

Review

# On the Mathematics of Coframe Formalism and Einstein–Cartan Theory—A Brief Review

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**Abstract:** This article is a review of what could be considered the basic mathematics of Einstein–Cartan theory. We discuss the formalism of principal bundles, principal connections, curvature forms, gauge fields, torsion form, and Bianchi identities, and eventually, we will end up with Einstein–Cartan–Sciama–Kibble field equations and conservation laws in their implicit formulation.

**Keywords:** general relativity; torsion-gravity; mathematical physics

## 1. Introduction

The formulation of torsion gravity and the consequent coupling with spin rely on a different formulation compared to the one of original works on General Relativity. This formulation regards geometrical objects called principal bundles. In this context, we can formulate General Relativity (or Einstein–Cartan–Sciama–Kibble (ECSK) theory in the presence of torsion) with a principal connection, which can be pulled back to the base manifold in a canonical way and further restricted to the only antisymmetric component, giving birth to the well-known spin connection. This process shows the possibility of formulating General Relativity as a proper gauge theory rather than using the affine formulation and Christoffel symbols  $\Gamma$ . What permits the equivalence of the two formulations is a bundle isomorphism called *tetrads* or *vierbein*, which is supposed to respect certain compatibility conditions. Then, we can define the associated torsion form and postulate the Palatini–Cartan action as a functional of such tetrads and spin connection. This leads to ECSK field equations.

We will first set up all the abstract tools of principal bundles, tetrads, and principal connection; secondly, we will derive the Einstein–Cartan–Sciama–Kibble theory in its implicit version; and finally, we will discuss conservation laws coming from local  $SO(3,1)$  and diffeomorphism invariance of ECSK theory.

Throughout the article, we will give theorems and definitions. However, we would like to stress that hypotheses for such theorems will often be slightly redundant: we will take spaces and functions to be differentiable manifolds and smooth, even though weaker statements would suffice. This is because we prefer displaying the setup for formalizing the theory rather than presenting theorems and definitions with weaker hypotheses that we will never use for the theory. Nonetheless, we will specify where such hypotheses are strengthened. In spite of this, the discussion will be rather general, probably more general than what is usually required in formulating ECSK (Einstein–Cartan–Sciama–Kibble) theory.

## 2. Bundle Structure

The introduction of a metric  $g$  and an orthogonality relation via a minkowskian metric  $\eta$  are two fundamental ingredients for building up a fiber bundle where we want the orthogonal group to act freely and transitively on the fibers. This will allow us to have a principal connection and to see the perfect analogy with an ordinary gauge theory ([1] chapter III).

Such a construction underlies the concept of *principal bundle*, and tetrads will be an isomorphism from the tangent bundle<sup>1</sup>  $TM$  to an associated bundle  $\mathcal{V}$ .

### 2.1. $G$ -Principal Bundle

We give some definitions<sup>2</sup>.

**Definition 1** ( $G$ -principal bundle<sup>3</sup>). *Let  $M$  be a differentiable manifold and  $G$  be a Lie group.*

*A  $G$ -principal bundle  $P$  is a fiber bundle  $\pi : P \rightarrow M$  together with a smooth (at least continuous) right action  $\mathfrak{P} : G \times P \rightarrow P$  such that  $\mathfrak{P}$  acts freely and transitively on the fibers<sup>4</sup> of  $P$  and such that  $\pi(\mathfrak{P}_g(p)) = \pi(p)$  for all  $g \in G$  and  $p \in P$ .*

We need to introduce a fundamental feature of fiber bundles.

**Definition 2** (Local trivialization of a fiber bundle). *Let  $E$  be a fiber bundle over  $M$ , a differentiable manifold, with fiber projection  $\pi : E \rightarrow M$ , and let  $F$  be a space<sup>5</sup>.*

*A local trivialization  $(U, \varphi_U)$  of  $E$ , is a neighborhood  $U \subset M$  of  $u \in M$  together with a local diffeomorphism.*

$$\varphi_U : U \times F \rightarrow \pi^{-1}(U) \tag{1}$$

such that  $\pi(\varphi_U(u, f)) = u \in U$  for all  $u \in U$  and  $f \in F$ .

This definition implies  $\pi^{-1}(u) \simeq F \forall u \in U$ .

**Definition 3** (Local trivialization of a  $G$ -principal bundle). *Let  $P$  be a  $G$ -principal bundle.*

*A local trivialization  $(U, \varphi_U)$  of  $P$  is a neighborhood  $U \subset M$  of  $u \in M$  together with a local diffeomorphism.*

$$\varphi_U : U \times G \rightarrow \pi^{-1}(U) \tag{2}$$

such that  $\pi(\varphi_U(u, g)) = u \in U$  for all  $u \in U$  and  $g \in G$  and such that

$$\varphi_U^{-1}(\mathfrak{P}_g(p)) = \varphi_U^{-1}(p)g = (u, g')g = (u, g'g). \tag{3}$$

<sup>1</sup> Disjoint union of tangent spaces:  $TM = \cup_{x \in M} \{x\} \times T_x M$

<sup>2</sup> References [2–4] are recommended for further details.

<sup>3</sup> We give the definition based on our purposes; in general, we can release some hypotheses. In particular,  $G$  needs to be only a locally compact topological group and  $M$  needs to be a topological Hausdorff space. This definition is a version with a stronger hypothesis than the one contained in Reference [5].

<sup>4</sup> Fibers are  $\pi^{-1}(x) \forall x \in M$ .

<sup>5</sup> In the present case,  $F$  will be a differentiable manifold, a vector space, a topological space, or a topological group. Furthermore, if we write “space”, we mean one among these.

**Observation 1:** A fiber bundle is said to be *locally trivial* in the sense that it admits a local trivialization for all  $x \in M$ , namely there exists an open cover  $\{U_i\}$  of  $M$  and a set of diffeomorphisms  $\varphi_i$  such that every  $\{U_i, \varphi_i\}$  is a local trivialization<sup>6</sup>.

Here, we recall the similarity with a differentiable manifold. For a manifold when we change charts, we have an induced diffeomorphism between the neighborhoods of the two charts, given by the composition of the two maps.

Thus, having two charts  $(U_i, \varphi_i)$  and  $(U_j, \varphi_j)$ , we define the following:

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j). \tag{4}$$

At a level up, we have an analogous thing when we change trivialization. Of course, here, we have one more element: the element of fiber.

Taking two local trivializations  $(U_i, \varphi_i)$  and  $(U_j, \varphi_j)$  and given a smooth left action  $\mathcal{T} : G \rightarrow \text{Diffeo}(F)$  of  $G$  on  $F$ , we then have

$$(\varphi_j^{-1} \circ \varphi_i)(x, f) = (x, \mathcal{T}(g_{ij}(x))(f)) \quad \forall x \in U_i \cap U_j, f \in F. \tag{5}$$

where the maps  $g_{ij} : U_i \cap U_j \rightarrow G$  are called the *transition functions* for this change of trivialization and  $G$  is called the structure group.

Such functions obey the following transition functions conditions for all  $x \in U_i \cap U_j$ :

- $g_{ii}(x) = id$
- $g_{ij}(x) = (g_{ji}(x))^{-1}$
- $g_{ij}(x) = g_{ik}(x)g_{kj}(x)$  for all  $x \in U_i \cap U_k \cap U_j$ .

The last condition is called the *cocycle condition*.

**Theorem 1** (Fiber bundle construction theorem). *Let  $M$  be a differentiable manifold,  $F$  be a space, and  $G$  be a Lie group with faithful smooth left action  $\mathcal{T} : G \rightarrow \text{Diffeo}(F)$  of  $G$  on  $F$ .*

*Given an open cover  $\{U_i\}$  of  $M$  and a set of smooth maps,*

$$t_{ij} : U_i \cap U_j \rightarrow G \tag{6}$$

*defined on each nonempty overlap, satisfying the transition function conditions.*

*Then, there exists a fiber bundle  $\pi : E \rightarrow M$  such that*

- $\pi^{-1}(x) \simeq F$  for all  $x \in M$
- its structure group is  $G$ , and
- it is trivializable over  $\{U_i\}$  with transition functions given by  $t_{ij}$ .

A proof of the theorem can be found in Reference [6] (Chapter 1).

## 2.2. Coframe Bundle and Minkowski Bundle

It is clear now that having  $E$  as a fiber bundle over  $M$  with fibers isomorphic to  $F$  and  $F'$  as a space equipped with the smooth action  $\mathcal{T}'$  of  $G$ , implies the possibility of building a bundle  $E'$  associated to

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<sup>6</sup> The bundle is said to be *trivial* if  $U = M$ .

$E$ , which shares the same structure group and the same transition functions  $g_{ij}$ . By the fiber bundle construction theorem, we have a new bundle  $E'$  over  $M$  with fibers isomorphic to  $F'$ .

This bundle is called the *associated bundle* to  $E$ .

Depending on the nature of the associated bundle<sup>7</sup>, we have the following two definitions:

**Definition 4** (Associated  $G$ -principal bundle). *Let  $\pi : E \rightarrow M$  be a fiber bundle over a differentiable manifold  $M$ ,  $G$  be a Lie group,  $F'$  be a topological space, and  $\mathfrak{A}$  be a smooth right action of  $G$  on  $F'$ . Let also  $E'$  be the associated bundle to  $E$  with fibers isomorphic to  $F'$ .*

*If  $F'$  is the principal homogeneous space<sup>8</sup> for  $\mathfrak{A}$ , namely  $\mathfrak{A}$  acts freely and transitively on  $F'$ , then  $E'$  is called the  $G$ -principal bundle associated to  $E$ .*

**Definition 5** (Associated bundle to a  $G$ -principal bundle). *Let  $P$  be a  $G$ -principal bundle over  $M$ ,  $F'$  be a space, and  $\rho : G \rightarrow \text{Diffeo}(F')$  be a smooth effective left action of the group  $G$  on  $F'$ .*

*We then have an induced right action of the group  $G$  over  $P \times F'$  given by*

$$(p, f') * g = (\mathfrak{A}_g(p), \rho(g^{-1})(f')). \tag{7}$$

*We define the associated bundle  $E$  to the principal bundle  $P$ , as an equivalence relation:*

$$E := P \times_{\rho} F' = \frac{P \times F'}{\sim}, \tag{8}$$

*where  $(p, f') \sim (\mathfrak{A}_g(p), \rho(g^{-1})(f'))$ ,  $p \in P$ , and  $f' \in F'$  with projection  $\pi_{\rho} : E \rightarrow M$  s.t.  $\pi_{\rho}([p, f']) = \pi(p) = x \in M$ .*

Therefore  $\pi_{\rho} : E \rightarrow M$  is a fiber bundle over  $M$  with  $\pi_{\rho}^{-1}(x) \simeq F'$  for all  $x \in M$ .

**Observation 2:** The new bundle, given by the latter definition, is what we expected from a general associated bundle: a bundle with the same base space, different fibers, and the same structure group.

**Idea:** We take a  $G$ -principal bundle  $P$  as an associated bundle to  $TM$ , and we build a vector bundle associated to  $P$  with a fiber-wise metric  $\eta$ . We shall call this associated bundle  $\mathcal{V}$ .

First of all, we display the  $G$ -principal bundle as the  $G$ -principal bundle associated to  $TM$ .

**Definition 6** (Orthonormal coframe). *Let  $(M, g)$  be a pseudo-riemannian  $n$ -dimensional differentiable manifold and  $(V, \eta)$  be an  $n$ -dimensional vector space with minkowskian metric  $\eta$ .*

*A coframe at  $x \in M$  is the linear isometry.*

$${}_x e := \{ {}_x e : T_x M \rightarrow V \mid {}_x e^* \eta := \eta_{ab} {}_x e^a {}_x e^b = g \}, \tag{9}$$

*equivalently  ${}_x e^a$  forms an ordered orthonormal basis in  $T_x^* M$ .*

*An orthonormal frame is defined as the dual of a coframe.*

<sup>7</sup> We will be dealing with two particular types of associated bundles: a principal bundle associated to a vector bundle and a vector bundle associated to a principal bundle.

<sup>8</sup> The space where the orbits of  $G$  span all the space.

**Observation 3:** Locally, coframes can be identified with local covector fields. A necessary and sufficient condition for identifying them with global covector fields (namely a coframe for each point of the manifold) is to have a *parallelizable* manifold, namely a trivial tangent bundle.

**Definition 7** (Orthonormal coframe bundle). *Let  $(M, g)$  be a differentiable  $n$ -dimensional manifold with pseudo-riemannian metric  $g$  and  $T^*M$  be its cotangent bundle (real vector bundle of rank  $n$ ).*

*We call the coframe bundle  $F_O^*(M)$  the  $G$ -principal bundle where the fiber at  $x \in M$  is the set of all orthonormal coframes at  $x$  and where the group  $G = O(n - 1, 1)$  acts freely and transitively on them.*

The dual bundle of this is the orthonormal frame bundle, and it is denoted by  $F_O(M)$ , made up of orthonormal frames (dual of orthonormal coframes).

**Observations 4:**

- i. The orthonormal frame bundle is an associated  $G$ -principal bundle to  $TM$ .
- ii. We can consider the *Minkowski bundle*  $\mathcal{V}$  the vector bundle over  $M$  with fibers  $V$ . It is clear that such a bundle and  $F_O(M)$  are one of the associated bundles of the other via action of the orthogonal group  $O(n - 1, 1)$ . Therefore,  $\mathcal{V} := F_O(M) \times_\rho V$ , where  $\rho$  is taken to be the fundamental representation of  $O(n - 1, 1)$ .
- iii. We stress that this bundle  $\mathcal{V}$  is not canonically isomorphic to  $TM$ ; in general, there is no canonical choice of a representative of  ${}_x e$  of the equivalence class  $[{}_x e, v] \in \mathcal{V}$ , of which the inverse  ${}_x e^{-1}(v)$  gives rise to a canonical identification of a vector in  $T_x M$ . Namely, fixed a  $v \in V$ , not all choices of  ${}_x e$  give rise to a fixed vector  $X \in T_x M$ . As a matter of fact, the reference metric fixed on  $V$  does not allow in general the existence of a canonical soldering (Section 7). In Reference [7], it is shown how to define the Minkowski bundle without deriving it from  $F_O(M)$ ; the authors refer to that as *fake tangent bundles*.
- iv. If the manifold is parallelizable, we have the bundle isomorphism  $e : TM \rightarrow \mathcal{V}$ , which is given by the identity map over  $M$  and  ${}_x e : T_x M \rightarrow V \forall x \in M$ . It can be regarded as a  $\mathcal{V}$ -valued 1-form  $e \in \Omega^1(M, \mathcal{V})$ . We can identify  $e$  with an element of  $\Omega^1(M, V)$ , thus with global sections of the cotangent bundle such that, at each point in  $M$ , the corresponding covectors  ${}_x e^a$  obey  $\eta_{ab} {}_x e^a {}_x e^b = g$ .

We are now ready to define tetrads.

**Definition 8** (Tetrads). *Let  $\rho : O(3, 1) \rightarrow \text{Aut}(V)$  be the fundamental representation.*

*Tetrads are the bundle isomorphisms  $e : TM \rightarrow \mathcal{V}$ . They are identifiable with elements  $e \in \Omega^1(M, \mathcal{V})$ , and if  $M$  is parallelizable, tetrads can be identified with  $\Omega^1(M, V) \ni e^a v_a$  such that  $\{v_a\}$  is an orthonormal basis of  $V$ ,  $e^a \in \Omega^1(M)$ , and  $\eta_{ab} e^a e^b = g$ .*

**3. Principal Connection**

*Is there any difference?*

In the ordinary formulation of General Relativity (as in the original Einstein’s work, for instance), we have objects called  $\Gamma$ s, which are coefficients of a linear connection  $\nabla$  and thus determined by a parallel transport of tangent vectors.

The biggest advantage of treating  $O(3, 1)$  as an “explicit symmetry” of the theory is that we have obtained the possibility of defining a *principal connection*, which is the same kind of entity we have in an ordinary gauge theory<sup>9</sup>.

### 3.1. Ehresmann Connection

If we consider a smooth fiber bundle  $\pi : E \rightarrow M$ , where fibers are differentiable manifolds, we can of course take tangent spaces at points  $e \in E$ . Having the tangent bundle  $TE$ , we may wonder if it is possible to separate the contributions coming from  $M$  to the ones from the fibers.

This cannot be done just by stating  $TE = TM \oplus TF$ , unless  $E = M \times F$  is the trivial bundle. Namely, we cannot split directly vector fields on  $M$  from vector fields on the fibers  $F$ .

We can formalize this idea: use our projection  $\pi$  for constructing a tangent map  $\pi_* = d\pi : TE \rightarrow TM$ , and consider its kernel.

**Definition 9** (Vertical bundle). *Let  $M$  be a differentiable manifold and  $\pi : E \rightarrow M$  be a smooth fiber bundle. We call the sub-bundle  $VE = \text{Ker}(\pi_* : TE \rightarrow TM)$  the vertical bundle.*

Following this definition, we have the natural extension to the complementary bundle of the vertical bundle, which is somehow the formalization of the idea we had of a bundle that takes care of tangent vector fields on  $M$ .

**Definition 10** (Ehresmann connection). *Let  $M$  be a differentiable manifold and  $\pi : E \rightarrow M$  be a smooth fiber bundle. Consider a complementary bundle  $HE$  such that  $TE = HE \oplus VE$ . We call this smooth sub-bundle  $HE$  the horizontal bundle or Ehresmann connection.*

Thus, vector fields will be called *vertical* or *horizontal* depending on whether they belong to  $\Gamma(VE)$  or  $\Gamma(HE)$ , respectively.

### 3.2. Ehresmann Connection and Horizontal Lift

We recall the case of the linear connection  $\nabla$ ; it was uniquely determined by a parallel transport procedure. In the case of a principal connection, we have an analogous.

**Definition 11** (Lift). *Let  $\pi : E \rightarrow M$  be a fiber bundle,  $M$  be a differentiable manifold,  $x \in M$  and  $e \in E$  such that  $\pi(e) = x$ .*

*Given a smooth curve  $\gamma : \mathbb{R} \rightarrow M$  such that  $\gamma(0) = x$ , we define a lift of  $\gamma$  through  $e$  as the curve  $\tilde{\gamma}$ , satisfying*

$$\tilde{\gamma}(0) = e \quad \text{and} \quad \pi(\tilde{\gamma}(t)) = \gamma(t) \quad \forall t. \tag{10}$$

*If  $E$  is smooth, then a lift is horizontal if every tangent to  $\tilde{\gamma}$  lies in a fiber of  $HE$ , namely*

$$\dot{\tilde{\gamma}}(t) \in HE_{\tilde{\gamma}(t)} \quad \forall t. \tag{11}$$

It can be shown that an Ehresmann connection uniquely determines a horizontal lift. Here, it is the analogy with parallel transport.

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<sup>9</sup> Think of  $(U(1), A_\mu)$  for electromagnetism.

### 3.3. Connection Form in a G-Principal Bundle

We now focus on the case where the smooth fiber bundle is a G-principal bundle with smooth action  $\mathfrak{P}$ .

Here, we need a group G, that we generally take to be a matrix Lie group. We then have the corresponding algebra  $\mathfrak{g}$ , a matrix vector space in the present case.

The action  $\mathfrak{P}$  defines a map  $\sigma : \mathfrak{g} \rightarrow \Gamma(VE)$  called the *fundamental map*<sup>10</sup>, where at  $p \in P$ , for an element  $\xi \in \mathfrak{g}$ , it is given via the exponential map  $Exp : \mathfrak{g} \rightarrow G$ .

$$\sigma_p(\xi) = \left. \frac{d}{dt} \mathfrak{P}_{e^{t\xi}}(p) \right|_{t=0}. \tag{12}$$

The map is vertical because

$$\pi_* \sigma_p(\xi) = \left. \frac{d}{dt} \pi(\mathfrak{P}_{e^{t\xi}}(p)) \right|_{t=0} = \left. \frac{d}{dt} \pi(p) \right|_{t=0} = 0. \tag{13}$$

Thus, the vector  $\sigma_p(\xi)$  is vertical and it is called the *fundamental vector*.

Before proceeding, we need some Lie group theory.

**Recall of Lie machinery:** Let G be a Lie group (a differentiable manifold) with  $\mathfrak{g}$  as its Lie algebra and  $\forall g, h \in G$ . We define:

- $L_g : G \rightarrow G$  and  $R_g : G \rightarrow G$ , such that  $L_g h = gh$  and  $R_g h = hg$  are the *left* and *right* actions, respectively;
- the adjoint map  $Ad_g : G \rightarrow G$  via such left and right actions is  $Ad_g := L_g \circ R_{g^{-1}}$ , namely  $Ad_g h = ghg^{-1}$ . It also acts on elements of the algebra  $\xi \in \mathfrak{g}$  as  $Ad_g : \mathfrak{g} \rightarrow \mathfrak{g}$  via the exponential map<sup>11</sup>

$$\begin{aligned} Ad_g \xi &= \left. \frac{d}{dt} ((L_g \circ R_{g^{-1}})(e^{t\xi})) \right|_{t=0} = \left. \frac{d}{dt} (ge^{t\xi}g^{-1}) \right|_{t=0} \\ &= g\xi g^{-1} \in \mathfrak{g}, \end{aligned} \tag{14}$$

where the last two equalities hold in the present case of matrix Lie groups. This is not to be confused with the adjoint action  $ad : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ , which is generated by the derivative of the adjoint map with  $g = e^{t\chi}$  and  $\chi \in \mathfrak{g}$ , such that  $ad_\chi \xi = [\chi, \xi]$ ;

- the left invariant vector fields  $v \in \Gamma(TG)$  as  $L_{g*} \circ v = v$ , namely  $v(g) = L_{g*} v(e)$ ;
- the *Maurer–Cartan form* is the left invariant  $\mathfrak{g}$ -valued 1-form  $\theta \in \Omega^1(G, \mathfrak{g})$  defined by its values at  $g$ .

$$\theta_g := L_{g^{-1}*} : T_g G \rightarrow T_e G \cong \mathfrak{g}. \tag{15}$$

For any left invariant vector field  $v$ , it holds  $\forall g \in G$  that  $\theta_g(v(g)) = v(e)$ . Therefore, left invariant vector fields are identified by their values over the identity thanks to the Maurer–Cartan form  $\theta$ . So we can state ([8]) that this identification  $v(e) \mapsto v$  defines an isomorphism between the space of left

<sup>10</sup> It turns out that it is an isomorphism, since  $\mathfrak{P}$  is regular.

<sup>11</sup> We stress that the exponential map is not an isomorphism for all Lie groups; thus, the elements generated by the exponential map belong, in general, to a connected subgroup of the total group, which is usually homeomorphic to its simply connected double cover. More in general, the isomorphism is between a subset of the algebra containing 0 and a subset of the group containing the identity. Moreover, for a compact, connected, and simply connected Lie group, the algebra always generates the whole group via the exponential map.

invariant vector fields on  $G$  and the space of vectors in  $T_eG$ , thus, the Lie algebra  $\mathfrak{g}$ . For matrix Lie groups, it holds that  $\theta_g = g^{-1}dg$ .

By definition, the action of an element of the group on  $P$  is  $\mathfrak{P}_g : P \rightarrow P$ , and therefore, it defines a tangent map  $\mathfrak{P}_{g*} : TP \rightarrow TP$ , for which the following Lemma holds:

**Lemma 1.**

$$\mathfrak{P}_{g*} \circ \sigma(\xi) = \sigma(\text{Ad}_{g^{-1}}\xi). \tag{16}$$

**Proof.** At  $p \in P$

$$\mathfrak{P}_{g*}\sigma_p(\xi) = \frac{d}{dt}((\mathfrak{P}_g \circ \mathfrak{P}_{e^{t\xi}})(p))|_{t=0} = \frac{d}{dt}((\mathfrak{P}_g \circ \mathfrak{P}_{e^{t\xi}} \circ \mathfrak{P}_{g^{-1}} \circ \mathfrak{P}_g)(p))|_{t=0}, \tag{17}$$

we then use the fact that  $\mathfrak{P}_g \circ \mathfrak{P}_{e^{t\xi}} \circ \mathfrak{P}_{g^{-1}} = \mathfrak{P}_{g^{-1}e^{t\xi}g} = \mathfrak{P}_{\text{Ad}_{g^{-1}}e^{t\xi}}$  and the identity for matrix groups  $\text{Ad}_g e^{t\xi} = e^{t\text{Ad}_g\xi}$  to get the following:

$$\mathfrak{P}_{g*}\sigma_p(\xi) = \frac{d}{dt}(\mathfrak{P}_{e^{t(\text{Ad}_{g^{-1}}\xi)}}(\mathfrak{P}_g(p)))|_{t=0} = \sigma_{\mathfrak{P}_g(p)}(\text{Ad}_{g^{-1}}\xi). \tag{18}$$

□

It is time to define what we were aiming to define at the beginning of the section: the *connection form*.

**Definition 12.** Let  $P$  be a smooth  $G$ -principal bundle and  $HE \subset TP$  be an Ehresmann connection. We call the  $\mathfrak{g}$ -valued 1-form  $\omega \in \Omega^1(P, \mathfrak{g})$ , satisfying

$$\omega(v) = \begin{cases} \xi & \text{if } v = \sigma(\xi), \xi \in \mathcal{C}^\infty(P, \mathfrak{g}) \\ 0 & \text{if } v \text{ horizontal,} \end{cases} \tag{19}$$

the connection 1-form.

**Proposition 1.**

$$\mathfrak{P}_g^*\omega = \text{Ad}_{g^{-1}} \circ \omega. \tag{20}$$

**Proof.** Suppose  $v = \sigma(\xi)$ , since the other case left is trivial.

We can carry out some calculations on the left-hand side, and following from the definition of pull-back and Lemma 1, we have

$$(\mathfrak{P}_g^*\omega)(\sigma(\xi)) = \omega(\mathfrak{P}_{g*} \circ \sigma(\xi)) = \omega(\sigma(\text{Ad}_{g^{-1}}(\xi))) = \text{Ad}_{g^{-1}}(\xi). \tag{21}$$

Then, we only need to manipulate the right-hand side as

$$\text{Ad}_{g^{-1}}(\omega(\sigma(\xi))) = \text{Ad}_{g^{-1}}(\xi). \tag{22}$$

Both times, we used just the given definition of connection 1-form (Equation (19)). □

**Remark 1.** This last Proposition is called  $G$ -equivariance. It can be imposed instead of by assuming that  $HE$  is an Ehresmann connection, and then  $HE$  can be shown to be such an Ehresmann connection.

Another fundamental concept is given in the following:

**Definition 13** (Tensorial form). Let  $\rho : G \rightarrow \text{Aut}(V)$  be a representation over a vector space  $V$  and  $\alpha \in \Omega^k(P, V)$  be a vector valued differential form.

We call  $\alpha$  a tensorial form if it is the following:

- horizontal, i.e.,  $\alpha(v_1, \dots, v_k) = 0$  if at least one  $v_i$  is a vertical vector field, and
- equivariant, i.e., for all  $g \in G$ ,  $\mathfrak{P}_g^* \alpha = \rho(g^{-1}) \circ \alpha$ .

We define horizontal and equivariant forms as maps belonging to  $\Omega_G^k(P, V)$ .

**Observation 5:** The connection form  $\omega$  is not, in general, horizontal; thus, it is not a tensorial form,  $\omega \notin \Omega_G^1(P, \mathfrak{g})$ . This will be clear when taking into account how the gauge field transforms under a change of trivialization in Section 4.

### 3.4. Curvature Forms

Given our connection 1-form  $\omega$ , we can proceed in two ways: the first consists in taking a map called the *horizontal projection* and in defining the curvature as this projection applied on the exterior derivative of  $\omega$ . In this way, we naturally see that curvature measures the displacement of the commutator of two vectors from being horizontal.

We will proceed in a different way though. We will define the curvature through a *structure equation*.

**Definition 14.** Given  $\omega \in \Omega^1(P, \mathfrak{g})$ , a principal connection 1-form, the 2-form  $\Omega \in \Omega_G^2(P, \mathfrak{g})$  satisfies the following:

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega] \tag{23}$$

whic is called curvature 2-form.

In Equation (23),  $[\omega \wedge \omega]$  denotes the bilinear operation on the Lie algebra  $\mathfrak{g}$  called differential Lie bracket. It is defined as follows:

$$[\omega \wedge \eta](u, v) = \frac{1}{2}([\omega(u), \eta(v)] - [\omega(v), \eta(u)]), \tag{24}$$

where  $u$  and  $v$  are vector fields.

It follows straightforwardly that, if we take two general horizontal vector fields  $u, v \in \Gamma(HE)$  and we use the ordinary formula<sup>12</sup> for the exterior derivative of a 1-form  $d\omega(u, v) = u\omega(v) - v\omega(u) - \omega([u, v])$ , since  $\omega(u) = \omega(v) = 0$ , we get

$$\Omega(u, v) = -\omega([u, v]). \tag{25}$$

We see that  $\Omega$  measures how the commutator of two horizontal vector fields is far from being horizontal as well.

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<sup>12</sup> Here, we regard  $\omega(u)$  as a function  $\omega(u) : P \rightarrow \mathfrak{g}$  belonging to the algebra of smooth functions to  $\mathfrak{g}$ ,  $C^\infty(P, \mathfrak{g})$ .

## 4. Exterior Covariant Derivative

### 4.1. For an Ehresmann Connection HE

**Observation 6:**  $\Omega_G^k(P, V)$  is not closed under the ordinary exterior derivative. In that sense, if  $\alpha \in \Omega_G^k(P, V)$ , then  $d\alpha \notin \Omega_G^{k+1}(P, V)$ . This is what a covariant differentiation will do instead.

The idea of a covariant exterior derivative for a connection HE is, given such an Ehresmann connection HE, the one of projecting vector fields onto this horizontal bundle and then feed our ordinary exterior derivative with such horizontal vector fields.

First of all, we define a map acting as a pull-back. Namely that, given a map  $h : TP \rightarrow HE$  such that, for all vertical vector fields  $v$ , we get  $h \circ v := hv = 0$  (called the *horizontal projection*), we define the dual map  $h^* : T^*P \rightarrow HE^*$  such that, for  $\alpha \in \Omega^1(P, V)$  and  $V$  a vector space, we have  $h^* \circ \alpha := h^*\alpha = \alpha \circ h$ .

**Definition 15** ( $d^h$ ). Let  $P$  be a  $G$ -principal bundle,  $V$  be a vector space, and  $\alpha \in \Omega^k(P, V)$  be an equivariant form. We define the exterior covariant derivative  $d^h$  as a map  $d^h : \Omega^k(P, V) \rightarrow \Omega_G^{k+1}(P, V)$  such that

$$d^h \alpha(v_0, \dots, v_k) := h^* d\alpha(v_0, \dots, v_k) = d\alpha(hv_0, \dots, hv_k), \tag{26}$$

where  $v_0, \dots, v_k$  are vector fields.

It depends on the choice of our Ehresmann connection HE, which reflects onto the horizontal projection  $h$ ; that is why we have the index  $h$ .

**Observation 7:** We can make our covariant derivative depend only on  $\omega$ , if we restrict it to only forms in  $\Omega_G^k(P, V)$  and if we consider the representation of the algebra induced by the derivative of  $\rho$  that we denote  $d\rho : \mathfrak{g} \rightarrow \text{End}(V)$ . Then, we have  $d\rho \circ \omega \in \Omega^k(P, \text{End}(V))$ .

### 4.2. For a Connection Form $\omega$

**Definition 16** ( $d_\omega$ ). Let  $P$  be a  $G$ -principal bundle,  $V$  be a vector space, and  $\alpha \in \Omega_G^k(P, V)$  be a tensorial form. We define the exterior covariant derivative  $d_\omega$  as a map  $d_\omega : \Omega_G^k(P, V) \rightarrow \Omega_G^{k+1}(P, V)$  such that<sup>13</sup>

$$\begin{aligned} d_\omega \alpha &:= d\alpha + \omega \wedge_{d\rho} \alpha \\ &:= d\alpha + d\rho \circ \omega \wedge \alpha. \end{aligned} \tag{27}$$

**Remark 2.**

– We observe that  $d_\omega^2 \alpha \neq 0$  for a general  $\alpha \in \Omega_G^k(P, V)$ , but it is easy to show that it holds<sup>14</sup>

$$d_\omega^2 \alpha = \Omega \wedge_{d\rho} \alpha, \tag{28}$$

Thus, for a flat connection such that  $\Omega = 0$ , we have  $d_\omega^2 \alpha = d^2 \alpha = 0$ .

<sup>13</sup> For a general  $k$ -form:

$$(\omega \wedge_{d\rho} \alpha)(v_1, \dots, v_{k+1}) = \frac{1}{(1+k)!} \sum_{\sigma} \text{sign}(\sigma) d\rho(\omega(v_{\sigma(1)}))(\alpha(v_{\sigma(2)}, \dots, v_{\sigma(k+1)}).$$

<sup>14</sup> See the first Bianchi identity in Equation (56) for the proof.

- We have observed that  $\omega \notin \Omega_G^1(P, \mathfrak{g})$ . Therefore,  $d_\omega\omega$  is not well defined. However, we can consider  $d^h\omega \in \Omega_G^2(P, \mathfrak{g})$ , and this is precisely our curvature  $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$ , where the anomalous  $\frac{1}{2}$  factor comes from the "non-tensoriality" of  $\omega$ . As a matter of fact, there is no representation that would make the  $\frac{1}{2}$  term arise if we considered  $d_\omega\omega$  instead.
- The fact that  $d_\omega$  is not well defined for non-tensorial forms does not mean that  $\omega$  defines a less general derivative than what  $d^h$  does. As a matter of fact, HE could be defined starting from  $\omega$ , as we mentioned above, since  $HE = \text{Ker } \omega$ .

## 5. Gauge Field and Field Strength

### 5.1. Make It Clear

**Definition 17** (Gauge field). Let  $P \rightarrow M$  be a  $G$ -principal bundle,  $G$  be a Lie group with  $\mathfrak{g}$  as the respective Lie algebra,  $\{U_\beta\}$  be a cover of  $M$ , and  $s_\beta : U_\beta \rightarrow P$  be a section.

We define the gauge field as the pull-back of the connection form  $\omega \in \Omega^1(P, \mathfrak{g})$  as

$$A_\beta = s_\beta^*\omega \in \Omega^1(U_\beta, \mathfrak{g}). \tag{29}$$

We notice that, under a change of trivialization, such a gauge field changes via the action of the adjoint map.

In fact, we have the following:

**Lemma 2.** The restriction of  $\omega$  to  $\pi^{-1}(U_\beta)$  agrees with

$$\omega_\beta = Ad_{g_\beta^{-1}} \circ \pi^* A_\beta + g_\beta^*\theta, \tag{30}$$

where  $g_\beta : \pi^{-1}(U_\beta) \rightarrow G$  is the map induced by the inverse of the trivialization map  $\varphi_\beta$  defined in Equation (2), and with  $Ad_{g_\beta^{-1}}$ , we intend for the adjoint map at the group element given by  $g_\beta(p)^{-1}$  at a point  $p \in \pi^{-1}(U_\beta)$ .

The proof comes from the observation that Equations (19) and (30) coincide in  $\pi^{-1}(U_\beta)$  for both a horizontal (for which they are zero) and a vertical vector field.

Thanks to this, we easily have the following:

**Proposition 2.** Let  $G$  be a matrix Lie group. Then it holds the following transformation for a gauge field:

$$A_\beta = g_{\beta\gamma} A_\gamma g_{\beta\gamma}^{-1} - dg_{\beta\gamma} g_{\beta\gamma}^{-1}. \tag{31}$$

**Proof.** Using Equations (29) and (30) for all  $x \in U_\beta \cap U_\gamma$ ,

$$\begin{aligned} A_\beta &= s_\beta^*\omega \\ &= s_\beta^*\omega_\beta = s_\beta^*\omega_\gamma \\ &= s_\beta^*(Ad_{g_\gamma^{-1}} \circ \pi^* A_\gamma + g_\gamma^*\theta) \\ &= Ad_{g_{\beta\gamma}^{-1}} \circ A_\gamma + g_{\beta\gamma}^*\theta \quad (\text{using } g_\gamma \circ s_\beta := g_{\beta\gamma} : U_\beta \cap U_\gamma \rightarrow G) \\ &= Ad_{g_{\beta\gamma}} \circ (A_\gamma - g_{\beta\gamma}^*\theta) \quad (Ad_{g_{\beta\gamma}} \circ g_{\beta\gamma}^*\theta = -g_{\beta\gamma}^*\theta), \end{aligned} \tag{32}$$

which reduces to the assert for matrix Lie groups.  $\square$

**Observations 8:**

- i. We observe that a local gauge transformation of the gauge field corresponds to a change of trivialization chart.
- ii. Non-tensoriality of  $\omega$  was given by the fact that it is, in general, not horizontal. For the gauge field  $A$ , we can generalize to forms on  $M$  the concept of tensoriality/non-tensoriality by noticing that a form obtained by the pull-back of a tensorial form, denoted with  $t \in \Omega_G^1(P, V)$ , would transform differently compared to  $A$ , namely as

$$t_\beta := s_\beta^* t = g_{\beta\gamma} t_\gamma g_{\beta\gamma}^{-1}. \tag{33}$$

- The Maurer–Cartan form  $\theta$  reflects the non-horizontality of  $\omega$  to the gauge field, from Equation (30).
- iii. A difference of two gauge fields like  $A - A'$  transforms as Equation (33). In fact, the transformation rule is one of a tensorial form, since the Maurer–Cartan forms simplify.
  - iv. We notice that (iii) is a particular case of a more general one. Indeed, it is possible to show with proof in Reference [2] (Chapter 5) that  $\Omega_G^k(P, V) \cong \Omega^k(M, P \times_\rho V)$ . This is essentially due to the fact that, thanks to the equivalence relation of the associated bundle and the gluing condition of sections on overlaps, the pull-backs by sections  $s_\beta : U_\beta \rightarrow P$  give a one-to-one correspondence between these two spaces. Therefore, we can obtain forms with a tensorial transformation like Equation (33) just by taking the pull-back of tensorial forms on  $P$ ; these will be forms on  $M$  with values into the associated bundle  $P \times_\rho V$ .
  - v. Observations (iii) and (iv) ensure that an object built with gauge fields  $A_\beta \in \Omega^1(U_\beta, \mathfrak{g})$  (which transform on overlaps by Equation (31)) will be in  $\Omega^2(M, P \times_{\text{Ad}} \mathfrak{g})$ ; see Observation 9.

**Claim.** *This gauge field defines an exterior covariant derivative for bundle-valued forms on  $M$ . We denote such a map with*

$$d_A : \Omega^k(M, P \times_\rho V) \rightarrow \Omega^{k+1}(M, P \times_\rho V). \tag{34}$$

Anyway, we will further develop this argument in Section 6.1.

We can proceed analogously and can define the pull-back of the curvature:

**Definition 18** (Field strength). *Let  $P \rightarrow M$  be a  $G$ -principal bundle,  $G$  be a Lie group with  $\mathfrak{g}$  as the respective Lie algebra,  $\{U_\beta\}$  be a cover of  $M$ , and  $s_\beta : U_\beta \rightarrow P$  be a section.*

*We define the field strength as the pull-back of the curvature form  $\Omega \in \Omega_G^2(P, \mathfrak{g})$  as*

$$F_\beta = s_\beta^* \Omega \in \Omega_G^2(U_\beta, \mathfrak{g}), \tag{35}$$

which, by definition of  $\Omega$ , is

$$F_\beta = dA_\beta + \frac{1}{2}[A_\beta \wedge A_\beta]. \tag{36}$$

Similarly to what we have done for the gauge field, we can show<sup>15</sup> that the field strength transforms as

$$F_\beta = \text{Ad}_{g_{\beta\gamma}} \circ F_\gamma = g_{\beta\gamma} F_\gamma g_{\beta\gamma}^{-1}, \tag{37}$$

---

<sup>15</sup> Using the Cartan structure equation for  $\theta$ ,  $d\theta = -\frac{1}{2}[\theta, \theta]$ .

where the last equality holds for matrix Lie groups with  $g$  and  $g^{-1}$  in  $G$ . This is indeed the transformation of a tensorial form, as in Equation (33).

**Observation 9:** Thanks to our previous observation, i.e., there is a canonical isomorphism between  $\Omega_G^k(P, V)$  and  $\Omega^k(M, P \times_\rho V)$ , we can relate  $\Omega$  and  $F_\beta$  with a form<sup>16</sup>  $F_A \in \Omega^2(M, \text{ad}P)$ . Namely there is a canonical isomorphism sending  $\Omega \in \Omega_G^2(P, \mathfrak{g})$  to  $F_A \in \Omega^2(M, \text{ad}P)$ . Indeed, given the transformation law for the field strength in Equation (37), we see that  $\{F_\beta\}$  is horizontal and equivariant and, thus, forms a global section belonging to  $\Omega^2(M, \text{ad}P)$ , which is usually denoted as  $F_A$ .

The notation  $F_A$  stresses that it is obtained from gauge fields in  $\Omega^1(U_\beta, \mathfrak{g})$ .

In the case of a trivial bundle, it is also possible to define a global gauge field  $A \in \Omega^1(M, \mathfrak{g})$ .

### 5.2. 2nd Bianchi Identity

Consider  $d_A : \Omega^k(M, P \times_\rho V) \rightarrow \Omega^{k+1}(M, P \times_\rho V)$  as the exterior covariant derivative and  $F_A \in \Omega^2(M, \text{ad}P)$  as the field strength.

Then, we have the following:

**Proposition 3.**

$$d_A F_A = 0. \tag{38}$$

This is the second Bianchi identity.

**Proof.** Given

$$F_A = dA + \frac{1}{2}[A \wedge A], \tag{39}$$

then

$$\begin{aligned} d_A F_A &= dF_A + [A \wedge F_A] \\ &= d^2 A + \frac{1}{2}d[A \wedge A] + [A \wedge dA] + \frac{1}{2}[A \wedge [A \wedge A]] \\ &= \frac{1}{2}[A \wedge [A \wedge A]] \quad (d^2 A = 0 \text{ and } \frac{1}{2}d[A \wedge A] = -[A \wedge dA]) \\ &= 0. \quad (\text{because of Jacobi identity}) \end{aligned} \tag{40}$$

□

## 6. Affine Formulation

In the usual formulation of General Relativity, one defines a covariant derivative  $\nabla$ , which is a map among tensors. Then, one can define curvature and torsion and eventually get the field equations for ECSK theory or General Relativity by setting torsion to zero.

One may wonder if this latter formulation is equivalent to the one we have been implementing through principal bundles and principal connection.

The answer is positive and is given in the next two sections.

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<sup>16</sup> Where we have introduced the notation  $\Omega^k(M, P \times_{\text{Ad}} \mathfrak{g}) := \Omega^k(M, \text{ad}P)$ .

### 6.1. Affine Covariant Derivative

We have built our setup by taking  $\rho$  to be the fundamental representation of  $O(3, 1)$ ,  $P = F_O(M)$ , and  $\mathcal{V} = F_O(M) \times_\rho V$  to be the Minkowski bundle, as in (ii) of Observations 4. Therefore, as mentioned above, the isomorphism between  $\Omega_G^k(P, V)$  and  $\Omega^k(M, P \times_\rho V)$  allows to define an exterior covariant derivative of forms in  $\Omega^1(M, \mathcal{V})$ :

$$d_A : \Omega^k(M, \mathcal{V}) \rightarrow \Omega^{k+1}(M, \mathcal{V}). \tag{41}$$

In this way, we have a covariant differentiation for  $\mathcal{V}$ -valued differential forms on  $M$  and, thus, also for tetrads, since  $e \in \Omega^1(M, \mathcal{V})$ .

Since we note that  $d\rho$  induces a one form  $d\rho(A) \in \Omega^1(M, \text{End}(V))$ , we can further define another kind of derivative that “takes care” of internal indices only; in particular, this will not be necessarily a map between differential forms.

This derivative in components reads, for  $\phi \in \Gamma(\mathcal{V})$ ,

$$(D_A \phi)_\mu^a = (\partial_\mu \phi^a + A_\mu^{ac} \eta_{cb} \phi^b) \tag{42}$$

and, for  $\alpha \in \Omega^k(M, \mathcal{V})$ ,

$$(D_A \alpha)_{\mu\nu_1 \dots \nu_k}^a = (\partial_\mu \alpha_{\nu_1 \dots \nu_k}^a + A_\mu^{ac} \eta_{cb} \alpha_{\nu_1 \dots \nu_k}^b), \tag{43}$$

which shows that it does not map  $\alpha$  to a differential form.

Now, we immediately apply the inverse of a tetrad to  $D_A \phi$  and identify it with  $\nabla$ .

In fact, we take a vector field  $X \in \Gamma(TM)$ , feed the tetrad  $e$  with it, then apply<sup>17</sup>  $D_A$  to get  $D_A(\iota_X e)$ , and finally pull it back with the inverse of the tetrad  $\bar{e}$ .

In components, this reads as follows:

$$(D_A(\iota_X e))_\mu^a = D_\mu(e_V^a X^\nu) = \partial_\mu(e_V^a X^\nu) + \omega_\mu^{ab} \eta_{bc} e_V^c X^\nu, \tag{44}$$

where, for reasons of metric compatibility<sup>18</sup> with  $\eta$ , we have the only antisymmetric part of the gauge field, for which we used the notation  $A_{\mu c}^a = \omega_\mu^{ab} \eta_{bc}$ .

**Be aware:** Do not get confuse. We shall refer to  $\omega \in \Omega^1(M, \Lambda^2 \mathcal{V})$  as the *spin connection*. To stress that we want  $D_A$  to depend on the spin connection only, we shall denote it with  $D_\omega$ .

Pulling back via  $\bar{e}$ , we obtain

$$\bar{e}_a^\sigma (D_\mu(e_V^a X^\nu)) = \bar{e}_a^\sigma (\partial_\mu(e_V^a X^\nu) + \omega_\mu^{ab} \eta_{bc} e_V^c X^\nu). \tag{45}$$

We define the Christoffel symbols  $\Gamma_{\mu\nu}^\sigma$  as

$$\begin{aligned} \Gamma_{\mu\nu}^\sigma &= \bar{e}_a^\sigma (D_\mu e_V^a) \\ &= \bar{e}_a^\sigma (\partial_\mu e_V^a + \omega_\mu^{ab} \eta_{bc} e_V^c) \end{aligned} \tag{46}$$

<sup>17</sup> Here, we use the so-called *interior product*, i.e., a map  $\iota_\xi : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ , such that  $(\iota_\xi \alpha)(X_1, \dots, X_{k-1}) = \alpha(\xi, X_1, \dots, X_{k-1})$ , for vector fields  $\xi, X_1, \dots, X_{k-1}$ . Furthermore it respects  $\iota_\xi(\alpha \wedge \beta) = (\iota_\xi \alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_\xi \beta)$ , where  $\alpha \in \Omega^k(M)$ . Therefore, it forms an antiderivation. The relation with the Lie derivative is given by the formula  $\mathcal{L}_\xi \alpha = d(\iota_\xi \alpha) + \iota_\xi d\alpha$ , called the Cartan identity. The interior product of a commutator satisfies  $\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y]$ , with  $X$  and  $Y$  vector fields.

<sup>18</sup> In fact, imposing the condition of  $D_A \eta = 0$  implies the antisymmetry of the gauge field.

and, thus, we get

$$\nabla_\mu X^\sigma := \bar{e}_a^\sigma (D_\mu (e^a_\nu X^\nu)) = \partial_\mu X^\sigma + \Gamma_{\mu\nu}^\sigma X^\nu, \tag{47}$$

which is the covariant derivative well known in General Relativity.

We can also see what the curvature form is in terms of the commutator of two derivatives, given by the only antisymmetric part of the connection.

Then,

$$F_{A^{antis}} := F_\omega \tag{48}$$

and it is given by

$$(D_{[\mu} D_{\nu]} \phi)^a = (\partial_{[\mu} \omega_{\nu]}^{ab} + \omega_{[\mu}^{ad} \eta_{de} \omega_{\nu]}^{eb}) \eta_{bc} \phi^c = F_{\mu\nu}^{ab} \eta_{bc} \phi^c, \tag{49}$$

where  $A_{[\mu} B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$  is our convention for the antisymmetrization. The fact that  $F_\omega$  is a 2-form shows that  $F_{\mu\nu}^{ab} = -F_{\nu\mu}^{ab}$ ; furthermore, metric compatibility ensures also  $F_{\mu\nu}^{ab} = -F_{\mu\nu}^{ba}$ , therefore  $F_\omega \in \Omega_G^2(M, \Lambda^2 \mathcal{V})$ .

**Observation 10:** We see<sup>19</sup> that, here, the bundle metric  $\eta$  acts as a map  $\eta : \Omega^2(M, \Lambda^2 \mathcal{V}) \rightarrow \Omega^2(M, \text{End}(V))$  isomorphically; thus, it permits to identify elements of the second exterior power  $\Lambda^2 \mathcal{V}$  with linear maps given by the fundamental representation of the algebra  $\mathfrak{g} = \mathfrak{so}(3, 1)$ . We can introduce the notation for the wedge product in the fundamental representation as  $\wedge_f$ ; namely for, say, an  $\alpha \in \Omega^1(M, \mathcal{V})$ , we have  $(\omega \wedge_f \alpha)^a = \omega^{ab} \eta_{bc} \wedge \alpha^c$ .

### 6.2. Riemann Curvature Tensor

We can now consider the commutator of two affine covariant derivatives and use Equation (47), getting

$$(\nabla_{[\mu} \nabla_{\nu]} X)^\sigma = \bar{e}_a^\sigma (D_{[\mu} D_{\nu]} (t_X e))^\sigma = \bar{e}_a^\sigma F_{\mu\nu}^{ab} \eta_{bc} e_\omega^c X^\omega. \tag{50}$$

We identify the Riemann tensor

$$R_{\mu\nu\omega}{}^\sigma = \bar{e}_a^\sigma F_{\mu\nu}^{ab} \eta_{bc} e_\omega^c, \tag{51}$$

the Ricci curvature tensor

$$R_{\mu\omega} = R_{\mu\sigma\omega}{}^\sigma = \bar{e}_a^\sigma F_{\mu\sigma}^{ab} \eta_{bc} e_\omega^c \tag{52}$$

and thus the Ricci scalar

$$R = g^{\mu\omega} R_{\mu\omega} = \bar{e}_d^\mu \bar{e}_e^\omega \eta^{de} \bar{e}_a^\sigma F_{\mu\sigma}^{ab} \eta_{bc} e_\omega^c = -\bar{e}_a^\mu \bar{e}_b^\omega F_{\mu\omega}^{ab}. \tag{53}$$

It follows the antisymmetry of the Riemann tensor in the indices  $\mu\nu$  and  $\omega\sigma$ , but it is important to note that we cannot ensure any symmetry in the Ricci curvature instead due to the presence of torsion.

### 7. Torsion

Here, we start focusing on the importance of torsion, which arises quite naturally as curvature does.

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<sup>19</sup> We denote the metric acting on the bundle and the metric acting on the fibers in the same way; however, we will usually deal with elements of the fibers.

### 7.1. Torsion Form

**Definition 19** (Solder form/soldering of a  $G$ -principal bundle). Let  $\pi : P \rightarrow M$  be a smooth  $G$ -principal bundle over a differentiable manifold  $M$ ,  $\rho : G \rightarrow \text{Aut}(V)$  be a representation, and  $G$  be a Lie group.

We define the solder form, or soldering, as the vector-valued 1-form  $\theta \in \Omega_G^1(P, V)$  such that  $\tilde{\theta} : TM \rightarrow P \times_\rho V$  is a bundle isomorphism, where  $\tilde{\theta} \in \Omega^1(M, P \times_\rho V)$  is the associated bundle map induced by the isomorphism of  $\Omega_G^1(P, V) \cong \Omega^1(M, P \times_\rho V)$ .

#### Observations 11:

- The choice of the solder form is not unique, in general.
- We can observe that, taking  $P = F_O(M)$ ,  $\rho$  as the fundamental representation of  $O(3, 1)$ , and  $V$  as the vector space with reference metric  $\eta$ ,  $\tilde{\theta}$  corresponds to our definition of tetrads. The different choices of soldering give rise to different tetrads.
- In the case that  $P = F_O(M)$  and that the associated bundle is simply chosen to be  $TM$ , the solder form is called *canonical* or *tautological*. Since the associated bundle  $TM$  sets the bundle isomorphism  $\tilde{\theta}$  to be the identity map  $\text{id} : TM \rightarrow TM$ .
- In Observations 4, we mentioned that the Minkowski bundle cannot be canonically identified with the tangent bundle itself; indeed, we fixed a reference metric  $\eta$ , which cannot be pulled back by the identity map to give the metric on  $TM$  in general, and thus, the solder form is not canonical.

The soldering of the principal frame bundle allows us to define the torsion form<sup>20</sup>.

**Definition 20** (Torsion form). Let  $P = F_O(M)$ ,  $\rho : O(3, 1) \rightarrow \text{Aut}(V)$  be the fundamental representation,  $V$  be a vector space with reference metric  $\eta$ , and  $\theta \in \Omega_G^1(P, V)$  be a solder form.

We define the torsion form  $\Theta \in \Omega_G^2(P, V)$  as follows:

$$\Theta = d_\omega \theta = d\theta + \omega \wedge_f \theta. \tag{54}$$

### 7.2. Torsion in a Local Basis

We would like to express the torsion form in terms of tetrads and the gauge field.

In Reference[9], a formula is given and it is obtained by applying the previous definition of the torsion form under the canonical isomorphism  $\Omega_G^k(P, V) \cong \Omega^k(M, \mathcal{V})$ ; therefore yielding

$$\tilde{\Theta}^a = (d_A e)^a = de^a + A^{ab} \eta_{bc} \wedge e^c. \tag{55}$$

### 7.3. 1st Bianchi Identity

**Proposition 4.** Following our previous definitions, we have

$$d_\omega \Theta = \Omega \wedge_f \theta, \tag{56}$$

which is called the first Bianchi identity.

**Proof.** For this proof, we prefer using Equation (26).

We consider three vector fields  $u, v, w \in \Gamma(TP)$ . By definition, it follows

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<sup>20</sup> Torsion can be defined for every principal bundle, but physics arises when considering the frame bundle.

$$\begin{aligned}
 d^h\Theta(u, v, w) &= d\Theta(hu, hv, hw) \\
 &= (d\omega \wedge_f \theta - \omega \wedge_f d\theta)(hu, hv, hw) \quad (\text{because of Equation (54)}) \\
 &= d\omega \wedge_f \theta(hu, hv, hw) \quad (\text{because of Equation (19)}) \\
 &= \Omega \wedge_f \theta(u, v, w).
 \end{aligned}
 \tag{57}$$

The last equality holds because of tensoriality of  $\theta$  and the second remark in Remark 2.  $\square$

This proposition is a natural consequence of the property of the covariant differential expressed in Equation (28).

#### 7.4. Torsion Tensor

**Definition 21** (Torsion tensor). *Given two vector fields  $X, Y \in \Gamma(TM)$  and a 1-form  $\tau \in \Omega^1(M)$ , we define the torsion tensor field  $Q$  as the tensor field of type- $\binom{1}{2}$  such that*

$$Q(X, Y; \tau) := \tau(Q(X, Y)) = \tau(\bar{e}(d_A e(X, Y))). \tag{58}$$

*It is evidently antisymmetric in  $X, Y$ , by definition.*

**Proposition 5.** *We have the following formula:*

$$Q(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{59}$$

*and, in components, it reads*

$$Q_{\mu\nu}{}^\sigma = \Gamma_{\mu\nu}^\sigma - \Gamma_{\nu\mu}^\sigma - C_{\mu\nu}^\sigma \tag{60}$$

*where  $C_{\mu\nu}^\sigma = 0$  in a holonomic basis for  $X$  and  $Y$  and  $\nabla$  is the covariant derivative<sup>21</sup>.*

**Proof.** Recalling the definition of torsion

$$Q = \bar{e} \cdot (d_\omega e) = \bar{e}_a (d_\omega e)^a, \tag{61}$$

it follows

$$\begin{aligned}
 \bar{e}_a (d_\omega e)^a &= \bar{e}_a^\sigma (\partial_{[\mu} e^a_{\nu]} + \omega^a_{[\mu b} e^b_{\nu]}) dx^\mu \wedge dx^\nu \otimes \partial_\sigma \\
 &= (\Gamma_{\mu\nu}^\sigma - \Gamma_{\nu\mu}^\sigma) dx^\mu \wedge dx^\nu \otimes \partial_\sigma \quad (\Gamma_{\mu\nu}^\sigma = \bar{e}_a^\sigma (D_\mu e_\nu^a)) \\
 &= Q_{\mu\nu}{}^\sigma dx^\mu \wedge dx^\nu \otimes \partial_\sigma,
 \end{aligned}
 \tag{62}$$

then  $Q_{\mu\nu}{}^\sigma = \Gamma_{\mu\nu}^\sigma - \Gamma_{\nu\mu}^\sigma$ .  $\square$

We have now set up all the background for building our theory and for discussing field equations of ECSK theory.

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<sup>21</sup> See Reference [10] for references about this.

### 8. Field Equations and Conservation Laws

We present here field equations for ECSK theory<sup>22</sup>. Thus, we will neither assume the possibility of a propagating torsion (and we will always keep non-identically vanishing Riemann curvature [14]) nor display a lagrangian for a totally independent torsion field; rather, we will only set the Palatini–Cartan lagrangian for gravity, as done in Reference [15], and a matter lagrangian as the source. This theory is known as Einstein–Cartan–Sciama–Kibble gravity (ECSK).

In the present case, torsion reduces to an algebraic constraint. This is a consequence of making torsion join the action of the theory as only contained in the Ricci scalar because of a non-torsion-free connection and not with an independent coupling coefficient. In works like References [16–19], torsion is present as an independent part (independent coupling coefficient) of the action and it does propagate.

This is why the ECSK is considered as the most immediate generalization of General Relativity with the presence of torsion.

Therefore, we wish to eventually obtain an action of two independent objects, tetrads and connection, where this latter action should give rise to equations for curvature when varying tetrads and for torsion when varying the connection.

We will focus more on the geometrical side of these equations and we will not dwell on deepening matter interaction (couplings, symmetry breaking, etc.), as done for instance in References [20–24].

#### 8.1. ECSK Equations

ECSK theory with cosmological constant belongs to the Lovelock–Cartan family, which describes the most general action in four dimensions such that this action is a polynomial on the tetrads and the spin connection (including derivatives), is invariant under diffeomorphisms and local Lorentz transformations, and is constructed without the Hodge dual<sup>23</sup>.

Recalling that we will refer to  $A$  as  $\omega$  and stressing that it must be only the antisymmetric part, the notation for  $d_A$  becomes  $d_\omega$ .

We will be dealing with a variational problem given by an action of the kind

$$S = S_{PC} + S_{matter}, \tag{63}$$

where the Palatini–Cartan action is

$$S_{PC}[e, \omega] = \int_M \text{Tr} \left[ \frac{1}{2} e \wedge e \wedge F_\omega + \frac{\Lambda}{4!} e^4 \right]. \tag{64}$$

The wedge product is defined over both space–time and internal indices as a map<sup>24</sup>  $\wedge : \Omega^k(M, \Lambda^p \mathcal{V}) \times \Omega^l(M, \Lambda^q \mathcal{V}) \rightarrow \Omega^{k+l}(M, \Lambda^{p+q} \mathcal{V})$  and the trace is a map  $\text{Tr} : \Lambda^4 \mathcal{V} \rightarrow \mathbb{R}$ , normalized such that (for  $v_i$  elements of a basis in  $\mathcal{V}$ )  $\text{Tr}[v_i \wedge v_j \wedge v_k \wedge v_l] = \varepsilon_{ijkl}$ . The choice of the normalization of the trace works as a choice of orientation for  $M$  (since the determinant of a matrix in  $O(3, 1)$  may be  $\pm 1$ ). Therefore, we reduce the total improper Lorentz group  $O(3, 1)$  to the only orientation preserving part, which is still not connected,  $SO(3, 1)$ . This gives an invariant volume form on  $M$ . In this way, we consider sections of  $\Lambda^k T^* M \otimes \Lambda^p \mathcal{V}$ .

<sup>22</sup> Some classical works about ECSK theory and General Relativity with torsion, like References [11–13].

<sup>23</sup> See Reference [25] for details.

<sup>24</sup> Such that, for  $\alpha \in \Omega^k(M, \Lambda^p \mathcal{V})$  and  $\beta \in \Omega^l(M, \Lambda^q \mathcal{V})$ , we have  $\alpha \wedge \beta = (-1)^{(k+p)(l+q)} \beta \wedge \alpha$ .

**Be aware:** Later on, we will make explicit some indices and keep implicit some others; for this purpose, we will specify what kind of wedge product we are dealing with, even though it will be evident because it will be among the implicit indices.

We recall the definition of  $F_\omega$  and deduce the identity for its variation

$$\delta_\omega F_\omega = d_\omega \delta\omega, \tag{65}$$

where we stress that, despite  $\omega$  being non-tensorial,  $\delta\omega$  is instead, and therefore, it transforms under the adjoint action (as  $F_\omega$  does) like in Equation (33). That holds because  $\delta\omega$  may be regarded as a difference of two spin connections.

The action for the matter is of the kind

$$S_{matter}[e, \omega, \varphi] = \kappa \int_M \text{Tr}[L(e, \omega, \varphi)], \tag{66}$$

where  $L$  is an invariant lagrangian density form with the proper derivative order in our variables,  $\varphi$  is a matter field, and  $\kappa$  is a constant.

Such matter lagrangian is supposed to be source for both curvature and torsion equations, namely it will be set for fulfilling some conditions fitting the theory.

Therefore, varying the actions in Equations (64) and (66) and considering Equation (65), we have<sup>25</sup>

$$\begin{aligned} \int_M \text{Tr}[\delta e \wedge (e \wedge F_\omega + \frac{\Lambda}{3!} e^3)] &= \int_M \text{Tr}[\kappa \frac{\delta L}{\delta e} \wedge \delta e] \\ \int_M \text{Tr}[\frac{1}{2} d_\omega(e \wedge e) \wedge \delta\omega] &= \int_M \text{Tr}[\kappa \frac{\delta L}{\delta \omega} \wedge \delta\omega], \end{aligned} \tag{67}$$

which is equivalent to

$$\begin{aligned} \varepsilon_{abcd} e^b \wedge F_\omega^{cd} + \frac{\Lambda}{3!} \varepsilon_{abcd} e^b \wedge e^c \wedge e^d &= \kappa \frac{\delta \text{Tr}[L]}{\delta e^a} := \kappa T_a \\ \frac{1}{2} \varepsilon_{abcd} d_\omega(e^c \wedge e^d) &= \kappa \frac{\delta \text{Tr}[L]}{\delta \omega^{ab}} := \kappa \Sigma_{ab} \end{aligned} \tag{68}$$

where the wedge product here is only between differential forms.

Setting  $\Lambda = 0$  and in performing the derivative, Equation (68) can be rewritten as

$$\begin{aligned} \varepsilon_{abcd} e^b \wedge F_\omega^{cd} &= \kappa T_a \\ \varepsilon_{abcd} \tilde{Q}^c \wedge e^d &= \kappa \Sigma_{ab}, \end{aligned} \tag{69}$$

where we have set  $\tilde{Q} = d_\omega e$ .

These are equations for the ECSK theory in their implicit form<sup>26</sup>, where  $T$  and  $\Sigma$  are related to, respectively, the *energy momentum* and the *spin* tensor, once pulled back.

By making all the indices explicit, as given in Reference [20], and properly setting  $\kappa$  according to natural units<sup>27</sup>, Equation (69) takes the following form

$$\begin{aligned} G_{\mu\nu} &= 8\pi T_{\mu\nu} \\ Q_{\mu\nu}{}^\sigma &= -16\pi \Sigma_{\mu\nu}{}^\sigma. \end{aligned} \tag{70}$$

<sup>25</sup> Omitting equations of motion  $\frac{\delta L}{\delta \varphi} = 0$  for the matter field, which have to be satisfied for conservation laws anyway.

<sup>26</sup> Without making space-time indices explicit.

<sup>27</sup> All fundamental constants = 1.

**Observations 12:**

- i.  $T_{\mu\nu}$  is not symmetric, as expected from the non-symmetry of the Ricci curvature  $R_{\mu\nu}$ .
- ii. We stress that, even though  $e$  is an isomorphism, the map  $e \wedge \cdot : \Omega^k(M, \Lambda^p \mathcal{V}) \rightarrow \Omega^{k+1}(M, \Lambda^{p+1} \mathcal{V})$  is not an isomorphism, in general. In fact, taking  $\frac{\delta L}{\delta e} = 0$  (with  $\Lambda = 0$ ) in Equation (67) does not imply  $F_\omega = 0$ , which would imply a flat connection.
- iii. Setting  $\frac{\delta L}{\delta \omega} = 0$  in Equation (67), one recovers the condition of vanishing torsion (hence, a Levi-Civita connection) and, therefore, the Einstein equations.
- iv. It is interesting to note that, requiring a totally antisymmetric spin tensor, sets the total antisymmetry of the torsion tensor. Namely, in the case of a totally antisymmetric  $\Sigma$ , we need to couple the only totally antisymmetric part of torsion into the geometrical lagrangian. This is further discussed in Reference [20].

8.2. Conservation Laws

We have two symmetries, i.e., local Lorentz transformations and diffeomorphisms. They are continuous symmetries, and as such, we expect two conservation laws. Since we are dealing with local symmetries, we shall not find two conserved currents but rather two relations for the variations of the matter lagrangian w.r.t.  $e$  and  $\omega$ .

These relations directly imply the Bianchi identities of Equations (38) and (56), but we could also do the converse, namely assuming Equations (38) and (56) and then deriving such conservation laws. This means that conservation laws are a consequence of symmetry on the one hand, implemented via the following symmetries (respectively diffeomorphisms and local  $SO(3, 1)$ )

$$\begin{aligned} \delta_\zeta e^a &= \mathcal{L}_\zeta e^a = \iota_\zeta d e^a + d \iota_\zeta e^a \\ \delta_\zeta \omega^{ab} &= \mathcal{L}_\zeta \omega^{ab} = \iota_\zeta d \omega^{ab} + d \iota_\zeta \omega^{ab}, \end{aligned} \tag{71}$$

where  $\zeta$  is the generator vector field,

$$\begin{aligned} \delta_\Lambda e^a &= \Lambda^{ab} \eta_{bc} e^c \\ \delta_\Lambda \omega^{ab} &= -d_\omega \Lambda^{ab} \quad \Lambda \in \mathfrak{so}(3,1), \end{aligned} \tag{72}$$

or a direct consequence if we impose field equations and, thus, gravitational dynamics and Bianchi identities on the other hand.

We will follow the shortest derivation, namely to implement the Bianchi identities of Equations (38) and (56) on field Equation (69).

Thanks to Bianchi identities, left hand side of field Equation (69) can be rewritten in the following way:

$$\begin{aligned} d_\omega (\varepsilon_{abcd} e^b \wedge F_\omega^{cd}) &= \iota_a \tilde{Q}^b \wedge (\varepsilon_{bcde} e^c \wedge F_\omega^{de}) + \iota_a F_\omega^{bc} \wedge (\varepsilon_{bcde} \tilde{Q}^d \wedge e^e) \\ d_\omega (\varepsilon_{abcd} \tilde{Q}^c \wedge e^d) &= -\frac{1}{2} (\varepsilon_{acde} e^c \wedge F_\omega^{de} \wedge e_b - \varepsilon_{bcde} e^c \wedge F_\omega^{de} \wedge e_a), \end{aligned} \tag{73}$$

where  $\iota_a = \iota_{\tilde{e}_a}$  and  $e_b = \eta_{bc} e^c$ .

However, because of the same field in Equation (69), they reduce to

$$\begin{aligned} d_\omega T_a &= \iota_a \tilde{Q}^b \wedge T_b + \iota_a F_\omega^{bc} \wedge \Sigma_{bc} \\ d_\omega \Sigma_{ab} &= \frac{1}{2} T_{[a} \wedge e_{b]}, \end{aligned} \tag{74}$$

In References [26,27], a more detailed discussion can be found. These are conservation laws for ECSK theory.

In components, as given in Reference [20], they read

$$\begin{aligned}\nabla_{\mu} T^{\mu\nu} + T_{\sigma\rho} Q^{\sigma\rho\nu} - \Sigma_{\mu\sigma\rho} R^{\mu\sigma\rho\nu} &= 0 \\ \nabla_{\mu} \Sigma_{\sigma\omega}{}^{\mu} + \frac{1}{2} T_{[\sigma\omega]} &= 0.\end{aligned}\tag{75}$$

## 9. Conclusions

We have set up all the mathematical background for building ECSK theory, eventually achieving field equations and conservation laws.

In ECSK theory, torsion is only an algebraic constraint and it does not propagate. This is a natural consequence of inserting torsion into the theory without an independent coupling coefficient but simply generalizing the Einstein–Hilbert action (or Palatini action in our formulation)  $\int R\sqrt{-g}d^4x$  to a non-torsion-free connection  $\nabla$  (or spin connection in our case). In this case, the Ricci scalar contains both curvature and torsion.

It is possible to immediately recover General Relativity by imposing the zero torsion condition, which, in the considered theory, translates to letting the matter field  $\varphi$  generate a null contribution to the spin tensor  $\Sigma_{\mu\nu}{}^{\sigma}$ . The most natural matter fields which would fit with the theory are spinors; indeed, spinors are the way in which we can have a non-vanishing spin tensor which is also dynamical because of equations of motion for the spinor field.

This review does not want to substitute the well-known literature but to just give a self-contained and mathematically rigorous introduction to ECSK theory, providing also some references for deepening knowledge in the present subjects. Also, we intentionally did not dive too deeply into physical applications to cosmology (like done in References [28–33]), that might be a valid argument for writing another review article.

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