re-expressed in scalar form by taking as independent momenta the magnitude $p_i = \sqrt{p_i^2}$ as the three independent variables. Defining $\mathcal{F}(p_1, p_2) = \Phi(p_1, p_2, \bar{p}_3)$ and using the relation

$$p_1^\alpha\frac{\partial\mathcal{F}}{p_1{}^\alpha} + p_2^\alpha\frac{\partial\mathcal{F}}{p_2{}^\alpha} = p_1\frac{\partial\Phi}{\partial p_1} + p_2\frac{\partial\Phi}{\partial p_2} + p_3\frac{\partial\Phi}{\partial p_3} \tag{4.1}$$

the anomalous scale equation becomes

$$\left(\Delta - 2d - \sum_{i=1}^{3} p_i\frac{\partial}{\partial p_i}\right)\Phi(p_1, p_2, \bar{p}_3) = 0. \tag{4.2}$$

The relation above is derived using the chain rule

$$\frac{\partial\Phi}{\partial p_i^\mu}=\frac{p_i^\mu}{p_i}\frac{\partial\Phi}{\partial p_i}-\frac{\bar{p}_3^\mu}{p_3}\frac{\partial\Phi}{\partial p_3}. \tag{4.3}$$

It is a straightforward but lengthy computation to show that the special (non anomalous) conformal transformation in d dimension takes the form, for the scalar component

$$K_{scalar}{}^\kappa\Phi=0 \tag{4.4}$$

with

$$K^\kappa_{scalar}=\sum_{i=1}^{3}p_i^\kappa\,K_i \tag{4.5}$$

$$K_i\equiv\frac{\partial^2}{\partial p_i\partial p_i}+\frac{d+1-2\Delta_i}{p_i}\frac{\partial}{\partial p_i} \tag{4.6}$$

with the expression (4.5) which can be split into the two independent equations

$$\frac{\partial^2\Phi}{\partial p_i\partial p_i}+\frac{1}{p_i}\frac{\partial\Phi}{\partial p_i}(d+1-2\Delta_1)-\frac{\partial^2\Phi}{\partial p_3\partial p_3}-\frac{1}{p_3}\frac{\partial\Phi}{\partial p_3}(d+1-2\Delta_3)=0\quad i=1,2. \tag{4.7}$$

Notice that in the derivation of (4.5) one needs at an intermediate step the derivative of the scaling WI

$$p_1\frac{\partial^2\Phi}{\partial p_3\partial p_1}+p_2\frac{\partial^2\Phi}{\partial p_3\partial p_2}=(\Delta-2d-1)\frac{\partial\Phi}{\partial p_3}-p_3\frac{\partial^2\Phi}{\partial p_3\partial p_3}. \tag{4.8}$$

Defining

$$K_{ij}\equiv K_i-K_j \tag{4.9}$$

Eqs. (4.7) take the form

$$K^\kappa_{13}\Phi=0\qquad\text{and}\qquad K^\kappa_{23}\Phi=0. \tag{4.10}$$

5. Transverse Ward identities

To fix the form of the correlator we need to impose the transverse WI on the vector lines and the conservation WI for $T^{\mu\nu}$. In this section we briefly discuss their derivation and their explicit expressions. We consider the functional

$$W[g,A]=\int D\bar{\psi}\,D\psi\,e^{-(S_0[g,\psi]+S_1[A,\psi])} \tag{5.1}$$

integrated over the fermions ψ, in the background of the metric $g_{\mu\nu}$ and of the gauge field A^a_μ. In the case of a nonabelian gauge theory the action is given by

$$S_0[g,A,\psi]=-\frac{1}{4}\int d^4x\sqrt{-g_x}F^a_{\mu\nu}F^{\mu\nu a}+\int d^4x\sqrt{-g_x}i\bar{\psi}\gamma^\mu D_\mu\psi$$

$$S_1[g,A]=\int\sqrt{-g_x}J^a_\mu A^{\mu a} \tag{5.2}$$

with

$$
\begin{aligned}
F^a_{\mu\nu} &= \nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + g_c f^{abc} A^b_\mu A^c_\nu \\
&= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_c f^{abc} A^b_\mu A^c_\nu, \\
\nabla_\mu A^{\nu a} &= \partial_\mu A^{\nu a} + \Gamma^\lambda_{\mu\nu} A^{\lambda a}
\end{aligned}
\tag{5.3}
$$

with $J^{\mu a} = g_c \bar{\psi} \gamma^\mu T^a \psi$ denoting the fermionic current, with T^a the generators of the theory and ∇_μ denoting the covariant derivative in the curved background on a vector field. The local Lorentz and gauge covariant derivative (D) on the fermions acts via the spin connection

$$
D_\mu \psi = \left(\partial_\mu \psi + A^a_\mu T^a + \frac{1}{4} \omega^{\underline{ab}}_{\underline{\mu}} \gamma_{\underline{ab}} \right) \psi \tag{5.4}
$$

having denoted with $\underline{ab}$ the local Lorentz indices. A local Lorentz covariant derivative (D) can be similarly defined for a vector field, say $V^{\underline{a}}$, via the Vielbein $e^{\underline{a}}_{\underline{\mu}}$ and its inverse $e_{\underline{a}}^{\underline{\mu}}$

$$
D_\mu V^{\underline{a}} = \partial_\mu V^{\underline{a}} + \omega^{\underline{a}}_{\underline{\mu b}} V^{\underline{b}} \tag{5.5}
$$

with

$$
\nabla_\mu V^\rho = e_{\underline{a}}^{\ \rho} D_\mu V^{\underline{a}} \tag{5.6}
$$

with the Christoffel and the spin connection related via the holonomic relation

$$
\Gamma^\rho_{\mu\nu} = e_{\underline{a}}^{\ \rho} \left(\partial_\mu e^{\underline{a}}_{\nu} + \omega^{\underline{a}}_{\underline{\mu b}} e^{\underline{b}}_{\nu} \right). \tag{5.7}
$$

Diffeomorphism invariance of the generating functional (5.1) gives

$$
\int d^d x \left(\frac{\delta W}{\delta g_{\mu\nu}} \delta g_{\mu\nu}(x) + \frac{\delta W}{\delta A^a_\mu} \delta A^a_\mu(x) \right) = 0 \tag{5.8}
$$

where the variation of the metric and the gauge fields are the corresponding Lie derivatives, for a change of variables $x^\mu \to x^\mu + \epsilon^\mu(x)$

$$
\begin{aligned}
\delta A^a_\mu(x) &= -\nabla_\alpha A^a_\mu \epsilon^\alpha - A^a_\alpha \nabla_\mu \epsilon^\alpha \\
\delta g_{\mu\nu} &= -\nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu
\end{aligned}
\tag{5.9}
$$

while for a gauge transformation with a parameter $\theta^a(x)$

$$
\delta A^a_\mu = \underline{D}_\mu \theta^a \equiv \partial_\mu \theta^a + g_c f^{abc} A^b_\mu \theta^c. \tag{5.10}
$$

Using (5.9), Eq. (5.8) becomes

$$
\begin{aligned}
0 &= \left\langle \int d^4 x \left(\frac{\delta (S_0 + S_1)}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\delta S_1}{\delta A^a_\mu} \delta A^a_\mu \right) \right\rangle \\
&= \left\langle \int d^4 x \sqrt{-g_x} \left[\nabla_\mu T^{\mu\nu} + (\nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu) J^{\mu a} + \nabla_\mu J^{\mu a} A^a_\nu \right] \epsilon^\nu(x) \right\rangle
\end{aligned}
\tag{5.11}
$$

while the condition of gauge invariance gives

$$
\int d^d x \frac{\delta W}{\delta A^a_\mu} \delta A^a_\mu = \left\langle \int d^4 x \sqrt{-g_x} J^\mu_a \underline{D}_\mu \theta^a \right\rangle = 0 \tag{5.12}
$$

which, in turn, after an integration by parts, generates the gauge WI

$$
\langle \nabla_\mu J^{\mu a} \rangle = g_c f^{abc} \langle J^b_\mu \rangle A^{\mu c}. \tag{5.13}
$$

Inserting this relation into (5.11) we obtain the conservation WI

$$\langle \nabla^{\mu} T_{\mu\nu} \rangle + F^{a}_{\mu\nu} \langle J^{\mu a} \rangle = 0. \tag{5.14}$$

In the abelian case, diffeomorphism and gauge invariance then give the relations

$$\begin{aligned} 0 &= \nabla_{\nu} \langle T^{\mu\nu} \rangle + F^{\mu\nu} \langle J_{\nu} \rangle \\ 0 &= \nabla_{\nu} \langle J^{\nu} \rangle \end{aligned} \tag{5.15}$$

with naive scale invariance gives the traceless condition

$$g_{\mu\nu} \langle T^{\mu\nu} \rangle = 0. \tag{5.16}$$

The functional differentiation of (5.15) and (5.16) allows to derive ordinary Ward identities for the various correlators. In the TJJ case we obtain, after a Fourier transformation, the conservation equation

$$\begin{aligned} & p_{1\nu_1} \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle \\ & \quad = 4 \left[\delta^{\mu_1\mu_2} p_{2\lambda} \langle J^{\lambda}(p_1+p_2) J^{\mu_3}(p_3) \rangle - p_2^{\mu_1} \langle J^{\mu_2}(p_1+p_2) J^{\mu_3}(p_3) \rangle \right] \\ & \quad + 4 \left[\delta^{\mu_1\mu_3} p_{3\lambda} \langle J^{\lambda}(p_1+p_3) J^{\mu_2}(p_2) \rangle - p_3^{\mu_1} \langle J^{\mu_3}(p_1+p_3) J^{\mu_2}(p_2) \rangle \right] \end{aligned} \tag{5.17}$$

and vector current Ward identities

$$p_{2\mu_2} \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle = 0 \tag{5.18}$$

$$p_{3\mu_3} \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle = 0, \tag{5.19}$$

while the naive identity (5.16) gives the non-anomalous condition

$$\delta_{\mu_1\nu_1} \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle = 0, \tag{5.20}$$

valid in the $d \neq 4$ case. We recall that the 2-point function of two conserved vector currents J_i $(i = 2, 3)$ [24] in any conformal field theory in d dimension is given by

$$\begin{aligned} & \langle J_2^{\alpha}(p) J_3^{\beta}(-p) \rangle = \delta_{\Delta_2 \Delta_3} (c_{123} \Gamma_J) \pi^{\alpha\beta}(p) (p^2)^{\Delta_2 - d/2}, \\ & \Gamma_J = \frac{\pi^{d/2}}{4^{\Delta_2 - d/2}} \frac{\Gamma(d/2 - \Delta_2)}{\Gamma(\Delta_2)}, \end{aligned} \tag{5.21}$$

with c_{123} an overall constant and $\Delta_2 = d - 1$. In our case $\Delta_2 = \Delta_3 = d - 1$ and Eq. (5.17) then takes the form

$$\begin{aligned} & p_{1\mu_1} \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle \\ & \quad = 4 c_{123} \Gamma_J \left(\delta^{\nu_1\mu_2} \frac{p_{2\lambda}}{(p_3^2)^{d/2-\Delta_2}} \pi^{\lambda\mu_3}(p_3) - \frac{p_2^{\nu_1}}{(p_3^2)^{d/2-\Delta_2}} \pi^{\mu_2\mu_3}(p_3) \right. \\ & \qquad \left. + \delta^{\nu_1\mu_3} \frac{p_{3\lambda}}{(p_2^2)^{d/2-\Delta_2}} \pi^{\lambda\mu_2}(p_2) - \frac{p_3^{\nu_1}}{(p_2^2)^{d/2-\Delta_2}} \pi^{\mu_3\mu_2}(p_2) \right). \end{aligned} \tag{5.22}$$

Explicit expressions of the secondary CWI's are determined using (5.18) and (5.20) and the explicit form (5.22).

6. TJJ reconstruction the BMS way

We are now going investigate the BMS approach, which is technically quite involved, highlighting several steps which are crucial in order to clarify the basic structure of the method. The method is exemplified in the case of the TJJ. Several intermediate steps, which we believe are necessary in order to characterize the approach, have been worked out independently and are based on the use of the Lorentz Ward identities.

Given the partial symmetry of the TJJ correlator, for instance, respect to the TTT case, one can choose as independent momenta either p_1 and p_2 or, more conveniently, p_2 and p_3, given the symmetry of the two J currents.

With the first choice, outlined below, the current $J(p_3)$ is singlet under the (spin) Lorentz generators. With the second choice, the two currents are treated symmetrically and the stress energy tensor is treated as a singlet under the same generators. The derivation of the CWI's in this second case will be outlined in section 7. The equations obtained in the two cases are obviously the same.

First of all we discuss the canonical Ward identities for the $\langle TJJ\rangle$ correlation function in momentum space. From the general definition of the global Ward identities in position space

$$\sum_{j=1}^{3} G_g(x_j)\,\langle T^{\mu_1\nu_1}(x_1)\,J^{\mu_2}(x_2)\,J^{\mu_3}(x_3)\rangle = 0, \tag{6.1}$$

where G_g is the generator of the infinitesimal symmetry transformation. The dilatation Ward identities take the form

$$0 = \left[\sum_{j=1}^{3}\Delta_j - (n-1)d - \sum_{j=1}^{2} p_j^{\alpha}\frac{\partial}{\partial p_j^{\alpha}}\right]\langle T^{\mu_1\nu_1}(p_1)\,J^{\mu_2}(p_2)\,J^{\mu_3}(\bar{p}_3)\rangle\,. \tag{6.2}$$

To proceed towards the analysis of the constraints, it is essential to introduce the Lorentz covariant Ward identities

$$\begin{aligned}
0 = \sum_{j=1}^{2}&\left[p_j^{\nu}\frac{\partial}{\partial p_{j\mu}} - p_j^{\mu}\frac{\partial}{\partial p_{j\nu}}\right]\langle T^{\mu_1\nu_1}(p_1)\,J^{\mu_2}(p_2)\,J^{\mu_3}(\bar{p}_3)\rangle\\
&+2\left(\delta^{\nu}_{\alpha_1}\delta^{\mu(\mu_1} - \delta^{\mu}_{\alpha_1}\delta^{\nu(\mu_1}\right)\langle T^{\nu_1)\alpha_1}(p_1)\,J^{\mu_2}(p_2)\,J^{\mu_3}(\bar{p}_3)\rangle\\
&+\left(\delta^{\nu}_{\alpha_2}\delta^{\mu\mu_2} - \delta^{\mu}_{\alpha_2}\delta^{\nu\mu_2}\right)\langle T^{\mu_1\nu_1}(p_1)\,J^{\alpha_2}(p_2)\,J^{\mu_3}(\bar{p}_3)\rangle\\
&+\left(\delta^{\nu}_{\alpha_3}\delta^{\mu\mu_3} - \delta^{\mu}_{\alpha_3}\delta^{\nu\mu_3}\right)\langle T^{\mu_1\nu_1}(p_1)\,J^{\mu_2}(p_2)\,J^{\alpha_3}(\bar{p}_3)\rangle\,,
\end{aligned} \tag{6.3}$$

where

$$\delta^{\nu(\mu_1}\,\langle T^{\nu_1)\alpha_1}\,J^{\mu_2}\,J^{\mu_3}\rangle \equiv \frac{1}{2}\left(\delta^{\nu\mu_1}\,\langle T^{\nu_1\alpha_1}\,J^{\mu_2}\,J^{\mu_3}\rangle + \delta^{\nu\nu_1}\,\langle T^{\mu_1\alpha_1}\,J^{\mu_2}\,J^{\mu_3}\rangle\right), \tag{6.4}$$

and finally the special conformal Ward identities

$$\begin{aligned}
0 = \sum_{j=1}^{2}&\left[2(\Delta_j - d)\frac{\partial}{\partial p_j^{\kappa}} - 2p_j^{\alpha}\frac{\partial}{\partial p_j^{\alpha}}\frac{\partial}{\partial p_j^{\kappa}} + (p_j)_{\kappa}\frac{\partial}{\partial p_j^{\alpha}}\frac{\partial}{\partial p_{j\alpha}}\right]\\
&\times\langle T^{\mu_1\nu_1}(p_1)\,J^{\mu_2}(p_2)\,J^{\mu_3}(\bar{p}_3)\rangle
\end{aligned}$$

$$+4\left(\delta^{\kappa(\mu_1}\frac{\partial}{\partial p_1^{\alpha_1}}-\delta^{\kappa}_{\alpha_1}\delta^{\lambda(\mu_1}\frac{\partial}{\partial p_1^{\lambda}}\right)\langle T^{\nu_1)\alpha_1}(p_1)\,J^{\mu_2}(p_2)\,J^{\mu_3}(\bar{p}_3)\rangle$$

$$+2\left(\delta^{\kappa\mu_2}\frac{\partial}{\partial p_2^{\alpha_2}}-\delta^{\kappa}_{\alpha_2}\delta^{\lambda\mu_2}\frac{\partial}{\partial p_2^{\lambda}}\right)\langle T^{\mu_1\nu_1}(p_1)\,J^{\alpha_2}(p_2)\,J^{\mu_3}(\bar{p}_3)\rangle\,, \tag{6.5}$$

which we will use in the next sections in order to determine the tensor structure of this correlator.

6.1. Projectors

The basic observation in the BMS approach is that the action of the special conformal, trace and conservation (longitudinal) WI's take a simpler form if we enforce a decomposition of the tensor correlators in terms of transverse traceless, longitudinal and trace parts and project the Ward identities on the same subspaces. Recall that for a symmetric tensor such as the EMT, this decomposition is performed using the following projectors

$$\pi^{\mu}_{\alpha}=\delta^{\mu}_{\alpha}-\frac{p^{\mu}p_{\alpha}}{p^2},\qquad \tilde{\pi}^{\mu}_{\alpha}=\frac{1}{d-1}\pi^{\mu}_{\alpha} \tag{6.6}$$

$$\Pi^{\mu\nu}_{\alpha\beta}=\frac{1}{2}\left(\pi^{\mu}_{\alpha}\pi^{\nu}_{\beta}+\pi^{\mu}_{\beta}\pi^{\nu}_{\alpha}\right)-\frac{1}{d-1}\pi^{\mu\nu}\pi_{\alpha\beta}, \tag{6.7}$$

$$\mathcal{I}^{\mu\nu}_{\alpha\beta}=\frac{1}{p^2}p^{\beta}\left(p^{\mu}\delta^{\nu}_{\alpha}+p^{\nu}\delta^{\mu}_{\alpha}-\frac{p_{\alpha}p_{\beta}}{p^2}(\delta^{\mu\nu}+(d-2)\frac{p^{\mu}p^{\nu}}{p^2})\right) \tag{6.8}$$

$$\mathcal{L}^{\mu\nu}_{\alpha\beta}=\frac{1}{2}\left(\mathcal{I}^{\mu\nu}_{\alpha\beta}+\mathcal{I}^{\mu\nu}_{\beta\alpha}\right)\qquad \tau^{\mu\nu}_{\alpha\beta}=\tilde{\pi}^{\mu\nu}\delta_{\alpha\beta} \tag{6.9}$$

with

$$\delta^{\mu\nu}_{\alpha\beta}=\Pi^{\mu\nu}_{\alpha\beta}+\Sigma^{\mu\nu}_{\alpha\beta} \tag{6.10}$$

$$\Sigma^{\mu\nu}_{\alpha\beta}\equiv\mathcal{L}^{\mu\nu}_{\alpha\beta}+\tau^{\mu\nu}_{\alpha\beta} \tag{6.11}$$

The previous identities allow to decompose a symmetric tensor into its transverse traceless (via Π), longitudinal (via $\mathcal{L}$) and trace parts (via τ), or on the sum of the combined longitudinal and trace contributions (via Σ). We are now going to illustrate the approach in the case of the TTT correlation function. The transverse traceless projections will be denoted as (t) and the trace parts with (s). We will be denoting such correlator as ϕ and try to resort to a condensed notation in order to characterize the algebraic structure of the procedure. For a rank-6 correlator of 3 T's, ϕ will denote the tensor

$$\phi\equiv\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle. \tag{6.12}$$

We can act on this correlator with the two projectors Π and Σ on each (combined) pair of indices and momenta. For instance, the transverse traceless part of ϕ is obtained by acting with 3 Π projectors on the indices of the EMT's

$$\phi_{ttt}\equiv\Pi_1\Pi_2\Pi_3\phi \tag{6.13}$$

$$\equiv\Pi_1{}^{\mu_1\nu_1}_{\alpha_1\beta_1}\Pi_2{}^{\mu_2\nu_2}_{\alpha_2\beta_2}\Pi_3{}^{\mu_3\nu_3}_{\alpha_3\beta_3}\langle T^{\alpha_1\beta_1}(p_1)T^{\alpha_2\beta_2}(p_2)T^{\alpha_3\beta_3}(p_3)\rangle, \tag{6.14}$$

where $\Pi_1{}^{\mu_1\nu_1}_{\alpha_1\beta_1}\equiv\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)$. Similarly, the remaining 7 components of the ϕ correlator can be obtained by acting with all the other combinations of projectors (Σ,Π), to obtain

$$\phi = \phi_{ttt} + \phi_{tst} + \phi_{tss} + \phi_{sss} + \phi_{stt} + \phi_{sst} + \phi_{sts}. \tag{6.15}$$

Acting with the special conformal transformation K^κ (both with the scalar and the spin parts) on ϕ we can again project the result onto the orthogonal subspaces ttt, tss, etc., and try to solve the equations separately in each of these 8 sectors. In our condensed notation the equation for the special conformal transformation takes the form

$$\phi' = K^\kappa \phi = 0, \tag{6.16}$$

and its projection into the 8 independent sectors, such as, for instance

$$\phi'_{ttt} \equiv \Pi_1 \Pi_2 \Pi_3 K^\kappa \phi = 0, \tag{6.17}$$

$$\phi'_{stt} \equiv \Sigma_1 \Pi_2 \Pi_3 K^\kappa \phi = 0, \qquad \phi'_{sst} \equiv \Sigma_1 \Sigma_2 \Pi_3 K^\kappa \phi = 0, \dots \tag{6.18}$$

and so on, can be obtained by the action of the Π's and Σ's on (6.16). It is important to realize that only the equation $\phi'_{ttt} = 0$ involves 3-point functions beside 2 point functions, and needs to be solved. The remaining sectors do not give any new equation, since they involve only 2-point functions, being related to the conservation and trace WI's. However they define consistency conditions for lower point functions that will introduce some constraint on the arbitrary constants appearing in the solutions of the primary WI's.

To illustrate these points we will treat the correlators ϕ, ϕ' as vectors in a functional space on which the K^κ operator will act both in differential form and algebraically via its spin rotation matrices. For instance we define

$$\phi_{ttt} \equiv P_{ttt}\phi, \text{ where } P_{ttt} \equiv \Pi_1 \Pi_2 \Pi_3, \ \phi_{tst} \equiv P_{tst}\phi \text{ with } P_{tst} \equiv \Pi_1 \Sigma_2 \Pi_3, \dots \tag{6.19}$$

and similarly in the other cases. In general, it is convenient to characterize the action of K^κ on each subspace via a projection, such as

$$\phi'^{(ttt)}_{ttt} \equiv P_{ttt} K^\kappa P_{ttt} \phi \qquad \phi'^{(ttt)}_{stt} = P_{stt} K^\kappa P_{ttt} \phi, \dots \tag{6.20}$$

and so on, for a total of $64 = 8 \times 8$ sectors. In the first expressions above, for example, the original transverse traceless projection (ttt) is acted upon by K^κ and then it is re-projected onto the ttt sector. There are several simplifications among these matrix elements. For example, a direct computation, that we will prove below, gives

$$P_{stt} K^\kappa P_{ttt} \phi = 0, \qquad P_{tst} K^\kappa P_{ttt} \phi = 0, \qquad P_{tts} K^\kappa P_{ttt} \phi = 0, \dots \tag{6.21}$$

showing that the ϕ_{ttt} amplitude is mapped only into another amplitude in the same ttt subspace by the action of K^κ.

6.1.1. Endomorphic action of K^κ on the transverse-traceless sector

K^κ acts as endomorphism on the transverse traceless sector of a tensor correlator. To illustrate this point we consider the case of the TTT, though the approach is generic. Define

$$Y^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} = \Pi_1{}^{\mu_1\nu_1}_{\alpha_1\beta_1} \Pi_2{}^{\mu_2\nu_2}_{\alpha_2\beta_2} \Pi_3{}^{\mu_3\nu_3}_{\alpha_3\beta_3} \langle T^{\alpha_1\beta_1} T^{\alpha_2\beta_2} T^{\alpha_3\beta_3} \rangle \tag{6.22}$$

to be the transverse traceless projection of the TTT. One can check the transversality of the action of $K^\kappa_{1scalar}$ (the other contribution of K^κ being similar) by contracting Y with p_1

$$
\begin{aligned}
& p_1^{\mu_1} K^{\kappa}_{1scalar} Y^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \\
&= \left(-2p_1^{\alpha} p_1^{\mu_1} \frac{\partial^2}{\partial p_1^{\alpha} \partial p_1^{\kappa}} + p_1^{\kappa} p_1^{\mu_1} \frac{\partial^2}{\partial p_1^{\alpha} \partial p_1^{\alpha}} \right) Y^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \\
&= -2p_1^{\alpha} \frac{\partial}{\partial p_1^{\alpha}} \left(p_1^{\mu_1} \frac{\partial}{\partial p_1^{\kappa}} Y^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \right) + 2p_1^{\mu_1} \frac{\partial}{p_1^{\kappa}} Y^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \\
&+ p_1^{\kappa} \frac{\partial}{\partial p_1^{\alpha}} \left(p_1^{\mu_1} \frac{\partial}{p_1^{\alpha}} Y^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \right) - p_1^{\kappa} \frac{\partial}{\partial p_1^{\mu_1}} Y^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \\
&= 2p_1^{\alpha} \frac{\partial}{\partial p_1^{\kappa}} Y^{\kappa\nu_1\mu_2\nu_2\mu_3\nu_3} - 2\, Y^{\kappa\nu_1\mu_2\nu_2\mu_3\nu_3} - 2p_1^{\kappa} \frac{\partial}{\partial p_1^{\mu_1}} Y^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}
\end{aligned} \tag{6.23}
$$

where we have rearranged the partial derivatives. For the spin part we obtain

$$
\begin{aligned}
p_1^{\mu_1} K^{\kappa}_{1spin} Y^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} &= 2p_1^{\mu_1} \left(\delta^{\kappa\mu_1} \frac{\partial}{\partial p_1^{\alpha}} - \delta^{\kappa\alpha} \frac{\partial}{p_1^{\mu_1}} \right) Y^{\alpha\nu_1\mu_2\nu_2\mu_3\nu_3} \\
&+ 2p_1^{\mu_1} \left(\delta^{\kappa\nu_1} \frac{\partial}{\partial p_1^{\alpha}} - \delta^{\kappa\alpha} \frac{\partial}{p_1^{\nu_1}} \right) Y^{\mu_1\alpha\mu_2\nu_2\mu_3\nu_3} \\
&= 2p_1^{\kappa} \frac{\partial}{\partial p_1^{\alpha}} Y^{\alpha\nu_1\mu_2\nu_2\mu_3\nu_3} - 2p_1^{\mu_1} \frac{\partial}{\partial p_1^{\mu_1}} Y^{\kappa\nu_1\mu_2\nu_2\mu_3\nu_3} \\
&+ 2Y^{\kappa\nu_1\mu_2\nu_2\mu_3\nu_3}.
\end{aligned} \tag{6.24}
$$

Adding (6.23) and (6.24) it is shown that

$$
p_1^{\mu_1} K_1^{\kappa} Y^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} = 0, \tag{6.25}
$$

which clearly holds for the entire K operator since Π_1 filters to the left of K_2, obtaining

$$
p_1^{\mu_1} K^{\kappa} Y^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} = 0. \tag{6.26}
$$

Notice that in the derivation of this result the nonlinear character of the action of K^{κ}_{scalar}, which induces mixed derivative terms does not play any role. Due to the trace-free property of the projectors, then we obtain in our condensed notation

$$
\Sigma_1 K^{\kappa} \phi_{ttt} = 0. \tag{6.27}
$$

The solution of the CWI's are then constructed, in this method, by acting on the entire correlator having parametrized its transverse traceless parts in therms of a minimal set of form factors plus trace/longitudinal terms (the semilocal or pinched terms). Semilocal terms are those containing a single delta function which will pinch two of the three external coordinates. The term ultralocal (or local) refers to the contribution of the anomaly itself, which is obtained when all the 3 point of the correlator coalesce.

6.2. Application to the T J J

Turning to the TJJ case, we can divide the 3-point function into two parts: the *transverse-traceless* part and the *semi-local* part (indicated by subscript loc) expressible through the transverse and trace Ward Identities. These parts are obtained by using the projectors Π and Σ, previously defined. We can then decompose the full 3-point function as follows

$$\begin{aligned}\langle T^{\mu_1\nu_1}\, J^{\mu_2}\, J^{\mu_3}\rangle &= \langle t^{\mu_1\nu_1}\, j^{\mu_2}\, j^{\mu_3}\rangle + \langle T^{\mu_1\nu_1}\, J^{\mu_2}\, j^{\mu_3}_{loc}\rangle + \langle T^{\mu_1\nu_1}\, j^{\mu_2}_{loc}\, J^{\mu_3}\rangle \\ &+ \langle t^{\mu_1\nu_1}_{loc}\, J^{\mu_2}\, J^{\mu_3}\rangle - \langle T^{\mu_1\nu_1}\, j^{\mu_2}_{loc}\, j^{\mu_3}_{loc}\rangle - \langle t^{\mu_1\nu_1}_{loc}\, j^{\mu_2}_{loc}\, J^{\mu_3}\rangle \\ &- \langle t^{\mu_1\nu_1}_{loc}\, J^{\mu_2}\, j^{\mu_3}_{loc}\rangle + \langle t^{\mu_1\nu_1}_{loc}\, j^{\mu_2}_{loc}\, j^{\mu_3}_{loc}\rangle\,. \end{aligned} \tag{6.28}$$

All the terms on the right-hand side, apart from the first one, may be computed by means of transverse and trace Ward Identities. The exact form of the Ward identities depends on the exact definition of the operators involved, but more importantly, all these terms depend on 2-point function only. The main goal now is to write the general form of the transverse-traceless part of the correlator and to give the solution using the Conformal Ward identities.

Using the projectors Π and π one can write the most general form of the transverse-traceless part as

$$\langle t^{\mu_1\nu_1}(p_1)\, j^{\mu_2}(p_2)\, j^{\mu_3}(p_3)\rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\pi^{\mu_2}_{\alpha_2}(p_2)\pi^{\mu_3}_{\alpha_3}(p_3)\; X^{\alpha_1\beta_1\,\alpha_3\alpha_3}, \tag{6.29}$$

where $X^{\alpha_1\beta_1\,\alpha_3\alpha_3}$ is a general tensor of rank four built from the metric and momenta. We can enumerate all possible tensor that can appear in $X^{\alpha_1\beta_1\,\alpha_3\alpha_3}$ preserving the symmetry of the correlator, as illustrated in [14]

$$\begin{aligned}&\langle t^{\mu_1\nu_1}(p_1) j^{\mu_2}(p_2) j^{\mu_3}(p_3)\rangle \\ &= \Pi_1{}^{\mu_1\nu_1}_{\alpha_1\beta_1}\pi_2{}^{\mu_2}_{\alpha_2}\pi_3{}^{\mu_3}_{\alpha_3}\Big(A_1\; p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_1^{\alpha_3} + A_2\;\delta^{\alpha_2\alpha_3} p_2^{\alpha_1} p_2^{\beta_1} + A_3\;\delta^{\alpha_1\alpha_2} p_2^{\beta_1} p_1^{\alpha_3} \\ &+ A_3(p_2\leftrightarrow p_3)\delta^{\alpha_1\alpha_3} p_2^{\beta_1} p_3^{\alpha_2} + A_4\;\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1}\Big)\,, \end{aligned} \tag{6.30}$$

where we have used the symmetry properties of the projectors, and the coefficients A_i $i = 1,\ldots,4$ are the form factors, functions of p_1^2, p_2^2 and p_3^2. This ansatz introduces a minimal set of form factors which will be later determined by the solutions of the CWI's. For future discussion, we will refer to this basis as to the A-basis.

We can now consider the dilatation Ward identities for the transverse-traceless part obtained by the decomposition of (6.2). We are then free to apply the projectors Π and π to this decomposition in order to obtain the final result

$$\begin{aligned}0 = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\pi^{\mu_2}_{\alpha_2}(p_2)\pi^{\mu_3}_{\alpha_3}(p_3)&\left[\sum_{j=1}^{3}\Delta_j - 2d - \sum_{j=1}^{2} p_j^{\alpha}\frac{\partial}{\partial p_j^{\alpha}}\right]\Big[A_1 p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_1^{\alpha_3} \\ &+ A_2\delta^{\alpha_2\alpha_3} p_2^{\alpha_1} p_2^{\beta_1} + A_3\delta^{\alpha_1\alpha_2} p_2^{\beta_1} p_1^{\alpha_3} + A_3(p_2\leftrightarrow p_3)\delta^{\alpha_1\alpha_3} p_2^{\beta_1} p_3^{\alpha_2} + A_4\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1}\Big]. \end{aligned} \tag{6.31}$$

It is possible to obtain from this projection a set of differential equations for all the form factors. These equations are expressed as

$$\left[2d + N_n - \sum_{j=1}^{3}\Delta_j + \sum_{j=1}^{2} p_j^{\alpha}\frac{\partial}{\partial p_j^{\alpha}}\right] A_n(p_1, p_2, p_3) = 0, \tag{6.32}$$

where N_n is the tensorial dimension of A_n, i.e. the number of momenta multiplying the form factor A_n and the projectors Π and π.

Turning to the special CWI's, $\langle T\,J\,J\rangle$ in (6.5), we can write the same equation in the form

$$K^{\kappa}\,\langle T^{\mu_1\nu_1}(p_1)\, J^{\mu_2}(p_2)\, J^{\mu_3}(p_3)\rangle = 0, \tag{6.33}$$

where K^κ is the special conformal generator. As before, we introduce the decomposition of the 3-point function to obtain

$$0 = K^\kappa \Big[\langle t^{\mu_1\nu_1}\, j^{\mu_2}\, j^{\mu_3}\rangle + \langle t^{\mu_1\nu_1}_{loc}\, j^{\mu_2}\, j^{\mu_3}\rangle + \langle t^{\mu_1\nu_1}\, j^{\mu_2}_{loc}\, j^{\mu_3}\rangle + \langle t^{\mu_1\nu_1}\, j^{\mu_2}\, j^{\mu_3}_{loc}\rangle + \langle t^{\mu_1\nu_1}_{loc}\, j^{\mu_2}_{loc}\, j^{\mu_3}\rangle + \langle t^{\mu_1\nu_1}_{loc}\, j^{\mu_2}\, j^{\mu_3}_{loc}\rangle + \langle t^{\mu_1\nu_1}\, j^{\mu_2}_{loc}\, j^{\mu_3}_{loc}\rangle + \langle t^{\mu_1\nu_1}_{loc}\, j^{\mu_2}_{loc}\, j^{\mu_3}_{loc}\rangle \Big]. \tag{6.34}$$

In order to isolate the equations for the form factors appearing in the decomposition, we are free to apply the projectors Π and π defined previously. Through a lengthy calculation we find

$$\begin{aligned}
&\Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}(p_1)\pi^{\rho_2}_{\mu_2}(p_2)\pi^{\rho_3}_{\mu_3}(p_3)\;\; K^\kappa \langle t^{\mu_1\nu_1}_{loc}\, j^{\mu_2}\, j^{\mu_3}\rangle \\
&\quad = \Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}\pi^{\rho_2}_{\mu_2}\pi^{\rho_3}_{\mu_3}\left[\frac{4d}{p_1^2}\,\delta^{\kappa\mu_1}\, p_{1\alpha_1}\,\langle T^{\alpha_1\nu_1} J^{\mu_2} J^{\mu_3}\rangle\right] \\
&\Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}(p_1)\pi^{\rho_2}_{\mu_2}(p_2)\pi^{\rho_3}_{\mu_3}(p_3)\;\; K^\kappa \langle t^{\mu_1\nu_1}\, j^{\mu_2}_{loc}\, j^{\mu_3}\rangle \\
&\quad = \Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}\pi^{\rho_2}_{\mu_2}\pi^{\rho_3}_{\mu_3}\left[\frac{2(d-2)}{p_2^2}\delta^{\kappa\mu_2}\, p_{2\alpha_2}\,\langle T^{\alpha_1\nu_1} J^{\alpha_2} J^{\mu_3}\rangle\right] \\
&\Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}(p_1)\pi^{\rho_2}_{\mu_2}(p_2)\pi^{\rho_3}_{\mu_3}(p_3)\;\; K^\kappa \langle t^{\mu_1\nu_1}\, j^{\mu_2}\, j^{\mu_3}_{loc}\rangle \\
&\quad = \Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}\pi^{\rho_2}_{\mu_2}\pi^{\rho_3}_{\mu_3}\left[\frac{2(d-2)}{p_3^2}\,\delta^{\kappa\mu_3}\, p_{3\alpha_3}\,\langle T^{\alpha_1\nu_1} J^{\mu_2} J^{\alpha_3}\rangle\right]
\end{aligned} \tag{6.35}$$

and all the terms with at least two insertion of local terms are zero. T We have verified, as expected, that the equations above remain invariant if we choose as independent momenta p_2 and p_3 while acting on p_1 indirectly by the derivative chain rule. More details on this analysis will be given in a section below. In this way we may rewrite (6.5) in the form

$$\begin{aligned}
0 &= \Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}(p_1)\pi^{\rho_2}_{\mu_2}(p_2)\pi^{\rho_3}_{\mu_3}(p_3)\left(K^\kappa\,\langle T^{\mu_1\nu_1}(p_1)\, J^{\mu_2}(p_2)\, J^{\mu_3}(p_3)\rangle\right) \\
&= \Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}(p_1)\pi^{\rho_2}_{\mu_2}(p_2)\pi^{\rho_3}_{\mu_3}(p_3)\Big\{K^\kappa\,\langle t^{\mu_1\nu_1}(p_1)\, j^{\mu_2}(p_2)\, j^{\mu_3}(p_3)\rangle \\
&\qquad + \frac{4d}{p_1^2}\,\delta^{\kappa\mu_1}\, p_{1\alpha_1}\,\langle T^{\alpha_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3)\rangle \\
&\qquad + \frac{2(d-2)}{p_2^2}\,\delta^{\kappa\mu_2}\, p_{2\alpha_2}\,\langle T^{\alpha_1\nu_1} J^{\alpha_2} J^{\mu_3}\rangle + \frac{2(d-2)}{p_3^2}\,\delta^{\kappa\mu_3}\, p_{3\alpha_3}\,\langle T^{\alpha_1\nu_1} J^{\mu_2} J^{\alpha_3}\rangle\Big\}.
\end{aligned} \tag{6.36}$$

The equation above is an independent derivation of the corresponding BMS result, which is not offered in [14]. Notice that our derivation, which details the various contributions coming from the local terms in the TJJ, has been derived using heavily the Lorentz Ward identities.

The last three terms may be re-expressed in terms of 2-point functions via the transverse Ward identities. After other rather lengthy computations, we find that the first term in the previous expression, corresponding to the transverse traceless contributions, can be written in the form

$$
\begin{aligned}
&\Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}(p_1)\pi^{\rho_2}_{\mu_2}(p_2)\pi^{\rho_3}_{\mu_3}(p_3)\left[\, K^\kappa\ \langle t^{\mu_1\nu_1}(p_1)\, j^{\mu_2}(p_2)\, j^{\mu_3}(p_3)\rangle\,\right]\\
&=\Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}(p_1)\pi^{\rho_2}_{\mu_2}(p_2)\pi^{\rho_3}_{\mu_3}(p_3)\times\\
&\times\Big[p_1^\kappa\big(C_{11}\,p_1^{\mu_3}p_2^{\mu_1}p_2^{\nu_1}p_3^{\mu_2}+C_{12}\,\delta^{\mu_2\mu_3}p_2^{\mu_1}p_2^{\nu_1}+C_{13}\delta^{\mu_1\mu_2}p_2^{\nu_1}p_1^{\mu_3}+C_{14}\delta^{\mu_1\mu_3}p_2^{\nu_1}p_3^{\mu_2}\\
&\quad+C_{15}\delta^{\mu_1\mu_2}\delta^{\nu_1\mu_3}\big)+p_2^\kappa\big(C_{21}\,p_1^{\mu_3}p_2^{\mu_1}p_2^{\nu_1}p_3^{\mu_2}+C_{22}\,\delta^{\mu_2\mu_3}p_2^{\mu_1}p_2^{\nu_1}+C_{23}\delta^{\mu_1\mu_2}p_2^{\nu_1}p_1^{\mu_3}\\
&\quad+C_{24}\delta^{\mu_1\mu_3}p_2^{\nu_1}p_3^{\mu_2}+C_{25}\delta^{\mu_1\mu_2}\delta^{\nu_1\mu_3}\big)+\delta^{\mu_1\kappa}\big(C_{31}\,p_1^{\mu_3}p_2^{\nu_1}p_3^{\mu_2}+C_{32}\,\delta^{\mu_2\mu_3}p_2^{\nu_1}\\
&\qquad+C_{33}\,\delta^{\mu_2\nu_1}p_1^{\mu_3}+C_{34}\,\delta^{\mu_3\nu_1}p_3^{\mu_2}\big)+\delta^{\mu_2\kappa}\big(C_{41}\,p_1^{\mu_3}p_2^{\mu_1}p_2^{\nu_1}+C_{42}\,\delta^{\mu_1\mu_3}p_2^{\nu_1}\big)\\
&\qquad+\delta^{\mu_3\kappa}\big(C_{51}\,p_3^{\mu_2}p_2^{\nu_1}p_3^{\mu_2}+C_{52}\,\delta^{\mu_1\mu_2}p_2^{\nu_1}\big)\Big]
\end{aligned}
\tag{6.37}
$$

where now C_{ij} are differential equations involving the form factors $A_1,\ A_2,\ A_3,\ A_4$ of the representation of the $\langle tjj\rangle$ in (6.30). For any 3-point function, the resulting equations can be divided into two groups, the *primary* and the *secondary* conformal Ward identities. The primary are second-order differential equations and appear as the coefficients of transverse or transverse-traceless tensor containing p_1^κ and p_2^κ, where κ is the special index related to the conformal operator K^κ. The remaining equations, following from all other transverse or transverse-traceless terms, are then secondary conformal Ward identities and are first-order differential equations.

6.3. Primary CWI's

From (6.36) and (6.37) one finds that the primary CWI's are equivalent to the vanishing of the coefficients C_{1j} and C_{2j} for $j=1,\dots,5$. The CWI's can be rewritten in terms of the operators defined in Eq. (4.9) as

$$
\begin{aligned}
0&=C_{11}=K_{13}A_1 & 0&=C_{21}=K_{23}A_1\\
0&=C_{12}=K_{13}A_2+2A_1 & 0&=C_{22}=K_{23}A_2\\
0&=C_{13}=K_{13}A_3-4A_1 & 0&=C_{23}=K_{23}A_3-4A_1\\
0&=C_{14}=K_{13}A_3(p_2\leftrightarrow p_3) & 0&=C_{24}=K_{23}A_3(p_2\leftrightarrow p_3)+4A_1\\
0&=C_{15}=K_{13}A_4-2A_3(p_2\leftrightarrow p_3) & 0&=C_{25}=K_{23}A_4+2A_3-2A_3(p_2\leftrightarrow p_3)
\end{aligned}
\tag{6.38}
$$

6.4. Secondary CWI's

The secondary conformal Ward identities are first-order partial differential equations and in principle involve the semi-local information contained in j^μ_{loc} and $t^{\mu\nu}_{loc}$. In order to write them compactly, one defines the two differential operators

$$
\begin{aligned}
L_N&=p_1(p_1^2+p_2^2-p_3^2)\frac{\partial}{\partial p_1}+2p_1^2\,p_2\frac{\partial}{\partial p_2}\\
&\quad+\big[(2d-\Delta_1-2\Delta_2+N)p_1^2+(2\Delta_1-d)(p_3^2-p_2^2)\big] &(6.39)\\
R&=p_1\frac{\partial}{\partial p_1}-(2\Delta_1-d)\,. &(6.40)
\end{aligned}
$$

The reason for introducing such operators comes from (6.37), once the action of K^κ is made explicit. The separation between the two sets of constraints comes from the same equation, and in particular from the terms trilinear in the momenta within the square bracket. One needs also the symmetric versions of such operators

$$L'_N = L_N, \quad \text{with } p_1 \leftrightarrow p_2 \text{ and } \Delta_1 \leftrightarrow \Delta_2, \tag{6.41}$$

$$R' = R, \quad \text{with } p_1 \mapsto p_2 \text{ and } \Delta_1 \mapsto \Delta_2. \tag{6.42}$$

These operators depend on the conformal dimensions of the operators involved in the 3-point function under consideration, and additionally on a single parameter N determined by the Ward identity in question. In the $\langle T\,J\,J\rangle$ case one finds considering the structure of Eqs. (6.36) and (6.37)

$$\begin{aligned}
C_{31} &= -\frac{2}{p_1^2}\left[L_4 A_1 + R A_3 - R A_3(p_2 \leftrightarrow p_3)\right] \\
C_{32} &= -\frac{2}{p_1^2}\left[L_2\, A_2 - p_1^2(A_3 - A_3(p_2 \leftrightarrow p_3))\right] \\
C_{33} &= -\frac{1}{p_1^2}\left[L_4\, A_3 - 2R\, A_4\right] \\
C_{34} &= -\frac{1}{p_1^2}\left[L_4\, A_3(p_2 \leftrightarrow p_3) + 2R\, A_4 - 4p_1^2 A_3(p_2 \leftrightarrow p_3)\right]
\end{aligned} \tag{6.43}$$

$$\begin{aligned}
C_{41} &= \frac{1}{p_2^2}\left[L'_3\, A_1 - 2R' A_2 + 2R' A_3\right] \\
C_{42} &= \frac{1}{p_2^2}\left[L'_1\, A_3(p_2 \leftrightarrow p_3) + p_2^2(4A_2 - 2A_3) + 2R' A_4\right]
\end{aligned} \tag{6.44}$$

$$\begin{aligned}
C_{51} &= \frac{1}{p_3}\left[(L_4 - L'_3)A_1 - 2(2d + R + R')A_2 + 2(2d + R + R')A_3(p_2 \leftrightarrow p_3)\right] \\
C_{52} &= \frac{1}{p_3^2}\left[(L_2 - L'_1)A_3 - 4p_3^2 A_2 + 2p_3^2 A_3(p_2 \leftrightarrow p_3) + 2(2d - 2 + R + R')A_4\right]
\end{aligned}$$

From (6.36) and (6.37) using (5.22) the secondary CWI's take the explicit form

$$\begin{aligned}
&C_{31} = C_{41} = C_{42} = C_{51} = C_{52} = 0, \qquad C_{32} = \frac{16\, d\, c_{123}\, \Gamma_J}{p_1^2}\left[\frac{1}{(p_3^2)^{\sigma_0}} - \frac{1}{(p_2^2)^{\sigma_0}}\right], \\
&C_{33} = \frac{16\, d\, c_{123}\, \Gamma_J}{p_1^2 (p_3^2)^{\sigma_0}}, \qquad C_{34} = -\frac{16\, d\, c_{123}\, \Gamma_J}{p_1^2\, (p_2^2)^{\sigma_0}},
\end{aligned} \tag{6.45}$$

where in our $\sigma_0 = d/2 - \Delta_2$. Expressed in this form all the scalar equations for the A_i are not apparently symmetric in the exchange of p_2 and p_3, and it may not be immediately evident that they can be recast in such a way that the symmetry is respected.

7. Symmetric treatment of the J currents

Let's now consider p_1 as dependent momentum, showing the equivalence of the CWI's with this second choice. As we have just mentioned above, this choice is the preferred one in the

search for the solutions of the TJJ. In this case, the action of the spin (Lorentz) part of the transformation will leave the stress energy tensor as a singlet, acting implicitly on p_1 via the chain rule. As we are going to show, the resulting equations will be linear combinations of the original part. This extends the analysis presented by BMS.

The structure of the decomposition in (6.30) of the $\langle TJJ\rangle$ correlator is still valid but now the explicit form of the special conformal operator K^κ has to be modified as

$$\begin{aligned}
K^\kappa\langle T^{\mu_1\nu_1}\,J^{\mu_2}\,J^{\mu_3}\rangle = \sum_{j=2}^{3}\Bigg[&2(\Delta_j-d)\frac{\partial}{\partial p_j^\kappa}-2p_j^\alpha\frac{\partial}{\partial p_j^\alpha}\frac{\partial}{\partial p_j^\kappa}+(p_j)_\kappa\frac{\partial}{\partial p_j^\alpha}\frac{\partial}{\partial p_{j\alpha}}\Bigg]\\
&\times\langle T^{\mu_1\nu_1}(\bar p_1)\,J^{\mu_2}(p_2)\,J^{\mu_3}(p_3)\rangle\\
&+2\left(\delta^{\kappa\mu_2}\frac{\partial}{\partial p_2^{\alpha_2}}-\delta^\kappa_{\alpha_2}\delta^{\lambda\mu_2}\frac{\partial}{\partial p_2^\lambda}\right)\langle T^{\mu_1\nu_1}(\bar p_1)\,J^{\alpha_2}(p_2)\,J^{\mu_3}(p_3)\rangle\\
&+2\left(\delta^{\kappa\mu_3}\frac{\partial}{\partial p_3^{\alpha_3}}-\delta^\kappa_{\alpha_3}\delta^{\lambda\mu_3}\frac{\partial}{\partial p_3^\lambda}\right)\langle T^{\mu_1\nu_1}(\bar p_1)\,J^{\mu_2}(p_2)\,J^{\alpha_3}(p_3)\rangle
\end{aligned}\tag{7.1}$$

where $\bar p_1^\mu=-p_2^\mu-p_3^\mu$. Considering the SCWI's for the 3-point function we can write

$$K^\kappa(p_2,p_3)\langle T^{\mu_1\nu_1}(\bar p_1)\,J^{\mu_2}(p_2)\,J^{\mu_3}(p_3)\rangle=0,$$

in which we have stress the p_2 and p_3 dependence of the special conformal operator. Then one has to take the decomposition of the 3-point function as in (6.30) and using the relations (6.35), that are still valid in this case, one derives (6.36), in which now the K operator is defined in terms of p_2 and p_3 only. As in the previous case, one finds CWI's which are similar to those given in (6.37)

$$\begin{aligned}
&\Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}(p_1)\pi^{\rho_2}_{\mu_2}(p_2)\pi^{\rho_3}_{\mu_3}(p_3)\left[K^\kappa\,\langle t^{\mu_1\nu_1}(p_1)\,j^{\mu_2}(p_2)\,j^{\mu_3}(p_3)\rangle\right]\\
&=\Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}(p_1)\pi^{\rho_2}_{\mu_2}(p_2)\pi^{\rho_3}_{\mu_3}(p_3)\times\\
&\times\Bigg[p_2^\kappa\Big(\tilde C_{11}\,p_1^{\mu_3}p_2^{\mu_1}p_2^{\nu_1}p_3^{\mu_2}+\tilde C_{12}\,\delta^{\mu_2\mu_3}p_2^{\mu_1}p_2^{\nu_1}+\tilde C_{13}\delta^{\mu_1\mu_2}p_2^{\nu_1}p_1^{\mu_3}+\tilde C_{14}\delta^{\mu_1\mu_3}p_2^{\nu_1}p_3^{\mu_2}\\
&\quad+\tilde C_{15}\delta^{\mu_1\mu_2}\delta^{\nu_1\mu_3}\Big)+p_3^\kappa\Big(\tilde C_{21}\,p_1^{\mu_3}p_2^{\mu_1}p_2^{\nu_1}p_3^{\mu_2}+\tilde C_{22}\,\delta^{\mu_2\mu_3}p_2^{\mu_1}p_2^{\nu_1}+\tilde C_{23}\delta^{\mu_1\mu_2}p_2^{\nu_1}p_1^{\mu_3}\\
&\quad+\tilde C_{24}\delta^{\mu_1\mu_3}p_2^{\nu_1}p_3^{\mu_2}+\tilde C_{25}\delta^{\mu_1\mu_2}\delta^{\nu_1\mu_3}\Big)+\delta^{\mu_1\kappa}\Big(\tilde C_{31}\,p_1^{\mu_3}p_2^{\nu_1}p_3^{\mu_2}+\tilde C_{32}\,\delta^{\mu_2\mu_3}p_2^{\nu_1}\\
&\quad+\tilde C_{33}\,\delta^{\mu_2\nu_1}p_1^{\mu_3}+\tilde C_{34}\,\delta^{\mu_3\nu_1}p_3^{\mu_2}\Big)+\delta^{\mu_2\kappa}\Big(\tilde C_{41}\,p_1^{\mu_3}p_2^{\mu_1}p_2^{\nu_1}+\tilde C_{42}\,\delta^{\mu_1\mu_3}p_2^{\nu_1}\Big)\\
&\quad+\delta^{\mu_3\kappa}\Big(\tilde C_{51}\,p_3^{\mu_2}p_2^{\nu_1}p_3^{\mu_2}+\tilde C_{52}\,\delta^{\mu_1\mu_2}p_2^{\nu_1}\Big)\Bigg].
\end{aligned}\tag{7.2}$$

In this case we obtain the primary WI's by imposing the vanishing of the coefficients $\tilde C_{ij}$, for $i=1,2$ and $j=1,\dots,5$. In this way we get

$$
\begin{aligned}
0 &= \tilde{C}_{11} = K_{21} A_1 & 0 &= \tilde{C}_{21} = K_{31} A_1 \\
0 &= \tilde{C}_{12} = K_{21} A_2 - 2A_1 & 0 &= \tilde{C}_{22} = K_{31} A_2 - 2A_1 \\
0 &= \tilde{C}_{13} = K_{21} A_3 & 0 &= \tilde{C}_{23} = K_{31} A_3 + 4A_1 \\
0 &= \tilde{C}_{14} = K_{21} A_3(p_2 \leftrightarrow p_3) + 4A_1 & 0 &= \tilde{C}_{24} = K_{31} A_3(p_2 \leftrightarrow p_3) \\
0 &= \tilde{C}_{15} = K_{21} A_4 + 2A_3 & 0 &= \tilde{C}_{25} = K_{31} A_4 + 2A_3(p_2 \leftrightarrow p_3)
\end{aligned}
\tag{7.3}
$$

and it is simple to verify that these equations are equivalent to those given in (6.38). In the case of the secondary WI's we have to consider some further properties of the form factors. For instance the coefficient $\tilde{C}_{31}$ has the explicit form

$$
\begin{aligned}
\tilde{C}_{31} = \frac{2}{p_1^2}\Big[& p_2(p_1^2 - p_2^2 + p_3^2)\frac{\partial}{\partial p_2} A_1 - p_1^2 p_3 \frac{\partial}{\partial p_3} A_1 - p_2^2 p_3 \frac{\partial}{\partial p_3} A_1 + p_3^3 \frac{\partial}{\partial p_3} A_1 \\
& - p_2 \frac{\partial}{\partial p_2} A_3 - p_3 \frac{\partial}{\partial p_3} A_3 + p_2 \frac{\partial}{\partial p_2} A_3(p_2 \leftrightarrow p_3) + p_3 \frac{\partial}{\partial p_3} A_3(p_2 \leftrightarrow p_3) \\
& - 6(p_2^2 - p_3^2) A_1 - 4(A_3 - A_3(p_2 \leftrightarrow p_3)) \Big]
\end{aligned}
\tag{7.4}
$$

in which it is possible to substitute the derivative with respect to p_3 in terms of derivatives with respect to p_2 and p_1 using the dilatation Ward identities

$$
\frac{\partial}{\partial p_3} A_n = \frac{1}{p_3}\Big[(d - 2 - N_n) A_n - \sum_{j=1}^{2} p_j \frac{\partial}{\partial p_j} A_n \Big]. \tag{7.5}
$$

Using the identity given above in (7.4), one derives the relation

$$
\tilde{C}_{31} = \frac{2}{p_1^2}\Big[L_4 A_1 + R A_3 - R A_3(p_2 \leftrightarrow p_3) \Big] \tag{7.6}
$$

with the identification of the differential operators L and R defined in (6.39) and (6.40). In this way it is possible to show that all the coefficients related to the secondary Ward identities are the same of those obtained with p_3 as the dependent momentum. This argument proves that in spite of the choice of the dependent momentum, the scalar equations for the form factors related to the CWI's remain identical.

8. The Fuchsian approach to the solutions of the primary CWI's and universality

In this section we are going to investigate the Fuchsian structure of the equations. The goal of the section is to present a new method of solution which differs from the one based on 3K integrals presented in [14]. We should mention that the number of integration constants introduced by the primary CWI's, using this method, may not necessarily coincide with those presented in [14], and the constraints imposed by the secondary CWIs, that we will not discuss, will obviously be different.

The goal of this section is twofold. We want to show first of all that the Fuchsian exponents (defined as (a_i, b_j) below), are universal and characterize the entire system of equations. In the scalar case, as well as for all the 3-point functions that we have investigated, we have verified that always the same set of exponents (a_i, b_j) are generated.

The second important feature is that the method allows to characterize particular solutions of 4 and higher point functions in some restricted kinematics, allowing a significant generalization

of the analysis presented here, with new special functions appearing in the solutions. Details of this study will be presented in a separate work.

Being the CWI's a system of equations, we will first solve for each of the form factors, starting from the equations for A_1, which are homogeneous, and then proceed towards the inhomogeneous ones, from A_2 to A_4. For each form factor we identify the general solution and a particular solution, which are added together. Then we impose the symmetry constraints on the two vector lines, due to Bose symmetry. For example, the solution for A_2 will constraint the constants appearing in the general solution of A_1, and so on for A_3 and A_4. The independent constants of integration are identified only at the end, once all the constraints from A_1 to A_4 are put together. We have included a small section where we summarize the final expressions of the form factors by this method.

8.1. Scalar 3-point functions

To illustrate our approach we start reviewing the case of the scalar correlator $\Phi(p_1, p_2, p_3)$, which is simpler, defined by the two homogeneous conformal equations

$$K_{31}\Phi = 0 \qquad K_{21}\Phi = 0 \tag{8.1}$$

combined with the scaling equation

$$\sum_{i=1}^{3} p_i \frac{\partial}{\partial p_i}\Phi = (\Delta - 2d)\Phi. \tag{8.2}$$

Following the approach presented in [24], the ansatz for the solution can be taken of the form

$$\Phi(p_1, p_2, p_3) = p_1^{\Delta-2d} x^a y^b F(x, y) \tag{8.3}$$

with $x = \frac{p_2^2}{p_1^2}$ and $y = \frac{p_3^2}{p_1^2}$. Here we are taking p_1 as "pivot" in the expansion, but we could equivalently choose any of the 3 momentum invariants. Φ is required to be homogenous of degree $\Delta - 2d$ under a scale transformation, according to (8.2), and in (8.3) this is taken into account by the factor $p_1^{\Delta-2d}$. The use of the scale invariant variables x and y takes to the hypergeometric form of the solution. One obtains

$$K_{21}\phi = 4p_1^{\Delta-2d-2} x^a y^b \Big(x(1-x)\frac{\partial}{\partial x \partial x} + (Ax+\gamma)\frac{\partial}{\partial x} - 2xy\frac{\partial^2}{\partial x \partial y} - y^2 \frac{\partial^2}{\partial y \partial y} + Dy\frac{\partial}{\partial y} + (E + \frac{G}{x})\Big) F(x, y) = 0 \tag{8.4}$$

with

$$A = D = \Delta_2 + \Delta_3 - 1 - 2a - 2b - \frac{3d}{2} \qquad \gamma(a) = 2a + \frac{d}{2} - \Delta_2 + 1$$
$$G = \frac{a}{2}(d + 2a - 2\Delta_2)$$
$$E = -\frac{1}{4}(2a + 2b + 2d - \Delta_1 - \Delta_2 - \Delta_3)(2a + 2b + d - \Delta_3 - \Delta_2 + \Delta_1). \tag{8.5}$$

Similar constraints are obtained from the equation $K_{31}\Phi = 0$, with the obvious exchanges $(a, b, x, y) \to (b, a, y, x)$

$$K_{31}\phi = 4p_1^{\Delta-2d-2}x^a y^b\Big(y(1-y)\frac{\partial}{\partial y\partial y} + (A'y+\gamma')\frac{\partial}{\partial y} - 2xy\frac{\partial^2}{\partial x\partial y} - x^2\frac{\partial^2}{\partial x\partial x} + D'x\frac{\partial}{\partial x} + (E' + \frac{G'}{y})\Big)F(x,y) = 0 \tag{8.6}$$

with

$$A' = D' = A \qquad \gamma'(b) = 2b + \frac{d}{2} - \Delta_3 + 1$$
$$G' = \frac{b}{2}(d + 2b - 2\Delta_3)$$
$$E' = E. \tag{8.7}$$

Notice that in (8.6) we need to set $G/x = 0$ in order to perform the reduction to the hypergeometric form of the equations, which implies that

$$a = 0 \equiv a_0 \qquad \text{or} \qquad a = \Delta_2 - \frac{d}{2} \equiv a_1. \tag{8.8}$$

From the equation $K_{31}\Phi = 0$ we obtain a similar condition for b by setting $G'/y = 0$, thereby fixing the two remaining indices

$$b = 0 \equiv b_0 \qquad \text{or} \qquad b = \Delta_3 - \frac{d}{2} \equiv b_1. \tag{8.9}$$

The four independent solutions of the CWI's will all be characterized by the same 4 pairs of indices (a_i, b_j) $(i, j = 1, 2)$. Setting

$$\alpha(a,b) = a + b + \frac{d}{2} - \frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1) \qquad \beta(a,b) = a + b + d - \frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3) \tag{8.10}$$

then

$$E = E' = -\alpha(a,b)\beta(a,b) \qquad A = D = A' = D' = -\left(\alpha(a,b) + \beta(a,b) + 1\right), \tag{8.11}$$

the solutions take the form

$$F_4(\alpha(a,b), \beta(a,b); \gamma(a), \gamma'(b); x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\alpha(a,b), i+j)\,(\beta(a,b), i+j)}{(\gamma(a), i)\,(\gamma'(b), j)} \frac{x^i}{i!}\frac{y^j}{j!} \tag{8.12}$$

where $(\alpha, i) = \Gamma(\alpha + i)/\Gamma(\alpha)$ is the Pochammer symbol. We will refer to $\alpha \ldots \gamma'$ as to the first, …, fourth parameters of F_4.

The 4 independent solutions are then all of the form $x^a y^b F_4$, where the hypergeometric functions will take some specific values for its parameters, with a and b fixed by (8.8) and (8.9). Specifically we have

$$\Phi(p_1, p_2, p_3) = p_1^{\Delta-2d}\sum_{a,b} c(a, b, \vec{\Delta})\, x^a y^b\, F_4(\alpha(a,b), \beta(a,b); \gamma(a), \gamma'(b); x, y) \tag{8.13}$$

where the sum runs over the four values a_i, b_i $i = 0, 1$ with arbitrary constants $c(a, b, \vec{\Delta})$, with $\vec{\Delta} = (\Delta_1, \Delta_2, \Delta_3)$. Notice that (8.13) is a very compact way to write down the solution. However, once this type of solutions of a homogeneous hypergeometric system are inserted into an

inhomogeneous system of equations, the sum over a and b needs to be made explicit. For this reason it is convenient to define

$$\alpha_0 \equiv \alpha(a_0, b_0) = \frac{d}{2} - \frac{\Delta_2 + \Delta_3 - \Delta_1}{2}, \qquad \beta_0 \equiv \beta(b_0) = d - \frac{\Delta_1 + \Delta_2 + \Delta_3}{2},$$
$$\gamma_0 \equiv \gamma(a_0) = \frac{d}{2} + 1 - \Delta_2, \qquad \gamma_0' \equiv \gamma(b_0) = \frac{d}{2} + 1 - \Delta_3, \tag{8.14}$$

to be the 4 basic (fixed) hypergeometric parameters, and define all the remaining ones by shifts respect to these. The 4 independent solutions can be re-expressed in terms of the parameters above as

$$S_1(\alpha_0, \beta_0; \gamma_0, \gamma_0'; x, y) \equiv F_4(\alpha_0, \beta_0; \gamma_0, \gamma_0'; x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\alpha_0, i+j)\,(\beta_0, i+j)}{(\gamma_0, i)\,(\gamma_0', j)} \frac{x^i}{i!}\frac{y^j}{j!} \tag{8.15}$$

and

$$S_2(\alpha_0, \beta_0; \gamma_0, \gamma_0'; x, y) = x^{1-\gamma_0} F_4(\alpha_0 - \gamma_0 + 1, \beta_0 - \gamma_0 + 1; 2 - \gamma_0, \gamma_0'; x, y)\,, \tag{8.16}$$
$$S_3(\alpha_0, \beta_0; \gamma_0, \gamma_0'; x, y) = y^{1-\gamma_0'} F_4(\alpha_0 - \gamma_0' + 1, \beta_0 - \gamma_0' + 1; \gamma_0, 2 - \gamma_0'; x, y)\,, \tag{8.17}$$
$$S_4(\alpha_0, \beta_0; \gamma_0, \gamma_0'; x, y) = x^{1-\gamma_0}\, y^{1-\gamma_0'} F_4(\alpha_0 - \gamma_0 - \gamma_0' + 2, \beta_0 - \gamma_0 - \gamma_0' + 2; 2 - \gamma_0, 2 - \gamma_0'; x, y)\,. \tag{8.18}$$

Notice that in the scalar case, one is allowed to impose the complete symmetry of the correlator under the exchange of the 3 external momenta and scaling dimensions, as discussed in [24]. This reduces the four constants to just one. We are going first to extend this analysis to the case of the $A_1 - A_4$ form factors of the TJJ.

8.2. *Form factors: the solution for A_1*

The solutions for the form factors $A_1 - A_4$ can be derived using a similar, but modified approach, being the equations also inhomogeneous. As previously we take as a pivot p_1^2, and assume a symmetry under the (P_{23}) exchange of (p_2, Δ_2) with (p_3, Δ_3) in the correlator. In the case of two photons $\Delta_2 = \Delta_3 = d - 1$.

We start from A_1 by solving the two equations from (6.38)

$$K_{21} A_1 = 0 \qquad K_{31} A_1 = 0. \tag{8.19}$$

In this case we introduce the ansatz

$$A_1 = p_1^{\Delta - 2d - 4} x^a y^b F(x, y) \tag{8.20}$$

and derive two hypergeometric equations, which are characterized by the same indices (a_i, b_j) as before in (8.8) and (8.9), but new values of the 4 defining parameters. We obtain

$$A_1(p_1, p_2, p_3) = p_1^{\Delta - 2d - 4} \sum_{a,b} c^{(1)}(a, b, \vec{\Delta})\, x^a y^b\, F_4(\alpha(a,b) + 2, \beta(a,b) + 2; \gamma(a), \gamma'(b); x, y) \tag{8.21}$$

with the expression of $\alpha(a,b), \beta(a,b), \gamma(a), \gamma'(b)$ as given before, with the obvious switching of the Δ_i in order to comply with the new choice of the pivot (p_1^2)

$$\alpha(a,b) = a + b + \frac{d}{2} - \frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1)$$
$$\beta(a,b) = a + b + d - \frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3) \tag{8.22}$$

which are P_{23} symmetric and

$$\gamma(a) = 2a + \frac{d}{2} - \Delta_2 + 1$$
$$\gamma'(b) = 2b + \frac{d}{2} - \Delta_3 + 1 \tag{8.23}$$

with $P_{23}\gamma(a) = \gamma'(b)$. If we require that $\Delta_2 = \Delta_3$, as in the TJJ case, the symmetry constraints are easily implemented. Given that the 4 indices, if we choose p_1 as a pivot, are given by

$$a_0 = 0, b_0 = 0, a_1 = \Delta_2 - \frac{d}{2}, b_1 = \Delta_3 - \frac{d}{2} \tag{8.24}$$

clearly in this case $a = b$ and $\gamma(a) = \gamma(b)$. F_4 has the symmetry

$$F_4(\alpha, \beta; \gamma, \gamma'; x, y) = F_4(\alpha, \beta; \gamma', \gamma; y, x), \tag{8.25}$$

and this reflects in the Bose symmetry of A_1 if we impose the constraint

$$c^{(1)}(a_1, b_0) = c^{(1)}(a_0, b_1). \tag{8.26}$$

8.3. The solution for A_2

The equations for A_2 are inhomogeneous. In this case the solution can be identified using some properties of the hypergeometric differential operators K_i, appropriately splitted. We recall that in this case they are

$$K_{21}A_2 = 2A_1 \tag{8.27}$$
$$K_{31}A_2 = 2A_1. \tag{8.28}$$

We take an ansatz of the form

$$A_2(p_1, p_2, p_3) = p_1^{\Delta - 2d - 2} F(x, y) \tag{8.29}$$

which provides the correct scaling dimensions for A_2. Observe that the action of K_{21} and K_3 on A_2 can be rearranged as follows

$$K_{21}A_2 = 4x^a y^b p_1^{\Delta - 2d - 4}\left(\bar{K}_{21}F(x,y) + \frac{\partial}{\partial x}F(x,y)\right) \tag{8.30}$$

$$K_{31}A_2 = 4x^a y^b p_1^{\Delta - 2d - 4}\left(\bar{K}_{31}F(x,y) + \frac{\partial}{\partial y}F(x,y)\right) \tag{8.31}$$

where

$$\bar{K}_{21}F(x,y)=\Big\{x(1-x)\frac{\partial^2}{\partial x^2}-y^2\frac{\partial^2}{\partial y^2}-2\,x\,y\frac{\partial^2}{\partial x\partial y}$$
$$+\big[(\gamma(a)-1)-(\alpha(a,b)+\beta(a,b)+3)x\big]\frac{\partial}{\partial x}$$
$$+\frac{a(a-a_1)}{x}-(\alpha(a,b)+\beta(a,b)+3)y\frac{\partial}{\partial y}-(\alpha+1)(\beta+1)\Big\}F(x,y), \tag{8.32}$$

and

$$\bar{K}_{31}A_2=\Big\{y(1-y)\frac{\partial^2}{\partial y^2}-x^2\frac{\partial^2}{\partial x^2}-2\,x\,y\frac{\partial^2}{\partial x\partial y}$$
$$+\big[(\gamma'(b)-1)-(\alpha(a,b)+\beta(a,b)+3)y\big]\frac{\partial}{\partial y}$$
$$+\frac{b(b-b_1)}{y}-(\alpha(a,b)+\beta(a,b)+3)x\frac{\partial}{\partial x}-(\alpha(a,b)+1)(\beta(a,b)+1)\Big\}$$
$$\times F(x,y). \tag{8.33}$$

At this point observe that the hypergeometric function solution of the equation

$$\bar{K}_{21}F(x,y)=0 \tag{8.34}$$

can be taken of the form

$$\Phi_1^{(2)}(x,y)$$
$$=p_1^{\Delta-2d-2}\sum_{a,b}c_1^{(2)}(a,b,\vec{\Delta})\,x^a y^b\,F_4(\alpha(a,b)+1,\beta(a,b)+1;\gamma(a)-1,\gamma'(b);x,y) \tag{8.35}$$

with $c_1^{(2)}$ a constant and the parameters a,b fixed at the ordinary values (a_i,b_j) as in the previous cases (8.8) and (8.9), in order to get rid of the $1/x$ and $1/y$ poles in the coefficients of the differential operators. The sequence of parameters in (8.35) will obviously solve the related equation

$$K_{31}\Phi_1^{(2)}(x,y)=0. \tag{8.36}$$

Eq. (8.34) can be verified by observing that the sequence of parameters $(\alpha(a,b)+1,\beta(a,b)+1\gamma(a)-1)$ allows to define a solution of (8.33) set to zero, for an arbitrary $\gamma'(b)$, since this parameter does not play any role in the solution of the corresponding equation. The sequence $(\alpha(a,b)+1,\beta(a,b)+1,\gamma'(b))$, on the other hand, solves the homogeneous equations associated to K_{31} (i.e. Eq. (8.36)) for any value of the third parameter of F_4, which in this case takes the value $\gamma(a)-1$. A similar result holds for the mirror solution

$$\Phi_2^{(2)}(x,y)$$
$$=p_1^{\Delta-2d-2}\sum_{a,b}c_2^{(2)}(a,b,\vec{\Delta})\,x^a y^b\,F_4(\alpha(a,b)+1,\beta(a,b)+1;\gamma(a),\gamma'(b)-1;x,y) \tag{8.37}$$

which satisfies

$$\bar{K}_{31}\Phi_2^{(2)}(x,y)=0\qquad K_{21}\Phi_2^{(2)}(x,y)=0. \tag{8.38}$$

As previously remarked, the values of the exponents a and b remain the same for any equation involving either a $K_{i,j}$ or a $\bar{K}_{ij}$, as can be explicitly verified. This implies that the fundamental solutions of the conformal equations are essentially the 4 functions of the type $S_1, \dots S_4$, for appropriate values of their parameters.

At this point, to show that F_1 and F_2 is a solution of Eqs. (8.27) we use the property

$$\frac{\partial^{p+q} F_4(\alpha,\beta;\gamma_1,\gamma_2;x,y)}{\partial x^p \partial y^q} = \frac{(\alpha,p+q)(\beta,p+q)}{(\gamma_1,p)(\gamma_2,q)} F_4(\alpha+p+q,\beta+p+q;\gamma_1+p;\gamma_2+q;x,y) \tag{8.39}$$

which gives (for generic parameters $\alpha,\beta,\gamma_1,\gamma_2$)

$$\frac{\partial F_4(\alpha,\beta;\gamma_1,\gamma_2;x,y)}{\partial x} = \frac{\alpha\beta}{\gamma_1} F_4(\alpha+1,\beta+1,\gamma_1+1,\gamma_2,x,y)$$
$$\frac{\partial F_4(\alpha,\beta;\gamma_1,\gamma_2;x,y)}{\partial y} = \frac{\alpha\beta}{\gamma_2} F_4(\alpha+1,\beta+1,\gamma_1,\gamma_2+1,x,y). \tag{8.40}$$

Obviously, such relations are valid whatever dependence the four parameters $\alpha,\beta,\gamma_1,\gamma_2$ may have on the Fuchsian exponents (a_i,b_j). The actions of K_{21} and K_{31} on the $\Phi_2^{(i)}$'s (i = 1, 2) in (8.37) are then given by

$$\begin{aligned}
K_{21}\Phi_1^{(2)}(x,y) &= 4p_1^{\Delta-2d-4}\sum_{a,b} c_1^{(2)}(a,b,\vec{\Delta})\, x^a y^b \frac{\partial}{\partial x} \\
&\quad \times F_4(\alpha(a.b)+1,\beta(a,b)+1;\gamma(a)-1,\gamma'(b);x,y) \\
&= 4p_1^{\Delta-2d-4}\sum_{a,b} c_1^{(2)}(a,b,\vec{\Delta})\, x^a y^b \frac{(\alpha(a,b)+1)(\beta(a,b)+1)}{(\gamma(a)-1)} \\
&\quad \times F_4(\alpha(a,b)+2,\beta(a,b)+2;\gamma(a),\gamma'(b);x,y) \\
K_{31}\Phi_1^{(2)}(x,y) &= 0 \qquad (8.41) \\
K_{31}\Phi_2^{(2)}(x,y) &= 4p_1^{\Delta-2d-4}\sum_{a,b} c_2^{(2)}(a,b,\vec{\Delta})\, x^a y^b \frac{\partial}{\partial y} \\
&\quad \times F_4(\alpha(a,b)+1,\beta(a,b)+1;\gamma(a),\gamma'(b)-1;x,y) \\
&= 4p_1^{\Delta-2d-4}\sum_{a,b} c_2^{(2)}(a,b,\vec{\Delta})\, x^a y^b \frac{(\alpha(a,b)+1)(\beta(a,b)+1)}{(\gamma'(b)-1)} \\
&\quad \times F_4(\alpha(a,b)+2,\beta(a,b)+2;\gamma(a),\gamma'(b);x,y) \\
K_{21}\Phi_2^{(2)}(x,y) &= 0, \qquad (8.42)
\end{aligned}$$

where it is clear that the non-zero right-hand-side of both equations are proportional to the form factor A_1 given in (8.21). Once this particular solution is determined, Eq. (8.21), by comparison, gives the conditions on $c_1^{(2)}$ and $c_1^{(2)}$ as

$$c_1^{(2)}(a,b,\vec{\Delta}) = \frac{\gamma(a)-1}{2(\alpha(a,b)+1)(\beta(a,b)+1)}\, c^{(1)}(a,b,\vec{\Delta})\,, \tag{8.43}$$

$$c_2^{(2)}(a,b,\vec{\Delta}) = \frac{\gamma'(b)-1}{2(\alpha(a,b)+1)(\beta(a,b)+1)}\, c^{(1)}(a,b,\vec{\Delta})\,. \tag{8.44}$$

Therefore, the general solution for A_2 in the TJJ case (in which $\gamma(a)=\gamma'(b)$) is given by superposing the solution of the homogeneous form of (8.21) and the particular one (8.35) and (8.37), by choosing the constants appropriately using (8.44). Its explicit form is written as

$$\begin{aligned} A_2 = p_1^{\Delta-2d-2}\sum_{ab} x^a y^b \Bigg[& c^{(2)}(a,b,\vec{\Delta})\, F_4(\alpha(a,b)+1,\beta(a,b)+1;\gamma(a),\gamma'(b);x,y) \\ & + \frac{(\gamma(a)-1)\,c^{(1)}(a,b,\vec{\Delta})}{2(\alpha(a,b)+1)(\beta(a,b)+1)}\Big(F_4(\alpha(a,b)+1,\beta(a,b)+1;\gamma(a)-1,\gamma'(b);x,y) \\ & + F_4(\alpha(a,b)+1,\beta(a,b)+1;\gamma(a),\gamma'(b)-1;x,y)\Big)\Bigg], \end{aligned} \tag{8.45}$$

since $\gamma(a)=\gamma'(b)$.

8.4. The solution for A_3

Using a similar strategy, the particular solution for the form factor A_3 of the equations

$$K_{21}A_3 = 0 \qquad K_{31}A_3 = -4A_1 \tag{8.46}$$

can be found in the form

$$\begin{aligned} &\Phi^{(3)}(x,y) \\ &\quad = p_1^{\Delta-2d-2}\sum_{ab} c_1^{(3)}(a,b,\vec{\Delta}) x^a y^b F_4(\alpha(a,b)+1,\beta(a,b)+1;\gamma(a),\gamma'(b)-1;x,y). \end{aligned} \tag{8.47}$$

Also in this case the inhomogeneous equation in (8.46) fixes the integration constants to be those appearing in A_1

$$c_1^{(3)}(a,b,\vec{\Delta}) = -\frac{\gamma'(b)-1}{(\alpha(a,b)+1)(\beta(a,b)+1)}\, c^{(1)}(a,b,\vec{\Delta}). \tag{8.48}$$

Therefore the general solution of the equations (8.46) can be written as

$$\begin{aligned} A_3 = p_1^{\Delta-2d-2}\sum_{ab} x^a y^b \Bigg[& c^{(3)}(a,b,\vec{\Delta})\, F_4(\alpha(a,b)+1,\beta(a,b)+1;\gamma(a),\gamma'(b);x,y) \\ & - \frac{(\gamma(a)-1)\,c^{(1)}(a,b,\vec{\Delta})}{(\alpha(a,b)+1)(\beta(a,b)+1)} \\ & \times F_4(\alpha(a,b)+1,\beta(a,b)+1;\gamma(a),\gamma'(b)-1;x,y)\Bigg] \end{aligned} \tag{8.49}$$

since $\gamma(a)=\gamma'(b)$ in the TJJ case.

8.5. The A_4 solution

The last pair of equations

$$K_{21}A_4 = -2A_3 \qquad K_{31}A_4 = -2A_3(p_2 \leftrightarrow p_3) \tag{8.50}$$

admit three particular solutions

$$\Phi_1^{(4)} = p_1^{\Delta-2d} \sum_{ab} x^a y^b c_1^{(4)}(a,b,\vec{\Delta})\, F_4(\alpha(a,b),\beta(a,b),\gamma(a)-1,\gamma'(b),x,y) \tag{8.51}$$

$$\Phi_2^{(4)} = p_1^{\Delta-2d} \sum_{ab} x^a y^b c_2^{(4)}(a,b,\vec{\Delta})\, F_4(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b)-1,x,y) \tag{8.52}$$

$$\Phi_3^{(4)} = p_1^{\Delta-2d} \sum_{ab} x^a y^b c_3^{(4)}(a,b,\vec{\Delta})\, F_4(\alpha(a,b),\beta(a,b),\gamma(a)-1,\gamma'(b)-1;x,y) \tag{8.53}$$

with the action of K_{21} and K_{31} on them as

$$K_{21}\Phi_1^{(4)} = 4p_1^{\Delta-2d-2} \sum_{ab} x^a y^b c_1^{(4)}(a,b,\vec{\Delta}) \frac{\alpha(a,b)\beta(a,b)}{(\gamma(a)-1)} \times F_4(\alpha(a,b)+1,\beta(a,b)+1,\gamma(a),\gamma'(b),x,y) \tag{8.54}$$

$$K_{31}\Phi_1^{(4)} = 0 \tag{8.55}$$

$$K_{21}\Phi_2^{(4)} = 0 \tag{8.56}$$

$$K_{31}\Phi_2^{(4)} = 4p_1^{\Delta-2d-2} \sum_{ab} x^a y^b c_2^{(4)}(a,b,\vec{\Delta}) \frac{\alpha(a,b)\beta(a,b)}{(\gamma'(b)-1)} \times F_4(\alpha(a,b)+1,\beta(a,b)+1,\gamma(a),\gamma'(b),x,y) \tag{8.57}$$

$$K_{21}\Phi_3^{(4)} = 4p_1^{\Delta-2d-2} \sum_{ab} x^a y^b c_3^{(4)}(a,b,\vec{\Delta}) \frac{\alpha(a,b)\beta(a,b)}{(\gamma(a)-1)} \times F_4(\alpha(a,b)+1,\beta(a,b)+1,\gamma(a),\gamma'(b)-1;x,y) \tag{8.58}$$

$$K_{31}\Phi_3^{(4)} = 4p_1^{\Delta-2d-2} \sum_{ab} x^a y^b c_3^{(4)}(a,b,\vec{\Delta}) \frac{\alpha(a,b)\beta(a,b)}{(\gamma'(b)-1)} \times F_4(\alpha(a,b)+1,\beta(a,b)+1,\gamma(a)-1,\gamma'(b);x,y) \tag{8.59}$$

The inhomogeneous equations (8.50) fix the integration constants to be those appearing in A_3 and $A_3(p_2 \leftrightarrow p_3)$ as

$$c_1^{(4)} = -\frac{(\gamma(a)-1)}{2\alpha(a,b)\beta(a,b)}\, c^{(3)}(a,b,\vec{\Delta}) \tag{8.60}$$

$$c_2^{(4)} = -\frac{(\gamma'(b)-1)}{2\alpha(a,b)\beta(a,b)}\, c^{(3)}(a,b,\vec{\Delta}) \tag{8.61}$$

$$c_3^{(4)} = \frac{(\alpha(a,b)+1)(\beta(a,b)+1)}{2\alpha(a,b)\beta(a,b)}\, c^{(1)}(a,b,\vec{\Delta}). \tag{8.62}$$

Finally, using the properties $\gamma(a) = \gamma'(b)$, we give the general solution for the A_4 as

$$\begin{aligned} A_4 = p_1^{\Delta-2d} \sum_{ab} x^a y^b \Bigg[& c^{(4)}(a,b,\vec{\Delta})\, F_4(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b);x,y) \\ & + \frac{(\alpha(a,b)+1)(\beta(a,b)+1)}{2\alpha(a,b)\beta(a,b)}\, c^{(1)}(a,b,\vec{\Delta}) \\ & \times F_4(\alpha(a,b),\beta(a,b),\gamma(a)-1,\gamma'(b)-1;x,y) \\ & - \frac{(\gamma(a)-1)}{2\alpha(a,b)\beta(a,b)}\, c^{(3)}(a,b,\vec{\Delta}) \Big(F_4(\alpha(a,b),\beta(a,b),\gamma(a)-1,\gamma'(b),x,y) \\ & + F_4(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b)-1,x,y) \Big) \Bigg] \end{aligned} \tag{8.63}$$

Notice that, differently from this case, number of free constants can be significantly reduced in the case of a fully symmetric correlator, such as the TTT, where the number of constants reduces to 4, as in the BMS case.

8.6. Summary

To summarize, the solutions of the primary WI's in the TJJ case are expressed as sums of 4 hypergeometrics of universal indicial points

$$a_0 = 0, \quad b_0 = 0, \quad a_1 = \Delta_2 - \frac{d}{2}, \quad b_1 = \Delta_3 - \frac{d}{2} \tag{8.64}$$

and parameters

$$\alpha(a,b) = a + b + \frac{d}{2} - \frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1)\,, \qquad \beta(a,b) = a + b + d - \frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3) \tag{8.65}$$

$$\gamma(a) = 2a + \frac{d}{2} - \Delta_2 + 1\,, \qquad \gamma'(b) = 2b + \frac{d}{2} - \Delta_3 + 1, \tag{8.66}$$

where $\Delta_2 = \Delta_3 = d - 1$ and $\Delta_1 = d$. In particular they are given by

$$A_1 = p_1^{\Delta-2d-4} \sum_{a,b} c^{(1)}(a,b,\vec{\Delta})\, x^a y^b\, F_4(\alpha(a,b)+2,\beta(a,b)+2;\gamma(a),\gamma'(b);x,y) \tag{8.67}$$

$$\begin{aligned} A_2 = p_1^{\Delta-2d-2} \sum_{ab} x^a y^b \Bigg[& c^{(2)}(a,b,\vec{\Delta})\, F_4(\alpha(a,b)+1,\beta(a,b)+1;\gamma(a),\gamma'(b);x,y) \\ & + \frac{(\gamma(a)-1)\, c^{(1)}(a,b,\vec{\Delta})}{2(\alpha(a,b)+1)(\beta(a,b)+1)} \Big(F_4(\alpha(a,b)+1,\beta(a,b)+1;\gamma(a)-1,\gamma'(b);x,y) \\ & + F_4(\alpha(a,b)+1,\beta(a,b)+1;\gamma(a),\gamma'(b)-1;x,y) \Big) \Bigg] \end{aligned} \tag{8.68}$$

$$A_3 = p_1^{\Delta-2d-2} \sum_{ab} x^a y^b \Big[c^{(3)}(a,b,\vec{\Delta})\, F_4(\alpha(a,b)+1, \beta(a,b)+1; \gamma(a), \gamma'(b); x, y) - \frac{(\gamma(a)-1)\, c^{(1)}(a,b,\vec{\Delta})}{(\alpha(a,b)+1)(\beta(a,b)+1)} \times F_4(\alpha(a,b)+1, \beta(a,b)+1; \gamma(a), \gamma'(b)-1; x, y) \Big] \quad (8.69)$$

$$A_4 = p_1^{\Delta-2d} \sum_{ab} x^a\, y^b \Big[c^{(4)}(a,b,\vec{\Delta})\, F_4(\alpha(a,b), \beta(a,b), \gamma(a), \gamma'(b); x, y) + \frac{(\alpha(a,b)+1)(\beta(a,b)+1)}{2\alpha(a,b)\beta(a,b)}\, c^{(1)}(a,b,\vec{\Delta}) \times F_4(\alpha(a,b), \beta(a,b), \gamma(a)-1, \gamma'(b)-1; x, y) - \frac{(\gamma(a)-1)}{2\alpha(a,b)\beta(a,b)}\, c^{(3)}(a,b,\vec{\Delta}) \Big(F_4(\alpha(a,b), \beta(a,b), \gamma(a)-1, \gamma'(b), x, y) + F_4(\alpha(a,b), \beta(a,b), \gamma(a), \gamma'(b)-1, x, y) \Big) \Big] \quad (8.70)$$

in terms of the constants $c^{(i)}(a,b)$ given above. The method has the advantage of being generalizable to higher point functions, in the search of specific solutions of the corresponding correlation functions.

9. Perturbative analysis in the conformal case: QED and scalar QED

In this section we turn to discuss the connection between the solutions of the CWI's presented by BMS and the perturbative TJJ vertex. The QED case has been previously studied in [4,7], where more details can be found. The expressions of the form factors had been given in the F-basis of 13 form factors, which will be reviewed in the next section. We will have to recompute them in order to present them expressed in terms of the two basic fundamental master integrals B_0 and C_0 of the tensor reduction rather than in their final form, given in [7].

Here we are also going to introduce the diagrammatic expansion for the TJJ in scalar QED, since it will be needed in the last part of the work when we are going to compare the general BMS solution against the perturbative one in $d = 3$ and $d = 5$.

The quantum actions for the fermion field is

$$S_{fermion} = \frac{i}{2} \int d^d x\; e\; e_a^\mu \left[\bar{\psi} \gamma^a (D_\mu \psi) - (D_\mu \bar{\psi}) \gamma^a \psi \right], \quad (9.1)$$

e_a^μ is the Vielbein and e its determinant, with its covariant derivative D_μ as

$$D_\mu = \partial_\mu + ie\, A_\mu + \Gamma_\mu = \partial_\mu + ie\, A_\mu + \frac{1}{2} \Sigma^{ab}\, e_a^\sigma \nabla_\mu\, e_{b\,\sigma}. \quad (9.2)$$

The Σ^{ab} are the generators of the Lorentz group in the case of a spin $1/2$-field. The gravitational field is expanded, as usual, in the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ around the flat background metric with fluctuations $h_{\mu\nu}$. As usual, the Latin and Greek indices are related to the locally flat and curved backgrounds respectively. We take the external momenta as incoming. In order to simplify the notation, we introduce the tensor components

$$A^{\mu_1\nu_1\mu\nu} \equiv \eta^{\mu_1\nu_1}\eta^{\mu\nu} - 2\eta^{\mu(\mu_1}\eta^{\nu_1)\nu} \tag{9.3}$$

where we indicate with the round brackets the symmetrization of the indices and the square brackets their anti-symmetrization

$$\eta^{\mu(\mu_1}\eta^{\nu_1)\nu} \equiv \frac{1}{2}\left(\eta^{\mu\mu_1}\eta^{\nu_1\nu} + \eta^{\mu\nu_1}\eta^{\mu_1\nu}\right) \tag{9.4}$$

and the vertices in the fermion sector are

$$V^{\mu}_{J\psi\bar{\psi}}(k_1, k_2) = -ie\,\gamma^{\mu} \tag{9.5}$$

$$V^{\mu_1\nu_1}_{T\psi\bar{\psi}}(p_1 k_1, k_2) = -\frac{i}{4}\,A^{\mu_1\nu_1\mu\nu}\,\gamma_{\nu}\,(k_1 + k_2)_{\mu} \tag{9.6}$$

$$V^{\mu_1\nu_1\mu_2}_{TJ\psi\bar{\psi}}(k_1, k_2) = \frac{i\,e}{2}A^{\mu_1\nu_1\mu_2\nu}\,\gamma_{\mu}. \tag{9.7}$$

In the one-loop approximation the contribution to the correlation functions are given by the diagrams in Fig. 1, with vertices shown in Fig. 2. We calculate all the diagram contributions in momentum space for the fermion sector as

$$\begin{aligned}\Gamma^{\mu_1\nu_1\mu_2\mu_3}(p_2, p_3) &\equiv \langle T^{\mu_1\nu_1}(p_1)\,J^{\mu_2}(p_2)\,J^{\mu_3}(p_3)\rangle_F \\ &= 2\left(\sum_{i=1}^{2} V^{\mu_1\nu_1\mu_2\mu_3}_{F,i}(p_1, p_2, p_3) + \sum_{i=1}^{2} W^{\mu_1\nu_1\mu_2\mu_3}_{F,i}(p_1, p_2, p_3)\right)\end{aligned} \tag{9.8}$$

where the $V_{F,i}$ terms are related to the triangle topology contributions, while the $W_{F,i}$ terms denote the two bubble contributions in Fig. 1. All these terms are explicitly given as

$$\begin{aligned}&V^{\mu_1\nu_1\mu_2\mu_3}_{F,1} \\ &\quad= -i^3\int\frac{d^d\ell}{(2\pi)^d} \\ &\qquad\times\frac{\mathrm{Tr}\left[V^{\mu_1\nu_1}_{T\psi\bar{\psi}}(\ell - p_2, \ell + p_3)\left(\ell\!\!\!/ + p\!\!\!/_3\right)V^{\mu_2}_{J\psi\bar{\psi}}(\ell, \ell - p_2)\,\ell\!\!\!/\,V^{\mu_3}_{J\psi\bar{\psi}}(\ell, \ell + p_3)\,\left(\ell\!\!\!/ - p\!\!\!/_2\right)\right]}{\ell^2\,(\ell - p_2)^2(\ell + p_3)^2}\end{aligned} \tag{9.9}$$

$$\begin{aligned}&V^{\mu_1\nu_1\mu_2\mu_3}_{F,2} \\ &\quad= -i^3\int\frac{d^d\ell}{(2\pi)^d} \\ &\qquad\times\frac{\mathrm{Tr}\left[V^{\mu_1\nu_1}_{T\psi\bar{\psi}}(\ell - p_3, \ell + p_2)\left(\ell\!\!\!/ + p\!\!\!/_2\right)V^{\mu_2}_{J\psi\bar{\psi}}(\ell, \ell - p_3)\,\ell\!\!\!/\,V^{\mu_3}_{J\psi\bar{\psi}}(\ell, \ell + p_2)\,\left(\ell\!\!\!/ - p\!\!\!/_3\right)\right]}{\ell^2\,(\ell - p_3)^2(\ell + p_2)^2}\end{aligned} \tag{9.10}$$

$$W^{\mu_1\nu_1\mu_2\mu_3}_{F,3} = -i^2\int\frac{d^d\ell}{(2\pi)^d}\,\frac{\mathrm{Tr}\left[V^{\mu_1\nu_1\mu_2}_{TJ\psi\bar{\psi}}(\ell + p_3, \ell)\left(\ell\!\!\!/ + p\!\!\!/_3\right)V^{\mu_3}_{J\psi\bar{\psi}}(\ell, \ell + p_3)\,\ell\!\!\!/\right]}{\ell^2\,(\ell + p_3)^2} \tag{9.11}$$

$$W^{\mu_1\nu_1\mu_2\mu_3}_{F,2} = -i^2\int\frac{d^d\ell}{(2\pi)^d}\,\frac{\mathrm{Tr}\left[V^{\mu_1\nu_1\mu_3}_{TJ\psi\bar{\psi}}(\ell + p_2, \ell)\left(\ell\!\!\!/ + p\!\!\!/_2\right)V^{\mu_2}_{J\psi\bar{\psi}}(\ell, \ell + p_2)\,\ell\!\!\!/\right]}{\ell^2\,(\ell + p_2)^2} \tag{9.12}$$

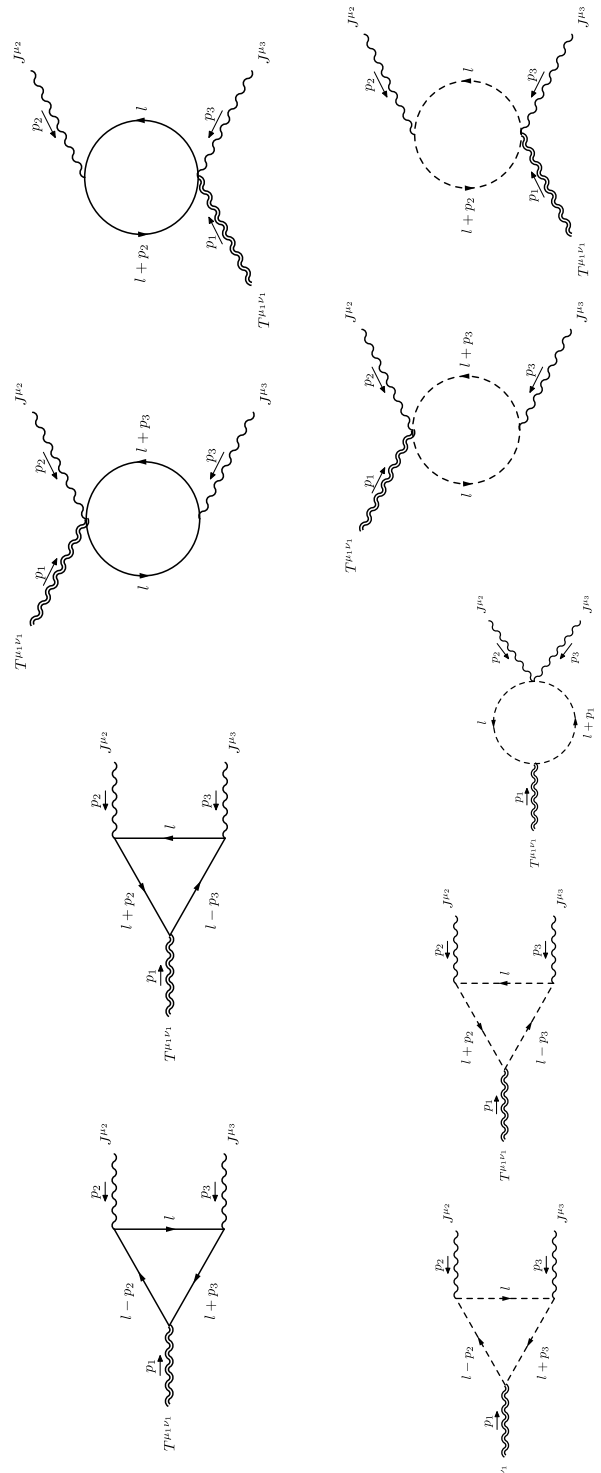

Fig. 1. One-loop diagrams for the TJJ in QED (continuous fermion lines). A similar set of diagrams describes the scalar QED case (dashed lines), by replacing the internal fermion with a charged scalar.

9.1. The T J J in scalar QED

Now we turn to consider scalar QED. The action, in this case, can be written as

$$S_{scalar} = \int d^d x \sqrt{-g} \left(\left| D_\mu \phi \right|^2 + \frac{(d-2)}{8(d-1)} R \left| \phi \right|^2 \right) \tag{9.13}$$

where R is the scalar curvature and ϕ denotes a complex scalar. We have explicitly reported the coefficient of the term of improvement, and with $D_\mu \phi = \partial_\mu \phi + ie A_\mu$ being the covariant derivative for the coupling to the gauge field A_μ. At one-loop the contribution to the TJJ is given by the diagram in Fig. 1, with the obvious replacement of a fermion by a scalar in the internal loop corrections. In this case they are given by

$$\langle T^{\mu_1\nu_1}(p_1)\, J^{\mu_2}(p_2)\, J^{\mu_3}(p_3)\rangle_S = 2\left(V_S^{\mu_1\nu_1\mu_2\mu_3}(p_1,p_2,p_3) + \sum_{i=1}^{3} W_{S,i}^{\mu_1\nu_1\mu_2\mu_3}(p_1,p_2,p_3) \right) \tag{9.14}$$

where the V_S terms are related to the triangle topology contribution and the $W_{S,i}$'s are the three bubble contributions in Fig. 1. All these are explicitly given as

$$V_S^{\mu_1\nu_1\mu_2\mu_3}(p_1,p_2,p_3) = i^3 \int \frac{d^d\ell}{(2\pi)^d} \frac{V_{T\phi\phi^*}^{\mu_1\nu_1}(\ell-p_2,\ell+p_3)\, V_{J\phi\phi^*}^{\mu_2}(\ell,\ell-p_2)\, V_{J\phi\phi^*}^{\mu_3}(\ell,\ell+p_3)}{\ell^2\,(\ell-p_2)^2(\ell+p_3)^2} \tag{9.15}$$

$$W_{S,1}^{\mu_1\nu_1\mu_2\mu_3}(p_1,p_2,p_3) = \frac{i^2}{2} \int \frac{d^d\ell}{(2\pi)^d} \frac{V_{T\phi\phi^*}^{\mu_1\nu_1}(\ell+p_1,\ell)\, V_{JJ\phi\phi^*}^{\mu_2\mu_3}(\ell,\ell+p_1)}{\ell^2\,(\ell+p_1)^2} \tag{9.16}$$

$$W_{S,2}^{\mu_1\nu_1\mu_2\mu_3}(p_1,p_2,p_3) = \frac{i^2}{2} \int \frac{d^d\ell}{(2\pi)^d} \frac{V_{TJ\phi\phi^*}^{\mu_1\nu_1\mu_2}(\ell+p_3,\ell)\, V_{J\phi\phi^*}^{\mu_3}(\ell,\ell+p_3)}{\ell^2\,(\ell+p_3)^2} \tag{9.17}$$

$$W_{S,3}^{\mu_1\nu_1\mu_2\mu_3}(p_1,p_2,p_3) = \frac{i^2}{2} \int \frac{d^d\ell}{(2\pi)^d} \frac{V_{TJ\phi\phi^*}^{\mu_1\nu_1\mu_3}(\ell+p_2,\ell)\, V_{J\phi\phi^*}^{\mu_2}(\ell,\ell+p_2)}{\ell^2\,(\ell+p_2)^2} \tag{9.18}$$

where we have included the symmetry factors and the vertices are given by

$$V_{J\phi\phi^*}^{\mu}(k_1,k_2) = ie\,(k_1^\mu + k_2^\mu) \tag{9.19}$$

$$V_{T\phi\phi^*}^{\mu_1\nu_1}(p_1 k_1,k_2) = \frac{i}{2} A^{\mu\nu\mu_1\nu_1} k_{1\mu} k_{2\nu} + i\,\chi\, \left(p_1^{\mu_1} p_1^{\nu_1} - \eta^{\mu_1\nu_1} p_1^2 \right) \tag{9.20}$$

$$V_{TJ\phi\phi^*}^{\mu_1\nu_1\mu_2}(k_1,k_2) = \frac{i\,e}{2}\, A^{\mu_1\nu_1\mu\mu_2}\,(k_{1\mu} + k_{2\mu}) \tag{9.21}$$

$$V_{JJ\phi\phi^*}^{\mu_1\mu_2}(k_1,k_2) = 2\,i\,e^2\,\eta^{\mu_1\mu_2} \tag{9.22}$$

where $\chi = (d-2)/[8(d-1)]$ is the coefficient for the term of improvement. They are shown in Fig. 2.

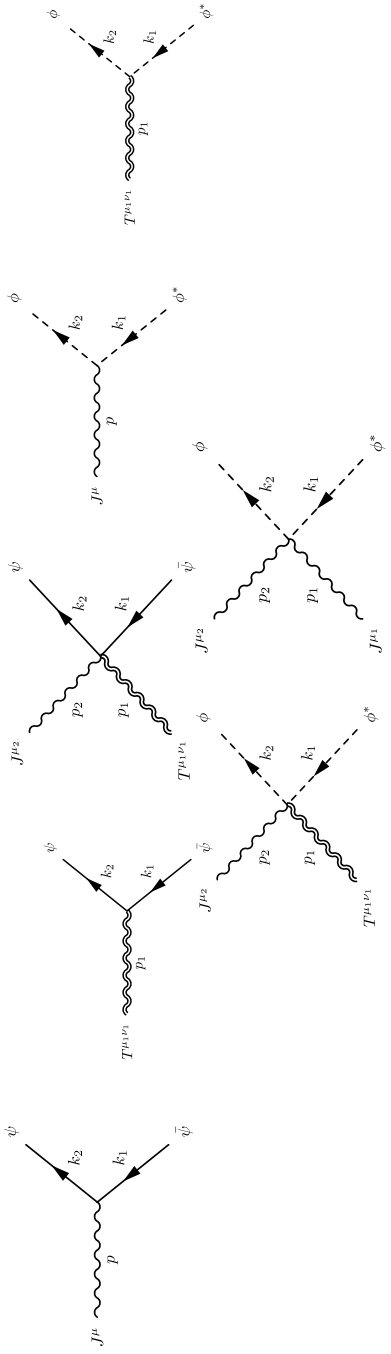

Fig. 2. The vertices in QED and scalar QED.

Table 1
Basis of 13 fourth rank tensors satisfying the vector current conservation on the external lines with momenta p and q.

i	$t_i^{\mu\nu\alpha\beta}(p,q)$
1	$\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) u^{\alpha\beta}(p.q)$
2	$\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) w^{\alpha\beta}(p.q)$
3	$\left(p^2 g^{\mu\nu} - 4p^\mu p^\nu\right) u^{\alpha\beta}(p.q)$
4	$\left(p^2 g^{\mu\nu} - 4p^\mu p^\nu\right) w^{\alpha\beta}(p.q)$
5	$\left(q^2 g^{\mu\nu} - 4q^\mu q^\nu\right) u^{\alpha\beta}(p.q)$
6	$\left(q^2 g^{\mu\nu} - 4q^\mu q^\nu\right) w^{\alpha\beta}(p.q)$
7	$\left[p\cdot q\, g^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)\right] u^{\alpha\beta}(p.q)$
8	$\left[p\cdot q\, g^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)\right] w^{\alpha\beta}(p.q)$
9	$\left(p\cdot q\, p^\alpha - p^2 q^\alpha\right)\left[p^\beta\left(q^\mu p^\nu + p^\mu q^\nu\right) - p\cdot q\,(g^{\beta\nu} p^\mu + g^{\beta\mu} p^\nu)\right]$
10	$\left(p\cdot q\, q^\beta - q^2 p^\beta\right)\left[q^\alpha\left(q^\mu p^\nu + p^\mu q^\nu\right) - p\cdot q\,(g^{\alpha\nu} q^\mu + g^{\alpha\mu} q^\nu)\right]$
11	$\left(p\cdot q\, p^\alpha - p^2 q^\alpha\right)\left[2q^\beta q^\mu q^\nu - q^2(g^{\beta\nu} q^\mu + g^{\beta\mu} q^\nu)\right]$
12	$\left(p\cdot q\, q^\beta - q^2 p^\beta\right)\left[2\,p^\alpha p^\mu p^\nu - p^2(g^{\alpha\nu} p^\mu + g^{\alpha\mu} p^\nu)\right]$
13	$\left(p^\mu q^\nu + p^\nu q^\mu\right) g^{\alpha\beta} + p\cdot q\left(g^{\alpha\nu} g^{\beta\mu} + g^{\alpha\mu} g^{\beta\nu}\right) - g^{\mu\nu} u^{\alpha\beta} - \left(g^{\beta\nu} p^\mu + g^{\beta\mu} p^\nu\right) q^\alpha - \left(g^{\alpha\nu} q^\mu + g^{\alpha\mu} q^\nu\right) p^\beta$

10. The F-basis of the expansion for the TJJ in QED

The TJJ amplitude can be expanded on the basis proposed by Giannotti and Mottola [4], in terms of 13 independent tensors structures given in Table 1. In this scheme, the amplitude can be written as

$$\Gamma^{\mu_1\nu_1\mu_2\mu_3}(p_2, p_3) = \sum_{i=1}^{13} F_i(s; s_1, s_2, 0)\, t_i^{\mu_1\nu_1\mu_2\mu_3}(p_2, p_3), \tag{10.1}$$

where the invariant amplitudes F_i are functions of the kinematic invariants $s = p_1^2 = (p_2 + p_3)^2$, $s_1 = p_2^2$, $s_2 = p_3^2$, and the $t_i^{\mu_1\nu_1\mu_2\mu_3}$ form the basis of independent tensor structures.

This set of 13 tensors is linearly independent in d dimensions, for generic k^2, p^2, q^2 different from zero. Five of the 13 are Bose symmetric,

$$t_i^{\mu\nu\alpha\beta}(p,q) = t_i^{\mu\nu\beta\alpha}(q,p), \qquad i = 1, 2, 7, 8, 13, \tag{10.2}$$

while the remaining eight tensors are Bose symmetric pairwise

$$t_3^{\mu\nu\alpha\beta}(p,q) = t_5^{\mu\nu\beta\alpha}(q,p), \tag{10.3}$$

$$t_4^{\mu\nu\alpha\beta}(p,q) = t_6^{\mu\nu\beta\alpha}(q,p), \tag{10.4}$$

$$t_9^{\mu\nu\alpha\beta}(p,q) = t_{10}^{\mu\nu\beta\alpha}(q,p), \tag{10.5}$$

$$t_{11}^{\mu\nu\alpha\beta}(p,q) = t_{12}^{\mu\nu\beta\alpha}(q,p). \tag{10.6}$$

In the set are present two tensor structures

$$u^{\alpha\beta}(p,q) \equiv (p\cdot q)g^{\alpha\beta} - q^{\alpha}p^{\beta}\,, \tag{10.7a}$$

$$w^{\alpha\beta}(p,q) \equiv p^2q^2g^{\alpha\beta} + (p\cdot q)p^{\alpha}q^{\beta} - q^2p^{\alpha}p^{\beta} - p^2q^{\alpha}q^{\beta}\,, \tag{10.7b}$$

which appear in t_1 and t_2 respectively. Each of them satisfies the Bose symmetry requirement,

$$u^{\alpha\beta}(p,q) = u^{\beta\alpha}(q,p)\,, \tag{10.8a}$$

$$w^{\alpha\beta}(p,q) = w^{\beta\alpha}(q,p)\,, \tag{10.8b}$$

and vector current conservation,

$$p_{\alpha}u^{\alpha\beta}(p,q) = 0 = q_{\beta}u^{\alpha\beta}(p,q)\,, \tag{10.9a}$$

$$p_{\alpha}w^{\alpha\beta}(p,q) = 0 = q_{\beta}w^{\alpha\beta}(p,q)\,, \tag{10.9b}$$

obtained from the variation of gauge invariant quantities $F_{\mu\nu}F^{\mu\nu}$ and $(\partial_{\mu}F^{\mu}_{\ \lambda})(\partial_{\nu}F^{\nu\lambda})$

$$u^{\alpha\beta}(p,q) = -\frac{1}{4}\int d^4x \int d^4y\, e^{ip\cdot x+iq\cdot y}\,\frac{\delta^2\{F_{\mu\nu}F^{\mu\nu}(0)\}}{\delta A_{\alpha}(x)A_{\beta}(y)}\,, \tag{10.10}$$

$$w^{\alpha\beta}(p,q) = \frac{1}{2}\int d^4x \int d^4y\, e^{ip\cdot x+iq\cdot y}\,\frac{\delta^2\{\partial_{\mu}F^{\mu}_{\ \lambda}\partial_{\nu}F^{\nu\lambda}(0)\}}{\delta A_{\alpha}(x)A_{\beta}(y)}\,. \tag{10.11}$$

All the t_i's are transverse in their photon indices

$$q^{\alpha}t_i^{\mu\nu\alpha\beta} = 0 \qquad p^{\beta}t_i^{\mu\nu\alpha\beta} = 0. \tag{10.12}$$

$t_2 \dots t_{13}$ are traceless, t_1 and t_2 are tracefull. With this decomposition, the two vector Ward identities are automatically satisfied by all the amplitudes, as well as the Bose symmetry.

Coming to the conservation WI for the graviton line, this is automatically satisfied by the two tensor structures t_1 and t_2, which are completely transverse, while it has to be imposed on the second set $(t_3 \dots t_{13})$ giving 3 constraints

$$\begin{aligned}
&-p^2F_3 + (3q^2 + 4p\cdot q)F_5 + (2p^2 + p\cdot q)F_7 - p^2q^2F_{10} - p^2(p^2 + p\cdot q)F_9 \\
&\quad + p^2q^2F_{11} = 0\,, \\
&p^2F_4 - (3q^2 + 4p\cdot q)F_6 - (2p^2 + p\cdot q)F_8 - p\cdot qF_{10} + (q^2 + 2p\cdot q)F_{11} = 0\,, \\
&-p\cdot q\,(p^2 + p\cdot q)F_9 - q^2(q^2 + p\cdot q)F_{11} + F_{13} + \Pi(p^2) = 0\,,
\end{aligned} \tag{10.13}$$

plus 3 symmetric additional ones, obtained by the exchange of the two photon momenta and the symmetries of the form factors corresponding to (10.3)

$$F_3(p,q) = F_5(q,p)\,, \tag{10.14}$$

$$F_4(p,q) = F_6(q,p)\,, \tag{10.15}$$

$$F_9(p,q) = F_{10}(q,p)\,, \tag{10.16}$$

$$F_{11}(p,q) = F_{12}(q,p)\,. \tag{10.17}$$

In other words, if we decided to identify from the $F_3 \dots F_{13}$ components a complete transverse traceless sector, using (10.14) and (10.13) we would identify only 4 components in this sector. Such 4 components, obviously, would be related to the transverse and traceless form factors

$(A_1 \dots A_4)$ introduced in the BMS parametrization. Their explicit expressions will be given below. An important aspect of the F-basis is that *only 1-form factor has to be renormalized* for dimensional reasons, the others being finite. Such form factor, F_{13}, plays an important role in the description of the behavior of the trace parts of the same expansion, which involve t_1 and t_2 (i.e. F_1 and F_2).

10.1. Dilatation Ward identities in the F-basis

The identification of the combination of form factors F_i which span the transverse traceless sector of the correlator can proceed in several ways. In this and in the next section we will proceed by starting from its general expansion in the F-basis, and perform a transverse traceless projection, after acting on it with the dilatation and the special conformal transformations. This allows to gather the result of the action of the dilatation in terms of coefficients D_i in the form

$$0 = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\pi^{\mu_2}_{\alpha_2}(p_2)\pi^{\mu_3}_{\alpha_3}(p_3)\big\{D_1\, p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_1^{\alpha_3} + D_2\,\delta^{\alpha_2\alpha_3} p_2^{\alpha_1} p_2^{\beta_1} + D_3\delta^{\alpha_1\alpha_2} p_2^{\beta_1} + p_1^{\alpha_3} D_4\,\delta^{\alpha_1\alpha_3} p_2^{\beta_1} p_3^{\alpha_2} + D_5\,\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1}\big\} \tag{10.18a}$$

where D_i, $i = 1, \dots, 5$ are differential operator acting on the F_i form factors. In order to verify the previous relation, the coefficients D_i multiplying the independent tensor structures have to vanish, giving a set of differential equation for particular combination of the F_i's. The first equation for D_1 will be of the form

$$D_1 = 4\left[\sum_{i=1}^{3} p_i \frac{\partial}{\partial p_i} - (d-6)\right](F_7 - F_3 - F_5) - 2p_2^2\left[\sum_{i=1}^{3} p_i \frac{\partial}{\partial p_i} - (d-8)\right]F_9 - 2p_3^2\left[\sum_{i=1}^{3} p_i \frac{\partial}{\partial p_i} - (d-8)\right]F_{10} = 0 \tag{10.19}$$

which can be rearranged as

$$D_1 = \left(\sum_{i=1}^{3} p_i \frac{\partial}{\partial p_i} - (d-6)\right)\big[4(F_7 - F_5 - F_3) - 2p_2^2 F_9 - 2p_3^2 F_{10}\big] = 0, \tag{10.20}$$

and similarly for the other D_i's, which correspond to

$$D_2\left(\sum_{i=1}^{3} p_i \frac{\partial}{\partial p_i} - (d-4)\right) \times \big[2(p_1^2 - p_2^2 - p_3^2)(F_7 - F_5 - F_3) - 4p_2^2 p_3^2(F_6 - F_8 + F_4) - 2F_{13}\big] = 0 \tag{10.21}$$

$$D_3 = \left(\sum_{i=1}^{3} p_i \frac{\partial}{\partial p_i} - (d-4)\right)\big[p_3^2(p_1^2 - p_2^2 - p_3^2)F_{10} - 2p_2^2\, p_3^2 F_{12} - 2F_{13}\big] = 0 \tag{10.22}$$

$$D_4 = \left(\sum_{i=1}^{3} p_i \frac{\partial}{\partial p_i} - (d-4)\right)\big[p_2^2(p_1^2 - p_2^2 - p_3^2)F_9 - 2p_2^2 p_3^2 F_{11} - 2F_{13}\big] = 0 \tag{10.23}$$

$$D_5 = \left(\sum_{i=1}^{3} p_i \frac{\partial}{\partial p_i} - (d-2)\right)\left[(p_1^2 - p_2^2 - p_3^2)F_{13}\right] = 0. \tag{10.24}$$

This allows us to identify specific combinations of the F's which will span the transverse traceless sector of the TJJ.

10.2. Special conformal Ward identities in the F-basis

A similar approach can be followed in the case of the primary and secondary CWI's. Also in this case we project the special CWI's onto the transverse traceless sector, obtaining

$$0 = \Pi^{\rho_1\sigma_1}_{\mu_1\nu_1}(p_1)\pi^{\rho_2}_{\mu_2}(p_2)\pi^{\rho_3}_{\mu_3}(p_3)\left\{ K^{\kappa}\ \langle t^{\mu_1\nu_1}(p_1)\, j^{\mu_2}(p_2)\, j^{\mu_3}(p_3)\rangle \right.$$
$$\left. + \frac{4d}{p_1^2}\,\delta^{\kappa\mu_1}\, p_{1\alpha_1}\ \langle T^{\alpha_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3)\rangle \right\} \tag{10.25}$$

where we have used the conservation Ward Identities

$$p_{2\mu_2}\,\langle T^{\mu_1\nu_1}(p_1)\, J^{\mu_2}(p_2)\, J^{\mu_3}(p_3)\rangle = 0$$
$$p_{3\mu_3}\,\langle T^{\mu_1\nu_1}(p_1)\, J^{\mu_2}(p_2)\, J^{\mu_3}(p_3)\rangle = 0. \tag{10.26a}$$

Also in this case one can express the first term in (10.25) in the form of (6.37) in order to isolate the primary and secondary WI's for the form factors.

10.3. Primary WI's

A first set of primary conformal WI's is given by

$$0 = K_{13}\left[4(F_7 - F_5 - F_3) - 2p_2^2 F_9 - 2p_3^2 F_{10}\right] \tag{10.27a}$$
$$0 = K_{13}\left[2(p_1^2 - p_2^2 - p_3^2)(F_7 - F_5 - F_3) - 4p_2^2 p_3^2(F_6 - F_8 + F_4) - 2F_{13}\right]$$
$$+ 2\left[4(F_7 - F_5 - F_3) - 2p_2^2 F_9 - 2p_3^2 F_{10}\right] \tag{10.27b}$$
$$0 = K_{13}\left[p_3^2(p_1^2 - p_2^2 - p_3^2)F_{10} - 2p_2^2\, p_3^2 F_{12} - 2F_{13}\right]$$
$$- 4\left[4(F_7 - F_5 - F_3) - 2p_2^2 F_9 - 2p_3^2 F_{10}\right] \tag{10.27c}$$
$$0 = K_{13}\left[p_2^2(p_1^2 - p_2^2 - p_3^2)F_9 - 2p_2^2 p_3^2 F_{11} - 2F_{13}\right] \tag{10.27d}$$
$$0 = K_{13}\left[(p_1^2 - p_2^2 - p_3^2)F_{13}\right] - 2\left[p_2^2(p_1^2 - p_2^2 - p_3^2)F_9 - 2p_2^2 p_3^2 F_{11} - 2F_{13}\right] \tag{10.27e}$$

and a second set as

$$0 = K_{23}\left[4(F_7 - F_5 - F_3) - 2p_2^2 F_9 - 2p_3^2 F_{10}\right] \tag{10.28a}$$
$$0 = K_{23}\left[2(p_1^2 - p_2^2 - p_3^2)(F_7 - F_5 - F_3) - 4p_2^2 p_3^2(F_6 - F_8 + F_4) - 2F_{13}\right] \tag{10.28b}$$
$$0 = K_{23}\left[p_3^2(p_1^2 - p_2^2 - p_3^2)F_{10} - 2p_2^2\, p_3^2 F_{12} - 2F_{13}\right]$$
$$- 4\left[4(F_7 - F_5 - F_3) - 2p_2^2 F_9 - 2p_3^2 F_{10}\right] \tag{10.28c}$$
$$0 = K_{23}\left[p_2^2(p_1^2 - p_2^2 - p_3^2)F_9 - 2p_2^2 p_3^2 F_{11} - 2F_{13}\right]$$
$$+ 4\left[4(F_7 - F_5 - F_3) - 2p_2^2 F_9 - 2p_3^2 F_{10}\right] \tag{10.28d}$$

$$0 = K_{23}\big[(p_1^2 - p_2^2 - p_3^2)F_{13}\big] + 2\big[p_3^2(p_1^2 - p_2^2 - p_3^2)F_{10} - 2p_2^2\,p_3^2F_{12} - 2F_{13}\big] - 2\big[p_2^2(p_1^2 - p_2^2 - p_3^2)F_9 - 2p_2^2p_3^2F_{11} - 2F_{13}\big]. \tag{10.28e}$$

It is clear from the way in which we have organized the contributions in square brackets ([ ]) that they correspond to the same structures identified in the projections of the dilatation WI's.

10.4. Secondary WI's

For completeness we list the secondary Ward Identities obtained in a similar way, which are given by

$$\begin{aligned} 0 = {} & L_3'\big[4(F_7 - F_3 - F_5) - 2p_2^2F_9 - 2p_3^2F_{10}\big] \\ & + 2R'\big[p_3^2(p_1^2 - p_2^2 - p_3^2)F_{10} - 2p_2^2\,p_3^2F_{12} - 2F_{13}\big] \\ & - 2R'\big[2(p_1^2 - p_2^2 - p_3^2)(F_7 - F_5 - F_3) - 4p_2^2p_3^2(F_6 - F_8 + F_4) - 2F_{13}\big] \end{aligned} \tag{10.29}$$

$$\begin{aligned} 0 = {} & L_1'\big[p_2^2(p_1^2 - p_2^2 - p_3^2)F_9 - 2p_2^2p_3^2F_{11} - 2F_{13}\big] \\ & - 2p_2^2\big[p_3^2(p_1^2 - p_2^2 - p_3^2)F_{10} - 2p_2^2\,p_3^2F_{12} - 2F_{13}\big] \\ & + 4p_2^2\big[2(p_1^2 - p_2^2 - p_3^2)(F_7 - F_5 - F_3) - 4p_2^2p_3^2(F_6 - F_8 + F_4) - 2F_{13}\big] \\ & + 2R'\big[(p_1^2 - p_2^2 - p_3^2)F_{13}\big] \end{aligned} \tag{10.30}$$

$$\begin{aligned} 0 = {} & -\frac{2}{p_1^2}\Big\{L_4\big[4(F_7 - F_3 - F5) - 2p_2^2F_9 - 2p_3^2F_{10}\big] \\ & + R\big[p_3^2(p_1^2 - p_2^2 - p_3^2)F_{10} - 2p_2^2\,p_3^2F_{12} - 2F_{13}\big] \\ & - R\big[p_2^2(p_1^2 - p_2^2 - p_3^2)F_9 - 2p_2^2p_3^2F_{11} - 2F_{13}\big]\Big\} \\ & + \frac{4d}{p_1^2}\Big[p_3^2(p_1\cdot p_3 - p_1\cdot p_2 - p_2\cdot p_3)F_{10} - 4F_3\,p_1\cdot p_2 \\ & \qquad - p_2^2p_3^2(F_{11} - F_{12}) + 4F_5\,p_1\cdot p_3 + 2(p_1\cdot p_2 - p_1\cdot p_3)F_7 \\ & + p_2^2(p_1\cdot p_3 - p_1\cdot p_2 + p_2\cdot p_3)F_9\Big] \end{aligned} \tag{10.31}$$

$$\begin{aligned} 0 = {} & -\frac{2}{p_2^2}\Big\{L_2\big[2(p_1^2 - p_2^2 - p_3^2)(F_7 - F_5 - F_3) - 4p_2^2p_3^2(F_6 - F_8 + F_4) - 2F_{13}\big] \\ & - p_1^2(p_1^2 - p_2^2 - p_3^2)(p_3^2F_{10} - p_2^2F_9) + 2p_3^2p_2^2p_1^2(F_{12} - F_{11})\Big\} \\ & + \frac{4d}{p_1^2}\big[(p_2^2 - p_3^2)F_{13} + (p_3^4 + p_1^4 - p_2^4 - 2p_1^2p_3^2)F_3 + 2p_2^2p_3^2(p_1^2 + p_2^2 - p_3^2)F_4 \\ & + (2p_1^2p_2^2 - p_1^4 - p_2^4 + p_3^4)F_5 + 2p_2^2p_3^2(p_2^2 - p_3^2 - p_1^2)F_6 + 2p_2^2p_3^2(p_3^2 - p_2^2)F_8 \\ & + (p_2^4 - p_3^4 + p_1^2p_3^2 - p_1^2p_2^2)F_7\big] \end{aligned} \tag{10.32}$$

$$0 = -\frac{1}{p_1^2}\Big\{L_4\big[p_3^2(p_1^2-p_2^2-p_3^2)F_{10} - 2p_2^2\,p_3^2F_{12} - 2F_{13}\big] - 2R\big[(p_1^2-p_2^2-p_3^2)F_{13}\big]\Big\}$$
$$+\frac{4d}{p_1^2}\big[p_1\cdot p_3 p_2\cdot p_3 p_3^2F_{10} + p.p_2p_2^2p_3^2F_{12} - p_2^2F_{13}\big] \tag{10.33}$$

$$0 = -\frac{1}{p_1^2}\Big\{L_4\big[p_2^2(p_1^2-p_2^2-p_3^2)F_9 - 2p_2^2p_3^2F_{11} - 2F_{13}\big] + 2R\big[(p_1^2-p_2^2-p_3^2)F_{13}\big]$$
$$-4p_1^2\big[p_2^2(p_1^2-p_2^2-p_3^2)F_9 - 2p_2^2p_3^2F_{11} - 2F_{13}\big]\Big\}$$
$$+\frac{4d}{p_1^2}\big[p_2^2p_3^2\,p_1\cdot p_3F_{11} + p_2^2F_{13} + p_1\cdot p_2p_2^2p_3\cdot p_2F_9\big]. \tag{10.34}$$

We are now going to use the results above in order to identify the link between the two transverse sections in the F-basis introduced by the perturbative expansion and the A-basis of the transverse traceless sector. Notice that the 13 form factors of the F-basis form a complete basis in d-dimensions, and have some nice properties, as we are going to emphasize below.

10.5. Connection between the A- and the F-basis

By a direct analysis of the previous primary and secondary constraint in the F-basis, using the equations given in Sections 6.3 and 6.4 for the A_i form factors, we obtain the relations which define the mapping between the transverse traceless sectors in the two basis, which is given by

$$A_1 = 4(F_7 - F_3 - F_5) - 2p_2^2F_9 - 2p_3^2F_{10} \tag{10.35a}$$
$$A_2 = 2(p_1^2-p_2^2-p_3^2)(F_7 - F_5 - F_3) - 4p_2^2p_3^2(F_6 - F_8 + F_4) - 2F_{13} \tag{10.35b}$$
$$A_3 = p_3^2(p_1^2-p_2^2-p_3^2)F_{10} - 2p_2^2\,p_3^2F_{12} - 2F_{13} \tag{10.35c}$$
$$A_3(p_2\leftrightarrow p_3) = p_2^2(p_1^2-p_2^2-p_3^2)F_9 - 2p_2^2p_3^2F_{11} - 2F_{13} \tag{10.35d}$$
$$A_4 = (p_1^2-p_2^2-p_3^2)F_{13}. \tag{10.35e}$$

It is worth noticing that the form factor A_3 and its corresponding $A_3(p_2\leftrightarrow p_3)$ are well-defined since

$$F_{10}(s;s_1,s_2,0) = F_9(s;s_2,s_1,0), \quad F_{12}(s;s_1,s_2,0) = F_{11}(s;s_2,s_1,0). \tag{10.36}$$

Going back to the full perturbative amplitude we can re-express the entire correlator as

$$\langle T^{\mu_1\nu_1}(p_1)\,J^{\mu_2}(p_2)\,J^{\mu_3}(p_3)\rangle$$
$$= \langle t^{\mu_1\nu_1}(p_1)\,j^{\mu_2}(p_2)\,j^{\mu_3}(p_3)\rangle + \langle t^{\mu_1\nu_1}_{loc}(p_1)\,J^{\mu_2}(p_2)\,J^{\mu_3}(p_3)\rangle \tag{10.37}$$

where the semi-local term is expressed exactly as

$$\langle t^{\mu_1\nu_1}_{loc}(p_1)\,J^{\mu_2}(p_2)\,J^{\mu_3}(p_3)\rangle =$$
$$= \frac{p_{1\beta_1}}{p_1^2}\left[2p_1^{(\mu_1}\delta^{\nu_1)}_{\alpha_1} - \frac{p_{1\alpha_1}}{(d-1)}\left(\delta^{\mu_1\nu_1} + (d-2)\frac{p_1^{\mu_1}p_1^{\nu_1}}{p_1^2}\right)\right]$$
$$\times\,\langle T^{\alpha_1\beta_1}(p_1)\,J^{\mu_2}(p_2)\,J^{\mu_3}(p_3)\rangle \tag{10.38}$$

and the transverse traceless part is reconstructed as

$$\langle t^{\mu_1\nu_1}(p_1)\, j^{\mu_2}(p_2)\, j^{\mu_3}(p_3)\rangle_{pert} = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\pi^{\mu_2}_{\alpha_2}(p_2)\pi^{\mu_3}_{\alpha_3}(p_3)\times$$
$$\times\Big\{\big[4(F_7-F_3-F_5)-2p_2^2F_9-2p_3^2F_{10}\big]\,p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_1^{\alpha_3}$$
$$+\big[2(p_1^2-p_2^2-p_3^2)(F_7-F_5-F_3)-4p_2^2p_3^2(F_6-F_8+F_4)-2F_{13}\big]\,\delta^{\alpha_2\alpha_3}p_2^{\alpha_1}p_2^{\beta_1}$$
$$+\big[\,p_3^2(p_1^2-p_2^2-p_3^2)F_{10}-2p_2^2\,p_3^2F_{12}-2F_{13}\big]\,\delta^{\alpha_1\alpha_2}p_2^{\beta_1}p_1^{\alpha_3}$$
$$+\big[p_2^2(p_1^2-p_2^2-p_3^2)F_9-2p_2^2p_3^2F_{11}-2F_{13}\big]\,\delta^{\alpha_1\alpha_3}p_2^{\beta_1}p_3^{\alpha_2}$$
$$+\big[(p_1^2-p_2^2-p_3^2)F_{13}\big]\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1}\Big\}. \tag{10.39}$$

Notice that neither F_1 nor F_2 will be part of the local contributions since they are both completely traceless. Therefore in the F-basis, the contributions appearing in (10.38) will be combinations of $F_3\ldots F_{13}$ which are independent from the 4 combinations of the F's identified by the mapping (10.35).

Since we are still defining the correlator in d-dimensions, and it is conformal in this case, then its d-dimensional trace has to vanish. This condition brings in two additional constraints on the two form factors F_1 and F_2, which now enter into the analysis,

$$F_1=\frac{(d-4)}{p_1^2(d-1)}\big[F_{13}-p_2^2\,F_3-p_3^2\,F_5-p_2\cdot p_3\,F_7\big] \tag{10.40}$$

$$F_2=\frac{(d-4)}{p_1^2(d-1)}\big[p_2^2\,F_4+p_3^2\,F_6+p_2\cdot p_3\,F_8\big], \tag{10.41}$$

which will be important for the renormalization procedure and the identification of the anomaly term.

We remark that the independent analysis of A_4 [13], which has essentially the same structure as F_{13}, as one can immediately realize from (10.35), shows that in a general conformal field theory the singularity of F_{13} can only be of order $1/(d-4)$ and not any higher. We are now going to test such general analysis to the specific case of QED at one-loop.

10.6. F_{13} in QED

The expressions of the 13 F_i form factors in d-dimensions are shown in the appendix. The renormalized results in $d=4$ have been given in [7]. Notice that we have expressed their expressions directly in terms of the two master integrals

$$B_0(s,0,0)=\int\frac{d^dl}{i\pi^2}\frac{1}{l^2(l-p_1)^2}\equiv B_0(s)$$
$$C_0(s,s_1,s_2)=\frac{1}{i\pi^2}\int d^dl\,\frac{1}{(l^2)\,((l-q)^2)\,((l+p)^2)} \tag{10.42}$$

which will be useful for the discussion of the action of the conformal generators on each of them. Introducing the variables $\sigma\equiv s^2-2s(s_1+s_2)+(s_1-s_2)^2$ and $\gamma\equiv s-s_1-s_2$, F_{13} takes the form

$$
\begin{aligned}
F_{13,d}(s,s_1,s_2) = {} & \frac{2\pi^2 e^2 s^2}{(d-2)(d-1)d\sigma^2}\Big\{-2s_2\big[(d^2-3d+4)(s+s1)-4(d-1)s_1\big] \\
& + (d^2-3d+4)\big[(s-s_1)^2+s_2^2\big]\Big\}B_0(s) - \frac{\pi^2 e^2 s_1}{(d-2)(d-1)d\sigma^2\gamma}\Big\{d^3\sigma^2 - d^2\big[3s^4 \\
& - 2s^3(7s_1+9s_2)+8s^2\big(3s_1^2+3s_1s_2+4s_2^2\big)-2s(s_1-s_2)^2(9s_1+11s_2)+5(s_1-s_2)^4\big] \\
& + 2d\big[2s^4-11s^3(s_1+s_2)+s^2\big(21s_1^2+24s_1s_2+19s_2^2\big)+s\big(-17s_1^3+5s_1^2s_2+25s_1s_2^2 \\
& - 13s_2^3\big)+(s_1-s_2)^3(5s_1-3s_2)\big]+8s_1\big(s^3-3s^2(s_1+s_2)+3s\big(s_1^2-s_2^2\big)-(s_1-s_2)^3\big)\Big\} \\
& \times B_0(s_1) - \frac{\pi^2 e^2 s_2}{(d-2)(d-1)d\sigma^2\gamma}\Big\{d^3\sigma^2 - d^2\big[3s^4-2s^3(9s_1+7s_2)+8s^2\big(4s_1^2+3s_1s_2 \\
& + 3s_2^2\big)-2s(s_1-s_2)^2(11s_1+9s_2)+5(s_1-s_2)^4\big]+2d\big[2s^4-11s^3(s_1+s_2)+s^2\big(19s_1^2 \\
& + 24s_1s_2+21s_2^2\big)+s\big(-13s_1^3+25s_1^2s_2+5s_1s_2^2-17s_2^3\big)+(s_1-s_2)^3(3s_1-5s_2)\big] \\
& + 8s_2\big[s^3-3s^2(s_1+s_2)-3s\big(s_1^2-s_2^2\big)+(s_1-s_2)^3\big]\Big\}B_0(s_2) \\
& + \frac{4\pi^2 e^2 s^2 s_1 s_2(d\sigma+8s_1s_2)}{(d-2)d\sigma^2\gamma}C_0(s,s_1,s_2),
\end{aligned}
\tag{10.43}
$$

which we will study in the $d \to 4$ limit. As discussed in [4] [7], the singularity of this form factor comes from the scalar form factor $\Pi(p^2)$ of the photon 2-point function $\langle JJ\rangle$. The singularity of this correlator will be at all orders of the form $1/\epsilon$ in a conformal theory and not higher. This is a crucial point in the proof which is clearly not satisfied in a non-conformal theory. In fact, the only available counterterm to regulate a conformal theory is given by

$$
\frac{1}{\epsilon}\int d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu} \tag{10.44}
$$

which renormalizes the 2-point function $\langle JJ\rangle$ and henceforth F_{13}. Explicit computations in QED at one-loop, where the theory is conformal, show that

$$
F_{13} = G_0(p_1^2,p_2^2,p_3^3) - \frac{1}{2}\left[\Pi(p_2^2)+\Pi(p_3^2)\right] \tag{10.45}
$$

We just recall that the structure of the two-point function of two conserved vector currents of scaling dimensions η_1 and η_2 is given by [24]

$$
G_V^{\alpha\beta}(p) = \delta_{\eta_1\eta_2}\, c_{V12}\, \frac{\pi^{d/2}}{4^{\eta_1-d/2}}\frac{\Gamma(d/2-\eta_1)}{\Gamma(\eta_1)}\left(\eta^{\alpha\beta}-\frac{p^\alpha p^\beta}{p^2}\right)(p^2)^{\eta_1-d/2}, \tag{10.46}
$$

with c_{V12} being an arbitrary constant. It will be nonvanishing only if the two currents share the same dimensions, and it is characterized just by a single pole (to all orders) $1/\epsilon$ in dimensional regularization. The divergence can be regulated with $d \to d-2\epsilon$, and expanding the product $\Gamma(d/2-\eta)\,(p^2)^{\eta-d/2}$ in 10.46 in a Laurent series around $d/2-\eta = -n$ (integer) one can extract the single pole in $1/\epsilon$ in the form [24]

$$
\Gamma\left(d/2-\eta\right)\,(p^2)^{\eta-d/2} = \frac{(-1)^n}{n!}\left(-\frac{1}{\epsilon}+\psi(n+1)+O(\epsilon)\right)(p^2)^{n+\epsilon}, \tag{10.47}
$$

where $\psi(z)$ is the logarithmic derivative of the Gamma function, and ϵ takes into account the divergence of the two-point correlator for particular values of the scale dimension η and of the

space–time dimension d. In the QED case, the renormalization involves only the master integrals $B_0(s_i)$, which gives $(d = 4 - 2\epsilon)$

$$F_{13} = \frac{2\pi^2 e^2}{3\epsilon} + F_{13}^{fin}, \tag{10.48}$$

implying that F_1 (from (10.41)) will be given by

$$F_1 = \left(\frac{2}{3}\frac{\epsilon}{s}\right)\left(\frac{2\pi e^2}{3\epsilon} + F_{13}^{fin}\right), \tag{10.49}$$

which in the $d \to 4$ limit gives

$$\lim_{d\to 4} F_1 = -\frac{4}{9}\frac{\pi^2}{s}, \tag{10.50}$$

showing the appearance of an anomaly pole in the single form factor which is responsible for the trace anomaly.

It is quite obvious that the non-perturbative analysis of [13] in the A-basis and the perturbative ones in the F-basis are consistent. There is some additional important information that we can extract in the latter basis if we go back to the two equations in (10.41).

1. From the finiteness of all the form factors, except for F_{13} which is regulated with a $1/\epsilon$ divergence, it is obvious that in the limit of $d \to 4$, as a result of (10.49), F_1 is nonvanishing and exhibits a $1/p_1^2 \equiv 1/s$ behavior. Therefore, the emergence of F_1 in $d = 4$ as a form factor which accounts for the anomaly is a nice feature if this general analysis. It shows how to link an anomaly pole to the renormalization of a single form factor in the expansion of the correlator.

2. At the same time, it is possible to check explicitly from its d-dimensional expression shown in (A.6) that the second form factor F_2 vanishes as $d \to 4$, proving that there will be one and only one tensor structure of nonzero trace.

3. Although the results above are fully confirmed by the previous perturbative analysis, they hold generically (non perturbatively) in the context of the conformal realizations of such correlator.

A natural question to ask is what happens to the tensor structures $t_2, \dots t_{13}$ as we move from d to 4 dimensions. The answer is quite immediate. We contract such structures with the d-dimensional metric $g_{\mu\nu}(d)$ and perform the $d \to 4$ limit. One can easily check that t_9, t_{10}, t_{11} and t_{12}, remain traceless in any dimensions, while the remaining ones become traceless in this limit. For instance

$$\begin{aligned} g^{\mu\nu}(d) t_1^{\mu\nu\alpha\beta} &= (d-4)k^2 u^{\alpha\beta}(p,q) \\ g^{\mu\nu}(d) t_2^{\mu\nu\alpha\beta} &= (d-4)k^2 w^{\alpha\beta}(p,q), \end{aligned} \tag{10.51}$$

and similarly for the others. Therefore, in the $d \to 4$ limit, the F-basis satisfies all the original constraints and the separation between traceless and trace-contributions which were described in Sect. 9.

10.7. Implications

We may summarize the result of this section by saying that the emergence of an anomaly pole in the TJJ is not limited to perturbation theory but is a specific feature of the non-perturbative solution as well. The perturbative description offers a simple view of why this phenomenon

Fig. 3. (a): Dispersive description of the singularity of the spectral density $\rho(s)$ as a spacetime process. (b): The exchange of a pole as the origin of the conformal anomaly in the TJJ viewed in perturbation theory.

takes place. In the dispersive representation of the unique form factor which is responsible for the appearance of the anomaly (F_1), this phenomenon is related to the exchange of a collinear fermion/antifermion pair in the s variable ($\rho(s) \sim \delta(s)$) (see Fig. 3). This configuration provides a contribution to the anomaly action of the form

$$\mathcal{S}_{pole} = -\frac{e^2}{36\pi^2}\int d^4x d^4y \left(\Box h(x) - \partial_\mu\partial_\nu h^{\mu\nu}(x)\right)\Box^{-1}_{x\,y} F_{\alpha\beta}(x)F^{\alpha\beta}(y) \tag{10.52}$$

In view of the equivalence between the perturbative and the nonperturbative solution for the A_i (and henceforth for the F-basis), which will be discussed in section 13, it is obvious that this phenomenon is lifted from its perturbative origin and acquires a general meaning. We refer to [4,25] for a general perturbative analysis of the spectral densities for such type of vertices.

There is no doubt that this is a one-loop phenomenon in QED which is obviously violated at higher orders, since the theory, in this case, ceases to be conformal. However, as we are going to show in the next section, the one-loop expression in QED reproduces the *entire* non perturbative conformal BMS solution, and this explains why our proof should be considered a definitive proof of the fact that, at least for this correlator, the exchange of effective massless interactions is the key signature of the conformal anomaly.

11. Conformal Ward identities and the perturbative master integrals

We now turn to illustrate the action of the conformal generators on the d-dimensional expressions of the F_i, showing that they are indeed solutions of the corresponding CWI's. We will first elaborate on the action of the conformal generators on the simple master integrals B_0, C_0. Such an action will be reformulated in terms of the external invariant of each master integral, starting from the original definitions of the special conformal K^κ and dilatation D operators.

Understanding the way the conformal constraints work on perturbative realizations of conformal correlators is indeed important for various reasons. For instance, one can find, by a direct perturbative analysis simpler realizations of the general hypergeometric solutions of the CWI's for the A_i, while, at the same time, one can test the consistency of the general approach implemented in the solution of the conformal constraints which does not rely on an explicit Lagrangian but just on the data content of a given CFT. We are going to show that indeed such is the case and that the d-dimensional form factors given in the appendix satisfy all the conformal constraints corresponding to the dilatation and special conformal WI's.

11.1. The conformal constraints on the external invariants

The action of the special conformal transformations K^κ and of the dilatations D on them, will require that K_{ij}, L_N, R in (4.9), (6.39) and (6.40) be expressed in terms of the three invariants s, s_1 and s_2.

Taking the four-momentum p_1^μ as independent, we will be using the chain rules

$$\frac{\partial}{\partial p_2^\mu} = -\frac{p_{1\mu}}{p_1}\frac{\partial}{\partial p_1} + \frac{p_{2\mu}}{p_2}\frac{\partial}{\partial p_2} \tag{11.1}$$

$$\frac{\partial}{\partial p_3^\mu} = -\frac{p_{1\mu}}{p_1}\frac{\partial}{\partial p_1} + \frac{p_{3\mu}}{p_3}\frac{\partial}{\partial p_3}, \tag{11.2}$$

and by taking appropriate linear combinations of these relations we obtain the system of equations

$$\begin{pmatrix} p_2^\mu \frac{\partial}{\partial p_2^\mu} - p_2^\mu \frac{\partial}{\partial p_3^\mu} \\ p_3^\mu \frac{\partial}{\partial p_2^\mu} - p_3^\mu \frac{\partial}{\partial p_3^\mu} \end{pmatrix} = \begin{pmatrix} p_2 & -\frac{p_2\cdot p_3}{p_3} \\ \frac{p_2\cdot p_3}{p_2} & -p_3 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial p_2} \\ \frac{\partial}{\partial p_3} \end{pmatrix}. \tag{11.3}$$

Solving the system above for the derivative of the magnitudes of the momenta we obtain the relations

$$\frac{\partial}{\partial p_2} = \left(\frac{p_2 p_3}{(p_2\cdot p_3)^2 - p_2^2 p_3^2}\right)\left[\frac{p_2\cdot p_3}{p_3}\left(p_3^\mu \frac{\partial}{\partial p_2^\mu} - p_3^\mu \frac{\partial}{\partial p_3^\mu}\right) - p_3\left(p_2^\mu \frac{\partial}{\partial p_2^\mu} - p_2^\mu \frac{\partial}{\partial p_3^\mu}\right)\right] \tag{11.4a}$$

$$\frac{\partial}{\partial p_3} = \left(\frac{p_2 p_3}{(p_2\cdot p_3)^2 - p_2^2 p_3^2}\right)\left[p_2\left(p_3^\mu \frac{\partial}{\partial p_2^\mu} - p_3^\mu \frac{\partial}{\partial p_3^\mu}\right) - \frac{p_2\cdot p_3}{p_2}\left(p_2^\mu \frac{\partial}{\partial p_2^\mu} - p_2^\mu \frac{\partial}{\partial p_3^\mu}\right)\right]. \tag{11.4b}$$

Afterwards, by taking other linear combinations we obtain

$$\begin{cases} p_2^\mu \frac{\partial}{\partial p_2^\mu} + p_3^\mu \frac{\partial}{\partial p_2^\mu} = p_1 \frac{\partial}{\partial p_1} - \frac{p_1\cdot p_2}{p_2}\frac{\partial}{\partial p_2} \\ p_3^\mu \frac{\partial}{\partial p_3^\mu} + p_2^\mu \frac{\partial}{\partial p_3^\mu} = p_1 \frac{\partial}{\partial p_1} - \frac{p_1\cdot p_3}{p_3}\frac{\partial}{\partial p_3} \end{cases}, \tag{11.5}$$

which, combined together, give

$$\begin{aligned} p_1 \frac{\partial}{\partial p_1} = \left(\frac{p_2\, p_3}{(p_2\cdot p_3)^2 - p_2^2\, p_3^2}\right)\Bigg[& \frac{p_2\cdot p_3}{p_2\, p_3}\big((p_2\cdot p_3) + p_2^2\big)\, p_3^\mu \frac{\partial}{\partial p_3^\mu} \\ & + \frac{p_2\cdot p_3}{p_2\, p_3}\big((p_2\cdot p_3) + p_3^2\big)\, p_2^\mu \frac{\partial}{\partial p_2^\mu} - \frac{p_2}{p_3}\big((p_2\cdot p_3) + p_3^2\big)\, p_3^\mu \frac{\partial}{\partial p_2^\mu} \\ & - \frac{p_3}{p2}\big((p_2\cdot p_3) + p_2^2\big)\, p_2^\mu \frac{\partial}{\partial p_3^\mu}\Bigg], \end{aligned} \tag{11.6}$$

with the four-vector forms of the derivatives rearranged as

$$p_2^\mu \frac{\partial}{\partial p_2^\mu} B_0(s) = \frac{(d-4)}{2s}(s + s_1 - s_2)\, B_0(s) \tag{11.7a}$$

$$p_3^\mu \frac{\partial}{\partial p_2^\mu} B_0(s) = \frac{(d-4)}{2s}(s - s_1 + s_2)\, B_0(s) \tag{11.7b}$$

$$p_2^\mu \frac{\partial}{\partial p_3^\mu} B_0(s) = \frac{(d-4)}{2s}(s - s_1 + s_2)\, B_0(s) \tag{11.7c}$$

$$p_3^\mu \frac{\partial}{\partial p_2^\mu} B_0(s) = \frac{(d-4)}{2s}(s + s_1 - s_2)\, B_0(s). \tag{11.7d}$$

Inserting these expressions in (11.4) and (11.6) we obtain the relations

$$p_1 \frac{\partial}{\partial p_1} B_0(s) = (d-4)\, B_0(s), \qquad \frac{\partial}{\partial p_2} B_0(s) = 0, \qquad \frac{\partial}{\partial p_3} B_0(s) = 0. \tag{11.8a}$$

The four-derivatives of $B_0(s_1)$ are computed in a similar way, obtaining

$$p_2^\mu \frac{\partial}{\partial p_2^\mu} B_0(s_1) = (d-4)\, B_0(s_1) \tag{11.9a}$$

$$p_3^\mu \frac{\partial}{\partial p_2^\mu} B_0(s_1) = \frac{(d-4)}{2s_1}(s - s_1 - s_2)\, B_0(s_1) \tag{11.9b}$$

$$p_2^\mu \frac{\partial}{\partial p_3^\mu} B_0(s_1) = p_3^\mu \frac{\partial}{\partial p_2^\mu} B_0(s_1) = 0, \tag{11.9c}$$

giving

$$p_1 \frac{\partial}{\partial p_1} B_0(s_1) = 0, \qquad \frac{\partial}{\partial p_2} B_0(s_1) = \frac{(d-4)}{p_2}\, B_0(s_1), \qquad \frac{\partial}{\partial p_3} B_0(s_1) = 0. \tag{11.10a}$$

Finally, for the scalar 2-point function $B_0(s_2)$ we obtain

$$p_2^\mu \frac{\partial}{\partial p_2^\mu} B_0(s_2) = p_3^\mu \frac{\partial}{\partial p_2^\mu} B_0(s_2) = 0 \tag{11.11a}$$

$$p_2^\mu \frac{\partial}{\partial p_3^\mu} B_0(s_1) = \frac{(d-4)}{2s_2}(s - s_1 - s_2)\, B_0(s_2) \tag{11.11b}$$

$$p_3^\mu \frac{\partial}{\partial p_2^\mu} B_0(s_1) = (d-4)\, B_0(s_2), \tag{11.11c}$$

and

$$p_1 \frac{\partial}{\partial p_1} B_0(s_2) = 0, \qquad \frac{\partial}{\partial p_2} B_0(s_2, 0, 0) = 0, \qquad \frac{\partial}{\partial p_3} B_0(s_2) = \frac{(d-4)}{p_3}\, B_0(s_2). \tag{11.12a}$$

Finally, we consider the action of these operators on the scalar integral $C_0(s, s_1, s_2)$, obtaining the relations

$$p_2^\mu \frac{\partial}{\partial p_2^\mu} C_0(s, s_1, s_2) = \frac{(d-3)}{s}\big[B_0(s_1) - B_0(s_2)\big] + \frac{(d-6)s + (d-4)(s_1 - s_2)}{2s} C_0(s, s_1, s_2) \tag{11.13a}$$

$$p_3^\mu \frac{\partial}{\partial p_2^\mu} C_0(s, s_1, s_2) = \frac{(d-3)}{s\, s_1}\big[(s_1 - s) B_0(s_2) + s B_0(s) - s_1 B_0(s_1)\big] + \frac{(d-4)(s - s_1)(s + s_1 - s_2)}{2s\, s_1} C_0(s, s_1, s_2) \tag{11.13b}$$

$$p_2^\mu \frac{\partial}{\partial p_3^\mu} C_0(s, s_1, s_2) = \frac{(d-3)}{s\, s_2}\big[(s_2 - s) B_0(s_1) + s B_0(s) - s_2 B_0(s_2)\big] + \frac{(d-4)(s - s_2)(s - s_1 + s_2)}{2s\, s_2} C_0(s, s_1, s_2) \tag{11.13c}$$

$$p_3^\mu \frac{\partial}{\partial p_2^\mu} C_0(s,s_1,s_2) = \frac{(d-3)}{s}\big[B_0(s_2) - B_0(s_1)\big] + \frac{(d-6)s - (d-4)(s_1-s_2)}{2s} C_0(s,s_1,s_2). \tag{11.13d}$$

Using the relations given above, after some algebra, we obtain the expressions

$$\frac{\partial}{\partial p_1} C_0 = \frac{p_1}{\sigma\, s}\Big\{2(d-3)\big[(s+s_1-s_2)\,B_0(s_1) + (s-s_1+s_2)\,B_0(s_2) - 2s\,B_0(s)\big] + \big[(d-4)(s_1-s_2)^2 - (d-2)s^2 + 2s(s_1+s_2)\big]\,C_0(s,s_1,s_2)\Big\} \tag{11.14a}$$

$$\frac{\partial}{\partial p_2} C_0 = \frac{p_2}{\sigma\, s_1}\Big\{2(d-3)\big[(s+s_1-s_2)\,B_0(s) + (s_1+s_2-s)\,B_0(s_2) - 2s_1\,B_0(s_1)\big] + \big[2s_2\big(s_1-(d-4)s\big) + (s-s_1)\big((d-4)s + (d-2)s_1\big) + (d-4)s_2^2\big] \times C_0(s,s_1,s_2)\Big\} \tag{11.14b}$$

$$\frac{\partial}{\partial p_3} C_0 = \frac{p_3}{\sigma\, s_2}\Big\{2(d-3)\big[(s-s_1+s_2)\,B_0(s) + (s_1+s_2-s)\,B_0(s_1) - 2s_2\,B_0(s_2)\big] + \big[(d-4)(s-s_1)^2 - (d-2)s_2^2 + 2s_2(s+s_1)\big]C_0(s,s_1,s_2)\Big\}. \tag{11.14c}$$

At this point, one can write the operators K, L_N and R of the special CWI's in terms of the three invariants s, s_1 and s_2 as

$$\frac{\partial}{\partial s} = \frac{1}{2p_1}\frac{\partial}{\partial p_1}, \qquad \frac{\partial}{\partial s_1} = \frac{1}{2p_2}\frac{\partial}{\partial p_2}, \qquad \frac{\partial}{\partial s_2} = \frac{1}{2p_3}\frac{\partial}{\partial p_3}, \tag{11.15}$$

and from the explicit form of the K_i operator in (4.9) we obtain

$$K_i = \delta_{i,j+1}\left[4s_j\frac{\partial^2}{\partial s_j^2} + 2(d+2-2\Delta_i)\frac{\partial}{\partial s_j}\right], \quad i=1,2,3, \tag{11.16}$$

where we have set $s_0 \equiv s$.

In the same way, we find that the operators L_N and R, from their defining expressions (6.39) and (6.40), take the form

$$L_N = 2(s+s_1-s_2)\,s\frac{\partial}{\partial s} + 4s\,s_1\frac{\partial}{\partial s_1} + \big[(2d-\Delta_1-2\Delta_2+N)\,s + (2\Delta_1-d)(s_2-s_1)\big] \tag{11.17}$$

$$R = 2s\frac{\partial}{\partial s} - (2\Delta_1 - d), \tag{11.18}$$

while their symmetric versions are given by

$$L'_N = L_N \ \text{ with } s \leftrightarrow s_1 \text{ and } \Delta_1 \leftrightarrow \Delta_2 \tag{11.19}$$

$$R' = R \ \text{ with } s \mapsto s_1 \text{ and } \Delta_1 \mapsto \Delta_2. \tag{11.20}$$

We then obtain

$$
\begin{aligned}
K_{12}\ B_0(s) &= K_{13}\ B_0(s) = -\frac{4(d-4)}{s}\, B_0(s,0,0), \qquad K_{23}\ B_0(s) = 0\\
K_{21}\ B_0(s_1) &= K_{23}\ B_0(s_1) = -\frac{2(d-4)}{s_1}\, B_0(s_1), \qquad K_{13}\ B_0(s_1) = 0\\
K_{31}\ B_0(s_2) &= K_{32}\ B_0(s_2) = -\frac{2(d-4)}{s_2}\, B_0(s_2), \qquad K_{12}\ B_0(s_2) = 0,
\end{aligned}
\tag{11.21a}
$$

and

$$
\begin{aligned}
&K_{12}\ C_0(s,s_1,s_2)\\
&= \frac{4(d-3)}{s\,s_1\,\sigma}\Big[-\big(s^2+(s-2s_1)(s_1-s_2)\big)\, B_0(s_2) + s(s+5s_1-s_2)\, B_0(s)\\
&\quad - 2s_1\,(2s+s_1-s_2)\, B_0(s_1)\Big]\\
&\quad + \frac{1}{s\,s_1\,\sigma}\big[4(d-3)(3s-3s_1-s_2)\,s\,s_1 + 2(d-4)(s-2s_1)\sigma\big]\, C_0(s,s_1,s_2)
\end{aligned}
\tag{11.22a}
$$

$$
\begin{aligned}
&K_{23}\ C_0(s,s_1,s_2)\\
&= \frac{4(d-3)}{s_1\,s_2\,\sigma}\Big[(s_1-s_2)(s-s_1-s_2)\, B_0(s) + s_1(s_1-s+2s_2)\, B_0(s_1)\\
&\quad + s_2(s-3s_1-s_2)\, B_0(s_2)\Big] + \frac{1}{s_1\,s_2\,\sigma}\big[2(d-4)(s_1-s_2)[(s-s_1)^2 - 2s\,s_2 + s_2^2]\\
&\quad + 4(d-2)(s_1-s2)\,s_1\,s_2\big]\, C_0(s,s_1,s_2)
\end{aligned}
\tag{11.22b}
$$

$$
\begin{aligned}
&K_{13}\ C_0(s,s_1,s_2)\\
&= \frac{4(d-3)}{s\,s_2\,\sigma}\Big[-\big(s^2+(s-2s_2)(s_2-s_1)\big)\, B_0(s_1) + s(s-s_1+5s_2)\, B_0(s)\\
&\quad - 2s_2(2s-s_1+s_2)\, B_0(s_2)\Big]\\
&\quad + \frac{1}{s\,s_2\,\sigma}\big[4(d-3)(3s-3s_2-s_1)\,s\,s_2 + 2(d-4)(s-2s_2)\sigma\big]\, C_0(s,s_1,s_2),
\end{aligned}
\tag{11.22c}
$$

while the action of the operators L_N and R on $B_0(s)$ is given by

$$
\begin{aligned}
L_N\ B_0(s) &= \big[(N-2)s - 4s_1 + 4s_2\big]\, B_0(s)\\
L'_N\ B_0(s) &= \big[(d+N-7)s_1 + (d-2)(s_2-s)\big]\, B_0(s)\\
R\ B_0(s) &= -4B_0(s)\\
R'\ B_0(s) &= -(d-2)B_0(s),
\end{aligned}
\tag{11.23a}
$$

and on $B_0(s_1)$

$$
\begin{aligned}
L_N\ B_0(s_1) &= \big[(d+N-6)s + d(s_2-s_1)\big]\, B_0(s_1)\\
L'_N\ B_0(s_1) &= \big[(N-3)s_1 - 2s + 2s_2\big]\, B_0(s_1)\\
R\ B_0(s_1) &= -d\ B_0(s_1)\\
R'\ B_0(s_1) &= -2\ B_0(s_1).
\end{aligned}
\tag{11.24a}
$$

Similarly, for $B_0(s_2)$ we obtain

$$
\begin{aligned}
L_N\, B_0(s_2) &= \big[(N-d+2)s + d(s_2-s_1)\big]\, B_0(s_2)\\
L'_N\, B_0(s_2) &= \big[(N-d+1)s_1 + (d-2)(s_2-s)\big]\, B_0(s_2)\\
R\, B_0(s_2) &= -d\, B_0(s_2)\\
R'\, B_0(s_2) &= -(d-2)\, B_0(s_2),
\end{aligned}
\tag{11.25a}
$$

and for $C_0(s,s_1,s_2)$

$$
\begin{aligned}
L_N\, C_0(s,s_1,s_2) &= 2(d-3)\big[B_0(s_1) - B_0(s_2)\big] + \big[(N-4)s - 4(s_1-s_2)\big]C_0(s,s_1,s_2)\\
L'_N\, C_0(s,s_1,s_2) &= 2(d-3)\big[B_0(s) - B_0(s_2)\big] + \big[(N-5)s_1 - 2(s-s_2)\big]C_0(s,s_1,s_2)\\
R\, C_0(s,s_1,s_2) &= \frac{2(d-3)}{\sigma}\big[(s+s_1-s_2)\, B_0(s_1) + (s-s_1+s_2)\, B_0(s_2) - 2s\, B_0(s)\big]\\
&\quad + \frac{2}{\sigma}\big[-(d-1)s^2 + (d+1)s(s_1+s_2) - 2(s_1-s_2)^2\big]C_0(s,s_1,s_2)\\
R'\, C_0(s,s_1,s_2) &= \frac{2(d-3)}{\sigma}\big[(s+s_1-s_2)\, B_0(s) + (-s+s_1+s_2)\, B_0(s_2) - 2s_1\, B_0(s_1)\big]\\
&\quad - \frac{2}{\sigma}\big[s(s_1 - d\, s_1 - 2s_2) + (s_1-s_2)(d-2)s_1 - s_2(s_1-s_2) + s^2\big]\\
&\quad \times C_0(s,s_1,s_2).
\end{aligned}
\tag{11.26a}
$$

Using the explicit expressions of the form factors F_i presented in the appendix, we have explicitly verified that they satisfy all the conformal constraints, once the action on the two master integrals is re-expressed according to the derivative rules that we have derived in this section in terms of the external invariants.

12. Renormalization and anomalous Ward identities in QED

In this final section we turn to the derivation of the anomalous CWI's for this correlator in QED. We recall that the singularity in the form factor F_{13}, which is the only one which is affected by the renormalization of the F-basis, can be removed by using the counterterm

$$
\int d^4x\, \sqrt{-g} F_{\mu\nu} F^{\mu\nu}. \tag{12.1}
$$

After the renormalization procedure in the F-basis, the renormalized expressions of the A_i form factors are obtained using the mapping (10.35). One can then derive renormalized anomalous (dilatation, primary and secondary) WI's. Using the explicit expressions of the A_i given in the appendix, and performing the renormalization, we obtain

$$
\left(\sum_i^3 p_i \frac{\partial}{\partial p_i} + 2\right) A_1 = 0 = -\mu \frac{\partial}{\partial \mu} A_1 \tag{12.2}
$$

$$
\left(\sum_i^3 p_i \frac{\partial}{\partial p_i}\right) A_2^R = \frac{8\pi^2 e^2}{3} = -\mu \frac{\partial}{\partial \mu} A_2^R \tag{12.3}
$$

$$
\left(\sum_i^3 p_i \frac{\partial}{\partial p_i}\right) A_3^R(p_2 \leftrightarrow p_3) = \frac{8\pi^2 e^2}{3} = -\mu \frac{\partial}{\partial \mu} A_3^R \tag{12.4}
$$

$$\left(\sum_i^3 p_i \frac{\partial}{\partial p_i} - 2\right) A_4^R = -\frac{4}{3}\pi^2 e^2 (s - s_1 - s_2) = -\mu \frac{\partial}{\partial \mu} A_4^R, \tag{12.5}$$

for the dilatation WI's, while for the primary CWI's we obtain

$$\begin{aligned}
&K_{13} A_1 = 0 && K_{23} A_1 = 0\\
&K_{13} A_2^R = -2A_1 && K_{23} A_2^R = 0\\
&K_{13} A_3^R = 4A_1 && K_{23} A_3^R = 4A_1\\
&K_{13} A_3^R(p_2 \leftrightarrow p_3) = 0 && K_{23} A_3^R(p_2 \leftrightarrow p_3) = -4A_1\\
&K_{13} A_4^R = 2A_3^R(p_2 \leftrightarrow p_3) - \frac{16\pi^2 e^2}{3} && K_{23} A_4^R = -2A_3^R + 2A_3^R(p_2 \leftrightarrow p_3).
\end{aligned} \tag{12.6}$$

In all the equations e is the renormalized charge and can be traded for $\beta(e)$ by the relation $\beta(e)/e = e^2/(12\pi^2)$. For the secondary CWI's we obtain

$$\begin{aligned}
&L_4 A_1 + R A_3^R - R A_3^R(p_2 \leftrightarrow p_3) = 0\\
&L_2\, A_2^R - s\,(A_3^R - A_3^R(p_2 \leftrightarrow p_3))\\
&\quad = \frac{16}{9}\pi^2 e^2 \left[3s1\, B_0^R(s_1, 0, 0) - 3s_2 B_0^R(s_2, 0, 0) - s1 + s2\right] + \frac{24}{9}\pi^2 e^2 s\\
&L_4\, A_3^R - 2R\, A_4^R = \frac{32}{9}\pi^2\, e^2 s_2 \left[1 - 3B_0^R(s_2, 0, 0)\right] + \frac{48}{9}\pi^2\, e^2\, s\\
&L_4\, A_3^R(p_2 \leftrightarrow p_3) + 2R\, A_4^R - 4\, s A_3^R(p_2 \leftrightarrow p_3) = \frac{32}{9}\,\pi^2\, e^2 s_1 \left[3B_0^R(s_1, 0, 0) - 1\right]\\
&L_3'\, A_1^R - 2R' A_2^R + 2R' A_3^R = 0\\
&L_1'\, A_3^R(p_2 \leftrightarrow p_3) + p_2^2(4A_2^R - 2A_3^R) + 2R' A_4^R = \frac{16}{3}\pi^2\, e^2\, s_1,
\end{aligned} \tag{12.7}$$

where we have used the relations

$$\frac{\partial}{\partial s_i} B_0^R(s_j, 0, 0) = -\frac{\delta_{ij}}{s_i}\; i = 0, 1, 2 \tag{12.8}$$

where $s_0 = s$ and

$$\begin{aligned}
\frac{\partial}{\partial s} C_0 &= \frac{1}{s\,\sigma}\left[s(s_1 + s_2 - s)C_0 + B_{0,R}(s_1)(s + s_1 - s_2) + B_{0,R}(s_2)(s - s_1 + s_2)\right.\\
&\quad \left. - 2s\, B_{0,R}(s)\right] \qquad (12.9)\\
\frac{\partial}{\partial s_1} C_0 &= \frac{1}{s_1\,\sigma}\left[s_1(s + s_2 - s_1)C_0 + B_{0,R}(s)(s + s_1 - s_2) + B_{0,R}(s_2)(s_1 - s + s_2)\right.\\
&\quad \left. - 2s_1\, B_{0,R}(s_1)\right] \qquad (12.10)\\
\frac{\partial}{\partial s_2} C_0 &= \frac{1}{s_2\,\sigma}\left[s_2(s + s_1 - s_2)C_0 + B_{0,R}(s_1)(s_2 + s_1 - s) + B_{0,R}(s)(s - s_1 + s_2)\right.\\
&\quad \left. - 2s_2\, B_{0,R}(s_2)\right]. \qquad (12.11)
\end{aligned}$$

We have defined $\sigma = s^2 - 2s(s_1+s_2) + (s_1-s_2)^2$, $B_{0,R}(s_i) \equiv B_{0,R}(s_i,0,0) = 2 - \log\left(-\frac{s_i}{\mu^2}\right)$ and, for simplicity, $C_0 \equiv C_0(s,s_1,s_2)$.

In $d=4$ the operator K_i and L_N take the forms

$$K_1 \equiv 4s\frac{\partial^2}{\partial s^2} - 4\frac{\partial}{\partial s}, \quad K_2 \equiv 4s_1\frac{\partial^2}{\partial s_1^2}, \quad K_3 \equiv 4s_2\frac{\partial^2}{\partial s_2^2} \tag{12.12}$$

$$L_N \equiv 2s\,(s+s_1-s_2)\frac{\partial}{\partial s} + 4s\,s_1\frac{\partial}{\partial s_1} + [(N-2)s + 4(s_2-s_1)], \qquad R \equiv 2s\,\frac{\partial}{\partial s} - 4 \tag{12.13}$$

$$L'_N \equiv 2s_1\,(s+s_1-s_2)\frac{\partial}{\partial s_1} + 4s\,s_1\frac{\partial}{\partial s} + [(N-3)s_1 + 2(s_2-s)], \qquad R' \equiv 2s_1\,\frac{\partial}{\partial s_1} - 2. \tag{12.14}$$

13. Comparing the $3K$ solutions of BMS with the perturbative result in QED

The perturbative analysis presented above provides a significant check of the consistency of the BMS approach for 3-point functions. However, at the same time, it carries a lot of insight about the connection between CFT's realized by free field theories and the structure of the corresponding nonperturbative solutions. In order to clarify this point we proceed with a direct comparison between the expressions of the A_i given in the appendix and the analogous results for the same correlator given in [14]. The A_i's given in the appendix have been obtained by the recomputed F_i's.

The final outcome of this analysis will be, by this direct check, that free field theories completely saturate the general nonperturbative solution, providing drastic simplifications of the results presented in [14]. As already mentioned, the latter have been presented in the form of parametric integrals of 3 Bessel functions, which obviously amount to linear combinations of Appell functions, originally introduced in studies of the AdS/CFT correspondence.

To make our treatment self-contained we need to provide some basic description of the structure of the solutions obtained by BMS in their analysis and then we will establish a direct link between these expressions and the simplified ones given in the appendix, corresponding to the perturbative A_i's. The BMS solutions take the form

$$\begin{aligned}
A_1 &= \alpha_1 J_{4[000]} \\
A_2 &= \alpha_1 J_{3[100]} + \alpha_3 J_{2[000]} \\
A_3 &= 2\alpha_1 J_{3[001]} + \alpha_3 J_{2[000]} \\
A_4 &= 2\alpha_1 J_{2[011]} + \alpha_3 J_{1[010]} + \alpha_4 J_{0[000]}
\end{aligned} \tag{13.1}$$

where the $J_{n[k_1k_2k_3]}$ are integrals corresponding to hypergeometric functions F_4 of 2 variables. In general they are defined as

$$J_{n[k_1k_2k_3]} = I_{d/2-1+n[\beta_1\beta_2\beta_3]} \tag{13.2}$$

where

$$I_{\alpha[\sigma_1\sigma_2\sigma_3]}(p_1,p_2,p_3) = \int_0^\infty dx x^\alpha \prod_{j=1}^3 p_j^{\sigma_j} K_{\sigma_j}(p_j x) \tag{13.3}$$

with $\beta_i = \Delta_i - d/2 + k_i$. In our case $\Delta_1 = d$ and $\Delta_2 = \Delta_3 = d-1$. The parametric integral is expressed in terms of products of modified Bessel functions $K_\nu(x)$ of second kind. The explicit expressions of such integrals have been worked out in [14]. All the J integrals appearing in the solution correspond to master integrals of the form [24]

$$J(\nu_1,\nu_2,\nu_3) = \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2)^{\nu_3}((l+p_1)^2)^{\nu_2}((l-p_2)^2)^{\nu_1}}\,, \tag{13.4}$$

which can be directly connected to 3-point functions of scalar operators, of suitable scaling dimensions Δ_i by the relations

$$\begin{aligned}
&\int \frac{d^d p_1}{(2\pi)^d}\frac{d^d p_2}{(2\pi)^d}\frac{d^d p_3}{(2\pi)^d}\,(2\pi)^d\delta^{(d)}(p_1+p_2+p_3)\,J(\nu_1,\nu_2,\nu_3)e^{-ip_1\cdot x_1 - ip_2\cdot x_2 - ip_3\cdot x_3}\\
&= \frac{1}{4^{\nu_1+\nu_2+\nu_3}\pi^{3d/2}}\frac{\Gamma(d/2-\nu_1)\Gamma(d/2-\nu_2)\Gamma(d/2-\nu_3)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)}\\
&\quad\times\frac{1}{(x_{12}^2)^{d/2-\nu_3}(x_{23}^2)^{d/2-\nu_1}(x_{31}^2)^{d/2-\nu_2}}\,,
\end{aligned} \tag{13.5}$$

with

$$\Delta_1 = d-\nu_2-\nu_3\,,\qquad \Delta_2 = d-\nu_1-\nu_3\,,\qquad \Delta_3 = d-\nu_1-\nu_2 \tag{13.6}$$

$$\nu_1 = \frac{1}{2}(d+\Delta_1-\Delta_2-\Delta_3)\qquad \nu_2 = \frac{1}{2}(d-\Delta_1+\Delta_2-\Delta_3)\,,$$

$$\nu_3 = \frac{1}{2}(d-\Delta_1-\Delta_2+\Delta_3) \tag{13.7}$$

An equivalent expression of the master integral $J(\nu_1,\nu_2,\nu_3)$ can be obtained combining the expressions above in the form

$$\begin{aligned}
J(\nu_1,\nu_2,\nu_3) &= \frac{\pi^{-d/2}2^{4-3d/2}}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(d-\nu_1-\nu_2-\nu_3)}\,p_1^{d/2-\nu_2-\nu_3}\,p_2^{d/2-\nu_1-\nu_3}\,p_3^{d/2-\nu_1-\nu_2}\\
&\quad\times\int_0^\infty dx\,x^{d/2-1}K_{d-\nu_1-\nu_3}(p_1x)K_{d-\nu_2-\nu_3}(p_2x)K_{d-\nu_1-\nu_2}(p_3x)
\end{aligned} \tag{13.8}$$

An alternative expression for this integral can be found in [24]. Notice that if we plug the scaling dimensions of $T^{\mu\nu}$ and of the current J^ρ, for instance in $J_{0[000]}$, we encounter divergences which need to be regulated using a rather complex scheme which has been discussed in [14]. This implies that the coefficients of the scalar 3-point functions contained in the J integrals correspond to generalized master integrals with indices ν_i which are *real* numbers rather than just integers, as in the case of ordinary perturbation theory, for instance in QCD. This sets them apart from the standard integrals appearing in massless theories at higher orders.

Because of this, it is not obvious that the general 3K solution can be directly related to the simple master integrals which appear in the A_i, given in the appendix, containing the two master integrals B_0 and C_0. In fact, all the relations presented in [14] which allow to connect various J integrals are, therefore, unrenormalized and need to be expanded in a regulator in order to generate the final expressions for each form factor. This, unfortunately does not allow to recognize that such solutions can be drastically simplified. By the same token, the solutions that we have provided for the A_i using the Fuchsian analysis of section 8 cannot be easily recognized for being related just to ordinary B_0 and C_0 master integrals.

We are now going to show that the same information provided by the J integrals is entirely reproduced by the perturbative solution in a simplified way. This clearly implies that there are significant cancellations among the contributions coming from different J integrals in the BMS solution, in whatever form they are written, which are far from being evident. The latter, obviously, remains essential in order to determine the minimal set of constants which characterize the general conformal solution and allow to establish a link between this and the perturbative result, but their significance probably stops here.

In order to establish this link, we consider different field theories in various dimensions characterized by a generic number of degenerate massless fermions, say n_ψ, which, as we are going to show, will be taking the role of α_1 in the final solution. Let's investigate this point.

As shown in [14], the secondary CWI's allow to express the 4 constants α_i in terms of α_1 and the normalization of the 2-point function C_J. These relations, in general, involve a regulator, except for specific dimensions. In $d = 3$ and $d = 5$, for instance, the correlator is finite and the relation between the perturbative and the non perturbative expressions of the A_i is transparent and can be worked out analytically.

We start from the two point functions, since the C_J presented [14] and in free field theory (QED) must be matched. In [14] the normalization of the JJ correlator is defined by the relation

$$\langle J^\mu(p)\, J^\nu(-p)\rangle = C_J\, \pi^{\mu\nu}(p)\Gamma\left(1-\frac{d}{2}+\frac{\epsilon}{2}\right) p^{d-2-\epsilon} \tag{13.9}$$

while in our case, in conformal QED, we find in dimensional regularization

$$\begin{aligned}\langle\, J^\mu(p)\, J^\nu(-p)\,\rangle &= \frac{2\pi^{\frac{D}{2}}\, e^2\,(d-2)}{d-1}\, p^2\, \pi^{\mu\nu}(p)\, \mathcal{B}_0(p^2,0,0)\\ &= \frac{2\pi^{\frac{d}{2}}\, e^2\,(d-2)\,\pi^{\mu\nu}(p)}{(d-1)\,\Gamma\,(d-2)}\, \left[\Gamma\left(\frac{d}{2}-1\right)\right]^2 \Gamma\left(2-\frac{D}{2}\right) p^{d-2},\end{aligned} \tag{13.10}$$

having used the explicit expression of the two point scalar integral (13.15).

In odd spacetime dimension n ($d = n + \epsilon$), the limit $\epsilon \to 0$ is finite and then the comparison between (13.9) and (13.10) gives the value for the normalization constant C_J of the two point function $\langle JJ\rangle$ as

$$C_J = \frac{2\pi^{\frac{d}{2}}\, e^2\,(d-2)}{(d-1)\,\Gamma\,(d-2)}\, \frac{\left[\Gamma\left(\frac{d}{2}-1\right)\right]^2 \Gamma\left(2-\frac{d}{2}\right)}{\Gamma\left(1-\frac{d}{2}\right)}. \tag{13.11}$$

It is simple to verify that the values of C_J in the $d = 3$ and $d = 5$ are those given by (13.11), in fact

$$C_J \overset{d=3}{=} \pi^{\frac{3}{2}} e^2 \left[\Gamma\left(\frac{1}{2}\right)\right]^3 \left[\Gamma\left(-\frac{1}{2}\right)\right]^{-1} = -\frac{\pi^{\frac{5}{2}}\, e^2}{2} \tag{13.12}$$

$$C_J \overset{d=5}{=} \frac{3\pi^{\frac{5}{2}} e^2}{2\,\Gamma(3)} \left[\Gamma\left(\frac{3}{2}\right)\right]^2 \Gamma\left(-\frac{1}{2}\right) \left[\Gamma\left(-\frac{3}{2}\right)\right]^{-1} = -\frac{9\pi^{\frac{7}{2}}\, e^2}{32} \tag{13.13}$$

With this information, we can proceed further, showing that the expressions of the A_i's presented in section A agree with those determined in the nonperturbative solution.

13.1. Explicit result in $d = 3$ (QED and scalar QED)

We consider the fermion and the scalar contribution combined together, selecting an arbitrary number of massless fermions and scalars running inside the loops, with constants n_F and n_S related to this condition.

In the particular case of $d = 3$ there are simplifications both in the exact and the perturbative solutions. The scalar integral B_0 is given by

$$B_0(s,0,0) = \frac{1}{i\pi^{\frac{d}{2}}}\int d^d\ell\,\frac{l}{\ell^2(\ell-p_1)^2} = \frac{\pi^{\frac{d}{2}}\left[\Gamma\left(\frac{d}{2}-1\right)\right]^2\Gamma\left(2-\frac{d}{2}\right)}{\pi^{\frac{d}{2}}\,\Gamma(d-2)\,(p_1^2)^{2-\frac{d}{2}}} \tag{13.14}$$

for which in $d = 3$ becomes

$$B_0(s,0,0) = \frac{\pi^{3/2}}{p_1} \tag{13.15}$$

where $p_1 = |p_1| = \sqrt{p_1^2}$. The scalar 3-point function C_0 can be simplified using the star-triangle rule for which

$$\begin{aligned}&\int \frac{d^dx}{[(x-x_1)^2]^{\alpha_1}\,[(x-x_2)^2]^{\alpha_2}\,[(x-x_3)^2]^{\alpha_3}}\\ &\overset{\sum_i\alpha_i=d}{=} \frac{i\pi^{d/2}\nu(\alpha_1)\nu(\alpha_2)\nu(\alpha_3)}{[(x_2-x_3)^2]^{\frac{d}{2}-\alpha_1}\,[(x_1-x_2)^2]^{\frac{d}{2}-\alpha_3}\,[(x_1-x_3)^2]^{\frac{d}{2}-\alpha_2}}\end{aligned} \tag{13.16}$$

where

$$\nu(x) = \frac{\Gamma\left(\frac{d}{2}-x\right)}{\Gamma(x)}, \tag{13.17}$$

which holds only if the condition $\sum_i \alpha_i = d$ is satisfied. In the case $d = 3$ (13.16) is proportional to the three point scalar integral, and in particular

$$\begin{aligned}C_0(p_1^2,p_2^2,p_3^2) &= \int \frac{d^d\ell}{i\pi^{\frac{d}{2}}}\frac{1}{\ell^2(\ell-p_2)^2(\ell+p_3)^2}\\ &= \int \frac{d^dk}{i\pi^{\frac{d}{2}}}\frac{1}{(k-p_1)^2(k+p_3)^2(k+p_3-p_2)^2}\\ &= \frac{\left[\Gamma\left(\frac{d}{2}-1\right)\right]^3}{(p_1^2)^{\frac{d}{2}-1}(p_2^2)^{\frac{d}{2}-1}(p_3^2)^{\frac{d}{2}-1}} \overset{d=3}{=} \frac{\pi^{3/2}}{p_1\,p_2\,p_3}.\end{aligned} \tag{13.18}$$

The explicit expression of the form factors in $d = 3$ using the perturbative approach to one loop order, can be found by substituting the expression of the scalar integral, using (13.15) and (13.18), and then considering the limit $d \to 3$ for all the form factors. The scalar and the fermion cases contribute equally, modulo an overall constant, giving

$$A_{1,D=3} = \left(\frac{\pi^3\,e^2(8n_F+n_S)}{6}\right)\frac{2\big(4p_1+p_2+p_3\big)}{(p_1+p_2+p_3)^4} \tag{13.19}$$

$$A_{2,D=3} = \left(\frac{\pi^3\,e^2(8n_F+n_S)}{6}\right)\frac{2\,p_1^2}{(p_1+p_2+p_3)^3} - \left(\frac{\pi^3\,e^2\,(8n_F-n_S)}{2}\right)$$

$$\times \frac{(2p_1+p_2+p_3)}{(p_1+p_2+p_3)^2} \tag{13.20}$$

$$A_{3,D=3} = \left(\frac{\pi^3 e^2(8n_F+n_S)}{6}\right)\frac{\left(-2p_1^2-3p_1p_2+3p_1p_3-p_2^2+p_3^2\right)}{(p_1+p_2+p_3)^3} - \left(\frac{\pi^3 e^2(8n_F-n_S)}{2}\right)\frac{(2p_1+p_2+p_3)}{(p_1+p_2+p_3)^2} \tag{13.21}$$

$$A_{4,D=3} = \left(\frac{\pi^3 e^2(8n_F+n_S)}{6}\right)\frac{(2p_1+p_2+p_3)\left(p_1^2-(p_2+p_3)^2+4p_2p_3\right)}{2(p_1+p_2+p_3)^2} + \left(\frac{\pi^3 e^2(8n_F-n_S)}{4}\right)\left(\frac{2p_1^2}{(p_1+p_2+p_3)-p_2-p_3}\right) \tag{13.22}$$

which coincide with the solution given by BMS in the limit $d=3$, modulo the identification of the constants

$$\alpha_1 = \left(\frac{\pi^3\, e^2(8n_F+n_S)}{6}\right), \qquad c_J = -\left(\frac{\pi^{\frac{5}{2}}\, e^2\,(8n_F-n_S)}{8}\right). \tag{13.23}$$

13.2. Explicit result in $d=5$ (QED and scalar QED)

The case $d=5$ is slightly more involved. In fact, while the result of B_0 is still the same, the explicit form of the C_0 needs some manipulations. We start from the star-triangle relation (13.16)

$$\int \frac{d^dx}{[(x-x_1)^2]^2\,[(x-x_2)^2]\,[(x-x_3)^2]^2} \overset{d=5}{=} \frac{i\pi^{d/2}\nu(2)\nu(1)\nu(2)}{[(x_2-x_3)^2]^{\frac{d}{2}-2}\,[(x_1-x_2)^2]^{\frac{d}{2}-2}\,[(x_1-x_3)^2]^{\frac{d}{2}-1}} \tag{13.24}$$

$$= \frac{i\pi^4}{2}\frac{1}{[(x_2-x_3)^2]^{1/2}\,[(x_1-x_2)^2]^{1/2}\,[(x_1-x_3)^2]^{3/2}} \tag{13.25}$$

and use an integration by parts to reduce the left-hand side. In particular, by setting $x_1 \to p_1$, $x_2 \to -p_3$, $x_3 \to p_1-p_3$ we obtain

$$\int \frac{d^dx}{[(x-x_1)^2]^2\,[(x-x_2)^2]\,[(x-x_3)^2]^2} = \int \frac{d^dx}{[(x-p_1)^2]^2\,[(x+p_3)^2]\,[(x-p_1+p_3)^2]^2} = \int \frac{d^d\ell}{[(\ell)^2]^2\,(\ell-p_2)^2\,[(\ell+p_3)^2]^2} = \frac{i\pi^4}{2\,p_1\,p_2\,p_3^3} \tag{13.26}$$

and

$$\int \frac{d^d\ell}{[(\ell)^2]^2\,(\ell-p_2)^2\,[(\ell+p_3)^2]^2} = -\frac{i\pi^{\frac{d}{2}}(d-4)\big[(d-6)(s-s_1)^2-(d-4)s_2^2+2s_2(s+s_1)\big]}{4s\,s_1\,s_2^2}\,C_0(s,s_1,s_2)$$

$$
\begin{aligned}
&- \frac{i\pi^{\frac{d}{2}}\,(d-3)\big[(d-6)(s-s_1)+(d-4)s_2\big]}{2s\,s_1\,s_2^2}\,B_0(s,0,0)\\
&+ \frac{i\pi^{\frac{d}{2}}\,(d-3)\big[(d-6)(s-s_1)-(d-4)s_2\big]}{2s\,s_1\,s_2^2}\,B_0(s_1,0,0)\\
&+ \frac{i\pi^{\frac{d}{2}}\,(d-3)\big[(d-6)(s+s_1)+(d-4)s_2\big]}{2s\,s_1\,s_2^2}\,B_0(s_2,0,0)
\end{aligned}
\tag{13.27}
$$

where $s=p_1^2$, $s_1=p_2^2$ and $s_2=p_3^2$ as previously. Inserting this expression into Eq. (13.25) and solving for C_0, in $d=5$ one finds

$$
C_0(s,s_1,s_2)=\frac{\pi^{3/2}}{p_1+p_2+p_3}. \tag{13.28}
$$

From (13.14) the B_0 is calculated in $d=5$ as

$$
B_0(s,0,0)=-\frac{\pi^{3/2}}{4}p_1. \tag{13.29}
$$

Plugging these results into the general form of the A_i given in appendix A we find

$$
\begin{aligned}
A_{1,d=5}=&\frac{\pi^4e^2(8n_F+n_S)}{120(p_1+p_2+p_3)^5}\Big[4p_1^4+20p_1^3(p_2+p_3)+4p_1^2\Big(7\big(p_2+p_3\big)^2+6p_2p_3\Big)\\
&+15p_1(p_2+p_3)\Big(\big(p_2+p_3\big)^2+p_2p_3\Big)+3(p_2+p_3)^2\Big(\big(p_2+p_3\big)^2+p_2p_3\Big)\Big]
\end{aligned}
\tag{13.30}
$$

$$
\begin{aligned}
A_{2,d=5}=&\frac{\pi^4e^2p_1^2(8n_F+n_S)}{120(p_1+p_2+p_3)^4}\\
&\times\Big[(p_1+p_2+p_3)^3+(p_1p_2+p_1p_3+p_2p_3)(p_1+p_2+p_3)+3p_1p_2p_3\Big]\\
&+\frac{\pi^4e^2(24n_F-n_S)}{48(p_1+p_2+p_3)^3}\Big[2p_1^4+6p_1^3(p_2+p_3)+2p_1^2\Big(5\big(p_2+p_3\big)^2+p_2p_3\Big)\\
&+9p_1\Big(p_2^3+2p_2^2p_3+2p_2p_3^2+p_3^3\Big)+3(p_2+p_3)^2\Big(\big(p_2+p_3\big)^2-p_2p_3\Big)\Big]
\end{aligned}
\tag{13.31}
$$

$$
\begin{aligned}
A_{3,d=5}=&\frac{\pi^4e^2(8n_F+n_S)}{240(p_1+p_2+p_3)^4}\Big[(-2p_1^5-8p_1^4(p_2+p_3)-8p_1^3p_2(2p_2+3p_3)\\
&+p_1^2\Big(-19p_2^3-40p_2^2p_3+24p_2p_3^2+15p_3^3\Big)\\
&-3(p_2-p_3)(p_2+p_3)\Big(p_2^2+3p_2p_3+p_3^2\Big)(4p_1+p_2+p_3)\Big]\\
&+\frac{\pi^4e^2(24n_F-n_S)}{48(p_1+p_2+p_3)^3}\Big[2p_1^4+6p_1^3(p_2+p_3)+2p_1^2\Big(5p_2^2+6p_2p_3+5p_3^2\Big)\\
&+9p_1\Big(p_2^3+2p_2^2p_3+2p_2p_3^2+p_3^3\Big)+3(p_2+p_3)^2\Big(p_2^2+p_2p_3+p_3^2\Big)\Big]
\end{aligned}
\tag{13.32}
$$

$$
\begin{aligned}
A_{4,d=5} = {} & \frac{\pi^4 e^2 (8n_F + n_S)}{480(p_1+p_2+p_3)^3}\Big[2p_1^6 + 6p_1^5(p_2+p_3) + 4p_1^4\left(2(p_2+p_3)^2 - p_2p_3\right) \\
& + p_1^2\left((p_2+p_3)^2 + p_2p_3\right)\left(3p_1(p_2+p_3) - 7(p_2+p_3)^2 + 32p_2p_3\right) \\
& - 3(p_2+p_3)\left((p_2+p_3)^2 - 4p_2p_3\right)\left((p_2+p_3)^2 + p_2p_3\right)(3p_1+p_2+p_3)\Big] \\
& - \frac{\pi^4 e^2 (24n_F - n_S)}{96(p_1+p_2+p_3)^2}\Big[2p_1^5 + 4p_1^4(p_2+p_3) + 4p_1^3\left(p_2^2 + p_2p_3 + p_3^2\right) \\
& - p_1^2(p_2+p_3)\left(p_2^2 - 5p_2p_3 + p_3^2\right) - 6p_1\left(p_2^4 + p_2^3p_3 + p_2p_3^3 + p_3^4\right) \\
& - 3(p_2+p_3)^3\left(p_2^2 - p_2p_3 + p_3^2\right)\Big]
\end{aligned}
\tag{13.33}
$$

in agreement with the solution given by BMS in the limit $d = 5$, with the identifications

$$
\alpha_1 = \left(\frac{\pi^4 e^2 (8n_F + n_S)}{240}\right), \qquad c_J = -\left(\frac{3\pi^{\frac{7}{2}} e^2 (24n_F - n_S)}{128}\right). \tag{13.34}
$$

This shows that, after fixing the normalization of the 2-point function, we are essentially left, both from the perturbative and the non perturbative side, with the same solution.

14. Comments

There are several obvious conclusions that we can draw from this comparative study of the TJJ that we are going to briefly summarize.

The A_i computed in perturbation theory satisfy the same anomalous conformal Ward identities as the nonperturbative ones, as shown by us in the previous sections for $d = 4$.

They both satisfy homogeneous (non anomalous) CWI's in general (d) dimensions.

In $d = 3$ and $d = 5$ the two solutions completely match, having consistently matched the normalization of the JJ 2-point function in the two separate cases.

In $d = 5$, where the corresponding field theory is nonrenormalizable, the perturbative computation still matches the nonperturbative one. This obviously occurs because the non-renormalizability of the theory does not play any role, being the matching between the two theories a purely one loop result (one loop saturation).

We conclude that, at least for the TJJ, free field theories in momentum space at one loop provide the same information derived from the non-perturbative solutions, and the two can be freely interchanged, being equivalent. The two analytical derivations are therefore expected to coincide for any dimension. Obviously, as already mentioned, this implies that there should be significant cancellations among the contributions of the 3K integrals or those given by us in 8.6 in such a way that they can be expressed in terms of the elementary master integrals B_0 and C_0.

The two expressions of A_1, which is affected only by a single integration constant α_1, in the two forms given in section A and in Eq. (13.1), show quite directly that $J_{4[000]}$ can be expressed in terms of the elementary master integrals present in the perturbative expansion. Direct checks for the other J integrals appearing in the expressions of A_2, A_3 ... and so on, are harder to perform, since each of these form factors depends on at least two J integrals, as evident from Eq. (13.1).

14.1. Coordinate space analysis

We should mention that this result *is not unexpected*, since it had been shown in ([20] section 3) that the solution for the TJJ (but also for the TOO and JJJ) in coordinate space

presented by Osborn and Petkou [1] could be *completely reproduced* by the simple one loop diagrams of scalar QED and QED using fermion and scalar sectors. These correlators had been shown to be "completely integrable", in the words of [20], which meant that it was possible for such expressions to proceed with a direct Fourier transform to momentum space without the need of introducing an extra regulator (ω) for the transform.

The fact that the nonperturbative solutions are mirrored by the perturbative ones, shows that result obtained at one loop in the perturbative description are automatically inherited by the nonperturbative one, being identical in the case of 3-point functions. The complex machinery of the general solutions, both in coordinate and in position space, can be completely reproduced by a simple one-loop analysis. Since conformal symmetry for 3-point functions is "saturated" at one loop, this explains why radiative corrections in such correlators which preserve the conformal symmetry are necessarily bound to be proportional to the one-loop result, as in the case of the AVV diagram, for instance.

We are then entitled to come to immediate conclusions concerning the analysis presented in [7] in regards to the emergence of anomaly poles in one-loop QED, which are not artifact of perturbation theory, but are naturally part of the nonperturbative solution and are *protected by the conformal symmetry*.

15. Summary and conclusions

We have presented a general discussion of the transition from position to momentum space in the analysis of tensor correlators, clarifying some aspects of the realization of the conformal generators in this space. We have illustrated how to handle the derivative of the δ functions of momentum conservation, giving a direct derivation of the correct forms of the equations. We have also shown how to proceed with the treatment of the dependent momentum, when acting with the conformal generator on each of the independent momenta.

Then we have moved toward a direct analysis of the TJJ vertex, detailing some of the involved intermediate steps, technically demanding, used in the BMS reconstruction of such correlator.

In a recent work [22] we have presented the physical motivations, derived from combined perturbative and non perturbative studies of the TJJ vertex, why anomaly poles in anomaly vertices should be considered the key signature of the conformal anomaly. The goal of this work has been to fill in the intermediate steps of our previous analysis. The presence of such massless degrees of freedom generated in the presence of anomalies shows that they are not an artifact of perturbation theory.

We have studied a simple instantiation of such vertex in massless QED, and confronted it with the general approach for solving the conformal constraints in momentum space for scalars [7,11] and tensor correlators [11,13]. Such solutions do not rely on any Lagrangian realization and are, therefore, very general. In this way it is possible to establish an important connection between perturbative and non-perturbative approaches, in the analysis of specific correlators, which brings to significant simplifications of the general result.

Even though massless QED is not a conformal theory, the violation of conformality is associated to the β-function of the running coupling constant and, exactly for this reason, specific correlation functions containing one or more insertions of stress–energy tensors play, in this context, an important role, due to the trace anomaly.

There has been considerable interest in the analysis of the breaking of the conformal symmetry in realistic non-conformal theories such as QED and QCD, and in their manifestations

through higher perturbative orders [26,27]. In this respect, the analysis of exact solutions, at least for 3-point functions, may shed light on the manifestation of the conformal breaking, which is obviously associated to a non-zero β function and, according to our analysis and to the analysis of [4,5,7], is accompanied by the appearance of an anomaly pole in the effective action.

We have shown that the one loop result in perturbation theory saturates the nonperturbative solutions. This implies that the main master integrals necessary to expand the nonperturbative result should turn out to be just B_0 and C_0, in agreement with former analysis in coordinate space. These analyses share the goal of establishing the form of the anomaly action and address its nonlocality, given their role in describing the breaking of conformal symmetry. It is not surprising, therefore, that such massless interactions have received attention in the investigation of the role of the chiral and of the conformal anomalies in topological insulators and in Weyl semi-metals, as recently emphasized in [8,9]. The study of such materials provides a direct application of the properties of the vertices that we have been interested in. Obviously, the parallel study of perturbative and non-perturbative methods stops at the level of 3-point functions, since higher point functions need to be bootstrapped. However, the connection found in our work clearly indicate that the possibility of studying a skeleton expansion of realistic field theories, with conformal vertices arrested to one-loop order, could provide a complementary way to investigate the bootstrap program for higher point functions is CFT's.

One obvious question, emerging from this analysis, is if this result can be generalized to more complex correlators, such as the TTT, where more integration constants are present and for which a coordinate space analysis of such correlator has not been related to the free field theory result. We plan to address this issue in related work.

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Appendix A. Form factors contributions

A.1. Form factors A_i in d dimensions

We present the expressions of the form factors A_i in the massless limit in D-dim. These relations will be functions of the three invariants s, s_1 and s_2 and we have defined $\sigma = s^2 - 2s(s_1+s_2) + (s_1-s_2)^2$ and $\gamma = s - s_1 - s_2$

$$
\begin{aligned}
A_{1,d}(s,s_1,s_2) = \frac{16\pi^2 e^2 s^2}{(d-2)(d-1)d\sigma^4}\Big\{ & 2\gamma\big[-2s_2(s+2s_1)+(s-s_1)^2+s_2^2\big]\big(s^2+s\,\gamma \\
& -(s_1-s_2)^2\big) - 2d^3 s_1 s_2 \gamma\big[(s_1-s_2)^2 - s^2\big] + 4d^2 s_1 s_2\big[s^2(s_1+s_2) - (\gamma+s)(s_1-s_2)^2 \\
& - 4s\,s_1\,s_2\big] + s_2 d\big(5s^2+12ss_1-5s_1^2\big)(s-s_1)^2 - 2s_2^2\big(5s^3+12s^2 s_1+17ss_1^2-2s_1^3\big) \\
& - 5s_2^4(s+s_1) - (s-s_1)^5 + s_2^5 + 2s_2^3(5s+s_1)(s+2s_1)\Big\} B_0(s) - \frac{4\pi^2 e^2 s_1}{(d-2)(d-1)d\,\sigma^4}
\end{aligned}
$$

$$\times \Big\{ 8\,d^3 s^2\, s_2 (s + s_1 - s_2)\gamma^2 + 2d\big[s^6 - 2s^5(s_1 + 3s_2) - s^4(s_1 - 3s_2)(5s_1 + 3s_2)$$
$$+ 4s^3 s_1\big(5s_1^2 + 2s_1 s_2 - 11 s_2^2\big) - s^2(s_1 - s_2)\big(25 s_1^3 + 37 s_1^2 s_2 + 59 s_1 s_2^2 - 9 s_2^3\big)$$
$$+ 2s(s_1 - s_2)^3\big(7s_1^2 + 12 s_1 s_2 - 3 s_2^2\big) - (s_1 - s_2)^5(3s_1 - s_2)\big] + d^2\big[s^6 - 2s^5(3s_1 + 5s_2)$$
$$+ (s_1 - s_2)^6 + s^4\big(15 s_1^2 + 50 s_1 s_2 + 31 s_2^2\big) - 4s^3\big(5s_1^3 + 17 s_1^2 s_2 - 9 s_1 s_2^2 + 11 s_2^3\big)$$
$$+ s^2(s_1 - s_2)\big(15 s_1^3 + 35 s_1^2 s_2 + 77 s_1 s_2^2 - 31 s_2^3\big) - 2s(s_1 - s_2)^4(3s_1 + 5s_2)\big] + 8s_1(\gamma$$
$$+ 2s_2)\big(2s^2 - s(s_1 + s_2) - (s_1 - s_2)^2\big)\big(\sigma - 2s s_2\big)\Big\} B_0(s_1) - \frac{4\pi^2 e^2 s_2}{(d-2)(d-1)d\sigma^4}$$
$$\times \Big\{ 8d^3 s^2 s_1 (s - s_1 + s_2)\gamma^2 + d^2\big[s^6 - 2s^5(5s_1 + 3s_2) + s^4\big(31 s_1^2 + 50 s_1 s_2 + 15 s_2^2\big)$$
$$- 4s^3\big(11 s_1^3 - 9 s_1^2 s_2 + 17 s_1 s_2^2 + 5 s_2^3\big) + s^2\big(31 s_1^4 - 108 s_1^3 s_2 + 42 s_1^2 s_2^2 + 20 s_1 s_2^3 + 15 s_2^4\big)$$
$$- 2s(s_1 - s_2)^4(5s_1 + 3s_2) + (s_1 - s_2)^6\big] + 2d\big[s^6 - 2s^5(3s_1 + s_2) + s^4\big(9 s_1^2 + 12 s_1 s_2$$
$$- 5 s_2^2\big) + 4s^3 s_2\big(-11 s_1^2 + 2 s_1 s_2 + 5 s_2^2\big) - s^2\big(9 s_1^4 - 68 s_1^3 s_2 + 22 s_1^2 s_2^2 + 12 s_1 s_2^3 + 25 s_2^4\big)$$
$$+ 2s(s_1 - s_2)^3\big(3 s_1^2 - 12 s_1 s_2 - 7 s_2^2\big) - (s_1 - 3s_2)(s_1 - s_2)^5\big] + 8s_2\big[2s^5 - 7s^4(s_1 + s_2)$$
$$+ s^3\big(-4 s_1^2 + 8 s_1 s_2 + 8 s_2^2\big) + s^2\big(8 s_1^3 - 6 s_1^2 s_2 - 2 s_2^3\big) + 2s(s_1 - s_2)^3(s_1 + s_2)$$
$$- (s_1 - s_2)^5\big]\Big\} B_0(s_2) + \frac{16\pi^2 e^2 s^2 s_1 s_2}{(d-2)d\sigma^4}\Big\{ d^2\big(s^2 - (s_1 - s_2)^2\big)\gamma^2 + 2d\big[s^3(s_1 + s_2)$$
$$+ s^2\big(-3 s_1^2 + 8 s_1 s_2 - 3 s_2^2\big) + s\big(3 s_1^3 - 7 s_1^2 s_2 - 7 s_1 s_2^2 + 3 s_2^3\big) - (s_1 - s_2)^2\big(s_1^2 + 4 s_1 s_2$$
$$+ s_2^2\big)\big] - 8 s_1 s_2\big(\sigma - 3\gamma s\big)\Big\} C_0(s, s_1, s_2) \tag{A.1}$$

$$A_{2,d}(s, s_1, s_2) = -\frac{4\pi^2 e^2 s^2}{(d-2)(d-1)d\sigma^3}\Big\{ -d\big[-2s_2(s + 3s_1) + (s - s_1)^2 + s_2^2\big]\sigma$$
$$+ 4d^2\sigma\, s_1 s_2 + d^3\big[-4s_2\big(s^2 + 4 s s_1 - 3 s_1^2\big)(s - s_1) - 4 s_2^3(s + 3 s_1) + 2 s_2^2(s + s_1)(3s$$
$$+ 11 s_1) + (s - s_1)^4 + s_2^4\big]\Big\} B_0(s) + \frac{2\pi^2 e^2 s_1}{(d-2)(d-1)d\sigma^3}\Big\{ -2d\sigma\big(s^3 - 12 s^2 s_2 - 3s(s_1$$
$$- s_2)(s_1 + 3s_2) + 2(s_1 - s_2)^3\big) + 2d^2\big[s^4(6s_1 - 8s_2) - s^5 + 2s^3\big(-7 s_1^2 + s_1 s_2 + 14 s_2^2\big)$$
$$+ 2s^2(s_1 - s_2)\big(8 s_1^2 + 9 s_1 s_2 + 15 s_2^2\big) - s(s_1 - s_2)^3(9s_1 + 13 s_2) + 2(s_1 - s_2)^5\big]$$
$$- 8s\, s_1(s - s_1 + s_2)(\sigma - 2s\, s_2) + d^3\sigma\big(s^3 - 3s^2 s_1 + 3 s^2 s_2 + 3 s s_1^2 + 2 s s_1 s_2 - 5 s s_2^2 - s_1^3$$
$$+ 3 s_1^2 s_2 - 3 s_1 s_2^2 + s_2^3\big)\Big\} B_0(s_1) + \frac{2\pi^2 e^2 s_2}{(d-2)(d-1)d\,\sigma^3}\Big\{ d^3\sigma\big[s^3 + 3s^2(s_1 - s_2) - s(s_1$$
$$- s_2)(5s_1 + 3s_2) + (s_1 - s_2)^3\big] - 2d^2\big[s^5 + s^4(8s_1 - 6s_2) - 2s^3\big(14 s_1^2 + s_1 s_2 - 7 s_2^2\big)$$
$$+ 2s^2(s_1 - s_2)\big(15 s_1^2 + 9 s_1 s_2 + 8 s_2^2\big) - s(s_1 - s_2)^3(13 s_1 + 9 s_2) + 2(s_1 - s_2)^5\big] - 2d\sigma$$
$$\times \big[s^3 - 12 s^2 s_1 + 3s(s_1 - s_2)(3s_1 + s_2) - 2(s_1 - s_2)^3\big] - 8s\, s_2(s + s_1 - s_2)(\sigma - 2s\, s_1)\Big\}$$
$$\times B_0(s_2) + \frac{8\pi^2 e^2 s^2}{(d-2)d\sigma^3} s_1 s_2\Big\{ -d^2\sigma\,\gamma + d\big(2s\,\gamma + \sigma\big)\gamma + 8 s s_1 s_2\Big\} C_0(s, s_1, s_2) \tag{A.2}$$

$$A_{3,d}(s,s_1,s_2) = -\frac{4\pi^2e^2s^2}{(d-2)(d-1)d\sigma^3}\Big\{4\gamma(s-s_1+s_2)\big(\sigma-2s_2s_1\big)-3d\sigma\big(-2s_2(s+3s_1)+(s-s_1)^2+s_2^2\big)+d^2\big[2s_2^2\big(3s^2+18ss_1+19s_1^2\big)-4s_2^3(s+7s_1)-4s_2(s+3s_1)\times(s-s_1)^2+(s-s_1)^4+s_2^4\big]+4d\sigma\,s_1s_2\Big\}B_0(s)+\frac{2\pi^2e^2s_1}{(d-2)(d-1)d\sigma^3}\Big\{d^3\sigma\big[s^3+3s^2\times(s_2-s_1)+s(s_1-s_2)(3s_1+5s_2)-(s_1-s_2)^3\big]+d^2\big[s^4(17s_1+7s_2)-3s^5-2s^3\times\big(19s_1^2+14s_1s_2-9s_2^2\big)+2s^2\big(21s_1^3+5s_1^2s_2+31s_1s_2^2-25s_2^3\big)-s(s_1-s_2)^3(23s_1+33s_2)+5(s_1-s_2)^5\big]+2d\sigma\big[2s^3-3s^2(3s_1+s_2)+2s(2s_1-s_2)(3s_1+5s_2)-(5s_1-3s_2)(s_1-s_2)^2\big]+8s_1\sigma(s-s_1+s_2)^2\Big\}B_0(s_1)+\frac{2\pi^2e^2s_2}{(d-2)(d-1)d\sigma^3}\Big\{d^3\sigma\big[s^3+3s^2(s_1-s_2)+s\big(-5s_1^2+2s_1s_2+3s_2^2\big)+(s_1-s_2)^3\big]-d^2\big[s^5+7s^4(s_1-s_2)-6s^3\big(5s_1^2+6s_1s_2-3s_2^2\big)+2s^2\big(19s_1^3+7s_1^2s_2+17s_1s_2^2-11s_2^3\big)-s(s_1-s_2)^3(19s_1+13s_2)+3(s_1-s_2)^5\big]+2d\sigma\big[s^2(3s_1+s_2)-2s\big(2s_1^2+5s_1s_2+s_2^2\big)+(s_1-s_2)^2(s_1+s_2)\big]-8s_2\times\big[-s^4+2s^3(2s_1+s_2)-2s(s_1-s_2)^2(2s_1+s_2)+(s_1-s_2)^4\big]\Big\}B_0(s_2)-\frac{8\pi^2e^2s^2s_1s_2}{(d-2)d\sigma^3}\times\Big\{d^2\big[\gamma\sigma+2(s_1+s_2)(s_1-s_2)^2\big]-d\big[s^3-s^2(3s_1+7s_2)+s\big(3s_1^2+2s_1s_2+11s_2^2\big)-s_1^3+5s_1^2s_2+s_1s_2^2-5s_2^3\big]+8s_1s_2(s-s_1+s_2)\Big\}C_0(s,s_1,s_2) \tag{A.3}$$

$$A_{4,d}(s,s_1,s_2) = \frac{2\pi^2e^2s^2\gamma}{(d-2)(d-1)d\sigma^2}\Big\{d^2\sigma+4(\sigma-2s_1s_2)-d\big[3\sigma-8s_1s_2\big]\Big\}B_0(s)-\frac{\pi^2e^2s_1}{(d-2)(d-1)d\sigma^2}\Big\{d^3\sigma^2+8s_1(s-s_1+s_2)(\sigma-2s\,s_2)-d^2\sigma\big(3s^2-4s(2s_1+3s_2)+5(s_1-s_2)^2\big)+2d\big[2s^4-11s^3(s_1+s_2)+s^2\big(21s_1^2+24s_1s_2+19s_2^2\big)-s(s_1-s_2)\times\big(17s_1^2+12s_1s_2-13s_2^2\big)+(s_1-s_2)^3(5s_1-3s_2)\big]\Big\}B_0(s_1,0,0)-\frac{\pi^2e^2s_2}{(d-2)(d-1)d\sigma^2}\times\Big\{d^3\sigma^2+d^2\sigma\big[-3s^2+4s(3s_1+2s_2)-5(s_1-s_2)^2\big]+2d\big[2s^4-11s^3(s_1+s_2)+s^2\big(19s_1^2+24s_1s_2+21s_2^2\big)+s\big(-13s_1^3+25s_1^2s_2+5s_1s_2^2-17s_2^3\big)+(s_1-s_2)^3(3s_1-5s_2)\big]+8s_2\big(s^3-3s^2(s_1+s_2)-3s\big(s_1^2-s_2^2\big)+(s_1-s_2)^3\big)\Big\}B_0(s_2,0,0)+\frac{4\pi^2e^2s^2s_1s_2}{(d-2)d\sigma^2}(d\sigma+8s_1s_2)C_0(s,s_1,s_2) \tag{A.4}$$

A.2. Form factors F_i in d-dim

We present here the expressions of the invariant amplitudes in the massless limit in d-dimensions, previously given in $d=4$. These expressions will be functions of the three invariants $s=p_1^2$, $s_2=p_1^2$ and $s_2=p_3^2$ and are explicitly given as

$$
\begin{aligned}
F_{1,d}(s,s_1,s_2,0) &= \frac{2\pi^2 e^2(d-4)}{3(d-2)(d-1)d\,\sigma^3}\Big\{2s_2^3\big[(8-9d)s^2+2s\,s_1 d(3d-8)+4s\,s_1+(2d\\
&-3)d\,s_1^2\big]-s_2(s-s_1)^2\big[(13d-16)s^2-2(d-4)(2d-1)s\,s_1+d(4d-7)s_1^2\big]\\
&+2s_2^2\big[(11d-12)s^3+3(5-2d)ds^2\,s_1+3(3-4d)d\,s\,s_1^2+12s\,s_1^2+d(2d-3)s_1^3\big]\\
&+s_2^4\big[(7d-4)s+(7-4d)ds_1\big]+(s-s_1)^4\big[(3d-4)s-d\,s_1\big]-d\,s_2^5\Big\}\,B_0(s)\\
&+\frac{2\pi^2 e^2(d-4)s_1}{3(d-2)(d-1)d\,s\,\sigma^3}\Big\{2s_2^3\big[-d^2(s^2+10s\,s_1+5s_1^2)+d(s+s_1)(s+9s_1)\\
&+4s_1(2s+s_1)\big]-2ds_2^2\big[(d-1)s^3+(13-7d)s^2\,s_1+(9-5d)s\,s_1^2+(11-5d)s_1^3\big]\\
&+s_2^4\big[3(d-1)d\,s+d\,s_1(5d-7)-4s_1\big]+(d-1)s_2(s-s_1)^2\big[d(s-s_1)(3s+5s_1)\\
&+8s_1(2s+s_1)\big]-(s-s_1)^4\big[(d-1)d\,s-s_1(d-3)-4s_1\big]-(d-1)d\,s_2^5\Big\}\,B_0(s_1)\\
&+\frac{2\pi^2 e^2(d-4)s_2}{3(d-2)(d-1)d\,s\,\sigma^3}\Big\{d^2\big[2s_2^3(5s^2+6s\,s_1+5s_1^2)-2s_2^2(5s^3+3s^2\,s_1-5s\,s_1^2+5s_1^3)\\
&+s_2(s-s_1)(5s^3+s^2\,s_1+15s\,s_1^2-5s_1^3)-5s_2^4(s+s_1)-(s-s_1)^4(s+s_1)+s_2^5\big]\\
&+d\sigma\big[7s_2^2(s+s_1)-5s_2(s-s_1)^2+(s-s_1)^2(s+s_1)-3s_2^3\big]+4s_2\gamma(s+s_1-s_2)\\
&\times\big[\sigma-2s\,s_1\big]\Big\}\,B_0(s_2)-\frac{4\pi^2 e^2(d-4)s_1\,s_2\,\gamma}{3(d-2)d\,\sigma^3}\Big\{d\big[-s^3+3s^2(s_1+s_2)\\
&-(s_1-s_2)^2(\gamma+2s)\big]+8s\,s_1\,s_2\Big\}\,C_0(s,s_1,s_2) \qquad \text{(A.5)}\\
F_{2,d}(s,s_1,s_2) &= \frac{4\pi^2 e^2(d-4)}{3(d-2)(d-1)d\,\sigma^3}\Big\{2s^2\big[2d^2\,s_1\,s_2-3(d-2)s_1^2-3(d-2)s_2^2\big]\\
&-(d-4)s^4+4(d-3)s^3(s_1+s_2)+4(d-1)s(s_1+s_2)\big[s_1^2-4s_1\,s_2+s_2^2\big]-d(s_1-s_2)^2\\
&\times\big[2(2d-3)s_1\,s_2+s_1^2+s_2^2\big]\Big\}\,B_0(s)-\frac{4\pi^2 e^2(d-4)s_1}{3(d-2)(d-1)d\,s\,\sigma^3}\Big\{4d^2\,s\,s_2\big[(s-s_2)^2-s_1^2\big]\\
&+d\,\sigma(\sigma-4s\,s_2)+4s_1(s-s_1+s_2)(\sigma-2s_2\,s)\Big\}\,B_0(s_1,0,0)-\frac{4\pi^2 e^2(d-4)s_2}{3(d-2)(d-1)d\,s\,\sigma^3}\\
&\times\Big\{4d^2 s\,s_1\big[(s-s_1)^2-s_2^2\big]+d\sigma(\sigma-4s\,s_1)+4s_2(s+s_1-s_2)(\sigma-2s\,s_1)\Big\}B_0(s_2,0,0)\\
&+\frac{8\pi^2(d-4)e^2\,s_1\,s_2}{3(d-2)d\sigma^3}\Big\{8s\,s_1\,s_2+d\,\gamma\,\big[s^2-(s_1-s_2)^2\big]\Big\}\,C_0(s,s_1,s_2) \qquad \text{(A.6)}\\
F_{3,d}(s,s_1,s_2) &= F_{5,d}(s,s_2,s_1)=\frac{\pi^2 e^2}{3(d-2)(d-1)d\sigma^4}\Big\{s_2^5\big[-3(d(5d-21)+8)s^2\\
&+4(d(11-d(4d+9))+8)s\,s_1+3(d-1)d(4d-7)s_1^2\big]-s_2(s-s_1)^4\big[\big((d-1)d+24\big)\\
&\times s^2+8\big(d((d-2)d+8)-4\big)s\,s_1+(d-1)d(4d-9)s_1^2\big]+s_2^4\big[2\big(5d(2d-11)+84\big)s^3\\
&+\big(d(d(32d-35)-121)+64\big)s^2\,s_1+2\big(d(3d(4d-17)+74)+4\big)s\,s_1^2-(d-1)d\\
&\times(8d-13)s_1^3\big]+s_2^2(s-s_1)^2\big[6\big((d-6)d+28\big)s^3+\big(d(d(4d-93)+37)+64\big)s^2\,s_1\\
&+2\big((149-84d)d+4\big)s\,s_1^2+3(d-1)d(4d-7)s_1^3\big]+8\big(d(d(d+33)-50)-8\big)s\,s_1^3
\end{aligned}
$$

$$
\begin{aligned}
&- (d-1)d(8d-13)s_1^4\big] + s_2^3\big[\big(5(19-3d)d-272\big)s^4 - 8d\big((d-18)d-7\big)s^3 s_1 \\
&+ 2\big(d(67-d(28d+111))+168\big)s^2 s_1^2 + s_2^6\big[2\big(d(3d-8)-4\big)s - (d-1)d(4d-9)s_1\big] \\
&+ (s-s_1)^6\big[2(d-4)s-(d-1)ds_1\big] - (d-1)ds_2^7\Big\} B_0(s,0,0) + \frac{\pi^2 e^2 s_1}{6(d-2)(d-1)ds\sigma^4} \\
&\times \Big\{ d^3\big[s_2^5\big(21s^2+46ss_1+21s_1^2\big) + s_2\big(7s^2+10ss_1+7s_1^2\big)(s-s_1)^4 + s_2^4\big(19s^3 \\
&- 105s^2 s_1 - 39ss_1^2 - 35s_1^3\big) - s_2^2\big(3s^3-17s^2 s_1+25ss_1^2+21s_1^3\big)(s-s_1)^2 + (s-s_1)^7 \\
&+ s_2^3\big(-29s^4+68s^3 s_1+50s^2 s_1^2+4ss_1^3+35s_1^4\big) - s_2^6(17s+7s_1) + s_2^7\big] \\
&+ d^2\big[-3s_2^5\big(47s^2+66ss_1+39s_1^2\big) + s_2(s-s_1)^4\big(25s^2-34ss_1-47s_1^2\big) + s_2^4\big(169s^3 \\
&- 85s^2 s_1 + 287ss_1^2 + 205s_1^3\big) - 3s_2^2(s-s_1)^2\big(7s^3-107s^2 s_1-47ss_1^2-45s_1^3\big) - s_2^3\big(71s^4 \\
&+ 20s^3 s_1 - 38s^2 s_1^2 + 116ss_1^3 + 215s_1^4\big) + s_2^6(49s+37s_1) - (5s-7s_1)(s-s_1)^6 - 5s_2^7\big] \\
&+ 2d\big[2s_2^5\big(51s^2+32ss_1+29s_1^2\big) - 2s_2(s-s_1)^4\big(11s^2-24ss_1-23s_1^2\big) - s_2^4\big(169s^3 \\
&- 2s^2 s_1 + 65ss_1^2 + 120s_1^3\big) - s_2^2(s-s_1)^2\big(3s^3+82s^2 s_1+123ss_1^2+112s_1^3\big) + 2s_2^3\big(55s^4 \\
&- 36s^3 s_1 - 34s^2 s_1^2 + 4ss_1^3 + 75s_1^4\big) - s_2^6(25s+16s_1) + (s-s_1)^6(5s-8s_1) + 2s_2^7\big] \\
&+ 16s_1\big(s^2+s(7s_2-2s_1)+(s_1-s_2)^2\big)\big(\sigma-2s\,s_2\big)\gamma(s-s_1+s_2)\Big\} B_0(s_1,0,0) \\
&+ \frac{\pi^2 e^2 s_2}{6(d-2)(d-1)ds\sigma^4}\Big\{ d^3\big[s^7 + 7s^6(s_1-s_2) - 3s^5(s_1-s_2)(17s_1+7s_2) + s^4\big(115s_1^3 \\
&- 73s_1^2 s_2 - 39s_1 s_2^2 - 35s_2^3\big) + s^3\big(-125s_1^4 + 116s_1^3 s_2 + 82s_1^2 s_2^2 - 44s_1 s_2^3 + 35s_2^4\big) \\
&+ s^2(s_1-s_2)\big(69s_1^4 - 36s_1^3 s_2 - 82s_1^2 s_2^2 - 36s_1 s_2^3 + 21s_2^4\big) - s(s_1-s_2)^3\big(17s_1^3+5s_1^2 s_2 \\
&+ 3s_1 s_2^2 + 7s_2^3\big) + (s_1-s_2)^7\big] + d^2\big[s^7 + s^6(s_2-35s_1) + s^5\big(165s_1^2+166s_1 s_2-27s_2^2\big) \\
&- s^4\big(335s_1^3+381s_1^2 s_2+169s_1 s_2^2-85s_2^3\big) + s^3\big(355s_1^4+100s_1^3 s_2+498s_1^2 s_2^2+4s_1 s_2^3 \\
&- 125s_2^4\big) - s^2(s_1-s_2)\big(201s_1^4 - 110s_1^3 s_2 + 452s_1^2 s_2^2 + 62s_1 s_2^3 + 99s_2^4\big) + s(s_1-s_2)^3 \\
&\times \big(55s_1^3 - 69s_1^2 s_2 + 5s_1 s_2^2 + 41s_2^3\big) - (5s_1-7s_2)(s_1-s_2)^6\big] + 2d\big[-s^7 + 2s^6(7s_1 \\
&+ 11s_2) - s^5\big(57s_1^2+176s_1 s_2+87s_2^2\big) + 2s^4\big(55s_1^3+190s_1^2 s_2+221s_1 s_2^2+70s_2^3\big) \\
&- s^3\big(115s_1^4+288s_1^3 s_2+502s_1^2 s_2^2+312s_1 s_2^3+95s_2^4\big) + 2s^2(s_1-s_2)\big(33s_1^4+40s_1^3 s_2 \\
&+ 162s_1^2 s_2^2 + 8s_1 s_2^3 - 3s_2^4\big) - s(s_1-s_2)^3\big(19s_1^3 - 7s_1^2 s_2 + 77s_1 s_2^2 + 23s_2^3\big) + 2(s_1-s_2)^5 \\
&\times \big(s_1^2 - 3s_1 s_2 + 4s_2^2\big)\big] + 16s_2\big(s^2+s(7s_2-2s_1)+(s_1-s_2)^2\big)\big(\sigma-2s\,s_1\big)\gamma\,(s+s_1 \\
&- s_2)\Big\} B_0(s_2,0,0) + \frac{2\pi^2 e^2 s_1 s_2}{3(d-2)d\sigma^4}\Big\{\big[s_2^4\big(-12(d-1)ds^2 + ((d+55)d+16)ss_1 \\
&- (d-1)ds_1^2\big) + s_2(s-s_1)^3\big(3(d-7)ds^2 - ((d+37)d+16)ss_1 + 2(d-1)ds_1^2\big) \\
&+ 2s_2^3\big((5d-23)ds^3 + 6((d+3)d+8)s^2 s_1 - (d+8)(5d+1)ss_1^2 + 2(d-1)ds_1^3\big) \\
&- s_2^2(s-s_1)\big(3(d-17)ds^3 + ((13d+59)d+96)s^2 s_1 + ((d-9)d-16)ss_1^2 \\
&- (d-1)ds_1^3\big) + ds_2^5(3(d+1)s - 2(d-1)s_1) - (d-1)d(s-s_1)^5(2s+s_1) \\
&+ (d-1)ds_2^6\Big\} C_0(s,s_1,s_2,0,0,0) \qquad \text{(A.7)}
\end{aligned}
$$

$$F_{4,d}(s,s_1,s_2)=F_{6,d}(s,s_2,s_1)=\frac{\pi^2e^2}{3(d-2)(d-1)d\sigma^4s_1}\Big\{2s^4\big[-\big((6d^2+45d-31)d+136\big)s_1s_2^2+5(d(3(2d-7)d+35)-8)s_1^3+4(d((d-9)d+9)-34)s_1^2s_2-30\times((d-3)d+4)s_2^3\big]+2s^2\big[2\big(-39d^2+d+44\big)s_1^2s_2^3+((3(5d-18)d+77)d+28)s_1^5-((3(8d-69)d+205)d+92)s_1^4s_2+2(((3d-64)d+19)d+60)s_1^3s_2^2+((3d(d+10)-67)d+28)s_1s_2^4-3(3(d-3)d+4)s_2^5\big]+3((d-3)d+8)s^7+2s^6((d((3d-16)d+35)-52)s_1-3(3(d-3)d+20)s_2)+s^5\big[((3(39-10d)d-217)d+160)s_1^2+2((-3(d-17)d-86)d+152)s_1s_2+15(3(d-3)d+16)s_2^2\big]+s^3\big[-5(((12d-41)d+63)d+8)s_1^4+4(((5d-61)d+74)d+48)s_1^3s_2+2(d(11(2d+5)d+61)-72)s_1^2s_2^2+4(((3d-5)d+36)d+8)s_1s_2^3+15(3(d-3)d+8)s_2^4\big]-s(s_1-s_2)^2\big[((3(2d-9)d+35)d+16)s_1^4-2(((11d-78)d+55)d+24)s_1^3s_2+2(d(d(13d+27)-34)-24)s_1^2s_2^2+2((3d(d+2)-5)d+8)s_1s_2^3-3(d-3)ds_2^4\big]-2(d-1)ds_1(s_1-s_2)^4\times\big[2(2d-3)s_1s_2+s_1^2+s_2^2\big]\Big\}B_0(s,0,0)+\frac{\pi^2e^2}{3(d-2)(d-1)ds\sigma^4}\Big\{2d^3s\big[s_2^4\big(9s^2-5ss_1+8s_1^2\big)-2ss_2^3\big(3s^2-12ss_1+s_1^2\big)+s_2\big(9s^2-ss_1-4s_1^2\big)(s-s_1)^3-2s_2^2\big(3s^4+s^3s_1+s^2s_1^2-9ss_1^3+4s_1^4\big)-s_2^5(3s+4s_1)-3s(s-s_1)^5\big]+d^2\big[12s^7-s^6(59s_1+48s_2)+2s^5\big(57s_1^2+23s_1s_2+36s_2^2\big)+s^4\big(-105s_1^3+162s_1^2s_2+23s_1s_2^2-48s_2^3\big)+4s^3\big(10s_1^4-69s_1^3s_2-45s_1^2s_2^2+13s_1s_2^3+3s_2^4\big)+s^2s_1(s_1-s_2)\big(3s_1^3+119s_1^2s_2+169s_1s_2^2+61s_2^3\big)-2ss_1(s_1-s_2)^3\big(3s_1^2+6s_1s_2-s_2^2\big)+s_1(s_1-s_2)^6\big]+d\big[-9s^7+16s^6(2s_1+3s_2)-s^5\big(17s_1^2+62s_1s_2+105s_2^2\big)-4s^4(s_1-s_2)\big(20s_1^2+53s_1s_2+30s_2^2\big)+s^3\big(165s_1^4+244s_1^3s_2+14s_1^2s_2^2-124s_1s_2^3-75s_2^4\big)-8s^2(s_1-s_2)\big(17s_1^4+22s_1^3s_2+8s_1^2s_2^2+10s_1s_2^3+3s_2^4\big)+s(s_1-s_2)^3\big(53s_1^3+57s_1^2s_2-s_1s_2^2+3s_2^3\big)-4s_1(s_1-s_2)^5(2s_1-s_2)\big]-8s_1(s-s_1+s_2)\big(s^2-2s(s_1+2s_2)+(s_1-s_2)^2\big)\big[3s^3-2s^2(2s_1+3s_2)+s\big(-s_1^2+8s_1s_2+3s_2^2\big)+2s_1(s_1-s_2)^2\big]\Big\}B_0(s_1,0,0)+\frac{\pi^2e^2}{6(d-2)(d-1)ds\sigma^4s_1}\Big\{d^3s\big[s_2^5\big(63s^2+90ss_1+79s_1^2\big)+21s_2^3(s-s_1)^2\big(5s^2+6ss_1+5s_1^2\big)+s_2(s-s_1)^4\big(21s^2-18ss_1+5s_1^2\big)-s_2^4\big(105s^3+87s^2s_1+23ss_1^2+137s_1^3\big)-s_2^2(s-s_1)^2\big(63s^3-63s^2s_1+ss_1^2+31s_1^3\big)-21s_2^6(s+s_1)-3(s-s_1)^6(s+s_1)+3s_2^7\big]+d^2\big[-3s_2^5\big(93s^3+100s^2s_1+89ss_1^2-10s_1^3\big)-s_2(s-s_1)^4\big(69s^3+28s^2s_1+65ss_1^2-2s_1^3\big)+s_2^4\big(435s^4+245s^3s_1+517s^2s_1^2+475ss_1^3-40s_1^4\big)+3s_2^2(s-s_1)^2\big(75s^4+s^3s_1-119s^2s_1^2+79ss_1^3-4s_1^4\big)+s_2^3\big(-405s^5+210s^4s_1+186s^3s_1^2+328s^2s_1^3-477ss_1^4+30s_1^5\big)+s_2^7(2s_1-15s)+3s_2^6(s+s_1)(33s-4s_1)+9s(s-s_1)^6(s+s_1)\big]+2d\big[s_2^6\big(-3s^2+17ss_1+44s_1^2\big)-s_2^2(s-s_1)^2\big(5s^2+23ss_1-4s_1^2\big)\big(9s^2-8ss_1+7s_1^2\big)+2s_2^5\big(9s^3+52s^2s_1+39ss_1^2-50s_1^3\big)+2s_2(s-s_1)^4\big(9s^3+16s^2s_1+15ss_1^2-2s_1^3\big)-s_2^4\big(45s^4+339s^3s_1+259s^2s_1^2+373ss_1^3-120s_1^4\big)+4s_2^3\big(15s^5+82s^4s_1-80s^3s_1^2-86s^2s_1^3+121ss_1^4-20s_1^5\big)-3s(s-s_1)^6(s+s_1)-8s_1s_2^7\big]-16s_2^2(s+s_1-s_2)\big(\sigma-2s\,s_1\big)\big[3s^3-2s^2(2s_1+3s_2)$$

$$+ s\big(-s_1^2+8s_1s_2+3s_2^2\big)+2s_1(s_1-s_2)^2\big]\Big\}B_0(s_2,0,0)+\frac{\pi^2e^2}{3(d-2)d\sigma^4}$$
$$\times\Big\{6ds_2^5\big[-(d-1)s^2+(d+11)ss_1-2(d-1)s_1^2\big]-2ds_2(s-s_1)^4\big(3(d-5)s^2$$
$$+(d-25)ss_1-2(d-1)s_1^2\big)-s_2^4\big[3(d+10)ds^3+2(5d(d+2)-48)s^2s_1+(d(11d$$
$$+38)-64)ss_1^2-8(d-1)ds_1^3\big]-s_2^2(s-s_1)^2\big[3(d+20)ds^3-6((3d+2)d+16)s^2s_1$$
$$-((13d+4)d+64)ss_1^2+12(d-1)ds_1^3\big]+4s_2^3\big[3(d+5)ds^4-(7(d+5)d+48)s^3s_1$$
$$+((7d+55)d+64)s^2s_1^2-((5d+17)d+32)ss_1^3+2(d-1)ds_1^4\big]+ds_2^6(3ds$$
$$+4(d-1)s_1)+3(d-2)ds(s-s_1)^6\Big\}C_0(s,s_1,s_2) \tag{A.8}$$

$$F_{7,d}(s,s_1,s_2)=-\frac{2\pi^2e^2}{3(d-2)(d-1)d\sigma^4\gamma}\Big\{4(d-4)s^8-2(d(d+10)-44)s^7(s_1+s_2)$$
$$+s^6\Big((d(11d+37)-192)s_1^2+2(d((4d-5)d+5)-136)s_1s_2+(d(11d+37)$$
$$-192)s_2^2\Big)-2s^5(s_1+s_2)\Big((d(12d+13)-100)s_1^2+2(d(4d(3d-8)-23)-8)s_1s_2$$
$$+(d(12d+13)-100)s_2^2\Big)+s^4\big[5(d(5d-1)-16)s_1^4+4((3d-7)(9d-1)d+8)s_1^3s_2$$
$$+2(d((20d-177)d+57)-128)s_1^2s_2^2+4((3d-7)(9d-1)d+8)s_1s_2^3+5(d(5d-1)$$
$$-16)s_2^4\big]-2s^3(s_1+s_2)\big[((5d-8)d+12)s_1^4+4(((14d-37)d+36)d+8)s_1^3s_2$$
$$-2(((44d-47)d+20)d+172)s_1^2s_2^2+4(((14d-37)d+36)d+8)s_1s_2^3+((5d-8)d$$
$$+12)s_2^4\big]-s^2\big[(d(3d+5)-32)s_1^6+2(((61-24d)d-129)d+8)s_1^5s_2+(((48d$$
$$-355)d+363)d+448)s_1^4s_2^2+4(d(147d+73)-376)s_1^3s_2^3+(((48d-355)d+363)d$$
$$+448)s_1^2s_2^4+2(((61-24d)d-129)d+8)s_1s_2^5+(d(3d+5)-32)s_2^6\big]+2s(s_1-s_2)^2$$
$$\times(s_1+s_2)\big[(d(2d-1)-4)s_1^4+8(1-4d)s_1^3s_2+2((69-2d(2d+25))d+12)s_1^2s_2^2$$
$$+8(1-4d)s_1s_2^3+(d(2d-1)-4)s_2^4\big]-(d-1)d(s_1-s_2)^4(s_1+s_2)^2\big(2(2d-3)s_1s_2$$
$$+s_1^2+s_2^2\big)\Big\}B_0(s,0,0)-\frac{e^2\pi^2s_1}{3(d-2)(d-1)ds\gamma^2\sigma^4}\Big\{\big[-s_2^9+(28s+5s_1)s_2^8-2\big(79s^2$$
$$+45s_1s+4s_1^2\big)s_2^7+2s\big(209s^2+214s_1s+101s_1^2\big)s_2^6-2\big(316s^4+427s_1s^3+195s_1^2s^2$$
$$+189s_1^3s-7s_1^4\big)s_2^5+2\big(293s^5+373s_1s^4-46s_1^2s^3+130s_1^3s^2+217s_1^4s-7s_1^5\big)s_2^4-2s$$
$$\times(s-s_1)\big(169s^4+236s_1s^3-2s_1^2s^2-68s_1^3s-143s_1^4\big)s_2^3+2(s-s_1)^3\big(59s^4+75s_1s^3$$
$$+33s_1^2s^2-67s_1^3s-4s_1^4\big)s_2^2-(s-s_1)^5\big(23s^3-3s_1s^2-39s_1^2s-5s_1^3\big)s_2+(s-s_1)^8$$
$$\times(2s+s_1)\big]d^3+\big[5s_2^9-(98s+27s_1)s_2^8+2\big(278s^2+229s_1s+24s_1^2\big)s_2^7-2\big(784s^3$$
$$+875s_1s^2+533s_1^2s+4s_1^3\big)s_2^6+2\big(1295s^4+1325s_1s^3+1790s_1^2s^2+861s_1^3s-39s_1^4\big)s_2^5$$
$$-2\big(1330s^5+618s_1s^4+1515s_1^2s^3+743s_1^3s^2+1023s_1^4s-45s_1^5\big)s_2^4+2(s-s_1)\big(854s^5$$
$$+317s_1s^4+185s_1^2s^3+19s_1^3s^2-803s_1^4s+4s_1^5\big)s_2^3-2(s-s_1)^3\big(328s^4+221s_1s^3$$
$$-358s_1^2s^2-407s_1^3s-24s_1^4\big)s_2^2+(s-s_1)^5\big(133s^3+39s_1s^2-211s_1^2s-33s_1^3\big)s_2$$
$$-(5s-7s_1)(s-s_1)^7(2s+s_1)\big]d^2+2\big[-2s_2^9+(41s+12s_1)s_2^8-\big(247s^2+198s_1s$$

$$+28s_1^2)s_2^7+\big(743s^3+755s_1s^2+706s_1^2s+20s_1^3\big)s_2^6-\big(1315s^4+1048s_1s^3+2423s_1^2s^2$$
$$+1470s_1^3s-32s_1^4\big)s_2^5+\big(1457s^5+101s_1s^4+2431s_1^2s^3+2011s_1^3s^2+1748s_1^4s-68s_1^5\big)$$
$$\times s_2^4-(s-s_1)\big(1021s^5-213s_1s^4+517s_1^2s^3-819s_1^3s^2-1302s_1^4s+28s_1^5\big)s_2^3+(s$$
$$-s_1)^3\big(437s^4-20s_1s^3-595s_1^2s^2-706s_1^3s-28s_1^4\big)s_2^2-(s-s_1)^5\big(103s^3-41s_1s^2$$
$$-224s_1^2s-30s_1^3\big)s_2+(s-s_1)^7(2s+s_1)(5s-8s_1)\big]d+16s_1\big(s^2-2(s_1+2s_2)s$$
$$+(s_1-s_2)^2\big)(s-s_1+s_2)\big[2s^5-7(s_1+s_2)s^4+2\big(4s_1^2+7s_2s_1+4s_2^2\big)s^3-2(s_1+s_2)^3$$
$$\times s^2-2\big(s_1^4+s_2s_1^3-24s_2^2s_1^2+s_2^3s_1+s_2^4\big)s+(s_1-s_2)^2(s_1+s_2)^3\big]\Big\}B_0(s_1)$$
$$-\frac{e^2\pi^2s_2}{3(d-2)(d-1)ds\gamma^2\sigma^4}\Big\{\big[s_2^9-(6s+5s_1)s_2^8+2\big(6s^2-7s_1s+4s_1^2\big)s_2^7+2ss_1(71s$$
$$+55s_1)s_2^6-2\big(21s^4+151s_1s^3+222s_1^2s^2+143s_1^3s+7s_1^4\big)s_2^5+2\big(42s^5+110s_1s^4$$
$$+221s_1^2s^3+75s_1^3s^2+217s_1^4s+7s_1^5\big)s_2^4+2s\big(-42s^5+35s_1s^4+66s_1^3s^2+130s_1^4s$$
$$-189s_1^5\big)s_2^3+2(s-s_1)^2\big(24s^5-55s_1s^4-149s_1^2s^3-5s_1^3s^2+93s_1^4s-4s_1^5\big)s_2^2$$
$$-(s-s_1)^4\big(15s^4-58s_1s^3-118s_1^2s^2+70s_1^3s-5s_1^4\big)s_2+(s-s_1)^6\big(2s^3-11s_1s^2$$
$$+22s_1^2s-s_1^3\big)\big]d^3+\big[-7s_2^9+(40s+33s_1)s_2^8-2\big(37s^2-23s_1s+24s_1^2\big)s_2^7-2\big(7s^3$$
$$+382s_1s^2+335s_1^2s+4s_1^3\big)s_2^6+2\big(140s^4+921s_1s^3+791s_1^2s^2+807s_1^3s+45s_1^4\big)s_2^5$$
$$-2\big(259s^5+835s_1s^4-98s_1^2s^3+822s_1^3s^2+1023s_1^4s+39s_1^5\big)s_2^4+2\big(245s^6+41s_1s^5$$
$$-1002s_1^2s^4-166s_1^3s^3-743s_1^4s^2+861s_1^5s-4s_1^6\big)s_2^3-2(s-s_1)^2\big(133s^5-196s_1s^4$$
$$-562s_1^2s^3-796s_1^3s^2+485s_1^4s-24s_1^5\big)s_2^2+(s-s_1)^4\big(79s^4-310s_1s^3-188s_1^2s^2$$
$$+350s_1^3s-27s_1^4\big)s_2-(s-s_1)^7\big(10s^2-63s_1s+5s_1^2\big)\big]d^2+2\big[8s_2^9-15(3s+2s_1)s_2^8$$
$$+\big(81s^2-74s_1s+28s_1^2\big)s_2^7+\big(21s^3+779s_1s^2+622s_1^2s+28s_1^3\big)s_2^6-\big(315s^4$$
$$+1632s_1s^3+1439s_1^2s^2+1330s_1^3s+68s_1^4\big)s_2^5+\big(567s^5+1165s_1s^4+325s_1^2s^3$$
$$+483s_1^3s^2+1748s_1^4s+32s_1^5\big)s_2^4+\big(-525s^6+350s_1s^5+582s_1^2s^4+1336s_1^3s^3$$
$$+2011s_1^4s^2-1470s_1^5s+20s_1^6\big)s_2^3+(s-s_1)^2\big(279s^5-453s_1s^4-409s_1^2s^3-1095s_1^3s^2$$
$$+650s_1^4s-28s_1^5\big)s_2^2-(s-s_1)^4\big(81s^4-232s_1s^3-83s_1^2s^2+150s_1^3s-12s_1^4\big)s_2$$
$$+(s-s_1)^6\big(10s^3-43s_1s^2+29s_1^2s-2s_1^3\big)\big]d+16\big(\sigma-2s\,s_1\big)(s+s_1-s_2)s_2\big[2s^5$$
$$-7(s_1+s_2)s^4+2\big(4s_1^2+7s_2s_1+4s_2^2\big)s^3-2(s_1+s_2)^3s^2-2\big(s_1^4+s_2s_1^3-24s_2^2s_1^2$$
$$+s_2^3s_1+s_2^4\big)s+(s_1-s_2)^2(s_1+s_2)^3\big]\Big\}B_0(s_2,0,0)+\frac{4\pi^2e^2s_1s_2}{3(d-2)d\sigma^4\gamma^2}\Big\{s_2^6\big[d(17-11d)s^2$$
$$+2((3d+19)d+8)ss_1-4(d-1)ds_1^2\big]+2ss_2^5\big(2(13d-22)ds^2-((11d+69)d$$
$$+16)ss_1+(5d+1)(d+8)s_1^2\big)+2ss_2(s-s_1)^4\big(2(4d-13)ds^2+((d+7)d+16)ss_1$$
$$+((3d+19)d+8)s_1^2\big)-d(s-s_1)^6\big[2(d-4)s^2-4(d-1)ss_1-(d-1)s_1^2\big]+s_2^4\big[5d(37$$
$$-19d)s^4+4(d(11d+30)-8)s^3s_1+(d(11d+63)-32)s^2s_1^2-2((7d+61)d$$
$$+16)ss_1^3+6(d-1)ds_1^4\big]+2ss_2^3\big[(47d-107)ds^4+2(d(35-19d)+32)s^3s_1-24((d$$
$$+9)d+2)s^2s_1^2+2((11d+61)d+192)ss_1^3-((7d+61)d+16)s_1^4\big]-s_2^2(s-s_1)^2\big[(53d$$

$$
\begin{aligned}
&- 143)ds^4 + 4((3d+11)d+28)s^3 s_1 - d(19d+239)s^2 s_1^2 - 2((d+45)d+8)ss_1^3 \\
&+ 4(d-1)ds_1^4\big] - 2(d-1)dss_2^7 + (d-1)ds_2^8\Big\} C_0(s,s_1,s_2) \qquad \text{(A.9)}
\end{aligned}
$$

$$
\begin{aligned}
F_{8,d}(s,s_1,s_2) = &-\frac{4\pi^2 e^2}{3(d-2)(d-1)d\sigma^4}\Big\{\big(d^2+d+16\big)s^6 + (d(3(d-4)d+11)-56) \\
&\times s^5(s_1+s_2) + s^4\big[((3(13-4d)d-55)d+64)s_1^2 + 2((d(4d+21)-21)d+56)s_1 s_2 \\
&+ ((3(13-4d)d-55)d+64)s_2^2\big] + 2s^3(s_1+s_2)\big[(d((9d-28)d+45)-8)s_1^2 - 4(((5d \\
&- 8)d+14)d+10)s_1 s_2 + (d((9d-28)d+45)-8)s_2^2\big] + s^2\big[-((3(4d-13)d+65)d \\
&+ 16)s_1^4 + 4((3(d-9)d+34)d+20)s_1^3 s_2 + 2(d(85d-7)-144)s_1^2 s_2^2 + 4((3(d-9)d \\
&+ 34)d+20)s_1 s_2^3 - ((3(4d-13)d+65)d+16)s_2^4\big] + s(s_1-s_2)^2(s_1+s_2)\big[((3(d-4)d \\
&+ 19)d+8)s_1^2 - 2(((d-36)d+37)d+16)s_1 s_2 + ((3(d-4)d+19)d+8)s_2^2\big] + (d-1) \\
&\times d(s_1-s_2)^4\big(2(2d-3)s_1 s_2 + s_1^2 + s_2^2\big)\Big\} B_0(s) + \frac{\pi^2 e^2}{3(d-2)(d-1)ds\sigma^4(s-s_1-s_2)} \\
&\times \Big\{s\big[3s_2^7 - 5(3s+s_1)s_2^6 + \big(27s^2+82s_1 s+47s_1^2\big)s_2^5 - \big(15s^3+287s_1 s^2+105s_1^2 s \\
&+ 137s_1^3\big)s_2^4 + \big(-15s^4+428s_1 s^3 - 66s_1^2 s^2 + 92s_1^3 s + 137s_1^4\big)s_2^3 + (s-s_1)\big(27s^4 \\
&- 280s_1 s^3 - 18s_1^2 s^2 + 32s_1^3 s + 47s_1^4\big)s_2^2 - (s-s_1)^3(s+s_1)\big(15s^2 - 68s_1 s + 5s_1^2\big)s_2 \\
&+ 3(s-s_1)^5(s+s_1)^2\big]d^3 + \big[-9s^8 + (53s_1+45s_2)s^7 - \big(155s_1^2+264s_2 s_1+81s_2^2\big)s^6 \\
&+ \big(297s_1^3+691s_2 s_1^2+599s_2^2 s_1+45s_2^3\big)s^5 - \big(385s_1^4+822s_2 s_1^3+196s_2^2 s_1^2+818s_2^3 s_1 \\
&- 45s_2^4\big)s^4 + \big(319s_1^5+275s_2 s_1^4 - 2s_2^2 s_1^3 - 1346s_2^3 s_1^2 + 707s_2^4 s_1 - 81s_2^5\big)s^3 - (s_1-s_2) \\
&\times \big(153s_1^5 - 59s_2 s_1^4 + 590s_2^2 s_1^3 + 990s_2^3 s_1^2 - 311s_2^4 s_1 + 45s_2^5\big)s^2 + (s_1-s_2)^3\big(35s_1^4 \\
&- 42s_2 s_1^3 + 116s_2^2 s_1^2 - 54s_2^3 s_1 + 9s_2^4\big)s - 2s_1(s_1-s_2)^6(s_1+s_2)\big]d^2 + 2\big[3s^8 - (13s_1 \\
&+ 15s_2)s^7 + \big(53s_1^2+32s_2 s_1+27s_2^2\big)s^6 - \big(183s_1^3+77s_2 s_1^2+29s_2^2 s_1+15s_2^3\big)s^5 \\
&+ \big(365s_1^4+108s_2 s_1^3 - 390s_2^2 s_1^2 + 60s_2^3 s_1 - 15s_2^4\big)s^4 + \big(-403s_1^5+79s_2 s_1^4+490s_2^2 s_1^3 \\
&+ 1030s_2^3 s_1^2 - 135s_2^4 s_1 + 27s_2^5\big)s^3 + (s_1-s_2)\big(243s_1^5 - 37s_2 s_1^4 + 94s_2^2 s_1^3 + 758s_2^3 s_1^2 \\
&- 113s_2^4 s_1 + 15s_2^5\big)s^2 - (s_1-s_2)^3\big(73s_1^4+30s_2 s_1^3+156s_2^2 s_1^2 - 38s_2^3 s_1 + 3s_2^4\big)s + 4s_1 \\
&\times (s_1-s_2)^5(2s_1-s_2)(s_1+s_2)\big]d + 32s_1^2\big(s^2 - 2(s_1+2s_2)s + (s_1-s_2)^2\big)(s-s_1+s_2) \\
&\times \big[2s^3 - 3(s_1+s_2)s^2 + 10s_1 s_2 s + (s_1-s_2)^2(s_1+s_2)\big]\Big\} B_0(s_1) \\
&+ \frac{\pi^2 e^2}{3(d-2)(d-1)ds\sigma^4(s-s_1-s_2)}\Big\{s\big[-3s_2^7 + (9s+5s_1)s_2^6 - \big(3s^2+78s_1 s+47s_1^2\big) \\
&\times s_2^5 + \big(-15s^3+151s_1 s^2+15s_1^2 s+137s_1^3\big)s_2^4 + (s-s_1)\big(15s^3 - 5s_1 s^2 + 45s_1^2 s \\
&+ 137s_1^3\big)s_2^3 + (s-s_1)^2\big(3s^3 - 135s_1 s^2 - 11s_1^2 s + 47s_1^3\big)s_2^2 - (s-s_1)^4\big(9s^2 - 62s_1 s \\
&+ 5s_1^2\big)s_2 + 3(s-s_1)^6(s+s_1)\big]d^3 + \big[-2s_2^8 + 5(7s+2s_1)s_2^7 - 3\big(51s^2+49s_1 s+6s_1^2\big) \\
&\times s_2^6 + \big(319s^3+212s_1 s^2+347s_1^2 s+10s_1^3\big)s_2^5 + \big(-385s^4+275s_1 s^3 - 649s_1^2 s^2 \\
&- 563s_1^3 s + 10s_1^4\big)s_2^4 + \big(297s^5 - 822s_1 s^4 - 2s_1^2 s^3 - 400s_1^3 s^2 + 561s_1^4 s - 18s_1^5\big)s_2^3
\end{aligned}
$$

$$
\begin{aligned}
&- (s-s_1)^2\big(155s^4 - 381s_1s^3 - 721s_1^2s^2 + 285s_1^3s - 10s_1^4\big)s_2^2 + (s-s_1)^4\big(53s^3 - 52s_1s^2 \\
&+ 73s_1^2s - 2s_1^3\big)s_2 - 9s(s-s_1)^6(s+s_1)\big]d^2 + 2\big[8s_2^8 - (73s+36s_1)s_2^7 + \big(243s^2 \\
&+ 189s_1s + 56s_1^2\big)s_2^6 - \big(403s^3 + 280s_1s^2 + 285s_1^2s + 20s_1^3\big)s_2^5 + \big(365s^4 + 79s_1s^3 \\
&+ 131s_1^2s^2 + 489s_1^3s - 40s_1^4\big)s_2^4 + \big(-183s^5 + 108s_1s^4 + 490s_1^2s^3 + 664s_1^3s^2 - 555s_1^4s \\
&+ 52s_1^5\big)s_2^3 + (s-s_1)^2\big(53s^4 + 29s_1s^3 - 385s_1^2s^2 + 231s_1^3s - 24s_1^4\big)s_2^2 - (s-s_1)^4\big(13s^3 \\
&+ 20s_1s^2 + 31s_1^2s - 4s_1^3\big)s_2 + 3s(s-s_1)^6(s+s_1)\big]d + 32\big(\sigma - 2s\,s_1\big)(s+s_1-s_2)s_2^2 \\
&\times \big[2s^3 - 3(s_1+s_2)s^2 + 10s_1s_2s + (s_1-s_2)^2(s_1+s_2)\big]\Big\}B_0(s_2,0,0) \\
&- \frac{2\pi^2e^2}{3(d-2)d\sigma^4(s-s_1-s_2)}\Big\{d^2\big[3s^5(s_1+s_2) - 4s^4\big(3s_1^2 - 2s_1s_2 + 3s_2^2\big) + 2s^3(s_1+s_2) \\
&\times \big(9s_1^2 - 20s_1s_2 + 9s_2^2\big) - 12s^2(s_1-s_2)^2\big(s_1^2 + s_1s_2 + s_2^2\big) + s(s_1-s_2)^2(s_1+s_2)\big(3s_1^2 \\
&- 2s_1s_2 + 3s_2^2\big) + 4s_1s_2(s_1-s_2)^4\big](-s+s_1+s_2)^2 - 2d\big[3s^7(s_1+s_2) - 2s^6\big(9s_1^2 \\
&+ 13s_1s_2 + 9s_2^2\big) + s^5(s_1+s_2)\big(45s_1^2 + 8s_1s_2 + 45s_2^2\big) - 2s^4\big(30s_1^4 + 5s_1^3s_2 + 74s_1^2s_2^2 \\
&+ 5s_1s_2^3 + 30s_2^4\big) + s^3(s_1+s_2)\big(45s_1^4 - 120s_1^3s_2 + 286s_1^2s_2^2 - 120s_1s_2^3 + 45s_2^4\big) - 2s^2 \\
&\times \big(9s_1^6 - 41s_1^5s_2 + 7s_1^4s_2^2 + 82s_1^3s_2^3 + 7s_1^2s_2^4 - 41s_1s_2^5 + 9s_2^6\big) + s(s_1-s_2)^2(s_1+s_2)^3 \\
&\times \big(3s_1^2 - 32s_1s_2 + 3s_2^2\big) + 2s_1s_2(s_1-s_2)^4(s_1+s_2)^2\big] + 64ss_1^2s_2^2\big(2s^3 - 3s^2(s_1+s_2) \\
&+ 10ss_1s_2 + (s_1-s_2)^2(s_1+s_2)\big)\Big\}C_0(s,s_1,s_2) \qquad \text{(A.10)}
\end{aligned}
$$

$$
\begin{aligned}
&F_{9,d}(s,s_1,s_2) = F_{10,d}(s,s_2,s_1) = \frac{4\pi^2e^2s}{(d-2)(d-1)d\sigma^3\gamma}\Big\{ - 2s_2(s-s_1)^2\big[\big(d^3 - 5d + 8\big)s \\
&- d((d-8)d+11)s_1\big] + 2s_2^2\big[(d+4)((d-3)d+4)s^2 + 2(d+2)((d-7)d+8)ss_1 \\
&- (d-4)d(3d-5)s_1^2\big] + 2s_2^3\big[(d((d-4)d+7)-8)s + (3(d-4)d+13)ds_1\big] + (d-3) \\
&\times d(s-s_1)^4 + ((5-2d)d-5)ds_2^4\Big\}B_0(s) - \frac{2\pi^2e^2}{(d-2)(d-1)d\sigma^3\gamma^2}\Big\{d^3\sigma\big[s^4 - 4s^3 \\
&\times (s_1+2s_2) + 2s^2\big(3s_1^2 + 6s_1s_2 + 7s_2^2\big) - 4s\big(s_1^3 + s_1s_2^2 + 2s_2^3\big) + (s_1-s_2)^4\big] \\
&+ d^2\big[- s_2^4\big(123s^2 + 112ss_1 + 69s_1^2\big) + 2s_2(s-s_1)^3\big(21s^2 - 19ss_1 - 18s_1^2\big) + 4s_2^3\big(43s^3 \\
&+ 10s^2s_1 + 11ss_1^2 + 24s_1^3\big) - s_2^2(s-s_1)\big(123s^3 - 53s^2s_1 - 23ss_1^2 - 79s_1^3\big) + 14s_2^5(3s \\
&+ 2s_1) - (5s-7s_1)(s-s_1)^5 - 5s_2^6\big] + 2d\sigma\big[s_2^2\big(26s^2 + 19ss_1 + 23s_1^2\big) + s_2\big(-16s^3 \\
&+ 33s^2s_1 + 2ss_1^2 - 19s_1^3\big) - s_2^3(16s + 13s_1) + 3(s-s_1)^3(s-2s_1) + 3s_2^4\big] - 32s_1^2s_2 \\
&\times (s-s_1+s_2)\big(s^2 - 2s(s_1+2s_2) + (s_1-s_2)^2\big)\Big\}B_0(s_1) + \frac{4\pi^2e^2s_2}{(d-2)(d-1)d\sigma^3\gamma^2} \\
&\times \Big\{2(d-1)s_2^3\big[((7d-19)d+24)s^2 + 4(d-4)dss_1 + ((5d-21)d+24)s_1^2\big] + ds_2 \\
&\times (s-s_1)^2\big[((d-6)d+13)s^2 - 2((3d-22)d+27)ss_1 + ((5d-22)d+25)s_1^2\big] - 2s_2^2 \\
&\times \big[(d((5d-18)d+29)-8)s^3 - (5d-3)((d-5)d+8)s^2s_1 + ((-5(d-6)d-57)d \\
&+ 24)ss_1^2 + (d((5d-24)d+35)-8)s_1^3\big] + s_2^4\big[(((30-7d)d-55)d+48)s - (d-3)
\end{aligned}
$$

$$\times ((5d-13)d+16)s_1\big] + (d-1)d(s-s_1)^4\big[(d-1)s-(d-3)s_1\big] + (d((d-6)d+13) - 16)s_2^5\Big\}B_0(s_2) - \frac{4\pi^2e^2ss_2}{(d-2)d\sigma^3\gamma^2}\Big\{d^2\sigma(s-s_1+s_2)\gamma^2 + 2d\big[-2s_2^2\big(s^3+5s^2s_1-ss_1^2 + 3s_1^3\big) + s_2^4(3s+4s_1) + 2ss_2^3(s_1-s) + s_2(s-s_1)^2(s+s_1)(3s+s_1) - (s-2s_1) \times (s-s_1)^4 - s_2^5\big] + 64ss_1^2s_2^2\Big\}C_0(s,s_1,s_2) \tag{A.11}$$

$$F_{11,d}(s,s_1,s_2) = F_{12,d}(s,s_2,s_1) = \frac{2\pi^2e^2s}{(d-2)(d-1)d\sigma^3s_2}\Big\{2s_2(s-s_1)\big[4\big(d^2+d-3\big)ss_1 + (d-3)((d-1)d+4)s^2 - d((d-8)d+11)s_1^2\big] - 2s_2^2\big[3(d-2)((d-1)d+2)s^2 + 2(d-1)((d-2)d+6)ss_1 + (3d-5)(d-4)ds_1^2\big] + 2s_2^3\big[(d-1)((3d-5)d+4)s + (3(d-4)d+13)ds_1\big] + (s-s_1)^3\big[((d-3)d+8)s-(d-3)ds_1\big] + ((5-2d)d-5) \times ds_2^4\Big\}B_0(s,0,0) - \frac{\pi^2e^2}{(d-2)(d-1)d\sigma^3s_2}\Big\{d^3\sigma\big[s_2\big(-s^2+6ss_1+3s_1^2\big) - s_2^2(s+3s_1) + (s-s_1)^3+s_2^3\big] + d^2\big[-2s_2^3\big(3s^2+10ss_1+19s_1^2\big) - 2s_2^2(s-3s_1)\big(3s^2+6ss_1+7s_1^2\big) + s_2(s-s_1)\big(9s^3-11s^2s_1+11ss_1^2+23s_1^3\big) + s_2^4(9s+17s_1) - (3s-5s_1)(s-s_1)^4 - 3s_2^5\big] + 2d\sigma\big[s^3-s^2(2s_1+s_2) + s\big(s_1^2-8s_1s_2-s_2^2\big) + s_2(s_1-s_2)^2\big] + 16s_1^2\big(s^2 - 2s(s_1+2s_2) + (s_1-s_2)^2\big)(s-s_1+s_2)\Big\}B_0(s_1) - \frac{2\pi^2e^2}{(d-2)(d-1)d\sigma^3}\Big\{2d^3s\sigma(s-s_1 - s_2) + d^2\big[-5s^4+8s^3(3s_1+2s_2) - 2s^2\big(17s_1^2+6s_1s_2+9s_2^2\big) + 8s(s_1-s_2)(2s_1-s_2) \times (s_1+s_2) - (s_1-s_2)^4\big] + d\sigma\big(5s^2-8s(2s_1+s_2)+3(s_1-s_2)^2\big) + 8s_2\big(s^2-2s(2s_1 + s_2) + (s_1-s_2)^2\big)(s+s_1-s_2)\Big\}B_0(s_2) + \frac{2\pi^2e^2s}{(d-2)d\sigma^3}\Big\{2ds_2^2\big(3(d-2)s^2+(d-5)ss_1 - 5s_1^2\big) + d(s-s_1)^2\big((d-2)s^2-(d-4)s_1^2+6ss_1\big) + 2s_2\big(-2(d-2)ds^3+(d-5) \times ds^2s_1 + 2(d+8)ss_1^2 + (d-1)ds_1^3\big) - 2ds_2^3(2(d-2)s+(d-5)s_1) + (d-2)ds_2^4\Big\} \times C_0(s,s_1,s_2) \tag{A.12}$$

$$F_{13,d}(s,s_1,s_2) = \frac{2\pi^2e^2s^2}{(d-2)(d-1)d\sigma^2}\Big\{-2s_2\big[((d-3)d+4)s + \big[(d-7)d+8\big]s_1\big] + ((d-3)d+4)(s-s_1)^2 + ((d-3)d+4)s_2^2\Big\}B_0(s) - \frac{\pi^2e^2s_1}{(d-2)(d-1)d\sigma^2(s-s_1-s_2)} \times \Big\{d^3\sigma^2 - d^2\big[3s^4-2s^3(7s_1+9s_2) + 8s^2\big(3s_1^2+3s_1s_2+4s_2^2\big) - 2s(s_1-s_2)^2(9s_1 + 11s_2) + 5(s_1-s_2)^4\big] + 2d\big[2s^4-11s^3(s_1+s_2) + s^2\big(21s_1^2+24s_1s_2+19s_2^2\big) + s\big(-17s_1^3+5s_1^2s_2+25s_1s_2^2-13s_2^3\big) + (s_1-s_2)^3(5s_1-3s_2)\big] + 8s_1\big(s^3-3s^2(s_1+s_2) + 3s\big(s_1^2-s_2^2\big) - (s_1-s_2)^3\big)\Big\}B_0(s_1) - \frac{\pi^2e^2s_2}{(d-2)(d-1)d\sigma^2\gamma}\Big\{d^3\sigma^2 - d^2\big[3s^4-2s^3(9s_1 + 7s_2) + 8s^2\big(4s_1^2+3s_1s_2+3s_2^2\big) - 2s(s_1-s_2)^2(11s_1+9s_2) + 5(s_1-s_2)^4\big] + 2d\big[2s^4 - 11s^3(s_1+s_2) + s^2\big(19s_1^2+24s_1s_2+21s_2^2\big) + s\big(-13s_1^3+25s_1^2s_2+5s_1s_2^2-17s_2^3\big)$$

$$+ (s_1 - s_2)^3(3s_1 - 5s_2)\big] + 8s_2\big[s^3 - 3s^2(s_1 + s_2) - 3s\big(s_1^2 - s_2^2\big) + (s_1 - s_2)^3\big]\Big\} B_0(s_2)$$

$$+ \frac{4\pi^2 e^2 s^2 s_1 s_2 (d\sigma + 8s_1 s_2)}{(d-2)d\sigma^2\gamma} C_0(s, s_1, s_2) \tag{A.13}$$

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