ASYMPTOTIC PROPERTIES OF THE SPECTRAL DISTRIBUTIONS OF DISORDERED SYSTEMS

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Let us first consider a discrete version of the Schrödinger operator with a random potential q :

(1) $\quad (H^{\omega}u)(a) = -(H^{0}u)(a) + q(a,\omega)u(a), \quad a \in Z^{\nu}.$

Here u is a function on the ν-dimensional lattice space Z^{ν}. H^{0} is a second order difference operator defined by

(2) $$(H^{0}u)(a) = \frac{\sigma^2}{2}\sum_{i=1}^{\nu}\{u(a_1,\cdots,a_i-1,\cdots,a_\nu) - 2u(a) +u(a_1,\cdots,a_i+1,\cdots,a_\nu)\}, \qquad a \in Z^{\nu},$$

with a positive constant σ. We assume that $\{q(a,\omega)\ ;\ a \in Z^{\nu}\}$ is an ergodic stationary random field defined over some probability space $(\Omega,\mathcal{B}, P)$.

Denote by $L^2(Z^{\nu})$ the space of all square summable functions on Z^{ν} with inner product $(u, v) = \sum_{a\in Z^{\nu}} u(a)v(a)$. For each $\omega \in \Omega$, the operator H^{ω} induces a self-adjoint operator A^{ω} on $L^2(Z^{\nu})$ by

(3) $$\begin{cases} \mathcal{D}(A^{\omega}) = \{u \in L^2(Z^{\nu})\ ;\ H^{\omega}u \in L^2(Z^{\nu})\} \\ A^{\omega}u = H^{\omega}u, \quad u \in \mathcal{D}(A^{\omega}). \end{cases}$$

Let $\{E_{\lambda}^{\omega}\ ;\ \lambda \in R^1\}$ be the resolution of the identity associated with A^{ω} : $A^{\omega} = \int_{-\infty}^{\infty}\lambda dE_{\lambda}^{\omega}$. We can then define the <u>spectral distribution function</u> $\rho(\lambda)$ of the ensemble of operators $\{H^{\omega}\ ;\ \omega \in \Omega\}$ by

(4) $\quad \rho(\lambda) = \langle(E_{\lambda}^{\omega}I_0, I_0)\rangle$

where $I_0(a) = \delta_0(a)$,$a \in Z^{\nu}$, and $\langle\ \rangle$ denotes the expectation in ω with respect to the probability measure P.

ρ can be identified with the so-called <u>integrated density of states</u> of a disordered physical system governed by $\{H^{\omega};\ \omega \in \Omega\}$. Indeed we have the following ergodic theorem : Let Λ be a rectangle containing the origin with sides parallel to axes. Let

$\lambda_1^\omega \leqq \lambda_2^\omega \leqq \lambda_3^\omega \leqq \cdots \leqq \lambda_N^\omega$ be the eigenvalues of the problem :

$$(H^\omega u)(a) = \lambda u(a), \quad a \in \Lambda - \partial\Lambda \; ; \quad u(a) = 0, \quad a \in \partial\Lambda .$$

We put, for each λ, $\mathcal{H}^\omega(\lambda \; ; \Lambda) = \sum_{\lambda_i^\omega \leqq \lambda} 1$. Then there exists a set $\Omega_0 \in \mathcal{B}$ with $P(\Omega_0) = 1$ such that, for each $\omega \in \Omega_0$,

$$(5) \qquad \lim_{\substack{L^{(i)}(\Lambda)\to\infty \\ i=1,2,\cdots,\nu}} \frac{\mathcal{H}^\omega(\lambda;\Lambda)}{|\Lambda|} = \rho(\lambda)$$

at every continuity point λ of $\rho(\lambda)$. Here $L^{(i)}(\Lambda)$ (resp. $|\Lambda|$) is the side length (resp. volume) of Λ.

Recent probabilistic investigations have revealed several asymptotic properties of the spectral distribution function near the end point of its spectrum ([1],[2],[3]). Here we state some of them.

Assume that $q(a)$, $a \in Z^\nu$, are non-negative valued independent and identically distributed random variables. In this case $\rho(\lambda)$ vanishes for negative λ and then increases from 0 exponentially :

Theorem 1. ([2])

(i) *If* $P(q(0) = 0) < 1$, *then* $\overline{\lim_{\lambda \downarrow 0}} \sqrt{\lambda} \log \rho(\lambda) < 0$.

(ii) *If* $P(q(0) = 0) > 0$ *and* $\nu = 1$, *then* $-\infty < \underline{\lim_{\lambda\downarrow 0}} \sqrt{\lambda} \log \rho(\lambda)$.

A similar phenomena occurs in the one dimensional Schrödinger operator with the potential q being the Poisson impulses :

$$(6) \qquad H^\omega u(a) = -\frac{1}{2} u''(a) + q(a,\omega)u(a), \qquad a \in R^1.$$

More precisely let $k(da,\omega)$ be the Poisson random measure with mean measure being the Lebesgue measure : $k(E,\omega)$ is distributed according to the Poisson law with mean $|E|$ and $k(E_1), k(E_2), \cdots, k(E_n)$ are independent for disjoint E_j's. The operator H^ω should be understood in a generalized sense([5]):

$$(6)' \qquad H^\omega u(a) = \frac{-\frac{1}{2}\, du'(a) + k(da,\omega)u(a)}{da} .$$

Then we can associate with the ensemble of operators $\{H^\omega\}$ its spectral distribution function $\rho(\lambda)$, $\lambda \in R^1$, just in the same manner as in (5).

Theorem 2. The spectral distribution function ρ of the random operator (6)' behaves as follows :

$$-\infty < \underline{\lim}_{\lambda \downarrow 0} \sqrt{\lambda} \log \rho(\lambda) \leq \overline{\lim}_{\lambda \downarrow 0} \sqrt{\lambda} \log \rho(\lambda) < 0.$$

This behaviour was first observed by I.M.Lifschitz ([4]), but the above theorem is due to S.Nakao ([3]).

Returning to the random difference operator (1), we mention some fine structure of the associated resolution of the identity $\{ E_\lambda^\omega, \lambda \in R^1 \}$. Suppose that $\nu = 1$ and that $q(a)$, $a \in Z^1$, are independent random variables with common distribution ϕ.

Theorem 3. If the support of ϕ contains at least two points and $\int_{-\infty}^{\infty} |x| \phi(dx) < \infty$, then there exists a set $\Omega_0 \in \mathcal{B}$ with $P(\Omega_0) = 1$ such that, for each $\omega \in \Omega$, the spectrum of the self-adjoint operator A^ω defined by (3) admits no absolutely continuous part : the one dimensional measure $d_\lambda (E_\lambda^\omega u, u)$ is singular for any $u \in L^2(Z^1)$.

This theorem was discovered by A.Casher and J.L.Lebowitz ([6]) in the case of semi-infinite lattice $Z_+^1 = \{0, 1, 2, \cdots\}$ with the fixed end boundary condition at -1. By extending an idea of K.Ishii ([7]), L.A.Pastur was recently able to get the result for the doubly infinite case Z^1.

The proof of Theorem 1 and Theorem 2 is carried out by making use of the Feynman-Kac expression of the Laplace transform of $\rho(\lambda)$ in terms of the associated auxiliary Markov process and then by invoking a Tauberian theorem of exponential type.

Theorem 3 is a consequence of an exponential growth character of the generalized eigenvector of the random operator H^ω. Such a character has been derived by H.Matsuda and K.Ishii ([8]) and Y.Yoshioka ([9]) from the Furstenberg limit theorem ([10]) on the product of random unimodular matrices.

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