M-theory , membrane instantons & T-duality

Natalia Saulina

A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Physics

November, 2003

Abstract

This thesis studies two independent aspects of M-theory. The focus of Chapter 2 is the phenomenologically promising model of our world which relies on the effective low energy dynamics of M-theory. The interesting background includes a Calabi-Yau 3-fold and an interval with M5-branes inserted at points along it. We obtain the effective potential for the scalars resulting from non-perturbative contributions due to open membrane instantons. We also discuss conditions under which the M5-branes may be attracted to the wall; and the chirality-changing phase transitions induced by an M5-brane hitting the wall.

In Chapter 3 we investigate the interplay between the proper quantization of RR fields in terms of K-theory and the T-duality group, an important subgroup of the U-duality group of M-theory. We pay particular attention to the effects of the topological phases in the supergravity action implied by the K-theoretic formulation of RR fields, and we use these to check the T-duality invariance of the partition function. We find that the partition function is only T-duality invariant when we take into account the T-duality anomalies in the RR sector, the fermionic path integral (including 4-fermi interaction terms), and 1-loop corrections including worldsheet instantons. We also discuss some issues which arise when one attempts to extend these considerations to checking the full U-duality invariance of the theory.

Acknowledgements

I would like to thank my thesis advisor, Gregory Moore, for his guidance and collaboration. I greatly appreciate the time he spent on discussions with me and on improving the drafts of our papers.

I am very grateful to Ori Ganor and Andrei Mikhailov for collaboration on the paper not included in this thesis.

I would also like to thank faculty at Princeton University, and especially Igor Klebanov and Alexander Polyakov for their invaluable insight.

I have learned a lot about the classics of string theory from our evening seminar. I would like to thank the organizers: Nissan Itzhaki, Leonardo Rastelli, John McGreevy and all participants for sharing their knowledge with me.

My last year in Princeton was supported by NSF grant PHY-9802484. I would like to thank them for their support.

Contents

Chapter 1

Why M-theory?

This thesis contains two independent parts. Each part studies certain aspects of M-theory and is endowed with its own introduction and conclusion. Now we would like to give an idea of M-theory and briefly summarize the results derived in Chapter2 and Chapter3.

M-theory is believed to be the theory that governs the laws of our Universe. This theory incorporates gravity, particle physics and quantum mechanics. M-theory has grown out of string theory. In string theory all fundamental particles are excitations of strings, the tiny lines moving in space-time. However, many basic questions about our world were not answered in the framework of string theory. For example, why our Universe presently has very small positive curvature? Why the observable masses of particles are so tiny in comparison to Plank mass, the characteristic scale of quantum gravity? To answer these questions we have to resolve the crucial problems of supersymmetry breaking and the choice of a true vacuum. String theory, being perturbative in nature, cannot resolve these issues. The quest for description of string dynamics at strong coupling gave rise to the concept of M-theory. So far we do not have M-theory in its final form but we know its low energy approximation and basic symmetries. The existence of these symmetries allows one to think of the five different string theories as various limits of M-theory, which are mapped into each other by duality transformations.

It is an outstanding open problem to construct quantum M-theory. Luckily, one can address some of the unresolved questions about our Universe by working with 11-dimensional supergravity, the low energy limit of M-theory. A promising model of our world relies on the effective low energy dynamics of the $E_8 \times E_8$ heterotic string theory. To understand this dynamics at strong string coupling we need M-theory. The phenomenologically interesting background includes a Calabi-Yau 3-fold and an interval with M5-branes inserted at points along it. In this model there are two worlds, our world and the hidden world. These two worlds live on the walls at the opposite ends of an interval. There is a hope that knowing the dynamics of this model it would be possible to reproduce observable physics. In Chapter 2 we obtain the effective potential for the scalars resulting from non-perturbative contributions due to open membrane instantons. We also discuss conditions under which the M5-branes may be attracted to the wall; and the chirality-changing phase transitions induced by an M5-brane hitting the wall. Our results were used in the analysis of modern cosmological scenarios [1].

The focus of Chapter 3 is the partition sum of M-theory on 11-manifolds $T^3 \times X$ where T^3 is a three-dimensional torus, and X is an 8-dimensional compact spin manifold. Duality symmetries, such as the U-duality symmetry of toroidally compactified M-theory, are key ingredients in the concept of M-theory. Topologically nontrivial effects associated with the RR sector have also played a crucial role in defining the theory. We investigate the interplay between the proper quantization of RR fields in terms of K-theory [80, 84, 86, 81, 111, 68, 72] and the T-duality group, an important subgroup of the full U-duality group. We demonstrate the cancellation of T-duality anomaly. We find that the partition sum is T-duality invariant only when the T-duality anomalies in the RR sector, the fermionic path integral, and one loop corrections are taken into account.

Chapter 2

Instabilities in heterotic M-theory induced by open membrane instantons.

2.1 Introduction

[1] In the past few years there have been significant advances in the study of strongly coupled heterotic string theory, thanks to the formulation in terms of M-theory on an interval S^1/Z_2 [2, 3]. In particular, the compactification of M-theory on a product of an interval with a Calabi-Yau 3-fold (denoted hereafter by $\mathcal{X}$) leads to qualitatively different physics from that of the weakly coupled heterotic string, as first noted in [4, 5, 16].

In heterotic string compactification one must choose an instanton configuration for gauge fields along $\mathcal{X}$. The so-called "standard embedding" identifies the gauge field with the spin connection of the metric. Other choices of gauge instantons, the so-called "nonstandard embeddings," are closely related, in the strongly coupled regime, to backgrounds obtained by including insertions of M5-branes wrapping a product of 4-dimensional spacetime with a holomorphic curve Σ in $\mathcal{X}$. At low energies, the physics of such backgrounds is summa-

[1] This chapter was done in collaboration with G.Moore and G.Peradze and is drawn from [115].

rized by a complicated $d = 4, N = 1$ supergravity theory. It has been shown in [17]-[21] that such backgrounds can lead to phenomenologically interesting gauge groups, and it is therefore of interest to understand more completely the full low energy supergravity in such backgrounds. While several aspects of the effective Lagrangian have been worked out in [4, 5],[17]-[21],[52, 53] (for a review see [46]) the Lagrangian is extremely complicated, and many details remain to be understood more thoroughly. The present chapter derives some further aspects of the low energy Lagrangian. Our main result is a formula for the potential energy for the moduli fields, valid in certain regions of moduli space. The detailed expression is given in eq. (2.102) et. seq. below, for the case when there is a single 5-brane insertion and $h^{1,1}(\mathcal{X}) = 1$. Since the derivation is rather long we explain here a few of the ingredients of this formula.

The chiral scalars in $d = 4$ supergravity take values in a target space which is Kähler-Hodge. These fields correspond to moduli for the Calabi-Yau metric on $\mathcal{X}$, moduli for the instanton gauge field along $\mathcal{X}$, and moduli for the positions of the M5 branes along the interval. In addition there are chiral fields charged under the gauge group H left unbroken by the $E_8 \times E_8$ instanton. These will generically be denoted by C^I.

The superpotential W is a section of a line bundle on the Kahler-Hodge target space for the chiral scalars. There are several sources for the superpotential in the effective supergravity. First of all, there is a perturbative term, cubic in the scalars C^I. In addition, there are several sources of nonperturbative effects. Some of these, such as heterotic worldsheet instantons, gluino condensation, and M5 instantons (wrapping $\mathcal{X}$) have been studied in many previous papers [7]-[16]. The inclusion of the effects of open membrane instantons, which have not been studied as thoroughly, is the main focus of this work.

There are three kinds of open membrane effects we must consider, since M2 branes can end both on M5-branes, or on the boundary "M9 brane" [29, 30]. Membranes stretching between the boundary M9-branes are the M-theory versions of heterotic worldsheet instantons, and as such have been studied in the context of (0,2) compactifications of heterotic string backgrounds [14, 15]. It is well-known that such effects often sum to zero, e.g., in backgrounds admitting a description by a linear sigma model [22, 23, 24]. The mechanism by which these contributions vanish is that a given homology class can contain many different holomorphic curves in $\mathcal{X}$. The instanton action depends only on the homology class, but the prefactor depends on the curve, and the sum of instanton amplitudes can, and often really does, vanish, as can already be seen in the case of the quintic. By contrast, the M2 instantons stretching between M5 and M9, or between M5 and M5 must wrap the particular holomorphic curve Σ already wrapped by the M5 brane. This is obvious for the part of the membrane worldvolume ending on the 5-brane. A study of the conditions for the supersymmetric instanton (based on [8]) reveals that the membrane must have a direct product structure $\Sigma \times I$ where I is an interval and $\Sigma \subset \mathcal{X}$ is a holomorphic curve. (The detailed argument is given in section 3 below.) Consequently, if Σ is a rigid holomorphic curve in $\mathcal{X}$ there will be no sum over instantons, and no integral over the moduli space for the curve. Moreover, if Σ is a rational curve there will be precisely two fermion zeromodes and the fermion 2-point function determining the superpotential will be nonzero. (Our calculation of the induced superpotential uses the technique discussed in [7, 8, 9].)

The backgrounds we study are in a regime of M-theory where we can do systematic expansions in the long wavelength expansion. It follows from [2, 3, 4] that this is an expansion in $R/V^{2/3}$ where R is the length of the interval S^1/Z_2 and V is the volume of $\mathcal{X}$ in 11-dimensional Planck units. We therefore assume $R/V^{2/3} \ll 1$. Now, gluino condensation and 5-brane instanton effects contribute terms of order $\Delta W \sim \exp[-c_1 V]$ to the superpotential W, where c_1 is of order 1. By contrast, open membrane effects contribute terms of order $\Delta W \sim \exp[-c_2 R V^{1/3}]$ where c_2 is of order 1 (or smaller). Thus, in the backgrounds under study in this work, *open membrane instantons are the leading source of nonperturbative effects.*

Our goal is to understand the physics of the moduli in heterotic M-theory, so we need the potential, rather than just the superpotential. The potential energy for scalars in $d = 4, N = 1$ supergravity is given by the famous formula [41, 40]

$$(\kappa_4)^4 U = e^K (K^{i\bar{j}} D_i W \overline{D_j W} - 3W\overline{\mathbf{W}}) + U_D \tag{2.1}$$

where $2\kappa_4^2 = 16\pi G_N$ is the (four-dimensional) Newton constant, K is the Kähler potential, $D_i W = \partial_i W + \partial_i K W$ is the covariant derivative, and U_D are "D-terms" for charged scalars $\sim \sum_a (\bar{C} T^a C)^2$.

The potential (2.1) is extremely complicated. Moreover, K is only approximately known only in some regions of moduli space. We are therefore forced to consider perturbation expansions in several quantities. First, we will expand in two dimensionless parameters

$$\mathcal{E}^{eff} \sim R/V^{2/3} \ll 1 \qquad \mathcal{E}_R^{eff} \sim V^{1/6}/R \ll 1 \tag{2.2}$$

which are necessary for the validity of the geometrical 11-dimensional picture (more precise formulae appear in eq. (2.11) below). Note that these imply that $V \gg 1$ and $R \gg 1$, and that the length of the interval is much larger than the scale set by $\mathcal{X}$. In addition we must expand in powers of the charged scalars C^I. The superpotential is a sum of two terms $W = W_{\rm pert} + W_{\rm nonpert}$, where $W_{\rm pert}$ is a cubic expression in the charged scalars C^I with coefficients that are functions of the complex structure and bundle moduli. We can organize the terms according to whether they are order $0, 1$, or 2 in $W_{\rm nonpert}$:

$$(\kappa_4)^4 U = (U_0 + U_1 + U_2) \tag{2.3}$$

We will now describe the leading expressions for the three terms in (2.3) in the case of a Calabi-Yau $\mathcal{X}$ with $h^{1,1}(\mathcal{X}) = 1$ together with a single 5-brane, inserted at x, where $0 \le x \le 1$ labels the position of the 5-brane along the M-theory interval. In addition to the charged scalars C^I the relevant chiral superfields are the "volume superfield" $S = V + i\sigma$, which determines the GUT coupling, the "Kähler superfield" $T = Ra + i\chi$, where a is the Kähler modulus for $\mathcal{X}$ (hence $V \sim a^3$), and the "position superfield" $Z = Rax + i\alpha$ for the 5-brane. The fields σ, χ and α are axions.

The first term, U_0, in (2.3) begins with the perturbative contribution to the potential. The leading order expression in an expansion in the charged scalars is a positive semidefinite quartic form:

$$U_0 = \frac{1}{VJ^2} U_{I\bar{J}K\bar{L}} C^I \bar{C}^{\bar{J}} C^K \bar{C}^{\bar{L}} \Big(1 + O(\frac{C^2}{J}, \mathcal{E}^{eff}, \mathcal{E}_R^{eff})\Big) \tag{2.4}$$

Here $J := Ra$. The coefficients $U_{I\bar{J}K\bar{L}}$ are functions of the complex structure and bundle moduli. We will give precise formulae for them, but will not be very explicit about their behavior.

The leading contribution to the second term in (2.3) is a one-instanton term resulting from cross terms between the perturbative and nonperturbative superpotentials. We find that the single instanton contribution has the form

$$U_1 = \frac{(1-x)}{VJ^2}\left\{e^{-Jx}\mathrm{Re}[U_{IJK}C^IC^JC^Ke^{i\alpha}] - e^{-J(1-x)}\mathrm{Re}[U_{IJK}C^IC^JC^Ke^{i(\chi-\alpha)}]\right\} + \cdots \quad (2.5)$$

The coefficients U_{IJK} are functions only of the complex structure and bundle moduli.

Finally, the third term U_2 in (2.3) begins with a 2-instanton effect

$$U_2 = \frac{E}{J^2}\Bigg\{e^{-2Jx} + e^{-2J(1-x)} - 2e^{-J}\cos(2\alpha-\chi) \quad (2.6)$$

$$+\frac{2J}{3V}(1-2x)e^{-2J(1-x)} + \frac{4Jx}{3V}e^{-J}cos(2\alpha-\chi) + \cdots\Bigg\}\left(1+\mathcal{O}(\frac{C^2}{J},\mathcal{E}^{eff},\mathcal{E}_R^{eff})\right)$$

where E is a positive definite function that depends only on the complex structure and bundle moduli. (We have kept some subleading terms in the second line. The reason for this is explained in detail in sections 5.4 and 5.5.)

A precise characterization of the region of validity of the above potential is given in section 5.4 below. The strongest constraints on the region of validity come from our ignorance of the exact Kähler potential. It is also important to bear in mind that the coefficients of the higher order terms in the expansion in $\frac{|C|^2}{J}, \mathcal{E}^{eff}, \mathcal{E}_R^{eff}$ are functions of the complex structure and bundle moduli. If these coefficients become singular somewhere in the moduli space then these "higher order" terms will dominate the physics. Our working assumption is that we are at a generic smooth point in bundle and complex structure moduli space.

Having determined the leading nonperturbative effects, and thereby the potential energy, we investigate briefly some of the resulting classical dynamics on moduli space, at a somewhat heuristic level. Although the M5 branes wrap all of spacetime, thanks to the central term in the superalgebra, their positions along the M-theory interval are in fact dynamical variables. In the regions where we can trust our answer we find two kinds of instabilities in the compactification, depending on whether the effects of vevs of the charged scalars C^I are important or not. When the charged scalar vevs are important, the leading x-dependent effect is a one-instanton effect. The axions will evolve to produce an attractive force between the M5 brane and the nearest M9-wall. This could possibly be interpreted as a consequence of the Witten effect: the axions evolve and continuously change an effective brane charge in order to produce "the most attractive channel," in particular producing an attraction between the 5-brane and the boundary. It would be interesting to understand the physics of this effect more fully.

The above discussion is valid for $Jx \gg 1$. As the five-brane moves towards the wall the approximations break down. The limit $x \to 0$ is extremely interesting and is related to the chirality-changing transitions discussed in [60, 45]. In order to study this limit one needs a multiple cover formula for the membrane instantons. This is discussed in section six below. We make some educated guesses and conclude that the physics depends on the (unknown) details of the covering formula.

A second kind of instability occurs when charged scalar vevs are small or zero. In this case the potential has a local minimum in x at $x = \frac{1}{2}$. The value of U at such points is small and of the form

$$U \sim \gamma \frac{e^{-J}}{JV}$$

where γ is a positive function of the complex structure and bundle moduli. The M2 branes lead to a repulsive interaction between the M5-brane and the M9-brane which induces decompactification of both the M-theory radius and the Calabi-Yau, while the M5-brane moves to the middle of the interval. Of course, in this instability new light modes appear as the theory becomes five-dimensional, and we should describe a matching to a description in terms of five-dimensional supergravity. (As the M5 moves to the middle of the interval there is a balancing of forces from the two boundaries and the leading terms in U_2 vanish. This is why we must include the subleading terms.)

The second kind of instability is an 11-dimensional manifestation of the Dine-Seiberg problem; it is hardly unexpected, and in the case of the standard embedding similar instabilities have already been pointed out by Banks and Dine in [5]. Nevertheless, it is interesting to note that in the 10-dimensional Dine-Seiberg instability the size of the M-theory interval S^1/Z_2 tends to *shrink*. There are thus different asymptotic regions of moduli space with qualitatively different dynamics, and hence different "basins of attraction" for the classical evolution of the moduli. One consequence is that there must be nontrivial stationary points for the potential in the middle of moduli space. The precise nature of such stationary points is of great interest, but remains out of reach so long as we cannot derive the Kähler potential in the interior of moduli space in a controlled approximation.

The Chapter2 is organized as follows. In section two we review briefly the M-theory geometry corresponding to strongly coupled $E_8 \times E_8$ heterotic strings with "nonstandard embedding." In section three we study supersymmetric M2-brane instantons in $\mathcal{X} \times S^1/Z_2$. In section four we derive the formula for the contribution to the superpotential from M2 instantons. In section five we find the potential and specify the region where we can trust it for the simplest case of a Calabi-Yau with $h^{(1,1)} = 1$. In section six we discuss the multiple covering formula and its relevance to chirality-changing transitions. In section seven we generalize the result to the case of N 5-branes on the interval. The final section contains a discussion of some possible extensions of the present work.

2.2 Review of heterotic M-theory background with M5-branes on the interval

In this section we review some of the results of ([4, 5],[17] - [21],[46]) which are needed for our subsequent computations.

Our conventions for the Lagrangian of 11D SUGRA are set by the Lagrangian:

$$2\kappa_{11}^2 S_{11D} = -\int eR - \frac{1}{2}\int G_4 \wedge * G_4 - \frac{1}{6}\int C_3 \wedge G_4 \wedge G_4 + \dots \tag{2.7}$$

where $G_{MNPQ} = 4\partial_{[M}C_{NPQ]}$. [2]

[2]We have a different normalization of fields compared to ([17]). $G^{here}_{MNPQ} = \sqrt{2}G^{[17]}_{MNPQ}$, $C^{here}_{MNP} = 6\sqrt{2}C^{[17]}_{MNP}$. We use the convention $2\kappa_{11}^2 = (2\pi)^8(M_{11})^{-9}$, and define the 11-dimensional Planck length by $l_{11} = 1/M_{11}$. Our signature is mostly plus.

The Lagrangian of the boundary $E8 \times E8$ theory is given by

$$2\kappa_{11}^2 S_{YM} = -\frac{1}{4\pi}\left(\frac{\kappa_{11}}{4\pi}\right)^{\frac{2}{3}} \int_{M_1^{10}} \sqrt{-g} tr\left(F^{(1)}\right)^2 - \frac{1}{4\pi}\left(\frac{\kappa_{11}}{4\pi}\right)^{\frac{2}{3}} \int_{M_2^{10}} \sqrt{-g} tr\left(F^{(2)}\right)^2 \quad (2.8)$$

where $F^{(1,2)}$ are the field strengths of the two E_8 gauge fields, to leading order in a long-wavelength expansion. In the above action and below tr means $\frac{1}{30}$ of the trace in the adjoint of E_8.

We begin by describing the background solution of M-theory on $R^4 \times \mathcal{X} \times S^1/Z_2$. Our coordinates on R^4 are x^μ, $\mu = 1, \dots, 4$. Complex coordinates along $\mathcal{X}$ have indices $m, \overline{m} = 1, \dots, 3$. The factor S^1/Z_2 in spacetime has coordinate X^{11}. In addition it will be convenient to set $X^{11} = \pi\rho y$ where y is a dimensionless coordinate $0 \leq y \leq 1$, and ρ is a dimensionful constant which sets a scale.

We must now specify the metric, four-form G_4, and boundary Yang-Mills fields. In order to write the background metric we introduce a basis of harmonic (1,1) forms on $\mathcal{X}$, $\omega_i, i = 1, \dots, h^{1,1}$ and denote the Kahler form on $\mathcal{X}$ by $\omega = a^i\omega_i$. Then, the background metric is a deformation of a metric of the form

$$ds_{11}^2 = V^{-1}R^{-1}g_{\mu\nu}dx^\mu dx^\nu + R^2(dX^{11})^2 - 2i\omega_{m\overline{m}}dx^m dx^{\overline{m}}. \quad (2.9)$$

In this formula R is dimensionless and $R\rho$ is the orbifold radius. Similarly, we introduce a fiducial, dimensionful, volume v for $\mathcal{X}$, and the volume of $\mathcal{X}$ in the metric (2.9) defines the dimensionless parameter V by $Vv := \frac{1}{3!}\int_{\mathcal{X}} \omega^3$. We will make a convenient choice of ρ, v in eq. (2.12) below; they will be of the order of l_{11}, l_{11}^6 and are independent of moduli. Because of the Weyl-rescaling in the first term in (2.9), $g_{\mu\nu}$ is the four-dimensional Einstein metric and the four-dimensional Newton constant is given by

$$\frac{1}{\kappa_4^2} = \frac{2\pi\rho v}{\kappa_{11}^2}. \quad (2.10)$$

As we have mentioned, the actual metric we will use is a deformation of eq. (2.9), and is only known to first order in a power series in two dimensionless expansion parameters

$$\mathcal{E}^{eff} = \frac{\epsilon R}{V^{\frac{2}{3}}} \ll 1, \quad \mathcal{E}_R^{eff} = \frac{\epsilon_R V^{\frac{1}{6}}}{R} \ll 1 \quad (2.11)$$

where we choose the constants

$$\epsilon = \left(\frac{\kappa_{11}}{4\pi}\right)^{\frac{2}{3}} \frac{2\pi^2\rho}{v^{\frac{2}{3}}} = 2, \qquad \epsilon_R = \frac{v^{\frac{1}{6}}}{\pi\rho} = (\frac{\pi}{2})^{\frac{1}{2}} \tag{2.12}$$

in order to simplify the normalization of the fields in the effective Lagrangian.[3]

The above inequalities (2.11) state, firstly, that the distortion of the background from (2.9) is small, and secondly that the interval is much larger than the length scale of $\mathcal{X}$. These expansion parameters can be related to the GUT scale and the 4-dimensional Newton constant [4, 5]. In our conventions the unified coupling $\alpha_{\mathrm{GUT}} \sim (\mathcal{E}^{eff}\mathcal{E}_R^{eff})^2 \sim 1/V$, while $(M_{\mathrm{GUT}}\kappa_4)^2 \sim (\mathcal{E}^{eff})^3(\mathcal{E}_R^{eff})^4 \sim 1/(RV^{4/3})$ determines the GUT scale in terms of the Newton constant. The latter formula follows by computing masses of gauge bosons and scalars associated with typical mechanisms of spontaneous symmetry breaking. [4] Unfortunately, it turns out that when we use the experimentally measured values of $\alpha_{\mathrm{GUT}}, M_{\mathrm{GUT}}$ and κ_4 the above expansion is not necessarily a good approximation. As discussed in [4, 5, 6, 17, 54], the experimentally measured values determine $\mathcal{E}_R^{eff} \ll \mathcal{E}^{eff} = O(1)$. Nevertheless, our focus in this work is on a systematic and controlled computation of nonperturbative effects; the restriction (2.11) is necessary since heterotic M-theory is only known as an effective theory to order $(\kappa_{11})^{\frac{2}{3}}$, and for this reason we will adopt it.

To lowest order in the expansion parameter the metric for the background takes the form

$$ds_{11}^2 = V^{-1}R^{-1}(1+\frac{B}{6})g_{\mu\nu}dx^\mu dx^\nu + R^2(1-\frac{2B}{3})(dx^{11})^2 - 2iJ_{m\overline{m}}dx^m dx^{\overline{m}}, \tag{2.13}$$

$$J_{m\overline{m}} = \omega_{m\overline{m}} + (B_{m\overline{m}} - \frac{1}{3}\omega_{m\overline{m}}B), \qquad B = 2\omega^{m\overline{m}}B_{m\overline{m}},$$

The deformation of the background is described by the (1,1) form $B_{n\bar{m}}$. In order to write it explicitly we must now introduce the M5 branes.

The backgrounds we study preserve $N=1$ supersymmetry. Therefore the 5-branes wrap a product of spacetime and a holomorphic curve in $\mathcal{X}$. If there are N 5-branes they will therefore have definite locations at $y = x_k$, $k = 1, \ldots, N$ along the interval. The k^{th} 5-brane

[3] We take $v = 8\pi^5 l_{11}^6$, $\pi\rho = 2\pi^{\frac{1}{3}} l_{11}$ to have $\epsilon = 2, {\epsilon_R}^2 = \frac{1}{2}\pi$

[4] We thank T. Banks for very helpful discussions on this point.

wraps a curve $\Sigma^{(k)}$ in $\mathcal{X}$ whose homology class may be expressed as $[\Sigma^{(k)}] = \beta_i^{(k)}[\Sigma_2^i]$ where $[\Sigma_2^i]$ is an integral basis of $H_2(X, \mathbf{Z})$, and $\beta_i^{(k)}$ is a collection of nonnegative integers. These integers are constrained by anomaly cancellation. Each of the M9 branes carries an E_8 vector bundle V_1, V_2, and to each bundle we associate a degree four integral characteristic class $c_2(V_i)$. Identifying $H_2(\mathcal{X}; Z)$ with $H^4(\mathcal{X}; Z)$ via Poincaré duality we may define

$$c_2(V_1) - \frac{1}{2}c_2(TX) = \beta_i^{(0)}[\Sigma_2^i] \qquad c_2(V_2) - \frac{1}{2}c_2(TX) = \beta_i^{(N+1)}[\Sigma_2^i] \tag{2.14}$$

The anomaly cancellation condition is then

$$\sum_{n=0}^{N+1} \beta_i^{(n)} = 0. \tag{2.15}$$

In terms of the above data, the formula for $B_{m\bar{m}}$ on the interval $(x_n, x_{n+1}), \quad n = 0, \ldots, N$ is given by

$$B_{m\bar{m}} = \frac{2R}{V} b_i \omega^i_{m\bar{m}}, \qquad b_j(y) = \sum_{k=0}^{n} \beta_j^{(k)}(y - x_k) - \frac{1}{2}\xi_j, \qquad \xi_j = \sum_{k=0}^{N+1}(1 - x_k)^2 \beta_j^{(k)}, \tag{2.16}$$

where $x_0 = 0, x_{N+1} = 1$ and the index i is raised with the inverse of the metric on the moduli space of Kahler structures on $\mathcal{X}$:

$$G_{ij} = \frac{1}{2vV}\int_{\mathcal{X}} \omega_i \wedge (*\omega_j) = -\frac{1}{2}\partial_i\partial_j ln\Big(d_{i_1 i_2 i_3} a^{i_1} a^{i_2} a^{i_3}\Big) \tag{2.17}$$

with

$$d_{i_1 i_2 i_3} = \int_{\mathcal{X}} \omega_{i_1} \wedge \omega_{i_2} \wedge \omega_{i_3}. \tag{2.18}$$

The choice of integration constant in the solution (2.16) fixes Vv to be equal to the volume of $\mathcal{X}$ averaged along the interval (to lowest order in $\mathcal{E}^{eff}$).

The flux of the 4-form G_4 is also given in terms of B:

$$G_{MNPQ} = \frac{1}{2}\epsilon_{MNPQEF}\partial_{11}B^{EF} \tag{2.19}$$

Note that it is discontinuous across the positions of the 5-branes.

Finally, we need to specify the E_8 gauge bundles V_1 and V_2. For simplicity we will follow [18] and take the bundle V_2 at $y = 1$ to be the trivial bundle. Accordingly, there is a "hidden sector" at $y = 1$ with unbroken E_8 gauge group. The bundle V_1 at $y = 0$ has an instanton whose holonomy lives in a subgroup $G \subset E_8$. The unbroken gauge symmetry is the commutant H of G in E_8. It is straightforward to extend our formulae to the case when both V_1 and V_2 are non-trivial bundles.

When we compactify M-theory on the above background, the physics at distances large compared to the M-theory interval is described by an effective $d = 4, N = 1$ supergravity theory. We now list the massless fields corresponding to small fluctuations around the above background. In addition to the superYang-Mills and supergravity multiplets there are a number of massless chiral scalar fields. To begin with, there are chiral superfields neutral under four dimensional gauge group H. These are:

$$T^i = Ra^i + i\chi^i, \tag{2.20}$$

$$S = V + i\sigma, \tag{2.21}$$

$$Z_n = R(\beta_i^{(n)} a^i) x_n - i\Big[\mathcal{A}_n(\beta_i^{(n)} a^i) - x_n(\beta_i^{(n)} \chi^i)\Big] \tag{2.22}$$

where

$$C_{m\overline{m}11} = \chi^i \omega_{i,m\overline{m}}, \qquad i = 1, \dots, h^{1,1}, \quad m, \overline{m} = 1, \dots, 3,$$

σ is a scalar dual to $C_{\mu\nu 11}$

$$3\partial_{[\mu} C_{\nu\rho]11} = V^{-2} \epsilon_{\mu\nu\rho\lambda} \partial^\lambda \sigma,$$

and Z_n is a holomorphic coordinate constructed out of the position x_n of the n-th 5-brane on the interval. The scalar $\mathcal{A}_n$ originates from the KK reduction of the 2-form living on the n-th 5-brane

$$A_n^{(2)} = \pi\rho \mathcal{A}_n f_n^*(\omega) \tag{2.23}$$

We have included the factor $\pi\rho$ in the above formula to make $\mathcal{A}_n$ dimensionless. In eq.(2.23) $f_n^*(\omega)$ is the pullback of the Kahler form to the cycle $\Sigma_2^{(n)}$. We denote by f_n the holomorphic embedding of the curve $\Sigma_2^{(n)}$ in $\mathcal{X}$. The pullback of each of the basis forms $f_n^*(\omega_i)$ is proportional to the pullback of the Kahler form ω

$$f_n^*(\omega_i) = \frac{\beta_i^{(n)}}{(\beta_j^{(n)} a^j)} f_n^*(\omega), \qquad \int_{\Sigma_2^{(n)}} f_n^*(\omega_i) = v^{\frac{1}{3}} \beta_i^{(n)}.$$

Finally, there are chiral multiplets charged under the unbroken gauge group H. Thanks to the Donaldson-Uhlenbeck-Yau theorem massless modes from small fluctuations of the gauge field can be associated with holomorphic deformations of holomorphic bundles on $\mathcal{X}$. The small fluctuations are parametrized by the space $H^{0,1}_{\bar\partial}(X,V)$ where V is the gauge bundle in the **248**. We assume the holonomy of the instanton is in G so the gauge bundle decomposes as $V = \oplus W_{\mathcal{R}} \otimes V_{\mathcal{S}}$ corresponding to the decomposition of the adjoint of E_8 under the embedding $H \times G \subset E_8$:

$$\mathbf{248} = \oplus \mathcal{R} \otimes \mathcal{S} \tag{2.24}$$

The charged scalars will be valued in $\oplus W_{\mathcal{R}} \otimes H^{0,1}_{\bar\partial}(X, V_{\mathcal{S}})$. In order to work out the Kaluza-Klein reduction we decompose the gauge field as:

$$A_{\overline{m}} = \frac{2^{\frac{3}{2}}\pi}{\kappa_4} u_{\hat{I},\overline{m}} C^{\hat{I}}, \quad \overline{m} = 1,2,3, \tag{2.25}$$

In (2.25) a summation is taken over the index $\hat{I}$ which labels

$$\hat{I} = (\mathcal{R}, I, p), \quad p = 1, \ldots, \dim \mathcal{R}, \quad I = 1, \ldots \dim H^1(X, V_{1\mathcal{S}}).$$

The normalization factor in (2.25) was chosen to make the charged scalar fields $C^{\hat{I}}$ dimensionless and to normalize their kinetic term conveniently.

When writing the perturbative superpotential below it will be convenient to define

$$u_{\hat{I},\overline{m}} = u^x_{I\overline{m}} T_{xp},$$

where x is an index for a basis for the representation $\mathcal{S}$ and $u^x_{I\overline{m}}$ is a basis of $H^1(X, V_{1\mathcal{S}})$. The factor T_{xp} is purely group-theoretic and corresponds to the generators of E_8 in the representation $\mathcal{R} \otimes \mathcal{S}$. The complex conjugate of these generators is denoted by T^{xp} and the normalization is chosen such that $tr\left(T_{xp}T^{yq}\right) = \delta^y_x \delta^q_p$.

We are not going to study four-dimensional gauge dynamics in this work. This has been studied, for example, in [4, 5, 18]. For completeness, and to fix our normalizations, we also give the gauge kinetic term in the 4D Lagrangian

$$S_{YM} = -\sum_{\alpha=1}^{2} \frac{1}{64\pi^2} \int_{M_4^{(\alpha)}} \sqrt{-g_4}\Big(Ref^{\alpha} trF^2 + \ldots \tag{2.26}$$

where on the brane at $y = 0$

$$Ref^{(1)} = V + Ra^i\Big(\beta_i^{(0)} + \sum_{n=1}^{N}(1 - x_n)^2\beta_i^{(n)}\Big) \tag{2.27}$$

and on the brane at $y = 1$

$$Ref^{(2)} = V + Ra^i\Big(\beta_i^{(N+1)} + \sum_{n=1}^{N}(x_n)^2\beta_i^{(n)}\Big) \tag{2.28}$$

Note, that due to the restrictions (2.11) on the moduli space, $Ref^{\alpha} = V + O(\mathcal{E}^{eff}), \quad \alpha = 1, 2$.

2.3 M2-brane instantons in $\mathcal{X} \times S^1/Z_2$

Open M2-branes ending on an M5 brane will play a crucial role in our calculation of the non-perturbative potential. These nonperturbative effects were first discussed in [29, 30, 43]. In this section we will derive the conditions for a supersymmetric open M2-brane instanton in the background described in the previous section. We will neglect the distortion of the background metric from a direct product metric in solving for the membrane configuration. This is valid in our approximation scheme.

The first step in finding the supersymmetric M2 configuration is to write the constant spinors corresponding to the supersymmetries unbroken by the background. We use a basis for the Γ-matrices in eleven dimensions of the form

$$\Gamma^{\mu} = (RV)^{\frac{1}{2}}\gamma^{\mu} \otimes \gamma^7 \quad \Gamma^m = 1 \otimes \gamma^m \quad \mu = 1, \ldots, 4, \quad \{\gamma^{\mu},\ \gamma^{\nu}\} = 2g^{\mu\nu} \tag{2.29}$$

$$\Gamma^{\overline{m}} = 1 \otimes \gamma^{\overline{m}} \quad \Gamma^{11} = \tfrac{1}{R}\gamma^5 \otimes \gamma^7 \quad m, \overline{m} = 1, \ldots, 3, \quad \{\gamma^{n},\ \gamma^{\overline{m}}\} = 2g_{CY}^{n\overline{m}} \tag{2.30}$$

where $(\gamma_m)^* = -\gamma_{\overline{m}} = (\gamma_m)^T$ and γ^{μ} is a weyl-basis in 4D.

Four dimensional anti-chiral (chiral) spinor indices are denoted by α ($\dot{\alpha}$) respectively. In this basis the surviving supersymmetry in the background $\mathcal{X} \times S^1/Z_2$ is of the form:

$$\epsilon = \Big(\epsilon^{\dot{\alpha}} \otimes \epsilon_1, \epsilon^{\alpha} \otimes \epsilon_2\Big), \tag{2.31}$$

where $\epsilon^{\dot{\alpha}}, \epsilon^{\alpha}$ are constant spinors on $R^4 \times S^1/Z_2$ and ϵ_1 ($\epsilon_2 = (\epsilon_1)^*$) is the chiral(anti-chiral) covariantly constant spinor on $\mathcal{X}$, normalized as in [8]:

$$\gamma_{\overline{m}}\epsilon_1 = 0, \quad \gamma_{n\overline{m}}\epsilon_1 = i\omega_{n\overline{m}}\epsilon_1, \quad \gamma_{mnp}\epsilon_1 = e^{-K}\Omega_{mnp}\epsilon_2, \quad \epsilon_1^{\dagger}\epsilon_1 = 1. \tag{2.32}$$

Here ω is the Kahler form, Ω is a holomorphic (3,0) form on $\mathcal{X}$ and $K = \frac{1}{2}(K_T - K_{cplx})$ with both Kahler functions K_T and K_{cplx} specified in section (5.2).

The surviving supersymmetry is consistent with having 5-branes wrapped over a holomorphic cycle $\Sigma \subset \mathcal{X}$, as shown in [17]. One cannot have anti-5-branes on the interval and preserve supersymmetry.

The presence of an M2-brane imposes an additional constraint on the supersymmetry parameter ϵ

$$\Gamma^{(2)}\epsilon = \epsilon, \tag{2.33}$$

where, (see for example [8]),

$$\Gamma^{(2)} = \frac{i}{3!\sqrt{g}}\epsilon^{ijk}\partial_i X^{\hat{M}}\partial_j X^{\hat{N}}\partial_k X^{\hat{K}}\Gamma_{\hat{M}\hat{N}\hat{K}} \tag{2.34}$$

In formula (2.34) $s^i, i = 1,2,3$ are coordinates on the world-volume of the M2-brane, $X^{\hat{M}}$, $\hat{M} = (\mu, m, \overline{m}, 11)$ are coordinates in the eleven dimensional target space and g is the determinant of the induced metric on the M2-brane.

Substituting (2.31) into (2.33) we find, first of all, that spinors of type $\epsilon^{\alpha}\otimes\epsilon_2$ lead to

$$\epsilon_2 = \Big(\frac{R}{\sqrt{g}}\epsilon^{ijk}\partial_i X^m\partial_j X^{\overline{n}}\partial_k X^{11}\omega_{m\overline{n}}\Big)\epsilon_2 + \Big(\frac{ie^{-K}}{3!\sqrt{g}}\epsilon^{ijk}\partial_i X^{\overline{m}}\partial_j X^{\overline{n}}\partial_k X^{\overline{p}}\Omega_{\overline{mnp}}\Big)\epsilon_1 \tag{2.35}$$

$$+\Big(\frac{\epsilon^{ijk}}{\sqrt{g}}\partial_i X^m\partial_j X^{\overline{n}}\partial_k X^{\overline{p}}\omega_{m\overline{n}}\Big)\gamma_{\overline{p}}\epsilon_2 + \Big(\frac{ie^{-K}}{4\sqrt{g}}\epsilon^{ijk}\partial_i X^{\overline{m}}\partial_j X^{\overline{n}}\partial_k X^{11}\Omega_{\overline{mnp}}g_{CY}^{\overline{p}q}\Big)\gamma_q\epsilon_1$$

Since the spinors $\epsilon_1, \epsilon_2, \gamma_m\epsilon_1, \gamma_{\overline{m}}\epsilon_2$ are linearly independent we get four equations

$$\partial_i X^{\overline{m}}\partial_j X^{\overline{n}}\partial_k X^{\overline{p}}\Omega_{\overline{mnp}} = 0 \tag{2.36}$$

$$R\partial_i X^m\partial_j X^{\overline{n}}\partial_k X^{11}\omega_{m\overline{n}} = \sqrt{g}\epsilon_{ijk} \tag{2.37}$$

$$\partial_i X^{\overline{m}}\partial_j X^{\overline{n}}\partial_k X^{11}\Omega_{\overline{mnp}}g_{CY}^{\overline{p}q} = 0 \tag{2.38}$$

$$\epsilon^{ijk}\partial_i X^m \partial_j X^{\overline{n}} \partial_k X^{\overline{p}} \omega_{m\overline{n}} = 0 \tag{2.39}$$

The constraints (2.36, 2.39) are automatically solved by the embedding

$$X^{11} = t, \quad X^m(y)$$

where $t = s^3$ is a coordinate along the orbifold interval and $y, \overline{y}$ are coordinates on a holomorphic 2-cycle. This is our basic instanton.

We claim that if the holomorphic curve $\Sigma \subset \mathcal{X}$ is isolated then the above membrane instanton is also. Moreover, we claim that the above instanton is the only instanton solution consistent with the boundary condition of having the M2 brane ending on Σ. Indeed, let us consider the possibility of having M2-branes starting and ending on holomorphic cycles inside $\mathcal{X}$ which differ from a direct product $\Sigma \times I$. Therefore we search for t-dependent embeddings $X^m(y,t), t \in [x_1, x_2]$ into $\mathcal{X}$. In this case equation (2.39) is not satisfied automatically and gives the constraint

$$\partial_{[i} X^m \partial_{j]} X^{\overline{n}} \omega_{m\overline{n}} = 0, \tag{2.40}$$

Taking the $i = y, j = \overline{y}$ component of this equation and evaluating it at the boundary $t = x_1$ or $t = x_2$ shows that the volume of the holomorphic cycle must be zero.

We conclude that an open M2-brane which starts and ends on a positive volume holomorphic curve preserves some supersymmetry iff it has the direct product form $\Sigma \times I$.

One can quite analogously prove that an M2-brane which starts and ends on a holomorphic curve should have the direct product form

$$X^{11} = -t, \quad X^m(y)$$

in order to preserve the other components $\epsilon^{\dot{\alpha}} \otimes \epsilon_1$ of the background supersymmetry.

Note that since the M2-brane instanton must start and end on the same 2-cycle in $\mathcal{X}$ there is a requirement on the 5-brane charges $\beta_i^{(n)} = \beta_i^{(k)}$ described in section 2 in order for there to be an M2-instanton stretched in the interval $[x_n, x_k]$.

2.4 Calculation of membrane-instanton-induced superpotentials

In this section we will give the derivation of the non-perturbative four-dimensional superpotential ΔW induced by open membranes.

We follow the procedure outlined in [8, 9]. The idea is to compute the 2-point correlation function of four-dimensional fermions with the instanton sector included in the supergravity path integral. An essential ingredient of this calcultaion is the coupling of the four-dimensional fermions to the world-volume degrees of freedom of the membrane through the so-called "membrane vertex operators." The computation of the superpotential follows from a computation of a 2-point correlation function of fermions in the four-dimensional effective theory $\langle\chi\chi\rangle_{inst}$, where χ are fermionic superpartners of Z. This in turn can be reduced to a membrane path integral with corresponding vertex operator insertions.

2.4.1 Summary of the computation of ΔW

Since the analysis is rather long let us summarize the computation here. Most of the work is devoted to finding the vertex operator, but the end result is very simple. The membrane theory has a chiral doublet of fermions $\vartheta^{\dot{\alpha}}$ transforming in the **2** of the 4 dimensional Lorentz group. These couple to the chiral fermions $\chi_{\dot{\alpha}}$ in the superfield Z via the vertex operator

$$V_\chi = \frac{i}{2}\vartheta^{\dot{\alpha}}\chi_{\dot{\alpha}}. \tag{2.41}$$

Using the above coupling we can compute $\langle\chi(\xi_1)\chi(\xi_2)\rangle$ in an instanton sector to be

$$\int \sqrt{-g_4}d^4\xi S_F(\xi_1-\xi)S_F(\xi_2-\xi)h\Phi exp(-Z). \tag{2.42}$$

Here ξ_1, ξ_2 are points in four dimensions and S_F is the 4-dimensional fermion propagator in the effective $d=4, N=1$ supergravity. This expression for the propagator is only valid for $(\xi_1-\xi), (\xi_1-\xi_2), (\xi_2-\xi) \gg l_{11}$. The integral of ξ^μ in eq.(2.42) should be regarded as an integral over the bosonic zero modes $X^\mu = \xi^\mu$ of the M2-instanton. The integral over the 2 fermion zero-modes ϑ^1, ϑ^2, on the M2-brane soaks up the $\vartheta^{\dot{\alpha}}$ from the vertex operator.

There are no other zero modes because the curve Σ is a rational curve and hence has no extra zero-modes associated with 1-forms. The prefactor $h\Phi$ stands for determinants of fluctuations in 11-dimensional supergravity together with 5-branes around the background (2.13,2.19), together with determinants associated with the degrees of freedom for the M2 instanton. While it is very complicated one can use holomorphy to extract the factor he^{-Z}, which depends holomorphically on the moduli. The factor h is a holomorphic section of a line bundle over complex structure moduli space and should properly be regarded as the true measure for the fermion zeromodes. In this work we will not be very explicit about it.

We can now extract ΔW by comparing (2.42) with the 2-point correlation function in the effective 4D supergravity

$$\langle \chi(\xi_1)\chi(\xi_2) exp[\int \sqrt{g_4} e^{\frac{1}{2}K} \partial_Z \partial_Z (\Delta W) \bar{\chi}\bar{\chi}] \rangle_{4D} \tag{2.43}$$

which is equal to

$$\int \sqrt{g_4} d^4\xi S_F(\xi_1 - \xi) S_F(\xi_2 - \xi) e^{\frac{1}{2}K_0} \partial_Z \partial_Z (\Delta W) \tag{2.44}$$

where $K_0 = K_T + K_S + K_{cplx} + K_{bundle}$ and we drop corrections of the order $O\Big(\mathcal{E}^{eff}, \mathcal{E}_R^{eff}, \frac{|C|^2}{Ra}\Big)$ to the mass term for a chiral fermion in the 4D, N=1 Lagrangian [41, 40].

Using holomorphy of the superpotential it now follows that

$$\Delta W = hexp(-Z), \quad \Phi = e^{\frac{1}{2}K_0}. \tag{2.45}$$

For the M2-brane stretched between the 5-brane and the other 9-brane at $y = 1$ analogous considerations give

$$\Delta W = hexp(-(\beta_i T^i - Z)). \tag{2.46}$$

2.4.2 Computation of the vertex operator

In this section we describe the computation of the vertex operator.

The vertex operators can be found by expanding the action of the $M2$ brane in the M-theory background fields. The action of an M2-brane ending on an M5-brane was written in

[36], using the superembedding approach of [31, 32]. In this approach the basic ingredients are:

- An (11|32) supermanifold M, giving the 11-dimensional supergravity background. The supercoordinates are denoted by $Z^{\underline{M}} = (X^{\hat{M}}, \Theta^{\hat{\rho}})$. where $\hat{\rho}$ is an index in the irreducible spinor representation of $so(1,10)$. Using the torsion constraints of [33] on the supervielbein one can expand an orthonormal frame for the cotangent space in powers of Θ. The expansion at low orders in Θ has been worked out in [9, 35, 10].

  It is convenient to introduce the notation for the vielbein:

$$E^{\underline{A}}(Z) = dZ^{\underline{M}} E_{\underline{M}}{}^{\underline{A}} = (E^{\underline{a}}, E^{\underline{\alpha}}) \tag{2.47}$$

  where in the second equality we have separated bosonic and fermionic cotangent vectors.

- A (6|16) supermanifold $\mathcal{M}$ describing the world-volume of the M5-brane. We denote supercoordinates on the worldvolume by $z^M = (y^M, \vartheta^\rho)$ and a cotangent frame on $\mathcal{M}$ by $e^A(z)$. A decomposition of the frame analogous to (2.47) is given by

$$e^A(z) = dz^M e_M{}^A = (e^a,\ e^{\beta q}) \tag{2.48}$$

  The index $a = 0, 1, ..., 5$ is the index of the vector representation of $so(1,5)$, while β and q are the indices of irreducible spinor representations of $so(1,5)$ and $so(5)$, respectively.

- A (3|0) manifold Σ, to be identified with the membrane worldvolume. The boundaries of Σ lie inside $\mathcal{M}$ or in ∂M. We denote the coordinates on Σ by $s^i, i, \ldots, 3$. Coordinates on the boundary surface are denoted by $\sigma^r, r = 1, 2$.

The relation of the pullback of the 11-dimensional supervielbein to the 5-brane to the supervielbein of the 5-brane itself defines the "embedding matrices" $E_A{}^{\underline{A}}$ via the equation

$$f^*(E^{\underline{A}}) = e^A E_A{}^{\underline{A}} \tag{2.49}$$

One may solve for these matrices in terms of the vielbeins

$$E_A{}^{\underline{A}} = e_A{}^M \partial_M Z^{\underline{M}} E_{\underline{M}}{}^{\underline{A}}. \tag{2.50}$$

The basic superembedding condition then says that

$$E_{\text{fermionic}}{}^{\text{bosonic}} = E_{\beta q}{}^{a} = 0. \tag{2.51}$$

This simple equation is extremely powerful, it leads to a complete set of covariant equations of motion for the 5-brane [31, 32].

The action of an M2-brane ending on an M5-brane, in Euclidean signature, is [47, 36]

$$S_{M2} = \tau_{M2} \int_{\Sigma} d^3 s \Big[\sqrt{det g_{ij}} + i f^* \mathbf{C}^{(3)}\Big] - i\tau_{M2} \int_{\partial\Sigma} d^2\sigma \phi^* B^{(2)}. \tag{2.52}$$

Here $\tau_{M2} = \frac{1}{(2\pi)^2} M_{11}^3$ is the M2-brane tension. Also, $B^{(2)}$ is the super 2-form living on $\mathcal{M}$ while $\mathbf{C}^{(3)}$ is the super 3-form living on the target superspace M. The pullback in eq.(2.52) under the embedding $f : s^i \to Z^{\underline{M}}$ is

$$f^*\mathbf{C}^{(3)} = \frac{1}{3!}\partial_i Z^{\underline{M}} \partial_j Z^{\underline{N}} \partial_k Z^{\underline{P}} \mathbf{C}^{(3)}_{\underline{MNP}} ds^i \wedge ds^j \wedge ds^k,$$

while the pullback under the embedding $\phi : \sigma^r \to z^M$ is

$$\phi^* B^{(2)} = \frac{1}{2}\partial_r z^M \partial_s z^N B^{(2)}_{MN} d\sigma^r \wedge d\sigma^s,$$

We specialize the action (2.52) to the case of an M2-brane stretched between $y = 0$ and $y = x$ in the background described in section 2. The membrane is a product $\Sigma \times [0, x]$ so it is convenient to define coordinates on the membrane $s^i = (t, \sigma, \bar{\sigma})$ where t is a coordinate on the interval and σ is a holomorphic coordinate along the curve Σ. The embedding coordinates of Σ into (11|32) superspace

$$Z^{\underline{M}}(s) = \Big(X^{\hat{M}}_{3,11}(s), \Theta_{3,11}(s)\Big), \tag{2.53}$$

have the following structure. First, the interval coordinate

$$X^{11}_{3,11}(s) = \pi\rho\Big(t + \frac{i}{2}\Theta_{3,11}\Gamma^{11}\overline{\Theta}_{3,11}\Big) \tag{2.54}$$

has an important correction quadratic in fermions, while the coordinates

$$X^m_{3,11}(s) = X^m(\sigma), \quad X^{\overline{m}}_{3,11}(s) = X^{\overline{m}}(\bar{\sigma}), \tag{2.55}$$

describe the holomorphic embedding. The coordinate $X^{\mu}_{3,11}(s)$ is unconstrained. The fermions $\Theta_{3,11}(s)$ satisfy the physical gauge condition

$$\Gamma^{(2)}\Theta_{3,11}(s) = -\Theta_{3,11}(s). \tag{2.56}$$

We have omitted the coordinates describing fluctuations of the membrane within $\mathcal{X}$ since we will restrict our consideration to an isolated curve Σ and hence these degrees of freedom will be massive.

The origin of the correction in (2.54) is continuity of embedding coordinates in superspace. That is, the embedding of the membrane into 11-dimensional superspace $(3|0) \rightarrow (11|32)$ must agree, on the boundary, with the embedding of the 5-brane into superspace $(6|16) \rightarrow (11|32)$ since the membrane ends on the 5-brane *in superspace*. We now derive this condition in more detail.

We choose bosonic coordinates on the M5-brane as $y^M = (y^\mu, y, \overline{y})$ where y^μ are real and y is complex, and consider the static embedding of the boundary of the M2-brane into the M5-brane

$$\phi : y = \sigma, \overline{y} = \bar{\sigma} \tag{2.57}$$

The superembedding $(6|16) \rightarrow (11|32)$ is described by superfields

$$Z^{\underline{M}} = (X^{\hat{M}}(z^M), \Theta(z^M))$$

Small fluctuations around static gauge are described by embeddings

$$X^{\hat{M}} = \left(y^m, X^{m'}(y,\vartheta)\right), \quad \Theta = \left(\vartheta, \psi(y,\vartheta)\right) \tag{2.58}$$

where ϑ is a fermionic coordinate in the $(6|16)$ superspace. The superfields $X^{m'}(y,\vartheta)$ and $\psi(y,\vartheta)$ have as their $\vartheta = 0$ component bosonic fluctuations transverse to the worldvolume of the M5-brane $X^{m'}(y^M)$ and physical fermions on the M5-brane $\psi(y^M)$ respectively. m' here denotes bosonic indices of coordinates transverse to the M5-brane.

As was discussed, for example, in [32], the basic superembedding condition (2.51) imposes a relation on the superfields $X^{m'}(y,\vartheta)$ and $\psi(y,\vartheta)$, such that

$$X^{m'}(y^M,\vartheta) = X^{m'}(y^M) + i\vartheta\Gamma^{m'}\psi(y) + ... \tag{2.59}$$

In particular, the superfield X^{11} up to linear order in ϑ is

$$X^{11}_{6,11} = \pi\rho\Big(x + i\vartheta\Gamma^{11}\overline{\psi} + ...\Big) \tag{2.60}$$

Recall that we introduced the factor $\pi\rho$ to make x dimensionless.

In the geometry of $\mathcal{X}$, the spinors ϑ and ψ can be decomposed as

$$\vartheta = \Big\{\vartheta^{\dot\alpha} \otimes \epsilon_1, \vartheta^{\alpha} \otimes \epsilon_2\Big\}. \tag{2.61}$$

$$\psi = \Big\{\psi^{\alpha} \otimes \epsilon_1, \psi^{\dot\alpha} \otimes \epsilon_2\Big\}, \tag{2.62}$$

Out of the 16-component spinors we only keep those components given by the covariantly constant spinor along $\mathcal{X}$. There are other physical degrees of freedom in the spinor ψ, but since we are considering a *rigid* curve in $\mathcal{X}$ only the above components lead to massless degrees of freedom.

In Euclidean space equation (2.60) becomes

$$X^{11}_{6,11} = \pi\rho\Big(x - \frac{i}{R}\big(\vartheta^{\dot\alpha}\psi_{\dot\alpha} + \vartheta^{\alpha}\psi_{\alpha}\big)\Big) \tag{2.63}$$

where now chiral and anti-chiral spinors are independent from each other. (To give the meaning to the fermionic bilinears in Euclidean space, we first define them in Minkowskii space, where

$$\vartheta_{\dot\alpha} = (\vartheta^{\alpha})^*, \quad \psi_{\dot\alpha} = (\psi^{\alpha})^*, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.64}$$

where the spinors indices are lowered via $\vartheta_{\dot\alpha} := \varepsilon_{\dot\alpha\dot\beta}\vartheta^{\dot\beta}$. Then we continue to Euclidean signature by dropping the reality conditions in eq.(2.64).)

From eq.(2.54) and eq.(2.63) we see that $X^{11}_{3,11}$ and $X^{11}_{6,11}$ match each other at the boundary of the M2-brane, i.e. at $t = x$, if the following boundary conditions are imposed on the physical fermions $\Theta_{3,11}$

$$\Theta^{\dot{\alpha}\dot{Y}}_{3,11}|_{t=x} = \vartheta^{\dot{\alpha}} \otimes \epsilon^{\dot{Y}}_1\,, \quad \Theta^{\dot{\alpha}Y}_{3,11}|_{t=x} = \psi^{\dot{\alpha}} \otimes \epsilon^{Y}_2\,, \tag{2.65}$$

where $\dot{Y}(Y)$ are chiral(anti-chiral) spinors indices on $\mathcal{X}$.

One can identify zero modes living on the boundary of the M2-brane $\vartheta^{\dot{\alpha}} \otimes \epsilon^{\dot{Y}}_1$ with the supersymmetry broken by the M2-brane. This is compatible with our considerations in section 3. Indeed, exactly these components of background supersymmetry are broken for the instanton described by the embedding eq.(2.54).

The bosonic part of the M2-action is

$$S_{M2} = Z, \quad Z = R\beta_i a^i x - i\hat{\mathcal{A}}, \quad \hat{\mathcal{A}} = \mathcal{A}(\beta_i a^i) - x(\beta_i \chi^i). \tag{2.66}$$

where, as in section two, $\beta_i[\Sigma^i_2]$ is the homology class of the boundary curve.

Now, by evaluating the embedding matrices for an M5-brane up to linear order in ϑ and solving the equation in the (6|16) superspace (see [36])

$$dB = H + f^*\mathbf{C}, \tag{2.67}$$

we obtain the expression for $B^{(2)}_{y\overline{y}}$ up to linear order in ϑ

$$B^{(2)}_{y\overline{y}} = A^{(2)}_{y\overline{y}} + i\partial_y X^n \partial_{\overline{y}} X^{\overline{m}} \Big(-\vartheta\Gamma_{n\overline{m}}\overline{\psi} + C_{n\overline{m}\hat{P}}\vartheta\Gamma^{\hat{P}}\overline{\psi}\Big). \tag{2.68}$$

In solving equation (2.67) we have used constraints on the superform H. Specifically, we have used the condition that the only non-vanishing components of the superform H, in the basis e^A, are the components H_{abc} with all three bosonic indices. This was derived in [36] by requiring κ-supersymmetry of the action of an M2-brane ending on an M5-brane.

In (2.68) we have dropped terms containing derivatives of the fluctuating fields such as $\partial_\mu \mathcal{A}$, since these terms in the vertex operator will not contribute to the fermion two-point function we wish to compute.

Now, from eq.(2.52) and eq.(2.68), and using the properties (2.32) of covariantly constant spinors on $\mathcal{X}$, we evaluate the interaction between zero-modes of fermions living on an M2-brane boundary $\vartheta_{\dot{\alpha}}$ and fermions $\psi_{\dot{\alpha}}$ to be

$$V_\psi = i(\beta_i a^i)\vartheta^{\dot{\alpha}}\psi_{\dot{\alpha}} \tag{2.69}$$

Note that the contribution to the interaction from the second boundary term in eq.(2.68) was cancelled by the term from the bulk, after integrating by parts, due to the presence of the piece in the embedding (2.54) which was quadratic in fermions.

The last step in deriving the "vertex operator" for the chiral fermion superpartner of Z, denoted $\chi_{\dot{\alpha}}$, is to relate $\psi_{\dot{\alpha}}$ and $\chi_{\dot{\alpha}}$. To achieve this we consider a supersymmetric variation of Z with supersymmetry parameter $(\epsilon^{\dot{\alpha}} \otimes \epsilon_1, \epsilon^{\alpha} \otimes \epsilon_2)$. The result is

$$\delta Z = (R\beta_i a^i)\delta x + x\delta(R\beta_i a^i) - i\delta\hat{\mathcal{A}} \tag{2.70}$$

where

$$\delta x = i\epsilon\Gamma^{(11)}\overline{\psi} = -\frac{i}{R}\left(\epsilon^{\dot{\alpha}}\psi_{\dot{\alpha}} + \epsilon^{\alpha}\psi_{\alpha}\right) \tag{2.71}$$

$$\delta\hat{\mathcal{A}} = (\beta_i a^i)\left(\epsilon^{\dot{\alpha}}\psi_{\dot{\alpha}} - \epsilon^{\alpha}\psi_{\alpha}\right) \tag{2.72}$$

Equation (2.72) is a direct consequence of eq.(2.68) and the definitions (2.23,2.66).

Denoting by $\lambda^i_{\dot{\alpha}}$ the superpartners of the bulk scalars T^i, we get the desired relation

$$\chi_{\dot{\alpha}} = 2(\beta_i a^i)\psi_{\dot{\alpha}} + x(\lambda^i_{\dot{\alpha}}\beta_i) \tag{2.73}$$

and hence the "vertex operator" for $\chi_{\dot{\alpha}}$ is

$$V_\chi = \frac{i}{2}\vartheta^{\dot{\alpha}}\chi_{\dot{\alpha}}. \tag{2.74}$$

2.5 The case of one M5-brane.

In this section we will discuss the scalar potential for the case of one M5-brane located at position $y = x$ on the M-theory interval. The general formula has been quoted above in (2.1). In order to evaluate this expression we need both K and W. We will describe first W and then K, and then put them together.

2.5.1 Superpotential

In the present setting the superpotential W can be written as a sum of 5 pieces

$$W = W_{\text{pert}} + W_2 + W_3 + W_4 + W_5. \tag{2.75}$$

which have the following origins:

- $W_{\rm pert}$ is the Yukawa superpotential for the charged chiral superfields, given in ([48, 18])

$$W_{\rm pert} = \frac{(4\pi)\sqrt{2}}{3}\lambda_{\hat{I}_1\hat{I}_2\hat{I}_3}C^{\hat{I}_1}C^{\hat{I}_2}C^{\hat{I}_3}, \tag{2.76}$$

The Yukawa couplings are given by

$$\lambda_{\hat{I}_1\hat{I}_2\hat{I}_3} = \int \Omega \wedge u^{x_1}_{I_1} \wedge u^{x_2}_{I_2} \wedge u^{x_3}_{I_3} f^{(\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3)}_{x_1x_2x_3} \tag{2.77}$$

and depend on the complex structure and bundle moduli, but are independent of T^i and S. $f^{(\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3)}_{x_1x_2x_3}$ projects onto the singlet in $\mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_3$ (if it exists), and Ω is a choice of nowhere zero holomorphic $(3,0)$ form on $\mathcal{X}$.

- W_2 is the sum of two pieces coming from an M2-brane stretched between the M5-brane and the boundary 9-brane at $y = 0$ or $y = 1$ respectively. We have shown in the previous section that

$$W_2 = h\Big\{exp\Big(-Z\Big) + exp\Big(-(\beta_i T^i - Z)\Big)\Big\}, \tag{2.78}$$

(Note that the relative sign can be changed by a shift in the imaginary part of T or Z.)

- W_3 is the contribution to the superpotential due to gaugino condensation studied in [16, 18, 5]. It is given by

$$W_3 \sim exp\Big(-\frac{3}{2b_0}S\Big), \tag{2.79}$$

where b_0 is a beta-function coefficient for the gauge group on the second 9-brane. We are working in a region of moduli space constrained by (2.11). It follows that

$$3V \gg 2b_0 R a^i \beta_i x, \quad 3V \gg 2b_0 R a^i \beta_i (1-x) \tag{2.80}$$

and hence the contribution of W_3 to the potential is much smaller than the contribution of W_2, and will henceforth be neglected.

- W_4 is the contribution from M2-brane instantons stretched between the two "M9-branes." The contribution from a single membrane wrapping a holomorphic curve $\Sigma \subset \mathcal{X}$ has the form

$$W_4 \sim exp(-\beta_i T^i),. \tag{2.81}$$

In the case of the standard embedding (with no 5-branes) the sum over all such curves in a fixed homology class vanishes. This happens because W_4 is just the world-sheet instanton contribution to the superpotential in the effective theory near a (2,2) vacuum of the weakly coupled heterotic string. Such superpotentials for moduli are known to be zero ([25, 26, 27]). W_4 is also zero for the special cases of the "non-standard embedding" arising in weakly coupled heterotic (0,2) vacua which are related to linear sigma models. For example, $W_4 = 0$ for the quintic in P^4. Nevertheless, it is expected that these corrections will be nonzero for generic $(0,2)$ compactifications [14, 15].

- W_5 is the superpotential coming from an M5-brane wrapping the whole $\mathcal{X}$.

$$W_5 \sim \exp\Big(-\tau_{M5} S v\Big) = \exp\Big(-\frac{1}{4}S\Big). \tag{2.82}$$

W_5 is of the same order as W_3 and again we can neglect it relative to the effects of open membranes.

2.5.2 Kahler potential for bulk moduli and charged scalars

To evaluate the scalar potential in (2.1) one also needs the Kahler potential [5]

$$K = K_S + K_T + K_m + K_{cplx} + K_5 + K_{\text{bundle}}, \tag{2.83}$$

The first four pieces in this expression have already been obtained in previous papers. We will derive a formula for K_5 below. It would be interesting to learn more about K_{bundle}, but we will not do so in this work. In this section we review the results for the first four terms, obtained in [17, 18, 49, 50, 51].

[5] Here we are assuming that the moduli space is a product space, as is valid in our approximation.

The first two terms in (2.83) are:

$$K_S = -ln(S+\overline{S}), \quad K_T = -ln\Big(\frac{1}{6}d_{ijk}(T^i+\overline{T}^i)(T^j+\overline{T}^j)(T^k+\overline{T}^k)\Big) \tag{2.84}$$

To leading order in an expansion in C^I the charged matter has a Kähler potential of the form

$$K_m = Z_{\hat{I}\hat{J}}C^{\hat{I}}\overline{C}^{\hat{J}} + \cdots \tag{2.85}$$

Here $Z_{\hat{I}\hat{J}}$ is constructed from the metric for bundle moduli $G_{B\hat{I}\hat{J}}$ as follows. First of all, G_B is defined by

$$G_{B\hat{I}\hat{J}} = \frac{1}{vV}\int_X \sqrt{g}g^{m\overline{m}}u_{Imx}u^x_{\overline{J}\overline{m}} \tag{2.86}$$

and depends on the Kahler moduli a^i, as well as on the complex structure and bundle moduli. Next we define $K_{B\hat{I}\hat{J}} := e^{\frac{K_T}{3}}G_{B\hat{I}\hat{J}}$. Note that the dependence on the Kahler moduli is only through the combination $T^i+\overline{T}^i$. Then we can define

$$Z_{\hat{I}\hat{J}} = G_{B\hat{I}\hat{J}} - \frac{e^{-\frac{K_T}{3}}2\xi_i}{S+\overline{S}}\tilde{\Gamma}^i_{B\hat{I}\hat{J}} \tag{2.87}$$

where ξ_i was defined in (2.16), and

$$\Gamma^i_{B\hat{I}\hat{J}} = K_T^{ij}\frac{\partial K_{B\hat{I}\hat{J}}}{\partial T^j}, \quad K_{Tij} = \frac{\partial^2 K_T}{\partial T^i \partial \overline{T}^j} \tag{2.88}$$

$$\tilde{\Gamma}^i_{B\hat{I}\hat{J}} = \Gamma^i_{B\hat{I}\hat{J}} - (T^i+\overline{T}^i)K_{B\hat{I}\hat{J}} - \frac{2}{3}(T^i+\overline{T}^i)(T^k+\overline{T}^k)K_{Tkj}\Gamma^j_{B\hat{I}\hat{J}},$$

K_T^{ij} denote the inverse of the matrix K_{Tij}.

In formulae (2.87) and (2.85) the following restrictions on the scalar fields are assumed

$$Z_{\hat{I}\hat{J}}C^{\hat{I}}\overline{C}^{\hat{J}} \ll 1 \tag{2.89}$$

$$\frac{2\xi_i}{S+\overline{S}}\tilde{\Gamma}^i_{B\hat{I}\hat{J}} \ll K_{B\hat{I}\hat{J}}. \tag{2.90}$$

The Kahler function for the complex structure moduli is

$$K_{cplx} = -ln\Big(\overline{\Pi^a}\mathcal{G}_a\Big), \quad \mathcal{G}_a = \partial_{\Pi^a}\mathcal{G}, \quad a = 1,\ldots,h_{21}+1 \tag{2.91}$$

and can be expressed in terms of the periods over the A-cycles Π^a and the prepotential $\mathcal{G}$, with complex structure moduli expressed as $\pi^\alpha = \frac{\Pi^\alpha}{\Pi^0}, \quad a = (0, \alpha)$.

2.5.3 Kahler potential for the M5-brane moduli

The last piece in (2.83) is K_5, the Kahler potential giving the kinetic terms for the 5-brane scalars x and $\mathcal{A}$. As we were finishing our project we found that ([42]) obtained K_5 in the special case of $h^{(1,1)} = 1$. Since we got our result independently and in a different way, we will explain the derivation below.

To find K_5 we start from the bosonic part of the Pasti-Sorokin-Tonin action for the M5-brane [38]

$$S_{M5} = \tau_{M5} \int_{W_6} d^6y \left(-\sqrt{-det(\gamma_{MN} + iH_{MN})} - \frac{1}{4}\sqrt{-\gamma} v_L H^{*LMN} H_{MNP} v^P \right) \qquad (2.92)$$

$$+\tau_{M5} \int_{W_6} \left(\hat{C}_6 + \frac{1}{2} dA_2 \wedge \hat{C}_3 \right),$$

Here the tension of the M5-brane is $\tau_{M5} = \frac{1}{(2\pi)^5} M_{11}^6$. The other terms in the action are defined by

$$\gamma_{MN} = \frac{\partial X^{\hat{M}}}{\partial y^M} \frac{\partial X^{\hat{N}}}{\partial y^N} g^{(11)}_{\hat{M}\hat{N}}, \quad H^{MN} = H^{*MNP} v_P, \quad H^{*MNP} = -\frac{1}{3!\sqrt{-\gamma}} \epsilon^{MNPLKQ} H_{LKQ}$$

$$H_{MNP} = 3\partial_{[M} A^{(2)}_{NP]} - (\hat{C}_3)_{MNP}, \quad \hat{C}_{MNP} = \frac{\partial X^{\hat{M}}}{\partial y^M} \frac{\partial X^{\hat{N}}}{\partial y^N} \frac{\partial X^{\hat{P}}}{\partial y^P} C_{\hat{M}\hat{N}\hat{P}} \qquad (2.93)$$

$$\hat{C}_{M_1 \ldots M_6} = \frac{\partial X^{\hat{M}_1}}{\partial y^{M_1}} \cdots \frac{\partial X^{\hat{M}_6}}{\partial y^{M_6}} C_{\hat{M}_1 \ldots \hat{M}_6} \quad v_N = \frac{\partial_N \Phi}{\sqrt{\partial_K \Phi \partial^K \Phi}}, \quad v_N v^N = 1.$$

where Φ is the PST scalar and $\hat{C}_6$ is the magnetic dual of $\hat{C}_3$.

We wish to do Kaluza-Klein reduction of the above action along the holomorphic curve Σ. We split the coordinates in the bulk as $X^{\hat{M}} = (\xi^\mu, X^a, X^{\bar{a}}, X^{11})$ where $a, \bar{a} = 1, \ldots, 3$ are indices for the complex coordinates, and ξ^μ are coordinates along the noncompact R^4. We choose μ to run over $\mu = 0, 1, 2, 5$. [6] The coordinates along the worldvolume W_6 of

[6]We have changed notation from section two for this computation.

the 5-brane are taken to be y^M which we split as 4 real coordinates y^μ, $\mu = 0,1,2,5$ and one complex coordinate y along the holomorphic curve Σ. A natural gauge choice for the PST scalar is [106]

$$v_M = \delta^5_M, \quad A_{5M} = 0 \tag{2.94}$$

While the gauge choice (2.94) breaks six dimensional covariance, after the KK reduction we will obtain a covariant 4-dimensional action.

The massless fluctuations of the M5 brane are described by fields

$$X^{11} = \pi\rho x(\xi^\mu), \quad A_{mn}(\xi^\mu), \quad \mathcal{A}(\xi^\mu), \quad \mu = (m,5),\, m = 0,1,2$$

where $\mathcal{A}(\xi^\mu)$ was defined in equation (2.23). Keeping only terms of quadratic order in derivatives we obtain the following 4-dimensional action

$$S_{M5} = -v^{\frac{1}{3}}\tau_{M5}\int_{W_4}\Big\{\frac{1}{2}(\pi\rho)^2\frac{e(a^i\beta_i)R}{V}(\partial_\mu x)(\partial^\mu x) - \frac{1}{2}\frac{(a^i\beta_i)RV}{eg^{55}}(H^{y\bar{y}})^2 \tag{2.95}$$

$$+\frac{1}{2}(\pi\rho)^2\frac{e}{(a^i\beta_i)RV}\Big[\partial_m\hat{\mathcal{A}} + x\partial_m(\beta_i\chi^i)\Big]g^{mn}_{(3)}\Big[\partial_n\hat{\mathcal{A}} + x\partial_n(\beta_i\chi^i)\Big]$$

$$+(\pi\rho)\frac{g^{5m}}{g^{55}}\Big[\partial_m\hat{\mathcal{A}} + x\partial_m(\beta_i\chi^i)\Big]H^{y\bar{y}} + (\pi\rho)\Big[\partial_5\hat{\mathcal{A}} + x\partial_5(\beta_i\chi^i)\Big]H^{y\bar{y}}$$

$$+(\pi\rho)^2\frac{ex}{V^2}\partial_5\sigma g^{55}\Big[\partial_5\hat{\mathcal{A}} + \frac{1}{2}x\partial_5(\beta_i\chi^i)\Big] + (\pi\rho)^2\frac{ex}{V^2}\partial^m\sigma\Big[\partial_m\hat{\mathcal{A}} + \frac{1}{2}x\partial_m(\beta_i\chi^i)\Big]\Big\}$$

where $e = \sqrt{-det g_{\mu\nu}}$, while g^{5m} and g^{55} are components of the inverse of the 4-dimensional metric $g^{\mu\nu}$. One should take care that $g^{mn}_{(3)}$ is the inverse of the 3-dimensional metric so that

$$g^{mn} = g^{mn}_{(3)} + \frac{g^{5m}g^{5n}}{g^{55}}.$$

Moreover, in (2.95) we have

$$H^{y\bar{y}} = \frac{1}{2}\epsilon^{mnp}\Big[\partial_m A_{np} - \partial_m X^{11}C_{11np}\Big],$$

(where we have used (2.93)) and finally we have also introduced

$$\hat{\mathcal{A}} = \mathcal{A}(\beta_i a^i) - x(\beta_i\chi^i), \quad \partial^m\sigma = g^{m\nu}\partial_\nu\sigma.$$

One can see from (2.95) that integrating out $H^{y\bar{y}}$ gives

$$H^{y\bar{y}} = (\pi\rho)\frac{e}{(\beta_i a^i)RV} g^{5\mu}\Big[\partial_\mu \hat{\mathcal{A}} + x\partial_\mu(\beta_i \chi^i)\Big]$$

Plugging the expression for $H^{y\bar{y}}$ back into (2.95) restores 4-dimensional covariance and results in the action

$$\kappa_4^2 S_{M5} = -\int_{W_4} e\Bigg\{\frac{1}{2}\frac{(a^i\beta_i)R}{V}(\partial_\mu x)(\partial^\mu x) + \frac{1}{2}\frac{1}{(a^i\beta_i)RV}\Big[\partial_\mu \hat{\mathcal{A}} + x\partial_\mu(\beta_i\chi^i)\Big]^2 \qquad (2.96)$$
$$+\frac{x}{V^2}\partial^\mu\sigma\Big[\partial_\mu\hat{\mathcal{A}} + \frac{1}{2}x\partial_\mu(\beta_i\chi^i)\Big]\Bigg\}$$

One can now extract the Kahler potential for the 5-brane moduli. The terms in the action (2.96) uniquely determine K_5 to be

$$K_5 = \frac{(Z+\overline{Z})^2}{(S+\overline{S})(\beta_i T^i + \beta_i \overline{T}^i)}, \qquad (2.97)$$

A check of the supergravity kinetic terms associated with K_5 shows that but there are extra terms coming from (2.97) and given by:

$$-\int_{W_4} e\Bigg\{-x(a^i\beta_i)RV^{-2}(\partial_\mu x)(\partial^\mu V) - \frac{1}{2}x^2 V^{-2}(\partial_\mu\Big(R(a^i\beta_i)\Big)(\partial^\mu V) \qquad (2.98)$$
$$+\frac{1}{2}x^2(a^i\beta_i)RV^{-3}\Big(\partial_\mu V\partial^\mu V + \partial_\mu\sigma\partial^\mu\sigma\Big).\Bigg\}$$

These terms are exactly cancelled by the terms coming from $K_S = -ln(S+\overline{S})$ after including an $x-$dependent correction to the definition of the chiral field S

$$S = V + R(a^i\beta_i)x^2 + i\sigma \qquad (2.99)$$

Note, that the above correction $R(a^i\beta_i)x^2$ is of order $\mathcal{E}^{eff}$ with respect to V. There are no $x-$dependent corrections to the other fields at this order.

The proper interpretation of these facts is that the complex structure on field space is corrected at the nonlinear level by (2.99) and that the Kähler potential $K_S + K_5$ should be written as

$$\widehat{K_S} = -\log\Big[S + \overline{S} - \frac{(Z+\overline{Z})^2}{(\beta_i T^i + \beta_i\overline{T}^i)}\Big] \qquad (2.100)$$

It would be interesting to learn if this expression is valid at higher order in the expansion in Z.

2.5.4 Potential in the case $h^{(1,1)} = 1$.

In the previous sections we have given formulae valid for a generic $\mathcal{X}$. We will now specialize to the case of $h^{(1,1)} = 1$ in order to simplify the analysis of the potential. As we have stressed above, in this case there is no net contribution to the non-perturbative potential from M2-branes stretched between the two boundary 9-branes, i.e. $W_4 = 0$.

When $h^{(1,1)} = 1$ the dependence on the Kahler parameter a of the metric (2.86) for the bundle moduli can be easily extracted by a scaling argument. We can choose a basis u^x_{Im} that does not depend on a. Then the inverse of Calabi-Yau metric scales like

$$g_{CY}^{m\overline{m}} = \frac{1}{a}\omega_{(1)}^{m\overline{m}}$$

where $\omega_{(1),n\overline{m}}$ is, say, an integral generator of $H^{(1,1)}(\mathcal{X}) \cap H^2(\mathcal{X}, Z)$. Under these conditions the Kahler metric for charged scalars (2.88) simplifies considerably and is given by

$$Z_{\hat{I}\hat{J}} = \Big(\frac{3}{T+\overline{T}} + \frac{2\xi}{S+\overline{S}}\Big)H_{\hat{I}\hat{J}}, \quad \xi = \beta^{(0)} + \beta(1-x)^2, \tag{2.101}$$

where $H_{\hat{I}\hat{J}}$ depends only on complex structure and bundle moduli. In the case at hand $H_2(\mathcal{X}, Z)$ is of rank 1 and generated by a rational curve Σ. We take $\beta = 1$, which corresponds to wrapping a 5-brane only once around Σ.

The perturbative potential for charged scalars was obtained in [18]. Using the formulae from the Appendix G we have calculated the non-perturbative potential including explicit leading dependence on Kahler moduli and charged scalars. As mentioned in the introduction we write the full potential in the form

$$(\kappa_4)^4 U = \Big(U_0 + U_1 + U_2\Big) \tag{2.102}$$

where, as mentioned in the introduction, we organize terms by the order in the nonperturbative superpotential. U_0 begins with the perturbative potential. U_1 results from mixing between the perturbative and nonperturbative contributions to W, while U_2 is the term of second order in the nonperturbative superpotential. The formula for the potential contains a prefactor e^K. We have used the explicit results for K_S and K_T, and we have dropped K_m and K_5 since they contribute subleading effects to the order we are working.

We now give the leading expressions for the three terms in (2.102) in more detail. The leading contributions to U_0 are given by

$$U_0 = \frac{4\pi^2}{3\tilde{d}} \frac{e^{K_{\rm cplx}+K_{\rm bundle}}}{VJ^2}|\lambda CC|^2 + U_D \tag{2.103}$$

In the above formulae $V = \tilde{d}a^3, \quad J := Ra$, where $d = 6\tilde{d}$ is the intersection number on $\mathcal{X}$. The expression

$$|\lambda CC|^2 = C^{\hat{I}_1}C^{\hat{I}_2}\lambda_{\hat{I}_1\hat{I}_2\hat{I}_3}H^{\hat{I}_3\hat{J}_3}\overline{\lambda}_{\hat{J}_1\hat{J}_2\hat{J}_3}\overline{C}^{\hat{J}_1}\overline{C}^{\hat{J}_2}, \tag{2.104}$$

comes from derivatives of W with respect to $C^{\hat{I}}$.

The D-term is given by

$$U_D = \frac{18\pi^2}{VJ^2}\sum_{(a)}\left(\overline{C}^{\hat{I}}H_{\hat{I}\hat{J}}T^{(a)}C^{\hat{J}}\right)^2 \tag{2.105}$$

where $T^{(a)}$ are generators of the unbroken four dimensional gauge group H, and we are assuming there are no induced FI parameters.

To leading order U_1 is given by:

$$U_1 = -\frac{e^{K_{\rm cplx}+K_{\rm bundle}}(1-x)}{2\tilde{d}VJ^2}\left\{e^{-Jx}{\rm Re}\left(W_{\rm pert}\bar{h}e^{i\alpha_1}\right) - e^{-J(1-x)}{\rm Re}\left(W_{\rm pert}\bar{h}e^{i\alpha_2}\right)\right\} \tag{2.106}$$

where we define the axion fields

$$\alpha_1 = ImZ, \quad \alpha_2 = Im(T-Z), \tag{2.107}$$

There will be corrections from terms higher order in the expansion in $\frac{|C|^2}{J}, \mathcal{E}^{eff}, \mathcal{E}_R^{eff}$. There are also corrections from multiply-wrapped membranes.

The leading contribution to U_2 is a two-instanton term

$$U_2 = \frac{e^{K_{\rm cplx}+K_{\rm bundle}}}{8\tilde{d}J^2}|h|^2\Big\{e^{-2Jx} + e^{-2J(1-x)} - 2e^{-J}\cos(\alpha_1-\alpha_2) \tag{2.108}$$
$$+\frac{2J}{3V}(1-2x)e^{-2J(1-x)} + \frac{4Jx}{3V}e^{-J}\cos(\alpha_1-\alpha_2)\Big\} + \cdots$$

The leading piece comes from $K^{Z\overline{Z}}|\partial_Z W|^2$. Note that in the second line of (2.108) we have kept terms which are formally higher order in our expression since they multiply $J/V \sim \mathcal{E}^{eff}$. We have kept these because, near $x = 1/2, \quad cos(\alpha_1 - \alpha_2) = 1$ the leading piece vanishes. At these points the order J/V corrections which come from $K^{Z\overline{Z}}$ and the prefactor e^K multiply zero and we can legitimately say that the leading term near $x = 1/2, \quad \cos(\alpha_1 - \alpha_2) = 1$ is given by the last term in the second line of (2.108). We will need these subleading terms in section 5.5 below. Of course, there are many other corrections of relative order $\mathcal{O}((\mathcal{E}^{eff})^p(\mathcal{E}_R^{eff})^q, \frac{|C|^2}{J})$, where $p \geq 1$ and $q > 0$.

The region of validity of our result for the potential, (2.102), is constrained by several considerations.

- We must assume that all sizes are much bigger than the 11D Plank length.

$$\pi\rho Rx \gg l_{11}, \quad \pi\rho R(1-x) \gg l_{11}, \quad a^{\frac{1}{2}}v^{\frac{1}{6}} \gg l_{11} \tag{2.109}$$

  and from these conditions it follows, in particular, that $Jx \gg 1, \quad J(1-x) \gg 1$.

- Since we are working to quadratic order in the Kähler potential in a series expansion in C we must have

$$|C|^2 := C^{\hat{I}} H_{\hat{I}\hat{J}} \overline{C}^{\hat{J}} \ll J \tag{2.110}$$

- The effective expansion parameters should be small, or, equivalently,

$$V \gg J, \quad J^2 \gg V \tag{2.111}$$

- We must be able to drop $\mathcal{E}^{eff}$ corrections to each of the 9 terms in the potential. We count all the terms wich have different sructure,i.e. 5 from U_2, 2 from U_1 and 2 from U_0.

$$U = \sum_{a=1}^{9} u_a(1 + f_a \mathcal{E}^{eff}),$$

  iff

$$\mathcal{E}^{eff}|\sum_{a=1}^{9} f_a u_a| \ll |\sum_{a=1}^{9} u_a|$$

Given our ignorance regarding f_a we will assume that they are $O(1)$ and impose the stronger condition

$$\mathcal{E}^{eff}\sum_{a=1}^{9} |u_a| \ll |\sum_{a=1}^{9} u_a| \tag{2.112}$$

- Finally, as mentioned in the introduction, we should stress that we are assuming that we are working at a generic smooth point in the complex structure and bundle moduli space.

2.5.5 5-brane dynamics

We can get some heuristic idea about the 5-brane dynamics by considering the theory on a finite volume of 3-dimensional space and keeping only the spatially homogeneous modes of the scalar fields. Even in this drastic approximation the resulting system is a very complicated dynamical system described by a particle mechanics Lagrangian with the (very) schematic form

$$\text{vol(space)} \int dt \left\{ (\frac{\dot{V}}{V})^2 + (\frac{\dot{J}}{J})^2 + \frac{(\dot{C})^2}{J} + \frac{(\dot{J}x + \dot{x}J)^2}{VJ} - U \right\} \tag{2.113}$$

where the potential energy is

$$U = \frac{1}{VJ^2}\Bigg(\alpha C^4 - \beta(1-x)|C|^3|e^{-Jx} \mp e^{-J(1-x)}| + \tag{2.114}$$

$$+\gamma V\Big[(e^{-Jx} \mp e^{-J(1-x)})^2 + \frac{2J}{3V}(1-2x)e^{-2J(1-x)} \pm \frac{4Jx}{3V}e^{-J}\Big]\Bigg)$$

We have only kept the real parts of the fields. The philosophy behind this is that by a "Born-Oppenheimer" type approximation we expect that the axions will relax much more rapidly than the real parts into the most attractive channel. The choice of $\pm$ depends on what term is dominating, U_1 or U_2.

The positive constants α, β and γ are functions of the complex structure and bundle moduli, but these are being held fixed in this discussion.

While the dynamical system we must study is rather complicated we can get some heuristic idea of what to expect in three distinct regions within the region of validity of our potential.

- In one region the charged scalar fields are zero, while J and V are large. The leading contributions to the potential are positive and decrease with increasing J, V. In this region the 5-brane leads to a *repulsive* interaction between the M9 walls. Setting $C = 0$ and choosing "- " sign to set axions in the most attractive channel in eq.(2.114) we get:

$$U \sim \gamma \frac{1}{J^2} \left\{ (e^{-Jx} - e^{-J(1-x)})^2 + \frac{2J}{3V}(1-2x)e^{-2J(1-x)} + \frac{4Jx}{3V}e^{-J} \right\} \tag{2.115}$$

  Note that U has a minimum in x at $x = \frac{1}{2}$. Expanding around the minimum $x = \frac{1}{2} + y$, the resulting potential is

$$U = \gamma \left\{ \frac{2}{3VJ} e^{-J} + 4y^2 e^{-J} \right\}. \tag{2.116}$$

  Now we can see the need for keeping the last two terms in the expression for U_2 in eq.(2.108). Although we are neglecting the $\mathcal{E}^{eff}$-corrections to the Kahler potential, such corrections multiply the leading three terms in eq.(2.108) which sum up to zero at $x = \frac{1}{2}$. On the other hand, the terms we have written, and which follow from the leading pieces in K become the leading terms at the stationary points $x = \frac{1}{2}$. Consequently, J flows towards infinity and x moves towards the middle of the interval. We must assume $V^{1/2} \ll J \ll V$ to stay within the region of validity.

- If the charged scalar fields are important in such a way that

$$|U_1| \gg U_2, \tag{2.117}$$

  then the leading x-dependent term in the potential is *attractive*:

$$U = \alpha C^4 - \beta(1-x)|C|^3 \left(e^{-Jx} + e^{-J(1-x)} \right)$$

  Note that in this case we have to choose the "+" sign in eq.(2.114) if the axions are in the most attractive channel. Thus, if $x < 1/2$ the 5-brane moves towards the wall at

$y = 0$, and if $x > 1/2$ the 5-brane moves towards the wall at $y = 1$. Note, that in each of these two subregions $U_0 \gg |U_1|$ as a direct consequence of (2.117). Since U_0 is the dominant term J and V will evolve to large values. Since the C field is simultaneously evolving a more careful analysis of the dynamical system would be highly desirable. But we will not do that here.

More generally, one can show that the potential (2.114) is non-negative at a generic point in the bundle and complex strucure moduli space, within in our region of validity (2.109-2.112), and thus predicts decompactification of both the Calabi-Yau and the orbifold interval. The argument, which is straightforward but long can be found in Appendix H.

Note that (2.114) is the leading potential only under our assumption that we work at a generic smooth point in bundle and complex structure moduli space. It would be interesting to incorporate singularities in complex structure and bundle moduli space in the discussion. There are potentially many new terms in the potential that must be reconsidered. It is possible that using the known results on complex structure and bundle moduli space one can address this problem.

2.5.6 Conflicting instabilities

One interesting consequence of the discussion in the previous section is that there is a strong coupling dual of the Dine-Seiberg problem where the M-theory interval (and the Calabi-Yau) tends to decompactify. In the case of heterotic M-theory with the standard embedding (i.e. no 5-branes) this has already been discussed by Banks and Dine [6], who noted that one can use holomorphy to extrapolate the weak coupling superpotential based on gluino condensation. In the presence of an M5-brane (in the case $h^{(1,1)} = 1$) the above formulae show that in the region specified by (2.109-2.112) there is a similar effect due to open membrane instantons.

It is of some interest to compare the above result with what we expect for the weakly coupled heterotic string, since our considerations are only valid at large heterotic string coupling. Indeed, the heterotic coupling is related to the length of the M-theory interval by

$$R\rho \sim l_{11}(g_s)^{\frac{2}{3}}$$

and we require $\pi\rho R \gg l_{11}$. In the regime of weak coupling and large V, the potential has been discussed in [28]. It was shown there that the effective potential is positive and behaves as

$$U_{eff} \sim e^{-\frac{V}{g_s^2}}. \tag{2.118}$$

This favors an evolution to weak coupling $g_s \to 0$ and large volume $V \to \infty$. One might worry that the calculations of ([28]) were performed in the case of the standard embedding, and in backgrounds with other $E_8 \times E_8$ gauge instantons one must take into account the contribution of world-sheet instantons as well [14, 15]. Nevertheless, as we have repeatedly mentioned, these effects often sum to zero [22, 23] so once again and we can use (2.118) in the region of small R and large V.

In view of the above, we can combine our result (2.102) with (2.118), to learn that the "true" potential goes to zero through positive values in both limits $R \to 0$ and $R \to \infty$. This indicates that there must be a stationary point somewhere in the intermediate region, i.e. at some finite value of R. The nature of the stationary points that lie in the middle of moduli space is unknown, and is, of course, an interesting and outstanding question.

2.6 Multiple covers and chirality changing transitions

It is of considerable interest to determine the nature of the low energy theory in the limit that the M5 moves into the the boundary 10-manifold. In this section we will make some comments on this limit. We will need to make some guesses and the results of this section are not as rigorous as in the previous sections. For definiteness we will consider the limit $x \to 0$.

One good reason for studying the limit $x \to 0$ is that there are strong indications that in such limits there can be very interesting chirality-changing phase transitions in the low energy theory. This was discovered, in the present context, by Kachru and Silverstein [60].

The theory of these transitions has been considerably extended to many new examples in [45].

The main new ingredient that is needed to discuss the $x \to 0$ behavior is a multiple-cover formula for the open membrane instantons. The fact that there must be nontrivial effects from multiply-wrapped M2 branes (at least for those stretching between two 5-branes) can be seen by considering the holomorphy of the full superpotential W as a function of $Z_i - Z_j$ [9]. The instanton effects must be suppressed by a factor proportional to the volume of the stretched membrane and therefore must behave like $\exp[\mp(Z_i - Z_j)]$ for $\pm Re(Z_i - Z_j) > 0$. This is only consistent with holomorphy if there is an infinite series with at best a finite radius of convergence.

Multiply-wrapped worldsheet instanton corrections to $d = 4, N = 2$ prepotentials are known to have a universal form $f(n, \Sigma)$ for an n-times wrapped curve Σ where f only depends on the topology of Σ [61, 62, 66, 63, 64, 65]. Since worldsheet instantons are special cases of M2-brane instantons we will make the working hypothesis that there is similarly a multiple cover factor $f(n, \Sigma)$ for M2-brane instanton corrections to the superpotential W. Some evidence for this can be found in [58, 59, 57]. Unfortunately, the topologies studied in the above papers do not contain our case of $P^1 \times [0,1]$. Therefore we will take

$$\Delta W = h \sum_{n=1}^{\infty} f(n) e^{-nZ} + e^{i\delta} h \sum_{n=1}^{\infty} f(n) e^{-n(T-Z)} \tag{2.119}$$

and make the rather weak assumption that the asymptotic behavior of $f(n)$ for large n is $f(n) \sim n^m e^{Jx_0 n}$ for some constants m and x_0. For simplicity we set $x_0 = 0$ although there could in principle be a shift in the location of the small instanton transition.

The constants m and δ above are unknown, but we can make some comments on them. First, the relative phase $e^{i\delta}$ was not important in the 1-instanton sector, where we can change the relative phase by shifting the axion ImT. It does become a nontrivial issue in the multi-instanton sectors. Nevertheless, for our analysis of the dynamics in the subregion $ReZ \ll 1$ the second piece in (2.119) is negligible, so the issue need not concern us here.

Next, let us consider the power m in the asymptotic behavior of $f(n)$. If we wish the chiral fermion mass term in standard supergravity to be a single-valued function of Z in a region surrounding $Z = 0$, then m cannot be a negative integer such that $|m| > 2$. Single-valuedness implies that the monodromy of $\partial_Z \partial_Z W$ around $Z = 0$ should be diagonalizable thus excluding a singularity of the type $Z^n \log Z, n \geq 2$ in W. [7]

Let us now re-consider the region of validity of our expression. The infinite series for ΔW can be obtained reliably in the region $ReZ \gg 1$ (where we can use 11-dimensional supergravity) and then analytically continued to the region where $ReZ = Jx \ll 1$. To ensure that corrections to the Kahler function are small one must still require that

$$V \gg 1, \quad V^{\frac{1}{2}} \ll J \ll V, \quad |C|^2 \ll J$$

Since these conditions do not imply $Jx \gg 1$ or $J(1-x) \gg 1$ we can study the physics of the 5-brane approaching the boundary.

For definiteness and simplicity let us now assume that $f(n) = n^m$ for some constant m. Then we have

$$\partial_Z \Delta W = -hLi_{-(m+1)}(e^{-Z}) \tag{2.120}$$

where Li is a polylogarithm function

$$Li_{-(m+1)}(t) = \sum_{n=1}^{\infty} n^{m+1} t^n$$

In this case the leading order contribution to U_1 is given by:

$$U_1 = -\frac{e^{K_{\rm cplx}+K_{\rm bundle}}(1-x)}{2\tilde{d}VJ^2} \mathrm{Re}\left[\overline{\mathbf{W}}_{\rm pert} hLi_{-(m+1)}(e^{-Z})\right] \tag{2.121}$$

while the leading contribution to U_2 is

$$U_2 = \frac{e^{K_{\rm cplx}+K_{\rm bundle}}}{8\tilde{d}J^2} |h|^2 \left|Li_{-(m+1)}(e^{-Z})\right|^2. \tag{2.122}$$

In all the cases below we will assume that the axion phases are in the maximally attractive channel.

One interesting limit is $Z \to 0$. Here we can use the behaviour of the polylogarithm

$$Li_{-(m+1)}(e^{-Z}) \sim Z^{-(m+2)}, \quad Z \to 0$$

[7] There are known examples of logarithmic superpotentials for n=0,1 that make good physical sense. This is usually related to some kind of pair creation phenomena. We thank K. Hori for very useful discussions on this issue.

(for $m = -2$ we replace Z^0 by $\log Z$) to write out (schematically) the leading potential at $Jx \ll 1$

$$U = \frac{1}{VJ^2}\left(\alpha C^4 - \beta(1-x)\frac{|C|^3}{(Jx)^{2+m}} + \gamma\frac{V}{(Jx)^{4+2m}}\right), \quad m > -2 \tag{2.123}$$

$$U = \frac{1}{VJ^2}\left(\alpha C^4 + \beta|C|^3(1-x)log(Jx) + \gamma V(logJx)^2\right), \quad m = -2 \tag{2.124}$$

where α, β, γ are positive functions of complex structure and bundle moduli as above and we have set $ImZ = 0$.

Therefore, for small enough Jx (holding the other moduli fixed) the leading term in the potential (2.123),(2.124) is

$$\frac{\gamma}{J^2(Jx)^{4+2m}}, \quad m > -2, \qquad \frac{\gamma(logJx)^2}{J^2}, \quad m = -2 \tag{2.125}$$

and there is a repulsive force on the 5-brane. Indeed there is an infinite energy barrier forbidding the 5 brane from hitting the wall.

One should not conclude from the above that there will be no chirality changing transition, since the axionic degree of freedom in Z can change the qualitative features of the potential drastically. Unfortunately, in order to study this question in detail just knowing the asymptotic behavior will not suffice and one needs a precise version of the formula for $f(n)$ in order to work out the analytic continuation from $Re(Z) \gg 1$ to $|e^{-Z}| = 1$. For definiteness we will consider $f(n) = n^m$ where $m \geq -2$ is integral.

Let us consider first the case $m \geq 0$. We take $Z = Jx + i\pi$, where $Jx \ll 1$ and expand $e^{-Z} = -1 + Jx - \frac{1}{2}(Jx)^2$. We use the Taylor expansion for the polylogarithm

$$Li_{-(m+1)}(-1+t) = Li_{-(m+1)}(-1) - Li_{-(m+2)}(-1)t + \frac{1}{2}\left(Li_{-(m+3)}(-1) - Li_{-(m+2)}(-1)\right)t^2$$

and the following useful relations

$$Li_{-m}(-1) = (2^{1+m} - 1)\zeta(-m), \quad \zeta(-2k) = 0, \quad \zeta(1-2k) = -\frac{B_{2k}}{2k}, \quad k = 1, 2, \ldots$$

where ζ is the Riemann ζ-function and B_k are Bernoulli numbers taken in the convention:

$$\frac{y}{e^y - 1} = 1 - \frac{1}{2}y + \frac{B_2}{2!}y^2 + \frac{B_4}{4!}y^4 + \ldots$$

Substituting $t = Jx - \frac{1}{2}(Jx)^2$ in the above Taylor expansion and keeping only terms up to $(Jx)^2$ we have

$$Li_{-(m+1)}(-1+t) = (-1)^{k+1}\Big(\nu_1^k - \nu_2^k(Jx)^2\Big), \quad m = 2k, \quad k = 0, \ldots \tag{2.126}$$

$$Li_{-(m+1)}(-1+t) = (-1)^{k+1}2\nu_2^k Jx, \quad m = 2k+1, \quad k = 0, \ldots \tag{2.127}$$

where we define positive numbers ν_1^k, ν_2^k

$$\nu_1^k = (2^{2k+2}-1)\frac{|B_{2k+2}|}{2(1+k)}, \quad \nu_2^k = (2^{2k+4}-1)\frac{|B_{2k+4}|}{4(2+k)}.$$

We now analyze the potential separately for the cases of even and odd m. For even $m = 2k$ we have the following leading potential at $Jx \ll 1$

$$U = \frac{1}{VJ^2}\left(\alpha C^4 - \beta(1-x)|C|^3\Big(\nu_1^k - \nu_2^k(Jx)^2\Big) + \gamma V\Big((\nu_1^k)^2 - 2\nu_1^k\nu_2^k(Jx)^2\Big)\right) \tag{2.128}$$

If $C \neq 0$ then for sufficiently small x the potential is attractive. If $C = 0$ the potential is repulsive.

Now, for odd $m = 2k+1$ the leading potential is

$$U = \frac{1}{VJ^2}\left(\alpha C^4 - 2\beta\nu_2^k(1-x)Jx|C|^3 + \gamma V(2\nu_2^k)^2(Jx)^2\right) \tag{2.129}$$

Now the situation is opposite to the previous case. For $C \neq 0$ there is a repulsive force and if $C = 0$ an attractive force.

Finally we analyze what happens for the cases $m = -1$ and $m = -2$, assuming $ImZ = \pi$ and $Jx \ll 1$. If $m = -1$ the leading potential is

$$U = \frac{1}{VJ^2}\left(\alpha C^4 - \frac{1}{2}\beta|C|^3(1-x)\Big(1 - \frac{1}{2}Jx\Big) + \frac{\gamma V}{4}\Big(1 - Jx\Big)\right) \tag{2.130}$$

The force on the 5-brane is attractive only if one allows for a large vev for $|C|^3$

$$\beta|C|^3 > \gamma V.$$

This is in principle possible since we only assume that $|C|^3 \ll V^{\frac{3}{2}}$ and for large V both inequalities can be satisfied. If $m = -2$ the leading potential is

$$U = \frac{1}{VJ^2}\left(\alpha C^4 - \beta|C|^3(1-x)\Big(log2 - \frac{1}{2}Jx\Big) + \gamma V log2\Big(log2 - Jx\Big)\right) \tag{2.131}$$

and the force is attractive only if

$$\beta|C|^3 > 2(log2)\gamma V.$$

In both cases $m = -1$ and $m = -2$, attraction is only possible for large vev of charged scalars.

The general conclusion based on the analysis of various cases is that the physics of what happens when the 5-brane approaches the wall depends strongly on the detailed form of the multiple-cover formula.

Finally, let us comment on the relevance of this compututation to the examples studied by Ovrut, Pantev, and Park in [45]. One might at first conclude that in these examples the superpotential must vanish since the five-brane wraps a high genus curve. However, the curve wrapped by the 5-brane is not irreducible and not isolated. It can very well happen that in the long-distance expansion of the M5 and M2 Lagrangians there are terms with many fermions (typically multiplying factors involving curvature tensors) which can lift the many fermion zeromodes associated with the nonisolated high genus curve. Thus, the question of whether or not a superpotential is generated is a complicated and difficult one, involving a discussion of the measure on the moduli space of the curve and the integral over that moduli space. Considerations based on the global form of the moduli space for these curves based on the results of [67] do not appear to exclude the generation of such superpotentials.

2.7 The case of N M5-branes.

We will now briefly consider the potential in the case that there are N M5-branes at positions $x_1 < x_2 < \ldots < x_N$. We will assume for simplicity that all the 5-branes are wrapped over the same rational curve Σ, so that open M2-instantons can be stretched between any pair of 5-branes. Moreover, to simplify the analysis we assume that the 5-branes are more or less evenly separated. Finally, we restrict our consideration only to the leading non-perturbative potential, so we do not take into account 2-instanton contributions to W and we need only consider M2-branes between neighboring 5-branes. Similarly, we only keep the contribution of 5-9 instantons coming from M2-instantons stretching between the boundary and the nearest M5-brane. Under these conditions we will have

$$R(x_n - x_{n-1})\pi\rho \gg l_{11}, \qquad \forall n = 1, \ldots, N+1.$$

Neglecting $\mathcal{E}^{eff}$ corrections due to the distortion of the background, the Kähler function for the collection of 5-branes will be just a sum of Kahler functions for each 5-brane.

The potential is again given by formula (2.102), with the same conditions on the region of validity. The 2-instanton terms in the potential U_2, which dominate at $C = 0$, are:

$$U_2 = \frac{e^{K_{\rm cplx}+K_{\rm bundle}}}{8\tilde{d}J^2}|h|^2 \sum_{n=1}^{N}\left\{e^{-2J(x_{n+1}-x_n)} + e^{-2J(x_n-x_{n-1})} - 2e^{-J(x_{n+1}-x_{n-1})}cos(\tilde{\alpha}_n)\right\} + \cdots \tag{2.132}$$

where we denote

$$\tilde{\alpha}_n = \left\{a(2\mathcal{A}_n - \mathcal{A}_{n+1} - \mathcal{A}_{n-1}) + \chi(x_{n+1} + x_{n-1} - 2x_n)\right\}, n = 1, \ldots, N \tag{2.133}$$

and $x_0 = 0, x_{N+1} = 1, \mathcal{A}_0 = \mathcal{A}_{N+1} = 0$. If instead we assume that

$$x_1 \gg x_n - x_{n-1}, \quad (1 - x_N) \gg x_n - x_{n-1}, \quad \forall 1 < n \leq N \tag{2.134}$$

and choose a special subregion where $\cos\tilde{\alpha}_n = 0, \quad \forall n$, then the potential has the form of a non-periodic Toda-chain potential. (The kinetic energies are the standard ones, in our approximation.) As is well known, Toda theory has an exact solution, where all particles move away from each other [44]. In heterotic M-theory this signals an instability in the time evolution of the positions of M5-branes along the orbifold interval: they tend to run away from each other. At the same time Ra evolves to infinity. In short, the system explodes.

Using again a "Born-Oppenheimer" type approximation we expect that the axions will relax much more rapidly than the real parts into the most attractive channel $cos\tilde{\alpha}_n = 1, \quad \forall n$. This implies that the evolution with a Toda-like potential is unstable because of the axions.

When the charged vevs are nonzero we should consider instead the term U_1 in the potential arising from cross terms between perturbative and nonperturbative pieces. This is given by

$$U_1 = -\frac{e^{K_{\text{cplx}}+K_{\text{bundle}}}|h|\zeta}{2\tilde{d}J^2V}\Big\{e^{-Jx_1}(1-x_1)cos(\tilde{\gamma}_1) - e^{-J(1-x_N)}(1-x_N)cos(\tilde{\gamma}_N) \tag{2.135}$$

$$-\sum_{n=1}^{N-1}(x_{n+1}-x_n)e^{-J(x_{n+1}-x_n)}cos(\tilde{\phi}_n)\Big\}$$

where $W_{\text{pert}} = \zeta e^{i\phi_1}, h = |h|e^{i\phi_h}$ are decompositions into modulus and phase and

$$\tilde{\gamma}_1 = ImZ_1 + \phi_1 - \phi_h, \quad \tilde{\gamma}_N = Im(T - Z_N) + \phi_1 - \phi_h, \tag{2.136}$$

$$\tilde{\phi}_n = Im(Z_{n+1} - Z_n) + \phi_1 - \phi_h, \quad n = 1, \ldots, N-1$$

2.8 Possible future directions and applications

A central question in heterotic M-theory is the existence of isolated minima of the potential for moduli. While most of our results predict runaway or unstable behavior (as expected) we have seen some encouraging hints. We have argued that the potential must have nontrivial stationary points in moduli space. We have also seen that a good place to look for interesting behavior of the potential is at singular loci in complex structure and bundle moduli space. For example, if one allows some of the coefficients α, β, γ in sections 5.5 and section 6 to vanish it is easy to imagine scenarios where the potential predicts compactification, rather than decompactification.

There are many technical issues raised by the above computations which should be solved and which moreover can be solved with presently available technology.

One circle of questions includes finding the appropriate generalization of the multiple-cover formula for worldsheet instantons. A related set of questions concerns effects associated with membranes wrapping higher genus curves and nonisolated curves in $\mathcal{X}$. As we have seen in section six, results on these questions would have very interesting physical applications.

A second circle of questions concerns the possibility of obtaining a more concrete understanding of the dependence of the membrane determinants as functions of the complex structure moduli. It might be possible to find classes of compactifications in which one can give fairly explicit formulae for the dependence on gauge bundle moduli, although this might prove to be challenging.

Beyond the extensions mentioned above, which we believe are within reach, there loom far more difficult questions. One of the most challenging issues is to give a proper definition of Horava-Witten theory in a regime outside the validity of the expansion in $(\kappa_{11})^{2/3}$. Another difficult, and pressing, problem is that of finding ways to make definite and quantitative statements about the Kähler potential of the effective supergravity theory in a wider range of validity.

Nevertheless, even given the limitations of our computations, the results do have some interesting ranges of validity. It might be quite interesting to study more thoroughly the dynamics, both classical and quantum mechanical of the moduli in the problem. In this work we have limited ourselves to some very heuristic and naive pictures of the dynamics. It might also be interesting to see if there are any distinctive features of the "modular cosmology" resulting from the above potential for moduli [55].

Chapter 3

T-Duality, and the K-Theoretic Partition Function of TypeIIA Superstring Theory

3.1 Introduction

[1] Duality symmetries, such as the U-duality symmetry of toroidally compactified M-theory, have been of central importance in the definition of string theory and M-theory. Topologically nontrivial effects associated with the RR sector have also played a crucial role in defining the theory. It is currently believed that RR fieldstrengths (and their D-brane charge sources) are classified topologically using K-theory [80, 84, 86, 81, 111, 68, 72]. Unfortunately, this classification is not U-duality invariant. Finding a U-duality invariant formulation of M-theory which at the same time naturally incorporates the K-theoretic formulation of RR fields remains an outstanding open problem.

With this problem as motivation, we investigate the interplay between the K-theoretic formulation of RR fields and the T-duality group, an important subgroup of the full U-duality group. While T-duality invariance of the theory was one of the guiding principles

[1]This chapter was done in collaboration with G.Moore and is drawn from [116].

in the definition of the K-theoretic theta function [86, 68] we will see that the full implementation of T-duality invariance of the low energy effective action of type II string theory is in fact surprisingly subtle, even on backgrounds as simple as $T^2 \times X$, where T^2 is a two-dimensional torus, and X is an 8-dimensional compact spin manifold. We will show that, in fact, in the RR sector there is a T-duality anomaly. This anomaly is cancelled by a compensating anomaly from fermion determinants together with quantum corrections to the 8D effective action. A by-product of our computation is a complete analysis of the 1-loop determinants of IIA supergravity on $X \times T^2$.

As an application of our discussion, we re-examine a proposal of C. Hull [73] for interpreting the Romans mass of IIA supergravity in the framework of M-theory. We will show that, while the interpretation cannot hold at the level of classical field theory, it might well hold as a quantum-mechanical equivalence. In section 10 we comment on some of the issues which arise in extending our computation to a fully U-duality invariant partition function. This includes the role of twisted K-theory in formulating the partition sum.

3.1.1 The effective eight-dimensional supergravity, and its partition function

Previous studies of the partition function in type II string theory [86, 68] considered the limit of a large 10-manifold. One chose a family of Riemannian metrics $g = t^2 g_0$ with $t \to \infty$ and g_0 fixed. Simultaneously, one took the string coupling to zero. The focus of these works was on the sum over classical field configurations of the RR fields. In this chapter we consider the limit where only 8 of the dimensions are large. The metric has the form

$$ds^2 = ds^2_{T^2} + t^2 ds^2_X \tag{3.1}$$

where $ds^2_{T^2}$ is flat when pulled back to T^2. The background dilaton $g^2_{\text{string}} = e^{2\xi}$ is constant. We will work in the limit

$$t \to \infty, \quad e^{-2\xi} := e^{-2\phi} V \to \infty \tag{3.2}$$

where V is the volume of T^2 and ϕ is 10-dimensional dilaton. Finally - and this is important -until section 10 we assume the background NSNS 3-form flux, $\hat{H}$, is identically zero. In particular, the 2-form potential, $\hat{B}$, is a globally well-defined harmonic form on $X \times T^2$.

As is well-known the background data for the toroidal compactification (3.1) include a pair of points $(\tau, \rho) \in \mathcal{H} \times \mathcal{H}$ where $\mathcal{H}$ is the upper half complex plane. τ is the Teichmuller parameter of the torus and $\rho := B_0 + iV$, where $B_0 d\sigma^8 \wedge d\sigma^9$ is an harmonic 2-form on T^2. While we work in the limit (3.2), within this approximation we work with exact expressions in the geometrical data (τ, ρ). In this way we go beyond [68].

It is extremely well-known that the low energy effective 8D supergravity theory obtained by Kaluza-Klein reduction of type II supergravity on T^2 has a "U-duality symmetry" which is classically $SL(3, R) \times SL(2, R)$, and is broken to $\mathcal{D} := SL(3, Z) \times SL(2, Z)$ by quantum effects [74, 75, 76, 77, 78]. These are symmetries of the equations of motion and not of the action. (The implementation of these symmetries at the level of the action involves a Legendre transformation of the fields.) What is perhaps less well-known is that the K-theoretic formulation of RR fields leads to an extra term in the supergravity action which is nonvanishing in the presence of nontrivial flux configurations. Indeed, the proper formulation of this term is unknown for arbitrary flux configurations with $[\hat{H}_3] \neq 0$, but for topologically trivial NSNS flux the extra term is known [68] and is recalled in equations (1.14) and (1.15) below. This term breaks naive duality invariance of the classical supergravity theory already for the T-duality subgroup of the U-duality group, and makes the discussion of T-duality nontrivial.

Let us now summarize the fields and T-duality transformation laws in the conventional description of the eight-dimensional effective supergravity theory on X. The T-duality group is $\mathcal{D}_T = SL(2, Z)_\tau \times SL(2, Z)_\rho$. The theory has the following bosonic fields. From the NSNS sector there is a scalar t, characterizing the size of X, a unit volume metric g_{MN}, a 2-form potential [2] $B_{(2)}$, with fieldstrength $H_{(3)}$, and a dilaton ξ, all of which are invariant under $\mathcal{D}_T$. In addition, there is a multiplet of 1-form potentials $\mathbf{A}_{(1)}^{m\alpha}$ transforming in the $(\mathbf{2}, \mathbf{2})$ of

[2] We will always indicate by the subscript (p) the degree p of a differential p-form on X

$\mathcal{D}_T$. Finally, the pair of scalars (τ, ρ), transform under $(\gamma_1, \gamma_2) \in \mathcal{D}_T$ as $(\tau, \rho) \to (\gamma_1 \cdot \tau, \gamma_2 \cdot \rho)$ where $\gamma\cdot$ is the action by a fractional linear transformation. We therefore call the factors $SL(2, Z)_\tau, SL(2, Z)_\rho$, respectively.

The fieldstrengths from the RR sector include a 0-form and a 2-form, $g^\alpha_{(p)}$, $p = 0, 2, \alpha = 1, 2$ transforming in the $(\mathbf{1}, \mathbf{2})$ of $\mathcal{D}_T$, and a 1-form and 3-form $g_{(p)m}$, $p = 1, 3, m = 8, 9$ transforming in the $(\mathbf{2}', \mathbf{1})$ of $\mathcal{D}_T$. Finally there is a 4-form fieldstrength $g_{(4)}$ on X. This field does not transform locally under T-duality, rather its equation of motion mixes with its Bianchi identity [78]. The fermionic partners are described in section 7 below.

The real part of the standard bosonic supergravity action takes the form

$$Re\left(S^{(8D)}_{\text{boson}}\right) = S_{NSNS} + \sum_{p=0}^{3} S_p\left(g_{(p)}\right) + S_4\left(g_{(4)}\right) \tag{3.3}$$

In the action (3.38) all of the terms except for the last term are manifestly T-duality invariant. The detailed forms of the actions are:

$$S_{NSNS} = \frac{1}{2\pi}\int_X e^{-2\xi}\Big\{t^6(\mathcal{R}(g) + 4d\xi \wedge *d\xi + 28t^{-2}dt \wedge * dt) + \frac{1}{2}t^2 H_{(3)} \wedge *H_{(3)}$$
$$+\frac{1}{2}t^6\frac{d\tau \wedge *d\tau}{(\mathrm{Im}\tau)^2} + \frac{1}{2}t^6\frac{d\rho \wedge *d\rho}{(\mathrm{Im}\rho)^2} + \frac{1}{2}t^4 g_{mn}\mathcal{G}_{\alpha\beta}\mathbf{F}^{m\alpha}_{(2)} \wedge * \mathbf{F}^{n\beta}_{(2)}\Big\} \tag{3.4}$$

where ϵ_{mn} and $\mathcal{E}_{\alpha\beta}$ are invariant antisymmetric tensors for $SL(2, \mathbf{Z})_\tau$ and $SL(2, \mathbf{Z})_\rho$ respectively, and $*$ stands for Hodge dual with the metric g_{MN}. We also denote

$$\mathbf{F}^{m\alpha}_{(2)} = d\mathbf{A}^{m\alpha}_{(1)}, \quad H_{(3)} = dB_{(2)} - \frac{1}{2}\epsilon_{mn}\mathcal{E}_{\alpha\beta}\mathbf{A}_{(1)m\alpha}\mathbf{F}^{n\beta}_{(2)} \tag{3.5}$$

and

$$g_{mn} = \mathcal{M}(\tau), \quad g^{mn} = \mathcal{M}(\tau)^{-1}, \quad \mathcal{G}_{\alpha\beta} = \mathcal{M}(\rho) \tag{3.6}$$

where it is convenient to define

$$\mathcal{M}(z) := \frac{1}{\mathrm{Im}z}\begin{pmatrix} 1 & \mathrm{Re}z \\ \mathrm{Re}z & |z|^2 \end{pmatrix}. \tag{3.7}$$

The real part of the RR sector action is given by

$$\sum_{p=0}^{3} S_p\left(g_{(p)}\right) = \pi\int_X\Big\{t^8\mathcal{G}_{\alpha\beta}g^\alpha_{(0)} \wedge * g^\beta_{(0)} + t^6 g^{mn} g_{(1)m} \wedge * g_{(1)n} + \tag{3.8}$$

$$t^4 \mathcal{G}_{\alpha\beta} g^{\alpha}_{(2)} \wedge * g^{\beta}_{(2)} + t^2 g^{mn} g_{(3)m} \wedge * g_{(3)n} \Bigg\}$$

together with

$$S_4\left(g_{(4)}\right) = \pi \int_X \mathrm{Im}(\rho) g_{(4)} \wedge * g_{(4)}. \tag{3.9}$$

3.1.2 The semiclassical expansion

The vevs of the two fields t and $e^{-2\xi}$ (the 8-dimensional length scale of X and the inverse-square 8D string coupling) define semiclassical expansions when they become large. We will expand around a solution of the equations of motion on X. To leading order in our expansion this means X admits a Ricci flat metric[3] g_{MN}. We also have constant scalars t, ξ, τ, ρ, and $\mathbf{F}^{m\alpha}_{(2)} = 0$, $H_{(3)} = 0$, so the background action S_{NSNS} is zero. Finally, we expand around a classical field configuration for the RR fluxes, and to leading order these fluxes $g_{(p)}$ are harmonic forms. Nonzero fluxes contribute terms to the partition function going like $\mathcal{O}(e^{-t^{8-2p}})$.

Let us consider the leading order contribution to the partition function. There are several sources of contributions even at leading order, but, since we are interested in questions of T-duality, most of these can be neglected. [4] The volume of X suppresses the contribution of fluxes $g_{(p)}, p = 0, 1, 2, 3$, and, to leading order in the $t \to \infty$ expansion these can be set to their classical values. Note, however, that neither the string coupling, nor the volume of X, suppress the action for $g_{(4)}$, and thus we must work in a fully quantum mechanical way with this field. This is just as well, since (not coincidentally) this is the term in the action which is not manifestly T-duality invariant. Fortunately, in our approximation, $g_{(4)}$ is a free, nonchiral field and hence quantization is straightforward. Including subleading terms

[3] Almost nothing in what follows relies on the Ricci flatness of the metric. We avoid using this condition since a T-duality anomaly on non-Ricci flat manifolds would signal an important inconsistency in formulating string theory on manifolds of topology $X \times T^2$.

[4] In particular we are negelecting determinants of KK and string modes, and perturbative corrections $\mathcal{O}(g^2_{\mathrm{string}})$. These are all T-duality invariant. The backreaction of nonzero RR fluxes on the NSNS action simply renormalizes V to V_{eff}, where $\rho = B_0 + iV_{eff}$ is the variable on which $SL(2, \mathbf{Z})_\rho$ acts by fractional linear transformations.

in the expansion parameter t involves (among other things) summing over the RR fluxes $g_{(p)}$, $p = 0, 1, 2, 3$.

Finally, in order to be consistent with our approximation scheme we must allow the possibility of *flat* potentials in the background. [5] These contribute nontrivially to the partition function through important phases and accordingly, we will generalize our background to include these. The real part of the action for the flat configurations vanishes, of course, and hence in the physical partition function one must integrate over these flat configurations. In the RR sector the flat potentials are thought to be classified by $K^1(X_{10}; U(1))$ [81]. These contribute no phase to the action and we will henceforth ignore them. [6] The space of flat NSNS potentials is $H^2(X; U(1)) \times (H^1(X; U(1)))^4$.

In this chapter we will work only with the identity component of this torus. Accordingly, we will identify the space of flat NSNS potentials with the torus

$$\frac{\mathcal{H}^2(X)}{\mathcal{H}^2_{\mathbf{Z}}(X)} \times \left(\frac{\mathcal{H}^1(X)}{\mathcal{H}^1_{\mathbf{Z}}(X)}\right)^4 \tag{3.10}$$

where $\mathcal{H}^p(X)$ is a space of harmonic p-forms on X and $\mathcal{H}^p_{\mathbf{Z}}(X)$ is the lattice of integrally normalized harmonic p-forms on X. The first factor is for $B_{(2)}$ and the second factor for the fields $\mathbf{A}^{m\alpha}_{(1)}$ transforming in the $(\mathbf{2}, \mathbf{2})$ of $\mathcal{D}_T$.

Putting all these ingredients together the partition function we wish to study can be schematically written as

$$Z(t, g_{MN}, \xi, \tau, \rho) = \int_{\text{flat potentials}} d\mu_{\text{flat}} \sum_{\text{RR fluxes}} \text{Det} \cdot e^{-S_{cl}} + \cdots \tag{3.11}$$

where $d\mu_{\text{flat}}$ is a T-duality invariant measure on the flat potentials, Det is a product of 1-loop determinants and S_{cl} is the classical action. Now, to investigate T-duality it is convenient to denote by $\mathcal{F}$ the collection of all fields occuring in (3.11) which transform locally and linearly under $\mathcal{D}_T$. These include the flat NSNS potentials above as well as the classical fluxes $g_{(p)}, p = 0, \ldots, 3$. We introduce a measure $[d\mathcal{F}]$ on $\mathcal{F}$ which includes integration over

[5] By "flat" we mean the DeRham representative of the relevant fieldstrength is zero.

[6] They do contribute an overall volume factor to the partition function. This volume includes a factor of $|K^0(X_{10})_{tors}|$ and should be T-duality invariant.

the flat potentials and summation over the fluxes for $p = 0, 1, 2, 3$. This measure is T-duality invariant, and we can write

$$Z(t, g_{MN}, \xi, \tau, \rho) = \int [d\mathcal{F}] Z(\mathcal{F}; t, g_{MN}, \xi, \tau, \rho). \tag{3.12}$$

The invariance of (3.68) under the subgroup $SL(2, Z)_\tau$ of the T-duality group is essentially trivial. The relevant actions and determinants are all based on $SL(2, Z)_\tau$-invariant differential operators. The invariance of the theory under $SL(2, Z)_\rho$ is, however, much more nontrivial. Therefore we simplify notation and just write $Z(\mathcal{F}, \rho)$ for the integrand of (3.68). Now, checking T-duality invariance is reduced to checking the invariance of $Z(\mathcal{F}, \rho)$. This function is constructed from

- a. The K-theoretic sum over RR fluxes of $g_{(4)}$ in the presence of $\mathcal{F}$.
- b. The integration over the Fermi zeromodes in the presence of $g_{(4)}$ and $\mathcal{F}$.
- c. The inclusion of 1-loop determinants, including determinants of the 8D supergravity fields and the quantum corrections due to worldsheet instantons.

In the following subsections we sketch how each of these elements enters $Z(\mathcal{F}, \rho)$. Briefly, the K-theoretic sum over RR fluxes $g_{(4)}$ leads to a theta function $\Theta(\mathcal{F}, \rho)$. This function turns out to transform anomalously under T-duality. The integration over the fermion zeromodes corrects this to a function $\widehat{\Theta}(\mathcal{F}, \rho)$. This function still transforms anomalously. The inclusion of 1-loop effects, including the string 1-loop effects finally cancels the anomaly.

3.1.3 The K-theoretic RR partition function

In order to write explicit formulae for the quantities in (3.68) we must turn to the K-theoretic formulation of RR fields. In practical terms the K-theoretic formulation alters the standard formulation of supergravity in two ways: First it restricts the allowed flux configurations through a "Dirac quantization condition" on the fluxes. Second, it changes the supergravity action by the addition of important topological terms in the action. [7]

[7] It also alters the overall normalization of the bosonic determinants by changing the nature of the gauge group for RR potentials, but we will not discuss this in the present work.

In more detail, the K-theoretic Dirac quantization condition states that the DeRham class of the total RR fieldstrength $[G/(2\pi)]$ is related to a K-theory class $x \in K^0(X_{10})$ via

$$[\frac{G}{2\pi}] = \mathrm{ch}(x)\sqrt{\hat{A}} \tag{3.13}$$

The topological terms in the action can be described as follows. On a general 10-manifold this term involves the mod-two index of a Dirac operator and cannot even be written as a traditional local term in the supergravity action [86, 81, 68]. In the case of zero NS-NS fluxes, the general expression for the phase in the supergravity theory is:

$$Im(S_{10D}) = -2\pi\Phi, \quad \Phi = \Phi_1 + \Phi_2 \tag{3.14}$$

where $e^{2\pi i\Phi_2}$ is the mod-two index and Φ_1 is given by the explicit expression

$$\begin{aligned} \Phi_1 = \int_{X_{10}} \Bigg\{ & -\frac{1}{15}\left(\frac{G_2}{2\pi}\right)^5 + \frac{1}{6}\left(\frac{G_2}{2\pi}\right)^3 \left[\left(\frac{G_4}{2\pi}\right) + \frac{p_1}{12}\left(1+\frac{G_0}{8\pi}\right)\right] \\ & - \left(\frac{G_2}{2\pi}\right)\left[\frac{p_1}{48}\left(\frac{G_4}{2\pi}\right) + \frac{\hat{A}_8}{2}\left(1+\frac{G_0}{2\pi}\right) + \frac{G_0}{4\pi}\left(\frac{p_1}{48}\right)^2\right]\Bigg\} \end{aligned} \tag{3.15}$$

where $G_{2j}, j = 0,1,2$ are RR fluxes on X_{10}, $p_1 = p_1(TX_{10})$ and $\hat{A}$ is expressed in terms of the Pontryagin classes of X_{10} as

$$\hat{A} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}\left(7p_1^2 - 4p_2\right). \tag{3.16}$$

In the case that we reduce to 8 dimensions, taking our manifold to be of the form $X \times T^2$ with the choice of supersymmetric spin structure on T^2 the above considerations simplify and can be made much more concrete.

Consider first the Dirac quantization condition. We reduce RR fieldstrengths as:

$$\begin{aligned} \frac{G_0}{2\pi} &= g^2_{(0)} \\ \frac{G_2}{2\pi} &= g^1_{(0)} d\sigma^8 \wedge d\sigma^9 + g_{(1)m}\wedge d\sigma^m + g^2_{(2)} \\ \frac{G_4}{2\pi} &= g_{(4)} + g_{(3)m}\wedge d\sigma^m + g^1_{(2)}\wedge d\sigma^8\wedge d\sigma^9 \end{aligned} \tag{3.17}$$

where $\sigma^m, m = 8,9$ are coordinates on T^2. Recall once again the subscript (p) on g's denotes form degree on X. The other sub(super)scripts indicate $\mathcal{D}_T$ transformation properties.

In the K-theoretic formulation of flux quantization the fieldstrengths $g_{(4)}, g_{(3)m}, g^\alpha_{(2)}, g_{(1)m}, g^\alpha_{(0)}$ are related to certain integral cohomology classes which we denote as

$$a \in H^4(X,\mathbf{Z}), \quad f_m \in H^3(X,\mathbf{Z}) \otimes \mathbf{Z}^2, \quad e^\alpha = \begin{pmatrix} e'' \\ e \end{pmatrix} \in H^2(X,\mathbf{Z}) \otimes \mathbf{Z}^2, \tag{3.18}$$

$$\gamma_m \in H^1(X,\mathbf{Z}) \otimes \mathbf{Z}^2, \quad n^\alpha = \begin{pmatrix} n_1 \\ n_0 \end{pmatrix} \in H^0(X,\mathbf{Z}) \otimes \mathbf{Z}^2$$

The explicit relation between these classes and the $g_{(p)}$ is somewhat complicated and given in equation (4.3) below. The K-theoretic Dirac quantization condition leaves all integral classes in (3.18) unconstrained except for f_m. One finds that $Sq^3(f_m)$ is fixed to be an integral class which depends only on the topology of X. Turning on flat NSNS potentials acts as an automorphism of the K-theory torus. These potentials modify the reduction formulae (3.17) according to equations (5.15) to (5.18) below.

Now let us consider the phase. It turns out that on 10-folds of the form $X \times T^2$ the phase $e^{2\pi i \Phi_2}$ arising from the mod 2 index may be expressed much more explicitly as

$$\exp[2\pi i \Phi_2] = \exp\Bigg[i\pi \int_X \Bigg\{ g_{(3)8} \cup Sq^2(g_{(3)9}) + g_{(3)8} \cup Sq^2(g_{(3)8}) + g_{(3)9} \cup Sq^2(g_{(3)9}) + \tag{3.19}$$

$$g^2_{(0)} \hat{A}_8 + \left(g_{(4)} + \frac{g^2_{(0)}}{48} p_1 - \frac{1}{2}\left(g^2_{(2)}\right)^2 \right)\left(\left[g^1_{(2)} - g^1_{(0)} g^2_{(2)} + g_{(1)8} g_{(1)9} \right]^2 + \frac{p_1}{2} \right)$$

$$+ \frac{p_1^2}{8} + g_{(1)8} g_{(1)9} \left(g^2_{(2)}\right)^3 - \left(g^2_{(2)}\right)^2 \epsilon^{mn} g_{(1)m} g_{(3)n} \Bigg\} \Bigg]$$

This expression is cohomological although it is still unconventional in supergravity theory since it involves Steenrod squares.

The above topological term (3.14) is deduced from the K-theory theta function Θ_K defined in [86, 81, 68], and reviewed below. As explained above, it is convenient to fix the fields $\mathcal{F}$. We can define a function $\Theta(\mathcal{F}, \rho)$ by writing Θ_K as a sum

$$\Theta_K = \sum e^{-S_B(\mathcal{F})} \Theta(\mathcal{F}, \rho) \tag{3.20}$$

The sum is over all integral classes except a. That is, we sum over $n^\alpha, \gamma_m, e^\alpha, f_m$ subject to the constraint on $Sq^3 f_m$. The action $S_B(\mathcal{F})$ is the manifestly T-duality invariant action for the fluxes given in (3.8). Θ_K is a function of g_{MN}, ρ, τ and the flat background NSNS fluxes. These flat fluxes have a K-theoretic interpretation in terms of automorphisms of the K-theory group $K^0(X) \otimes R$ and act naturally on the theta function. More concretely, the inclusion of nonzero flat NSNS fields $B_0, B_{(2)}, \mathbf{A}^{m\alpha}_{(1)}$ modifies the phase Φ as shown in equations (5.20)-(5.24) below.

Since the K-theoretic constraint $Sq^3 a = 0, a \in H^4(X, \mathbf{Z})$ is automatically satisfied on spin 8-folds X it turns out that $\Theta(\mathcal{F})$ is, essentially, a Siegel-Narain theta function for the lattice $H^4(X; \mathbf{Z})$. More precisely, there is a quadratic form on $H^4(X; \mathbf{R})$ given by $Q = \text{Im}(\rho) HI - i\text{Re}(\rho) I$ where H is the action of Hodge $*$ and I is the integral intersection pairing on $H^4(X, \mathbf{Z})$. Then

$$\Theta(\mathcal{F}, \rho) = e^{i2\pi \Delta\widetilde{\Phi}(\mathcal{F})} \Theta \begin{bmatrix} \vec{\tilde{\alpha}} \\ \vec{\tilde{\beta}} \end{bmatrix} (Q) \tag{3.21}$$

Here $\Theta \begin{bmatrix} \vec{\tilde{\alpha}} \\ \vec{\tilde{\beta}} \end{bmatrix} (Q)$ is the Siegel-Narain theta function with characteristics. The characteristics are written explicitly in (5.20) below. Finally, the prefactor $\Delta\widetilde{\Phi}(\mathcal{F})$ in (3.21) is defined in (5.23) below.

3.1.4 T-duality transformations

One of the more subtle aspects of the K-theoretic formulation of RR fluxes, is that the very formulation of the action depends crucially on a choice of polarization of the K-theory lattice $K(X_{10})$ with respect to the quadratic form defined by the index. In the above discussion we have chosen the "standard polarization" for IIA theory, i.e Γ_2 is the sublattice of $K(X_{10})$ with vanishing G_4, G_2, G_0. Γ_1 is then a complementary Lagrangian sublattice such that $K(X_{10}) = \Gamma_1 + \Gamma_2$. The standard polarization is distinguished for any large 10-manifold in the following sense. When the metric of X_{10} is scaled up $\hat{g}_{\hat{M}\hat{N}} \to t^2 \hat{g}_{\hat{M}\hat{N}}$ the action $\int_{X_{10}} \sqrt{\hat{g}} |G_{2p}|^2$ of the Type IIA RR 2p-form scales as t^{10-4p}. This allows the sensible approximation of first summing only over G_4, with $G_2 = G_0 = 0$, then including G_2 with $G_0 = 0$, and finally summing over all classical fluxes G_4, G_2, G_0.

In the case of $X_{10} = T^2 \times X$ with the metric (3.1) the standard polarization is no longer distinguished. Various equally good choices are related by the action of the T-duality group $\mathcal{D}_T$ on $\Gamma_K := K(X \times T^2)$. [8] In section 4 we explain how the duality group $\mathcal{D}_T$ acts as a subgroup of symplectic transformations on the K-theory lattice and we give an explicit embeding $\mathcal{D}_T \subset Sp(2N, \mathbf{Z})$, where $2N = rank(\Gamma_K)$. As explained in section 4.2, since $\mathcal{D}_T$ acts symplectically, the function $\Theta(\mathcal{F}, \rho)$ must transform under T-duality as $\Theta(\gamma \cdot \mathcal{F}, \gamma \cdot \rho) = j(\gamma, \rho)\Theta(\mathcal{F}, \rho)$ where $j(\gamma, \rho)$ is a standard transformation factor for modular forms. Nevertheless, this transformation law leaves open the possibility of a T-duality anomaly through a multiplier system in $j(\gamma, \rho)$. In order to investigate this potential anomaly more closely we must choose an explicit duality frame and perform the relevant modular transformations.

We find that, in fact, the function $\Theta(\mathcal{F}, \rho)$ does transform as a modular form with a nontrivial "multiplier system" under $SL(2, Z)_\rho$. That is, using the standard generators T, S of $SL(2, Z)_\rho$ we have:

$$\Theta(T \cdot \mathcal{F}, \rho + 1) = \mu(T)\Theta(\mathcal{F}, \rho) \tag{3.22}$$

$$\Theta(S \cdot \mathcal{F}, -1/\rho) = \mu(S)(-i\rho)^{\frac{1}{2}b_4^+}(i\bar{\rho})^{\frac{1}{2}b_4^-}\Theta(\mathcal{F}, \rho)$$

where $T \cdot \mathcal{F}, S \cdot \mathcal{F}$ denotes the linear action of $\mathcal{D}_T$ on the fluxes. Here b_4^+, b_4^- is the dimension of the space of self-dual and anti-self-dual harmonic forms on X and the multiplier system is

$$\mu(T) = \exp[\frac{i\pi}{4}\int_X \lambda^2] \tag{3.23}$$

$$\mu(S) = \exp[\frac{i\pi}{2}\int_X \lambda^2]$$

where λ is the integral characteristic class of the spin bundle on X. (So, $2\lambda = p_1$). The multiplier system is indeed nontrivial on certain 8-manifolds, for example on all Calabi-Yau 4-folds CY_4 with Euler characteristic χ not divisible by 12. This follows from the relations valid for any CY_4

$$\frac{1}{4}\int_X \lambda^2 = 62\int_X \hat{A}_8 - 4 + \frac{1}{12}\chi \tag{3.24}$$

[8] There is also a polarization on manifolds of the type $S^1 \times X_9$, (in our case $X_9 = S^1 \times X$) where the measure is is purely real and the imaginary part of the action is an integral multiple of $i\pi$ (without flat NSNS potentials). However, this polarization does not lead to a good long-distance approximation scheme.

An example of such a Calabi-Yau, considered in [88], is a homogenuos polynomial of degree 6 in P^5, which has $\chi = 2610$.

In more physical language, the "multiplier system" signals a potential T-duality anomaly. Such an anomaly would spell disaster for the theory since the T-duality group should be regarded as a *gauge symmetry* of M-theory. Accordingly, we turn to the remaining functional integrals in the supergravity theory. We will find that the anomalies cancel, of course, but this cancellation is surprisingly intricate.

3.1.5 Inclusion of 1-loop effects

We first turn to the 1-loop functional determinants of the quantum fluctuations of the bosonic fields. We show that these are all manifestly T-duality invariant functions of $\mathcal{F}$ except for the quantum fluctuations of $g_{(4)}$. The full bosonic 1-loop determinant Det_B is given in equation (6.20) below. The net effect of inlcuding the bosonic determinants is thus to replace

$$e^{-S_B(\mathcal{F})}\Theta(\mathcal{F},\rho) \rightarrow Z_B(\mathcal{F},\rho) := \mathrm{Det}_B e^{-S_B(\mathcal{F})}\Theta(\mathcal{F},\rho) \tag{3.25}$$

Inclusion of this determinant alters the modular weight so that $Z_B(\mathcal{F},\rho)$ transforms with weight $(\frac{1}{4}(\chi+\sigma), \frac{1}{4}(\chi-\sigma))$, in close analogy to the theory of abelian gauge potentials on a 4-manifold [89, 90]. Here χ, σ are the Euler character and signature of the 8-fold X. The multiplier system (3.119)is left unchanged.

Now let us consider modifications from the fermionic path integral. Recall that we may always regard a modular form as a section of a line bundle over the modular curve $\mathcal{H}/SL(2,Z)_\rho$. On general grounds, we expect the fermionic path integral to provide a trivializing line bundle. The gravitino and dilatino in the 8d theory transform as modular forms under the T-duality group $\mathcal{D}_T$ with half-integral weights and consequently they too are subject to potential T-duality anomalies.

The inclusion of the fermions modifies the bosonic partition function in two ways: through zeromodes and through determinants. The fermion action in the 8D supergravity has the form

$$S^{(8)}_{\text{Fermi}} = S_{\text{kinetic}} + S_{\text{fermi-flux}} + S_{4-\text{fermi}} \tag{3.26}$$

where kinetic terms S_{kinetic} as well as fermion-flux couplings $S_{\text{fermi-flux}}$ are quadratic in fermions and $S_{4-\text{fermi}}$ denotes the four-fermion coupling. S_{kinetic} is T-duality invariant but $S_{\text{fermi-flux}}$ and $S_{4-\text{fermi}}$ contain some non-invariant terms. The non-invariant fermion zeromode couplings are collected together in the form

$$S^{(zm)ninv} = 4\pi \text{Im}\rho \; g_{(4)} \wedge * Y_{(4)} + 2\pi \text{Im}\rho \; Y_{(4)} \wedge * Y_{(4)} \tag{3.27}$$

where the harmonic 4-form $Y_{(4)}$ is bilinear in the fermion zeromodes. The explicit expression for $Y_{(4)}$ can be found in equations (7.21) and (7.41) below.

The inclusion of the integral over the fermionic zeromodes of S_{kinetic} modifies the partition function by replacing the expression $\Theta(\mathcal{F}, \rho)$ in (3.21) by

$$\widehat{\Theta}(\mathcal{F}, \rho) = \int d\mu_F^{(zm)} \; e^{i2\pi \widehat{\Delta\Phi}(\mathcal{F})} \Theta \begin{bmatrix} \vec{\tilde{\alpha}} \\ \vec{\tilde{\beta}} \end{bmatrix} (Q) \tag{3.28}$$

Here

$$\Theta \begin{bmatrix} \vec{\tilde{\alpha}} \\ \vec{\tilde{\beta}} \end{bmatrix} (Q)$$

is a supertheta function for a superabelian variety based on the K-theory theta function. (This is explained in Appendix F.) In particular, the characteristics $\vec{\tilde{\alpha}}, \vec{\tilde{\beta}}$ differ from $\vec{\alpha}, \vec{\beta}$ by expressions bilinear in the fermion zeromodes. Similarly, the prefactor $\widehat{\Delta\Phi}$ differs from $\Delta\tilde{\Phi}$ by an expression quartic in the fermion zeromodes. Finally, $d\mu_F^{(zm)}$ is a T-duality invariant measure for the finite dimensional integral over fermion and ghost zeromodes. It includes the T-duality invariant term $e^{-S^{(zm)inv}}$ from the action.

Including the one-loop fermionic determinants of the non-zero modes we finally arrive at

$$Z_{B+F}(\mathcal{F}, \rho) := \text{Det}'_B \text{Det}'_F e^{-S_B(\mathcal{F})} \widehat{\Theta}(\mathcal{F}, \rho) \tag{3.29}$$

The formula we derive for (3.29) allows a relatively straightforward check of the T-duality transformation laws and we find:

$$Z_{B+F}(T \cdot \mathcal{F}, \rho + 1) = \mu(T) Z_{B+F}(\mathcal{F}, \rho) \tag{3.30}$$

$$Z_{B+F}(S\cdot\mathcal{F},-1/\rho)=(-i\rho)^{\frac{1}{4}\chi+\frac{1}{8}\int_X(p_2-\lambda^2)}(i\bar{\rho})^{\frac{1}{4}\chi-\frac{1}{8}\int_X(p_2-\lambda^2)}Z_{B+F}(\mathcal{F},\rho)$$

Perhaps surprisingly, the fermion determinants have *not* completely trivialized the RR contribution to the path integral measure. However, there is one final ingredient we must take into account: In the low energy supergravity there are quantum corrections which contribute to leading order in the $t\to\infty$ and $\xi\to-\infty$ limit. From the string worldsheet viewpoint these consist of a 1-loop term in the α' expansion together with worldsheet instanton corrections. From the M-theory viewpoint we must include the one-loop correction $\int C_3X_8$ in M-theory together with the effect [79] of membrane instantons. The net effect is to modify the action by the quantum correction

$$S_{\text{quant}}=\Big[\frac{1}{2}\chi+\frac{1}{4}\int_X(p_2-\lambda^2)\Big]log\,[\eta(\rho)]+\Big[\frac{1}{2}\chi-\frac{1}{4}\int_X(p_2-\lambda^2)\Big]log\,[\eta(-\bar{\rho})] \qquad (3.31)$$

Where $\eta(\rho)$ is the Dedekind function. The final combination

$$Z(\mathcal{F},\rho)=e^{-S_{\text{quant}}}Z_{B+F}(\mathcal{F},\rho) \qquad (3.32)$$

is the fully T-duality invariant low energy partition function.

3.1.6 Applictions

As a by-product of the above results we will make some comments on the open problem of the relation of M-theory to massive IIA string theory. In [73] C. Hull made an interesting suggestion for an 11-dimensional interpretation of certain backgrounds in the Romans theory. One version of Hull's proposal states that massive IIA string theory on $T^2\times X$ is equivalent to M-theory on a certain 3-manifold, the nilmanifold.

In section 9 we review Hull's proposal. We note that when proper account is taken of the various phases of the supergravity action the equivalence of classical actions required in this proposal cannot be true. Nevertheless, it is not difficult to modify Hull's proposal so that the partition function $Z(\mathcal{F},\rho)$ can be identified with a certain sum over M-theory geometries involving the nilmanifold. The detailed proposal can be found in section 9.3.

3.1.7 U-duality and M-theory

In the final section of Chapter3 we comment on some of the issues which arise in trying to extend these considerations to writing the fully U-duality-invariant partition function. We summarize briefly the M-theory partition function on $X \times T^3$, we comment on the $SL(2,Z)_\rho$ duality invariance, and we make some preliminary remarks on how one can see K-theory theta functions for twisted K-theory from the M theory formulation.

3.2 Review of T-duality invariance in the standard formulation of type IIA supergravity

We start by reviewing bosonic part of the standard 10D IIA supergravity action [91]. Fermions will be incorporated into the discussion in section 7.

3.2.1 Bosonic action of the standard 10D IIA supergravity

The 10D NSNS fields are the dilaton ϕ, 2-form potential $\hat{B}_2$ and string frame metric $\hat{g}_{\hat{M}\hat{N}}$, where $\hat{M},\hat{N} = 0,\ldots 9$. The 10D RR fieldstrenghts are the 4-form G_4, 2-form G_2 and 0-form G_0.

We measure all dimensionful fields in units of 11D Planck length l_p and set $k_{11} = \pi$, so

$$S_{bos}^{(10)} = \frac{1}{2\pi}\int_{X_{10}} e^{-2\phi}\left(\sqrt{g_{10}}\mathcal{R}(\hat{g}) + 4d\phi\wedge\hat{*}d\phi + \frac{1}{2}\hat{H}_3\wedge\hat{*}\hat{H}_3\right) \tag{3.33}$$
$$+\frac{1}{4\pi}\int_{X_{10}}\left(\tilde{G}_4\wedge\hat{*}\tilde{G}_4 + i\hat{B}\wedge\tilde{G}_4\wedge\tilde{G}_4 + \tilde{G}_2\wedge\hat{*}\tilde{G}_2 + \sqrt{g_{10}}G_0^2\right)$$

where $\hat{*}$ stands for the 10D Hodge duality operator. The fields in (3.33) are defined as

$$\tilde{G}_2 = G_2 + \hat{B}_2 G_0, \quad \tilde{G}_4 = G_4 + \hat{B}_2 G_2 + \frac{1}{2}\hat{B}_2\hat{B}_2 G_0, \quad \hat{H}_3 = d\hat{B}_2.$$

We explain the relation between our fields and those of [91] in Appendix(B).

3.2.2 Reduction of IIA supergravity on a torus

We now recall some basic facts about the reduction of the bosonic part of the 10D action on T^2. Let us consider $X_{10} = T^2 \times X$ and split coordinates as $X^{\hat{M}} = (x^M, \sigma^m)$, where $M = 0, \ldots, 7, \quad m = 8, 9$.

The standard ansatz for the reduction of the 10d metric has the form:

$$ds^2_{10} = t^2 g_{MN} dx^M dx^N + V g_{mn} \omega^m \otimes \omega^n \tag{3.34}$$

where g_{mn} is defined in (3.6), $t^2 g_{MN}$ is 8D metric, $det g_{MN} = 1$. V is the volume of T^2 and $\omega^m = d\sigma^m + A^m_{(1)}$. The other bosonic fields of the $8D$ theory are listed below.

- 1. $g^\alpha_{(0)}, g^\alpha_{(2)}, \quad \alpha = 1, 2 \quad g_{(1)m}, g_{(3)m} \quad m = 8, 9 \quad$ and $g_{(4)}$ are defined from[9]

$$\begin{aligned} \frac{G_0}{2\pi} &= g^2_{(0)} \\ \frac{\tilde{G}_2}{2\pi} &= \left(g^1_{(0)} + g^2_{(0)} B_0\right) \frac{1}{2} \epsilon_{mn} \omega^m \omega^n + g_{(1)m} \omega^m + g^2_{(2)} \\ \frac{\tilde{G}_4}{2\pi} &= g_{(4)} + g_{(3)m} \omega^m + \left(B_0 g^2_{(2)} + g^1_{(2)}\right) \frac{1}{2} \epsilon_{mn} \omega^m \omega^n \end{aligned} \tag{3.35}$$

- 2. The $8D$ dilaton ξ is defined by

$$e^{-2\xi} = e^{-2\phi} V \tag{3.36}$$

- 3. $B_{(2)}, B_{(1)m}, B_0$ are obtained from the KK reduction of the NSNS 2-form potential in the following way

$$\hat{B}_2 = \frac{1}{2} B_0 \epsilon_{mn} \omega^m \omega^n + B_{(1)m} \omega^m + B_{(2)} + \frac{1}{2} A^m_{(1)} B_{(1)m} \tag{3.37}$$

Now, the real part of the $8D$ bosonic action obtained by the above reduction is

$$Re\left(S^{(8D)}_{\text{boson}}\right) = S_{NS} + \sum_{p=0}^{3} S_p\left(g_{(p)}\right) + S_4\left(g_{(4)}\right) \tag{3.38}$$

[9] $\epsilon^{89} = 1, \quad \epsilon_{89} = 1$

where

$$S_{NS} = \frac{1}{2\pi}\int e^{-2\xi}\Big\{t^6(\mathcal{R}(g) + 4d\xi\wedge *d\xi + 28t^{-2}dt\wedge * dt) + \frac{1}{2}t^2 H_{(3)}\wedge *H_{(3)} \tag{3.39}$$

$$+\frac{1}{2}t^6\frac{d\tau\wedge *d\tau}{(\mathrm{Im}\tau)^2} + \frac{1}{2}t^6\frac{d\rho\wedge *d\rho}{(\mathrm{Im}\rho)^2} + \frac{1}{2}t^4 g_{mn}\mathcal{G}_{\alpha\beta}\mathbf{F}^{m\alpha}\wedge *\mathbf{F}^{n\beta}\Big\}$$

where $\mathcal{G}_{\alpha\beta}$ is defined in (3.6) and $\mathcal{A}^m_{(1)}$ and $B_{(1)m}$ are combined into 1-form as a collection of

$$\mathbf{A}^{m\alpha}_{(1)} = \begin{pmatrix} \epsilon^{mn}B_{(1)n} \\ \mathcal{A}^m_{(1)} \end{pmatrix} \tag{3.40}$$

Also, we denote[10]

$$H_{(3)} = dB_{(2)} - \frac{1}{2}\epsilon_{mn}\mathcal{E}_{\alpha\beta}\mathbf{A}_{(1)m\alpha}\mathbf{F}^{n\beta}_{(2)} \tag{3.41}$$

$$\sum_{p=0}^{3} S_p\left(g_{(p)}\right) = \pi\int_X\Big\{t^8\mathcal{G}_{\alpha\beta}g^\alpha_{(0)}\wedge * g^\beta_{(0)} + t^6 g^{mn}g_{(1)m}\wedge * g_{(1)n}+ \tag{3.42}$$

$$t^4\mathcal{G}_{\alpha\beta}g^\alpha_{(2)}\wedge * g^\beta_{(2)} + t^2 g^{mn}g_{(3)m}\wedge * g_{(3)n}\Big\}$$

Finally we have

$$S_4\left(g_{(4)}\right) = \pi\int_X \mathrm{Im}(\rho)g_{(4)}\wedge *g_{(4)} \tag{3.43}$$

It is convenient to introduce the notation $S_B(\mathcal{F}) = \sum_{p=0}^{3} S_p\left(g_{(p)}\right)$ for the value of the actions evaluated on a background flux field configuration. $S_B(\mathcal{F})$ will enter the partition sum $Z_{B+F}(\mathcal{F},\tau,\rho)$ in equation (8.1) below.

3.2.3 T-duality action on 8D bosonic fields

The T-duality group of the $8D$ effective theory obtained by reduction on T^2 is known to be $\mathcal{D}_T = SL(2,\mathbf{Z})_\tau \times SL(2,\mathbf{Z})_\rho$, where the first factor is mapping class group of T^2 which acts on τ

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{3.44}$$

and the second factor acts on $\rho = B_0 + iV$

$$\rho \to \frac{\alpha\rho + \beta}{\gamma\rho + \delta} \tag{3.45}$$

[10] $\mathcal{E}_{12} = 1, \quad \mathcal{E}_{21} = -1$

Let us denote generators of $SL(2,\mathbf{Z})_\rho$ by

$$S : \rho \to -1/\rho, \quad T : \rho \to \rho + 1$$

and generators of $SL(2,\mathbf{Z})_\tau$ by

$$\tilde{S} : \tau \to -1/\tau, \quad \tilde{T} : \tau \to \tau + 1$$

We now recall how T-duality acts on the remaining bosonic fields of the 8D theory [78]. First, ξ, t, g_{MN} are T-duality invariant. Next, there is the collection of fields $\mathcal{F}$ mentioned in the introduction. These transform linearly under T-duality. They include the NS potential $B_{(2)}$, which is T-duality invariant, as well as $\mathbf{A}^{m\alpha}_{(1)}$, which transform in the $(\mathbf{2},\mathbf{2})$. The other components of $\mathcal{F}$ are the RR fieldstrengths $g^\alpha_{(0)}, g^\alpha_{(2)}, \alpha = 1,2$ which transform in the $(\mathbf{1},\mathbf{2})$ of $\mathcal{D}_T$ and $g_{(1)m}, g_{(3)m}, m = 8,9$ which transform in the $(\mathbf{2}',\mathbf{1})$ of $\mathcal{D}_T$.

Finally, the field $g_{(4)}$ is singled out among all the other fields since according to the conventional supergravity [78] $SL(2,\mathbf{Z})_\rho$ mixes $g_{(4)}$ with its Hodge dual $*g_{(4)}$ and hence $g_{(4)}$ does not have a local transformation. More concretely,

$$\begin{pmatrix} -\mathrm{Re}\rho g_{(4)} + i\mathrm{Im}\rho * g_{(4)} \\ g_{(4)} \end{pmatrix} \tag{3.46}$$

transforms in the $(\mathbf{1},\mathbf{2})$ of $\mathcal{D}_T$. Due to this non-trivial transformation the classical bosonic 8D action $S^{(8D)}_{\text{boson}}$ is not manifestly invariant under $SL(2,\mathbf{Z})_\rho$.

3.3 Review of the K-theory theta function

In this section we review the basic flux quantization law of RR fields and the definition of the K-theory theta function. We follow closely the treatment in [86, 81, 68].

3.3.1 K-theoretic formulation of RR fluxes

As found in [80]-[86] RR fields in IIA superstring theory are classified topologically by an element $x \in K^0(X_{10})$. The relation for $\hat{B}_2 = 0$ is

$$\left[\frac{G}{2\pi}\right] = \sqrt{\hat{A}}chx, \quad G = \sum_{j=0}^{10} G_j \tag{3.47}$$

where ch is the total Chern character and $\hat{A}$ is expressed in terms of the Pontryagin classes as

$$\hat{A} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}\left(7p_1^2 - 4p_2\right) \tag{3.48}$$

In (3.47), the right hand side refers to the harmonic differential form in the specified real cohomology class. The quantization of the RR background fluxes is understood in the sense that they are derived from an element of $K^0(X_{10})$.

3.3.2 Definition of the K-theory theta function

Let us recall the general construction of a K-theory theta function, which serves as the RR partition function in Type IIA. One starts with the lattice $\Gamma_K = K^0(X_{10})/K^0(X_{10})_{tors}$. This lattice is endowed with an integer-valued unimodular antisymmetric form by the formula

$$\omega(x, y) = I(x \otimes \bar{y}), \tag{3.49}$$

where for any $z \in K^0(X_{10})$, $I(z)$ is the index of the Dirac operator with values in z.

Given a metric on X_{10}, one can define a metric on Γ_K

$$g(x, y) = \int_{X_{10}} \frac{G(x)}{2\pi} \wedge \frac{\hat{*}G(y)}{2\pi} \tag{3.50}$$

where $\hat{*}$ is the 10D Hodge duality operator.

Let us consider the torus $\mathbf{T} = (\Gamma_K \otimes_{\mathbf{Z}} \mathbf{R})/\Gamma_K$. The quantities ω and g can be interpreted as a symplectic form and a metric,respectively, on $\mathbf{T}$. To turn $\mathbf{T}$ into a Kahler manifold one defines the complex structure J on $\mathbf{T}$ as

$$g(x, y) = \omega(Jx, y) \tag{3.51}$$

Now, if it is possible to find a complex line bundle $\mathcal{L}$ over $\mathbf{T}$ with $c_1(\mathcal{L}) = \omega$, then $\mathbf{T}$ becomes a "principally polarized abelian variety." $\mathcal{L}$ has, up to a constant multiple, a unique[11] holomorphic section which is the contribution of the sum over fluxes to the RR partition function.

[11]The uniqueness follows from the index theorem on $\mathbf{T}$ using unimodularity of ω and the fact that for any complex line bunlde M over $\mathbf{T}$ with positive curvature we have $H^i(\mathbf{T}; M) = 0, \quad i > 0$.

As explained in detail in [85] , holomorphic line bundles $\mathcal{L}$ over $\mathbf{T}$ with constant curvature ω are in one-one correspondence with U(1)-valued functions Ω on Γ_K such that

$$\Omega(x+y) = \Omega(x)\Omega(y)(-1)^{\omega(x,y)} \tag{3.52}$$

In the case of weakly coupled Type II superstrings one can take Ω to be valued in $\mathbf{Z}_2$. Motivated in part by T-duality Witten proposed that the natural $\mathbf{Z}_2-$ valued function Ω for the RR partition function is given by a mod two index ??. For any $x \in K^0(X_{10})$, $x \otimes \bar{x} \in KO(X_{10})$, lies in the real K-theory group on X_{10}, and for any $v \in KO(X_{10})$, there is a well-defined mod 2 index $q(v)$??. We take

$$\Omega(x) = (-1)^{j(x)} \tag{3.53}$$

where $j(x) = q(x \otimes \bar{x})$.

As explained in [86, 81, 68] there is an anomaly in the theory unless $\Omega(x)$ is identically 1 on the torsion subgroup of $K(X_{10})$. In the absence of this anomaly it descends to a function on $\Gamma_K = K^0(X_{10})/K^0(X_{10})_{tors}$ and can be used to define a line bundle $\mathcal{L}$ and hence the RR partition function.

To define the theta function one can pick an arbitrary splitting of Γ_K as a sum $\Gamma_1 \oplus \Gamma_2$, where Γ_1 and Γ_2 are "maximal Lagrangian" sublattices. ω establishes a duality between Γ_1 and Γ_2, and therefore there exists $\theta_K \in \Gamma_1/2\Gamma_1$ such that

$$\Omega(y) = (-1)^{\omega(\theta_K,y)}, \quad \forall y \in \Gamma_2 \tag{3.54}$$

Following [68] we choose the standard polarization:the sublattice Γ_2^{std} is defined as the set of x with vanishing G_0, G_2, G_4. This choice implies that G_0, G_2, G_4 are considered as independent variables. This is a distinguished choice for every large 10-manifold in the sense that it allows for a good large volume semiclassical approximation scheme on any 10-manifold (see sec.5).

It was demonstrated in [68] that Γ_1^{std} in the standard polarization consists of K-theory classes of the form $x = n_0\mathbf{1} + x(c_1, c_2)$. $\mathbf{1}$ is a trivial complex line bundle and $x(c_1, c_2)$ is defined for $c_1 \in H^2(X_{10},\mathbf{Z})$ and $c_2 \in H^4(X_{10},\mathbf{Z})$ with $Sq^3c_2 = 0$, as

$$ch(x(c_1,c_2)) = c_1 + (-c_2 + \frac{1}{2}c_1^2) + \dots . \qquad (3.55)$$

The higher Chern classes indicated by ... are such that $x(c_1,c_2)$ is in a maximal Lagrangian sublattice Γ_1^{std} complementary to Γ_2^{std}. Then, θ_K for the standard polarization can be chosen to satisfy

$$ch_0(\theta_K) = 0, \quad ch_1(\theta_K) = 0, \quad ch_2(\theta_K) = -\lambda + 2\hat{a}_0, \quad I(\theta_K) = 0 \qquad (3.56)$$

where $\lambda = \frac{1}{2}p_1$ and $\hat{a}_0$ is a fixed element of $H^4(X_{10},\mathbf{Z})$ such that

$$\forall \hat{c} \in L' \quad f(\hat{c}) = \int_{X_{10}} \hat{c} \cup Sq^2\hat{a}_0 \qquad (3.57)$$

where $L' = \left\{\hat{c} \in H^4_{tors}(X_{10},\mathbf{Z})/2H^4_{tors}(X_{10},\mathbf{Z}), \quad Sq^3(\hat{c}) = 0\right\}$ and $f(\hat{a})$ stands for the mod 2 index of the Dirac operator coupled to an E_8 bundle on the 11D manifold $X_{10}\times S^1$ with the characteristic class $\hat{a} \in H^4(X_{10},\mathbf{Z})$ and supersymmetric spin structure on the S^1.

The K-theory theta function in the standard polarization is

$$\Theta_K = e^{iu}\sum_{x\in\Gamma_1} e^{i\pi\tau_K(x+\frac{1}{2}\theta_K)}\Omega(x) \qquad (3.58)$$

where $u = -\frac{\pi}{4}\int_{X_{10}} ch_2(\theta_K)ch_3(\theta_K)$ and the explicit form of the period matrix τ_K is given by

$$Re\tau_K(x+\frac{1}{2}\theta_K) = \frac{1}{(2\pi)^2}\int_{X_{10}}(G_0G_{10} - G_2G_8 + G_4G_6) \qquad (3.59)$$

$$Im\tau_K(x+\frac{1}{2}\theta_K) = \sum_{p=0}^{2}\frac{1}{(2\pi)^2}\int_{X_{10}} G_{2p}\wedge\hat{*}G_{2p} \qquad (3.60)$$

The RR fields which enter (3.59,3.60) are:

$$\begin{aligned} \frac{1}{2\pi}G_0(x+\frac{1}{2}\theta_K) &= n_0 \\ \frac{1}{2\pi}G_2(x+\frac{1}{2}\theta_K) &= \hat{e} \\ \frac{1}{2\pi}G_4(x+\frac{1}{2}\theta_K) &= \hat{a} + \frac{1}{2}\hat{e}^2 - \frac{1}{2}(1+n_0/12)\lambda \end{aligned} \qquad (3.61)$$

where we denote $\hat{e} = c_1(x), \quad \hat{a} = -c_2(x) + \hat{a}_0$.

From (3.58) and (3.59,3.60) the following topological term was found in [68] to be the K-theoretic corrections to the 10D IIA supergravity action.

$$e^{2\pi i\Phi(n_0,\hat{e},\hat{a})} = exp\Big[-2\pi i n_0 \int_{X_{10}} \hat{e}\left(\sqrt{\hat{A}}\right)_8\Big] (\Omega(1))^{n_0} e^{2\pi i\Phi(\hat{e},\hat{a})} \tag{3.62}$$

$$e^{2\pi i\Phi(\hat{e},\hat{a})} = (-1)^{f(\hat{a}_0)}(-1)^{f(\hat{a})} exp\left[2\pi i \int_{X_{10}} \left(\frac{\hat{e}^5}{60} + \frac{\hat{e}^3\hat{a}}{6} - \frac{11\hat{e}^3\lambda}{144} - \frac{\hat{e}\hat{a}\lambda}{24} + \frac{\hat{e}\lambda^2}{48} - \frac{1}{2}\hat{e}\hat{A}_8\right)\right] \tag{3.63}$$

3.3.3 Turning on NSNS 2-form flux with $[\hat{H}_3] = 0$

Let us turn on $\hat{B}_2 \in H^2(X_{10}, R)$. We normalize $\hat{B}_2$ so that it is defined mod $H^2(X_{10}, \mathbf{Z})$ under global tensorfield gauge transformation.

We assume that the RR fields are still classified topologically by $x \in K^0(X_{10})$. The standard coupling to the D-branes implies that the cohomology class of the RR field is [104]

$$\frac{\tilde{G}(x)}{2\pi} = e^{\hat{B}_2} ch(x)\sqrt{\hat{A}} \tag{3.64}$$

If we assume that

$$\frac{\overline{\tilde{G}(x)}}{2\pi} := e^{-\hat{B}_2} ch(\bar{x})\sqrt{\hat{A}} \tag{3.65}$$

then the bilinear form on $\Gamma_K = K^0(X_{10})/K^0(X_{10})_{tors}$ is still given by the index:

$$\omega(x,y) = \frac{1}{(2\pi)^2}\int_{X_{10}} \tilde{G}(x)\wedge\overline{\tilde{G}(y)} = I(x\otimes\bar{y}) \tag{3.66}$$

The metric on Γ_K is modified to be

$$\tilde{g}(x,y) = \frac{1}{(2\pi)^2}\int_{X_{10}} \tilde{G}(x)\wedge\hat{*}\tilde{G}(y) \tag{3.67}$$

and $\mathbf{Z}_2$ valued function $\Omega(x)$ is unchanged. If we continue to use the standard polarization then $\theta_K \in \Gamma_1/2\Gamma_1$ is unchanged as well.

The net effect to modify (3.58) is that the period matrix τ_K should be substituted for $\widetilde{\tau_K} = \tau_K(G \to \tilde{G})$.

$$\Theta_K\left(\hat{B}_2\right) = e^{iu}\sum_{x\in\Gamma_1} e^{i\pi\widetilde{\tau_K}(x+\frac{1}{2}\theta_K)}\Omega(x) \tag{3.68}$$

Note, that the constant phase e^{iu} in front of the sum remains the same as in (3.58).

The imaginary part of the 10D Type IIA supergravity action now becomes $Im(S_{10D}) = -2\pi\tilde{\Phi}$, where

$$\tilde{\Phi} = \Phi + \frac{1}{8\pi^2}\Big[\hat{B}_2 G_4^2 + \hat{B}_2^2 G_2 G_4 + \frac{1}{3}\hat{B}_2^3\left(G_2^2 + G_0 G_4\right) + \frac{1}{4}\hat{B}_2^4 G_0 G_2 + \frac{1}{20}\hat{B}_2^5 G_0^2\Big], \qquad (3.69)$$

Φ is defined in (3.62,3.63) and $G_{2p}(x + \frac{1}{2}\theta_K)$, $\quad p = 0, 1, 2$ are given in (3.61).

From (3.69) we find that corrections to Φ depending on $\hat{B}_2$ coincide with the imaginary part of the standard supergravity action (see, for example [76].)

Note, that $\tilde{G}$ defined in (3.65) is a gauge invariant field if the global tensorfield gauge transformation

$$\hat{B}_2 \to \hat{B}_2 + f_2, \quad f_2 \in H^2(X_{10}, \mathbf{Z}) \qquad (3.70)$$

also acts on $K^0(X_{10})$ as:

$$x \to L(-f_2) \otimes x, \quad x \in K^0(X_{10}) \qquad (3.71)$$

where the line bundle $L(-f_2)$ has $c_1\left(L(-f_2)\right) = -f_2$.

Thus, according to (3.71) a tensorfield gauge transformation acts as an automorphism of Γ_K, preserving the symplectic form ω. (3.71) acts on theta function (3.68) by the multiplication by a constant phase:

$$\Theta_K\left(\hat{B}_2 + f_2\right) = e^{i\frac{\pi}{4}\int_{X_{10}} f_2(\lambda - 2\hat{a}_0)^2}\,\Theta_K\left(\hat{B}_2\right) \qquad (3.72)$$

3.4 Action of T-duality in K-theory

In this section we consider $X_{10} = T^2 \times X$ and describe the action of T-duality on the K-theory variables.

Let us recall [68] that the standard polarization is distinguished for any large 10-manifold in the following sense. When the metric of X_{10} is scaled up $\hat{g}_{\hat{M}\hat{N}} \to t^2\hat{g}_{\hat{M}\hat{N}}$ the action $\int_{X_{10}} \sqrt{\hat{g}}|G_{2p}|^2$ of the Type IIA RR 2p-form scales as t^{10-4p}. This allows the sensible approximation of keeping only G_4 whose periods have the smallest action, then including G_2 and finally keeping all G_4, G_2, G_0.

In the case of $X_{10} = T^2 \times X$ with the metric (3.1), the standard polarization is no longer distinguished. Various equally good choices are related by the action of the T-duality group $\mathcal{D}_T$ on $\Gamma_K = K^0(T^2 \times X)/K^0_{tors}(T^2 \times X)$.

We argue below that $\mathcal{D}_T$ can be considered as a subgroup of $Sp(2N, \mathbf{Z})$, where N denotes the complex dimension of the K-theory torus $\mathbf{T} = K^0(T^2 \times X) \otimes_{\mathbf{Z}} \mathbf{R}/\Gamma_K$ and $Sp(2N, \mathbf{Z})$ stands for the group of symplectic transformations of the lattice Γ_K.

3.4.1 Background RR fluxes in terms of integral classes on X.

To describe the action of $\mathcal{D}_T$ on K-theory variables, we will write RR fields in terms of integral classes on X. Let us start from the standard polarization [12] and write a general element of Γ_1^{std} as

$$x = n_0\mathbf{1} + \Big(L(n_1 e_0 + e + \gamma_m d\sigma^m) - \mathbf{1}\Big) + x(e_0 e' + a + h_m d\sigma^m) + \Delta \tag{3.73}$$

where $e_0 = d\sigma^8 \wedge d\sigma^9$, so that $\int_{T^2} e_0 = 1$. $L(\hat{e})$ is a line bundle with $c_1(L) = \hat{e} \in H^2(X_{10}; \mathbf{Z})$, $\mathbf{1}$ is a trivial line bundle, and for any $\hat{a} \in H^4(X_{10}; Z)$, $x(\hat{a})$ is a K-theory lift (if it exists). In 3.73) Δ puts x into the Lagrangian lattice Γ_1^{std} and we also introduce the notations:

$$a \in H^4(X;\mathbf{Z}), \quad e, e' \in H^2(X;\mathbf{Z}), \quad h_m \in H^3(X,\mathbf{Z}), \quad \gamma_m \in H^1(X;\mathbf{Z}) \quad m = 8,9 \tag{3.74}$$

Note that if $\hat{a}_0 \in H^4(X \times T^2, \mathbf{Z})$, defined in (3.57), is nonzero, it has the property $Sq^3 \hat{a}_0 \neq 0$, so that it must have the form $\hat{a}_0 = a_m d\sigma^m, a_m \in H^3(X, \mathbf{Z})$. It is convinient to redefine h_m by including a_m, which will be assumed from now on.

The RR fields entering (3.59,3.60) are given by

$$\begin{aligned} \frac{1}{2\pi} G_0(x + \frac{1}{2}\theta_K) &= n_0, \\ \frac{1}{2\pi} G_2(x + \frac{1}{2}\theta_K) &= n_1 e_0 + e + \gamma_m d\sigma^m, \\ \frac{1}{2\pi} G_4(x + \frac{1}{2}\theta_K) &= a + \frac{1}{2}e^2 + e_0 e'' + f_m d\sigma^m - \frac{1}{2}(1 + n_0/12)\lambda \end{aligned} \tag{3.75}$$

[12] Γ_1^{std} and Γ_2^{std} are defined on page 19.

where

$$e'' = n_1 e + e' - \gamma_1\gamma_2, \quad f_m = h_m + a_m + e\gamma_m \tag{3.76}$$

From the 10D constraint $Sq^3\hat{a} = Sq^3\hat{a}_0$, valid in the case $[\hat{H}_3] = 0$, we find the constraints on the integral cohomology classes: $Sq^3 f_m = Sq^3 a_m, \quad m = 8, 9$.

3.4.2 The embedding $\mathcal{D}_T \subset Sp(2N, \mathbf{Z})$

From the transformation rules of the RR fields under the T-duality group [92] we find that f_m and γ_m transform in the $(\mathbf{2}', \mathbf{1})$ of $\mathcal{D}_T$ and we can form a representation $(\mathbf{1}, \mathbf{2})$ out of n_0, n_1 and e, e'' in the following way:

$$n^\alpha = \begin{pmatrix} n_1 \\ n_0 \end{pmatrix}, \quad e^\alpha = \begin{pmatrix} e'' \\ e \end{pmatrix} \tag{3.77}$$

We would like to reformulate the transformation rules for RR fields in terms of the action on Γ_K.

Firstly, we note that $SL(2, \mathbf{Z})_\tau$ does not act on Γ_K. Secondly, the action of T on Γ_K is a particular case of the global gauge transformation (3.70,3.71) with $f_2 = e_0$ and for this reason $T \in Sp(2N, \mathbf{Z})$. Moreover, T preserves standard polarization since it maps $\Gamma_2^{std} \to \Gamma_2^{std}$:

$$G_{2p}\left(y \otimes L(-e_0)\right) = 0, \quad \forall y \in \Gamma_2^{std} \quad p = 0, 1, 2 \tag{3.78}$$

More interesting is the action of the generator S on Γ_K: S is a symplectic transformation which maps standard polarization into another polarization.

To demonstrate this let us write a generic element $y \in \Gamma_2^{std}$ as

$$y = x(\tilde{a}) \otimes \Big(L(e_0) - \mathbf{1}\Big) + z_1 + z_2 + z_3 \otimes \Big(L(e_0) - \mathbf{1}\Big), \quad \tilde{a} \in H^4(X, \mathbf{Z}) \tag{3.79}$$

In (3.79) z_1, z_2, z_3 are such that

$$\frac{G}{2\pi}(z_1) = j_m d\sigma^m, \quad \frac{G}{2\pi}(z_2) = k, \quad \frac{G}{2\pi}(z_3) = k' \tag{3.80}$$

where $j_m \in H^5(X, \mathbf{R}) \oplus H^7(X, \mathbf{R}), \quad k, k' \in H^6(X, \mathbf{R}) \oplus H^8(X, \mathbf{R})$

According to the transformation rules of RR fields [92] S acts on y as

$$S : y \to y', \quad y' = x(\tilde{a}) + z_1 + z_3 - z_2 \otimes \Big(L(e_0) - \mathbf{1}\Big) \tag{3.81}$$

From (3.81) we find that $\Gamma_2' := S(\Gamma_2^{std})$ (an image of the Γ_2^{std} under S) is a maximal Lagrangian sublattice of Γ_K, and as such completely determines the new polarization.

Since we have an embedding $\mathcal{D}_T \subset Sp(2N, \mathbf{Z})$, we can deduce the existence of well-defined transformation laws under $\mathcal{D}_T$ of the function $\Theta(\mathcal{F}, \rho)$, related by (3.20) to the K-theory theta function Θ_K. This follows from the fact that Θ_K is an holomorphic section of the the line bundle $\mathcal{L}$ over the K-theory torus with $c_1(\mathcal{L}) = \omega$. Since $\mathcal{L}$ is not affected by symplectic transformations, and has a one-dimensional space of holomorphic sections, it follows that under T-duality transformatons Θ_K can at worst be multiplied by a constant. Nevertheless, this leaves open the possibility of a T-duality anomaly, as indeed takes place.

To conclude this section we show how the multiplier system is related to the standard 8^{th} roots of unity appearing in theta function transformation laws. Recall the general transformation rule under $Sp(2N, \mathbf{Z})$ for the theta function $\theta[m]\,(\tau)$ of principally polarized lattice $\Lambda = \Lambda_1 + \Lambda_2$ of rank $2N$. Here $m = \begin{pmatrix} m' \\ m'' \end{pmatrix} \in R^{2N}$ are the characterstics and the period matrix $\tau \in M_N(\mathbf{C})$, $\quad \tau^{\mathbf{T}} = \tau$ is a quadratic form on Λ_1.

It was found in [100] that under symplectic transformations

$$\sigma \cdot \tau = \frac{A\tau + B}{C\tau + D}, \quad \sigma \in Sp(2N, \mathbf{Z}) \tag{3.82}$$

the general $\theta[m](\tau)$ transforms as

$$\vartheta[\sigma \cdot m]\,(\sigma \cdot \tau) = \kappa(\sigma) e^{2\pi i \phi(m,\sigma)} det(C\tau + D)^{1/2} \vartheta[m](\tau) \tag{3.83}$$

where

$$\sigma \cdot m = m\sigma^{-1} + \frac{1}{2}\begin{pmatrix} \left(C^T D\right)_d \\ \left(A^T B\right)_d \end{pmatrix}$$

$$\phi(m, \sigma) = -\frac{1}{2}\left(m'^T D B^T m' - 2m'^T B C^T m'' + m''^T C A^T m''\right) +$$
$$+\frac{1}{2}\left(m'^T D - m''^T C\right)\left(A^T B\right)_d$$

where $(A)_d$ denotes a vector constructed out of diagonal elements of matrix A.

The factor $\kappa(\sigma)$ in (3.83) has quite nontrivial properties [100]. In particular $\kappa^2(\sigma)$ is a character of $\Gamma(1,2) \subset Sp(2N, \mathbf{Z})$, where

$$\sigma \in \Gamma(1,2) \quad iff \quad \left(A^T B\right)_d \in 2\mathbf{Z}, \quad \left(C^T D\right)_d \in 2\mathbf{Z} \tag{3.84}$$

One can easily check that $SL(2,Z)_\rho \subset \Gamma(1,2)$ by writing out explicit representations $\sigma(S)$ and $\sigma(T)$ in $Sp(2N, \mathbf{Z})$. We give $\sigma(S)$ and $\sigma(T)$ in Appendix(A).

Using the explicit expressions for $\sigma(S)$ and $\sigma(T)$ as well as the definition of τ_K (3.59,3.60) we find that in (3.83)

$$det(C(S)\tau_K + D(S))^{1/2} = e^{i\frac{\pi}{4}b_4}(-i\rho)^{\frac{1}{2}b_4^+}(i\bar{\rho})^{\frac{1}{2}b_4^-}, \quad \phi(m, \sigma(S)) = 0 \tag{3.85}$$

$$det(C(T)\tau_K + D(T))^{1/2} = 1, \quad \phi(m, \sigma(T)) = 0 \tag{3.86}$$

Now comparing (3.83) and the explicit formulae (5.31) for the transformation laws of $\Theta(\mathcal{F}, \rho)$ derived in the next section we find the relation between $\kappa(\sigma)$ and the multiplier system $\mu(S), \mu(T)$

$$\kappa(S)e^{i\frac{\pi}{4}b_4} = \mu(S), \quad \kappa(T) = \mu(T) \tag{3.87}$$

3.5 $\Theta(\mathcal{F}, \rho)$ as a modular form

In this section we derive an explicit expression for $\Theta(\mathcal{F}, \rho)$ using its relation (3.20) to the K-theory theta function Θ_K and we check that $\Theta(\mathcal{F}, \rho)$ transforms under the T-duality group $\mathcal{D}_T$ as a modular form.

3.5.1 Zero NSNS fields

We first assume that all NSNS background fields are zero. In this case $\Theta(\mathcal{F}, \rho)$, defined in (3.20) is given by

$$\Theta(\mathcal{F}, \rho) = \sum_{a \in H^4(X, \mathbf{Z})} e^{i2\pi\Phi(a, \mathcal{F})} e^{-\pi \int_X Im(\rho) g_{(4)} \wedge * g_{(4)}} \tag{3.88}$$

To find the imaginary part of the 8D effective action $2\pi\Phi(a, \mathcal{F})$ we substitute

$$\hat{a} = a + e_0 e' + h_m d\sigma^m, \quad \hat{e} = e + n_1 e_0 + \gamma_m d\sigma^m \tag{3.89}$$

into the definition (3.62) of $e^{i2\pi\Phi(n_0,\hat{e},\hat{a})}$.

We need to evaluate $f(a + e_0 e' + h_m d\sigma^m)$. Let us use bilinear identity from [68]

$$f(u+v) = f(u) + f(v) + \int_{X_{10}} u Sq^2 v, \quad \forall u, v \in H^4(X_{10}; \mathbf{Z}) \tag{3.90}$$

to find

$$f(a + e_0 e' + h_m d\sigma^m) = f(a + e_0 e') + f(h_m d\sigma^m) \tag{3.91}$$

Let us consider $f(h_m d\sigma^m)$ first. Again using the bilinear identity we obtain:

$$f(h_m d\sigma^m) = f(h_8 d\sigma^8) + f(h_9 d\sigma^9) + \int_X h_8 Sq^2(h_9) \tag{3.92}$$

From (3.90) it follows that $f(hd\sigma^m), m = 8, 9$ are linear functions of h. Moreover, from the diffeomorphism invariance of the mod two index we see that $f(h_8 d\sigma^8 + h_9 d\sigma^9) = f(h_8 d\sigma^8 + (h_9 + \ell h_8) d\sigma^9)$, for any integer ℓ and, using the bilinear identity once more we find that $f(hd\sigma^m) = r(h), m = 8, 9$ where

$$r(h) = \int_X h Sq^2 h, \quad h \in H^3(X, \mathbf{Z}) \tag{3.93}$$

is a spin-cobordism invariant Z_2 valued function. $r(h)$ is nonzero since for $X = SU(3)$ and $h = x_3$ the generator of $H^3(SU(3), \mathbf{Z})$ we have $r(h) = 1$. Then, using (3.76) we obtain:

$$f(h_m d\sigma^m) = \int_X \Big(f_8 Sq^2(f_9) + f_8 Sq^2(f_8) + f_9 Sq^2(f_9) + e^2(\gamma_9 f_8 - \gamma_8 f_9) + e^3 \gamma_8 \gamma_9 \Big) \tag{3.94}$$

Now we consider $f(a + e_0 e')$:

$$f(e_0 e' + a) = f(a) + f(e_0 e') + \int_{X_{10}} e_0 e' Sq^2 a = \tag{3.95}$$

$$= \int_X (a)^2 - \frac{1}{2}(e')^2 \lambda + (e')^2 a = \int_X a\lambda + (e')^2 (a - \frac{1}{2}\lambda)$$

This uses the bilinear identity (3.90), the reduction of the mod two index along T^2, and the formula eq.(8.40) for $f(u \cup v)$ from [68]. Taking into account (3.94) and (3.95) we find the total phase $\Phi(a, \mathcal{F})$:

$$\Phi(a, \mathcal{F}) = \Delta\Phi + \int_X (a + \alpha)\beta, \tag{3.96}$$

where characteristics are defined as:

$$\alpha = \frac{1}{2}(e)^2 + \frac{1}{2}(1 - n_0/12)\,\lambda + \frac{1}{2}(e'' + e)\,\epsilon^{mn}\gamma_m\gamma_n$$
$$\beta = \frac{1}{2}(e'')^2 + \frac{1}{2}(1 - n_1/12)\,\lambda + \frac{1}{2}(e'' - e)\,\epsilon^{mn}\gamma_m\gamma_n \tag{3.97}$$

and we recall that $e'' = n_1 e + e' - \frac{1}{2}\epsilon^{mn}\gamma_m\gamma_n$. Note that for convenience we have made a shift of the summation variable in (3.88) $a \to a + \lambda + \frac{1}{2}(e + e'')\,\epsilon^{mn}\gamma_m\gamma_n$.

The prefactor $\Delta\Phi$ is given by

$$exp[2\pi i\Delta\Phi] = (-1)^{f(\hat{a}_0)} exp\left[\pi i \int_X \left(f_8 Sq^2(f_9) + f_8 Sq^2(f_8) + f_9 Sq^2(f_9)\right)\right] \tag{3.98}$$

$$exp\Bigg[2\pi i \int_X \Bigg(-\frac{1}{4}(e''e)^2 - \frac{1}{24}e''e\lambda + \frac{1}{6}e^3 e'' - \frac{1}{4}e^2\lambda + \frac{1}{48}n_0\lambda\,(e'')^2 + \frac{1}{4}(1 + n_0/12)\lambda^2 +$$
$$+\frac{1}{2}(n_0 - n_1)\hat{A}_8 - \frac{1}{2}n_0 n_1\left[\hat{A}_8 + \left(\frac{\lambda}{24}\right)^2\right] + \frac{\lambda}{24}\epsilon^{mn}\gamma_m f_n +$$
$$+\frac{1}{48}\left[n_0(e'' - e)\lambda - 12e^2 e'' - 4e\lambda - 4e^3\right]\epsilon^{mn}\gamma_m\gamma_n\Bigg)\Bigg]$$

In deriving $\Delta\Phi$ we have used

$$(\sqrt{\hat{A}})_8 = \frac{1}{2}\left[\hat{A}_8 - \left(\frac{\lambda}{24}\right)^2\right]$$

Also, in bringing $\Delta\Phi$ to the form (3.98) we have used important congruences

$$\int_X \frac{1}{6}\left[(e'')^3 e + e''e^3\right] + \frac{1}{4}(e'')^2 e^2 - \frac{1}{12}\lambda e''e \in \mathbf{Z} \tag{3.99}$$

$$\int_X (e''e)^2 \in 2\mathbf{Z}, \quad \int_X e''e\lambda \in 2\mathbf{Z} \tag{3.100}$$

following from the index theorem on X :

$$\int_X \frac{1}{24}e^4 - \frac{1}{24}\lambda e^2 \in \mathbf{Z}, \forall e \in H^2(X, \mathbf{Z}) \tag{3.101}$$

3.5.2 Including flat NSNS potentials

Let us now take into account globally defined NSNS fields:

$$\hat{B}_2 = \frac{1}{2}B_0\epsilon_{mn}\omega^m\omega^n + B_{(1)m}\omega^m + B_{(2)} + \frac{1}{2}\mathcal{A}^m_{(1)}B_{(1)m}, \quad \mathcal{A}^m_{(1)}$$

and recall that $\mathcal{A}^m_{(1)}$ and $B_{(1)m}$ are combined into the $(\mathbf{2},\mathbf{2})$ of $\mathcal{D}_T$ as in (3.40).

We define a gauge invariant fieldstrength $\tilde{G} = e^{\hat{B}_2}G$ as in (3.64) where G are given in (3.75) and expand $\tilde{G}\left(x + \frac{1}{2}\theta_K\right)$ as

$$\begin{aligned} \frac{\tilde{G}_0}{2\pi}\left(x+\frac{1}{2}\theta_K\right) &= g^2_{(0)} \\ \frac{\tilde{G}_2}{2\pi}\left(x+\frac{1}{2}\theta_K\right) &= \left(g^1_{(0)} + g^2_{(0)}B_0\right)\frac{1}{2}\epsilon_{mn}\omega^m\omega^n + g_{(1)m}\omega^m + g^2_{(2)} \\ \frac{\tilde{G}_4}{2\pi}\left(x+\frac{1}{2}\theta_K\right) &= g_{(4)} + g_{(3)m}\omega^m + \left(B_0 g^2_{(2)} + g^1_{(2)}\right)\frac{1}{2}\epsilon_{mn}\omega^m\omega^n \end{aligned} \tag{3.102}$$

The first effect of including flat NSNS fields is to modify fields which enter $S_B(\mathcal{F})$. These fields $g^\alpha_{(0)}, g_{(1)m}, g^\alpha_{(2)}, g_{(3)m}$ are now linear combinations of the integral classes $\gamma_m, f_m, e^\alpha, n^\alpha$ defined in (3.74,3.76) with coefficients constructed from $\mathbf{A}^{m\alpha}_{(1)}$ and $B_{(2)}$.

$$g^\alpha_{(0)} = \begin{pmatrix} n_1 \\ n_0 \end{pmatrix}, \quad g_{(1)m} = \gamma_m + \xi_{(1)m}, \quad g^\alpha_{(2)} = e^\alpha + \mathbf{A}^{m\alpha}_{(1)}\left(\gamma_m + \frac{1}{2}\xi_{(1)m}\right) + B_{(2)}g^\alpha_{(0)} \tag{3.103}$$

$$g_{(3)m} = f_m + B_{(2)}g_{(1)m} + \lambda_{(3)m} + \frac{1}{2}k_{(3)m} + \frac{1}{6}\epsilon_{mn}\mathcal{E}_{\alpha\beta}\mathbf{A}^{p\alpha}_{(1)}\xi_{(1)p}\mathbf{A}^{n\beta}_{(1)} \tag{3.104}$$

where we denote

$$\xi_{(1)m} = \epsilon_{mn}\mathcal{E}_{\alpha\beta}g^\alpha_{(0)}\mathbf{A}^{n\beta}_{(1)}, \quad \lambda_{(3)m} = \epsilon_{mn}\mathcal{E}_{\alpha\beta}e^\alpha\mathbf{A}^{n\beta}_{(1)}, \quad k_{(3)m} = \epsilon_{mn}\mathcal{E}_{\alpha\beta}\mathbf{A}^{p\alpha}_{(1)}\gamma_p\mathbf{A}^{n\beta}_{(1)} \tag{3.105}$$

The other effect of including flat NSNS fields is to introduce a term $i\pi\int_X Re(\rho)g_{(4)}\wedge g_{(4)}$ and to shift the characteristics and the prefactor of $\Theta(\mathcal{F},\rho)$. Now $\Theta(\mathcal{F},\rho)$ has the form:

$$\Theta(\mathcal{F},\rho) = e^{2\pi i\Delta\tilde{\Phi}}\sum_{a\in H^4(X,\mathbf{Z})} exp\left[\int_X\left(-\pi Im(\rho)g_{(4)}\wedge * g_{(4)} + i\pi Re(\rho)g_{(4)}\wedge g_{(4)} + 2\pi i g_{(4)}\tilde{\beta}\right)\right] \tag{3.106}$$

where $g_{(4)} = a + \tilde{\alpha}, \quad a \in H^4(X, \mathbf{Z})$. The shifted characteristics $\tilde{\alpha}, \tilde{\beta}$ are

$$\tilde{\alpha} = \alpha + \varphi^2, \quad \tilde{\beta} = \beta + \varphi^1 \tag{3.107}$$

where α, β are defined in terms of integral classes $n_0, n_1, \gamma_m, e^\alpha$ in (3.97), while φ^α transform in the $(\mathbf{1}, \mathbf{2})$ of $\mathcal{D}_T$. Explicitly,

$$\varphi^\alpha = \mathbf{A}^{m\alpha}_{(1)} \left(f_m + \frac{1}{2}\lambda_{(3)m} + \frac{1}{6}k_{(3)m} \right) + B_{(2)} \left[e^\alpha + \mathbf{A}^{m\alpha}_{(1)} \left(\gamma_m + \frac{1}{2}\xi_{(1)m} \right) \right] + \tag{3.108}$$
$$+ \frac{1}{2} B_{(2)} B_{(2)} g^\alpha_{(0)} - \zeta_{(4)} g^\alpha_{(0)}$$

where $\xi_{(1)m}, \lambda_{(3)m}, k_{(3)m}$ are given in (3.105) and we also denote

$$\zeta_{(4)} = \frac{1}{64} \mathcal{E}_{\beta_1\beta_2} \mathcal{E}_{\beta_3\beta_4} \mathbf{A}^{n_1\beta_1}_{(1)} \epsilon_{n_1 n_2} \mathbf{A}^{n_2\beta_3}_{(1)} \mathbf{A}^{m_1\beta_2}_{(1)} \epsilon_{m_1 m_2} \mathbf{A}^{m_2\beta_4}_{(1)} \tag{3.109}$$

The shifted prefactor $\Delta\tilde{\Phi}$ in (3.106) is given by

$$\Delta\tilde{\Phi} = \Delta\Phi - \int_X \left[\beta \wedge \varphi^2 + \frac{1}{2} \varphi^1 \wedge \varphi^2 \right] + (\Delta\Phi)_{inv} \tag{3.110}$$

where $\Delta\Phi$ is defined in terms of integral classes $n_0, n_1 \gamma_m, e^\alpha, f_m$ in (3.98) and $(\Delta\Phi)_{inv}$ is a part of the phase which is manifestly invariant under the T-duality group $\mathcal{D}_T$.

$$(\Delta\Phi)_{inv} = \int_X B^3_{(2)} \left[\frac{1}{12} \mathcal{E}_{\alpha\beta} g^\alpha_{(0)} e^\beta - \frac{1}{6} \epsilon^{mn} \gamma_m c_n - \frac{1}{4} \epsilon^{mn} \xi_{(1)m} c_n - \frac{1}{8} \epsilon^{mn} \xi_{(1)m} \xi_{(1)n} \right] + \tag{3.111}$$
$$\int_X B^2_{(2)} \left[-\frac{1}{4} \epsilon^{mn} \xi_{(1)m} f_n - \frac{1}{2} \epsilon^{mn} \lambda_{(3)m} \gamma_n - \frac{3}{8} \epsilon^{mn} \lambda_{(3)m} \xi_{(1)n} - \frac{1}{24} \epsilon^{mn} k_{(3)m} \xi_{(1)n} \right] +$$
$$\int_X B_{(2)} \left[-\frac{1}{2} \epsilon^{mn} f_m f_n - \frac{1}{2} \epsilon^{mn} \lambda_{(3)m} f_n - \frac{1}{4} \epsilon^{mn} \lambda_{(3)m} \lambda_{(3)n} - \frac{1}{6} \epsilon^{mn} \lambda_{(3)m} k_{(3)n} + \right.$$
$$\left. + \frac{1}{12} \xi_{(1)m} q^m_{(5)} + \frac{1}{2} \zeta_{(4)} \mathcal{E}_{\alpha\beta} e^\alpha g^\beta_{(0)} + \zeta_{(4)} \epsilon^{mn} \gamma_m \gamma_n \right] + \int_X \left[\frac{1}{12} \lambda_{(3)m} q^m_{(5)} + \zeta_{(4)} \epsilon^{mn} \gamma_m f_n \right]$$

where $q^m_{(5)} = \mathcal{E}_{\alpha\beta} \mathbf{A}^{p\alpha}_{(1)} f_p \mathbf{A}^{m\beta}_{(1)}$

3.5.3 Derivation of T-duality transformations.

Let us study transformations of $\Theta(\mathcal{F}, \rho)$ defined in (3.106) under $\mathcal{D}_T$. First, we note that $\Theta(\mathcal{F}, \rho)$ is invariant under $SL(2, Z)_\tau$. Next,we consider the action of the generator S.

For any function $h(\mathcal{F})$ of fluxes $\mathcal{F}$, we denote

$$S\Big[h(\mathcal{F})\Big] = h(S \cdot \mathcal{F})$$

where $S\cdot\mathcal{F}$ means a linear action on fluxes according to their representation under $SL(2,\mathbf{Z})_\rho$ and

$$\delta_S[h] = S[h] - h.$$

To check the transformation under S we need to do a Poisson resummation on the self-dual lattice $H^4(X,\mathbf{Z})$. The basic transformation law is:

$$\vartheta\begin{bmatrix}\theta\\ \phi\end{bmatrix}(0|-1/\tau) = (-i\tau)^{1/2} e^{2\pi i\theta\phi}\vartheta\begin{bmatrix}-\phi\\ \theta\end{bmatrix}(0|\tau) \tag{3.112}$$

and its generalization to self-dual lattices (3.83).

After the Poisson resummation and shift of the summation variable $a \to a + e^2 + \lambda$ we find that $\Theta(\mathcal{F},\rho)$ transforms under S as

$$\Theta(S\cdot\mathcal{F}, -1/\rho) = e^{2\pi i\{\int_X S[\tilde{\alpha}]S[\tilde{\beta}] + \delta_S[\Delta\tilde{\Phi}]\}}(-i\rho)^{\frac{1}{2}b_4^+}(i\bar{\rho})^{\frac{1}{2}b_4^-}\Theta(\mathcal{F},\rho) \tag{3.113}$$

Now using the definitions of $\tilde{\alpha},\tilde{\beta}$ (3.107, 3.108) and $\Delta\tilde{\Phi}$ (3.110) as well as the transformation rules for $\mathcal{F}$, we find after some tedious algebra

$$\delta\left[\Delta\tilde{\Phi}\right] = -\int_X S[\tilde{\alpha}]S[\tilde{\beta}] + \int_X \frac{\lambda^2}{4} + \mathbf{Z} \tag{3.114}$$

We conclude that the generator S acts as

$$\Theta(S\cdot\mathcal{F}, -1/\rho) = e^{i\pi\int_X \lambda^2/2}(-i\rho)^{\frac{1}{2}b_4^+}(i\bar{\rho})^{\frac{1}{2}b_4^-}\Theta(\mathcal{F},\rho) \tag{3.115}$$

To check how $\Theta(\mathcal{F},\rho)$ transforms under the generator T we use its relation (3.20) to the K-theory theta function Θ_K as well as the transformation of Θ_K under global gauge transformation $\hat{B}_2 \to \hat{B}_2 + f_2$ (3.72) where the action of the generator T corresponds to $f_2 = e_0$.

In this way we find from (3.72) that

$$\Theta(T\cdot\mathcal{F}, \rho+1) = e^{i\pi\int_X \lambda^2/4}\Theta(\mathcal{F},\rho) \tag{3.116}$$

where we used that $\hat{a}_0 = a_m d\sigma^m$.

3.5.4 Summary of T-duality transformation laws

Below we summarize the transformation laws of the function $\Theta(\mathcal{F},\rho)$ under the generators of T-duality group $\mathcal{D}_T$.

$\Theta(\mathcal{F},\rho)$ is invariant under $SL(2,Z)_\tau$:

$$\Theta(\tilde{T}\cdot\mathcal{F},\rho)=\Theta(\mathcal{F},\rho)$$

$$\Theta(\tilde{S}\cdot\mathcal{F},\rho)=\Theta(\mathcal{F},\rho) \tag{3.117}$$

$\Theta(\mathcal{F},\rho)$ transforms as a modular form with a nontrivial "multiplier system" under $SL(2,Z)_\rho$. That is, using the standard generators T,S of $SL(2,Z)_\rho$ we have:

$$\Theta(T\cdot\mathcal{F},\rho+1)=\mu(T)\Theta(\mathcal{F},\rho)$$

$$\Theta(S\cdot\mathcal{F},-1/\rho)=\mu(S)(-i\rho)^{\frac{1}{2}b_4^+}(i\bar{\rho})^{\frac{1}{2}b_4^-}\Theta(\mathcal{F},\rho) \tag{3.118}$$

where $T\cdot\mathcal{F},S\cdot\mathcal{F}$ denotes the linear action of $\mathcal{D}_T$ on the fluxes. Here b_4^+,b_4^- is the dimension of the space of self-dual and anti-self-dual harmonic forms on X and the multiplier system is

$$\mu(T)=\exp[\frac{i\pi}{4}\int_X\lambda^2],\quad \mu(S)=\exp[\frac{i\pi}{2}\int_X\lambda^2] \tag{3.119}$$

where $p_1=p_1(TX)$. These define the "T-duality anomaly of RR fields."

3.6 The bosonic determinants

In this section we compute bosonic quantum determinants around the background specified in section 2 (below (3.43)).

Let us factorize bosonic quantum determinants as: $Det_B=\mathcal{D}_{RR}\mathcal{D}_{NS}$, where $\mathcal{D}_{RR}(\mathcal{D}_{NS})$ denotes contribution from RR (NSNS) fields.

3.6.1 Quantum determinants $\mathcal{D}_{RR}$ for RR fields

Quantum determinants $\mathcal{D}_{RR}$ for RR fields have the form

$$\mathcal{D}_{RR} = \prod_{p=1}^{4} Z_{RR,p} \tag{3.120}$$

where $Z_{RR,p}$ is the quantum determinant for $g_{(p)}$. First, we present the contribution $Z_{RR,4}$ from the fluctuation $dC_{(3)}$ of $g_{(4)}$. From (3.43) we find the kinetic term for $C_{(3)}$

$$S_{3,cl} = \pi Im(\rho)(dC_{(3)}, dC_{(3)}) \tag{3.121}$$

where $(,)$ denotes standard inner product on a space of p-forms on X constructed with the background metric g_{MN}.

We use the standard procedure [93, 102] for path-integration over p-forms, which can be summarized as follows. Starting from the classical action for the p-form $S_{p,cl} = \alpha(dC_{(p)}, dC_{(p)})$ one constructs the quantum action as[13] :

$$S_{p,qu} = \alpha(C_{(p)}, \Delta_p C_{(p)}) + \sum_{m=1}^{p} \alpha^{\frac{1}{m+1}} \sum_{k=1}^{m+1} \left(u^k_{(p-m)}, \Delta_{p-m} u^k_{(p-m)}\right) \tag{3.122}$$

where $u^k_{(p-m)}$, $k = 1, \ldots m+1, m = 1, \ldots p$ are ghosts of alternating statistics. For example, $u^k_{(p-1)}$, $k = 1,2$ are fermions, $u^k_{(p-2)}$, $k = 1,2,3$ are bosons, etc. In (3.122) Δ_p is the Laplacian acting on p-forms and constructed with g_{MN}[14] .

To compute $Z_{RR,4}$ we take (3.122) for $p = 3$, $\alpha = \pi \mathrm{Im}(\rho)$ and use the measure $[DC_p]$ normalized as $\int [DC_p] e^{-(C_p, C_p)} = 1$:

$$Z_{RR,4} = (\alpha)^{-\frac{1}{2}(B'_3 - B'_2 + B'_1 - B'_0)} \left[\frac{det'\Delta_3}{V_3}\right]^{-\frac{1}{2}} \left[\frac{det'\Delta_2}{V_2}\right] \left[\frac{det'\Delta_1}{V_1}\right]^{-3/2} \left[\frac{det'\Delta_0}{V_0}\right]^{2} \tag{3.123}$$

where $det'\Delta_p$ is a regularized determinant of nonzero modes of the Laplacian acting on p-forms. $B'_p = B_p - b_p$, where B_p denotes the (infinite) number of eigen-p-forms and b_p and V_p are the dimension and the determinant of the metric of the harmonic torus $T^p_{harm} = \mathcal{H}^p / \mathcal{H}^p_{\mathbf{Z}}$. The appearance of V_p in (3.123) is due to the appropriate treatment of zeromodes and is explained in Appendix(E).

[13] Factors $\alpha^{\frac{1}{m+1}}$ should be understood as a mnemonic rule to keep track of the dependence on α which follows from the analysis of various cancellations between ghosts and gauge-fixing fields

[14] $\Delta = dd^\dagger + d^\dagger d$

The infinite powers depending on B_p, here and below, require regularization and renormalization, of course. These can be handled using the techniques of [112]. In particular the expression

$$q(Im\rho) := (\mathrm{Im}\rho)^{-\frac{1}{2}(B_3-B_2+B_1-B_0)} \tag{3.124}$$

is a local counterterm of the form $e^{-\pi \mathrm{Im}\rho \int_X (u\lambda^2+vp_2)}$, and the numbers u, v depend on the regularization. From now on we will assume that $\pi Im\rho \int_X (u\lambda^2 + vp_2)$ is included into the 1-loop action:

$$S_{1-loop} = \pi \mathrm{Im}\rho \int_X (u\lambda^2 + vp_2) + \frac{i\pi}{24}\mathrm{Re}\rho \int_X (p_2 - \lambda^2) \tag{3.125}$$

In section 8 we will show that T-duality invariance determines u and v uniquely.

Next, we consider the contributions to $\mathcal{D}_{RR}$ from $dC_{(2)m}$, $dC^{\alpha}_{(1)}$, $d\tilde{C}_{(0)m}$ which are respectively the fluctuations for $g_{(3)m}$, $g^{\alpha}_{(2)}$, $g_{(1)m}$. Let us also make field redefinition of the quantum fields $\tilde{C}_{(0)m}, m = 8,9$ to the fields $C_{(0)m}, m = 8,9$ which have well defined transformation properties under the full U-duality group[15]

$$C_{(0)8} = \sqrt{\tau_2}e^{\xi}\tilde{C}_{(0)8}, \quad C_{(0)9} = \frac{1}{\sqrt{\tau_2}}e^{\xi}\tilde{C}_{(0)9} \tag{3.126}$$

From (3.42) we find classical action quadratic in the above specified fluctuations:

$$S_{0,cl} = \pi\tilde{t}^6 g'^{mn}\left(C_{(0)m}, d^{\dagger}dC_{(0)n}\right), \quad S_{1,cl} = \pi t^4 \mathcal{G}_{\alpha\beta}\left(C^{\alpha}_{(1)}, d^{\dagger}dC^{\beta}_{(1)}\right)$$

$$S_{2,cl} = \pi t^2 g^{mn}\left(C_{(2)m}, d^{\dagger}dC_{(2)n}\right)$$

where $\tilde{t} = te^{-\xi/3}$ is U-duality invariant, and $g'^{88} = \frac{1}{\tau_2}g^{88}$, $g'^{99} = \tau_2 g^{99}$, $g'^{89} = g^{89}$. Now, using (3.122) with $a = \pi\tilde{t}^6 g'^{mn}, \pi t^4\mathcal{G}_{\alpha\beta}, \pi t^2 g^{mn}$ and $p = 0,1,2$ correspondingly we find:

$$Z_{RR,1} = \left(\pi\tilde{t}^6\right)^{-B_0'}\left[\frac{det'\Delta_0}{V_0}\right]^{-1} \tag{3.127}$$

$$Z_{RR,2} = \left(\pi t^4\right)^{B_0'-B_1'}\left[\frac{det'\Delta_1}{V_1}\right]^{-1}\left[\frac{det'\Delta_0}{V_0}\right]^{2} \tag{3.128}$$

$$Z_{RR,3} = \left(\pi t^2\right)^{-B_2'+B_1'-B_0'}\left[\frac{det'\Delta_2}{V_2}\right]^{-1}\left[\frac{det'\Delta_1}{V_1}\right]^{2}\left[\frac{det'\Delta_0}{V_0}\right]^{-3} \tag{3.129}$$

[15] For the discussion of U-duality see sec.10

In computing (3.127-3.129) we also used that $det_{m,n} g^{mn} = 1, \quad det_{m,n} g'^{mn} = 1$ and $det_{\alpha,\beta} \mathcal{G}_{\alpha\beta} = 1$.

Collecting together (3.123) and (3.127-3.129) we find that $\mathcal{D}_{RR}$ has the form:

$$\mathcal{D}_{RR} = r_{RR}(t,\rho)\Big[\frac{det'\Delta_3}{V_3}\Big]^{-\frac{1}{2}}\Big[\frac{det'\Delta_1}{V_1}\Big]^{-\frac{1}{2}} \tag{3.130}$$

where

$$r_{RR}(t,\rho) = (e^{\xi})^{2B_0'}\Big(Im\rho\Big)^{\frac{1}{2}(b_3-b_2+b_1-b_0)} t^{-2B_2'-2B_1'-4B_0'}(\pi)^{-\frac{1}{2}(B_0'+B_1'+B_2'+B_3')}$$

and we recall that $q(Im\rho)$ was included into S_{1-loop}.

We have computed the quantum determinants $\mathcal{D}_{RR}$ treating RR fluctuations as differential forms. It would be more natural if these determinants had a K-theoretic formulation. This might be an interesting application to physics of differential K-theory.

3.6.2 Quantum determinants for NSNS fields

Let us first consider fluctuations $d\mathbf{a}_{(1)}^{m\alpha}$ and $db_{(2)}$ of the NSNS field $\mathbf{F}_{(2)}^{m\alpha}$ and $H_{(3)}$. From (3.39) we find the classical action quadratic in this fluctuation:

$$S_{cl} = \frac{1}{4\pi}e^{-2\xi}\Big\{t^4 g_{mn}\mathcal{G}_{\alpha\beta}\left(\mathbf{a}_{(1)}^{m\alpha}, d^\dagger d\mathbf{a}_{(1)}^{n\beta}\right) + t^2\left(b_{(2)}, d^\dagger db_{(2)}\right)\Big\} \tag{3.131}$$

Now, again using (3.122) we find

$$Z_{NS,2} = \left(\frac{t^4}{4\pi}e^{-2\xi}\right)^{2(B_0'-B_1')}\Big[\frac{det'\Delta_1}{V_1}\Big]^{-2}\Big[\frac{det'\Delta_0}{V_0}\Big]^{4} \tag{3.132}$$

and

$$Z_{NS,3} = \left(\frac{t^2}{4\pi}e^{-2\xi}\right)^{\frac{1}{2}(B_1'-B_2'-B_0')}\Big[\frac{det'\Delta_2}{V_2}\Big]^{-\frac{1}{2}}\Big[\frac{det'\Delta_1}{V_1}\Big]\Big[\frac{det'\Delta_0}{V_0}\Big]^{-3/2} \tag{3.133}$$

Let us now consider fluctuations of scalars: $\delta\xi, \delta\tau, \delta\rho$. From (3.39) we write the action quadratic in these fluctuations:

$$S_{scal} = \beta\int_X\Big\{8\partial^M\delta\xi\partial_M\delta\xi + \frac{1}{(\tau_2)^2}\partial^M\delta\tau\partial_M\delta\bar\tau + \frac{1}{(\rho_2)^2}\partial^M\delta\rho\partial_M\delta\bar\rho\Big\} \tag{3.134}$$

where $\beta = \frac{1}{4\pi}e^{-2\xi}t^6$. Now using the scalar measures defined as

$$\int[D\delta\rho][D\delta\bar\rho]e^{-\int_X\frac{\delta\rho\wedge*\delta\bar\rho}{(\mathrm{Im}\rho)^2}} = 1, \quad \int[D\delta\tau][D\delta\bar\tau]e^{-\int_X\frac{\delta\tau\wedge*\delta\bar\tau}{(\mathrm{Im}\tau)^2}} = 1 \tag{3.135}$$

$$\int [D\delta\xi] e^{-8\int_X \delta\xi\wedge *\delta\xi} = 1 \tag{3.136}$$

we find the quantum determinants for the NSNS scalars $Z_{NS,0}$:

$$Z_{NS,0} = \beta^{-\frac{5}{2}B_0'} \Big[\frac{det'\Delta_0}{V_0}\Big]^{-\frac{5}{2}} \tag{3.137}$$

Finally, we consider fluctuation h_{MN} of the metric $t^2 g_{MN}$. Recall that we work in the limit $e^{-\xi} \to \infty$ so that in computing the quantum determinant for the metric we drop couplings to RR background fluxes.

From (3.39) we find quadratic terms in the action:

$$S_{metr} = \beta \int_X \Big\{ (D_N h_{MP}) P^{MPQS} \left(D^N h_{QS}\right) + h^{MP} \mathcal{R}_{MNPQ} h^{NQ} \tag{3.138}$$

$$- \left(D^M h_{MN} - \frac{1}{2} D_N h\right)^2 \Big\}$$

where $h = g^{MN} h_{MN}$ and

$$P^{MPQS} = \frac{1}{2} g^{MQ} g^{PS} - \frac{1}{4} g^{MP} g^{QS}$$

In (3.138) $\mathcal{R}_{MNPQ}$ is the Riemann tensor of the Ricci-flat[16] background metric g_{MN}. The covariant derivative D_M is performed with the background metric, and indices are raised and lowered with this metric.

Following standard procedure [94, 95] we first insert gauge fixing condition into the path-integral $\delta\left(\kappa_N - (D^M h_{MN} - \frac{1}{2} D_N h)\right)$ Then, we insert the unit

$$1 = \sqrt{det(\beta \mathbf{1}_1)} \int D\kappa_{(1)} e^{-\beta\left(\kappa_{(1)},\kappa_{(1)}\right)} \tag{3.139}$$

and integrate over $\kappa_{(1)}$ in the path-integral. This procedure brings the kinetic term for the fluctuation h_{MN} to the form

$$\beta \int_X h_{MP} P^{MPNR} \mathcal{K}_{NR}^{QS} h_{QS}, \quad \mathcal{K}_{NR}^{QS} = -\delta_N^Q \delta_R^S D_L D^L + 2\mathcal{R}_N{}^Q{}_R{}^S \tag{3.140}$$

Gauge fixing also introduces fermionic ghosts $k_{(1)}, l_{(1)}$ with the action

$$S_{gh} = \beta^{1/2} \left(l_{(1)}, \Delta_1 k_{(1)}\right) \tag{3.141}$$

[16] If the background metric is not Ricci-flat there are terms involving Ricci-tensor in (3.138) as well as in (6.22) below.

Using the measure $\int[Dh_{MN}]e^{-\int_X h_{MN}P^{MNPQ}h_{PQ}} = 1$ we obtain the result for the quantum determinant Z_{metr} of the metric:

$$Z_{metr} = (\beta)^{-\frac{1}{2}(N'_{\mathcal{K}}-B'_1)}\left[det'\mathcal{K}\right]^{-\frac{1}{2}}\frac{det'\Delta_1}{V_1} \tag{3.142}$$

where $det'\mathcal{K}$ is a regularized determinant of nonzero modes of the operator $\mathcal{K}$ defined in (3.140) and $N'_{\mathcal{K}} = N_{\mathcal{K}} - n_{\mathcal{K}}$, where $N_{\mathcal{K}}$ stands for the dimension (infinite) of the space of the second rank symmetric tensors and $n_{\mathcal{K}}$ is the number of zeromodes of the operator $\mathcal{K}$. We will explain how we regularize $det'\mathcal{K}$ shortly.

Let us combine all NSNS determinants together:

$$\mathcal{D}_{NS} = r_{NS}(t,\xi)\left[det'\mathcal{K}\right]^{-\frac{1}{2}}\left[\frac{det'\Delta_2}{V_2}\right]^{-\frac{1}{2}} \tag{3.143}$$

where

$$r_{NS}(t,\xi) = (4\pi)^{\frac{1}{2}N'_{\mathcal{K}}+B'_0+B'_1+\frac{1}{2}B'_2}\left(e^{\xi}\right)^{N'_{\mathcal{K}}+B'_2+2B'_1+2B'_0}t^{-3N'_{\mathcal{K}}-B'_2-4B'_1-8B'_0} \tag{3.144}$$

Finally, from (3.130) and (3.143) we find the full expression for bosonic determinants

$$Det_B = Q(t,g_{MN})\left(Im\rho\right)^{\frac{1}{2}(b_3-b_2+b_1-b_0)} \tag{3.145}$$

where Q is a function of the T-duality invariant objects g_{MN}, t and ξ. Explicitly,

$$Q(t,g_{MN}) = r_{tot}\left[det'\mathcal{K}\right]^{-\frac{1}{2}}\left[\frac{det'\Delta_3}{V_3}\right]^{-\frac{1}{2}}\left[\frac{det'\Delta_2}{V_2}\right]^{-\frac{1}{2}}\left[\frac{det'\Delta_1}{V_1}\right]^{-\frac{1}{2}} \tag{3.146}$$

where we regularized $det'\mathcal{K}$ in a way that eliminates dependence on ifinite numbers B_p and $N_{\mathcal{K}}$ so that

$$r_{tot} = (\tilde{t})^{3(n_{\mathcal{K}}+b_2+2b_1+4b_0)} \tag{3.147}$$

where we recall $\tilde{t} = te^{-\xi/3}$.

Now, let us check the transformation laws of Det_B under $\mathcal{D}_T$. From (3.145) it is obvious that Det_B is manifestly invariant under all generators of $\mathcal{D}_T$ except generator S.

Using,

$$Im(-1/\rho) = \frac{Im(\rho)}{\rho\bar{\rho}} \tag{3.148}$$

we find that under S Det_B transforms as

$$Det_B(-1/\rho) = s_B Det_B(\rho), \quad s_B = (\rho\bar{\rho})^{\frac{1}{2}(b_0-b_1+b_2-b_3)} \tag{3.149}$$

3.7 Inclusion of the fermion determinants

In this section we include the effects of the fermionic path integral. We recall the fermion content in the 10-dimensional and 8-dimensional supergravity theories and derive their actions. In the presence of nontrivial fluxes these fermionic path integrals are nonvanishing, even for the supersymmetric spin structure on T^2.

3.7.1 Fermions in 8D theory and their T-duality transformations.

Let us begin by listing the fermionic content in the 8-dimensional supergravity theory (this content will be derived from the 10-dimensional theory below.)

The fermions in the 8D theory include two gravitinos $\psi^A, \quad \eta^A, \quad A = 0, \ldots, 7$ and spinors $\Sigma, \quad \Lambda, \quad l, \quad \mu, \quad \tilde{l}, \quad \tilde{\mu}$.[17] The relation of these fields to the 10D fields is explained in (7.13),(7.14) below. There are also bosonic spinor ghosts b_1, c_1, Υ_2 and b_2, c_2, Υ_1 which accompany ψ^A and η^A respectively.

The fermions and ghosts transform under T-duality generators as follows. The generators $T, \tilde{T}, \tilde{S}$ act trivially on fermions and ghosts while the under the generator S they transform as

$$\psi^A \to e^{i\alpha\bar{\Gamma}}\psi^A, \quad \eta^A \to \eta^A, \quad \Lambda \to e^{-i\alpha\bar{\Gamma}}\Lambda, \quad \Sigma \to \Sigma \tag{3.150}$$

$$l \to e^{2i\alpha\bar{\Gamma}}l, \quad \tilde{l} \to e^{-2i\alpha\bar{\Gamma}}\tilde{l}, \quad \mu \to e^{i\alpha\bar{\Gamma}}\mu, \quad \tilde{\mu} \to e^{i\alpha\bar{\Gamma}}\tilde{\mu} \tag{3.151}$$

and ghosts transform as

$$\Upsilon_1 \to \Upsilon_1, \quad \Upsilon_2 \to e^{-i\alpha\bar{\Gamma}}\Upsilon_2 \tag{3.152}$$

$$\{c_1, b_1\} \to e^{i\alpha\bar{\Gamma}}\{c_1, b_1\} \quad \{c_2, b_2\} \to \{c_2, b_2\} \tag{3.153}$$

where α is defined by

$$\alpha = \nu + \frac{1}{2}\pi, \quad i\bar{\rho} = e^{i\nu}|\rho| \tag{3.154}$$

[17] We supress 16 component spinor indices below

and $\bar{\Gamma}$ is the $8D$ chirality matrix.

The above transformation rules for space-time fermions follow from the transformation rules for the appropriate vertex operators on the world-sheet (as discussed for example in [75]). The only generator of $\mathcal{D}_T$ acting non-trivially on fermions is S. The components $\mathcal{V}^a_{NS}, a = 8, 9$ of the right-moving NS vertex are rotated by 2α, while the components $\mathcal{V}^A_{NS}$ are invariant. This follows since S does not act on the left-moving components of vertex operators. In this way we find the transformation rules for $\eta^A, b_2, c_2, \Sigma, \Upsilon_1, l, \tilde{l}$, which originate from $R \otimes NS$ sector. To account for the transformation rules for $\psi^A, b_1, c_1, \Lambda, \Upsilon_2, \mu, \tilde{\mu}$ we recall that these fields originate from $NS \otimes R$ sector and that the right-moving R vertex $\mathcal{V}_R$ transforms under S as

$$S : \mathcal{V}_R \to e^{i\alpha\bar{\Gamma}}\mathcal{V}_R. \tag{3.155}$$

3.7.2 10D fermion action

We start from the part of the $10D$ IIA supergravity action quadratic in fermions[91]. We work in the string frame. [18]

$$\begin{aligned}
S^{(10)}_{ferm} &= \int \sqrt{-g_{10}}e^{-2\phi}\left[\frac{1}{2}\bar{\hat{\psi}}_{\hat{A}}\hat{\Gamma}^{\hat{A}\hat{N}\hat{B}}D_{\hat{N}}\hat{\psi}_{\hat{B}} + \frac{1}{2}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{N}}D_{\hat{N}}\hat{\Lambda} - \frac{1}{\sqrt{2}}(\partial_{\hat{N}}\phi)\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{A}}\hat{\Gamma}^{\hat{N}}\hat{\psi}_{\hat{A}}\right] \\
&+\frac{1}{16}\int \sqrt{-g_{10}}e^{-\phi}\tilde{G}_{\hat{A}\hat{C}}\left[\bar{\hat{\psi}}^{\hat{E}}\hat{\Gamma}_{[\hat{E}}\hat{\Gamma}^{\hat{A}\hat{C}}\hat{\Gamma}_{\hat{F}]}\hat{\Gamma}^{11}\hat{\psi}^{\hat{F}} + \frac{3}{\sqrt{2}}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{E}}\hat{\Gamma}^{\hat{A}\hat{C}}\hat{\Gamma}^{11}\hat{\psi}_{\hat{E}} + \frac{5}{4}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{A}\hat{C}}\hat{\Gamma}^{11}\hat{\Lambda}\right] \\
&+\int \sqrt{-g_{10}}e^{-\phi}G_0\left[\frac{1}{8}\bar{\hat{\psi}}_{\hat{A}}\hat{\Gamma}^{\hat{A}\hat{B}}\hat{\psi}_{\hat{B}} + \frac{5}{8\sqrt{2}}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{A}}\hat{\psi}_{\hat{A}} - \frac{21}{32}\bar{\hat{\Lambda}}\hat{\Lambda}\right]+ \\
&+\frac{1}{192}\int \sqrt{-g_{10}}e^{-\phi}\tilde{G}_{\hat{A}\hat{B}\hat{C}\hat{D}}\left[\bar{\hat{\psi}}^{\hat{E}}\hat{\Gamma}_{[\hat{E}}\hat{\Gamma}^{\hat{A}\hat{B}\hat{C}\hat{D}}\hat{\Gamma}_{\hat{F}]}\hat{\psi}^{\hat{F}} + \frac{1}{\sqrt{2}}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{E}}\hat{\Gamma}^{\hat{A}\hat{B}\hat{C}\hat{D}}\hat{\psi}_{\hat{E}} + \frac{3}{4}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{A}\hat{B}\hat{C}\hat{D}}\hat{\Lambda}\right] \\
&+\frac{1}{48}\int \sqrt{-g_{10}}e^{-2\phi}H_{\hat{A}\hat{B}\hat{C}}\left[\bar{\hat{\psi}}^{\hat{E}}\hat{\Gamma}_{[\hat{E}}\hat{\Gamma}^{\hat{A}\hat{B}\hat{C}}\hat{\Gamma}_{\hat{F}]}\hat{\Gamma}^{11}\hat{\psi}^{\hat{F}} + \sqrt{2}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{E}}\hat{\Gamma}^{\hat{A}\hat{B}\hat{C}}\hat{\Gamma}^{11}\hat{\psi}_{\hat{E}}\right]
\end{aligned} \tag{3.156}$$

where $\hat{\Lambda}$ and $\hat{\psi}^{\hat{A}}$ are dilatino and gravitino and covariant derivatives act on them as

$$D_{\hat{N}}\hat{\psi}^{\hat{A}} = \partial_{\hat{N}}\hat{\psi}^{\hat{A}} + \omega_{\hat{N}}{}^{\hat{A}}{}_{\hat{B}}\hat{\psi}^{\hat{B}} + \frac{1}{4}\omega_{\hat{N}\hat{B}\hat{C}}\Gamma^{\hat{B}\hat{C}}\hat{\psi}^{\hat{A}}$$

$$D_{\hat{N}}\hat{\Lambda} = \partial_{\hat{N}}\hat{\Lambda} + \frac{1}{4}\omega_{\hat{N}\hat{B}\hat{C}}\Gamma^{\hat{B}\hat{C}}\hat{\Lambda}$$

[18] We explain the relation between our conventions and those of [91] in Appendix(B).

There are also terms quartic in fermions in the action. It turns out that it is important to take them into account to check the T-duality invariance of partition sum. We recall the 4-fermionic terms in Appendix(C).

3.7.3 Reduction on T^2.

To make the reduction of the fermionic action to 8D we choose the gauge for the 10D veilbein as

$$\hat{E}_{\hat{M}}^{\hat{A}} = \begin{pmatrix} tE_M^A & \mathcal{A}_M^m V^{\frac{1}{2}} e_m^a \\ 0 & V^{\frac{1}{2}} e_m^a \end{pmatrix}, \tag{3.157}$$

and use the following basis of 10D 32×32 matrices $\hat{\Gamma}^{\hat{A}}$,

$$\hat{\Gamma}^A = \sigma_2 \otimes \Gamma^A \quad A = 0, \ldots 7, \quad \hat{\Gamma}^8 = \sigma_1 \otimes \mathbf{1}_{16}, \quad \hat{\Gamma}^9 = \sigma_2 \otimes \bar{\Gamma}, \quad \bar{\Gamma} = \Gamma^0 \ldots \Gamma^7 \tag{3.158}$$

Here Γ^A are symmetric 8D Dirac matrices, which in Euclidean signature can be all chosen to be real and $\sigma_{1,2,3}$ are Pauli matrices.

In this basis 10D chirality $\hat{\Gamma}^{11}$ and charge conjugation $C^{(10)}$ matrices have the form

$$\hat{\Gamma}^{11} = \sigma_3 \otimes \mathbf{1}_{16}, \qquad C^{(10)} = i\sigma_2 \otimes \mathbf{1}_{16} \tag{3.159}$$

The 8D fermions listed in section 7.1 are related to 10D fields $\hat{\psi}^{\hat{A}}$ and $\hat{\Lambda}$ in the following way [19] :

$$\begin{pmatrix} \psi^A \\ \eta^A \end{pmatrix} = \hat{\psi}^A + \frac{1}{6}\hat{\Gamma}^A\hat{\Gamma}_a\hat{\psi}^a, \quad \begin{pmatrix} \Sigma \\ \Lambda \end{pmatrix} = \frac{3}{4}\hat{\Lambda} + \frac{\sqrt{2}}{4}\hat{\Gamma}_a\hat{\psi}^a, \tag{3.160}$$

$$\begin{pmatrix} l \\ \mu \end{pmatrix} = \hat{\Gamma}_a\hat{\psi}^a - \frac{\sqrt{2}}{2}\hat{\Lambda}, \quad \begin{pmatrix} \tilde{\mu} \\ \tilde{l} \end{pmatrix} = \hat{\psi}_8 - \hat{\Gamma}^{89}\hat{\psi}_9 \tag{3.161}$$

3.7.4 8D fermion action

Now we present the 8D action $S_{quad}^{(8)} = S_{kin} + S_{fermi-flux}$ quadratic in fermionic fluctuations [20] over the 8D background specified in section 2.2. The kinetic term is standard

[19] $\hat{\Lambda}$ and $\hat{\Gamma}_a\hat{\psi}^a$ are mixed to give 8d "dilatino", the superpartner of $e^{-2\xi} = e^{-2\phi}V$

[20] In Minkowski signature $\bar{\psi}_A = \psi_A^\dagger\Gamma^0$. In Euclidean signature $\bar{\psi}_A$ and ψ_A are treated as independent fields.

$$S_{kin} = \int_X e^{-2\xi} t^7 \Big\{ \frac{1}{2}\bar{\psi}_A \Gamma^{AMB} D_M \psi_B + \frac{1}{2}\bar{\eta}_A \Gamma^{AMB} D_M \eta_B + \frac{2}{3}\bar{\Sigma}\Gamma^M D_M \Sigma + \frac{2}{3}\bar{\Lambda}\Gamma^M D_M \Lambda \tag{3.162}$$

$$+\frac{1}{4}\bar{l}\Gamma^M D_M l + \frac{1}{4}\bar{\mu}\Gamma^M D_M \mu + \frac{1}{4}\bar{\tilde{l}}\Gamma^M D_M \tilde{l} + \frac{1}{4}\bar{\tilde{\mu}}\Gamma^M D_M \tilde{\mu} \Big\}$$

The coupling of fluxes to fermion bilinears is:

$$S_{fermi-flux} = \frac{\pi}{4}\int_X e^{-\xi}\Big\{ t^8 \Big[\frac{n_0\rho + n_1}{\sqrt{\mathrm{Im}\rho}} X_{(0)} - \frac{n_0\bar{\rho} + n_1}{\sqrt{\mathrm{Im}\rho}} \tilde{X}_{(0)}\Big] + t^7 g_{(1)m} \wedge * X^m_{(1)} + \tag{3.163}$$

$$+t^6 \Big[\frac{g^2_{(2)}\rho + g^1_{(2)}}{\sqrt{\mathrm{Im}\rho}} \wedge * X_{(2)} - \frac{g^2_{(2)}\bar{\rho} + g^1_{(2)}}{\sqrt{\mathrm{Im}\rho}} \wedge * \tilde{X}_{(2)}\Big] + t^5 g_{(3)m} \wedge * X^m_{(3)}$$

$$+t^4 \sqrt{\mathrm{Im}\rho}\, g_{(4)} \wedge * \Big[X_{(4)} + \tilde{X}_{(4)}\Big]\Big\}$$

where the harmonic fluxes $g_{(p)}, p = 0, \ldots, 4$ were defined in (3.103,3.104). These harmonic fields couple to differential p-forms $X_{(p)}, \tilde{X}_{(p)}$ constructed out of fermi bilinears. We now give explicit formulae for $X_{(p)}$:

$$X_{(0)} = -\overline{\mathbf{\Psi}}^{(-)}_A \Gamma^{AB} \mathbf{W}^{(-)}_B - \overline{\mathbf{W}}^{(-)}_B \Gamma^{AB} \mathbf{\Psi}^{(-)}_A + i\sqrt{2}\bar{\Lambda}^{(+)}\Gamma^A \mathbf{W}^{(+)}_A \tag{3.164}$$

$$-i\sqrt{2}\overline{\mathbf{W}}^{(-)}_A \Gamma^A \Lambda^{(+)} + i\sqrt{2}\bar{\Sigma}^{(+)}\Gamma^A \mathbf{\Psi}^{(-)}_A - i\sqrt{2}\overline{\mathbf{\Psi}}^{(-)}_A \Gamma^A \Sigma^{(+)}$$

$$+\frac{i}{2}\bar{l}^{(-)}\Gamma^A \mathbf{\Psi}^{(+)}_A - \frac{i}{2}\overline{\mathbf{\Psi}}^{(+)}_A \Gamma^A l^{(-)} + \frac{i}{2}\bar{\mu}^{(-)}\Gamma^A \mathbf{W}^{(+)}_A - \frac{i}{2}\overline{\mathbf{W}}^{(+)}_A \Gamma^A \mu^{(-)}$$

$$+4\bar{\Sigma}^{(+)}\Lambda^{(+)} - 4\bar{\Lambda}^{(+)}\Sigma^{(+)} - \frac{1}{2}\bar{\tilde{l}}^{(+)}\tilde{\mu}^{(+)} + \frac{1}{2}\bar{\tilde{\mu}}^{(+)}\tilde{l}^{(+)}$$

$$(X_{(2)})_{MN} = \overline{\mathbf{\Psi}}^{(-)}_A \Gamma^{[A}\Gamma_{MN}\Gamma^{B]} \mathbf{W}^{(-)}_B + \overline{\mathbf{W}}^{(-)}_A \Gamma^{[A}\Gamma_{MN}\Gamma^{B]} \mathbf{\Psi}^{(-)}_B + \tag{3.165}$$

$$i\sqrt{2}\bar{\Lambda}^{(+)}\Gamma_{MN}\Gamma^A \mathbf{W}^{(-)}_A - i\sqrt{2}\overline{\mathbf{W}}^{(-)}_A \Gamma^A \Gamma_{MN}\Lambda^{(+)} + i\sqrt{2}\bar{\Sigma}^{(+)}\Gamma_{MN}\Gamma^A \mathbf{\Psi}^{(-)}_A$$

$$+i\sqrt{2}\overline{\mathbf{\Psi}}^{(-)}_A \Gamma^A \Gamma_{MN}\Sigma^{(+)} + \frac{i}{2}\bar{l}^{(-)}\Gamma^A \Gamma_{MN}\mathbf{\Psi}^{(+)}_A + \frac{i}{2}\overline{\mathbf{\Psi}}^{(+)}_A \Gamma_{MN}\Gamma^A l^{(-)}$$

$$-\frac{i}{2}\bar{\mu}^{(-)}\Gamma^A \Gamma_{MN}\mathbf{W}^{(+)}_A - \frac{i}{2}\overline{\mathbf{W}}^{(+)}_A \Gamma_{MN}\Gamma^A \mu^{(-)} + 4\bar{\Sigma}^{(+)}\Gamma_{MN}\Lambda^{(+)}$$

$$+4\bar{\Lambda}^{(+)}\Gamma_{MN}\Sigma^{(+)} - \frac{1}{2}\bar{\tilde{l}}^{(+)}\Gamma_{MN}\tilde{\mu}^{(+)} - \frac{1}{2}\bar{\tilde{\mu}}^{(+)}\Gamma_{MN}\tilde{l}^{(+)}$$

where $\psi_A^{(\pm)} = \frac{1}{2}\left(\mathbf{1}_{16} \pm \bar{\Gamma}\right)\psi_A$,etc. and we use the combinations of $8D$ fields

$$\mathbf{\Psi}_A = \psi_A + i\frac{\sqrt{2}}{3}\Gamma_A\Lambda, \quad \mathbf{W}_A = \eta_A - i\frac{\sqrt{2}}{3}\Gamma_A\Sigma$$

to make the expressions for $X_{(0)}, X_{(2)}$ have nicer coefficients.

The forms $\tilde{X}_{(0)}, \tilde{X}_{(2)}$ can be obtained from $X_{(0)}, X_{(2)}$ by exchange of 8D chiralities $(-) \leftrightarrow (+)$.

Under the T-duality generator S the above forms transform as

$$\left\{X_{(0)}, X_{(2)}\right\} \to e^{-i\alpha}\left\{X_{(0)}, X_{(2)}\right\}, \quad \left\{\tilde{X}_{(0)}, \tilde{X}_{(2)}\right\} \to e^{i\alpha}\left\{\tilde{X}_{(0)}, \tilde{X}_{(2)}\right\} \tag{3.166}$$

so that the combinations $\frac{1}{\sqrt{\mathrm{Im}\rho}}(n_0\rho + n_1)X_{(p)}$, $\frac{1}{\sqrt{\mathrm{Im}\bar{\rho}}}(n_0\bar{\rho} + n_1)\tilde{X}_{(p)}$ for $p = 0, 2$ which appear in the action (3.162) are invariant under S.

Also we have defined the 1-form

$$\begin{aligned}
(X_{(1)}^m)_M = e_+^m\Big[&\overline{\mathbf{\Psi}}_A^{(-)}\Gamma^{[A}\Gamma_M\Gamma^{B]}\mathbf{W}_B^{(+)} - \overline{\mathbf{W}}_A^{(+)}\Gamma^{[A}\Gamma_M\Gamma^{B]}\mathbf{\Psi}_B^{(-)} \\
&-i\sqrt{2}\overline{\Lambda}^{(+)}\Gamma_M\Gamma^A\mathbf{W}_A^{(+)} + i\sqrt{2}\overline{\mathbf{W}}_A^{(+)}\Gamma^A\Gamma_M\Lambda^{(+)} - i\sqrt{2}\overline{\Sigma}^{(-)}\Gamma_M\Gamma^A\mathbf{\Psi}_A^{(-)} \\
&+i\sqrt{2}\overline{\mathbf{\Psi}}_A^{(-)}\Gamma^A\Gamma_M\Sigma^{(-)} - \frac{i}{2}\bar{l}^{(+)}\Gamma^A\Gamma_M\mathbf{\Psi}_A^{(+)} + \frac{i}{2}\overline{\mathbf{\Psi}}_A^{(+)}\Gamma_M\Gamma^A\tilde{l}^{(+)} \\
&-\frac{i}{2}\bar{\tilde{\mu}}^{(-)}\Gamma^A\Gamma_M\mathbf{W}_A^{(-)} + \frac{i}{2}\overline{\mathbf{W}}_A^{(-)}\Gamma_M\Gamma^A\tilde{\mu}^{(-)} - 4\overline{\Sigma}^{(-)}\Gamma_M\Lambda^{(+)} \\
&+4\overline{\Lambda}^{(+)}\Gamma_M\Sigma^{(-)} - \frac{1}{2}\bar{\mu}^{(+)}\Gamma_M l^{(-)} + \frac{1}{2}\bar{l}^{(-)}\Gamma_M\mu^{(+)}\Big] + e_-^m\Big[(+) \leftrightarrow (-)\Big]
\end{aligned} \tag{3.167}$$

and the 3-form

$$\begin{aligned}
(X_{(3)}^m)_{MNP} = e_+^m\Big[&-\overline{\mathbf{\Psi}}_A^{(-)}\Gamma^{[A}\Gamma_{MNP}\Gamma^{B]}\mathbf{W}_B^{(+)} - \overline{\mathbf{W}}_A^{(+)}\Gamma^{[A}\Gamma_{MNP}\Gamma^{B]}\mathbf{\Psi}_B^{(-)} \\
&+i\sqrt{2}\overline{\Lambda}^{(+)}\Gamma_{MNP}\Gamma^A\mathbf{W}_A^{(+)} + i\sqrt{2}\overline{\mathbf{W}}_A^{(+)}\Gamma^A\Gamma_{MNP}\Lambda^{(+)} - i\sqrt{2}\overline{\Sigma}^{(-)}\Gamma_{MNP}\Gamma^A\mathbf{\Psi}_A^{(-)} \\
&-i\sqrt{2}\overline{\mathbf{\Psi}}_A^{(-)}\Gamma^A\Gamma_{MNP}\Sigma^{(-)} - \frac{i}{2}\bar{l}^{(+)}\Gamma^A\Gamma_{MNP}\mathbf{\Psi}_A^{(+)} - \frac{i}{2}\overline{\mathbf{\Psi}}_A^{(+)}\Gamma_{MNP}\Gamma^A\tilde{l}^{(+)} \\
&+\frac{i}{2}\bar{\tilde{\mu}}^{(-)}\Gamma^A\Gamma_{MNP}\mathbf{W}_A^{(-)} + \frac{i}{2}\overline{\mathbf{W}}_A^{(-)}\Gamma_{MNP}\Gamma^A\tilde{\mu}^{(-)} - 4\overline{\Sigma}^{(-)}\Gamma_{MNP}\Lambda^{(+)} \\
&-4\overline{\Lambda}^{(+)}\Gamma_{MNP}\Sigma^{(-)} + \frac{1}{2}\bar{\mu}^{(+)}\Gamma_{MNP}l^{(-)} + \frac{1}{2}\bar{l}^{(-)}\Gamma_{MNP}\mu^{(+)}\Big] + e_-^m\Big[(+) \leftrightarrow (-)\Big]
\end{aligned} \tag{3.168}$$

where we denote $e_\pm^m = e_8^m \mp i e_9^m$.

The forms $X^m_{(1)}$ and $X^m_{(3)}$ transform in the **2** of $SL(2,\mathbf{Z})_\tau$. Also from (3.167,3.168) we find that $X^m_{(1)}$ and $X^m_{(3)}$ are invariant under $SL(2,\mathbf{Z})_\rho$ if we accompany the action of generator S by the U(1) rotation of e^m_a

$$e^m_\pm \to e^{\pm i\alpha} e^m_\pm \tag{3.169}$$

The most important objects in (3.163) are the self-dual[21] form $X_{(4)}$ and the anti-self-dual form $\tilde{X}_{(4)}$ which couple to the flux $g_{(4)}$. $X_{(4)}$ is defined by

$$\begin{aligned}(X_{(4)})_{MNPQ} = & -i\overline{\Psi}^{(+)}_A\Gamma^{[A}\Gamma_{MNPQ}\Gamma^{B]}\mathbf{W}^{(+)}_B + i\overline{\mathbf{W}}^{(+)}_A\Gamma^{[A}\Gamma_{MNPQ}\Gamma^{B]}\Psi^{(+)}_B \\ & -\sqrt{2}\overline{\Lambda}^{(-)}\Gamma_{MNPQ}\Gamma^A\mathbf{W}^{(+)}_A + \sqrt{2}\overline{\mathbf{W}}^{(+)}_A\Gamma^A\Gamma_{MNPQ}\Lambda^{(-)} \\ & -\sqrt{2}\overline{\Sigma}^{(-)}\Gamma_{MNPQ}\Gamma^A\Psi^{(+)}_A + \sqrt{2}\overline{\Psi}^{(+)}_A\Gamma^A\Gamma_{MNPQ}\Sigma^{(-)} - \frac{1}{2}\bar{l}^{(+)}\Gamma^A\Gamma_{MNPQ}\Psi^{(-)}_A \\ & +\frac{1}{2}\overline{\Psi}^{(-)}_A\Gamma_{MNPQ}\Gamma^A l^{(+)} - \frac{1}{2}\bar{\mu}^{(+)}\Gamma^A\Gamma_{MNPQ}\mathbf{W}^{(-)}_A + \frac{1}{2}\overline{\mathbf{W}}^{(-)}_A\Gamma_{MNPQ}\Gamma^A\mu^{(+)} \\ & +4i\overline{\Sigma}^{(-)}\Gamma_{MNPQ}\Lambda^{(-)} - 4i\overline{\Lambda}^{(-)}\Gamma_{MNPQ}\Sigma^{(-)} - \frac{i}{2}\bar{\tilde{l}}^{(-)}\Gamma_{MNPQ}\tilde{\mu}^{(-)} + \frac{i}{2}\bar{\tilde{\mu}}^{(-)}\Gamma_{MNPQ}\tilde{l}^{(-)}\end{aligned} \tag{3.170}$$

and $\tilde{X}_{(4)}$ can be obtained from $X_{(4)}$ by the exchange of 8D chiralities $(+) \leftrightarrow (-)$.

Under the T-duality generator S these forms transform as

$$X_{(4)} \to e^{i\alpha}X_{(4)}, \quad \tilde{X}_4 \to e^{-i\alpha}\tilde{X}_{(4)} \tag{3.171}$$

We have also checked using Appendix(C) that the 4-fermion terms in the 8D action can be written as

$$S^{(8D)}_{4-ferm} = S'_{4-ferm} + S''_{4-ferm}, \quad S'_{4-ferm} = \frac{\pi}{128}\int_{X_8} e^{-2\xi}t^8\left[X_{(4)}\wedge * X_{(4)} + \tilde{X}_{(4)}\wedge * \tilde{X}_{(4)}\right] \tag{3.172}$$

While S''_{4-ferm} is manifestly invariant under T-duality, we will see that the non-invariant term S'_{4-ferm} is required for T-duality invariance of the total partition sum $Z^{(total)}_{B+F}(\mathcal{F},\tau,\rho)$.

The classical 8D action obtained from the reduction of 10D IIA supergravity on T^2 is invariant under local supersymmetry (all 32 components survive the reduction).

To construct the quantum action we have to impose a gauge fixing condition on the gravitino $\hat{\psi}_{(8D)} := \begin{pmatrix}\psi_A \\ \eta_A\end{pmatrix}$ and include ghosts. Since the susy transformation laws involve

[21] In our conventions $\Gamma_{A_1A_2A_3A_4} = -\frac{1}{4!}\epsilon_{A_1A_2A_3A_4B_1B_2B_3B_4}\Gamma^{B_1B_2B_3B_4}\bar{\Gamma}$

fluxes, there is a potential T-duality anomaly from the ghost sector. In fact no such anomaly will occur as we now demonstrate. There are two generic properties of supergravity theories:

- 1.) In addition to a pair of Faddeev-Popov ghosts associated to the local susy gauge transformation $\hat{\psi}_{(8D)} \to \hat{\psi}^A_{(8D)} + \delta_{\hat{\epsilon}}\hat{\psi}^A_{(8D)}$ a "third ghost" appears [97].

- 2.) Terms quartic in Faddeev-Popov ghosts are required [98].

Let us recall first how the "third ghost" appears. Following the standard procedure we fix the local susy gauge by inserting $\delta\left(f - \hat{\Gamma}_A\hat{\psi}^A_{(8D)}\right)$ into the path integral. Then we also insert the unit[22]

$$1 = \frac{1}{\sqrt{det\left(\frac{1}{2}e^{-2\xi t^7}\hat{D}\right)}} \int [df] e^{\frac{i}{2}\int_X e^{-2\xi t^7}\bar{f}\hat{D}f}, \quad \hat{D} = i\hat{\Gamma}^N D_N \tag{3.173}$$

and integrate over $[df]$. (If $\hat{D}$ has zeromodes this expression is formally 0/0, but (7.27) below still makes sense.)

As a result we first find that the gravitino kinetic term gets modified to

$$-\frac{i}{2}\int_X e^{-2\xi t^7}\left\{\bar{\psi}^A \mathcal{M}_{AB}\psi^B + \bar{\eta}^A\mathcal{M}_{AB}\eta^B\right\} \tag{3.174}$$

where the operator $\mathcal{M}_{AB}$ acts on sections of the bundle [23] $Spin(X)\otimes TX$ as

$$\mathcal{M}_{AB} = \delta_{AB} i\Gamma^M D_M - 2i\Gamma_A D_B \tag{3.175}$$

where $D_A = E^M_A D_M$. The determinant in (3.173) is expressed as the partition function for the "third ghost" $\hat{\Upsilon}$ with action

$$S_{\hat{\Upsilon}} = -\frac{i}{2}\int_X e^{-2\xi t^7}\bar{\hat{\Upsilon}}\hat{D}\hat{\Upsilon} \tag{3.176}$$

$\hat{\Upsilon}$ is bosonic 32 component spinor, which we decompose into 16 component spinors as

$$\hat{\Upsilon} = \begin{pmatrix}\Upsilon_1\\ \Upsilon_2\end{pmatrix}$$

[22]We use the measure $\int[df]e^{i\int_X \bar{f}f} = 1$.
[23]$Spin(X)$ and TX are spinor and tangent bundles on X

Now we come to the most interesting part of quantum action which involves Faddeev-Popov ghosts $\hat{b}, \hat{c}$.

$$S_{bc} = S_{bc}^{(2)} + S_{bc}^{(4)} \tag{3.177}$$

where $S_{bc}^{(2)}$ $\left(S_{bc}^{(4)}\right)$ denotes the parts of the action quadratic (quartic) in FP ghosts. Let us discuss the quadratic part first. According to the standard FP procedure we have

$$S_{bc}^{(2)} = \int_{X_8} t^7 e^{-2\xi} \bar{\hat{b}} \hat{\Gamma}_A \delta_{\hat{c}} \hat{\psi}^A_{(8D)} \tag{3.178}$$

We decompose bosonic 32 component spinors $\hat{b}, \hat{c}$ as

$$\hat{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \hat{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

We can write the action as a sum of two pieces

$$S_{bc}^{(2)} = S_{bc}^{(2)0} + S_{bc}^{(2)2}$$

Here $S_{bc}^{(2)0}$ does not contain fermionic matter fields while $S_{bc}^{(2)2}$ is quadratic in fermions. We now present $S_{bc}^{(2)0}$ and put $S_{bc}^{(2)2}$ in Appendix(D).

$$S_{bc}^{(2)0} = \int_{X_8} t^7 e^{-\xi} \bar{\hat{b}}(-i\hat{D})\hat{c} - \pi e^{-\xi} \Big\{ \frac{2}{3} t^8 \Big[\frac{n_0 \rho + n_1}{\sqrt{\mathrm{Im}\rho}} X^{gh}_{(0)} - \frac{n_0 \bar{\rho} + n_1}{\sqrt{\mathrm{Im}\rho}} \tilde{X}^{gh}_{(0)} \Big] \tag{3.179}$$
$$+ \frac{1}{2} t^7 g_{(1)m} \wedge * X^{gh\ m}_{(1)} + \frac{1}{3} t^6 \mathrm{Im} \Big[\frac{g^2_{(2)} \rho + g^1_{(2)}}{\sqrt{\mathrm{Im}\rho}} \wedge * X^{gh}_{(2)} - \frac{g^2_{(2)} \bar{\rho} + g^1_{(2)}}{\sqrt{\mathrm{Im}\rho}} \wedge * \tilde{X}^{gh}_{(2)} \Big]$$
$$+ \frac{1}{8} t^5 g_{(3)m} \wedge * X^{gh\ m}_{(3)} \Big\}$$

where we define forms bilinear in FP ghosts as

$$X^{gh}_{(0)} = \frac{1}{2} \Big\{ \overline{b_2}^{(-)} c_1^{(-)} - \overline{c_1}^{(-)} b_2^{(-)} - \overline{b_1}^{(-)} c_2^{(-)} + \overline{c_2}^{(-)} b_1^{(-)} \Big\} \tag{3.180}$$

$$(X^{gh}_{(2)})_{MN} = \frac{1}{2} \Big\{ \overline{b_2}^{(-)} \Gamma_{MN} c_1^{(-)} + \overline{c_1}^{(-)} \Gamma_{MN} b_2^{(-)} + \overline{b_1}^{(-)} \Gamma_{MN} c_2^{(-)} + \overline{c_2}^{(-)} \Gamma_{MN} b_1^{(-)} \Big\} \tag{3.181}$$

$$(X^{gh\ m}_{(1)})_M = \frac{1}{2} e^m_+ \Big[\overline{b_2}^{(+)} \Gamma_M c_1^{(-)} - \overline{c_1}^{(-)} \Gamma_M b_2^{(+)} - \overline{b_1}^{(-)} \Gamma_M c_2^{(+)} + \overline{c_2}^{(+)} \Gamma_M b_1^{(-)} \Big] \tag{3.182}$$

$$+\frac{1}{2}e_-^m\Big[(+)\leftrightarrow(-)\Big]$$

$$(X_{(3)}^{gh\ m})_{MNP} = \frac{1}{2}e_+^m\Big[\overline{b_2}^{(+)}\Gamma_{MNP}c_1^{(-)} + \overline{c_1}^{(-)}\Gamma_{MNP}b_2^{(+)} + \overline{b_1}^{(-)}\Gamma_{MNP}c_2^{(+)} \tag{3.183}$$

$$+\overline{c_2}^{(+)}\Gamma_{MNP}b_1^{(-)}\Big] + \frac{1}{2}e_-^m\Big[(+)\leftrightarrow(-)\Big]$$

The forms $\tilde{X}_{(0)}^{gh}, \tilde{X}_{(2)}^{gh}$ can be obtained from $X_{(0)}^{gh}, X_{(2)}^{gh}$ by exchange of 8D chiralities $(-)\leftrightarrow(+)$. Note, that $\hat{b}, \hat{c}$ do not couple to $g_{(4)}$ flux.

Let us now present the quartic in ghosts part of the quantum 8D action, obtained by following the procedure of [98].

$$S_{bc}^{(4)} = e^{-2\xi}t^8\Big\{\frac{1}{84}\left(\bar{\hat{b}}\hat{\Gamma}^{ABC}\hat{c}\right)\left(\bar{\hat{b}}\hat{\Gamma}_{ABC}\hat{c}\right) + \frac{1}{3}\left(\bar{\hat{b}}\hat{\Gamma}^{A}\hat{c}\right)\left(\bar{\hat{b}}\hat{\Gamma}_{A}\hat{c}\right)\Big\} \tag{3.184}$$

The presence of the quartic in FP ghosts part in the action is due to the fact that gauge symmetry algebra is open in supergravity: $[\delta_{\hat{\epsilon}_1}, \delta_{\hat{\epsilon}_1}]\,\hat{\psi}_{(8D)}^A$ contains a term proportional to the equation of motion of $\hat{\psi}_{(8D)}^A$.

T-duality invariance of $S_{bc}^{(4)}, S_{bc}^{(2)0}$ and $S_{\hat{\Upsilon}}$ is manifest and we have also checked that $S_{bc}^{(2)2}$ is T-duality invariant, so we conclude that the part of the 8D quantum action which contains ghosts is T-duality invariant.

We can now compute the fermionic quantum determinants including ghosts. Let us expand the fields $\Lambda, \Sigma, l, \tilde{l}, \mu, \tilde{\mu}, b_1, b_2, c_1, c_2, \Upsilon_1, \Upsilon_2$ and ψ_A, η_A in the full orthonormal basis of the operators $\breve{D} = i\Gamma^N D_N$ and $\mathcal{M}$ respectively, where the operator $\mathcal{M}$ was defined in (3.175). Note that since we are assuming that background fluxes are harmonic, fermionic non-zero modes do not couple to them. Moreover,we can rescale non-zero modes by a factor of $e^{-\xi}t^{7/2}$ so that kinetic terms appear without any dependence on ξ and t, but four-fermionic terms are supressed as $e^{2\xi}t^{-6}$ with respect to the kinetic terms. Since kinetic terms are manifestly T-duality invariant the integration over nonzero modes will just give a factor Det'_F depending only on the Ricci flat metric g_{MN} and the constants t and ξ, all of which are T-duality invariant. Det'_F has the form

$$\mathrm{Det}'_F = r_F(\xi, t)det'\mathcal{M} \tag{3.185}$$

where $det'\mathcal{M}$ is determinant of the operator $\mathcal{M}$ defined in (3.175) regularized in a way that

$$r_F(\xi,t) = const\left(e^{-2\xi}t^7\right)^{-n_{\mathcal{M}}} \tag{3.186}$$

where $n_{\mathcal{M}}$ denotes the number of zero modes of $\mathcal{M}$.

Note, that determinants of nonzero modes of the fermions $\Sigma, \Lambda, l, \mu, \tilde{l}, \tilde{\mu}$ and bosons $\Upsilon_1, \Upsilon_2, b_1, b_2, c_1, c_2$ cancel each other and do not contribute to Det'_F.

For zero-modes the situation is quite different: the kinetic terms are zero but there is nonzero coupling to harmonic fluxes, so that if we rescale fermion zeromodes by $e^{-\frac{1}{2}\xi}t^2$ we make both fermion coupling to $g_{(4)}$ and fermion quartic terms independent of ξ and t. We will also rescale ghost zeromodes by $e^{-\frac{1}{2}\xi}t^2$ and include the factor $\left(e^{-\xi}t^4\right)^{n_{\mathcal{M}}}$ which comes from the rescaling of fermion and ghost zeromodes into the definition of Det'_F, i.e. we define new r_F:

$$r_F^{new}(\xi,t) := r_F(\xi,t)\left(e^{-\xi}t^4\right)^{n_{\mathcal{M}}} = const(t)^{-3n_{\mathcal{M}}}(e^{\xi})^{n_{\mathcal{M}}} \tag{3.187}$$

Note, that from (3.147) and (3.187) we find that the full quantum determinants depend on t and ξ in the following way

$$(\tilde{t}^{-3})^{n_{\mathcal{M}}-n_{\mathcal{K}}-b_2-2b_1-4b_0} \tag{3.188}$$

where we recall that $\tilde{t} = te^{-\xi/3}$ is U-duality invariant combination and for any Ricci-flat spin 8-manifold the numbers $n_{\mathcal{M}}$ and $n_{\mathcal{K}}$ can be expressed in terms of topological invariants.

We can split the action of the rescaled fermion and (gravitino)ghost zeromodes as

$$S^{(zm)} = S^{(zm)inv} + S^{(zm)ninv}.$$

Here the part $S^{(zm)inv}$ is invariant under T-duality and includes all the ghost zeromode interactions, the coupling of the fermion zeromodes to all RR fluxes except for $g_{(4)}$ and the invariant part of the 4-fermion zeromode couplings, denoted $S^{(zm)''}_{4-ferm}$.

$S^{(zm)ninv}$ transforms non-trivially under the generator S of T-duality and can be recast in the following way:

$$S^{(zm)ninv} = 4\pi\mathrm{Im}\rho g_{(4)}\wedge * Y_{(4)} + 2\pi\mathrm{Im}\rho Y_{(4)}\wedge * Y_{(4)} \tag{3.189}$$

where we define the harmonic 4-form $Y_{(4)}$ as

$$Y_{(4)} = \frac{1}{16}\frac{1}{\sqrt{\mathrm{Im}\rho}}\left[X^{(zm)}_{(4)} + \tilde{X}^{(zm)}_{(4)}\right] \quad (3.190)$$

which transforms under S as

$$S \cdot Y_{(4)} = -\mathrm{Re}\rho Y_{(4)} + i\mathrm{Im}\rho * Y_{(4)} \quad (3.191)$$

Let us expand the harmonic 4-forms in the basis ω_i of $H^4(X, \mathbf{Z})$

$$g_{(4)} = (n^i + \tilde{\alpha}^i)\omega_i, \quad Y_{(4)} = y^i\omega_i, \quad \tilde{\beta} = \tilde{\beta}^i\omega_i$$

where the chracteristics $\tilde{\alpha}, \tilde{\beta}$ are given in (3.107). We now define a new object

$$\widehat{\Theta}(\mathcal{F}, \rho) = \int d\mu_F^{(zm)}\hat{h}e^{i2\pi\widehat{\Delta\Phi}}\Theta\left[\begin{matrix}\widehat{\alpha}\\ \widehat{\beta}\end{matrix}\right](Q) \quad (3.192)$$

where shifted characterstics are defined as $\widehat{\alpha}^i = \tilde{\alpha}^i + y^i$, $\widehat{\beta}^i = \tilde{\beta}^i + S \cdot y^i$, $d\mu_F^{(zm)}$ denotes the measure of rescaled fermion and ghost (which accompany gravitino) zeromodes and we also recall that $Q(\rho) = [H\mathrm{Im}\rho - i\mathrm{Re}\rho]I$. In (3.192) $\hat{h} = e^{-S^{(zm)inv}}$ is the expression which depends on τ, ρ, t, g_{MN} as well as fermion and ghost zeromodes in a T-duality invariant way, where the dependence on τ, ρ, t, g_{MN} comes entirely from the coupling of the rescaled zeromodes (of fermions and ghosts) to the fluxes $g_{(p)}, p = 0, 1, 2, 3$.

Another new object in (3.192) is

$$\widehat{\Delta\Phi}(\mathcal{F}, \rho, \vec{y}) = \Delta\tilde{\Phi} - \frac{1}{2}\vec{y}IS \cdot \vec{y} - \vec{y}I\vec{\beta}$$

where $\Delta\tilde{\Phi}$ was defined in (3.110).

$\widehat{\Theta}(\mathcal{F}, \rho)$ is invariant under $SL(2, \mathbf{Z})_\tau$ and transforms under $SL(2, \mathbf{Z})_\rho$ as

$$\widehat{\Theta}(S \cdot \mathcal{F}, -1/\rho) = s_F\mu(S)(-i\rho)^{\frac{1}{2}b_4^+}(i\bar{\rho})^{\frac{1}{2}b_4^-}\widehat{\Theta}(\mathcal{F}, \rho) \quad (3.193)$$

$$\widehat{\Theta}(T \cdot \mathcal{F}, \rho + 1) = \mu(T)\widehat{\Theta}(\mathcal{F}, \rho) \quad (3.194)$$

We do Poisson ressumation to find (3.193) and the extra phase s_F is due to the transformation[24] of $d\mu_F^{zm}$

$$s_F = \left(e^{i\alpha}\right)^{I(\mathcal{M})} = (i)^{I(\mathcal{M})}(-i\rho)^{-\frac{1}{2}I(\mathcal{M})}(i\bar{\rho})^{\frac{1}{2}I(\mathcal{M})} \quad (3.195)$$

[24] We are using that 10D fermions are Majorana fermions in Minkowski signature.

where $I(\mathcal{M})$ is the index of the operator $\mathcal{M}$ defined in (3.175). As in the standard computation of chiral anomaly [99], only the zeromodes contribute to the transformation of fermionic measure.

Note that the contribution of non-trivially transforming bosonic ghosts c_1, b_1, Υ_2 to the transformation of the measure cancel the contribution of the fermions $\mu, \tilde{\mu}, \Lambda, l, \tilde{l}$.

3.8 T-duality invariance

3.8.1 Transformation laws for $Z_{B+F}(\mathcal{F}, \tau, \rho)$

Now we study transformation laws for

$$Z_{B+F}(\mathcal{F}, \tau, \rho) = Det_B Det'_F e^{-S_B(\mathcal{F})} \widehat{\Theta}(\mathcal{F}, \rho) \tag{3.196}$$

where $\widehat{\Theta}(\mathcal{F}, \rho)$ is defined in (3.192), Det_B and Det'_F are defined in (3.145) and (3.185, 3.187) respectively. We also recall that $S_B(\mathcal{F})$ is the real part of the classical action evaluated on the background field configuration.

First, we note that $Z_{B+F}(\mathcal{F}, \tau, \rho)$ is invariant under $SL(2, Z)_\tau$. Second, we learn how $Z_{B+F}(\mathcal{F}, \tau, \rho)$ transforms under $SL(2, Z)_\rho$ by using the transformation rules of Det_B (3.149) and $\widehat{\Theta}(\mathcal{F}, \rho)$ (3.193,3.194). We find:

$$Z_{B+F}(S \cdot \mathcal{F}, \tau, -1/\rho) = s_B s_F \mu(S) (-i\rho)^{\frac{1}{2} b_4^+} (i\bar{\rho})^{\frac{1}{2} b_4^-} Z_{B+F}(\mathcal{F}, \tau, \rho) \tag{3.197}$$

$$Z_{B+F}(T \cdot \mathcal{F}, \tau, \rho + 1) = \mu(T) Z_{B+F}(\mathcal{F}, \tau, \rho) \tag{3.198}$$

where s_B is taken from the transformation of D_B.

Now, using the definition of χ and σ

$$\frac{1}{2}(b_0 - b_1 + b_2 - b_3 + b_4^{\pm}) = \frac{1}{4}(\chi \pm \sigma), \quad \sigma = b_4^+ - b_4^- \tag{3.199}$$

as well as the index theorem:

$$I(\mathcal{M}) + \int_X \lambda^2 = \int_X 248 \hat{A}_8$$

we obtain the final result for the transformation under the generator S

$$Z_{B+F}(S\cdot\mathcal{F},\tau,-1/\rho)=(-i\rho)^{\frac{1}{4}\chi+\frac{1}{8}(p_2-\lambda^2)}(i\bar{\rho})^{\frac{1}{4}\chi-\frac{1}{8}(p_2-\lambda^2)}Z_{B+F}(\mathcal{F},\tau,\rho) \qquad (3.200)$$

From (3.198) and (3.200) we find that there is a T-duality anomaly.

Note, that transformations (3.198,3.200) are consistent for any 8-dimensional spin manifold. This can be seen by computing [25]

$$Z_{B+F}((ST)^6\cdot\mathcal{F},\tau,\rho)=e^{i\frac{\pi}{4}\int_X(7\lambda^2-p_2)}Z_{B+F}(\mathcal{F},\tau,\rho) \qquad (3.201)$$

$$Z_{B+F}(S^4\cdot\mathcal{F},\tau,\rho)=Z_{B+F}(\mathcal{F},\tau,\rho)$$

and then noting that the index theorem for 8-dimensional spin manifolds implies

$$\int_X(7\lambda^2-p_2)\in 1440\mathbf{Z}. \qquad (3.202)$$

Incidentally, when X admits a nowhere-vanishing Majorana spinor of $\pm$ chirality the Euler characteristic is given by [82]:

$$\chi=\pm\frac{1}{2}\int_X(p_2-\lambda^2) \qquad (3.203)$$

and the transformation rule (3.200) simplifies to:

$$Z_{B+F}(S\cdot\mathcal{F},\tau,-1/\rho)=(-i\rho)^{\frac{1}{2}\chi}Z_{B+F}(\mathcal{F},\tau,\rho) \qquad (3.204)$$

$$Z_{B+F}(S\cdot\mathcal{F},\tau,-1/\rho)=(i\bar{\rho})^{\frac{1}{2}\chi}Z_{B+F}(\mathcal{F},\tau,\rho) \qquad (3.205)$$

for the positive and negative chirality respectively.

3.8.2 Including quantum corrections

Now we recall that there is a 1-loop correction to the effective $8D$ action:

$$S_{1-loop}=\pi\mathrm{Im}\rho\int_X(u\lambda^2+vp_2)+\frac{i\pi}{24}\mathrm{Re}\rho\int_X(p_2-\lambda^2) \qquad (3.206)$$

[25]The branches for the $8-th$ roots of unity are chosen in such a way that $S^2=(-)^{F_R}$, where F_R is a space-time fermion number in right-moving sector of type IIA string

where we remind that $\pi \mathrm{Im}\rho \int_X (u\lambda^2 + vp_2)$ comes from the regularization of $q(\mathrm{Im}\rho)$ in (3.124) and the numbers u and v depend on the regularization.

We now demonstrate that to construct a T-duality invariant partition function this term should be replaced with

$$S_{quant} = \Big[\frac{1}{2}\chi + \frac{1}{4}\int_X (p_2 - \lambda^2)\Big] log\left[\eta(\rho)\right] + \Big[\frac{1}{2}\chi - \frac{1}{4}\int_X (p_2 - \lambda^2)\Big] log\left[\eta(-\bar{\rho})\right] \qquad (3.207)$$

where $\eta(\rho)$ is Dedekind function. Taking the limit $Im\rho \to \infty$ one can uniquely determine $u = -\frac{1}{24}$ and $v = \frac{1}{24}$ in (3.206).

η has the following transformation laws:

$$\eta\left(-1/\rho\right) = (-i\rho)^{\frac{1}{2}}\eta(\rho), \quad \eta(\rho+1) = e^{\frac{\pi i}{12}}\eta(\rho) \qquad (3.208)$$

so that $e^{-S_{quant}}$ transforms as

$$e^{-S_{quant}}\left(-1/\rho\right) = (-i\rho)^{-\frac{1}{4}\chi - \frac{1}{8}\int_X (p_2 - \lambda^2)} (i\bar{\rho})^{-\frac{1}{4}\chi + \frac{1}{8}\int_X (p_2 - \lambda^2)} e^{-S_{quant}}\left(\rho\right) \qquad (3.209)$$

$$e^{-S_{quant}}\left(\rho + 1\right) = e^{-i\frac{\pi}{24}\int_X (p_2 - \lambda^2)} e^{-S_{quant}}\left(\rho\right) \qquad (3.210)$$

Using (3.202) the total partition function we find that

$$Z(\mathcal{F}, \tau, \rho) := e^{-S_{quant}} Z_{B+F}(\mathcal{F}, \tau, \rho) \qquad (3.211)$$

is invariant:

$$Z(T \cdot \mathcal{F}, \tau, \rho + 1) = Z(\mathcal{F}, \tau, \rho), \qquad (3.212)$$

$$Z(S \cdot \mathcal{F}, \tau, -1/\rho) = Z(\mathcal{F}, \tau, \rho). \qquad (3.213)$$

This is our main result.

As a consistency check we consider(for simplicity) the case when X admits a nowehere-vanishing spinor of positive chirality and take the limit $Im\rho = V \to \infty$

$$S_{quant} \to \left(\frac{i\pi}{12}\rho + \sum_{n\geq 1}\sum_{m\geq 1}\frac{1}{m}e^{2\pi i n m \rho}\right)\chi \qquad (3.214)$$

and we recognize the multiple cover formula for world-sheet instantons on T^2 from [79].

3.9 Application: Hull's proposal for interpreting the Romans mass in the framework of M-theory

As a by-product of the above results we will make some comments on an interesting open problem in the relation of M-theory to IIA string theory.

It is well known that IIA supergravity admits a massive deformation, leading to the Romans theory. The proper interpretation of this massive deformation in 11-dimensional terms is an intriguing open problem. In [73] C. Hull made an interesting suggestion for an 11-dimensional interpretation of certain backgrounds in the Romans theory. His interpretation involved T-duality in an essential way, and in the light of the above discussion we will make some comments on Hull's proposal.

3.9.1 Review of the relation of M-theory to IIA supergravity

Naive Kaluza-Klein reduction says that for an appropriate transformation of fields

$$\{g_{M-theory}, C_{M-theory}\} \to \{g_{IIA}, H_{IIA}, \phi_{IIA}, C_{IIA}\} \tag{3.215}$$

we have

$$S_{M-theory} = S_{IIA} \tag{3.216}$$

One of the main points of [68] was that, in the presence of topologically nontrivial fluxes equation (3.216) is not true! Indeed, given our current understanding of these fields, there is not even a 1-1 correspondence between classical M-theory field configurations and classical IIA field configurations. Rather, certain *sums* of IIA-theoretic field configurations were asserted to be equal to certain sums of M-theoretic field configurations. In this sense, the equivalence of type IIA string theory and M-theory on a circle fibration is a quantum equivalence.

To be more precise, in [68] it was shown that for product manifolds $Y = X_{10} \times S^1$, the sum over K-theory lifts $x(\hat{a})$ of a class $\hat{a} \in H^4(X_{10}; \mathbf{Z})$ is proportional to the sum over torsion shifts of the M-theory 4-form of Y. We have:

$$\frac{N(-)^{\mathrm{Arf}(q)+f(\hat{a}_0)}}{\sqrt{N_2}N_K}\sum_{x(\hat{a})} e^{-S_{IIA}} = exp\left(-||G_{M-theory}(\hat{a})||^2\right) \sum_{\hat{c}\in H^4_{tors}(X_{10},\mathbf{Z})} (-1)^{f(\hat{a}+\hat{c})} \quad (3.217)$$

The above formula is the main technical result of [68]. We recall that $[G_{M-theory}(\hat{a})] = 2\pi(\hat{a}-\frac{1}{2}\lambda)$ and the equivalence class of $\hat{a}$ is defined to contain M-theory field configurations with fixed kinetic energy

$$||G_{M-theory}(\hat{a})||^2 = \frac{1}{4\pi}\int_{X_{10}} G_{M-theory}(\hat{a})\wedge\hat{*}G_{M-theory}(\hat{a}),$$

from which follows that these fields are characterized by $\hat{a}' = \hat{a}+\hat{c}, \quad \hat{c}\in H^4_{tors}(X_{10},\mathbf{Z})$. Also, in (3.217) N_K and N is the order of $K^0_{tors}(X_{10})$ and $H^4_{tors}(X_{10};\mathbf{Z})$ respectively, N_2 stands for the number of elements in the quotient $L'' = L/L'$, where $L = H^4_{tors}(X_{10};\mathbf{Z})/2H^4_{tors}(X_{10};\mathbf{Z})$ and $L' = \left\{\hat{c}\in L, \quad Sq^3\hat{c} = 0\right\}$. Finally, Arf$(q)$ is the Arf invariant of the quadratic form $q(\hat{c}) = f(\hat{c}) + \int_{X_{10}}\hat{c}\cup Sq^2\hat{a}_0$ on L''. The identity (3.217) extends to the case where Y is a nontrivial circle bundle over X_{10} [68].

We interpret the fact that we must sum over field configurations in (3.217) as a statement that IIA-theory on X_{10} and M-theory on $Y = X_{10}\times S^1$ are really only quantum-equivalent. This point might seem somewhat tenuous, relying, as it does, on the fact that the torsion groups in cohomology and K-theory are generally different. Nevertheless, as we will now show, a precise version of Hull's proposal again requires equating sums over IIA and M-theory field configurations. In this case, however, the sums are over non-torsion cohomology classes, and in this sense the fact that IIA-theory and M-theory are only quantum equivalent becomes somewhat more dramatic.

3.9.2 Review of Hull's proposal

One version of Hull's proposal states that massive IIA string theory on $T^2\times X$ is equivalent to M-theory on a certain 3-manifold which is a nontrivial circle bundle over a torus. The proposal is based on T-duality invariance, which allows one to transform away G_0 at the expense of introducing G_2 along the torus, combined with the interpretation of G_2 flux as the first Chern class of a nontrivial M-theory circle bundle [68]. We now describe this in more detail.

Hull's proposal is based on the result [76] that dimensional reduction of massive IIA supergravity with mass m a circle of radius R, (denoted S^1_R), gives the same theory as reduction of IIB supergravity on $S^1_{1/R}$ with a twist using the $SL(2,R)$ symmetry of IIB supergravity, that is, the fields are twisted by

$$g(\theta) = \begin{pmatrix} 1 & m\theta \\ 0 & 1 \end{pmatrix} \tag{3.218}$$

where the coordinate on $S^1_{1/R}$ is $z = \frac{2\pi}{R}\theta, \quad \theta \in [0,1]$ and we use Scherk-Schwarz reduction with monodromy

$$g(1)g(0)^{-1} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \tag{3.219}$$

Schematically:

$$\frac{IIA_m}{S^1_R \times X_9} = \left(\frac{IIB}{S^1_{1/R} \times X_9}\right)_{g(\theta)} \tag{3.220}$$

where X_9 is an arbitrary 9-manifold. Note, in particular, that the twist acts on the IIB axiodil $\tau_B = C_0 + ie^{-\phi_B}$ as

$$\tau_B(\theta) = \tau_B(0) + m\theta \tag{3.221}$$

which implies that the IIB RR field G_1 has a nonzero period.

Let us also recall the duality between IIB on a circle and M-theory on T^2:

$$\frac{IIB}{S^1_{R'} \times S^1_{1/R} \times X} = \frac{M}{T^2\left(\tau_M, A_M\right) \times S^1_{1/R} \times X} \tag{3.222}$$

where the $T^2(\tau_M, A_M)$ on the M-theory side has complex structure $\tau_M = \tau_B(0)$ and area $A_M = e^{\frac{\phi_B}{3}}(R')^{-\frac{4}{3}}$.

Now, invoking the adiabatic argument we have:

$$\left(\frac{IIB}{S^1_{1/R} \times S^1_{R'} \times X}\right)_{g(\theta)} = \frac{M}{B(m;R',R) \times X} \tag{3.223}$$

where $B(m;R',R)$ is a 3-manifold with metric:

$$ds^2 = \left(\frac{2\pi}{R}\right)^2 (d\theta)^2 + A_M\left[\frac{1}{\mathrm{Im}\tau_M}(dx + (\mathrm{Re}\tau_M + m\theta)dy)^2 + \mathrm{Im}\tau_M dy^2\right] \tag{3.224}$$

where x, y are periodic $x \sim x+1$ and $y \sim y+1$.

Combining (3.220) with (3.223) we get the basic statement of Hull's proposal:

$$\frac{IIA_m}{S^1_R \times S^1_{R'} \times X} = \frac{M}{B(m; R', R) \times X} \tag{3.225}$$

3.9.3 A modified proposal

In view of what we have discussed in the present work, the equivalence of classical actions - when proper account is taken of the various phases of the supergravity action - cannot be true. This is reflected, for example, in the asymmetry of the phase (3.98) in exchanging n_0 for n_1. However, we follow the lead of (3.217) and therefore modify Hull's proposal by identifying sums over certain geometries on the IIA and M-theory side.

A modified proposal is to identify $Z(\mathcal{F}, \rho, \tau)$ defined in (3.196,3.211) with a sum over M-theory geometries as follows. Recall first that in the 8D theory there is a doublet of zeroforms $g^{\alpha}_{(0)}$, arising from G_0 and G_2. Next, let us factor $g_{(0)} = \ell \begin{pmatrix} p \\ q \end{pmatrix}$ where p, q are relatively prime integers and ℓ is an integer. Then we take a matrix $\mathcal{N} \in SL(2, \mathbf{Z})_\rho$

$$\mathcal{N} = \begin{pmatrix} r & -s \\ -q & p \end{pmatrix} \qquad rp - sq = 1 \tag{3.226}$$

such that

$$\mathcal{N} g_{(0)} = \begin{pmatrix} \ell \\ 0 \end{pmatrix} \tag{3.227}$$

This is the T-duality transformation that eliminates Romans flux.

Now, thanks to the invariance of $Z(\mathcal{F}, \tau, \rho)$ under T-duality transformations (see (3.212,3.213) above) we find:

$$Z(\mathcal{F}, \tau, \rho) = Z\left(\mathcal{N} \cdot \mathcal{F}, \tau, \frac{p\rho + s}{q\rho + r}\right) \tag{3.228}$$

By the results of [68] the right hand side of (3.228), having $G_0 = 0$, *does* have an interpretation as a sum over M-theory geometries. The M-theory geometry is indeed a circle bundle over $T^2 \times X$ defined by $c_1 = \ell e_0 + pe - qe'' + \gamma_m d\sigma^m$ (as in Hull's proposal), but in addition it is necessary to sum over E_8 bundles on the 11-manifold $B \times X$. While it is essential to sum over $g_{(4)}$, all other fluxes may be treated as classical - that is, they may be fixed and it is not necessary to sum over them.

Both sides of (3.228) should be regarded as wavefunctions in the quantization of self-dual fields. For this reason we propose that there is only an intrinsically *quantum mechanical* equivalence between IIA theory and M-theory in the presence of G_0.

3.10 Comments on the U-duality invariant partition function

The present work has been based on weakly coupled string theory. However, our motivation was understanding the relationship between K-theory and U-duality. In generalizing our considerations to the full U-duality group $\mathcal{D} = SL(3, \mathbf{Z}) \times SL(2, \mathbf{Z})_\rho$ of toroidally compactified IIA theory it is necessary to go beyond the weak coupling expansion. Thus, it is appropriate to start with the M-theory formulation. In the present section we make a few remarks on the U-duality of the M-theory partition function and its relation to the K-theory partition functions of type IIA strings. In particular, we will address the following points:

a.) The invariance of the M-theory partition function under the nongeometrical $SL(2, Z)_\rho$ is not obvious and appears to require surprising properties of η invariants.

b.) We will sketch how one can recover "twisted K-theory theta functions," at weak coupling cusps when the H-flux is nonzero.

We believe that one can clarify the relation between K-theory and U-duality by studying the behavior of the M-theory partition function at different cusps of the M-theory moduli space. At a given cusp the summation over fluxes is supported on fluxes which can be related to K-theory. (See, for example, (3.217).) A U-duality invariant formulation of the theory must map the equations defining the support at one cusp to those at any other cusp. This should define the U-duality invariant extension of the K-theory constraints.

3.10.1 The M-theory partition function

Let us consider the contribution to the M-theory partition function from a background Y which is a T^3 fibration over X.

$$ds_{11}^2 = V^{-\frac{1}{3}}\tilde{t}^2 g_{MN} dx^M dx^N + V^{\frac{2}{3}}\tilde{g}_{\mathbf{mn}}\theta^{\mathbf{m}}\theta^{\mathbf{n}} \tag{3.229}$$

where $\theta^{\mathbf{m}} = dx^{\mathbf{m}} + \mathcal{A}^{\mathbf{m}}_{(1)}$ and $x^{\mathbf{m}} \in [0,1]$. $\tilde{t}^2 g_{MN}$ is an Einstein 8D metric, $det g_{MN} = 1$. $\tilde{g}_{\mathbf{mn}}$ and V are the shape and the volume of the T^3 fiber. We denote world indices on T^3 by $\mathbf{m} = (m, 11), m = 8, 9$ and $M = 0, \ldots, 7$ as before.

Topologically, one can specify the T^3 fibration over X by a triplet of line bundles $L^{\mathbf{m}}$ which transform in the representation $\mathbf{3}$ of $SL(3, \mathbf{Z})$ and have first Chern classes $c_1(L^{\mathbf{m}}) = \mathcal{F}^{\mathbf{m}}_{(2)}$, where $\mathcal{F}^{\mathbf{m}}_{(2)} = d\theta^{\mathbf{m}}$. Such a specification is valid up to possible monodromies. These are characterized by a homomorphism $\pi_1(X) \to SL(3, \mathbf{Z})$.

On a manifold Y of the type (3.229) we reduce the M-theory 4-form $G_{M-theory}$ as

$$\frac{G_{M-theory}}{2\pi} = G_{(4)} + G_{(3)\mathbf{m}}\theta^{\mathbf{m}} + \frac{1}{2}\left(F_{(2)\mathbf{mn}} + \varepsilon_{\mathbf{mnk}} B_0 \mathcal{F}^{\mathbf{k}}_{(2)}\right)\theta^{\mathbf{m}}\theta^{\mathbf{n}} \tag{3.230}$$

and we also must include the flat potential

$$c_{(0)} = \frac{1}{6} B_0 \varepsilon_{\mathbf{mnk}} \theta^{\mathbf{m}}\theta^{\mathbf{n}}\theta^{\mathbf{k}} \tag{3.231}$$

in the Kaluza-Klein reduction.[26] (We will list the full set of flat potentials in this background below.)

From the Bianchi identity $dG_{M-theory} = 0$ we have

$$dG_{(4)} = \mathcal{F}^{\mathbf{m}}_{(2)} G_{(3)\mathbf{m}}, \quad dG_{(3)\mathbf{m}} = \mathcal{F}^{\mathbf{n}}_{(2)} F_{(2)\mathbf{mn}} \quad dF_{(2)\mathbf{mn}} = 0 \quad d\mathcal{F}^{\mathbf{m}}_{(2)} = 0 \tag{3.232}$$

which implies that fluxes $G_{(4)}$ and $G_{(3)\mathbf{m}}$ are in general not closed forms.[27]

Let us recall how various fields transform under $\mathcal{D} = SL(3, \mathbf{Z}) \times SL(2, \mathbf{Z})_\rho$ [78].

- $\tilde{t}, g_{MN}$ are U-duality invariant.

[26] $\varepsilon^{11,8,9} = \varepsilon_{11,8,9} = 1$.

[27] In IIA at weak coupling we assumed $G_{(3)11} = 0$ and $\mathcal{F}^{n}_{(2)} = 0, n = 8, 9$, so that all background fluxes are closed forms.

- $SL(2,\mathbf{Z})_\rho$ acts on $\rho = B_0 + iV \in \mathcal{H}$ by fractional linear transformations.

- $SL(3,\mathbf{Z})$ acts on the scalars $\tilde{g}_{\mathrm{mn}}$ parametrizing $SL(3,R)/SO(3)$ via the mapping class group of T^3.

- $\mathbf{F}^{m\alpha}_{(2)} = \begin{pmatrix} F^{\mathrm{m}}_{(2)} \\ \mathcal{F}^{\mathrm{m}}_{(2)} \end{pmatrix}$ transform in the $(\mathbf{3},\mathbf{2})$ of $\mathcal{D}$, where $F^{\mathrm{m}}_{(2)} := \frac{1}{2}\varepsilon^{\mathrm{mnk}} F_{(2)\mathrm{nk}}$.

- $G_{(3)\mathrm{m}}$ transform in the $(\mathbf{3}',\mathbf{1})$ of $\mathcal{D}$

- $G_{(4)}$ is singled out among all the other fields since according to conventional supergravity [78] $SL(2,\mathbf{Z})_\rho$ mixes $G_{(4)}$ with its Hodge dual $*G_{(4)}$. More concretely,

$$\begin{pmatrix} -\mathrm{Re}\rho G_{(4)} + i\mathrm{Im}\rho * G_{(4)} \\ G_{(4)} \end{pmatrix}$$

transforms in the $(\mathbf{1},\mathbf{2})$ of $\mathcal{D}$. Due to this non-trivial transformation the classical bosonic 8D action is not manifestly invariant under $SL(2,\mathbf{Z})_\rho$.

Now, let us consider the real part of the bosonic 8D action

$$Re(S_{8D}) = \pi \int_X \left\{ \mathrm{Im}\rho G_{(4)} \wedge * G_{(4)} + \tilde{t}^2 \tilde{g}^{\mathrm{mn}} G_{(3)\mathrm{m}} \wedge * G_{(3)\mathrm{n}} + \tilde{t}^4 \tilde{g}_{\mathrm{mn}} \mathcal{G}_{\alpha\beta} \mathbf{F}^{\mathrm{m}\alpha}_{(2)} \wedge * \mathbf{F}^{\mathrm{n}\beta}_{(2)} \right\} \tag{3.233}$$

$$+ \frac{1}{2\pi} \int_X \tilde{t}^6 \left\{ \mathcal{R} + 28\tilde{t}^{-2} \partial_M \tilde{t} \partial_M \tilde{t} + \frac{1}{2\rho_2^2} \partial_M \rho \partial^M \bar{\rho} + \frac{1}{3} \tilde{g}^{\mathrm{mn}} \tilde{g}^{\mathrm{kl}} \partial_M \tilde{g}_{\mathrm{mk}} \partial^M \tilde{g}_{\mathrm{nl}} \right\}$$

where $\mathcal{G}_{\alpha\beta}$ is defined in (3.6), $\tilde{g}^{\mathrm{kl}}$ is inverse of $\tilde{g}_{\mathrm{mk}}$ and $\mathcal{R}$ is the Ricci-scalar of the metric g_{MN}.

The imaginary part of the 8D bosonic action follows from the reduction of M-theory phase $\Omega_M(C)$. This phase is subtle to define in topologically nontrivial field configurations of the G-field. It may be formulated in two ways. The first formulation was given in [83]. It uses Stong's result that the spin-cobordism group $\Omega_{11}(K(\mathbf{Z},4)) = 0$ [113]. That is, given a spin 11-manifold Y and a 4-form flux $\frac{G}{2\pi}$ one can always find a bounding spin 12-manifold Z and an extension $\tilde{G}$ of the the flux to Z. In these terms the M-theory phase $\Omega_M(C)$ is given as:

$$\Omega_M(C) = \epsilon \exp\left[\frac{2\pi i}{6} \int_Z \tilde{G}^3 - \frac{2\pi i}{48} \int_Z \tilde{G}(p_2 - \lambda^2) \right] \tag{3.234}$$

Here ϵ is the sign of the Rarita-Schwinger determinant. The phase does not depend on the choice of bounding manifold Z, but does depend on the "trivializing" C-field at the boundary Y.

A second formulation [68, 70, 103] proceeds from the observation of [83] that the integrand of (3.234) may be identified as the index density for a Dirac operator coupled to an E_8 vector bundle. The M-theory 4-form can be formulated in the following terms [68, 70, 103]. We set:

$$\frac{G_{M-theory}}{2\pi} = \bar{G} + dc \tag{3.235}$$

where $\bar{G} = \frac{1}{60}Tr_{248}\frac{F^2}{8\pi^2} + \frac{1}{32\pi^2}TrR^2$, F is the curvature of a connection A on an E_8 bundle V on Y and R is the curvature of the metric connection on TY. $G_{M-theory}$ is a real differential form, and $c \in \Omega^3(Y, R)/\Omega^3_Z(Y)$, where $\Omega^3_Z(Y)$ are 3-forms with integral periods. The pair (A, c) is subject to an equivalence relation. In these terms the M-theory phase is expressed as:

$$\Omega_M(C) = exp\left[2\pi i\Big(\frac{\eta\,(D_V) + h\,(D_V)}{4} + \frac{\eta\,(D_{RS}) + h\,(D_{RS})}{8}\Big)\right]\omega(c) \tag{3.236}$$

where D_V is the Dirac operator coupled to the connection A on the E_8 bundle V, which enters the definition of M-theory 4-form in (3.235). D_{RS} is the Rarita-Schwinger operator, $h(D)$ is the number of zeromodes of the operator D on Y, and $\eta(D)$ is the η invariant of Atiyah-Patodi-Singer. The phase $\omega(c)$ is given by

$$\omega(c) = exp\Big[\pi i \int_Y \Big(c(\bar{G}^2 + X_8) + cdc\bar{G} + \frac{1}{3}c(dc)^2\Big)\Big] \tag{3.237}$$

3.10.2 The semiclassical expansion

For large $\tilde{t}$ there is a well-defined semiclassical expansion of the M-theory partition function, which follows from the appearance of kinetic terms in the action (3.233) scaling as $\tilde{t}^{2k}$ for $k = 0, 1, 2, 3$. In the leading approximation we can fix all the fields except $G_{(4)}$, but this last field must be treated quantum mechanically. Note that this semiclassical expansion can differ from that described above because we do not necessarily require weak string coupling. In the second approximation we treat $G_{(4)}$ and $G_{(3)\mathbf{n}}$ as quantum fields, and so on.

In the leading approximation in addition to the sum over fluxes $G_{(4)}$ we must integrate over the flat potentials. These include flat connection $A^{\mathbf{m}}_{(1)}$ of the T^3 fibration and potentials coming from KK reduction of c

$$c = C'_{(3)} + C'_{(2)\mathbf{m}}\theta^{\mathbf{m}} + \frac{1}{2}C_{(1)\mathbf{mn}}\theta^{\mathbf{m}}\theta^{\mathbf{n}} + c_{(0)} \tag{3.238}$$

where $C'_{(2)\mathbf{m}} = C_{(2)\mathbf{m}} - \frac{1}{2}C_{(1)\mathbf{pm}}\mathcal{A}^{\mathbf{p}}_{(1)}$ and $C'_{(3)} = C_{(3)} - C'_{(2)\mathbf{m}}\mathcal{A}^{\mathbf{m}}_{(1)}$, and $c_{(0)}$ is defined in (3.231). $C_{(3)}$ is invariant under U-duality, $C_{(2)\mathbf{m}}$ transforms in the $(\mathbf{3},\mathbf{1})$ of $\mathcal{D}$. We can combine the flat potentials $C_{(1)\mathbf{mn}}$ and $\mathcal{A}^{\mathbf{m}}_{(1)}$ in the U-duality multiplet of $\mathcal{D}$ transforming as $(\mathbf{3},\mathbf{2})$ by writing

$$\mathbf{A}^{\mathbf{m}\alpha}_{(1)} = \begin{pmatrix} \frac{1}{2}\varepsilon^{\mathbf{mnk}}C_{(1)\mathbf{nk}} \\ \mathcal{A}^{\mathbf{m}}_{(1)} \end{pmatrix}. \tag{3.239}$$

The duality invariance in the leading approximation is straightforward to check. We keep only $G_{(4)}$. The flux is quantized by $[G_{(4)}] = a - \frac{1}{2}\lambda$, where $a \in H^4(X,\mathbf{Z})$ is the characteristic class of the E_8 bundle and λ is the characteristic class of the spin bundle. We sum over $a \in H^4(X,\mathbf{Z})$. The 8D action, including the imaginary part is $SL(3,\mathbf{Z})$ invariant. The imaginary part of the 8D effective action in this case takes a simple form which can be found from (3.236) :

$$Im(S_{8D}) = -\pi \int_X \left(a \cup \lambda + B_0 \left(a - \frac{1}{2}\lambda\right)^2\right) \tag{3.240}$$

The invariance under $SL(2,\mathbf{Z})_\rho$ then follows in the same way as our discussion in weak string coupling regime.

Let us now try to go beyond the first approximation. In the second approximation $[G_{(4)}] = a - \frac{1}{2}\lambda + [\mathcal{A}^{\mathbf{m}}_{(1)}G_{(3)\mathbf{m}}]$. We allow nonzero fluxes $G_{(3)\mathbf{m}}$, but still set to zero the fieldstrengths $F_{(2)}$ and $\mathcal{F}_{(2)}$. We thus have a family of tori with flat connections. Already in the second approximation, when we switch on nonzero fluxes $G_{(3)\mathbf{m}}$ there does not appear to be a simple expression for the M-theory phase.

Nevertheless, one can get some information about the M-theory phase from the requirement of U-duality invariance. We know that $SL(3,\mathbf{Z})$ invariance is again manifest from the definition of $\Omega_M(C)$ and $\mathrm{Re}(S_{8D})$. But the expected $SL(2,\mathbf{Z})_\rho$ invariance gives nontrivial information about $\Omega_M(C)$.

To state these nontrivial properties of $\Omega_M(C)$ let us write M-theory partition function in the second approximation

$$Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, \rho) = \int d\mu_{flat} \sum_{G_{(3)\mathbf{m}}} Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, G_{(3)\mathbf{m}}, \rho) \tag{3.241}$$

where $d\mu_{flat}$ stands for the integration over

$$\frac{\mathcal{H}^3(X)}{\mathcal{H}^3_{\mathbf{Z}}(X)} \times \left(\frac{\mathcal{H}^2(X)}{\mathcal{H}^2_{\mathbf{Z}}(X)}\right)^3 \times \left(\frac{\mathcal{H}^1(X)}{\mathcal{H}^1_{\mathbf{Z}}(X)}\right)^6 \tag{3.242}$$

where $\mathcal{H}^p(X)$ is a space of harmonic p-forms on X and $\mathcal{H}^p_{\mathbf{Z}}(X)$ is the lattice of integrally normalized harmonic p-forms on X. The first factor is for $C_{(3)}$, the second factor for $C_{(2)\mathbf{m}}$ and the third factor is for the fields $\mathbf{A}^{\mathbf{m}\alpha}_{(1)}$ transforming in the $(\mathbf{3},\mathbf{2})$ of $\mathcal{D}$. The integration measure $d\mu_{flat}$ is U-duality invariant.

The summand in (3.241) with fixed $G_{(3)\mathbf{m}}$ is given by

$$Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, G_{(3)\mathbf{m}}, \rho) = \sum_{a \in H^4(X,\mathbf{Z})} Det(G_{(4)}, G_{(3)\mathbf{m}}) e^{-S_{quant}} e^{-S_{cl}} \tag{3.243}$$

where

$$e^{-S_{cl}} = \Omega_M\left(G_{(4)}, G_{(3)\mathbf{m}}, B_0\right) e^{-\pi \int_X \left(Im(\rho) G_{(4)} \wedge * G_{(4)} + \tilde{t}^2 \tilde{g}^{\mathbf{mn}} G_{(3)\mathbf{m}} \wedge * G_{(3)\mathbf{n}}\right)}$$

and $Det(G_{(4)}, G_{(3)\mathbf{m}})$ denotes 1-loop determinants. These depend implicitly on the scalars $\rho, \tilde{g}_{\mathbf{mn}}, \tilde{t}$ as well as on the metric g_{MN}.

The M-theory phase Ω_M in (3.243) depends on the fieldstrenths $G_{(4)}, G_{(3)\mathbf{m}}$ and the flat potentials, but it is metric-independent, and hence should be a topological invariant. The dependence of Ω_M on flat potentials is explicit from (3.237) for c as in (3.238). For example dependence of Ω_M on B_0 has the form

$$e^{i\pi \int_X B_0 G_{(4)} G_{(4)}} \tag{3.244}$$

(We choose to include 1-loop corrections $\int_X B_0 X_8$ together with effect of membrane instantons in S_{quant}.) The nontrivial question is dependence on $G_{(4)}$ and $G_{(3)\mathbf{m}}$ which also comes from $\eta(D_V) + h(D_V)$.

The independence of Ω_M on the metric on $Y = X \times T^3$ (in the second approximation) follows from the standard variation formula for η-invariant. To show this let us fix the connection on the E_8 bundle V with curvature F and consider the family of veilbeins $e(s)$ on $Y = X \times T^3$ parametrized by $s \in [0,1]$ such that the metric on T^3 remains flat and independent of the coordinates on X. The corresponding family of Rieman tensors $\mathcal{R}(s)$ gives an A-roof genus $\hat{A}(s)$ which is a pullback from $X \times [0,1]$. Now we can write the standard formula for the change in η-invariant under the variation of veilbein [110]:

$$\eta(e(1)) - \eta(e(0)) = j + \int_{Y\times[0,1]} ch(V)\hat{A}(s) \tag{3.245}$$

where integer j is a topological invariant of $Y \times [0,1]$ and $ch(V) := \frac{1}{30}Tr_{248}e^{\frac{iF_C}{2\pi}}$ is defined in terms of the curvature F_C on the complexification of the bundle V. In the second approximation we only switch on $\bar{G} = G_{(4)} + G_{(3)\mathbf{m}}dx^m$ so that neither $ch_2(V) = -2(\bar{G} + \frac{1}{2}\lambda)$ nor $ch_4(V) = \frac{1}{5}(\bar{G} + \frac{1}{2}\lambda)^2$ have a piece $\sim dx^8dx^9dx^{11}$ and integral in (3.245) vanishes.

Now we come to the main point. The requirement of the invariance under the standard generators S, T of $SL(2,\mathbf{Z})_\rho$

$$Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, -1/\rho) = Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, \rho) \tag{3.246}$$

$$Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, \rho+1) = Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, \rho) \tag{3.247}$$

gives a nontrivial statement about the properties of the function $\Omega_M(G_{(4)}, G_{(3)\mathbf{m}}, B_0)$.

The sum over fluxes $G_{(3)\mathbf{m}} \in H^3(X,\mathbf{Z})$ in (3.241) might be entirely supported by classes which satisfy a system of $SL(3,\mathbf{Z})$ invariant constraints. These constraints can in principle be determined by summing over torsion classes once the phase Ω_M is known in sufficently explicit terms. In the simple case when $G_{(3)\mathbf{m}}$ are all 2-torsion classes, one can act by the generators of $SL(3,\mathbf{Z})$ on the constraint

$$Sq^3(G_{(3)9}) + Sq^3(G_{(3)11}) + G_{(3)9} \cup G_{(3)11} = 0 \tag{3.248}$$

which follows from [68] and get

$$G_{(3)\mathbf{m}} \cup G_{(3)\mathbf{n}} = 0, \quad \mathbf{m},\mathbf{n} = 8, 9, 11 \tag{3.249}$$

3.10.3 Comment on the connection with twisted K-theory

In this section we discuss the behavior of the partition function near a weak-coupling cusp. There is a twisted version of K-theory which is thought to be related to the classification of D-brane charges in the presence of nonzero NSNS H-flux [84, 109, 87, 108]. It is natural to ask if the contributions to the M-theory partition function $Z_{M-theory}(\tilde{g}_{\mathbf{mn}}, \rho)$ from fluxes with nonzero $H_{(3)} := G_{(3)11} \in H^3(X, \mathbf{Z})$ are related, in the weak string-coupling cusp, to some kind of twisted K-theory theta function.

The weak-coupling cusp may be described by relating the fields in (3.229) to the fields in IIA theory. First, the scale $\tilde{t}$ is related to the expansion parameter used in our previous sections by $\tilde{t}^2 = e^{-\frac{2}{3}\xi}t^2$. Next, we parametrize the shape of T^3 as $\tilde{g}_{\mathbf{mn}} = e^{\mathbf{a}}{}_{\mathbf{m}} e^{\mathbf{b}}{}_{\mathbf{n}} \delta_{\mathbf{ab}}$ where

$$e^{\mathbf{a}}{}_{\mathbf{m}} = \begin{pmatrix} \frac{e^{-\xi/3}}{\sqrt{\tau_2}} & 0 & 0 \\ 0 & e^{-\xi/3}\sqrt{\tau_2} & 0 \\ 0 & 0 & e^{2\xi/3} \end{pmatrix} \begin{pmatrix} 1 & \tau_1 & C_{(0)8} \\ 0 & 1 & C_{(0)9} \\ 0 & 0 & 1 \end{pmatrix} \tag{3.250}$$

We denote frame indices by $\mathbf{a} = (a, 11), a = 8, 9$. The weak coupling cusp may be written as

$$R \times R^2 \times SL(2, R)/SO(2) \tag{3.251}$$

where the first factor is for the dilaton ξ, the second for $C_{(0)8}, C_{(0)9}$,[28] and the third for the modular parameter τ of the IIA torus.

As far as we know, nobody has precisely defined what should be meant by the " K_H theta function." Since the Chern character has recently been formulated in [104], this should be possible. Nevertheless, even without a precise definition we do expect it to be a sum over a "Lagrangian" sublattice of $K_H(X_8 \times T^2)$. At the level of DeRham cohomology, this should be a "maximal Lagrangian" sublattice of $\ker d_3/\mathrm{Im} d_3$ where $d_3 : H^*(X_{10}, \mathbf{Z}) \to H^*(X_{10}, \mathbf{Z})$ is the differential $d_3(\omega) = \omega \wedge H_{(3)}$. Using the filtration implied by the semiclassical expansion, and working to the approximation of e^{-t^2} this means that we should first define a sublattice of the cohomology lattice by the set of integral cohomology classes $(a, G_{(3)8}, G_{(3)9})$ such

[28]These are related to the RR potentials $\tilde{C}_{(0)m}$ transforming in the $\mathbf{2}'$ of $SL(2, \mathbf{Z})_\tau$ as $C_{(0)8} = e^{\xi}\sqrt{\tau_2}\tilde{C}_{(0)8}$, $C_{(0)9} = e^{\xi}\frac{1}{\sqrt{\tau_2}}\tilde{C}_{(0)9}$

that $(G_{(4)}, G_{(3)8}, G_{(3)9})$ are in the kernel of d_3:

$$H_{(3)} \wedge G_{(4)} = 0, \quad H_{(3)} \wedge G_{(3)m} = 0, \quad m = 8, 9 \tag{3.252}$$

Then the theta function should be a sum over the quotient lattice obtained by modding out by the image of d_3

$$G_{(3)8} \sim G_{(3)8} - pH_{(3)}, \quad G_{(3)9} \sim G_{(3)9} - sH_{(3)}, \quad G_{(4)} \sim G_{(4)} - \omega_{(1)}H_{(3)}. \tag{3.253}$$

Here $p, s \in \mathbf{Z}$ and $\omega_{(1)} \in H^1(X, \mathbf{Z})$. Thus, our exercise is to describe how a sum over this quotient lattice emerges from (3.241).

Let us consider the couplings of flat potentials $C_{(1)89}$ and $C_{(2)m}$ to the fluxes which follow from (3.237):

$$e^{i2\pi \int_X C_{(1)89} H_{(3)} G_{(4)}} e^{i2\pi \int_X \epsilon^{mn} C_{(2)m} G_{(3)n} H_{(3)}} \tag{3.254}$$

Integrating over $C_{(1)89}$ and $C_{(2)m}$ gives $H_{(3)} \wedge G_{(4)} = 0$ and $\epsilon^{mn} H_{(3)} \wedge G_{(3)n} = 0$ respectively.

Next, we note that, due to the $SL(3, \mathbf{Z})$ invariance of the M-theory action we have (suppressing many irrelevant variables)

$$Z_{M-theory}(C_{(0)m}, G_{(3)m} - p_m H_{(3)}, \mathcal{A}^{11}_{(1)}, G_{(4)} - \omega_{(1)} H_{(3)}) = \tag{3.255}$$

$$Z_{M-theory}(C_{(0)m} + p_m, G_{(3)m}, \mathcal{A}^{11}_{(1)} + \omega_{(1)}, G_{(4)})$$

Now we use (3.255) to write the sum over all fluxes $G_{(4)}, G_{(3)m}, \quad m = 8, 9$ in the kernel of d_3 as

$$Z_H = \sum_{d_3-kernel} Z_{M-theory}(C_{(0)m}, G_{(3)m}, \mathcal{A}^{11}_{(1)}, G_{(4)}) = \sum_{\mathcal{M}_{fund}} W \tag{3.256}$$

where $\mathcal{M}_{fund}$ stands for the fluxes in the fundamental domain for the image of d_3 within the kernel of d_3 and

$$W = \sum_{p_m \in \mathbf{Z}^2} \sum_{\omega_{(1)} \in H^1(X, \mathbf{Z})} Z_{M-theory}(C_{(0)m} + p_m, G_{(3)m}, \mathcal{A}^{11}_{(1)} + \omega_{(1)}, G_{(4)}) \tag{3.257}$$

Now, we can recognize that Z_H descends naturally to the quotient of the weak-coupling cusp.

$$\Gamma'_\infty \backslash \left[R \times R^2 \times SL(2, R)/SO(2) \right] \tag{3.258}$$

where Γ'_∞ is the subgroup of the parabolic group Γ_∞ and given by:

$$L_{\mathbf{m}}{}^{\mathbf{n}} = \begin{pmatrix} 1 & 0 & p \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \quad p, s \in Z \tag{3.259}$$

Written this way, Z_H is clearly a sum over a Lagrangian sublattice of the $K_H(X \times T^2)$ lattice. (in the approximation of working in the DeRham theory, with the filtration appropriate to the second approximation.)

The interesting point that we learn from this exercise is that in formulating the K_H theta function, the weighting factor for the contribution of a class in K_H should be given by (3.257). The dependence of the action on the integers p_m and $\omega_{(1)} \in H^1(X, \mathbf{Z})$ is $\exp[-Q(p_m, \omega_{(1)})]$ where Q is quadratic form. Therefore W is itself already a theta function. This follows since the dependence on $C_{(0)m}$ and $\mathcal{A}^{11}_{(1)}$ comes entirely from the real part of the classical action (3.233), since, as we have shown, the phase is independent of the metric on $X \times T^3$. The dependence on $C_{(0)m}$ comes from $\int_X \tilde{t}^2 \tilde{g}^{\mathbf{mn}} G_{(3)\mathbf{m}} \wedge * G_{(3)\mathbf{n}}$ and the dependence on $\mathcal{A}^{11}_{(1)}$ from $\int_X Im\rho G_{(4)} \wedge * G_{(4)}$, where we recall that $G_{(4)} = a - \frac{1}{2}\lambda + \mathcal{A}^{\mathbf{m}}_{(1)} G_{(3)\mathbf{m}}$.

It would be very interesting to see if the function Z_H defined in (3.256) is in accord with a mathematically natural definition of a theta function for twisted K-theory. But we will leave this for future work.

As an example, let us consider $X = SU(3)$. Let x_3 generate $H^3(X, \mathbf{Z})$. Then fixing $H_{(3)} = kx_3$ we find that the fundamental domain of the image of d_3 within the kernel of d_3 is given by

$$G_{(3)8} = rx_3, \quad G_{(3)9} = px_3, \quad 0 \leq r, p \leq k-1 \tag{3.260}$$

so that the sum over RR fluxes in (3.256) is finite and in this sense RR fluxes are k-torsion. This example of $X = SU(3)$ is especially interesting since it is well known [107, 106, 108] that at weak string coupling D-brane charges on $SU(3)$ in the presence of $H_{(3)} = kx_3$ are classified by twisted K-theory groups of $SU(3)$, and these groups are k-torsion. As argued in [81], from Gauss's law it is then natural to expect that RR fluxes are also k-torsion. This is indeed what we find in (3.260).

Appendix A

Here we give the explicit expressions for representations of S and T in $Sp(2N, \mathbf{Z})$. Let us choose the following basis of the lattice Γ

$$\vec{\mathbf{x}} = \left(\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2\right) \tag{A.1}$$

$$\vec{\mathbf{x}}_1 = \Big(y_l\mathbf{1}, y_l \otimes (L(e_0) - \mathbf{1}), (L(e_s) - \mathbf{1}), (L(e_s) - \mathbf{1}) \otimes (L(e_0) - \mathbf{1}), (L(\gamma_r d\sigma^m) - \mathbf{1}), \quad x(f_k d\sigma^m), x(\omega_i)\Big) \tag{A.2}$$

$$\vec{\mathbf{x}}_2 = \Big(x(\omega_i) \otimes (L(e_0) - \mathbf{1}), x(d_k d\sigma^m), x(w_r d\sigma^m), x(u_s), x(u_s) \otimes (L(e_0) - \mathbf{1}), \quad x(h_l), x(h_l) \otimes (L(e_0) - \mathbf{1})\Big) \tag{A.3}$$

where we introduce

$$y_l \in H^0(X, \mathbf{Z}), \quad h_l \in H^8(X, \mathbf{Z}), \quad l = 1, \ldots, b_0$$

$$\gamma_r \in H^1(X, \mathbf{Z}), \quad w_r \in H^7(X, \mathbf{Z}), \quad r = 1, \ldots b_1$$

$$e_s \in H^2(X, \mathbf{Z}), \quad u_s \in H^6(X, \mathbf{Z}) \quad s = 1, \ldots, b_2,$$

$$f_k \in H^3(X, \mathbf{Z}), \quad d_k \in H^5(X, \mathbf{Z}), k = 1, \ldots, \mathbf{b_3}, \quad \omega_i \in H^4(X, \mathbf{Z}), i = 1, \ldots, b_4,$$

where b_p is the rank of $H^p(X, \mathbf{Z})$ and $\mathbf{b_3}$ is the rank of the sublattice of $H^3(X, \mathbf{Z})$ which is span by classes f such that $Sq^3 f = Sq^3 \hat{a}_0$.

In the above basis the generators S and T are represented by

$$\sigma(S) = \begin{pmatrix} A(S) & B(S) \\ C(S) & D(S) \end{pmatrix}, \quad \sigma(T) = \begin{pmatrix} A(T) & B(T) \\ C(T) & D(T) \end{pmatrix} \tag{A.4}$$

$$A(S)=\begin{pmatrix} & \mathbf{1}_{b_0} & & & & & \\ -\mathbf{1}_{b_0} & & & & & & \\ & & & \mathbf{1}_{b_2} & & & \\ & & -\mathbf{1}_{b_2} & & & & \\ & & & & \mathbf{1}_{2b_1} & & \\ & & & & & \mathbf{1}_{2\mathbf{b_3}} & \\ & & & & & & \mathbf{0}_{b_4} \end{pmatrix} \quad B(S)=\begin{pmatrix} \\ \\ \\ \\ \\ \\ \mathbf{1}_{b_4} \end{pmatrix} \tag{A.5}$$

$$C(S)=\begin{pmatrix} -\mathbf{1}_{b_4} & & & & & & \end{pmatrix} \quad D(S)=\begin{pmatrix} \mathbf{0}_{b_4} & & & & & & \\ & \mathbf{1}_{2\mathbf{b_3}} & & & & & \\ & & \mathbf{1}_{2b_1} & & & & \\ & & & & \mathbf{1}_{b_2} & & \\ & & & -\mathbf{1}_{b_2} & & & \\ & & & & & & \mathbf{1}_{b_0} \\ & & & & & -\mathbf{1}_{b_0} & \end{pmatrix} \tag{A.6}$$

$$A(T)=\begin{pmatrix} \mathbf{1}_{b_0} & & & & & & \\ -\mathbf{1}_{b_0} & \mathbf{1}_{b_0} & & & & & \\ & & \mathbf{1}_{b_2} & & & & \\ & & -\mathbf{1}_{b_2} & \mathbf{1}_{b_2} & & & \\ & & & & \mathbf{1}_{2b_1} & & \\ & & & & & \mathbf{1}_{2\mathbf{b_3}} & \\ & & & & & & \mathbf{1}_{b_4} \end{pmatrix} \quad B(T)=\mathbf{0}_N \tag{A.7}$$

$$C(T)=\mathbf{0}_N, \quad D(T)=\begin{pmatrix} \mathbf{1}_{b_4} & & & & & & \\ & \mathbf{1}_{2\mathbf{b_3}} & & & & & \\ & & \mathbf{1}_{2b_1} & & & & \\ & & & \mathbf{1}_{b_2} & & & \\ & & & -\mathbf{1}_{b_2} & \mathbf{1}_{b_2} & & \\ & & & & & \mathbf{1}_{b_0} & \\ & & & & & -\mathbf{1}_{b_0} & \mathbf{1}_{b_0} \end{pmatrix} \tag{A.8}$$

Appendix B

The 10D fields that we use are related to the fields in [91] as:

$$\frac{\tilde{G}_4}{\sqrt{2\pi}} = e^{-\frac{3\phi}{4}} F_4^{Rom}, \quad \frac{G_2}{\sqrt{2\pi}} = -e^{-\frac{9\phi}{4}} F_2^{Rom}, \qquad \frac{\hat{B}_2}{\sqrt{2\pi}} = -e^{\frac{3\phi}{2}} B_2^{Rom}, \quad m = G_0 e^{\frac{15\phi}{4}},$$

$$\hat{\psi}_{\hat{A}} = e^{-\frac{\phi}{8}} \psi_{\hat{A}}^{(Rom)}, \quad \hat{\Lambda} = e^{-\frac{\phi}{8}} \lambda^{(Rom)}, g_{\hat{M}\hat{N}} = e^{\frac{1}{2}\phi} g_{\hat{M}\hat{N}}^{(Rom)}$$

We also remind that we set $k_{11} = \pi$ while in [91] $k_{11} = \sqrt{2\pi}$ was assumed.

Appendix C

Below we collect 4-fermionic terms in D=10 IIA supergravity action which are obtained from circle reduction of the D=11 action of [96].

$$S^{(10)}_{4-ferm} = \frac{\pi}{2}\int \sqrt{-g_{10}}e^{-2\phi}\Big\{-\frac{1}{64}\left[\bar{\chi}_{\mathbf{E}}\hat{\Gamma}^{\mathbf{ABCDEF}}\chi_{\mathbf{F}} + 12\bar{\chi}^{[\mathbf{A}}\hat{\Gamma}^{\mathbf{BC}}\chi^{\mathbf{D}]}\right]\bar{\chi}_{[\mathbf{A}}\hat{\Gamma}_{\mathbf{BC}}\chi_{\mathbf{D}]} \tag{C.1}$$

$$+\frac{1}{32}\left(\bar{\chi}_{\mathbf{E}}\hat{\Gamma}^{\mathbf{ABCEF}}\chi_{\mathbf{F}}\right)\left(\bar{\chi}_{\mathbf{A}}\hat{\Gamma}_{\mathbf{B}}\chi_{\mathbf{C}}\right) + \frac{1}{4}\left(\bar{\chi}_{\mathbf{A}}\hat{\Gamma}^{\mathbf{A}}\chi_{\mathbf{C}}\right)\left(\bar{\chi}_{\mathbf{B}}\hat{\Gamma}^{\mathbf{B}}\chi^{\mathbf{C}}\right)$$

$$-\frac{1}{8}\left(\bar{\chi}_{\mathbf{A}}\hat{\Gamma}^{\mathbf{B}}\chi_{\mathbf{C}}\right)\left(\bar{\chi}_{\mathbf{B}}\hat{\Gamma}^{\mathbf{A}}\chi^{\mathbf{C}}\right) - \frac{1}{16}\left(\bar{\chi}_{\mathbf{A}}\hat{\Gamma}_{\mathbf{B}}\chi_{\mathbf{C}}\right)\left(\bar{\chi}^{\mathbf{A}}\hat{\Gamma}^{\mathbf{B}}\chi^{\mathbf{C}}\right)\Big\}$$

where

$$\chi_{\hat{A}} = \left[\hat{\psi}_{\hat{A}} + \frac{1}{6\sqrt{2}}\hat{\Gamma}_{\hat{A}}\hat{\Lambda}\right]$$

$$\chi_{11} = -\frac{2\sqrt{2}}{3}\hat{\Gamma}^{11}\hat{\Lambda}$$

and $\mathbf{A} = (\hat{A}, 11)$.

Recall that the graviton $\mathbf{E}^{\mathbf{A}}_{\mathbf{M}}$ and the gravitino $\psi^{(11)}_{\mathbf{A}}$ of 11D supergravity are related to 10D fields as [96]:

$$\mathbf{E}^{\hat{A}}_{\hat{M}} = e^{-\frac{\phi}{3}}\hat{E}^{\hat{A}}_{\hat{M}}, \quad \mathbf{E}^{11}_{11} = e^{\frac{2\phi}{3}}, \quad \mathbf{E}^{11}_{\hat{M}} = e^{\frac{2\phi}{3}}C_{\hat{M}}$$

$$\psi^{(11)}_{\mathbf{A}} = \frac{1}{\sqrt{2\pi}}e^{\frac{\phi}{6}}\chi_{\mathbf{A}}$$

Appendix D

Below we collect terms in the 8D quantum action which are bilinear in FP ghosts and bilinear in fermions:

$$S_{bc}^{(2)2} = \frac{\pi}{2}\int_X t^8 e^{-2\xi}\Bigg\{\frac{1}{8}\left(\bar{\chi}_{\mathbf{B}}\hat{\Gamma}_A\chi_{\mathbf{C}} + 2\bar{\chi}_A\hat{\Gamma}_{\mathbf{B}}\chi_{\mathbf{C}}\right)\left(\bar{\hat{b}}\hat{\Gamma}^A\hat{\Gamma}^{\mathbf{BC}}\hat{c}\right) + \tag{D.1}$$

$$\frac{1}{6}\left(\bar{\chi}_{\mathbf{B}}\hat{\Gamma}_{\bar{a}}\chi_{\mathbf{C}} + 2\bar{\chi}_{\bar{a}}\hat{\Gamma}_{\mathbf{B}}\chi_{\mathbf{C}}\right)\left(\bar{\hat{b}}\hat{\Gamma}^{\bar{a}}\hat{\Gamma}^{\mathbf{BC}}\hat{c}\right)$$

$$+\frac{1}{6}\left(\bar{\chi}_A\hat{\Gamma}_{\mathbf{BC}}\chi_{\mathbf{D}}\right)\left(\bar{\hat{b}}\hat{\Gamma}^{A\mathbf{BCD}}\hat{c}\right) + \frac{2}{9}\left(\bar{\chi}_{\bar{a}}\hat{\Gamma}_{\mathbf{BC}}\chi_{\mathbf{D}}\right)\left(\bar{\hat{b}}\hat{\Gamma}^{\bar{a}\mathbf{BCD}}\hat{c}\right)$$

$$-\frac{1}{48}\bar{\hat{b}}\left[\hat{\Gamma}_A\hat{\Gamma}^{A\mathbf{BCDE}} + \frac{4}{3}\hat{\Gamma}_{\bar{a}}\hat{\Gamma}^{\bar{a}\mathbf{BCDE}}\right]\hat{c}\left(\bar{\chi}_{\mathbf{B}}\hat{\Gamma}_{\mathbf{CD}}\chi_{\mathbf{E}}\right)$$

$$-\left(\bar{\hat{c}}\hat{\Gamma}^{\mathbf{B}}\chi_A\right)\left(\bar{\hat{b}}\hat{\Gamma}^A\chi_{\mathbf{B}}\right) - \frac{4}{3}\left(\bar{\hat{c}}\hat{\Gamma}^{\mathbf{B}}\chi_{\bar{a}}\right)\left(\bar{\hat{b}}\hat{\Gamma}^{\bar{a}}\chi_{\mathbf{B}}\right) + \frac{1}{4}\left(\bar{\hat{c}}\hat{\Gamma}^{11}\chi_{11}\right)\left[\bar{\hat{b}}\hat{\Gamma}^A\chi_A + \frac{4}{3}\bar{\hat{b}}\hat{\Gamma}^{\bar{a}}\chi_{\bar{a}}\right]$$

$$+L_{A\bar{a}}\left(\bar{\hat{b}}\hat{\Gamma}^A\chi^{\bar{a}}\right) + \frac{4}{3}L_{\bar{a}\mathbf{D}}\left(\bar{\hat{b}}\hat{\Gamma}^{\bar{a}}\chi^{\mathbf{D}}\right) + \frac{1}{4}L_{\mathbf{DE}}\bar{\hat{b}}\left(\hat{\Gamma}^A\hat{\Gamma}^{\mathbf{DE}}\chi_A + \frac{4}{3}\hat{\Gamma}^{\bar{a}}\hat{\Gamma}^{\mathbf{DE}}\chi_{\bar{a}}\right)\Bigg\}$$

where we now split indices as $\mathbf{A} = (A, \bar{a})$, $A = 0, \ldots 7$, $\bar{a} = (a, 11)$, $a = 8, 9$. Nonzero components of $L_{\mathbf{DE}}$ are given by:

$$L_{A\bar{d}} = -\bar{\hat{c}}\hat{\Gamma}_A\chi_{\bar{d}}, \quad L_{a11} = -\bar{\hat{c}}\hat{\Gamma}_a\chi_{11}$$

$S_{bc}^{(2)2}$ is obtained by relating 8D gauge field $\hat{\psi}^A_{(8D)}$(gauge parameter $\hat{\epsilon}$) to 11D gravitino $\psi_{\mathbf{A}}^{(11)}$ (gauge parameter $\epsilon^{(11)}$) as

$$\hat{\psi}^A_{(8D)} = \sqrt{2\pi}e^{-\frac{\phi}{6}}\left[\psi_A^{(11)} + \frac{1}{6}\hat{\Gamma}_A\hat{\Gamma}^{\bar{a}}\psi_{\bar{a}}^{(11)}\right], \quad \hat{\epsilon} = \sqrt{2\pi}e^{\frac{\phi}{6}}\epsilon^{(11)}$$

Let us also remind a standard fact that to keep the gauge

$$\mathbf{E}_{11}^{\hat{A}} = 0, \quad \mathbf{E}_m^A = 0$$

used in reduction from 11D one has to accompany supersymmetry transformations of [96] with field dependent Lorentz transformations.

The last line in the action $S_{bc}^{(2)2}$ originates from such Lorentz transformations.

To write out $S_{bc}^{(2)2}$ in terms of 8D fields

$$\hat{\psi}^A_{(8D)} := \begin{pmatrix} \psi^A \\ \eta^A \end{pmatrix}, \quad \hat{\Lambda}_{(8D)} := \begin{pmatrix} \Sigma \\ \Lambda \end{pmatrix}, \quad \hat{\theta}_{(8D)} := \begin{pmatrix} l \\ \mu \end{pmatrix}, \quad \hat{\nu}_{(8D)} := \begin{pmatrix} \tilde{l} \\ \tilde{\mu} \end{pmatrix}$$

one should substitute

$$\chi^A = \hat{\psi}^A_{(8D)} + \frac{1}{12}\hat{\Gamma}^A\hat{\theta}_{(8D)} + \frac{\sqrt{2}}{6}\hat{\Gamma}^A\hat{\Lambda}_{(8D)}, \quad A = 0, \dots, 7$$

$$\chi^8 = \frac{1}{2}\hat{\nu}_{(8D)} + \frac{1}{3}\hat{\Gamma}^8\left(\hat{\theta}_{(8D)} + \sqrt{2}\hat{\Lambda}_{(8D)}\right), \quad \chi^9 = \frac{1}{2}\hat{\Gamma}^{89}\hat{\nu}_{(8D)} + \frac{1}{3}\hat{\Gamma}^9\left(\hat{\theta}_{(8D)} + \sqrt{2}\hat{\Lambda}_{(8D)}\right)$$

$$\chi_{11} = -\frac{2\sqrt{2}}{3}\hat{\Gamma}^{11}\left(\hat{\Lambda}_{(8D)} - \frac{\sqrt{2}}{4}\hat{\theta}_{(8D)}\right)$$

We do not present the final expression but we have checked that $S_{bc}^{(2)2}$ is T-duality invariant.

Appendix E

Here we explain why $det'\Delta_p$ are divided by V_p in (3.123). This is related to the integration over zeromodes.

Introducing a basis $a^i_{(p)}$, $i = 1, \ldots, b^p$ in $\mathcal{H}^p_{\mathbf{Z}}$ let us denote

$$V_p^{ij} = \int_X a^i_{(p)} \wedge * a^j_{(p)}, \quad V_p = det_{i,j} V_p^{ij} \tag{E.1}$$

Note, that V_p is invariant under the choice of basis in $\mathcal{H}^p_{\mathbf{Z}}$.

To explain integration over fermionic zero modes let us consider the following path-integral over fermionic p-forms u and v.

$$\int DuDv \left[\prod_{i=1}^{b^p} \int_{\gamma_i} u \prod_{j=1}^{b^p} \int_{\gamma_j} v \right] e^{-(v, \Delta_p u)} \tag{E.2}$$

where $\gamma_i, i = 1, \ldots b^p$ is a basis of $H_p(X, \mathbf{Z})$.

In (E.2) we have inserted $\prod_{i=1}^{b^p} \int_{\gamma_i} u \prod_{j=1}^{b^p} \int_{\gamma_j} v$, to get non-zero answer, i.e. to saturate fermion zero modes.

To perform the integration in (E.2) we expand u and v in an orthonormal basis $\{\psi_n\}$ of eigen p-forms of Δ_p.

$$u = \sum_n u_n \psi_n, \quad v = \sum_n v_n \psi_n, \quad (\psi_n, \psi_m) = \delta_{n,m} \tag{E.3}$$

Let us choose the basis $a^i_{(p)}$, $i = 1, \ldots, b^p$ of the lattice $\mathcal{H}^p_{\mathbf{Z}}$, dual to the basis $\gamma_i \in H_p(X, \mathbf{Z})$, i.e

$$\int_{\gamma_i} a^j_{(p)} = \delta_{ij}$$

Then, orthonormal zero-modes are expressed as

$$\psi^i_{zm} = a^j_{(p)} \left(W_p^{-1}\right)^i_j \tag{E.4}$$

where W_p^{-1} is the inverse of the vielbein for the metric on $\mathcal{H}^p_{\mathbf{Z}}$: $(V_p)^{ij} = \left(W_p^T W_p\right)^{ij}$.

Now, we integrate (E.2) and obtain

$$\left[\frac{det'\Delta_p}{V_p}\right] \tag{E.5}$$

In the case of bosonic p-forms u and v we do not need to insert anything to get a non-zero answer:

$$\int DuDve^{-(u,\Delta_p v)} = \left[\frac{det'\Delta_p}{V_p}\right]^{-1} \tag{E.6}$$

where in (E.6) the integration over bosonic zero-modes was performed

$$\int \prod_{i=1}^{b^p} Du^i_{zm} \prod_{j=1}^{b^p} Dv^j_{zm} = \frac{1}{\left(det_{i,j} \int_{\gamma_i} \psi^j_{zm}\right)^2} = V_p \tag{E.7}$$

Appendix F

Here we explain why $\widehat{\Theta}(\mathcal{F},\rho)$ defined in (3.28) is a supertheta function for a family of principally polarized superabelian varieties. To show this we use the results of [105], where supertheta functions were studied.

A generic complex supertorus is defined as a quotient of the affine superspace with even coordinates $z_i, \quad i = 1,\ldots,N_{even}$ and odd coordinates $\xi_a, \quad a = 1,\ldots,N_{odd}$ by the action of the abelian group generated by $\{\lambda_i, \lambda_{i+N_{even}}\}$

$$\lambda_i : z_j \to z_j + \delta_{ij}, \quad \xi_a \to \xi_a \tag{F.1}$$

$$\lambda_{i+N_{even}} : z_j \to z_j + (\Omega_{even})_{ij}, \quad \xi_a \to \xi_a + (\Omega_{odd})_{ia} \tag{F.2}$$

We will restrict to the special case $(\Omega_{odd})_{ia} = 0$ relevant for our discussion. Let us also assume that the reduced torus (obtained from the supertorus by forgetting all odd coordinates) has a structure of a principally polarized abelian variety and denote its Kahler form by ω.

It follows from the results of [105], that a complex line bundle L on the supertorus with $c_1(L) = \omega$ has a unique section (up to constant multiple) iff $\Omega_{even}^T = \Omega_{even}$ together with the positivity of the imaginary part of the reduced matrix. This section is a supertheta function.

Now we can find a family of principally polarized superabelian varieties relevant to our case simply by setting $N_{even} = N$ and $N_{odd} = N_{ferm.zm}$ and by *defining* symmetric Ω_{even} as

$$Re(\Omega_{even})_{ij} = Re\tau_K(x_i, x_j), \tag{F.3}$$

$$Im(\Omega_{even})_{ij} = Im\tau_K(x_i, x_j) + \tag{F.4}$$
$$\sum_{p=0}^{2} \int_{X_{10}} \Big(G_{2p}(x_i) + G_{2p}(x_j) \Big) \wedge \hat{*} \mathcal{J}_{2p}(zm) + \delta_{ij} F(zm)$$

where $x_i, i = 1, \dots, N$ is a basis of Γ_1. In (F.3) $\mathcal{J}_{2p}(zm)$ is a 2p-form on X_{10} constructed as a bilinear expression in fermion(and ghosts) zeromodes and $F(zm)$ is a functional quartic in fermion(and ghosts) zeromodes, both $\mathcal{J}_{2p}(zm)$ and $F(zm)$ can in principle be found from the 10D fermion action (7.10),(14.1) as well as from the ghost action (7.35),(7.40),(15.1). The modified characteristics $\hat{\vec{\alpha}}, \hat{\vec{\beta}}$ and prefactor $\widehat{\Delta\Phi}(\mathcal{F})$ in (3.28) all originate from the shift of the imaginary part of the period matrix described in (F.4). It would be very nice if one could formulate this superabelian variety in a more natural way, without reference to a Lagrangian splitting of Γ_K.

Appendix G

For the convenience of the reader we list here the leading expressions for Kahler potential in the case $h^{(1,1)} = 1$, together with formulae for the Kahler metric and inverse metric. The Kahler potential is:

$$K = K_S + K_T + K_m + K_{cplx} + K_5 + K_{\text{bundle}},$$

$$K_S = -ln(S+\overline{S}), \quad K_T = -ln\Big(\tilde{d}(T+\overline{T})^3\Big)$$

$$K_5 = \frac{(Z+\overline{Z})^2}{(S+\overline{S})(\beta_i T^i + \beta_i \overline{T}^i)}$$

$$K_m = \Big(\frac{3}{T+\overline{T}} + \frac{2\xi}{S+\overline{S}}\Big) H_{\hat{I}\hat{J}} C^{\hat{I}} \overline{C}^{\hat{J}}$$

We now give the components of the Kahler metric on the space of scalars which have been used in section 5.4. We keep only leading terms in each of the component, neglecting corrections of the relative order $O(\mathcal{E}^{eff}, \mathcal{E}_R^{eff}, \frac{|C|^2}{Ra})$.

$$K_{S\overline{S}} = \frac{1}{4V^2}, \quad K_{S\overline{T}} = \frac{x^2}{4V^2}, \quad K_{S\hat{J}} = -\frac{\xi H_{\hat{I}\hat{J}} C^{\hat{I}}}{2V^2}, \quad K_{S\overline{\alpha}} = -\frac{\xi \partial_{\overline{\alpha}} H_{\hat{I}\overline{\hat{J}}} C^{\hat{I}} \overline{C}^{\overline{\hat{J}}}}{2V^2},$$

$$K_{S\overline{Z}} = -\frac{x}{2V^2}, \quad K_{T\overline{T}} = \frac{3}{(2Ra)^2}, \quad K_{\overline{T}I} = -\frac{3H_{\hat{I}\hat{J}}\overline{C}^{\hat{J}}}{(2Ra)^2}$$

$$K_{\overline{T}\alpha} = -\frac{3C^{\hat{I}} \partial_\alpha H_{\hat{I}\hat{J}} \overline{C}^{\hat{J}}}{(2Ra)^2} \quad K_{Z\overline{T}} = -\frac{x}{2VRa}, \quad K_{\hat{I}\hat{J}} = \frac{3}{2Ra} H_{\hat{I}\hat{J}}, \quad K_{\hat{I}\overline{\alpha}} = \frac{3\partial_{\overline{\alpha}} H_{\hat{I}\hat{J}} \overline{C}^{\hat{J}}}{2Ra}$$

$$K_{Z\hat{I}} = -\frac{(1-x)H_{\hat{I}\hat{J}}C^{\hat{J}}}{VRa}, \quad K_{\overline{Z}\alpha} = -\frac{(1-x)C^{\hat{I}}\partial_\alpha H_{\hat{I}\hat{J}}\overline{C}^{\hat{J}}}{VRa}$$

$$K_{\alpha\overline{\beta}} = K^{(cplx)}_{\alpha\overline{\beta}}, \quad K_{Z\overline{Z}} = \frac{1}{2VRa}$$

Now, we solve the matrix equation

$$KK^{-1} = 1 + O(\mathcal{E}^{eff}, \mathcal{E}^{eff}_R, \frac{|C|^2}{Ra})$$

The inverse metric solving this equation is

$$K^{S\overline{S}} = 4V^2, \quad K^{T\overline{T}} = \frac{(2Ra)^2}{3}, \quad K^{\overline{T}\hat{J}} = C^{\hat{J}}\frac{2Ra}{3},$$

$$K^{\hat{I}\hat{J}} = \frac{2Ra}{3}H^{\hat{I}\hat{J}}, \quad K^{\overline{\alpha}\beta} = K^{\overline{\alpha}\beta}_{cplx}, \quad K^{\hat{J}\beta} = -\partial_{\overline{\delta}}H_{\hat{I}\hat{K}}\overline{C}^{\hat{K}}H^{\hat{I}\hat{J}}K^{\overline{\delta}\beta}$$

$$K^{Z\overline{Z}} = 2RaV, \quad K^{Z\overline{S}} = 4RaVx, \quad K^{Z\overline{T}} = \frac{(2Ra)^2}{3}x, \quad K^{\hat{J}\overline{Z}} = \frac{(2Ra)}{3}C^{\hat{J}}(2-x)$$

where the components not listed above are zero in our approximation.

Appendix H

In section 5.5 of Chapter2 we asserted that, within the region of validity of our computations, the potential is always positive. Here we give the detailed proof of that claim.

The only potentially negative term in the potential is U_1. We will show that it cannot be larger in magnitude than both of U_0 and U_2 in our region of validity.

First, imposing

$$|U_1| \geq U_2$$

means

$$\beta(1-x)|C|^3\left(e^{-Jx}+e^{-J(1-x)}\right) \geq \gamma V\left(e^{-Jx}+e^{-J(1-x)}\right)^2 \tag{H.1}$$

It follows immediately that

$$|C| \geq \left(\frac{\gamma V}{\beta}[e^{-Jx}+e^{-J(1-x)}]\right)^{\frac{1}{3}}.$$

Now, at a generic point in bundle and complex moduli space, we have

$$\alpha C^4 \geq \alpha|C|^3\left(\frac{\gamma V}{\beta}[e^{-Jx}+e^{-J(1-x)}]\right)^{\frac{1}{3}} \gg \beta|C|^3(1-x)\left(e^{-Jx}+e^{-J(1-x)}\right)$$

and we see that $U_0 \gg |U_1|$.

Let us now assume

$$|U_1| \geq U_0.$$

From this it follows that

$$\beta(1-x)\left(e^{-Jx}+e^{-J(1-x)}\right) \geq \alpha|C|$$

and hence

$$|U_1| \leq \frac{1}{J^2V}\frac{\beta^4}{\alpha^3}(1-x)^4\Big(e^{-Jx} + e^{-J(1-x)}\Big)^4 \tag{H.2}$$

Let us consider, first,the region far enough from $x = 1/2$. Then, for $x < \frac{1}{2}$, we have

$$|U_1| \leq \frac{1}{J^2V}\frac{\beta^4}{\alpha^3}(1-x)^4 e^{-4Jx}$$

and

$$U_2 \sim \frac{1}{J^2}e^{-2Jx}$$

As a consequence, $U_2 \gg |U_1|$.

In the region close to $x = \frac{1}{2}$ we have instead, for sign "+" in eq.(H.2)

$$|U_1| \leq \frac{1}{J^2V}\frac{\beta^4}{\alpha^3}e^{-2J}$$

and

$$U_2 \sim \frac{1}{J^2}e^{-J}$$

and it follows immediately that

$$U_2 \gg |U_1|.$$

For sign "-" in eq.(H.2) the last statement is obvious.

References

[1] A.Linde,"Inflationary Theory versus Ekpyrotic/Cyclic Scenario", hep-th/0205259 R.Kallosh,l.Kofman, A.Linde,A.Tseytlin, "BPS branes in Cosmology",Phys.Rev. D64(2001)123524, hep-th/0106241.

[2] P. Hořava, E. Witten, "Heterotic and Type I String Dynamics from Eleven Dimensions" Nucl.Phys. B460 (1996) 506-524, hep-th/9510209

[3] P. Hořava, E. Witten, "Eleven-Dimensional Supergravity on a Manifold with Boundary", Nucl.Phys. B475 (1996) 94-114,hep-th/9603142

[4] E.Witten, " Strong Coupling Expansion Of Calabi-Yau Compactification", Nucl.Phys. B471 (1996) 135-158,hep-th/9602070

[5] T.Banks and M. Dine, "Phenomenology of Strongly Coupled Heterotic String Theory", Nucl.Phys.B479(1996)173.hep-th/9609046

[6] T.Banks and M. Dine, "Couplings and Scales in Strongly Coupled Heterotic String Theory", (hep-th/9605136)

[7] E.Witten, "Non-Perturbative Superpotentials In String Theory", Nucl.Phys. B474 (1996) 343-360,hep-th/9604030

[8] K.Becker, M.Becker, A.Strominger, Five-branes, Membranes and Non-perturbative String Theory. Nucl.Phys.B456:130-152,1995,hep-th/9507158

[9] J. Harvey, G.Moore, Superpotentials and Membrane Instanons, hep-th/9907026

[10] B. de Wit, K.Peeters, J. Plefka, " Open and Closed Supermembranes with Winding," Nucl.Phys.Proc.Suppl. 68 (1998) 206-215

[11] E. Witten,World-Sheet Corrections Via D-Instantons, hep-th/9907041

[12] S. Kachru, S. Katz, A. Lawrence, J. McGreevy,Open string instantons and superpotentials Phys.Rev. D62 (2000) 026001; hep-th/9912151

[13] B.S. Acharya, M theory, Joyce Orbifolds and Super Yang-Mills,hep-th/9812205

[14] M.Dine, N.Seiberg, X.G.Wen, E.Witten, "Nonperturbative effects on the string worldsheet", Nucl.Phys. B278(1986)769-789

[15] M.Dine, N.Seiberg, X.G.Wen, E.Witten, "Nonperturbative effects on the string worldsheet (II)", Nucl.Phys. B289(1987)319-363

[16] P. Hořava , " Gluino Condensation in Strongly Coupled Heterotic String Theory", Phys.Rev. D54 (1996) 7561-7569,hep-th/9608019

[17] A. Lukas, B. A. Ovrut, D. Waldram, "Non-standard embedding and five-branes in heterotic M-Theory", Phys.Rev. D59 (1999) 106005, hep-th/9808101

[18] A. Lukas, B. A. Ovrut, D. Waldram, "Five-branes and Supersymmetry Breaking in M-theory", JHEP 9904 (1999) 009;hep-th/9901017

[19] A. Lukas, B. A. Ovrut, D. Waldram,K.Stelle, " Heterotic M-Theory in Five Dimensions,hep-th/9806051", Nucl.Phys. B552 (1999) 246-290

[20] R. Donagi, A. Lukas, B. A. Ovrut, D. Waldram " Non-Perturbative Vacua and Particle Physics in M-Theory", JHEP 9905 (1999) 018, hep-th/9811168

[21] R. Donagi, A. Lukas, B. A. Ovrut, D. Waldram, Holomorphic Vector Bundles and Non-Perturbative Vacua in M-Theory" JHEP 9906 (1999) 034,hep-th/9901009

[22] E.Silverstein, E.Witten, "Criteria for Conformal Invariance of (0,2) Models", Nucl.Phys. B444 (1995) 161-190,hep-th/9503212

[23] J.Distler, S.Kachru,"Singlet Couplings and (0,2) Models", Nucl. Phys. B430 (1994) 13, hep-th/9406090

[24] P. Berglund, P. Candelas, X. de la Ossa, E. Derrick, J. Distler, T. Hubsch, On the Instanton Contributions to the Masses and Couplings of E_6 Singlets,Nucl. Phys. B454 (1995) 127;hep-th/9505164

[25] P. Candelas, X. de la Ossa, Moduli Space of Calabi-Yau Manifolds. Nucl.Phys.B355:455-481,1991

[26] A.Strominger," Special Geometry", Comm.Math.Phys.1333(1990)163.

[27] E.Witten, "Phases of N=2 Theories in two Dimensions", Nucl.Phys. B403(1993)159.hep-th/9301042

[28] M.Dine, N.Seiberg, " Is the superstring weakly coupled?", Phys.Lett.B162(1985)299-302; M.Dine, R.Rohm, N.Seiberg, E. Witten, Phys.Lett.B156(1985)55; M.Dine, N.Seiberg, Phys.Rev. Lett.55(1985)366

[29] P.Townsend, Brane Surgery Nucl. Phys. B., Proc. Suppl. 58 (1997) 163-175.[HEP-TH 9609217]

[30] A. Strominger Open P-Branes, Phys.Lett.B383:44-47,1996 ;hep-th/9512059

[31] P.S. Howe, E.Sezgin, Superbranes, Phys.Lett. B390 (1997) 133-142;hep-th/9607227; D=11, p=5, Phys.Lett. B394 (1997) 62-66;hep-th/9611008; P.S. Howe, E. Sezgin, P.C. West, Covariant Field Equations of the M Theory Five-Brane, Phys.Lett. B399 (1997) 49-59;hep-th/9702008

[32] D.Sorokin,Superbranes and Superembeddings, Phys.Rept. 329 (2000) 1-101 (hep-th/9906142)

[33] E. Cremmer and S. Ferrara, Phys. Lett. **91B** (1980) 61;
L. Brink and P. S. Howe, Phys. Lett. **91B** (1980) 384.

[34] B. de Wit, K. Peeters, and J. Plefka, Superspace geometry for supermembrane backgrounds," Nucl.Phys. B532 (1998) 99;

[35] Y. Shibusa, "11-dimensional curved backgrounds for supermembrane in superspace,"

[36] E. Sezgin, P. Sundell," Aspects of the M5-Brane", hep-th/9902171

[37] M.Fabinger, P.Hořava.,"Casimir effect between world-branes in heterotic M-theory, hep-th/0002073

[38] P. Pasti, D. Sorokin and M. Tonin, Physics Lett.B3989741, (hep-th/9701037);
I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin and M. Tonin, Phys. Rev. Lett.78974332, (hep-th/9701149).

[39] M. Aganagic, J. Park, C. Popescu, J. Schwarz, "World-Volume Action of the M Theory Five-Brane", Nucl.Phys. B496 (1997) 191-214;hep-th/9701166

[40] J.Wess and J. Bagger,Supersymmetry and Supergravity,Second ed.,Princeton University Press 1992.

[41] E. Cremmer, S. Ferrara, L. Girardello, A. Van Proeyen, " Yang-Milles theories with local supersymmetry: Lagrangian, transformation laws and superhiggs effect," Nucl.Phys.B212:413,1983

[42] J.P.Derendinger, R.Sauser, "A five-brane Modulus in the Effective N=1 Supergravity of M-theory", hep-th/0009054

[43] Ph.Brax, J.Mourad, " Open Supermembranes Coupled to M-Theory Five-Branes", Phys.Lett. B416 (1998) 295-302; hep-th/9707246

[44] Perelomov.A, Integrable systems of classical mechanics and Lie algebras. Birkhuser Verlag, 1990.

[45] B.Ovrut, T.Pantev, J.Park: Small Instanton Transitions in Heterotic M-Theory, JHEP 0005 (2000) 045; hep-th/0001133

[46] C. Munoz, "Effective Supergravity from Heterotic M-Theory and its Phenomenological Implications," hep-th/9906152; D.G. Cerdeno and C. Munoz, "Phenomenology of Non-Standard Embedding and Five-branes in M-Theory," hep-ph/9904444

[47] E. Bergshoeff, E. Sezgin, and P.K. Townsend, "Supermembranes and 11-dimensional supergravity," Phys. Lett. **189B**(1987)75; "Properties of the Eleven-Dimensional Supermembrane Theory," Ann. Phys. **185**(1988)330.

[48] M. Green, J.H. Schwarz, and E. Witten, *Superstring Theory*, Cambridge University Press, Cambridge (1987)

[49] T.Li, J.L.Lopez and D.V.Nanopoulos, Phys.Rev.D56,2602,(1997)

[50] E.Dudas,C.Grojean, Nucl.Phys.B507,553 (1997)

[51] H.P.Nilles, M.Olechowski, M.Yamaguchi, Phys.Lett.B415,24 (1997); Nucl.Phys.B530,43 (1998)

[52] S. Stieberger," (0,2) Heterotic Gauge Couplings and their M-Theory Origin", Nucl.Phys. B541 (1999) 109-144

[53] K. Choi, H. B. Kim, H. Kim, " Moduli Stabilization in Heterotic M-theory", Mod.Phys.Lett. A14 (1999) 125-134, hep-th/9808122

[54] K. Benakli, " Phenomenology of Low Quantum Gravity Scale Models," Phys.Rev. D60 (1999) 104002

[55] T. Banks, M. Berkooz, G. Moore, S.H. Shenker, and P. Steinhardt, "Modular Cosmology," hep-th/9503114; Phys.Rev. D52 (1995) 3548-3562

[56] M. Atiyah, J. Maldacena, C. Vafa, " An M-theory Flop as a Large N Duality,"hep-th/0011256

[57] C. Vafa,"Superstrings and Topological Strings at Large N", hep-th/0008142

[58] H. Ooguri, C. Vafa, " Knot Invariants and Topological Strings",Nucl.Phys. B577 (2000) 419-438

[59] S. Kachru, S. Katz, A. Lawrence, J. McGreevy, "Mirror symmetry for open strings", Phys.Rev. D62 (2000) 126005, hep-th/0006047

[60] S.Kachru, E.Silverstein, "Chirality Changing Phase Transitions in 4d String Vacua", Nucl.Phys. B504 (1997) 272-284, hep-th/9704185

[61] P.Candelas, X. C. de la Ossa. P.Green, L.Parkes,"A pair of Calabi-Yau Manifolds as an exactly soluble superconformal theory", Essays on Mirror Manifolds, ed. S.T.Yau

[62] P.S. Aspinwall and D.R. Morrison, "Topological field theory and rational curves," Commun. Math. Phys. **151**(1993)245

[63] M. Marino, G. Moore, "Counting higher genus curves in a Calabi-Yau manifold", Nucl.Phys. B543 (1999) 592-614, hep-th/9808131

[64] C. Faber and R. Pandharipande, "Hodge Integrals and Gromov-Witten Theory," math.AG/9810173

[65] R. Gopakumar, C. Vafa, " M-Theory and Topological Strings–I",hep-th/9809187; "M-Theory and Topological Strings–II", hep-th/9812127

[66] D. R. Morrison, M. R. Plesser, " Summing the Instantons: Quantum Cohomology and Mirror Symmetry in Toric Varieties", Nucl. Phys. B440 (1995) 279-354, hep-th/9412236

[67] R. Donagi, B.A. Ovrut, and D. Waldram, "Moduli spaces of fivebranes on elliptic Calabi-Yau threefolds," hep-th/9904054

[68] D.E.Diaconescu, G.Moore, E. Witten, "E8 Gauge Theory, and a Derivation of K-Theory from M-Theory", hep-th/005090

[69] D.E.Diaconescu, G.Moore, E. Witten, unpublished.

[70] D.E. Diaconescu and G. Moore, unpublished.

[71] G.Moore, E. Witten, "Integration over the u-plane in Donaldson theory",hep-th/9709193

[72] D. Freed, Dirac Charge Quantization and Generalized Differential Cohomology," hep-th/0011220

[73] C.M. Hull,"Massive String Theories From M-Theory and F-Theory", JHEP 9811 (1998) 027,hep-th/ 9811021

[74] C.Hull,P.Townsend,"Unity of Superstring Dualities," Nucl.Phys. B438 (1995) 109-137,hep-th/9410167

[75] A. Giveon, M. Porrati, E. Rabinovici, " Target Space Duality in String Theory", hep-th/9401139

[76] E.Bergshoeff,M.de Roo,M.B.Green, G.Papadopoulos, P.K.Townsend,"Duality of II 7-branes and 8-branes", hep-th/9601150

[77] I.Lavrinenko, H.Lu, C.Pope, T.Tran,"U-duality as general Coordinate Transformations, and Spacetime Geometry", hep-th/9807006

[78] M. Cvetic, H. Lu, C.N. Pope, K.S. Stelle," T-Duality in the Green-Schwarz Formalism, and the Massless/Massive IIA Duality Map",Nucl.Phys. B573 (2000) 149-176, hep-th/9907202

[79] B. Pioline, H. Nicolai, J. Plefka, A. Waldron, R^4 couplings, the fundamental membrane and exceptional theta correspondences, hep-th/0102123

[80] R. Minasian and G. Moore,"K Theory and Ramond-Ramond Charge," JHEP **9711**:002, 1997; hep-th/9710230.

[81] G. Moore and E. Witten, "Self-Duality, Ramond-Ramond Fields, and K-Theory," hep-th/9912279;JHEP 0005 (2000) 032

[82] C.Isham,C.Pope,N.Warner, "Nowhere-vanishing spinors and triality rotations in 8-manifolds," Class.Quantum Grav.5(1988)1297

[83] E. Witten, "On Flux Quantization in M-Theory and the Effective Action," hep-th/9609122; Journal of Geometry and Physics, **22** (1997) 1.

[84] E. Witten, "D-Branes And K-Theory," JHEP **9812**:019, 1998; hep-th/9810188.

[85] E.Witten," Five-brane effective action in M-theory," J.Geom.Phys.,22 (1997)103,hep-th/9610234

[86] E. Witten, "Duality Relations Among Topological Effects In String Theory," hep-th/9912086;JHEP 0005 (2000) 031

[87] E. Witten, "Overview of K-theory applied to strings," hep-th/0007175.

[88] S. Sethi, C. Vafa, E. Witten, "Constraints on Low-Dimensional String Compactifications", hep-th/9606122, Nucl.Phys. B480 (1996) 213-224

[89] E.Witten,"On S-Duality in Abelian Gauge Theory", hep-th/9505186

[90] E. Verlinde,"Global Aspects of Electric-Magnetic Duality", Nucl.Phys. B455 (1995) 211-228,hep-th/9506011

[91] L. Romans,"Massive IIA Supergravity in Ten Dimensions', Phys.Let.169B(1986)374

[92] P. Meessen, T. Ortin, "An Sl(2,Z) Multiplet of Nine-Dimensional Type II Supergravity Theories", Nucl.Phys. B541 (1999) 195-245, hep-th/9806120

[93] W.Siegel,"Hidden ghosts", Phys.Lett.B93(1980),170

[94] S.Deser,P.van Nieuwenhuizen,"One-loop divergences of quantized Einstein-maxwell fields",Phys.Rev.D(1974), v.10,p.401

[95] G.'t Hooft, M.Veltman,"One loop divergences in the theory of gravitation", Annales Inst.Henri Poincare ,vol.20,(1974),69

[96] E.Cremmer, B.Julia, J.Scherk, "Supergravity theory in 11 dimensions", Phys.Lett. B76(1978)409

[97] N.Nielsen, "Ghost counting in supergravity", NPB 140(1978)499

[98] R.Kallosh,"Modified Feynman rules in supergravity",NPB 141(1978)141

[99] K.Fujikawa," Path Integral for gauge theories with fermions", Phys. Rev.D.21(1980) 2848

[100] J.Igusa, *Theta-functions*, Berlin, New York, Springer-Verlag, 1972.

[101] E.D'Hoker,D.Phong,"Conformal scalar fields and chiral splitting on SuperRiemann surfaces",Comm.Math.Phys.125 (1989)469;

K.Aoki,E.D'Hoker,D.Phong, "Unitarity of closed superstring Perturbation Theory", NPB 342(1990)149-230

[102] J.Gegenborg, G.Kunstatter,"The partition function for topological field theories",hep-th/9304016

[103] G.Moore,"Some comments on Branes, G-flux, and K-theory", Int.J.Mod.Phys.A16 (2001)936-944, hep-th/0012007

[104] V. Mathai, D. Stevenson, "Chern character in twisted K-theory: equivariant and holomorphic cases" hep-th/0201010

[105] J.Rabin, M.Bergvelt,"Super curves, their Jacobians, and super KP equations",alg-geom/9601012

[106] S.Fredenhagen, V.Schomerus, Non-commutative geometry, gluon condensates, and K-theory",JHEP 0004(2001)007, hep-th/0012164

[107] S.Stanciu,"An illustrated guide to D-branes in SU(3)", hep-th/0111221

[108] J.Maldacena,G.Moore, N.Seiberg, "D-brane instantons and K-theory charges", JHEP 0111(2001)062; hep-th/0108100

[109] P.Bouwknegt, V.Mathai,"D-branes, B-fields and twisted K-theory", JHEP 0003 (2000)007, hep-th/0002023

[110] M.Atiyah,V.Patodi, I.Singer, Math.Proc.Cambridge Phil.Soc.77(1975)43;405.

[111] D. S. Freed, M. J. Hopkins, "On Ramond-Ramond fields and K-theory," JHEP 0005 (2000) 044; hep-th/0002027

[112] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge Univ. Press, 1982

[113] R. Stong, "Calculation of $\Omega_{11}^{spin}(K(\mathbf{Z},4))$" in *Unified String Theories*, 1985 Santa Barbara Proceedings, M. Green and D. Gross, eds. World Scientific 1986.

[114] M. F. Atiyah, I. M. Singer, "The index of elliptic operators: V" Ann. Math. **93** (1971) 139.

[115] G.Moore, G.Peradze, N.Saulina, "Instabilities in heterotic M-theory induced by open membrane instantons," Nucl.Phys.B607(2001)117, hep-th/0012104.

[116] G.Moore, N.Saulina, "T-Duality, and the K-Theoretic Partition Function of Type IIA Superstring Theory", hep-th/0206092.