Discrete and continuous symmetries in α-cluster nuclei

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Discrete and continuous symmetries in α-cluster nuclei

Roelof Bijker

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A.P. 70-543, 04510 México, D.F., México

E-mail: bijker@nucleares.unam.mx

Abstract. I discuss some properties of the algebraic cluster model which is based on the introduction of a spectrum generating algebra for the relative motion of the clusters. Particular attention is paid to the permutation symmetry of identical clusters. As an example, it is shown how the method can be used to study α-cluster configurations in ^{12}C and ^{16}O with point group symmetries, $\mathcal{D}_{3h}$ and $\mathcal{T}_d$, respectively.

1. Introduction

The binding energy per nucleon for light nuclei shows large oscillations with maxima for nuclei with $A = 4n$ and $Z = N$, indicating the importance of α-cluster phenomena in these nuclei. There exists a large literature on α cluster models to describe the spectral properties of nuclei like ^{12}C and ^{16}O [1, 2, 3, 4] (for a recent review on cluster models see [5]).

Recently, there has been a lot of renewed interest in the structure of ^{12}C with the measurement of a new lowlying 2^+ state around 10 MeV with a width of approximately 0.8 MeV [6, 7, 8] which was interpreted as a rotational excitation of the Hoyle state, thus providing the first evidence of a collective band structure built on top of the Hoyle state.

In view of the new experimental information, in this contribution I revisit an algebraic treatment of α-cluster nuclei, called the Algebraic Cluster Model [9], discuss the permutation symmetry of identical clusters, as well as two special solutions which are relevant to the α-cluster nuclei ^{12}C and ^{16}O.

2. The Algebraic Cluster Model

The Algebraic Cluster Model (ACM) is an interacting boson model to describe the relative motion of n clusters based on the spectrum generating algebra of $U(\nu + 1)$ with $\nu = 3(n - 1)$. As special cases the ACM contains the vibron model [10] for two-body problems ($n = 2$), the $U(7)$ model for three-body clusters ($n = 3$) with applications in hadronic physics [11], molecular physics [12] and nuclear physics [9], and the $U(10)$ model for four-body clusters ($n = 4$) [13].

The building blocks of the ACM consist of a dipole boson $b_k^\dagger$ for each independent relative Jacobi coordinate $k = 1, \ldots, n - 1$ and an auxiliary scalar boson $s^\dagger$ which is added under the restriction that the total number of bosons $N = n_s + n_b = n_s + \sum_k n_k$ is conserved. The bilinear products of creation and annihilation operators span the Lie algebra of $U(3n-2)$. All operators of interest, such as the Hamiltonian and electromagnetic transition operators, are expanded

into elements of this algebra. This procedure leads to a compact spectrum generating algebra of $U(3n-2)$ whose model space contains the oscillator shells with $n_b = 0, 1, 2, \ldots, N$. The introduction of the scalar boson makes it possible to investigate the mixing of oscillator shells.

For n identical objects, the Hamiltonian has to be invariant under the permuation group S_n. The transformation properties under S_n of all operators in the model follow from those of the building blocks. The scalar boson $s^\dagger$ transforms as the symmetric representation $[n]$, whereas the dipole bosons $b_k^\dagger$ transform as the $n-1$ components of the mixed symmetry representation $[n-1,1]$ [14]. Next, one can use the multiplication rules for S_n to construct physical operators with the appropriate symmetry properties. As a result, the most general one- and two-body Hamiltonian that describes the relative motion of a system of n identical clusters, is a scalar under the permutation group S_n, is rotationally invariant, conserves parity as well as the total number of bosons, is given by

$$\begin{aligned} H &= \epsilon_0\, s^\dagger \tilde{s} - \epsilon_1 \sum_k b_k^\dagger \cdot \tilde{b}_k + u_0\, s^\dagger s^\dagger \tilde{s}\tilde{s} - u_1 \sum_k s^\dagger b_k^\dagger \cdot \tilde{b}_k \tilde{s} + v_0 \left[\sum_k b_k^\dagger \cdot b_k^\dagger\, \tilde{s}\tilde{s} + \text{h.c.} \right] \\ &\quad + \sum_L \Big[w_L\, T^{\dagger(L)}_{[n]} \cdot \tilde{T}^{(L)}_{[n]} + x_L\, T^{\dagger(L)}_{[n-1,1]} \cdot \tilde{T}^{(L)}_{[n-1,1]} \\ &\qquad + y_L\, T^{\dagger(L)}_{[n-2,1,1]} \cdot \tilde{T}^{(L)}_{[n-2,1,1]} + z_L\, T^{\dagger(L)}_{[n-2,2]} \cdot \tilde{T}^{(L)}_{[n-2,2]} \Big] \ , \end{aligned} \tag{1}$$

with $\tilde{b}_{k,m} = (-1)^{1-m} b_{k,-m}$ and $\tilde{s} = s$. The coefficients ϵ_0, ϵ_1, u_0, u_1, v_0, w_L, x_L, y_L and z_L parametrize the interactions. The operators $T^{(L)}_{[f]}$ are spherical tensor operators with respect to the rotation group as well as irreducible tensor operators with respect to the permutation group

$$T^{\dagger(L)}_{[f]k} = \sum_{i,j=1}^{n-1} c_{ij}^{[f]k}\, (b_i^\dagger \times b_j^\dagger)^{(L)} \ . \tag{2}$$

Here L represents the angular momentum ($L = 0, 2$ for $i = j$ and $L = 0, 1, 2$ for $i \neq j$) and the coefficients $c^{[f]}$ denote the Clebsch-Gordan coefficients for S_n

$$c_{ij}^{[f]k} = \langle [n-1,1]i, [n-1,1]j | [f]k \rangle \ , \tag{3}$$

with

$$[n-1,1] \otimes [n-1,1] = [n] \oplus [n-1,1] \oplus [n-2,1,1] \oplus [n-2,2] \qquad \text{for } n \geq 4 \ . \tag{4}$$

For $n = 3$ the last term is absent. Recently, the Clebsch-Gordan coefficients of Eq. (3) were derived in closed analytic form [15], in agreement with the expressions obtained in [14].

The eigenvalues and corresponding eigenvectors are obtained numerically by diagonalizing the Hamiltonian in a coupled harmonic oscillator basis. By construction, the wave functions are characterized by the total number of bosons N, angular momentum and parity L^P, and their transformation property $[f]$ under the permutation group S_n. For the case of the harmonic oscillator the construction of states with good permutation symmetry was studied by Kramer and Moshinsky [14]. In particular, it was shown that for the n-body problem, the reduction $U(3n-3) \supset U(3) \otimes U(n-1)$ with $U(3) \supset SO(3)$ and $U(n-1) \supset O(n-1) \supset S_n$ provides a basis for an explicit construction of states with good permutation symmetry. Here $U(3)$ denotes the symmetry group for the coupled harmonic oscillator and $U(n-1)$ is the symmetry group associated with the transformation of the indices of the $n-1$ relative Jacobi vectors.

However, in the application to the ACM the number of oscillator shells can be large ($n_b = 0, 1, \ldots, N$ with $N \sim 10 - 20$), and moreover the v_0 term in Eq. (1) mixes different

oscillator shells with $\Delta n_b = \pm 2$. For this reason, there is need for a general procedure to create a set of basis states with good permutation symmetry which is easy to implement. One possible solution is to generate these symmetry-adapted basis states numerically by diagonalizing S_n invariant interactions for each oscillator shell separately and to determine the permutation symmetry $[f]$ of a given basis state from the transformation properties under the interchange $P(12)$ and the cyclic permutation $P(12\cdots n)$. Since the Hamiltonian of Eq. (1) is invariant under $P(12)$, basis states with n_1 even and n_1 odd do not mix, and can therefore be treated separately. In operator form, one has

$$\left\langle \psi_{[f]k} \right| P(12) \left| \psi_{[f]k} \right\rangle \;=\; \left\langle \psi_{[f]k} \right| \mathrm{e}^{i\pi b_1^\dagger b_1} \left| \psi_{[f]k} \right\rangle \;. \tag{5}$$

Next, the permutation symmetry $[f]$ of the wave functions can be determined by evaluating the matrix elements of the cyclic permutation $P(12\cdots n)$

$$\left\langle \psi_{[f]k} \right| P(12\cdots n) \left| \psi_{[f]k} \right\rangle \;=\; \left\langle \psi_{[f]k} \right| \mathrm{e}^{i\pi \sum_{i=1}^{n-1} b_i^\dagger b_i} \prod_{j=1}^{n-2} \mathrm{e}^{\theta_j (b_j^\dagger b_{j+1} - b_{j+1}^\dagger b_j)} \left| \psi_{[f]k} \right\rangle \;, \tag{6}$$

with $\theta_j = \arctan\sqrt{j(j+2)}$. The first term depends on the total number of oscillator quanta n_b and gives rise to a factor $+1$ (-1) for states with even (odd) parity. The other terms can be interpreted as a change of oscillator coordinates, and hence their matrix elements can be expressed in terms of Talmi-Moshinksy brackets [16, 17] which are calculated with the program TMB developed by Dobeš [18]. Finally, the relative phases of the degenerate representations can be determined from the off-diagonal matrix elements

$$\left\langle \psi_{[f]k} \right| P(12\cdots n) \left| \psi_{[f]k'} \right\rangle \tag{7}$$

by requiring that they transform like the components of the degenerate representations.

This numerical technique was developed for systems of three identical clusters [11] and later generalized to four identical clusters [13].

3. Special solutions

The ACM has a rich algebraic structure, which includes both continuous and discrete symmetries. It is of general interest to study limiting cases of the Hamiltonian in which the energy spectra can be obtained in closed form. The S_n invariant ACM Hamiltonian of Eq. (1) has dynamic symmetries corresponding to the group chains

$$U(3n-2) \;\supset\; \begin{cases} U(3n-3) \\ \\ SO(3n-2) \end{cases} . \tag{8}$$

These dynamic symmetries were studied in general for the n-body problem and were shown to correspond to the $(3n-3)$-dimensional (an)harmonic oscillator and a deformed oscillator, respectively [13].

In addition, there are several other limiting cases of S_n invariant Hamiltonians for which approximate solutions can be obtained in a semiclassical mean-field analysis and which are of interest for the application to α-cluster nuclei. For the case of three-body clusters I discuss the oblate symmetric top with three particles located at the vertices of an equilateral triangle, and for four-body clusters the spherical top with four particles at the vertices of a regular tetrahedron.

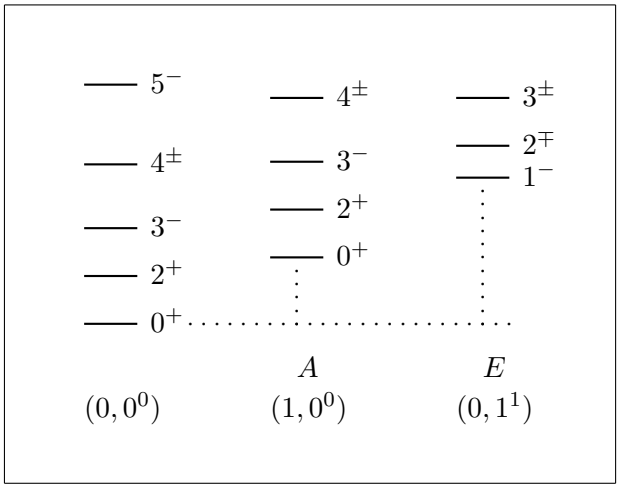

Figure 1. Schematic spectrum of an oblate symmetric top. The rotational bands are labeled by (v_1, v_2^l) (bottom). All states are symmetric under $S_3 \sim \mathcal{D}_3$.

3.1. Three-body clusters: oblate top

For the case of three identical clusters the spectrum generating algebra of the ACM is given by $U(7)$. Here I consider a Hamiltonian which is a special case of Eq. (1)

$$\begin{aligned} H_1 &= \xi_1 \left(R^2\, s^\dagger s^\dagger - b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger\right) \left(R^2\, \tilde{s}\tilde{s} - \tilde{b}_1 \cdot \tilde{b}_1 - \tilde{b}_2 \cdot \tilde{b}_2\right) \\ &\quad + \xi_2 \left[\left(b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger\right) \left(\tilde{b}_1 \cdot \tilde{b}_1 - \tilde{b}_2 \cdot \tilde{b}_2\right) + 4 \left(b_1^\dagger \cdot b_2^\dagger\right) \left(\tilde{b}_2 \cdot \tilde{b}_1\right)\right] \\ &\quad + 2\kappa_1 \left(b_1^\dagger \times \tilde{b}_1 + b_2^\dagger \times \tilde{b}_2\right)^{(1)} \cdot \left(b_1^\dagger \times \tilde{b}_1 + b_2^\dagger \times \tilde{b}_2\right)^{(1)} \\ &\quad + 3\kappa_2 \left(b_1^\dagger \times \tilde{b}_2 - b_2^\dagger \times \tilde{b}_1\right)^{(0)} \cdot \left(b_2^\dagger \times \tilde{b}_1 - b_1^\dagger \times \tilde{b}_2\right)^{(0)} . \end{aligned} \tag{9}$$

For $R^2 = 0$, this Hamiltonian has $U(7) \supset U(6)$ symmetry (anharmonic oscillator), whereas for $R^2 = 1$ and $\xi_2 = 0$ it has $U(7) \supset SO(7)$ symmetry (deformed oscillator). The general case with $R^2 \neq 0$ and $\xi_1,\ \xi_2 > 0$ corresponds to a geometric configuration in which the α particles are located at the vertices of an equilateral triangle [9]. The corresponding point group symmetry is $\mathcal{D}_{3h}$ whose subgroup $\mathcal{D}_3$ is isomorphic to the permutation group S_3. Even though in this case the energy eigenvalues cannot be derived in closed form, an approximate expression for the energy levels can be obtained by making use of the method of intrinsic or coherent states (valid in the limit of large N) [11, 12]

$$E_1 \approx \omega_1 \left(v_1 + \frac{1}{2}\right) + \omega_2 \left(v_2 + 1\right) + \kappa_1\, L(L+1) + \kappa_2 \left(K \pm 2l\right)^2 , \tag{10}$$

with frequencies $\omega_1 = 4NR^2\xi_1$ and $\omega_2 = 4NR^2\xi_2/(1+R^2)$. The quantum numbers have the following meaning: v_1, v_2 are vibrational quantum numbers: v_1 corresponds to a symmetric one-dimensional stretching vibration, and v_2 represents a two-dimensional bending vibration; $l = v_2, v_2-2, \ldots, 1$ or 0 is the vibrational angular momentum of the doubly degenerate vibration; L is the angular momentum, and K its projection on a body-fixed axis [9].

Fig. 1 shows the rotation-vibration spectrum of a triangular configuration with $\mathcal{D}_{3h}$ symmetry which is characterized by a series of rotational bands labeled by (v_1, v_2^l). It is assumed that the spin of the identical clusters is zero, as is relevant for the description of the ^{12}C nucleus as a

cluster of three α particles. As a consequence, all states in Fig. 1 are symmetric under S_3. For this case, the bands with $(v_1, 0^0)$ can have angular momenta and parity $L^P = 0^+, 2^+, 3^-, 4^\pm, 5^-, \ldots$, whereas the angular momentum content of the doubly degenerate vibrations $(v_1, 1^1)$ is given by $L^P = 1^-, 2^\mp, 3^\mp, \ldots$, in agreement with Ref. [19]. The structure of the rotational bands can be considered as the fingerprint of the underlying $\mathcal{D}_{3h}$ symmetry of three identical structureless particles at the vertices of an equilateral triangle.

3.2. Four-body clusters: spherical top

As a second example, I discuss the spherical top with tetrahedral symmetry $\mathcal{T}_d$ as a special case of the ACM for four identical clusters. The spectrum generating algebra is given by $U(10)$. Let us consider a S_4 invariant Hamiltonian of the form

$$\begin{aligned} H_2 &= \xi_1 \left(R^2\, s^\dagger s^\dagger - b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger - b_3^\dagger \cdot b_3^\dagger\right) (\text{h.c.}) \\ &\quad + \xi_2 \Big[(-2\sqrt{2}\, b_1^\dagger \cdot b_3^\dagger + 2 b_1^\dagger \cdot b_2^\dagger) (\text{h.c.}) \\ &\qquad\qquad + (-2\sqrt{2}\, b_2^\dagger \cdot b_3^\dagger + (b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger)) (\text{h.c.}) \Big] \\ &\quad + \xi_3 \Big[(2 b_1^\dagger \cdot b_3^\dagger + 2\sqrt{2}\, b_1^\dagger \cdot b_2^\dagger) (\text{h.c.}) \\ &\qquad\qquad + (2 b_2^\dagger \cdot b_3^\dagger + \sqrt{2}\, (b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger)) (\text{h.c.}) \\ &\qquad\qquad + (b_1^\dagger \cdot b_1^\dagger + b_2^\dagger \cdot b_2^\dagger - 2 b_3^\dagger \cdot b_3^\dagger) (\text{h.c.}) \Big] \\ &\quad + \kappa_1\, \vec{L} \cdot \vec{L} + \kappa_2\, (\vec{L} \cdot \vec{L} - \vec{I} \cdot \vec{I})^2 \ , \end{aligned} \tag{11}$$

where $\vec{L}$ denotes the angular momentum in coordinate space and $\vec{I}$ the angular momentum in index space. For $R^2 = 0$, this Hamiltonian has $U(10) \supset U(9)$ symmetry (anharmonic oscillator), whereas for $R^2 = 1$ and $\xi_2 = \xi_3 = 0$ it has $U(10) \supset SO(10)$ symmetry (deformed oscillator). The general case with $R^2 \neq 0$ and ξ_1, ξ_2, $\xi_3 > 0$ corresponds to a geometric configuration in which the four clusters are located at the vertices of a regular tetrahedron [13]. The corresponding point group symmetry is $\mathcal{T}_d$ which is isomorphic to the permutation group S_4. Just as in the previous case of the oblate top, also for the Hamiltonian of Eq. (11) an approximate energy formula can be derived in a semiclassical mean-field analysis by making use of the methods of intrinsic or coherent states which is valid in the limit of large N. The resulting energy spectrum is that of the vibrational and rotational excitations of a spherical top with tetrahedral symmetry

$$E_2 \approx \omega_1 (v_1 + \frac{1}{2}) + \omega_2 (v_2 + 1) + \omega_3 (v_3 + \frac{3}{2}) + \kappa_1\, L(L+1) \ , \tag{12}$$

with frequencies $\omega_1 = 4NR^2\xi_1$, $\omega_2 = 8NR^2\xi_2/(1+R^2)$ and $\omega_3 = 8NR^2\xi_3/(1+R^2)$. Here v_1 represents the vibrational quantum number for a symmetric stretching A vibration, v_2 denotes a doubly degenerate E vibration, and v_3 a three-fold degenerate F vibration.

Whereas the angular momentum L is an exact symmetry of H_2 of Eq. (11), the angular momentum in index space I in general does not commute with the Hamiltonian. Only if $\xi_2 = \xi_3$ in Eq. (11), does I become a good quantum number. The rotational excitations of the ground state vibrational band of H_2 with $(v_1, v_2, v_3) = (0, 0, 0)$ are characterized by $L = I$. This property is a consequence of the fact that the operator $\vec{L} \cdot \vec{L} - \vec{I} \cdot \vec{I}$ annihilates the coherent (or intrinisic) state corresponding to the rigid equilibrium shape of a regular tetrahedron.

Fig. 2 shows the structure of the spectrum of a spherical top with tetrahedral symmetry according to the approximate energy formula of Eq. (12). The energy spectrum consists of a series of rotational bands labeled by (v_1, v_2, v_3). It is assumed that the spin of the identical clusters is zero, as is relevant for the description of the ^{16}O nucleus as a cluster of four α

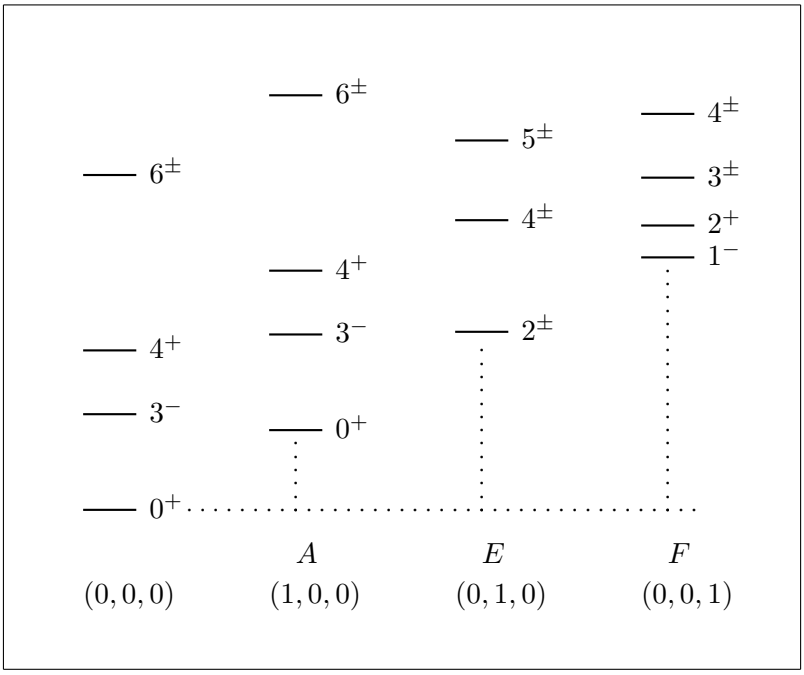

Figure 2. Schematic spectrum of a spherical top with tetrahedral symmetry. The rotational bands are labeled by (v_1, v_2, v_3) (bottom). All states are symmetric under $S_4 \sim \mathcal{T}_d$.

particles. As a consequence, all states in Fig. 2 are symmetric under S_4. For this case, the bands with $(v_1, 0, 0)$ can have angular momenta and parity $L^P = 0^+, 3^-, 4^+, 6^{\pm}, \ldots$, whereas the angular momentum content of the doubly degenerate E vibration $(v_1, 1, 0)$ is given by $L^P = 2^{\pm}, 4^{\pm}, 5^{\pm}, \ldots$, and for the triply degenerate F vibration $(v_1, 0, 1)$ by $L^P = 1^-, 2^+, 3^{\pm}, 4^{\pm}, \ldots$, in agreement with Ref. [19]. The structure of the rotational bands can be considered as the fingerprint of the underlying $\mathcal{T}_d$ tetrahedral symmetry of four identical structureless particles at the vertices of a regular tetrahedron.

4. Summary and conclusions

In this contribution, I discussed the interplay of continuous and discrete symmetries in the framework of the Algebraic Cluster Model (ACM). The ACM incorporates all vibrational and rotational degrees of freedom from the outset. In addition, the permutation symmetry of identical clusters can be taken into account in an exact manner.

It was shown that for systems with identical clusters the structure of rotational bands is determined by the underlying point-group symmetry of the geometric configuration of the α particles. For the case of three-body clusters with the three α particles located at the vertices of an equilateral triangle (oblate top), the rotational structure of the ground state band is given by the sequence $L^P = 0^+, 2^+, 3^-, 4^{\pm}, 5^-, \ldots$, whereas for the case of four-body clusters with the four α particles located at the vertices of a regular tetrahedron (spherical top), the rotational sequence is given by $L^P = 0^+, 3^-, 4^+, 6^{\pm}, \ldots$.

If a physical system, like an α-cluster nucleus, is claimed to consist of identical structureless particles at the vertices of an equilateral triangle (as in ^{12}C) or a regular tetrahedron (as in ^{16}O), then its spectrum *must* be as shown in Figs. 1 or 2, respectively. The rotational structure provides a fingerprint of the underlying point-group symmetry of the geometric configuration of

the α particles.

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