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p-Adic open string amplitudes with Chan-Paton factors coupled to a constant B-field

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Abstract

We establish rigorously the regularization of the p-adic open string amplitudes, with Chan-Paton rules and a constant B-field, introduced by Ghoshal and Kawano. In this study we use techniques of multivariate local zeta functions depending on multiplicative characters and a phase factor which involves an antisymmetric bilinear form. These local zeta functions are new mathematical objects. We attach to each amplitude a multivariate local zeta function depending on the kinematic parameters, the B-field and the Chan-Paton factors. We show that these integrals admit meromorphic continuations in the kinematic parameters. This result allows us to regularize the Ghoshal-Kawano amplitudes. The regularized amplitudes do not have ultraviolet divergencies. Due to the need for a certain symmetry, the theory works only for prime numbers which are congruent to 3 modulo 4. We also discuss the limit $p \to 1$ in the noncommutative effective field theory and in the Ghoshal-Kawano amplitudes. We show that in the case of four points, the limit $p \to 1$ of the regularized Ghoshal-Kawano amplitudes coincides with the Feynman amplitudes attached to the limit $p \to 1$ of the noncommutative Gerasimov-Shatashvili Lagrangian.

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Contents

1. Introduction

The deep connections between p-adic analysis and physics are a natural consequence of the emergence of ultrametricity in physics, which means the occurrence of ultrametric spaces in physical models, see e.g. [1–9] and the references therein. The existence of a Planck length implies that the spacetime considered as a topological space is completely disconnected. The points (which are the connected components) play the role of spacetime quanta. This is precisely the Volovich conjecture on the non-Archimedean nature of the spacetime below the Planck scale, [2,3], [8, Chapter 6]. Ultrametric spaces have also appeared in models of complex systems. A central paradigm in the physics of certain complex systems (for instance proteins), asserts that the dynamics of such systems can be modeled as a random walk over the leaves of a rooted tree. This tree is a finite ultrametric space constructed out of the energy landscape. Mean-field approximations of these models drive naturally to models involving p-adic numbers, see e.g. [7,10–12], and the references therein.

In the last forty years, the above mentioned ideas have motivated many developments in quantum field theory and string theory, see e.g. [1,5,13,14], and more recently, [15–28].

In string theory, the scattering amplitudes are obtained integrating over the moduli space of Riemann surfaces. Even for tree-level amplitudes (on the sphere for closed strings and on the disk for open strings) N-point amplitudes are difficult to compute beyond four points. Moreover, the convergence of these integrals is not evident by itself [29,30]. Recently, in [31] was established in a rigorous mathematical way that Koba-Nielsen amplitudes are bona fide integrals, which admit meromorphic continuations when considered as complex functions of the kinematic parameters.

String theory with a p-adic world-sheet was proposed and studied for the first time in [32]. Later this theory was formally known as p-adic string theory. The Adelic scattering amplitudes which are related to the Archimedean ones were studied in [33]. The tree-level string amplitudes were explicitly computed in the case of p-adic string world-sheet in [34] and [35]. These amplitudes can be formally obtained from a suitable action using general principles [36]. A general treatment, starting by describing a discrete field theory on a Bruhat-Tits tree and obtaining the tree-level string amplitudes ([34]), was established in [37]. Similarly as in the standard string theory, in p-adic string theories it is difficult to determine the convergence region in the momentum space, however this was done precisely for the N-point tree amplitudes in [38]. In this article we show (in a rigorous mathematical way) that the p-adic open string $N-$point tree amplitudes are bona fide integrals that admit meromorphic continuations as rational functions, by relating them with multivariate local zeta functions (also called multivariate Igusa local zeta functions [39–41]).

In p-adic string theory the limit $p \to 1$ is very intriguing since it seems to be related to the real versions of these theories [36,42]. This limit is special since the effective theory shows that it is related to physical string theories such as the boundary string field theory [43]. Another interpretation of the limit $p \to 1$ was given in terms of the renormalization group scaling transformation in the Bruhat-Tits tree for some suitable p [44]. In the worldsheet theory we cannot forget the nature of p as a prime number, thus the analysis of the limit is more subtle. The correct way of taking the limit $p \to 1$ involves the introduction of finite extensions of the p-adic field $\mathbb{Q}_p$. In [45] the limit $p \to 1$ was discussed at the tree-level string amplitudes. We provided a rigorous definition of this limit using the theory of topological zeta functions due to Denef and Loeser [46,47].

In ordinary string theory the effective action for bosonic open strings in gauge field backgrounds was discussed many years ago in [48]. The analysis incorporating a Neveu-Schwarz B field in the target space leads to a noncommutative effective gauge theory on the world-volume of D-branes [49]. The study of the p-adic open string tree amplitudes including Chan-Paton factors was started in [34]. However the incorporation of a B-field in the p-adic context and the computation of the tree level string amplitudes was discussed in [50,51]. In these works it was reported that the tree-level string amplitudes are affected by a noncommutative factor. In [50] Ghoshal and Kawano introduced new amplitudes involving multiplicative characters and a noncommutative factor. These amplitudes coincide with the ones obtained directly from the noncommutative effective action [52].

In the present article we study the p-adic string amplitudes, with Chan-Paton rules and a constant B-field, introduced in [50], by using techniques of multivariate local zeta functions. The N-point, p-adic, open string amplitudes, with Chan-Paton rules in a constant B-field, have the form

$$\int_{\mathbb{Q}_p^N} \prod_{1\leq i<j\leq N} \left|x_i - x_j\right|_p^{\boldsymbol{k}_i\boldsymbol{k}_j} H_\tau\left(x_i - x_j\right)$$
$$\times \exp\left\{-\frac{\sqrt{-1}}{2}\left(\sum_{1\leq i<j\leq N}(\boldsymbol{k}_i\theta\boldsymbol{k}_j)\mathrm{sgn}_\tau(x_i - x_j)\right)\right\}\prod_{i=1}^{N}dx_i, \tag{1}$$

where $N \geq 4$, $\boldsymbol{k} = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N)$, $\boldsymbol{k}_i = \left(k_{0,i}, \ldots, k_{l,i}\right)$, $i = 1, \ldots, N$, is the momentum vector of the i-th tachyon vertex operator (with Minkowski product $\boldsymbol{k}_i\boldsymbol{k}_j = -k_{0,i}k_{0,j} + k_{1,i}k_{1,j} + \cdots + k_{l,i}k_{l,j}$) obeying

$$\sum_{i=1}^{N}\boldsymbol{k}_i = \boldsymbol{0}, \quad \boldsymbol{k}_i\boldsymbol{k}_i = 2 \text{ for } i = 1, \ldots, N, \tag{2}$$

$H_\tau(x_i) = \frac{1}{2}(1 + \mathrm{sgn}_\tau(x))$, $\mathrm{sgn}_\tau(x)$ is a p-adic version of the sign function, θ is a fixed antisymmetric bilinear form, and $\prod_{i=1}^{N}dx_i$ is the normalized Haar measure of $\mathbb{Q}_p^N$. A symmetry requirement for this theory is that the sign function be odd, meaning $\mathrm{sgn}_\tau(-x) = -\mathrm{sgn}_\tau(x)$.

Unfortunately, this theory is not invariant under projective Möbius transformations and consequently the normalization $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$ can not be carried out. This is a consequence of the fact that $\mathbb{Q}_p$ is not an ordered field. Anyway, in [50] the authors assumed a such normalization, which is equivalent to assuming that the vertex operators are inserted in the boundary of the Bruhat-Tits tree at 'non-generic points'. Taking the normalization $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$, the amplitude takes the form

$$A^{(N)}(\boldsymbol{k}, \theta, \tau) = \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{\boldsymbol{k}_1\boldsymbol{k}_i} |1 - x_i|_p^{\boldsymbol{k}_{N-1}\boldsymbol{k}_i} H_\tau(x_i) H_\tau(1 - x_i)$$
$$\times \prod_{2\leq i<j\leq N-2} \left|x_i - x_j\right|_p^{\boldsymbol{k}_i\boldsymbol{k}_j} H_\tau\left(x_i - x_j\right)$$
$$\times \exp\left\{-\frac{\sqrt{-1}}{2}\left(\sum_{1\leq i<j\leq N-1}(\boldsymbol{k}_i\theta\boldsymbol{k}_j)\mathrm{sgn}_\tau(x_i - x_j)\right)\right\}\prod_{i=2}^{N-2}dx_i. \tag{3}$$

We have called such integrals Ghoshal-Kawano amplitudes. The main goal of this article is the study of the amplitude $A^{(N)}(\boldsymbol{k}, \theta, \tau)$ using twisted multivariate Igusa's local zeta functions. We attach to $A^{(N)}(\boldsymbol{k}, \theta, \tau)$ the following Igusa type integral:

$$Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau) = \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} |1 - x_i|_p^{s_{(N-1)i}} H_\tau(x_i) H_\tau(1 - x_i)$$

$$\times \prod_{2\leq i<j\leq N-2} \left|x_i - x_j\right|_p^{s_{ij}} H_\tau\left(x_i - x_j\right)$$
$$\times \exp\left\{-\frac{\sqrt{-1}}{2}\left(\sum_{1\leq i<j\leq N-1} \widetilde{s}_{ij}\,\mathrm{sgn}_\tau(x_i - x_j)\right)\right\}\prod_{i=2}^{N-2} dx_i, \tag{4}$$

where the s_{ij} are complex symmetric variables and the $\widetilde{s}_{ij}$ are real antisymmetric variables. We have called integrals $Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$ Ghoshal-Kawano local zeta functions. As a consequence of the presence of the Chan-Paton factors, the normalization $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$ and the requirement that $\mathrm{sgn}_\tau(-x) = -\mathrm{sgn}_\tau(x)$, the integration in (4) is actually on $\mathbb{Z}_p^{N-3}$ (the $N-3$-dimensional unit ball). This fact implies that turning off the background B-field, the amplitude $A^{(N)}(\boldsymbol{k},\theta,\tau)$ does not reduce to the p-adic open string amplitude at the tree level. This fact was already noticed in [50], in the case $N = 4$. We show that integrals $Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$ can be expressed as finite sums of a new type of twisted multivariate Igusa's local zeta functions, and by using the techniques of [39,53], we establish that $Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$ admits a meromorphic continuation as a rational function in the variables $p^{-s_{1j}}$, $p^{-s_{(N-1)j}}$, $p^{-s_{ij}}$. Furthermore, $Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$ is holomorphic in

$$\bigcap_{\mathcal{H}}\left\{s_{ij}\in\mathbb{C}^{\mathbf{d}};\ \sum_{i,j\in M} N_{ij,k}\,\mathrm{Re}\left(s_{ij}\right)+\gamma_k>0,\ \text{for } k\in T\right\}, \tag{5}$$

where $N_{ij,k}\in\mathbb{N}$, $\gamma_k, \mathbf{d}\in\mathbb{N}\smallsetminus\{0\}$, and M, T are finite sets, and the real parts of its poles belong to the finite union of hyperplanes of type

$$\mathcal{H}=\left\{s_{ij}\in\mathbb{C}^{\mathbf{d}};\ \sum_{ij\in M} N_{ij,k}\,\mathrm{Re}\left(s_{ij}\right)+\gamma_k=0,\ \text{for } k\in T\right\}.$$

We regularize the amplitude $A^{(N)}(\boldsymbol{k},\theta,\tau)$ by redefining it as

$$A^{(N)}(\boldsymbol{k},\theta,\tau) = Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)\mid_{\substack{s_{ij}=\boldsymbol{k}_i\boldsymbol{k}_j\\ \widetilde{s}_{ij}=\boldsymbol{k}_i\theta\boldsymbol{k}_j}},$$

in this way $A^{(N)}(\boldsymbol{k},\theta,\tau)$ is a well-defined meromorphic function of the kinematic parameters $\boldsymbol{k}_i\boldsymbol{k}_j$, which agrees with integral (3), if it exists. As a consequence of the description of the poles of $Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$, $A^{(N)}(\boldsymbol{k},\theta,\tau)$ is defined for arbitrary large momenta, since in (5) the values $\boldsymbol{k}_i\boldsymbol{k}_j$ can take arbitrarily large values. This fact is not valid for the p-adic Koba-Nielsen amplitudes, see [38], and [31], since Ghoshal-Kawano amplitudes are supposed to be generalizations of the p-adic open amplitudes at the tree level, we conclude that the normalization $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$ is not possible in the presence of a background B-field. In a forthcoming article we expect to study amplitudes (1). It is worth to mention here that the Ghoshal-Kawano local zeta functions are new Igusa-type integrals coming from p-adic string theory.

The construction of a physical theory over a $p-$adic spacetime (worldsheet in our case) raises the question about the physical meaning of the prime p. The spacetime is a quadratic space $(\mathbb{Q}_p^N,\mathfrak{q})$, where $\mathfrak{q}$ is a quadratic form, and consequently, the spacetime depends on the pair $(p,\mathfrak{q})$. In this article, we require $p\equiv 3 \bmod 4$ and $\tau\in\{p,\varepsilon p\}$ in order to have the symmetry $\mathrm{sgn}_\tau(-x) = -\mathrm{sgn}_\tau(x)$.

This article is organized as follows. In section 2, we study the limit $p\to 1$ in the noncommutative version of the effective action discussed in [52]. We describe the noncommutative version of the Gerasimov-Shatashvili action and find explicitly its four-point amplitudes. In section 3, we review the basic aspects of the twisted, multivariate Igusa's local zeta functions. The local zeta functions required here are a variation of the ones considered in [53]. Sections 4-5 are dedicated to establish the meromorphic continuation of Ghoshal-Kawano local zeta functions. Sections 6 and 7 are devoted to give the explicit calculation for the 4-point and 5-point amplitudes. The 4-point amplitude was already obtained by Ghoshal and Kawano in [50] under certain hypotheses and the 5-point amplitude is new. In section 8, we compute the $p \to 1$ limit of the p-adic 4-point and 5-point amplitudes. We verified that the $p \to 1$ limit of 4-point amplitude coincides with the Feynman amplitude computed from the noncommutative Gerasimov-Shatashvili action in section 2. The final remarks are collected in section 9. Finally in the Appendix, we review the basic aspects of the p-adic analysis, and introduce some notations and conventions used along this article.

2. The limit $p \to 1$ in the effective action with a B-field

2.1. The limit $p \to 1$ in the noncommutative effective action

In [52], it was considered a noncommutative action as the effective action of the theory of p-adic open strings with a B-field. The corresponding action in the D-dimensional spacetime is given by

$$S(\phi) = \frac{1}{g^2}\frac{p^2}{p-1}\int d^Dx\left(-\frac{1}{2}\phi \star p^{-\frac{1}{2}\Delta}\phi + \frac{1}{p+1}(\star\phi)^{p+1}\right), \tag{6}$$

where g and Δ are the coupling constant and the Laplacian, respectively, and $(\star\phi)^p$ is defined by $\phi \star \phi \star \cdots \star \phi$ p-times. Here $\star$ is the Moyal star product, which is defined for any suitable pair of smooth functions f and g as

$$(f \star g)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial z^\nu}\right) f(x+y)g(x+z)\bigg|_{y=z=0}.$$

The corresponding equation of motion is given by

$$p^{-\frac{1}{2}\Delta}\phi = (\star\phi)^p. \tag{7}$$

The solutions of this equation are defined in the target space $\mathbb{R}^D$, where p plays the role of a real parameter. In particular the limit p approaches to one makes sense.

Now following [42], by considering the Taylor expansion of $\exp(-\frac{1}{2}\Delta \log p)$ and $\exp(p\log(\star\phi))$ at $p = 1$, and keeping only the linear term, we get

$$\Delta\phi = -2\phi \star \log(\star\phi), \tag{8}$$

where $\log(\star\phi) = \phi - \frac{1}{2}\phi \star \phi + \frac{1}{3}\phi \star \phi \star \phi - \cdots$. Thus the heuristic $p \to 1$ limit leads to a noncommutative version of the Gerasimov and Shatashvili Lagrangian:

$$S(\phi) = \int d^Dx\Big((\partial\phi)^2 - V(\star\phi)\Big), \tag{9}$$

where

$$V(\phi) = (\star\phi)^2 \star \log\left[\frac{(\star\phi)^2}{e}\right].$$

In noncommutative field theory, it is well known that the nontrivial noncommutative effect comes from the potential energy of the Lagrangian. The propagators associated with the kinetic energy of the Lagrangian are the same as the ones of the commutative theory. Thus the free Lagrangian with an external source $J(x)$ is

$$S_0(\phi) = \int d^D x\big[(\partial\phi)^2 + \phi^2(x) + J(x)\phi(x)\big].$$

The propagators are given by $x_{ij} = \frac{1}{\boldsymbol{k}_i \cdot \boldsymbol{k}_{j+1}}$, where k_i, with $i = 1, \dots, N$, are the external momenta of the particles. The Feynman rule for the interaction vertex can be obtained in the noncommutative theory by considering the cubic, quartic, etc. interaction terms and computing the correlation functions, see for instance, [54,55].

2.2. *Amplitudes from the noncommutative Gerasimov-Shatashvili Lagrangian*

In this subsection we show how to extract the four-point amplitudes from the noncommutative Gerasimov-Shatashvili Lagrangian (9). In order to do that, we first require to study the interacting theory. The generating functional of the correlation function for the free theory is given by

$$\mathcal{Z}_0[J] = \mathcal{N}[\det(\Delta - 1)]^{-1/2} \exp\left\{ -\frac{i}{2\hbar} \int d^D x \int d^D x' J(x) G_F(x - x') J(x') \right\},$$

where $G_F(x-x')$ is the Green function of time-ordered product of two fields of the theory, $\mathcal{N}$ is a normalization constant, $[\det(\Delta - 1)]^{-1/2}$ is a suitable regularization of the divergent determinant bosonic operator. The noncommutative action is given by

$$S(\phi) = \int d^D x\big[(\partial\phi)^2 + \phi^2 - U(\star\phi)\big], \tag{10}$$

where $U(\star\phi) = 2(\star\phi)^2 \star \log(\star\phi)$. We expand $U(\star\phi)$ in Taylor series as follows:

$$U(\star\phi) = A\phi\star\phi + B\phi\star\phi\star\phi + C\phi\star\phi\star\phi\star\phi + \cdots, \tag{11}$$

where $A = -\frac{25}{6}$, $B = 8$ and $C = -6$.

The generating $Z[J]$ functional incorporating the interaction is given by

$$\begin{aligned}
\mathcal{Z}[J] = \exp\Bigg\{ & \frac{25i}{6\hbar} \int d^D x \Big(-i\hbar \frac{\delta}{\delta J(x)} \Big) \star \Big(-i\hbar \frac{\delta}{\delta J(x)} \Big) \\
& - \frac{8i}{\hbar} \int d^D x \Big(-i\hbar \frac{\delta}{\delta J(x)} \Big) \star \Big(-i\hbar \frac{\delta}{\delta J(x)} \Big) \star \Big(-i\hbar \frac{\delta}{\delta J(x)} \Big) \\
& + \frac{6i}{\hbar} \int d^D x \Big(-i\hbar \frac{\delta}{\delta J(x)} \Big) \star \Big(-i\hbar \frac{\delta}{\delta J(x)} \Big) \star \Big(-i\hbar \frac{\delta}{\delta J(x)} \Big) \\
& \star \Big(-i\hbar \frac{\delta}{\delta J(x)} \Big) + \cdots \Bigg\} \mathcal{Z}_0[J].
\end{aligned} \tag{12}$$

We are interested in checking whether connected tree-level scattering amplitudes of this theory match exactly with the corresponding p-adic amplitudes in the limit when p tends to one. The computation of the field theory performed here will be compared to the computation of the p-adic string amplitudes at section 8.

2.3. Four-point amplitudes

In this subsection we consider the quartic term from the potential (11). The expansion of the exponential function of this term in the interacting generating functional is expressed as

$$\begin{aligned}\mathcal{Z}[J] &= \cdots + 6i\hbar^3 \int d^Dx \left\{ \left(\frac{\delta}{\delta J(x)}\right) \star \left(\frac{\delta}{\delta J(x)}\right) \star \left(\frac{\delta}{\delta J(x)}\right) \star \left(\frac{\delta}{\delta J(x)}\right) \right\} \mathcal{Z}_0[J] + \cdots \\ &= \cdots + 6i\hbar^3 \lim_{x=y_1=y_2=y_3=y_4} \lim_{w_1=w_2=w_3=w_4=0} \int d^Dy_1 d^Dy_2 d^Dy_3 d^Dy_4 \\ &\quad \times \exp\left\{ \frac{i}{2}\theta^{\mu_1\nu_1} \frac{\partial}{\partial w_1^{\mu_1}} \frac{\partial}{\partial w_2^{\nu_1}} \right\} \exp\left\{ \frac{i}{2}\theta^{\mu_2\nu_2} \frac{\partial}{\partial w_3^{\mu_2}} \frac{\partial}{\partial w_4^{\nu_2}} \right\} \\ &\quad \times \left\{ \left(\frac{\delta}{\delta J(y_1+w_1)}\right)\left(\frac{\delta}{\delta J(y_2+w_2)}\right)\left(\frac{\delta}{\delta J(y_3+w_3)}\right)\left(\frac{\delta}{\delta J(y_4+w_4)}\right) \right\} \mathcal{Z}_0[J] \\ &\quad + \cdots . \end{aligned} \tag{13}$$

A straightforward computation of the 4-point vertex gives

$$\begin{aligned} &\left.\frac{\delta^4 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0} \\ &= \cdots + 768i\hbar^3 \int d^Dx \left\{ \left\{ \cos\left(\frac{\partial_1\theta\partial_2}{2}\right)\cos\left(\frac{\partial_3\theta\partial_4}{2}\right) + \cos\left(\frac{\partial_1\theta\partial_3}{2}\right)\cos\left(\frac{\partial_2\theta\partial_4}{2}\right) \right.\right. \\ &\quad \left. + \cos\left(\frac{\partial_1\theta\partial_4}{2}\right)\cos\left(\frac{\partial_2\theta\partial_3}{2}\right) \right\} \left[-\frac{i}{2\hbar} G_F(x-x_1) \right] \left[-\frac{i}{2\hbar} G_F(x-x_2) \right] \\ &\quad \times \left[-\frac{i}{2\hbar} G_F(x-x_3) \right] \left[-\frac{i}{2\hbar} G_F(x-x_4) \right] + \cdots , \end{aligned} \tag{14}$$

where $G_F(x-y)$ is the propagator and $\partial_{1,2,3,4}$ are the partial derivative with respect to the coordinates x_1, x_2, x_3 and x_4, respectively.

The interaction term $8(\star\phi)^3$ in the Lagrangian gives also a non-vanishing contribution to the 4-point tree amplitudes of the second order in perturbation theory. They are described by Feynman diagrams with two vertices located at points y and z connected by a propagator $G_F(y-z)$ and with two external legs attached to each vertex. In this case the amplitude is computed from the relevant part of the generating functional:

$$\mathcal{Z}[J] = \cdots + 64\hbar^4 \int d^Dy \int d^Dz \left(\star \frac{\delta}{\delta J(y)}\right)^3 \left(\star \frac{\delta}{\delta J(z)}\right)^3 \mathcal{Z}_0[J] + \cdots . \tag{15}$$

This expression can be written explicitly in terms of the Moyal product as

$$\begin{aligned} &\mathcal{Z}[J] \\ &= \cdots + 64\hbar^4 \lim_{y=y_1=y_2=y_3} \lim_{z=z_1=z_2=z_3} \lim_{w_1=w_2=w_3=w_4=0} \int d^Dy_1 d^Dy_2 d^Dy_3 d^Dz_1 d^Dz_2 d^Dz_3 \\ &\quad \times \exp\left\{ \frac{i}{2}\theta^{\mu_1\nu_1} \frac{\partial}{\partial w_1^{\mu_1}} \frac{\partial}{\partial w_2^{\nu_1}} \right\} \exp\left\{ \frac{i}{2}\theta^{\mu_2\nu_2} \frac{\partial}{\partial w_3^{\mu_2}} \frac{\partial}{\partial w_4^{\nu_2}} \right\} \\ &\quad \times \left\{ \left(\frac{\delta}{\delta J(y_1+w_1)}\right)\left(\frac{\delta}{\delta J(y_2+w_2)}\right)\left(\frac{\delta}{\delta J(y_3)}\right) \right\} \end{aligned}$$

$$\times \left\{ \left(\frac{\delta}{\delta J(z_1 + w_3)} \right) \left(\frac{\delta}{\delta J(z_2 + w_4)} \right) \left(\frac{\delta}{\delta J(z_3)} \right) \right\} \mathcal{Z}_0[J] + \cdots .$$

The connected 4-point amplitudes at the second order coming from the cubic interaction $8(\star\phi)^3$ yields to

$$\begin{aligned}
&\frac{\delta^4 \mathcal{Z}[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0} \\
&= \cdots + 8192\hbar^4 \int d^D y \int d^D z \left[-\frac{i}{2\hbar} G_F(y-z) \right] \\
&\quad \times \left\{ \cos\left(\frac{\partial_1 \theta \partial_2}{2}\right) \cos\left(\frac{\partial_3 \theta \partial_4}{2}\right) \right. \\
&\quad \times \left\{ \left[-\frac{i}{2\hbar} G_F(y-x_1) \right] \left[-\frac{i}{2\hbar} G_F(y-x_2) \right] \left[-\frac{i}{2\hbar} G_F(z-x_3) \right] \left[-\frac{i}{2\hbar} G_F(z-x_4) \right] \right. \\
&\quad + \left. \left[-\frac{i}{2\hbar} G_F(z-x_1) \right] \left[-\frac{i}{2\hbar} G_F(z-x_2) \right] \left[-\frac{i}{2\hbar} G_F(y-x_3) \right] \left[-\frac{i}{2\hbar} G_F(y-x_4) \right] \right\} \\
&\quad + \cos\left(\frac{\partial_1 \theta \partial_3}{2}\right) \cos\left(\frac{\partial_2 \theta \partial_4}{2}\right) \\
&\quad \times \left\{ \left[-\frac{i}{2\hbar} G_F(y-x_1) \right] \left[-\frac{i}{2\hbar} G_F(z-x_2) \right] \left[-\frac{i}{2\hbar} G_F(y-x_3) \right] \left[-\frac{i}{2\hbar} G_F(z-x_4) \right] \right. \\
&\quad + \left. \left[-\frac{i}{2\hbar} G_F(z-x_1) \right] \left[-\frac{i}{2\hbar} G_F(y-x_2) \right] \left[-\frac{i}{2\hbar} G_F(z-x_3) \left[-\frac{i}{2\hbar} G_F(y-x_4) \right] \right] \right\} \\
&\quad + \cos\left(\frac{\partial_1 \theta \partial_4}{2}\right) \cos\left(\frac{\partial_2 \theta \partial_3}{2}\right) \\
&\quad \times \left\{ \left[-\frac{i}{2\hbar} G_F(z-x_1) \right] \left[-\frac{i}{2\hbar} G_F(y-x_2) \right] \left[-\frac{i}{2\hbar} G_F(y-x_3) \right] \left[-\frac{i}{2\hbar} G_F(z-x_4) \right] \right. \\
&\quad + \left. \left. \left[-\frac{i}{2\hbar} G_F(y-x_1) \right] \left[-\frac{i}{2\hbar} G_F(z-x_2) \right] \left[-\frac{i}{2\hbar} G_F(z-x_3) \left[-\frac{i}{2\hbar} G_F(y-x_4) \right] \right] \right\} \right\} \\
&\quad + \cdots . \qquad (16)
\end{aligned}$$

This total amplitude corresponds exactly to the sum of the partial amplitudes associated to the channels s, t and u. The sum of (16) and (14) constitutes the 4-point amplitude (at the tree-level). This amplitude agrees with the limit $p \to 1$ of the sum over the permutations of the momenta $\boldsymbol{k}_i$ of the 4-point p-adic amplitudes computed in section 6. The details of this calculation are given in section 8. Moreover, five-point non-commutative amplitudes (and higher-order amplitudes) in the limit $p \to 1$ can be computed following a similar procedure, but it will not be performed here.

3. Multivariate local zeta functions

For the notation and the definition of basic objects such as multiplicative characters, sign functions, Haar measure, etc., the reader may consult the Appendix. In this section we review some basic aspects of the twisted multivariate local zeta functions. The meromorphic continuation of the local zeta functions play a central role in sections 4 and 5.

Let $f_1(x), \ldots, f_m(x) \in \mathbb{Q}_p[x_1, \ldots, x_n]$ be non-constant polynomials, we denote by $\mathbb{D} := \cup_{i=1}^m f_i^{-1}(0)$ the divisor attached to them. Let $\chi_1, \ldots, \chi_m$ be multiplicative characters. We set $\boldsymbol{f} := (f_1, \ldots, f_m)$, $\boldsymbol{\chi} := (\chi_1, \ldots, \chi_m)$, and $\boldsymbol{s} := (s_1, \ldots, s_m) \in \mathbb{C}^m$. The multivariate local zeta function attached to $(\boldsymbol{f}, \boldsymbol{\chi}, \Theta)$, with Θ a test function (i.e. a locally constant function with compact support), is defined as

$$Z_\Theta(\boldsymbol{s}, \boldsymbol{\chi}, \boldsymbol{f}) = \int\limits_{\mathbb{Q}_p^n \smallsetminus \mathbb{D}} \Theta(x) \prod_{i=1}^m \chi_i(ac(f_i(x))) \, |f_i(x)|_p^{s_i} \prod_{i=1}^n dx_i, \tag{17}$$

with $\mathrm{Re}(s_i) > 0$ for all i. Integrals of type (17) are holomorphic functions in $\boldsymbol{s}$, which admit meromorphic continuations to the whole $\mathbb{C}^m$, [53, Théorème 1.1.4.], see also [39]. More precisely, the integrals $Z_\Theta(\boldsymbol{s}, \boldsymbol{\chi}, \boldsymbol{f})$ admit meromorphic continuations as rational functions in the variables $p^{-s_1}, \ldots, p^{-s_m}$. Let us emphasize that the notation $\chi_i(ac(x))$, $x \neq 0$, means that the character χ_i depends only on the angular component of x, see Appendix.

We need a variation of the Loeser result [53, Théorème 1.1.4.], more precisely, when each $\chi_i \circ ac$ is the trivial character $\chi_{\text{triv}}(x)$ or $\mathrm{sgn}_\tau(x)$. We denote by χ_i one of these characters. This last function is a multiplicative character on $\mathbb{Q}_p^\times$, but it depends on the angular component of x and on the order of x. By using Hironaka's resolution of singularities theorem, see e.g. [39], $Z_\Theta(\boldsymbol{s}, \boldsymbol{\chi}, \boldsymbol{f})$ can be written as linear combination of integrals of type

$$\int\limits_{c + p^e \mathbb{Z}_p^n} \prod_{j=1}^r \left\{ |y_j|_p^{\sum_{i=1}^m N_{i,j} s_i + v_j - 1} \chi_i^{N_{i,j}}(y_j) \right\} dy_j,$$

where $c = (c_1, \ldots, c_n) \in \mathbb{Q}_p^n$, $1 \le r \le n$, $N_{i,j}$ are nonnegative integers for $i \in \{1, \ldots, m\}$, $j \in \mathcal{T}$, and v_j a positive integer, for $j \in \mathcal{T}$ (a finite set), see proof of [39, Theorem 8.2.1] and [53, Théorème 1.1.4.].

Then, we have to study the meromorphic continuation of an integral of type

$$I(s) := \int\limits_{c_j + p^e \mathbb{Z}_p} |y_j|_p^{\sum_{i=1}^m N_{i,j} s_i + v_j - 1} \mathrm{sgn}_\tau^{N_{i,j}}(y_j) \, dy_j,$$

since the one corresponding to the trivial character is already known, see e.g. [39, Lemma 8.2.1]. Several cases occur. If $c_j \notin p^e \mathbb{Z}_p$, by using the fact that $|\cdot|_p$ and $\mathrm{sgn}_\tau(\cdot)$ are locally constant functions we get that

$$I(s) = p^{-e} \, |c_j|_p^{\sum_{i=1}^m N_{i,j} s_i + v_j - 1} \mathrm{sgn}_\tau^{N_{i,j}}(c_j).$$

In the case $c_j \in p^e \mathbb{Z}_p$, we have

$$\begin{aligned} I(s) &= \sum_{l=e}^{\infty} \int\limits_{p^l \mathbb{Z}_p^\times} |y_j|_p^{\sum_{i=1}^m N_{i,j} s_i + v_j - 1} \mathrm{sgn}_\tau^{N_{i,j}}(y_j) \, dy_j \\ &= \left\{ \sum_{l=e}^{\infty} p^{-l(\sum_{i=1}^m N_{i,j} s_i + v_j)} \mathrm{sgn}_\tau^{N_{i,j}}\left(p^l\right) \right\} \int\limits_{\mathbb{Z}_p^\times} \mathrm{sgn}_\tau^{N_{i,j}}(u) \, du \\ &=: J(s) \int\limits_{\mathbb{Z}_p^\times} \mathrm{sgn}_\tau^{N_{i,j}}(u) \, du, \end{aligned}$$

where $y_j = p^l u$.

Now if $\tau = \varepsilon$, $\mathrm{sgn}_\tau(u) = (-1)^{ord(u)} \equiv 1$ for any $u \in \mathbb{Z}_p^\times$, then $\int_{\mathbb{Z}_p^\times} \mathrm{sgn}_\varepsilon^{N_{i,j}}(u)\, du = 1 - p^{-1}$. In the case $\tau \neq \varepsilon$,

$$\int\limits_{\mathbb{Z}_p^\times} \mathrm{sgn}_\tau^{N_{i,j}}(u)\, du = \begin{cases} 1 - p^{-1} & \text{if} \quad N_{i,j} \text{ is even} \\ 0 & \text{if} \quad N_{i,j} \text{ is odd.} \end{cases}$$

By using that

$$\mathrm{sgn}_\tau^{N_{i,j}}\left(p^l\right) = \mathrm{sgn}_\tau^{lN_{i,j}}(p) = \begin{cases} 1 & \text{if} \quad l \text{ is even} \\ \mathrm{sgn}_\tau^{N_{i,j}}(p) & \text{if} \quad l \text{ is odd,} \end{cases}$$

we have

$$\begin{aligned} J(s) &= \sum_{l=e}^{\infty} p^{-l\left(\sum_{i=1}^m N_{i,j}s_i + v_j\right)} \mathrm{sgn}_\tau^{lN_{i,j}}(p) \\ &= \sum_{k=0}^{\infty} p^{-(k+e)\left(\sum_{i=1}^m N_{i,j}s_i + v_j\right)} \mathrm{sgn}_\tau^{N_{i,j}}\left(p^{k+e}\right) \\ &= p^{-e\left(\sum_{i=1}^m N_{i,j}s_i + v_j\right)} \mathrm{sgn}_\tau^{N_{i,j}}\left(p^e\right) \sum_{k=0}^{\infty} p^{-k\left(\sum_{i=1}^m N_{i,j}s_i + v_j\right)} \mathrm{sgn}_\tau^{kN_{i,j}}(p). \end{aligned}$$

If $N_{f_i,j}$ is even

$$\begin{aligned} J(s) &= p^{-e\left(\sum_{i=1}^m N_{i,j}s_i + v_j\right)} \sum_{k=0}^{\infty} p^{-k\left(\sum_{i=1}^m N_{i,j}s_i + v_j\right)} \\ &= \frac{p^{-e\left(\sum_{i=1}^m N_{i,j}s_i + v_j\right)}}{1 - p^{-\sum_{i=1}^m N_{i,j}s_i - v_j}}. \end{aligned}$$

If $N_{i,j}$ is odd, we have $I(s) = 0$. In conclusion, since $Z_\Theta(\boldsymbol{s}, \boldsymbol{\chi}, \boldsymbol{f})$ is a finite linear combination of products of integrals of type $I(s)$, then $Z_\Theta(\boldsymbol{s}, \boldsymbol{\chi}, \boldsymbol{f})$ admits a meromorphic continuation as a rational function in the variables $p^{-s_1}, \ldots, p^{-s_m}$. More precisely,

$$Z_\Theta(\boldsymbol{s}, \boldsymbol{\chi}, \boldsymbol{f}) = \frac{L_{\Theta,\boldsymbol{\chi}}(\boldsymbol{s})}{\prod\limits_{j\in\mathcal{T}} \left(1 - p^{-\sum_{i=1}^m N_{i,j}s_j - v_j}\right)}, \tag{18}$$

where $L_{\Theta,\boldsymbol{\chi}}(\boldsymbol{s})$ is a polynomial in the variables $p^{-s_1}, \ldots, p^{-s_m}$, and the real parts of its poles belong to the finite union of hyperplanes

$$\sum_{i=1}^m N_{i,j}\,\mathrm{Re}(s_i) + v_j = 0, \qquad \text{for } j \in \mathcal{T}.$$

This result is a variation of [53, Théorème 1.1.4.].

4. The Ghoshal-Kawano local zeta function

From now on, we use θ to denote a fixed antisymmetric bilinear form. In [50] Ghoshal and Kawano proposed the following amplitude (for the N-point tree-level, p-adic open string amplitude, with Chan-Paton rules in a constant B-field):

$$A^{(N)}\left(\boldsymbol{k},\theta,\tau;x_1,x_{N-1}\right):=\int_{\mathbb{Q}_p^{N-3}\smallsetminus\mathbb{D}}\prod_{i=2}^{N-2}|x_i|_p^{\boldsymbol{k}_1\boldsymbol{k}_i}\,|1-x_i|_p^{\boldsymbol{k}_{N-1}\boldsymbol{k}_i}\,H_\tau\left(x_i\right)H_\tau\left(1-x_i\right)$$
$$\times\prod_{2\leq i<j\leq N-2}\left|x_i-x_j\right|_p^{\boldsymbol{k}_i\boldsymbol{k}_j}H_\tau\left(x_i-x_j\right)$$
$$\times\ \exp\left\{-\frac{\sqrt{-1}}{2}\left(\sum_{1\leq i<j\leq N-1}(\boldsymbol{k}_i\theta\boldsymbol{k}_j)\mathrm{sgn}_\tau(x_i-x_j)\right)\right\}\prod_{i=2}^{N-2}dx_i,$$

where $N\geq 4$, $\boldsymbol{k}=(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N)$, $\boldsymbol{k}_i=\left(k_{0,i},\ldots,k_{l,i}\right)$, $i=1,\ldots,N$, is the momentum vector of the i-th tachyon (with Minkowski product $\boldsymbol{k}_i\boldsymbol{k}_j=-k_{0,i}k_{0,j}+k_{1,i}k_{1,j}+\cdots+k_{l,i}k_{l,j}$) obeying to (2) and $\prod_{i=2}^{N-2}dx_i$ is the normalized Haar measure of $\mathbb{Q}_p^{N-3}$, and

$$\mathbb{D}:=\left\{(x_2,\ldots,x_{N-2})\in\mathbb{Q}_p^{N-3};\ \prod_{i=2}^{N-2}x_i\left(1-x_i\right)\prod_{2\leq i<j\leq N-2}\left(x_i-x_j\right)=0\right\}.$$

In the bosonic string theory $l=26$, however, this choice of the dimension does not play any role in our calculations.

In order to study the amplitude $\boldsymbol{A}^{(N)}\left(\underline{\boldsymbol{k}},\theta,\tau;x_1,x_{N-1}\right)$, we introduce

$$\boldsymbol{s}=\left(s_{ij}\right)=\cup_{i=2}^{N-2}\left\{s_{1i},s_{(N-1)i}\right\}\cup\cup_{2\leq i<j\leq N-2}\left\{s_{ij}\right\}\in\mathbb{C}^{\mathbf{d}}$$

a list consisting of $\mathbf{d}$ complex variables, satisfying $s_{ij}=s_{ji}$ for any i and j, where

$$\mathbf{d}:=\begin{cases}2(N-3)+\binom{N-3}{2} & \text{if}\quad N\geq 5\\ 2 & \text{if}\quad N=4\end{cases}=\frac{N\left(N-3\right)}{2}.$$

Furthermore, we introduce the variables $\widetilde{s}_{ij}\in\mathbb{R}$, for $1\leq i<j\leq N-1$. We denote by $\widetilde{\boldsymbol{s}}=\left(\widetilde{s}_{ij}\right)$ for $1\leq i<j\leq N-1$. We set

$$F(\boldsymbol{x},\boldsymbol{s},\tau):=\prod_{i=2}^{N-2}|x_i|_p^{s_{1i}}|1-x_i|_p^{s_{(N-1)i}}H_\tau(x_i)H_\tau(1-x_i)\prod_{2\leq i<j\leq N-2}|x_i-x_j|_p^{s_{ij}}H_\tau(x_i-x_j),$$

and

$$E(\boldsymbol{x},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1}):=\exp\left\{\frac{-\sqrt{-1}}{2}\left(\sum_{2\leq j\leq N-1}\widetilde{s}_{1j}\mathrm{sgn}_\tau(x_1-x_j)\right)\right\}\times$$
$$\exp\left\{\frac{-\sqrt{-1}}{2}\left(\sum_{2\leq i\leq N-2}\widetilde{s}_{i(N-1)}\mathrm{sgn}_\tau(x_i-x_{N-1})\right)\right\}\times\qquad(19)$$

$$\exp\left\{\frac{-\sqrt{-1}}{2}\left(\sum_{2\leq i<j\leq N-2}\widetilde{s}_{ij}\operatorname{sgn}_{\tau}(x_i-x_j)\right)\right\}.$$

Later on, we will use the convention $x_1=0$, $x_{N-1}=1$ and $x_N=\infty$. Now, we define the Ghoshal-Kawano local zeta function as

$$Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1})=\int\limits_{\mathbb{Q}_p^{N-3}\smallsetminus\mathbb{D}}F(\boldsymbol{x},\boldsymbol{s},\tau)E(\boldsymbol{x},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1})\prod_{i=2}^{N-2}dx_i. \tag{20}$$

For the sake of simplicity, from now on, we will use $\mathbb{Q}_p^{N-3}$ as domain of integration in (20).

By using that $|E(\boldsymbol{x},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1})|=1$, $|H_\tau(x_i)|\leq 1$, $|H_\tau(1-x_i)|\leq 1$, for any i, and that $\left|H_\tau(x_i-x_j)\right|\leq 1$, for any i, j, we have

$$\begin{aligned}&\left|Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1})\right|\\&\leq\int\limits_{\mathbb{Q}_p^{N-3}}\prod_{i=2}^{N-2}|x_i|_p^{\operatorname{Re}(s_{1i})}\,|1-x_i|_p^{\operatorname{Re}\left(s_{(N-1)i}\right)}\prod_{2\leq i<j\leq N-2}\left|x_i-x_j\right|_p^{\operatorname{Re}\left(s_{ij}\right)}\prod_{i=2}^{N-2}dx_i\\&=Z^{(N)}\left(\operatorname{Re}(\boldsymbol{s})\right),\end{aligned}$$

where $Z^{(N)}(\boldsymbol{s})$ is the Koba-Nielsen string amplitude studied in [38], see also [31]. Since this last integral is holomorphic in an open set $\mathcal{K}\subset\mathbb{C}^{\mathbf{d}}$, we conclude that

$$Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)\text{ is holomorphic in }\boldsymbol{s}\in\mathcal{K}\text{ for any }\widetilde{\boldsymbol{s}},\,\tau,\,x_1,\,x_{N-1}.$$

We set $T:=\{2,\ldots,N-2\}$, and define for $I\subseteq T$, *the sector attached to* I as

$$Sect(I)=\left\{(x_2,\ldots,x_{N-2})\in\mathbb{Q}_p^{N-3};\,|x_i|_p\leq 1\ \Leftrightarrow i\in I\right\}.$$

Then $\mathbb{Q}_p^{N-3}=\bigsqcup_{I\subseteq T}Sect(I)$ and

$$Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1})=\sum_{I\subseteq T}Z_I^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1}), \tag{21}$$

where

$$Z_I^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1}):=\int\limits_{Sect(I)}F(\boldsymbol{x},\boldsymbol{s},\tau)E(\boldsymbol{x},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1})\prod_{i=2}^{N-2}dx_i.$$

We now notice that $Z_I^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1})\equiv 0$ if $I\neq T$. Indeed, in the case $I^c=T\smallsetminus I\neq\emptyset$, $F(\boldsymbol{x},\boldsymbol{s},\tau)\equiv 0$ due to the fact $H_\tau(x)H_\tau(-x)$ appears as a factor in $F(\boldsymbol{x},\boldsymbol{s},\tau)$, and that $H_\tau(x)H_\tau(-x)=0$. For this reason, we redefine the Ghoshal-Kawano local zeta function as

$$Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1})=\int\limits_{\mathbb{Z}_p^{N-3}}F(\boldsymbol{x},\boldsymbol{s},\tau)E(\boldsymbol{x},\widetilde{\boldsymbol{s}},\tau;x_1,x_{N-1})\prod_{i=2}^{N-2}dx_i. \tag{22}$$

5. Meromorphic continuation of Ghoshal-Kawano local zeta function

5.1. Some formulae

For $\widetilde{s} \in \mathbb{R}$ and $x \in \mathbb{Q}_p \smallsetminus \{0\}$,

$$\exp\left\{\frac{-\sqrt{-1}\widetilde{s}}{2}\operatorname{sgn}_\tau(x)\right\} = \cos\left(\frac{\widetilde{s}}{2}\right) - \sqrt{-1}\,\operatorname{sgn}_\tau(x)\sin\left(\frac{\widetilde{s}}{2}\right). \tag{23}$$

By using this formula, and the convention $x_1 = 0$, $x_{N-1} = 1$, we obtain that

$$\exp\left\{\frac{-\sqrt{-1}}{2}\left(\sum_{2\le j\le N-1}\widetilde{s}_{1j}\operatorname{sgn}_\tau(-x_j)\right)\right\} = \sum_{I\subseteq\{2,\ldots,N-1\}} C_I(\widetilde{\boldsymbol{s}})\prod_{j\in I}\operatorname{sgn}_\tau(x_j);$$

$$\exp\left\{\frac{-\sqrt{-1}}{2}\left(\sum_{2\le j\le N-1}\widetilde{s}_{i(N-1)}\operatorname{sgn}_\tau(x_j-1)\right)\right\} = \sum_{J\subseteq\{2,\ldots,N-1\}} D_J(\widetilde{\boldsymbol{s}})\prod_{j\in J}\operatorname{sgn}_\tau(1-x_j);$$

$$\exp\left\{\frac{-\sqrt{-1}}{2}\left(\sum_{2\le i<j\le N-2}\widetilde{s}_{ij}\operatorname{sgn}_\tau(x_i-x_j)\right)\right\}$$
$$= \sum_{K\subseteq\{2\le i<j\le N-2\}} D_K(\widetilde{\boldsymbol{s}})\prod_{i,j\in K}\operatorname{sgn}_\tau(x_i-x_j),$$

with the convention that $\prod_{j\in\emptyset}\equiv 1$. Here, using (23), we can see that C_I, D_J and D_K are complex functions depending on $\cos\left(\frac{\widetilde{s_{ij}}}{2}\right)$ and $\sin\left(\frac{\widetilde{s_{ij}}}{2}\right)$. In conclusion,

$$E(\boldsymbol{x},\widetilde{\boldsymbol{s}},\tau) := \sum_{I,J,K} E_{I,J,K}(\widetilde{\boldsymbol{s}})\prod_{j\in I}\operatorname{sgn}_\tau(x_j)\prod_{j\in J}\operatorname{sgn}_\tau(1-x_j)\prod_{i,j\in K}\operatorname{sgn}_\tau(x_i-x_j). \tag{24}$$

In a similar way, we obtain that

$$\prod_{i=2}^{N-2} H_\tau(x_i)H_\tau(1-x_i)\prod_{2\le i<j\le N-2} H_\tau(x_i-x_j)$$
$$= \sum_{I,J,K} e_{I,J,K}\prod_{j\in I}\operatorname{sgn}_\tau(x_j)\prod_{j\in J}\operatorname{sgn}_\tau(1-x_j)\prod_{i,j\in K}\operatorname{sgn}_\tau(x_i-x_j), \tag{25}$$

where the $e_{I,J,K}$s are constants.

5.2. Meromorphic continuation of $Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$

We assume the normalization $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$, and denote the corresponding Ghoshal-Kawano zeta function as $Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$. By using formulae (23)-(25) and (22), $Z^{(N)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$ is a finite sum of integrals of type

$$C(\widetilde{\boldsymbol{s}})\int_{\mathbb{Z}_p^{N-3}}\prod_{i=2}^{N-2}|x_i|_p^{s_{1i}}|1-x_i|_p^{s_{(N-1)i}}\prod_{2\le i<j\le N-2}|x_i-x_j|_p^{s_{ij}}\prod_{j\in I}\chi_\tau(x_j)\prod_{j\in J}\chi_\tau(1-x_j)$$

$$\times \prod_{i,j\in K} \chi_\tau(x_i - x_j) \prod_{i=2}^{N-2} dx_i,$$

where $C(\widetilde{s})$ is an $\mathbb{R}$-analytic function, χ_τ denotes the trivial character or sgn_τ. This formula implies that $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ is a linear combination of multivariate Igusa local zeta functions with coefficients in the ring of $\mathbb{R}$-analytic functions in the variables $\widetilde{\boldsymbol{s}}$. Consequently, by (18), $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ admits a meromorphic continuation as a rational function in the variables $p^{-s_{1j}}$, $p^{-s_{(N-1)j}}$, $p^{-s_{ij}}$ and the real parts of its poles belong to the finite union of hyperplanes of type

$$\mathcal{H} = \left\{ s_{ij} \in \mathbb{C}^{\mathbf{d}};\ \sum_{ij\in M} N_{ij,k} \operatorname{Re}\left(s_{ij}\right) + \gamma_k = 0,\ \text{for } k \in T \right\},$$

where $N_{ij,k} \in \mathbb{N}$, $\gamma_k \in \mathbb{N} \smallsetminus \{0\}$, and M, T are finite sets. Furthermore, $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ is holomorphic in

$$\bigcap_{\mathcal{H}} \left\{ s_{ij} \in \mathbb{C}^{\mathbf{d}};\ \sum_{ij\in M} N_{ij,k} \operatorname{Re}\left(s_{ij}\right) + \gamma_k > 0,\ \text{for } k \in T \right\}.$$

5.3. Meromorphic continuation of $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ without the normalization $x_1 = 0$, $x_{N-1} = 1$

The Ghoshal-Kawano local zeta function depends on x_1, x_{N-1}, i.e. $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1})$. In [50], the corresponding amplitude was considered in the case $x_1 = 0, x_{N-1} = 1, x_N = \infty$. Our result about the meromorphic continuation of $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$ is also valid for $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1})$. Indeed, by using that

$$\mathbb{Z}_p^{N-3} = \bigsqcup_{i=1}^{W} \boldsymbol{a}_i + p^L \mathbb{Z}_p^{N-3},$$

where W, L are positive integers, later on we will require that L be sufficiently large, and $\boldsymbol{a}_i \in \mathbb{Z}_p^{N-3}$ for any i. With this notation, we have

$$Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1}) = \sum_{i=1}^{W} Z_{\boldsymbol{a}_i}^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1}),$$

where

$$Z_{\boldsymbol{b}}^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1}) := \int_{\boldsymbol{b}+p^L\mathbb{Z}_p^{N-3}} F(\boldsymbol{x}, \boldsymbol{s}, \tau) E(\boldsymbol{x}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1}) \prod_{i=2}^{N-2} dx_i,$$

see (19). The meromorphic continuation of $Z_{\boldsymbol{b}}^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau; x_1, x_{N-1})$ can be obtained by the methods presented in Sections 5.1-5.2, by computing a Taylor expansion of the polynomial $\prod_{i=2}^{N-2} x_i (1 - x_i) \prod_{2\le i<j\le N-2} \left(x_i - x_j\right)$ near $\boldsymbol{b}$.

6. Explicit computation of $Z^{(4)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$

In this section we compute the Ghoshal-Kawano local zeta function for four points:

$$Z^{(4)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau) = \exp\left\{\frac{i}{2}\widetilde{s}_{13}\right\} \int_{\mathbb{Z}_p} |x_2|_p^{s_{12}} |1 - x_2|_p^{s_{32}} H_\tau(x_2) H_\tau(1 - x_2) E^{(4)}(x_2, \widetilde{\boldsymbol{s}}, \tau) dx_2,$$

where

$$E^{(4)}(x_2, \widetilde{s}, \tau) := E^{(4)}(x_2, \widetilde{s}_{12}, \widetilde{s}_{32}, \tau) = \exp\left\{ \frac{i}{2}\left(\widetilde{s}_{12}\mathrm{sgn}_\tau(x_2) + \widetilde{s}_{23}\mathrm{sgn}_\tau(1-x_2) \right) \right\}.$$

We recall that Ghoshal and Kawano take $x_1 = 0, x_3 = 1, x_4 = \infty$. By using the fact that $\mathrm{sgn}_\tau(y) \in \{1, -1\}$ and $H_\tau(y) \in \{0, 1\}$, one verifies that

$$\exp\left\{ \frac{i}{2}\left(\widetilde{s}_{12}\mathrm{sgn}_\tau(x_2) \right) \right\} H_\tau(x_2) = \exp\left(\frac{i}{2}\widetilde{s}_{12} \right) H_\tau(x_2),$$

$$\exp\left\{ \frac{i}{2}\left(\widetilde{s}_{23}\mathrm{sgn}_\tau(1-x_2) \right) \right\} H_\tau(1-x_2) = \exp\left(\frac{i}{2}\widetilde{s}_{23} \right) H_\tau(1-x_2),$$

and consequently,

$$E^{(4)}(\widetilde{s}_{12}, \widetilde{s}_{32}) = \exp\left\{ \frac{i}{2}\left(\widetilde{s}_{12} + \widetilde{s}_{23} \right) \right\},$$

and

$$Z^{(4)}(\boldsymbol{s}, \widetilde{s}, \tau) = \exp\left\{ \frac{i}{2}\left(\widetilde{s}_{13} + \widetilde{s}_{12} + \widetilde{s}_{23} \right) \right\} \int_{\mathbb{Z}_p} |x_2|_p^{s_{12}} |1-x_2|_p^{s_{32}} H_\tau(x_2) H_\tau(1-x_2)\, dx_2.$$

We first compute some p-adic integrals needed in this section.

6.1. Some p-adic integrals

6.1.1. Formula 1

Assume that $S \subset Z_p \smallsetminus \{0\}$ satisfies $-S = S$. Then, for $\tau \in \{p, \varepsilon p\}$

$$\int_S |x_2|_p^{s_{12}} \mathrm{sgn}_\tau(x_2) dx_2 = 0.$$

This formula follows from changing variables as $x_2 = -y$ and using the fact that $\mathrm{sgn}_\tau(-y) = -\mathrm{sgn}_\tau(y)$.

6.1.2. Formula 2

If $p \equiv 3 \bmod 4$ and $\tau \in \{p, \varepsilon p\}$, then

$$S(\tau, p) := \frac{1}{p} \sum_{j=2}^{p-1} H_\tau(j) H_\tau(1-j) = \frac{p-3}{4p}$$

From table (58), for $j = 2, \ldots, p-1$,

$$H_\tau(j) H_\tau(1-j) = \frac{1}{4}\left\{ 1 + \left(\frac{j}{p} \right) \right\} \left\{ 1 - \left(\frac{j-1}{p} \right) \right\}$$

and thus

$$S(\tau, p) := \frac{1}{4p}\left\{ p - 2 + \sum_{j=2}^{p-1} \left(\frac{j}{p} \right) - \sum_{j=2}^{p-1} \left(\frac{j-1}{p} \right) - \sum_{j=2}^{p-1} \left(\frac{j}{p} \right)\left(\frac{j-1}{p} \right) \right\}.$$

Now by using that $\sum_{k=1}^{p-1} \left(\frac{k}{p} \right) = 0$, we get that

$$\sum_{j=2}^{p-1}\left(\frac{j}{p}\right)=-1 \text{ and } \sum_{j=2}^{p-1}\left(\frac{j-1}{p}\right)=\sum_{k=1}^{p-2}\left(\frac{k}{p}\right)=-\left(\frac{p-1}{p}\right)=1,$$

and thus

$$S(\tau,p)=\frac{1}{4p}\left\{p-4-\sum_{k=1}^{p-2}\left(\frac{k+1}{p}\right)\left(\frac{k}{p}\right)\right\}.$$

To compute

$$L(\tau,p):=\sum_{k=1}^{p-2}\left(\frac{k+1}{p}\right)\left(\frac{k}{p}\right),$$

we define

$$A_{ij}=\left\{a\in\{1,\dots,p-2\};\left(\frac{a}{p}\right)=(-1)^i \text{ and } \left(\frac{a+1}{p}\right)=(-1)^j\right\},$$

then $\{1,\dots,p-2\}=A_{00}\bigsqcup A_{01}\bigsqcup A_{10}\bigsqcup A_{11}$ and

$$L(\tau,p)=\#A_{00}-\#A_{01}-\#A_{10}+\#A_{11}.$$

Now, if $p\equiv 3 \bmod 4$, then

$$\#A_{00}=\#A_{10}=\#A_{11}=\frac{p-3}{4}, \text{ and } \#A_{01}=\frac{p+1}{4}, \tag{26}$$

see e.g. [57, Chapter 9, Exercise 5 in p. 201], and therefore

$$L(\tau,p)=\#A_{00}-\#A_{01}=-1, \text{ and } S(\tau,p)=\frac{1}{4p}(p-3).$$

6.1.3. Formula 3

Set

$$I(\boldsymbol{s},\tau)=\int_{\mathbb{Z}_p}|x_2|_p^{s_{12}}\,|1-x_2|_p^{s_{32}}\,H_\tau(x_2)\,H_\tau(1-x_2)\,dx_2.$$

Then

$$I(\boldsymbol{s},\tau)=\frac{p-3}{4p}+\frac{p^{-1-s_{12}}\left(1-p^{-1}\right)}{2\left(1-p^{-1-s_{12}}\right)}+\frac{p^{-1-s_{32}}\left(1-p^{-1}\right)}{2\left(1-p^{-1-s_{32}}\right)} \tag{27}$$

By using the partition

$$\mathbb{Z}_p=\bigsqcup_{j=0}^{p-1} j+p\mathbb{Z}_p \tag{28}$$

and the fact that

$$H_\tau(x_2)\,|_{j+p\mathbb{Z}_p}=H_\tau(j) \text{ for } j\neq 0 \text{ and } H_\tau(1-x_2)\,|_{j+p\mathbb{Z}_p}=H_\tau(1-j) \text{ for } j\neq 1, \tag{29}$$

we have

$$I(\boldsymbol{s},\tau)=\sum_{j=0}^{p-1} I_j(\boldsymbol{s},\tau), \tag{30}$$

where

$$I_j(\boldsymbol{s},\tau)=\int\limits_{j+p\mathbb{Z}_p}|x_2|_p^{s_{12}}\,|1-x_2|_p^{s_{32}}\,H_\tau(x_2)\,H_\tau(1-x_2)\,dx_2.$$

If $j\neq 0,1$, then

$$I_j(\boldsymbol{s},\tau)=p^{-1}H_\tau(j)\,H_\tau(1-j)\,. \tag{31}$$

If $j=0$, then by using Formula 1,

$$\begin{aligned}I_0(\boldsymbol{s},\tau)&=\int\limits_{p\mathbb{Z}_p}|x_2|_p^{s_{12}}\,H_\tau(x_2)\,dx_2=\frac{1}{2}\int\limits_{p\mathbb{Z}_p}|x_2|_p^{s_{12}}\,dx_2+\frac{1}{2}\int\limits_{p\mathbb{Z}_p}|x_2|_p^{s_{12}}\,\mathrm{sgn}_\tau(x_2)dx_2\\&=\frac{1}{2}\int\limits_{p\mathbb{Z}_p}|x_2|_p^{s_{12}}\,dx_2=\frac{1}{2}\frac{p^{-1-s_{12}}\left(1-p^{-1}\right)}{1-p^{-1-s_{12}}}.\end{aligned} \tag{32}$$

The case $j=1$ is similar to the case $j=0$,

$$I_1(\boldsymbol{s},\tau)=\int\limits_{1+p\mathbb{Z}_p}|1-x_2|_p^{s_{32}}\,H_\tau(1-x_2)\,dx_2=\frac{1}{2}\frac{p^{-1-s_{32}}\left(1-p^{-1}\right)}{1-p^{-1-s_{32}}}. \tag{33}$$

Formula (27) follows from (30) by using (31)-(33) and Formula 2.

6.2. *Computation of* $Z^{(4)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$

In conclusion,

$$\begin{aligned}&Z^{(4)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)\\&=\exp\left\{\frac{i}{2}\left(\widetilde{s}_{13}+\widetilde{s}_{12}+\widetilde{s}_{23}\right)\right\}I(\boldsymbol{s},\tau)\\&=\exp\left\{\frac{i}{2}\left(\widetilde{s}_{13}+\widetilde{s}_{12}+\widetilde{s}_{23}\right)\right\}\left(\frac{p-3}{4p}+\frac{p^{-1-s_{12}}\left(1-p^{-1}\right)}{2\left(1-p^{-1-s_{12}}\right)}+\frac{p^{-1-s_{32}}\left(1-p^{-1}\right)}{2\left(1-p^{-1-s_{32}}\right)}\right)\end{aligned} \tag{34}$$

is holomorphic in

$$\mathrm{Re}(s_{12})>-1 \text{ and } \mathrm{Re}(s_{32})>-1. \tag{35}$$

The above formula for $Z^{(4)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$ was also obtained in [50].

7. Explicit computation of $Z^{(5)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$

In this section, using the normalization $x_1=0$, $x_4=1$ and $x_5=\infty$, we compute the amplitude for five points:

$$Z^{(5)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)=\int\limits_{\mathbb{Z}_p^2}E^{(5)}(x_2,x_3,\widetilde{\boldsymbol{s}},\tau)F^{(5)}(x_2,x_3,\boldsymbol{s},\tau)dx_2dx_3,$$

with

$$
\begin{aligned}
&E^{(5)}(x_2, x_3, \widetilde{\boldsymbol{s}}, \tau) \\
&\quad = \exp\left\{\frac{-\sqrt{-1}}{2}\Big(\widetilde{s}_{14}\mathrm{sgn}_\tau(-1) + \widetilde{s}_{12}\mathrm{sgn}_\tau(-x_2) + \widetilde{s}_{13}\mathrm{sgn}_\tau(-x_3)\Big)\right\} \\
&\quad \times \exp\left\{\frac{-\sqrt{-1}}{2}\Big(\widetilde{s}_{42}\mathrm{sgn}_\tau(1-x_2) + \tilde{s}_{43}\mathrm{sgn}_\tau(1-x_3) + \tilde{s}_{23}\mathrm{sgn}_\tau(x_2-x_3)\Big)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
F^{(5)}(x_2, x_3, \boldsymbol{s}, \tau) = {} & |x_2|_p^{s_{12}} |x_3|_p^{s_{13}} |1-x_2|_p^{s_{42}} |1-x_3|_p^{s_{43}} |x_2-x_3|_p^{s_{23}} \\
& \times H_\tau(x_2) H_\tau(x_3) H_\tau(1-x_2) H_\tau(1-x_3) H_\tau(x_2-x_3).
\end{aligned}
$$

Using the reasoning given at the beginning of the previous section we have

$$
E^{(5)}(\widetilde{\boldsymbol{s}}) = \exp\left\{\frac{\sqrt{-1}}{2}\Big(\widetilde{s}_{14} + \widetilde{s}_{12} + \widetilde{s}_{13} + \widetilde{s}_{24} + \tilde{s}_{34} + \tilde{s}_{32}\Big)\right\}
$$

and then

$$
Z^{(5)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau) = E^{(5)}(\widetilde{\boldsymbol{s}}) L(\boldsymbol{s}, \tau), \tag{36}
$$

where

$$
L(\boldsymbol{s}, \tau) = \int_{\mathbb{Z}_p^2} F^{(5)}(x_2, x_3, \boldsymbol{s}, \tau) dx_2 dx_3. \tag{37}
$$

First we give some formulae needed in the following calculations.

7.1. More p-adic sums and integrals

7.1.1. Formula 4

For $A \subset \{1, 2, \dots, p-1\}$, by using that $H_\tau(x) = \frac{1}{2}(1 + \mathrm{sgn}_\tau(x))$, where the sign function sgn_τ is given in Table (58), and that $\mathrm{sgn}_\tau(-x) = -\mathrm{sgn}_\tau(x)$, for $p \equiv 3 \bmod 4$ and $\tau \in \{p, \varepsilon p\}$, we have

$$
V(A, p, \tau) := \sum_{\substack{i,j \in A \\ i \neq j}}^{p-1} H_\tau(i-j) = \frac{(\#A)(\#A-1)}{2} = \binom{\#A}{2}.
$$

Indeed

$$
\begin{aligned}
V(A, p, \tau) &= \sum_{\substack{i,j \in A \\ j<i}} H_\tau(i-j) + \sum_{\substack{i,j \in A \\ i<j}} H_\tau(-(j-i)) = \sum_{\substack{i,j \in A \\ j<i}} \left[H_\tau(i-j) + H_\tau(-(i-j))\right] \\
&= \frac{1}{2} \sum_{\substack{i,j \in A \\ j<i}} \left[1 + \left(\frac{i-j}{p}\right) + 1 - \left(\frac{i-j}{p}\right)\right] = \sum_{\substack{i,j \in A \\ j<i}} 1 = \binom{\#A}{2}.
\end{aligned}
$$

7.1.2. Formula 5

If $p \equiv 3 \bmod 4$ and $\tau \in \{p, \varepsilon p\}$, then

$$\begin{aligned} T(p,\tau) &:= \frac{1}{p^2} \sum_{\substack{i,j=2 \\ i \neq j}}^{p-1} H_\tau(i) H_\tau(1-i) H_\tau(j) H_\tau(1-j) H_\tau(i-j) \\ &= \frac{(p-3)(p-7)}{32p^2}. \end{aligned}$$

We define $B := \{k \in \{2, 3, \ldots, p-1\}\,;\, H_\tau(k) H_\tau(1-k) = 1\}$. Then, by using the results and notation given in the proof of Formula 2, we have $\#B = \#A_{10} = \frac{p-3}{4}$, and

$$T(p,\tau) = \frac{1}{p^2} \sum_{\substack{i,j \in B \\ i \neq j}} H_\tau(i-j) = \frac{1}{p^2} \binom{\#B}{2} = \frac{1}{2p^2} \left(\frac{p-3}{4}\right) \left(\frac{p-7}{4}\right) = \frac{(p-3)\,(p-7)}{32p^2}.$$

7.1.3. Formula 6

We set for $a, b, c \in \mathbb{C}$,

$$L_{00}(a,b,c) := \frac{1}{8} \int\limits_{(p\mathbb{Z}_p)^2} |x_2|_p^a |x_3|_p^b |x_2 - x_3|_p^c dx_2 dx_3, \text{ for } \operatorname{Re}(a),\ \operatorname{Re}(b),\ \operatorname{Re}(c) > 0.$$

Then $L_{00}(a,b,c)$ has a meromorphic continuation to the whole complex plane given by

$$\begin{aligned} &L_{00}(a,b,c) \\ &\quad = \frac{1}{8} \frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}} \left\{ p^{-1}(p-2) + \frac{p^{-1-a}\left(1-p^{-1}\right)}{1-p^{-1-a}} + \frac{p^{-1-b}\left(1-p^{-1}\right)}{1-p^{-1-b}} \right. \\ &\qquad \left. + \frac{p^{-1-c}\left(1-p^{-1}\right)}{1-p^{-1-c}} \right\}. \end{aligned}$$

In order to compute $L_{00}(a,b,c)$, we introduce the following subsets:

$$\begin{aligned} A &:= \left\{ (x_2, x_3) \in \left(p\mathbb{Z}_p\right)^2 ; \left|\frac{x_2}{x_3}\right|_p \le 1 \right\}, \\ B &:= \left\{ (x_2, x_3) \in \left(p\mathbb{Z}_p\right)^2 ; \left|\frac{x_3}{x_2}\right|_p < 1 \right\}. \end{aligned}$$

Then

$$\left(p\mathbb{Z}_p\right)^2 \smallsetminus \left\{ (x_2,x_3) \in \left(p\mathbb{Z}_p\right)^2 ; x_2 x_3 = 0 \right\} = A \bigsqcup B, \tag{38}$$

and $L_{00}(a,b,c) = L_{00}^{(A)}(a,b,c) + L_{00}^{(B)}(a,b,c)$, where

$$L_{00}^{(A)}(a,b,c) := \frac{1}{8} \int\limits_A |x_2|_p^a |x_3|_p^b |x_2 - x_3|_p^c dx_2 dx_3,$$

and

$$L_{00}^{(B)}(a,b,c) := \frac{1}{8}\int\limits_{B} |x_2|_p^a |x_3|_p^b |x_2 - x_3|_p^c dx_2 dx_3.$$

We compute first $L_{00}^{(A)}(a,b,c)$, by using the following change of variables:

$$x_2 = uv, \;\; x_3 = u. \tag{39}$$

Then $dx_2dx_3 = |u|_p \, dudv$ and

$$\begin{aligned} L_{00}^{(A)}(a,b,c) &= \frac{1}{8}\int\limits_{p\mathbb{Z}_p\times\mathbb{Z}_p} |u|_p^{a+b+c+1}|v|_p^a|v-1|_p^c dudv \\ &= \frac{1}{8}\left\{\int\limits_{p\mathbb{Z}_p}|u|_p^{a+b+c+1}du\right\}\left\{\int\limits_{\mathbb{Z}_p}|v|_p^a|v-1|_p^c dv\right\} \\ &=: \frac{1}{8}\frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}} J\left(a,c\right). \end{aligned} \tag{40}$$

By using partition (28),

$$J\left(a,c\right) = \sum_{i=0}^{p-1} J_i\left(a,c\right).$$

For $i \neq 0, 1$,

$$J_i\left(a,c\right) = \int\limits_{i+p\mathbb{Z}_p} |v|_p^a|v-1|_p^c dv = p^{-1},$$

and the contribution of all these integrals is

$$\sum_{i=2}^{p-1} J_i\left(a,c\right) = p^{-1}\left(p-2\right). \tag{41}$$

For $i = 0$,

$$J_0\left(a,c\right) = \int\limits_{p\mathbb{Z}_p} |v|_p^a dv = \frac{p^{-1-a}\left(1-p^{-1}\right)}{1-p^{-1-a}}. \tag{42}$$

For $i = 1$,

$$J_1\left(a,c\right) = \int\limits_{1+p\mathbb{Z}_p} |v-1|_p^c dv = \frac{p^{-1-c}\left(1-p^{-1}\right)}{1-p^{-1-c}}. \tag{43}$$

Therefore, from (40)-(43),

$$\begin{aligned} &L_{00}^{(A)}(a,b,c) \\ &\quad = \frac{1}{8}\frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}}\left\{p^{-1}\left(p-2\right) + \frac{p^{-1-a}\left(1-p^{-1}\right)}{1-p^{-1-a}} + \frac{p^{-1-c}\left(1-p^{-1}\right)}{1-p^{-1-c}}\right\}. \end{aligned}$$

Now we compute $L_{00}^{(B)}(a,b,c)$, by using the following change of variables:

$$x_2 = t, \quad x_3 = zt. \tag{44}$$

Then $dx_2 dx_3 = |t|_p\, dz dt$ and

$$L_{00}^{(B)}(a,b,c) = \frac{1}{8} \int\limits_{(p\mathbb{Z}_p)^2} |t|_p^{a+b+c+1} |z|_p^b |1-z|_p^c dz dt = \frac{1}{8} \int\limits_{(p\mathbb{Z}_p)^2} |t|_p^{a+b+c+1} |z|_p^b |dz dt$$
$$= \frac{1}{8} \frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}} \frac{p^{-1-b}\left(1-p^{-1}\right)}{1-p^{-1-b}}.$$

7.1.4. Formula 7

For $a, b, c \in \mathbb{C}$,

$$L_{00}^{(1)}(a,b,c,\tau) := \frac{1}{8} \int\limits_{(p\mathbb{Z}_p)^2} |x_2|_p^a |x_3|_p^b |x_2 - x_3|_p^c \operatorname{sgn}_\tau(x_2) \operatorname{sgn}_\tau(x_3) dx_2 dx_3$$
$$= \frac{1}{8} \frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}} \left\{ -p^{-1} + \frac{p^{-1-c}\left(1-p^{-1}\right)}{1-p^{-1-c}} \right\},$$

for $\operatorname{Re}(a)$, $\operatorname{Re}(b)$, $\operatorname{Re}(c) > 0$. By using partition (38), we get that $L_{00}^{(1)}(a,b,c,\tau) = L_{00}^{(1,A)}(a,b,c,\tau) + L_{00}^{(1,B)}(a,b,c,\tau)$. We compute integral $L_{00}^{(1,A)}(a,b,c,\tau)$, respectively $L_{00}^{(1,B)}(a,b,c,\tau)$, by using change of variables (39), respectively (44), as follows:

$$L_{00}^{(1,A)}(a,b,c,\tau) = \frac{1}{8} \left\{ \int\limits_{p\mathbb{Z}_p} |u|_p^{a+b+c+1} du \right\} \left\{ \int\limits_{\mathbb{Z}_p} |v|_p^a |v-1|_p^c \operatorname{sgn}_\tau(v) dv \right\}$$
$$= \frac{1}{8} \frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}} K(a,c,\tau).$$

By using partition (28),

$$K(a,c,\tau) = \sum_{j=0}^{p-1} K_j(a,c,\tau).$$

For $j \neq 0, 1$, $K_j(a,c,\tau) = p^{-1} \operatorname{sgn}_\tau(j)$, thus, the contribution of all these integrals is

$$\sum_{j=2}^{p-1} K_j(a,c,\tau) = p^{-1} \sum_{j=2}^{p-1} \operatorname{sgn}_\tau(j) = p^{-1} \sum_{j=2}^{p-1} \left(\frac{j}{p}\right) = -p^{-1}.$$

For $j = 0$, by using the Formula 1, $K_0(a,c,\tau) = 0$. For $j = 1$,

$$K_1(a,c,\tau) = \int\limits_{1+p\mathbb{Z}_p} |v-1|_p^c \operatorname{sgn}_\tau(v) dv = \int\limits_{1+p\mathbb{Z}_p} |v-1|_p^c dv$$
$$= \int\limits_{p\mathbb{Z}_p} |v|_p^c dv = \frac{p^{-1-c}\left(1-p^{-1}\right)}{1-p^{-1-c}}.$$

In conclusion,

$$L_{00}^{(1,A)}(a,b,c,\tau)=\frac{1}{8}\frac{p^{-a-b-c-2}\left(1-p^{-1}\right)}{1-p^{-a-b-c-2}}\left\{-p^{-1}+\frac{p^{-1-c}\left(1-p^{-1}\right)}{1-p^{-1-c}}\right\}.$$

Now, by Formula 1,

$$L_{00}^{(1,B)}(a,b,c,\tau)=\left\{\frac{1}{8}\int\limits_{p\mathbb{Z}_p}|t|_p^{a+b+c+1}dt\right\}\left\{\int\limits_{p\mathbb{Z}_p}|z|_p^{b}\ \mathrm{sgn}_\tau(z)dz\right\}=0.$$

7.1.5. Formula 8

For $a,b,c\in\mathbb{C}$, we set

$$L_{00}^{(2)}(a,b,c,\tau):=\frac{1}{8}\int\limits_{(p\mathbb{Z}_p)^2}|x_2|_p^a|x_3|_p^b|x_2-x_3|_p^c\ \mathrm{sgn}_\tau(x_2)\mathrm{sgn}_\tau(x_2-x_3)dx_2dx_3.$$

Then

$$L_{00}^{(2)}(a,b,c,\tau)=L_{00}^{(1)}(a,c,b,\tau).$$

This identity is obtained by changing variables as $u=x_2$, $v=x_2-x_3$, and using Formula 7.

7.1.6. Formula 9

For $a,b,c\in\mathbb{C}$, we set

$$L_{00}^{(3)}(a,b,c,\tau):=\frac{1}{8}\int\limits_{(p\mathbb{Z}_p)^2}|x_2|_p^a|x_3|_p^b|x_2-x_3|_p^c\ \mathrm{sgn}_\tau(x_3)\mathrm{sgn}_\tau(x_2-x_3)dx_2dx_3.$$

Then

$$L_{00}^{(3)}(a,b,c,\tau)=-L_{00}^{(2)}(b,a,c,\tau).$$

This formula follows from Formula 8 by changing variables as $(x_2,x_3)\to(x_3,x_2)$.

7.2. Computation of $Z^{(5)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$

The computation of $Z^{(5)}(\boldsymbol{s},\widetilde{\boldsymbol{s}},\tau)$ is reduced to the computation of integral $L(\boldsymbol{s},\tau)$, see (36)-(37). By using the partition

$$\mathbb{Z}_p^2=\bigsqcup_{i,j=0}^{p-1}\left(i+p\mathbb{Z}_p\right)\times\left(j+p\mathbb{Z}_p\right),$$

we have

$$L(\boldsymbol{s},\tau)=\sum_{i,j=0}^{p-1}L_{ij}(\boldsymbol{s},\tau)$$

where

$$L_{ij}(\boldsymbol{s},\tau)=\int\limits_{i+p\mathbb{Z}_p\times j+p\mathbb{Z}_p}F^{(5)}(x_2,x_3,\boldsymbol{s},\tau)dx_2dx_3.$$

The calculation of these integrals is achieved by considering several cases.

Case $i, j \in \{2, 3, \dots, p-1\}$ **and** $i \neq j$.

In this case, by using that $H_\tau |_{i+p\mathbb{Z}_p} = H_\tau(i)$ for $i \in \{1, \dots, p-1\}$,

$$L_{ij}(\boldsymbol{s}, \tau) = p^{-2} H_\tau(i) H_\tau(1-i) H_\tau(j) H_\tau(1-i) H_\tau(i-j).$$

Now by using Formula 5, the contribution of all these integrals is

$$\sum_{\substack{i,j=2 \\ i \neq j}}^{p-1} L_{ij}(\boldsymbol{s}, \tau) = \frac{(p-3)(p-7)}{32p^2}. \tag{45}$$

Case $i, j \in \{2, 3, \dots, p-1\}$ **and** $i = j$.

In this case, by using (32),

$$\begin{aligned} L_{ii}(\boldsymbol{s}, \tau) &= H_\tau^2(i) H_\tau^2(1-i) \int_{i+p\mathbb{Z}_p \times i+p\mathbb{Z}_p} |x_2 - x_3|_p^{s_{23}} H_\tau(x_2 - x_3) dx_2 dx_3 \\ &= H_\tau(i) H_\tau(1-i) \int_{(p\mathbb{Z}_p)^2} |x_2 - x_3|_p^{s_{23}} H_\tau(x_2 - x_3) dx_2 dx_3 \\ &= p^{-1} H_\tau(i) H_\tau(1-i) \int_{p\mathbb{Z}_p} |x_2|_p^{s_{23}} H_\tau(x_2) dx_2 = p^{-1} H_\tau(i) H_\tau(1-i) I_0(s_{23}, \tau) \\ &= p^{-1} H_\tau(i) H_\tau(1-i) \frac{p^{-1-s_{23}}(1-p^{-1})}{2(1-p^{-1-s_{23}})}. \end{aligned}$$

Now, by using that $p \equiv 3 \bmod 4$, $\tau \neq \varepsilon$, and Formula 2, the contribution of all these integrals is

$$\frac{p^{-1-s_{23}}(1-p^{-1})}{2(1-p^{-1-s_{23}})} \frac{1}{p} \sum_{j=2}^{p-1} H_\tau(j) H_\tau(1-j) = \left(\frac{p-3}{8p}\right) \frac{p^{-1-s_{23}}(1-p^{-1})}{(1-p^{-1-s_{23}})}. \tag{46}$$

Case $i = 1$ **and** $j = 0$.

In this case by using Formula 1,

$$\begin{aligned} L_{10}(\boldsymbol{s}, \tau) &= \int_{1+p\mathbb{Z}_p \times p\mathbb{Z}_p} |1 - x_2|_p^{s_{42}} |x_3|_p^{s_{13}} H_\tau(1-x_2) H_\tau(x_3) dx_2 dx_3 \\ &= \left\{ \int_{1+p\mathbb{Z}_p} |1-x_2|_p^{s_{42}} H_\tau(1-x_2) dx_2 \right\} \left\{ \int_{p\mathbb{Z}_p} |x_3|_p^{s_{13}} H_\tau(x_3) dx_3 \right\} \\ &= \left\{ \int_{p\mathbb{Z}_p} |x_2|_p^{s_{42}} H_\tau(-x_2) dx_2 \right\} \left\{ \int_{p\mathbb{Z}_p} |x_3|_p^{s_{13}} H_\tau(x_3) dx_3 \right\} \\ &= \left\{ \frac{1}{2} \int_{p\mathbb{Z}_p} |x_2|_p^{s_{42}} dx_2 \right\} \left\{ \frac{1}{2} \int_{p\mathbb{Z}_p} |x_3|_p^{s_{13}} dx_3 \right\} \end{aligned}$$

$$= \frac{\left(1-p^{-1}\right)^2}{4} \frac{p^{-2-s_{42}-s_{13}}}{\left(1-p^{-1-s_{42}}\right)\left(1-p^{-1-s_{13}}\right)}. \tag{47}$$

Case $i=0$ **and** $j=1$.

Since

$$H_\tau(x_2-x_3)\Big|_{p\mathbb{Z}_p\times 1+p\mathbb{Z}_p} = H_\tau(-1) = 0,$$

we have $L_{01}(\boldsymbol{s},\tau)=0$.

Case $i=j=0$.

In this case,

$$L_{00}(\boldsymbol{s},\tau) = \int\limits_{(p\mathbb{Z}_p)^2} |x_2|_p^{s_{12}}|x_3|_p^{s_{13}}|x_2-x_3|_p^{s_{23}} H_\tau(x_2)H_\tau(x_3)H_\tau(x_2-x_3)dx_2dx_3.$$

By using that

$$H_\tau(x_2)H_\tau(x_3)H_\tau(x_2-x_3) = \frac{1}{8}\left\{1+\operatorname{sgn}_\tau(x_2)+\operatorname{sgn}_\tau(x_3)+\operatorname{sgn}_\tau(x_2-x_3)\right.$$
$$+\operatorname{sgn}_\tau(x_2)\operatorname{sgn}_\tau(x_3)+\operatorname{sgn}_\tau(x_2)\operatorname{sgn}_\tau(x_2-x_3)+\operatorname{sgn}_\tau(x_3)\operatorname{sgn}_\tau(x_2-x_3)$$
$$\left.+\operatorname{sgn}_\tau(x_2)\operatorname{sgn}_\tau(x_3)\operatorname{sgn}_\tau(x_2-x_3)\right\},$$

and the notation introduced in Formulae 6 to 9, we have

$$\begin{aligned} L_{00}(\boldsymbol{s},\tau) &= L_{00}(s_{12},s_{13},s_{23}) + L_{00}^{(1)}(s_{12},s_{13},s_{23},\tau) + L_{00}^{(2)}(s_{12},s_{13},s_{23},\tau) \\ &\quad + L_{00}^{(3)}(s_{12},s_{13},s_{23},\tau) \\ &= L_{00}(s_{12},s_{13},s_{23}) + L_{00}^{(1)}(s_{12},s_{13},s_{23},\tau) + L_{00}^{(1)}(s_{12},s_{23},s_{13},\tau) \\ &\quad - L_{00}^{(1)}(s_{13},s_{23},s_{12},\tau), \end{aligned} \tag{48}$$

the integrals involving an odd number of sign functions vanish. This fact can be established by a suitable change of variables as in Formula 1.

Case $i=j=1$.

In this case,

$$L_{11}(\boldsymbol{s},\tau)$$
$$= \int\limits_{(1+p\mathbb{Z}_p)^2} |1-x_2|_p^{s_{42}}|1-x_3|_p^{s_{43}}|x_2-x_3|_p^{s_{23}} H_\tau(1-x_2)H_\tau(1-x_3)H_\tau(x_2-x_3)dx_2dx_3.$$

Now by changing variables as $u=1-x_2$, $v=1-x_3$, we get

$$\begin{aligned} &L_{11}(\boldsymbol{s},\tau) \\ &= \int\limits_{(p\mathbb{Z}_p)^2} |u|_p^{s_{42}}|v|_p^{s_{43}}|u-v|_p^{s_{23}} H_\tau(u)H_\tau(v)H_\tau(v-u)dudv \\ &= L_{00}(s_{42},s_{43},s_{23}) + L_{00}^{(1)}(s_{42},s_{43},s_{23},\tau) - L_{00}^{(2)}(s_{42},s_{43},s_{23},\tau) - L_{00}^{(3)}(s_{42},s_{43},s_{23},\tau) \\ &= L_{00}(s_{42},s_{43},s_{23}) + L_{00}^{(1)}(s_{42},s_{43},s_{23},\tau) - L_{00}^{(1)}(s_{42},s_{23},s_{43},\tau) + L_{00}^{(1)}(s_{43},s_{23},s_{42},\tau) \end{aligned} \tag{49}$$

Cases $i=0$ **and** $j\in\{2,3,\dots,p-1\}$ **or** $i\in\{2,3,\dots,p-1\}$ **and** $j=1$.
In these cases,

$$L_{0j}(\boldsymbol{s},\tau)=L_{i1}(\boldsymbol{s},\tau)=0. \tag{50}$$

The vanishing of the integral $L_{0j}(\boldsymbol{s},\tau)$ follows from

$$H_\tau(x_3)H_\tau(x_2-x_3)\Big|_{p\mathbb{Z}_p\times j+p\mathbb{Z}_p}=H_\tau(j)H_\tau(-j)=0.$$

The other case is treated in a similar way.

Case $i\in\{2,3,\dots,p-1\}$ **and** $j=0$.
By using (32),

$$\begin{aligned}L_{i0}(\boldsymbol{s},\tau)&=H_\tau^2(i)H_\tau(1-i)H_\tau(1)\int\limits_{i+p\mathbb{Z}_p\times p\mathbb{Z}_p}|x_3|_p^{s_{13}}H_\tau(x_3)dx_2dx_3\\&=p^{-1}H_\tau(i)H_\tau(1-i)\int\limits_{p\mathbb{Z}_p}|x_3|_p^{s_{13}}H_\tau(x_3)dx_3=p^{-1}H_\tau(i)H_\tau(1-i)I_0(s_{13},\tau)\\&=p^{-1}H_\tau(i)H_\tau(1-i)\frac{p^{-1-s_{13}}(1-p^{-1})}{2(1-p^{-1-s_{13}})}.\end{aligned} \tag{51}$$

Now, using Formula 2, the contribution of all these integrals is

$$\sum_{i=2}^{p-1}L_{i0}(\boldsymbol{s},\tau)=\frac{p^{-1-s_{13}}(1-p^{-1})}{2(1-p^{-1-s_{13}})}\frac{1}{p}\sum_{i=2}^{p-1}H_\tau(i)H_\tau(1-i)=\left(\frac{p-3}{8p}\right)\frac{p^{-1-s_{13}}(1-p^{-1})}{(1-p^{-1-s_{13}})}. \tag{52}$$

Case $i=1$ **and** $j\in\{2,3,\dots,p-1\}$.
This case is similar to the previous one,

$$\sum_{j=2}^{p-1}L_{1j}(\boldsymbol{s},\tau)=p^{-1}I_0(s_{42},\tau)\sum_{j=2}^{p-1}H_\tau(j)H_\tau(1-j)=\left(\frac{p-3}{8p}\right)\frac{p^{-1-s_{42}}(1-p^{-1})}{(1-p^{-1-s_{42}})}. \tag{53}$$

In conclusion, from (36), (37), and (45)-(53), we have

$$\begin{aligned}Z^{(5)}(\boldsymbol{s},\widetilde{\boldsymbol{s}})=E^{(5)}(\widetilde{\boldsymbol{s}})\Bigg\{&\frac{(p-3)(p-7)}{32p^2}+\left(\frac{p-3}{8p}\right)\Bigg[\frac{p^{-1-s_{23}}(1-p^{-1})}{(1-p^{-1-s_{23}})}\\&+\frac{p^{-1-s_{13}}(1-p^{-1})}{(1-p^{-1-s_{13}})}+\frac{p^{-1-s_{42}}(1-p^{-1})}{(1-p^{-1-s_{42}})}\Bigg]+\frac{\left(1-p^{-1}\right)^2}{4}\frac{p^{-2-s_{13}-s_{42}}}{\left(1-p^{-1-s_{13}}\right)\left(1-p^{-1-s_{42}}\right)}\\&+\frac{1}{4}\frac{p^{-s_{12}-s_{13}-s_{23}-2}\left(1-p^{-1}\right)}{1-p^{-s_{12}-s_{13}-s_{23}-2}}\left[\frac{1}{2}-\frac{3}{2p}+\frac{p^{-1-s_{23}}\left(1-p^{-1}\right)}{1-p^{-1-s_{23}}}+\frac{p^{-1-s_{13}}\left(1-p^{-1}\right)}{1-p^{-1-s_{13}}}\right]\\&+\frac{1}{4}\frac{p^{-s_{42}-s_{43}-s_{23}-2}\left(1-p^{-1}\right)}{1-p^{-s_{42}-s_{43}-s_{23}-2}}\left[\frac{1}{2}-\frac{3}{2p}+\frac{p^{-1-s_{23}}\left(1-p^{-1}\right)}{1-p^{-1-s_{23}}}+\frac{p^{-1-s_{42}}\left(1-p^{-1}\right)}{1-p^{-1-s_{42}}}\right]\Bigg\}.\end{aligned} \tag{54}$$

$Z^{(5)}(\boldsymbol{s},\widetilde{\boldsymbol{s}})$ is a holomorphic function in

$$\mathrm{Re}(s_{13}) > -1; \quad \mathrm{Re}(s_{23}) > -1; \quad \mathrm{Re}(s_{42}) > -1;$$

$$\mathrm{Re}(s_{12} + s_{13} + s_{23}) > -2; \quad \mathrm{Re}(s_{42} + s_{43} + s_{23}) > -2.$$

8. The limit $p \to 1$ of the Ghoshal-Kawano amplitudes

In [45] we established that the limit p approaches to one of p-adic open string amplitudes at the tree-level can be defined rigorously by using the Denef and Loeser theory of topological zeta functions [46]. Notice that the calculations involving the limit $p \to 1$ in the case of the effective action are performed in $\mathbb{R}^D$, meanwhile the calculations involving the limit $p \to 1$ in the case of p-adic string amplitudes are performed in $\mathbb{Q}_p^D$, and in the p-adic topology the limit $p \to 1$ does not make sense. However, surprisingly, the computation of the limit $p \to 1$ (considering p as a real parameter) of the p-adic open string amplitudes gives the right answer! In this subsection we compute limit $p \to 1$ (considering p as a real parameter) in the cases $N = 4, 5$. The computation of the limit $p \to 1$ in the general case require the so called explicit formulas, see [45] for further details.

The limit $p \to 1$ of $Z^{(N)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau)$, $N = 4, 5$, with $p \equiv 3 \bmod 4$, are given by

$$\lim_{p \to 1} Z^{(4)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}, \tau) = \exp\left\{ \frac{\sqrt{-1}}{2} (\widetilde{s}_{13} + \widetilde{s}_{12} + \widetilde{s}_{23}) \right\} \left\{ -\frac{1}{2} + \frac{1}{2(s_{12}+1)} + \frac{1}{2(s_{32}+1)} \right\}, \tag{55}$$

for $\tau \in \{p, \varepsilon p\}$, and

$$\begin{aligned}\lim_{p \to 1} Z^{(5)}(\boldsymbol{s}, \widetilde{\boldsymbol{s}}) = E^{(5)}(\widetilde{\boldsymbol{s}}) \Bigg\{ &\frac{3}{16} - \frac{1}{4(s_{23}+1)} - \frac{1}{4(s_{13}+1)} - \frac{1}{4(s_{42}+1)} \\ &+ \frac{1}{4\,(s_{42}+1)\,(s_{13}+1)} + \frac{1}{4\,(s_{12}+s_{13}+s_{23}+2)} \left[-1 + \frac{1}{(s_{23}+1)} + \frac{1}{(s_{13}+1)} \right] \\ &+ \frac{1}{4\,(s_{42}+s_{43}+s_{23}+2)} \left[-1 + \frac{1}{(s_{23}+1)} + \frac{1}{(s_{42}+1)} \right] \Bigg\}. \end{aligned} \tag{56}$$

In the case $N = 4$, after the appropriate sum over the permutations of the momenta $\boldsymbol{k}_i$, the amplitude agrees with the Feynman amplitude obtained from the noncommutative version of the Gerasimov-Shatashvili action with a logarithmic potential (9).

9. Final remarks

In the present article, we study the Ghoshal-Kawano amplitudes for p-adic open strings at tree-level [50]. These amplitudes include Chan-Paton factors and an external B-field.

In section 2 starting from the noncommutative effective action (6) discussed in [50,52], in the present article, we obtain the corresponding tree-level four-point amplitudes (34) in the limit $p \to 1$. This result was achieved by adapting the heuristic approach given in [42] for the noncommutative case. By an explicit computation using the noncommutative field theory [54, 55], we determine the four-point amplitude at the tree level coming from the noncommutative Gerasimov-Shatashvili Lagrangian. This amplitude is the sum of the expressions (14) and (16). The first one represents the noncommutative vertex four-point function and the second one is the superposition of the amplitudes corresponding to the noncommutative channels s, t and u. The calculated tree-level amplitude is completely described by planar Feynman diagrams and consequently the noncommutativity effect arises as a global phase factor in front of the amplitude. Five-point amplitudes (or higher-order amplitudes) can be also computed in a straightforward way following the same procedure.

The study of the p-adic Ghoshal-Kawano amplitudes requires the use of multivariate local zeta functions involving multiplicative characters and a phase factor including the noncommutative parameter θ. These are new mathematical objects. We call these objects Ghoshal-Kawano zeta functions. In sections 4 and 5, by using Hironaka's resolution of singularities theorem, we prove that these integrals admit meromorphic continuation as complex functions in the external momenta of the N external particles.

Four and five point amplitudes were computed explicitly in sections 6 and 7, see (34) and (54). The four-point amplitude (34) coincides with the one obtained in [50]. The five-point amplitude was not obtained previously.

In section 8 we study again the amplitudes from the worldsheet view point. We compute the limit $p \to 1$ for four and five point amplitudes resulting in the formulae (55) and (56), respectively. The four-point amplitude (55) agrees with the heuristic computation given by the superposition of formulae (14) and (16).

As we mentioned before, in the computation of Ghoshal-Kawano amplitudes at the tree-level, the noncommutative effect coming from the constant B-field arises only as a global phase factor because only planar diagrams are involved. In the computation of amplitudes at one-loop or multi-loops, non-planar diagrams systematically arise. It would be very interesting to study the possibility of finding a non-trivial noncommutative effect, as the IR/UV mixing, as a result of the contribution of one-loop non-planar diagrams. Probably the multi-loop analysis of the p-adic string theory studied in [56], will play an important role for the analysis of the IR/UV mixing and other interesting effects of the B-field in p-adic string theory amplitudes.

On the other hand, we think that the study of the amplitudes (1) without the ad hoc normalization $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$ may provide new insights on the effects of the B-field in p-adic string theory amplitudes. However, the study of these amplitudes is more involved than the one done here. Some of this work is in progress and will be reported elsewhere.

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Appendix A. Basic aspects of the p-adic analysis

In this appendix we collect some basic results about p-adic analysis that will be used in the article. For an in-depth review of the p-adic analysis the reader may consult [5,58,59].

A.1. The field of p-adic numbers

Along this article p will denote a prime number different from 2. The field of $p-$adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p-$adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if} \quad x = 0 \\ p^{-\gamma} & \text{if} \quad x = p^{\gamma} \frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the $p-$adic order of x.

Any $p-$adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j,$$

where $x_j \in \{0, \ldots, p-1\}$ and $x_0 \neq 0$. In addition, any non-zero $p-$adic number can be represented uniquely as $x = p^{ord(x)} ac\,(x)$ where $ac\,(x) = \sum_{j=0}^{\infty} x_j p^j$, $x_0 \neq 0$, is called the angular component of x. Notice that $|ac\,(x)|_p = 1$.

We extend the $p-$adic norm to $\mathbb{Q}_p^n$ by taking

$$||x||_p := \max_{1 \leq i \leq n} |x_i|_p, \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

We define $ord(x) = \min_{1 \leq i \leq n}\{ord(x_i)\}$, then $||x||_p = p^{-ord(x)}$. The metric space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a separable complete ultrametric space. For $r \in \mathbb{Z}$, denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n; ||x-a||_p \leq p^r\}$ the ball of radius p^r with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B_r^n(0) := B_r^n$. Note that $B_r^n(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p; |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 = \mathbb{Z}_p$, the ring of $p-$adic integers of $\mathbb{Q}_p$. We also denote by $S_r^n(a) = \{x \in Q_p^n; ||x-a||_p = p^r\}$ the sphere of radius p^r with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $S_r^n(0) := S_r^n$. We notice that $S_0^1 = \mathbb{Z}_p^{\times}$ (the group of units of $\mathbb{Z}_p$), but $\left(\mathbb{Z}_p^{\times}\right)^n \subsetneq S_0^n$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$. In addition, two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other.

As a topological space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. A subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$, see e.g. [5, Section 1.3], or [58, Section 1.8]. The balls and spheres are compact subsets. Thus $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a locally compact topological space.

A.2. Integration

Since $(\mathbb{Q}_p, +)$ is a locally compact topological group, there exists a measure dx, which is invariant under translations, i.e. $d(x+a) = dx$. If we normalize this measure by the condition $\int_{\mathbb{Z}_p} dx = 1$, then dx is unique. A such measure is called the Haar measure of $(\mathbb{Q}_p, +)$. In the n-dimensional case, $(\mathbb{Q}_p^n, +)$ is also a locally compact topological group. We denote by $d^n x$ the Haar measure normalized by the condition $\int_{\mathbb{Z}_p^n} d^n x = 1$. This measure agrees with the product measure $dx_1 \cdots dx_n$, and it also satisfies that $d^n(x+a) = d^n x$, for $a \in \mathbb{Q}_p^n$

A function $h : U \to \mathbb{Q}_p$ is said to be analytic on an open subset $U \subseteq \mathbb{Q}_p^n$, if for every $b = (b_1, \ldots, b_n) \in U$ there exists an open subset $\widetilde{U} \subset U$, with $b \in \widetilde{U}$, and a convergent power series $\sum_{i \in \mathbb{N}^n} a_i\,(x-b)^i$ for $x = (x_1, \ldots, x_n) \in \widetilde{U}$, such that $h\,(x) = \sum_{i \in \mathbb{N}^n} a_i\,(x-b)^i$ for $x \in \widetilde{U}$, with $i = (i_1, \ldots, i_n)$ and $(x-b)^i = \prod_{j=1}^n \left(x_j - b_j\right)^{i_j}$. In this case, $\frac{\partial}{\partial x_l} h\,(x) = \sum_{i \in \mathbb{N}^n} a_i \frac{\partial}{\partial x_l}\,(x-b)^i$ is a convergent power series.

Let U, V be open subsets in $\mathbb{Q}_p^n$. A mapping $H : U \to V$, $H = (H_1, \ldots, H_n)$ is called analytic if each H_i is analytic. The mapping H is said to be bi-analytic if H and H^{-1} are analytic.

A.2.1. Change of variables formula

Let $K_0, K_1 \subset \mathbb{Q}_p^n$ be open compact subsets, and let $H = (H_1, \ldots, H_n) : K_1 \to K_0$ be a bi-analytic map such that

$$\det\left[\frac{\partial H_i}{\partial y_j}(y)\right] \neq 0, \text{ for } \ y \in K_1.$$

If f is a continuous function on K_0, then

$$\int_{K_0} f(x)\ d^n x = \int_{K_1} f(H(y)) \left|\det\left[\frac{\partial H_i}{\partial y_j}(y)\right]\right|_p d^n y, \qquad (x = H(y)).$$

For the proof of this theorem the reader may consult [39, Prop. 7.4.1] or [60, Section 10.1.2].

A.3. Some arithmetic functions

In this section we review some arithmetic functions that we shall use throughout this article.

A.3.1. Multiplicative characters

A multiplicative character (or quasi-character) of the group $\left(\mathbb{Q}_p^\times, \cdot\right)$ is a continuous homomorphism $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ satisfying $\chi(xy) = \chi(x)\chi(y)$. Every multiplicative character has the form

$$\chi(x) = |x|_p^s\, \chi_0(ac(x)), \text{ for some } s \in \mathbb{C},$$

where χ_0 is the restriction of χ to $\mathbb{Z}_p^\times$, which is a continuous multiplicative character of $\left(\mathbb{Z}_p^\times, \cdot\right)$ into the complex unit circle.

A.3.2. The Legendre symbol

For a an integer number and p a prime number, the Legendre symbol is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if} \quad x^2 \equiv a \ mod\ p \text{ has a solution} \\ -1 & \text{if} \quad \text{otherwise.} \end{cases}$$

The following formulas are used in several calculations in this article:

$$\left(\frac{1}{p}\right) = 1; \qquad \left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if} \quad p \equiv 1\ mod\ 4 \\ -1 & \text{if} \quad p \equiv 3\ mod\ 4. \end{cases}$$

Take for $x \in \mathbb{Q}_p^\times$, $ac(x) = x_0 + x_1 p + \ldots \in \mathbb{Z}_p^\times$, then

$$\begin{array}{ccc} \mathbb{Q}_p^\times & \to & \{\pm 1\} \\ x & \to & (\frac{x_0}{p}) \end{array}$$

is a unitary multiplicative character.

A.3.3. The sign function

We review the p-adic sign function, which is a multiplicative character with values in $\{\pm 1\}$. We denote $\left[\mathbb{Q}_p^\times\right]^2$ the multiplicative subgroup of squares in $\mathbb{Q}_p^\times$, i.e.

$$\left[\mathbb{Q}_p^\times\right]^2 = \{a \in \mathbb{Q}_p; a = b^2 \text{ for some } b \in \mathbb{Q}_p^\times\}.$$

For $p \neq 2$, and $\varepsilon \in \{1, \dots, p-1\}$ satisfying $(\frac{\varepsilon}{p}) = -1$, we have

$$\mathbb{Q}_p^\times / \left[\mathbb{Q}_p^\times\right]^2 = \{1, \varepsilon, p, \varepsilon p\}.$$

This means that any nonzero p-adic number can be written uniquely as

$$x = \tau a^2, \text{ with } a \in \mathbb{Q}_p^\times \text{ and } \tau \in \mathbb{Q}_p^\times / \left[\mathbb{Q}_p^\times\right]^2.$$

For a fixed $\tau \in \{\varepsilon, p, \varepsilon p\}$, and $x \in \mathbb{Q}_p^\times$, we set

$$\operatorname{sgn}_\tau(x) := \begin{cases} 1 & \text{if } x = a^2 - \tau b^2 \text{ for } a, b \in \mathbb{Q}_p \\ -1 & \text{otherwise.} \end{cases} \tag{57}$$

The following is the list of all the possible p-adic sign functions:

$p \equiv 1 \mod 4$	$p \equiv 3 \mod 4$
$\operatorname{sgn}_\varepsilon(x) = (-1)^{ord(x)}$	$\operatorname{sgn}_\varepsilon(x) = (-1)^{ord(x)}$
$\operatorname{sgn}_p(x) = \left(\frac{x_0}{p}\right)$	$\operatorname{sgn}_p(x) = (-1)^{ord(x)}\left(\frac{x_0}{p}\right)$
$\operatorname{sgn}_{\varepsilon p}(x) = (-1)^{ord(x)}\left(\frac{x_0}{p}\right)$	$\operatorname{sgn}_{\varepsilon p}(x) = \left(\frac{x_0}{p}\right)$,

(58)

see [23]. Then sgn_τ is a multiplicative character, and a locally constant function in $\mathbb{Q}_p^\times$, more precisely, $\operatorname{sgn}_\tau(x-y) = \operatorname{sgn}_\tau(x)$ if $ord(y) > ord(x)$.

We take $p \equiv 3 \mod 4$ and $\tau \in \{p, \varepsilon p\}$, to have $\operatorname{sgn}_\tau(-y) = -\operatorname{sgn}_\tau(y)$, for any $y \in \mathbb{Q}_p^\times$. In all the calculations involving sgn_τ we assume that $p \equiv 3 \mod 4$ and $\tau \in \{p, \varepsilon p\}$. We define the Heaviside step function as

$$H_\tau(x) := H_\tau^+(x) = \frac{1}{2}(1 + \operatorname{sgn}_\tau(x)) = \begin{cases} 1 & \text{if } \operatorname{sgn}_\tau(x) = 1 \\ 0 & \text{if } \operatorname{sgn}_\tau(x) = -1, \end{cases} \tag{59}$$

for any $x \in \mathbb{Q}_p^\times$. It is convenient to set

$$H_\tau^\pm(x) := \frac{1}{2}(1 \pm \operatorname{sgn}_\tau(x)).$$

The following properties are useful:

$$H_\tau(x)H_\tau(x) = H_\tau(x); \quad H_\tau(x) + H_\tau^-(x) = 1; \quad H_\tau(x)H_\tau^-(x) = 0;$$
$$H_\tau(xy) = H_\tau(x)H_\tau(y) + H_\tau^-(x)H_\tau^-(y).$$

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