Torsion geometry and scalar functions

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Til mine forældre!

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Abstract

ONE ILLUMINATING EXAMPLE of the interplay between mathematics and physics is the relation between symplectic geometry and mechanics. A symplectic manifold is characterised by a closed, non-degenerate form of degree two. In modern physics higher degree differential forms play an important role too. In this thesis, we study geometries that are either completely or partly specified in terms of a differential form.

In the first part of the thesis, three-forms play the main role. When the form is closed, we call the geometry *strong*. One particular class of examples comes from torsion geometry, where the three-form appears as the torsion of a metric connection. Our first main result is a classification of invariant strong Kähler with torsion structures on four-dimensional solvable Lie groups.

We then pass on to study strong geometries in general. When these come with a Lie group action which preserves the strong structure, we introduce a notion of moment map. While the basic ideas come from the theory of symplectic moment maps, the adaption to strong geometry with symmetry group requires several fundamentally new approaches. We show existence of our multi-moment maps in many circumstances, including mild topological assumptions on the underlying manifold. Such maps are also shown to exist for all groups whose second and third Lie algebra Betti numbers are zero. We show that these form a special class of solvable Lie groups and provide a structural characterisation. We give many examples of multi-moment maps for different geometries, including strong hyperKähler manifolds with torsion and strict nearly Kähler six-manifolds.

By generalising the arguments, we obtain a notion of multi-moment map for geometries with closed forms of higher degree. As in the three-form case, these maps often exist, for instance, under mild topological assumptions on the underlying manifold, or if the Lie group of symmetries has a vanishing pair of Lie algebra Betti numbers.

One intriguing application of multi-moment maps addresses the classification of Riemannian manifolds with exceptional holonomy and an isometric action of a torus. We explore the cases when the multi-moment map is a scalar function. Via a reduction procedure, the study of these exceptional holonomy spaces is related to tri-symplectic geometry in dimension four.

In the last part of the thesis, we introduce a Calabi-Yau problem for hyperKähler manifolds with torsion, and we take the first steps towards a solution via the continuity method.

Torsionsgeometri og skalarfunktioner

Sammenfatning

Et illustrativt eksempel på samspillet mellem matematik og fysik udgøres af koblingen mellem symplektisk geometri og mekanik. En symplektisk mangfoldighed er karakteriseret ved tilstedeværelsen af en lukket, ikkedegenereret toform. I moderne fysik spiller differentialformer af højere grad også en vigtig rolle. I denne afhandling studeres geometrier, som enten helt eller delvist er karakteriseret ved hjælp af en differentialform.

I den første del af afhandlingen udspilles hovedrollen af treformer. Når formen er lukket, kaldes geometrien *stærk*. En vigtig kilde til eksempler udgøres af torsionsgeometrier, hvor treformen optræder som torsionen af en metrisk konnektion. Vores første hovedresultat er en klassifikation af invariante stærke Kähler-med-torsion strukturer på firedimensionale opløselige Lie grupper.

Dernæst vendes blikket mod generelle stærke geometrier. Når disse er udstyret med en Lie gruppevirkning, som bevarer den stærke struktur, indføres et momentafbildningsbegreb. Inspirationskilden er symplektiske momentafbildninger, men tilpasningen til stærk geometri er baseret på en række fundamentalt nye observationer. Vi beviser eksistens af vores multi-momentafbildninger i en række situationer, blandt andet under milde topologiske antagelser om den underliggende mangfoldighed. Afbildningerne eksisterer også, hvis symmetrigruppen har andet og tredje Lie algebra Betti tal lig med nul. Vi viser, at sådanne grupper udgør en underklasse af opløselige Lie grupper og beskriver dem strukturelt. Endelig giver vi adskillige eksempler på multi-momentafbildninger for forskellige geometrier, heriblandt hyperKähler-med-torsion mangfoldigheder og strengt næsten-Kähler mangfoldigheder.

Ved at generalisere argumenterne opnås et multi-momentafbildningsbegreb for geometrier med lukkede differentialformer af højere grad. Ligesom i treformstilfældet viser vi, at disse afbildninger ofte eksisterer, blandt andet hvis den underliggende mangfoldighed har visse topologiske egenskaber, eller hvis symmetrigruppen har et par af Lie algebra Betti tal lig med nul.

En særlig interessant anvendelse af multi-momentafbildninger vedrører klassifikationen af Riemannske mangfoldigheder med exceptionel holonomi og en isometrisk torusvirkning. Vi udforsker situationen, hvor multi-momentafbildningen er en skalarfunktion. Via en reduktionsprocedure relateres studiet af sådanne mangfoldigheder til trisymplektisk geometri i fire dimensioner.

I afhandlingens sidste del indføres et Calabi-Yau problem for hyperKähler-med-torsion mangfoldigheder, og vi tager de første skridt mod en løsning via kontinuitetsmetoden.

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Thomas Bruun Madsen
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THE CURRENT VERSION of my thesis is a minor revision of the original. In addition to correcting typos, the only changes consist of amendments of Chapter 5 where I have corrected the (partial) characterisation of $(3,4)$-trivial Lie algebras.

Thomas Bruun Madsen
London, February 2013.

Contents

Chapter 1

Introduction

In this thesis we study different aspects of three-form geometry. While the most important ideas and results already appeared in the papers [MS11b, MS10, MS11a, Mad11], see Appendix A, we hope this collected work succeeds in bridging the gaps between these references, and that it may serve as a comprehensive introduction to strong geometry and related notions, in particular multi-moment maps. In addition, the thesis extends our previous work. Most importantly, we generalise the notion of multi-moment maps to closed geometries.

Our approach is purely mathematical. However, it is worth emphasising that many of the geometric structures encountered owe their existence to developments in theoretical physics. One illuminating example of this interplay between the two disciplines is the relation between symplectic geometry and mechanics. A symplectic manifold is characterised by a closed, non-degenerate form of degree two. In modern physics higher degree forms play an important role too. While some authors have looked at extensions of field theories, closed three-forms appear to be particularly relevant in supersymmetric theories with Wess-Zumino terms, string theory and one-dimensional quantum mechanics [MS00, Str86, GHR84, BHR10, DI11]. They have been studied mathematically in a number of contexts including stable forms [Hit01], strong geometries with torsion [FPS04], gerbes [Bry93] and generalized geometry [Hit03, Gua04].

In the first part of the thesis, three-forms appear as the torsion of a metric connection. Specifically, we study Hermitian manifolds that admit a compatible connection whose torsion is a closed three-form. Our main result in this direction is Theorem 3.8 which classifies the invariant SKT structures on four-dimensional solvable Lie groups. The classification includes solutions on groups that do not admit compact four-dimensional quotients and therefore supplements a known result by Gauduchon regarding existence and uniqueness of standard metrics on compact four-manifolds [Gau84]. Moreover, our description of invariant SKT structures is very explicit and has therefore been useful in related studies of the Hermitian curvature flow for pluriclosed metrics and taming symplectic forms, cf. [Enr10, EF11].

Passing on from a particular type of three-form geometry, we turn to develop a new tool applicable in a more general setting. To motivate this, recall that one construction illustrating the aforementioned link between symplectic geometry and physics is that of moment maps. A moment map is an equivariant map from a symplectic manifold into the dual of the Lie algebra of a Lie group acting by symplectomorphisms. It captures the concepts of linear and angular momentum from mechanics. In the second part of the thesis the main objective is to explain that a similar type of map exists when we are given a manifold M with a closed three-form c and a Lie group G that acts on M preserving c. We call the pair (M, c) a strong geometry and refer to the Lie group G as a group of symmetries. We write $\mathfrak{g}$ for the Lie algebra of G.

An important feature of our construction is that the resulting multi-moment map is a map from M to a vector subspace $\mathcal{P}^*_{\mathfrak{g}}$ of $\Lambda^2 \mathfrak{g}^*$, with $\mathcal{P}^*_{\mathfrak{g}}$ independent of M. This is in contrast to previous considerations [CCI91, GIMM98] of so-called covariant moment maps $\sigma\colon M \to \Omega^1(M, \mathfrak{g}^*)$, which are defined via the relation

$$d\langle \sigma, \mathsf{X} \rangle = X \lrcorner c, \qquad \text{for all } \mathsf{X} \in \mathfrak{g}, \tag{1.1}$$

where X is the vector field on M generated by $\mathsf{X} \in \mathfrak{g}$. Here the target space $\Omega^1(M, \mathfrak{g}^*)$ is an infinite-dimensional space depending both on M and on $\mathfrak{g}$. We also note that finding covariant moment maps can be hard; equation (1.1) has a solution $\langle \sigma, \mathsf{X} \rangle$ only if the cohomology class $[X \lrcorner c]$ vanishes in $H^2(M)$. Thus, existence of covariant moment maps often requires some non-trivial topological assumption such as $b_2(M) = 0$.

In contrast, we will show that multi-moment maps exist under mild topological assumptions: if M is simply-connected and either G is compact or M is compact with G-invariant volume form. This is analogous to symplectic moment maps, and enables us to give many examples.

In the symplectic case, there is also a general existence theorem for moment maps in the case that the symmetry group is semi-simple; it is a result that does not require any topological assumptions on the manifold. Note that semi-simplicity of a Lie group is characterised algebraically by the vanishing of the first and second Betti numbers of the Lie algebra cohomology. In this direction, we prove that multi-moment maps exist whenever the second and third Betti numbers $b_2(\mathfrak{g})$ and $b_3(\mathfrak{g})$ of the Lie algebra cohomology of G vanish. We call Lie algebras of this type $(2, 3)$-trivial. The weaker setting of Lie algebras with $b_2(\mathfrak{g}) = 0$, where multi-moment maps are unique if defined, provides many examples of homogeneous strong geometries, including examples that are 2-plectic in the terminology of [BHR10]; of particular interest are the strict nearly Kähler six-manifolds, classified by Butruille [But05].

As far as we know, $(2, 3)$-trivial algebras have not been studied before. We show that these are solvable Lie algebras, that are not products of smaller dimensional algebras. Their derived algebra is of codimension one, and is necessarily nilpotent. From this one may classify the low-dimensional examples, and further study leads to a characterisation of the allowed solvable extensions of nilpotent algebras. The structure theory shows that many examples exist, including some that are unimodular. On the other hand one finds that some nilpotent algebras can not be realised as the derived algebra of a $(2,3)$-trivial algebra.

While the most interesting strong geometries carry additional structure, our approach clearly illustrates the usefulness of regarding the closed three-form as being the essential building block. From this point of view it seems reasonable to ask whether the ideas developed in Chapter 4 generalise to higher degree closed forms $\alpha \in \Omega^{k+1}(M)$; the pair (M,α) is now referred to as a closed geometry. An affirmative answer is given in Chapter 5. We develop a notion of multi-moment maps for closed geometries that subsumes the concepts of moment maps in the symplectic and strong settings. For a closed geometry with symmetry group G, a multi-moment map is a map from M to a vector subspace $\mathcal{P}^*_{\mathfrak{g}}$ of $\Lambda^k \mathfrak{g}^*$, with $\mathcal{P}^*_{\mathfrak{g}}$ independent of M.

Multi-moment maps for closed geometries are guaranteed to exist under mild topological conditions, similar to those discussed in Chapter 4. We also provide an algebraic existence criterion. This leads to a generalisation of the notion of $(2,3)$-triviality. Generally, it makes sense to talk about $(k_1,\dots,k_\ell)$-trivial Lie algebras. Along these lines we describe the structure of $(3,4)$-trivial algebras and also observe that most compact simple Lie algebras are $(1,2,4,5,6)$-trivial.

Geometries with closed forms of higher degree appear regularly in the physics literature. While recent developments in black hole physics [GGP11b, GGP11a] indicate some relevance of models with five- or higher degree form fluxes, we expect that our generalisation of multi-moment maps will be more useful in the four-form setting. Firstly because the rigidity of closed form geometries weakens as k becomes larger. Secondly because we already know of several interesting applications in the four-form case. For instance Theorem 5.25 tells us how to exhibit the inverse of the Swann bundle construction in terms of a quaternionic analogue of the Marsden-Weinstein quotient.

The final chapter of part two is devoted to an intriguing application of multi-moment maps. Specifically, we will use multi-moment maps to study seven-manifolds with holonomy contained in G_2 and eight-manifolds with holonomy in $Spin(7)$, when these have a free isometric action of a two-torus and a three-torus, respectively. In both situations we find that the geometry is determined by a conformal structure on a four-manifold specified by a certain triple of symplectic two-forms. Our main results are the theorems 6.11 and 6.30. These give a local classification of exceptional holonomy metrics with torus symmetry similar to the Gibbons-Hawking ansatz for hyperKähler surfaces with circle symmetry. In the G_2 case this extends the work of Apostolov and Salamon [AS04], and both descriptions fit with the perspective of Donaldson [Don06]. While the four-dimensional tri-symplectic manifolds are obtained via a reduction procedure for multi-moment maps, the inverse construction is based on a modification of Hitchin's evolution equations for half-flat $SU(3)$-structures and cosymplectic G_2-structures [Hit01]. The solvability of these flows rely on real-analyticity of the data. A delicate observation ensures that the analyticity criterion fits naturally into our framework. This contrasts with earlier studies of the Hitchin flow [Bry10, CLSSH11].

The concluding part of the thesis outlines a future research project, and is best characterised as speculations on HKT geometry. We address some general aspects of HKT manifolds emphasising the similarities with Kähler geometry. Most importantly, we introduce an HKT Calabi-Yau problem in Question 7.7. We argue that this should be solved via the continuity method. Hence we consider a one-parameter family of equations with parameter $t \in S \subset [0,1]$, and solvability of the problem is then equivalent to the set S being open and closed. In Theorem 7.19 we prove the openness. In order to prove the closedness, one has to establish a series of a priori estimates, which is a highly non-trivial analytic task. By bridging the gap between our problem and a related study of quaternionic Monge-Ampère equations [AV10], we obtain a first a priori estimate in Theorem 7.18. Our study of the HKT Calabi-Yau problem is mainly motivated by a quest for canonical HKT metrics compatible with a given hypercomplex structure. This aspect is briefly discussed in the final part of Chapter 7.

Geometry with torsion

Chapter 2

Geometry with torsion

One type of geometry that is partly characterised in terms of a three-form is a metric geometry with torsion. Of particular interest are examples with one or multiple complex structures. These generalise the more studied Kähler and hyperKähler manifolds.

A Riemannian manifold (M, g) comes equipped with a unique metric and torsion-free connection, the Levi-Civita connection ∇^{LC}. Any other connection ∇ has torsion measured by the $(2,1)$-tensor

$$T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

or, equivalently, by the $(3,0)$-tensor c^{∇} defined by

$$c^{\nabla}(X,Y,Z) = (T^{\nabla})^{\flat}(X,Y,Z) = g(T^{\nabla}(X,Y),Z). \tag{2.1}$$

Given a three-form $c \in \Omega^3(M)$ we may use equation (2.1) to define what we call a *skew-symmetric* $(2,1)$*-tensor* T. Given such a tensor, direct calculations show that the expression

$$\nabla_X Y = \nabla^{\mathrm{LC}}_X Y + \frac{1}{2} T(X,Y) \tag{2.2}$$

defines connection which preserves the metric and has torsion $T^{\nabla} = T$. In fact, ∇ is uniquely determined by these two properties, since they ensure that the connection $\nabla - \frac{1}{2}T$ is metric and torsion-free, and thus equals ∇^{LC}.

In summary, we have the following well-known [Car25, AF04] extension of the fundamental theorem of Riemannian geometry.

Theorem 2.1. *Let* (M, g) *be a Riemannian manifold, and* T *a skew-symmetric* $(2,1)$*-tensor on* M*. Then there exists a unique metric connection* ∇ *on* M *such that* $T^{\nabla} = T$*. Explicitly,* ∇ *is given by* (2.2)*. Moreover,* ∇ *has the same geodesics as* ∇^{LC}*.* □

Following [Swa07] we refer to the triple (M, g, c) as a *metric geometry with torsion*. While this notion is not particularly rigid on its own, things change drastically once complex structures are involved.

2.1 Strong Kähler manifolds with torsion

Any Hermitian manifold (M, g, J) has a unique Hermitian connection [Gau97], called the Bismut connection, which has torsion a three-form. Explicitly the Bismut connection is given by

$$\nabla^{\mathrm{B}} = \nabla^{\mathrm{LC}} + \tfrac{1}{2}T^{\mathrm{B}}, \qquad c^{\mathrm{B}} = \left(T^{\mathrm{B}}\right)^{\flat} = -Jd\omega, \tag{2.3}$$

where $\omega = g(J\cdot,\cdot)$ is the fundamental two-form and $Jd\omega = -d\omega(J\cdot, J\cdot, J\cdot)$. If the torsion three-form c^{B} is closed, we have a *strong Kähler manifold with torsion*, or briefly an SKT *manifold*. The study of SKT structures has received notable attention over recent years, see [FT09] for a survey and for an approach through generalized geometry, see [Cav06]. This has been motivated partly by the quest for canonical choices of metric compatible with a given complex structure and partly by the relevance of such geometries to super-symmetric theories from physics [GHR84, HP88, HLR+09, MS00, Str86].

Kähler manifolds are precisely the SKT manifolds with torsion three-form identically zero. However, most SKT manifolds are non-Kähler. For example compact semi-simple Lie groups cannot be Kähler since they have second Betti number equal to zero, but any even-dimensional compact Lie group can be endowed with the structure of an SKT manifold. The existence of SKT structures on compact even-dimensional Lie groups is briefly indicated in the introduction to [FPS04], and attributed to [SSTVP88]. However, the result is not explicit in the latter reference and neither specifies the complex structures. We therefore give a proof for reference.

Proposition 2.2. *Any even-dimensional compact Lie group G admits a left-invariant* SKT *structure.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{t}^{\mathbb{C}}$ a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. By [Sam53], left-invariant complex structures J on G are in one-to-one correspondence with pairs $(J_{\mathfrak{t}}, P)$, where $J_{\mathfrak{t}}$ is any complex structure on $\mathfrak{t}$, skew-symmetric for B, and $P \subseteq \Delta$ is a system of positive roots: one defines

$$\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \bigoplus_{\alpha\in P} \mathfrak{g}^{\mathbb{C}}_{\alpha}\,. \tag{2.4}$$

Extend the negative of the Killing form on $[\mathfrak{g},\mathfrak{g}]$ to a J-compatible positive definite inner product g on $\mathfrak{g}$ such that we have an orthogonal decomposition $Z(\mathfrak{g})\oplus[\mathfrak{g},\mathfrak{g}]$. The associated Levi-Civita connection on G has $\nabla^{\mathrm{LC}}_X Y = \frac{1}{2}[X,Y]$, for $X, Y \in \mathfrak{g}$. Consider now the left-invariant connection given by

$$\nabla_X Y = 0, \quad \text{for } X, Y \in \mathfrak{g}. \tag{2.5}$$

This connection preserves the metric g and the complex structure J and has torsion $T^{\nabla}(X,Y) = -[X,Y]$, so $(T^{\nabla})^{\flat}(X,Y,Z) = -g([X,Y],Z)$, which is a closed three-form. Thus (G, g, J) is an SKT manifold. □

The SKT geometry of nilpotent Lie groups was studied by Fino, Parton and Salamon [FPS04], who provided a full classification in dimension six. In Chapter 3, we classify SKT structures on four-dimensional solvable Lie groups, showing that there are a number of new examples; see Table 3.1, only the first two entries belong to the nilpotent classification. The greater variety and complexity of this case is already seen from the classification results for complex structures: Salamon [Sal01] classified the integrable complex structures on six-dimensional nilpotent Lie groups, whereas in the solvable case there is a classification only in dimension four [ABDO05, Ova04, Sno90]. Recently, Enrietti, Fino and Vezzoni [EFV10] studied the SKT condition on nilmanifolds in dimension eight and above. They showed that a nilmanifold endowed with an invariant complex structure can admit an SKT metric only if it is at most two-step. Using this observation, they then classified the eight-dimensional nilmanifolds endowed with an invariant SKT structure.

2.2 HyperKähler manifolds with torsion

We now turn to manifolds with multiple complex structures. Recall that an almost hyperHermitian manifold is a Riemannian manifold (M, g) endowed with three metric compatible almost complex structures I, J, K that satisfy the quaternion relations $K = IJ = -JI$. If each of the almost complex structures is integrable, then we have a hyperHermitian manifold. Geometrically, one may think of a hyperKähler manifold with torsion as a hyperHermitian manifold on which the Bismut connections associated with the Hermitian structures (g, I), (g, J), (g, K) coincide.

Cabrera and Swann showed [MCS08] that there is an equivalent definition which is usually easier to check, since the integrability of (I, J, K) comes for free.

Definition 2.3. An almost hyperHermitian manifold (M, g, I, J, K) is called a *hyperKähler manifold with torsion*, or briefly an HKT *manifold*, if

$$Id\omega_I = Jd\omega_J = Kd\omega_K, \tag{2.6}$$

where $\omega_I(X, Y) = g(IX, Y)$, etc., and $Id\omega_I(X, Y, Z) = -d\omega_I(IX, IY, IZ)$, etc.

As in the Hermitian case, we use the terminology *strong* HKT, or briefly SHKT, to refer to HKT geometries with a closed torsion three-form $c = -Id\omega_I$.

Example 2.4. An example of a homogeneous HKT manifold is the compact simple Lie group $SU(3)$. In fact, this group admits a left-invariant SHKT structure.

In order to endow $SU(3)$ with a left-invariant HKT structure, we describe the corresponding data on its Lie algebra $\mathfrak{su}(3)$. To this end we write E_{pq} for the elementary 3×3-matrix with a 1 at position (p, q), and then introduce the following $\mathfrak{su}(3)$ basis consisting of eight complex matrices:

$$A_1 = i(E_{11} - E_{22}), \quad A_2 = i(E_{22} - E_{33}),$$
$$B_{pq} = E_{pq} - E_{qp}, \quad C_{pq} = i(E_{pq} + E_{qp}),$$

for $p = 1, 2 < q = 2, 3$. We write $a_1, \ldots, c_{23}$ for the dual basis.

Using the formula

$$d\alpha(X, Y) = -\alpha([X, Y]) \tag{2.7}$$

and Table (2.1), we now find that

$$\begin{aligned} da_1 &= -2b_{12}c_{12} - 2b_{13}c_{13},\ da_2 = -2b_{13}c_{13} - 2b_{23}c_{23}, \\ db_{12} &= (2a_1 - a_2)c_{12} + b_{13}b_{23} + c_{13}c_{23},\ dc_{12} = (-2a_1 + a_2)b_{12} - b_{13}c_{23} - b_{23}c_{13}, \\ db_{13} &= (a_1 + a_2)c_{13} - b_{12}b_{23} + c_{12}c_{23},\ dc_{13} = (-a_1 - a_2)b_{13} - b_{12}c_{23} + b_{23}c_{12}, \\ db_{23} &= (-a_1 + 2a_2)c_{23} + b_{12}b_{13} + c_{12}c_{13},\ dc_{23} = (a_1 - 2a_2)b_{23} + b_{12}c_{13} + b_{13}c_{12}, \end{aligned} \tag{2.8}$$

where $\wedge$ signs have been omitted.

A positive definite inner product g on $\mathfrak{su}(3)$ is provided by minus the Killing form on $\mathfrak{su}(3)$. In concrete terms, this means that we consider the map $(X, Y) \mapsto -\frac{1}{2}\operatorname{Tr}(XY)$, which expressed in the above basis becomes

$$2g = 2a_1^2 - a_1a_2 + 2a_2^2 + 2(b_{12}^2 + b_{13}^2 + b_{23}^2 + c_{12}^2 + c_{13}^2 + c_{23}^2).$$

Joyce proved the existence of hypercomplex structures on certain compact Lie groups [Joy92, Thm. 4.2] including $SU(2n+1)$. For $SU(3)$, Joyce's hypercomplex structure comes from a particular decomposition of its Lie algebra $\mathfrak{su}(3)$. One takes a highest root $\mathfrak{su}(2)^{\mathbb{C}}$, e.g., the complex span of A_1, B_{12}, C_{12}, and think of the complement as $\mathbb{H} + \mathbb{R}$, where $\mathbb{H} \cong \langle B_{13}, C_{13}, B_{23}, C_{23}\rangle$ and $\mathbb{R} \cong \langle A_1 + 2A_2\rangle$. With this concrete decomposition in mind, let us write $\boldsymbol{I} = A_1$, $\boldsymbol{J} = B_{12}$ and $\boldsymbol{K} = C_{12}$. We then define I on $\mathbb{H}$ to be $\operatorname{ad}_{\boldsymbol{I}}$. Similarly J and K act on $\mathbb{H}$ by $\operatorname{ad}_{\boldsymbol{J}}$ and $\operatorname{ad}_{\boldsymbol{K}}$, respectively. On $\mathbb{R} + \mathfrak{su}(2)$ the actions of I, J and K are given by $IV = \boldsymbol{I}$, $JV = \boldsymbol{J}$ and $KV = \boldsymbol{K}$, respectively. Here V is chosen to be the following linear combination of A_1 and A_2:

$$V = (A_1 + 2A_2)/\sqrt{3}.$$

The action of I, etc., on the modified $\mathfrak{su}(3)$ basis is thus

$$\begin{aligned} I(V) = A_1, \quad I(A_1) = -V, \quad I(B_{12}) = C_{12}, \quad I(C_{12}) = -B_{12}, \\ I(B_{13}) = C_{13}, \quad I(C_{13}) = -B_{13}, \quad I(B_{23}) = -C_{23}, \quad I(C_{23}) = B_{23}, \end{aligned}$$

and so forth.

Direct computations now show that I, J and K satisfy the quaternion relations $IJ = K = -JI$, and that they are metric compatible, meaning $g(X, Y) = g(IX, IY)$, etc., for all $X, Y \in \mathfrak{su}(3)$.

By an appropriate basis change of the subspace $\langle a_1, a_2\rangle$, concretely put $2a_1' = 2a_1 - a_2$ and $2a_2' = \sqrt{3}a_2$, we find that the non-degenerate two-forms $\omega_I = g(I\cdot, \cdot)$, etc., are given by

$$\begin{aligned} \omega_I &= -a_1'a_2' + b_{12}c_{12} + b_{13}c_{13} - b_{23}c_{23}, \\ \omega_J &= a_2'b_{12} - a_1'c_{12} - b_{13}b_{23} - c_{13}c_{23}, \\ \omega_K &= a_2'c_{12} + a_1'b_{12} + b_{13}c_{23} + b_{23}c_{13}. \end{aligned} \tag{2.9}$$

Combining these formulae with (2.8), we then compute

$$\begin{aligned} d\omega_I &= -\sqrt{3}a_1'(b_{13}c_{13} + b_{23}c_{23}) + a_2'(2b_{12}c_{12} + b_{13}c_{13} - b_{23}c_{23}) \\ &\quad - b_{12}b_{13}c_{23} - b_{12}b_{23}c_{13} - b_{13}b_{23}c_{12} - c_{12}c_{13}c_{23}, \\ d\omega_J &= 2a_1'a_2'c_{12} + a_1'(b_{13}c_{23} + b_{23}c_{13}) - a_2'(b_{13}b_{23} + c_{13}c_{23}) \\ &\quad - \sqrt{3}b_{12}b_{13}c_{13} - \sqrt{3}b_{12}b_{23}c_{23} + b_{13}c_{12}c_{13} - b_{23}c_{12}c_{23}, \\ d\omega_K &= -2a_1'a_2'b_{12} + a_1'(b_{13}b_{23} + b_{23}c_{13}) + a_2'(b_{13}c_{23} + b_{23}c_{13}) \\ &\quad + \sqrt{3}b_{13}c_{12}c_{13} + \sqrt{3}b_{23}c_{12}c_{23} + b_{12}b_{13}c_{13} - b_{12}b_{23}c_{23}. \end{aligned}$$

From the above descriptions of $d\omega_I, d\omega_J, d\omega_K$ and the actions of I, J, K, we can now verify the HKT condition:

$$\begin{aligned} -d\omega_I(I\cdot, I\cdot, I\cdot) &= a_1(2b_{12}c_{12} + b_{13}c_{13} - b_{23}c_{23}) - a_2(b_{12}c_{12} - b_{13}c_{13} - 2b_{23}c_{23}) \\ &\quad - b_{23}c_{12}c_{13} - b_{13}c_{12}c_{23} - b_{12}c_{13}c_{23} - b_{12}b_{13}b_{23} \\ &= -d\omega_J(J\cdot, J\cdot, J\cdot) = -d\omega_K(K\cdot, K\cdot, K\cdot). \end{aligned}$$

Finally, using (2.4) and (2.8), we check that the torsion three-form $c = d\omega(I\cdot, I\cdot, I\cdot)$ is closed. We have thus shown that $(SU(3), g, I, J, K)$ is an SHKT manifold, as claimed. ♢

Remark 2.5. Throughout the thesis, we will frequently adopt the notation of the previous example, meaning that we usually omit $\wedge$ signs when there is no risk of confusion. △

Howe and Papadopoulos introduced HKT manifolds in the physics literature [HP96]. Later Grantcharov and Poon [GP00] gave the first mathematical description. Their work was followed by a series of papers investigating the subject. The early results included both general aspects, such as a potential theory [BS04], and aspects of Hodge theory [Ver02] as well as explicit constructions and (counter) examples, with a particular focus on nilmanifolds [FG04, DF02]. It seems that HKT nilmanifolds [BDV09, Bar09] and twists of these [Swa10b] are quite well understood. Contrasting with this, there are still surprisingly few known examples of compact SHKT manifolds. The most interesting class is still the one derived from Joyce's hypercomplex structures on compact Lie groups; Example 2.4 generalises to $SU(2n+1)$ and similar constructions hold for products $T^\ell \times G$, where G is a compact Lie group and the rank ℓ of the torus factor depends on the Joyce decomposition of G, see [PP99] or Table 4.3. Moreover, Barberis and Fino recently showed [BF11] that these Joyce SHKT manifolds give rise to further examples via a construction on so-called tangent Lie algebras. As SHKT manifolds are particular examples of strong geometries, the tools developed in Chapter 4 apply in the study of such manifolds.

Example 2.6. Based on the work of Gualtieri, see in particular [Gua04, Chapter 6.4], we will now explore Example 2.4 from a generalized viewpoint. We showed that the eight-manifold $SU(3)$ carries a left-invariant SHKT structure (g_-, I_-, J_-, K_-). Note that a basis free expression for the torsion three-form

	A_2	B_{12}	C_{12}	B_{13}	C_{13}	B_{23}	C_{23}
A_1	0	$2C_{12}$	$-2B_{12}$	C_{13}	$-B_{13}$	$-C_{23}$	B_{23}
A_2		$-C_{12}$	B_{12}	C_{13}	$-B_{13}$	$2C_{23}$	$-2B_{23}$
B_{12}			$2A_1$	$-B_{23}$	$-C_{23}$	B_{13}	C_{13}
C_{12}				C_{23}	$-B_{23}$	C_{13}	$-B_{13}$
B_{13}					$2(A_1 + A_2)$	$-B_{12}$	C_{12}
C_{13}						$-C_{12}$	$-B_{12}$
B_{23}							$2A_2$

Table 2.1: Our preferred basis for $\mathfrak{su}(3)$ satisfies the above commutation relations.

$c_- = d\omega_{I_-}(I_-\cdot, I_-\cdot, I_-\cdot)$ follows from the last part of the proof of Proposition 2.2; c_- is, up to scaling, obtained by left-translating the Cartan three-form $g([X,Y],Z)$. The same example provides us with a right-invariant SHKT structure (g_+, I_+, J_+, K_+). Moreover, the torsion three-forms c_+ and c_- are easy to relate: the metric is in fact bi-invariant, so $g_- = g_+ =: g$, and left and right Lie algebras are anti-isomorphic. Hence, we find that $-c_- = c_+ =: c$. From the biHermitian structure $(g, I_\pm)$ on $SU(3)$, we may construct a pair $(\mathbb{I}_+, \mathbb{I}_-)$ of commuting endomorphisms of $\mathbb{T} := T\,SU(3) \oplus T^*\,SU(3)$, such that $\mathbb{I}_\pm^2 = -1$. Explicitly, put

$$\mathbb{I}_\pm = \frac{1}{2}\begin{pmatrix} I_+ \pm I_- & -((\omega_I^+)^{-1} \mp (\omega_I^-)^{-1}) \\ \omega_I^+ \mp \omega_I^- & -(I_+^* \pm I_-^*) \end{pmatrix},$$

where $I_\pm^*$ denotes the transpose of $I_\pm$, and $\omega_I^\pm = g(I_\pm\cdot,\cdot)$; also note that $(\omega_I^+)^{-1} = -I_+(\cdot)^\sharp$, and so forth. Calculations show that $\mathbb{I}_+$ and $\mathbb{I}_-$ are orthogonal with respect to the natural $(8,8)$-signature pairing

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y))$$

on $\mathbb{T}$. To see this, we compute:

$$\begin{aligned} 4\|\mathbb{I}_+(X+\xi)\|^2 &= g(I_+X, I_+X) + g(I_+X, I_-X) + g(I_+X, I_+\xi^\sharp) - g(I_+X, I_-\xi^\sharp) \\ &\quad - g(I_-X, I_+X) - g(I_-X, I_-X) - g(I_-X, I_+\xi^\sharp) + g(I_-X, I_-\xi^\sharp) \\ &\quad - \xi(I_+^2X) - \xi(I_+I_-X) - \xi(I_+^2\xi^\sharp) + \xi(I_+I_-\xi^\sharp) \\ &\quad - \xi(I_-I_+X) - \xi(I_-^2X) - \xi(I_-I_+\xi^\sharp) + \xi(I_-^2\xi^\sharp) \\ &= 4\xi(X) = 4\|X+\xi\|^2. \end{aligned}$$

A similar calculation shows that $\|\mathbb{I}_-(X+\xi)\| = \|X+\xi\|$, so that the claimed orthogonality follows by the polarization identity. The data $g, \mathbb{I}_\pm$ thus specify an almost generalized Kähler structure on $(\mathbb{T}, c)$. Moreover, integrability of this structure can be phrased as the conditions that I_+, I_- are integrable and $c_+ = -c_-$, and these are clearly satisfied.

Similarly the triples $(g, \mathbb{J}_\pm)$ and $(g, \mathbb{K}_\pm)$ provide us with generalized Kähler structures, and direct inspection shows that the triple $(\mathbb{I}_\pm, \mathbb{J}_\pm, \mathbb{K}_\pm)$ satisfies the following additional relations:

$$\mathbb{I}_\pm\mathbb{J}_\pm = -\mathbb{J}_\pm\mathbb{I}_\pm = \mathbb{K}_+ \quad \text{and} \quad \mathbb{I}_\pm\mathbb{J}_\mp = -\mathbb{J}_\mp\mathbb{I}_\pm = \mathbb{K}_-.$$

Altogether our observations may be summarised by saying that $(g, \mathbb{I}_\pm, \mathbb{J}_\pm, \mathbb{K}_\pm)$ defines a *generalized hyperKähler structure* on $(\mathbb{T}, c)$, cf. [BCG06, EG07, Bre07]. In order to appreciate this terminology, one may note that the above data reduce the structure group $SO(8,8)$ of $(\mathbb{T}, \langle\cdot,\cdot\rangle)$ to a maximal compact subgroup $Sp(2) \times Sp(2)$ of $Sp(2,2) \subset SO(8,8)$.

Finally, let us remark that our arguments hold for any even-dimensional compact Lie group $T^\ell \times G$ admitting one of Joyce's hypercomplex structures. ◇

Chapter 3

Lie theoretic approach

In dimension four, a Hermitian manifold (M, g, J) is an SKT manifold precisely when the associated Lee one-form $\theta = Jd^*\omega$ is co-closed. When M is compact, Gauduchon [Gau84] showed that, up to homothety, there is a unique such metric in each conformal class of Hermitian metrics. The situation for non-compact manifolds is less clear. In this chapter we obtain a classification of left-invariant SKT structures on four-dimensional solvable Lie groups. Our result includes non-compact SKT manifolds that admit no compact quotient, and also shows that there are invariant complex structures that admit no compatible invariant SKT metric.

3.1 Solvable Lie algebras

Since we are interested in invariant structures on a simply-connected Lie group G, it is sufficient to study the corresponding structures on the Lie algebra $\mathfrak{g}$. To $\mathfrak{g}$ one associates two series of ideals: the *lower central series*, which is given by $\mathfrak{g}_1 = \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_k = [\mathfrak{g}, \mathfrak{g}_{k-1}]$ and the *derived series* defined by $\mathfrak{g}^1 = \mathfrak{g}'$, $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}^{k-1}]$. The Lie algebra is *nilpotent* (resp. *solvable*) if its lower (resp. derived) series terminates after finitely many steps.

One has that $\mathfrak{g}^j \subseteq \mathfrak{g}_j$, so that nilpotent algebras are solvable. On the other hand, consider a solvable Lie algebra $\mathfrak{g}$. Lie's theorem applied to the adjoint representation of the complexification $\mathfrak{g}_{\mathbb{C}}$, gives a complex basis for $\mathfrak{g}_{\mathbb{C}}$ with respect to which each ad_X is upper triangular. One then has the well-known:

Lemma 3.1. *A finite-dimensional Lie algebra $\mathfrak{g}$ is solvable if and only if its derived algebra $\mathfrak{g}'$ is nilpotent.* □

Remark 3.2. For $\mathfrak{g}$ solvable of dimension four, $\mathfrak{g}'$ has dimension at most three and so is one of a known list. Lemma 3.1 then implies that $\mathfrak{g}'$ is either Abelian or the Heisenberg algebra $\mathfrak{h}_3$, which has basis elements E_1, E_2, E_3 with only one non-trivial Lie bracket $[E_1, E_2] = E_3$. △

Identifying $\mathfrak{g}$ with left-invariant vector fields on G, and $\mathfrak{g}^*$ with left-invariant one-forms one has the relation (2.7), i.e.,

$$da(X,Y) = -a([X,Y])$$

for all $X, Y \in \mathfrak{g}$ and $a \in \mathfrak{g}^*$. We may describe for example $\mathfrak{h}_3$ by letting e_1, e_2, e_3 be the dual basis in $\mathfrak{g}^*$ to E_1, E_2, E_3 and computing $de_1 = 0$, $de_2 = 0$, $de_3 = e_2 \wedge e_1$. We will use the compact notation $\mathfrak{h}_3 = (0, 0, 21)$ to encode these relations.

Let $\Lambda^* \mathfrak{g}^*$ be the exterior algebra on $\mathfrak{g}^*$ and write $\mathcal{I}(A)$ for the ideal in $\Lambda^* \mathfrak{g}^*$ generated by a subset A. We interpret the condition for $\mathfrak{g}$ to be solvable dually via the elementary:

Lemma 3.3. *A finite-dimensional Lie algebra $\mathfrak{g}$ is solvable if and only if there are maximal subspaces $\{0\} = W_0 < W_1 < \cdots < W_r = \mathfrak{g}^*$ such that*

$$dW_i \subseteq \mathcal{I}(W_{i-1}) \tag{3.1}$$

for each i. □

Concretely $W_1 = \ker(d\colon \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*)$ (cf. [Sal01]) and W_i is defined inductively to be the maximal subspace satisfying (3.1). We will sometimes find it useful to choose a filtration $\{0\} = V_0 < V_1 < \cdots < V_n = \mathfrak{g}^*$ with

$$\dim_{\mathbb{R}} V_i = i \quad \text{and} \quad dV_i \subseteq \mathcal{I}(V_{i-1}) \qquad \text{for each } i. \tag{3.2}$$

One way to construct such filtrations is to refine the spaces W_i, however in some cases other choices may be possible and useful.

3.1.1 Unimodular Lie algebras

The map $\chi\colon \mathfrak{g} \to \mathbb{R}$, $\chi(x) = \operatorname{Tr}(\operatorname{ad}(x))$, is a Lie algebra homomorphism. Its kernel $\mathfrak{u}(\mathfrak{g})$, the *unimodular kernel of* $\mathfrak{g}$, is an ideal in $\mathfrak{g}$ containing the derived algebra $\mathfrak{g}'$. The Lie algebra $\mathfrak{g}$ is said to be *unimodular* if $\chi \equiv 0$. Note that if G admits a co-compact discrete subgroup then $\mathfrak{g}$ is necessarily unimodular [Mil76].

Remark 3.4. There are useful alternative ways of characterising unimodularity of an n-dimensional Lie algebra $\mathfrak{g}$, cf. [SH10]. One finds that $\mathfrak{g}$ is unimodular if and only if all $(n-1)$-forms are closed, or equivalently $b_n(\mathfrak{g}) = 1$; here $b_k(\mathfrak{g}) = \dim H^k(\mathfrak{g})$. △

It is well-known, from the disseration work of Jean-Louis Koszul, that any unimodular n-dimensional Lie algebra $\mathfrak{g}$ satisfies *Hodge duality*, cf. [GHV73, Chapter IV.5]. As we will need this result in Chapter 4, we now give a precise statement and a proof for reference. The argument is essentially the same as the one applied in a more general context in [ACK99, Theorem 2.1].

Proposition 3.5. *If $\mathfrak{g}$ is a unimodular n-dimensional Lie algebra, then $b_k(\mathfrak{g}) = b_{n-k}(\mathfrak{g})$, for $0 \leqslant k \leqslant n$.*

Proof. The $\wedge$ product defines a non-degenerate bilinear pairing $Q\colon \Lambda^k \mathfrak{g}^* \times \Lambda^{n-k} \mathfrak{g}^* \to \Lambda^n \mathfrak{g}^*$ given by $Q(a,b) = a \wedge b$. Note that for any pair of closed elements $a \in \Lambda^k \mathfrak{g}^*$, $b \in \Lambda^{n-k} \mathfrak{g}^*$ and any pair $\alpha \in \Lambda^{k-1} \mathfrak{g}^*$, $\beta \in \Lambda^{n-k-1} \mathfrak{g}^*$, we have that

$$(a + d\alpha) \wedge (b + d\beta) = a \wedge b + d(\alpha \wedge b + (-1)^k a \wedge \beta + \alpha \wedge d\beta).$$

Hence Q induces a pairing on Lie algebra cohomology, $\widehat{Q}\colon H^k(\mathfrak{g}) \times H^{n-k}(\mathfrak{g}) \to H^n(\mathfrak{g})$. In order to prove the statement of the proposition, it suffices to show that $\widehat{Q}$ is non-degenerate. Indeed, in that case the pairing establishes a linear isomorphism $H^k(\mathfrak{g}) \cong H^{n-k}(\mathfrak{g})$, so that $b_k(\mathfrak{g}) = b_{n-k}(\mathfrak{g})$, as required.

To prove non-degeneracy of $\widehat{Q}$, we first identify Q with a positive definite inner product on $\Lambda^k \mathfrak{g}^*$ as follows. Pick a basis $E_1, \dots, E_n$ for $\mathfrak{g}$, and declare it to be oriented and orthonormal. Denote by $e_1, \dots, e_n$ the dual basis in $\mathfrak{g}^*$, and extend the associated inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ to $\Lambda^k \mathfrak{g}^*$ via the formula

$$\langle a, b \rangle = \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} a(E_{i_1}, \dots, E_{i_k}) b(E_{i_1}, \dots, E_{i_k}),$$

for $a, b \in \Lambda^k \mathfrak{g}^*$. Having chosen an inner product and an orientation, we get an operator $*\colon \Lambda^k \mathfrak{g}^* \to \Lambda^{n-k} \mathfrak{g}^*$, which is uniquely characterised by the property that

$$Q(a, *b) = \langle a, b \rangle e_1 \wedge \dots \wedge e_n,$$

for $a, b \in \Lambda^k \mathfrak{g}^*$, and $0 \leqslant k \leqslant n$. Moreover, $*$ satisfies the relation $*^2 = (-1)^{k(n-k)}\colon \Lambda^k \mathfrak{g}^* \to \Lambda^k \mathfrak{g}^*$.

We now define a linear map $d^*\colon \Lambda^k \mathfrak{g}^* \to \Lambda^{k-1} \mathfrak{g}^*$ by the formula $d^* := -(-1)^{n(k+1)} {*d*}$. We claim that $\langle da, b \rangle = \langle a, d^* b \rangle$, for $a \in \Lambda^k \mathfrak{g}^*$ and $b \in \Lambda^{k+1} \mathfrak{g}^*$. To prove this assertion, note that as $a \wedge *b \in \Lambda^{n-1} \mathfrak{g}^*$ we have that $d(a \wedge *b) = 0$, by unimodularity of $\mathfrak{g}$. Hence

$$\begin{aligned} 0 &= (da) \wedge *b + (-1)^k a \wedge d(*b) = (da) \wedge *b + (-1)^{n(k+2)-2n-k(k-1)} a \wedge *^2 d(*b) \\ &= Q(da, *b) - Q(a, *(d^* b)) = (\langle da, b \rangle - \langle a, d^* b \rangle) e_1 \wedge \dots \wedge e_n. \end{aligned}$$

Below we will use this observation to show that any closed element $a \in \Lambda^k \mathfrak{g}^*$, admits an orthogonal decomposition

$$a = d\alpha_1 + \alpha_2, \tag{3.3}$$

where $(dd^* + d^*d)\alpha_2 = 0$ and the element $*\alpha_2 \in \Lambda^{n-k} \mathfrak{g}^*$ satisfies $d(*\alpha_2) = 0$ and $Q(a, *\alpha_2) = \|\alpha_2\|^2 e_1 \wedge \dots \wedge e_n$. In particular, this will imply that if $[a] \neq 0$, then we have that $\widehat{Q}([a], [\alpha_2]) \neq 0$. Hence the induced pairing on Lie algebra cohomology is non-degenerate, as required.

It remains to verify the decomposition (3.3). Firstly, we observe that

$$\ker d \cap \ker d^* = \ker(dd^* + d^*d). \tag{3.4}$$

The non-trivial inclusion $\ker d \cap \ker d^* \supset \ker(dd^* + d^*d)$ is implied by the computation

$$\|(dd^* + d^*d)a\|^2 = \|dd^*a\|^2 + 2\langle dd^*a, d^*da \rangle + \|d^*da\|^2 = \|dd^*a\|^2 + \|d^*da\|^2, \tag{3.5}$$

where we have used that $d^2 = 0$. One easily checks that the vanishing of (3.5) implies that $da = 0$ and $d^*a = 0$. As one obviously has that $\ker d \perp \operatorname{Im} d^*$, (3.3) will follow if we can show that $\Lambda^* \mathfrak{g}^* = \operatorname{Im} d \oplus \operatorname{Im} d^* \oplus \ker(dd^* + d^*d)$. Since $\operatorname{Im} d \perp \operatorname{Im} d^*$ and $(\operatorname{Im} d \oplus \operatorname{Im} d^*) \perp \ker(dd^* + d^*d)$, this assertion is implied, once we have shown that

$$(\operatorname{Im} d \oplus \operatorname{Im} d^*)^{\perp} \subset \ker(dd^* + d^*d).$$

But this inclusion is an immediate consequence of the observations that $(\operatorname{Im} d)^{\perp} \subset \ker d^*$ and $(\operatorname{Im} d^*)^{\perp} \subset \ker d$, combined with (3.4).

This completes the proof of the Hodge duality. □

3.2 The SKT structural equations

A left-invariant almost Hermitian structure on G is determined by an inner product g on the Lie algebra $\mathfrak{g}$ and a linear endomorphism J of $\mathfrak{g}$ such that $J^2 = -1$ and $g(JX, JY) = g(X, Y)$ for all $X, Y \in \mathfrak{g}$. The SKT condition consists of the requirement that J be integrable and that $dJd\omega = 0$ where $\omega(X, Y) = g(JX, Y)$. In the differential algebra, integrability of J may be expressed as the condition that $d\Lambda^{1,0} \subseteq \Lambda^{2,0} + \Lambda^{1,1}$. If $\mathfrak{g}$ is four-dimensional and solvable, we now show that there is one of two choices of possible good bases $\{a, Ja, b, Jb\}$ for $\mathfrak{g}^*$. We will later determine the SKT condition in each case.

Lemma 3.6. *Let $\mathfrak{g}$ be a solvable Lie algebra of dimension four. If (g, J) is an integrable Hermitian structure on $\mathfrak{g}$ then there is an orthonormal set $\{a, b\}$ in $\mathfrak{g}^*$ such that $\{a, Ja, b, Jb\}$ is a basis for $\mathfrak{g}^*$ and either*

Complex case: *$\mathfrak{g}$ has structural equations*

$$\begin{aligned} da = 0, \quad d(Ja) = x_1 aJa, \quad db = y_1 aJa + y_2 ab + y_3 aJb + z_1 bJa + z_2 JaJb, \\ d(Jb) = u_1 aJa + u_2 ab + u_3 aJb + v_1 bJa + v_2 JaJb + w_1 bJb, \end{aligned} \tag{3.6}$$

or

Real case: *$\mathfrak{g}$ has structural equations*

$$\begin{aligned} da = 0, \quad d(Ja) &= x_1 aJa + x_2(ab + JaJb) + x_3(aJb + bJa) + y_2 bJb, \\ db &= z_1 aJa + z_2 ab + z_3 aJb, \\ d(Jb) &= u_1 aJa + u_2 ab + u_3 aJb + v_1 bJa + v_2 bJb + w_1 JaJb. \end{aligned} \tag{3.7}$$

In the complex case, $\{a, Ja, b, Jb\}$ may be chosen orthonormal and $\omega = aJa + bJb$. In the real case, $\omega = aJa + bJb + t(ab + JaJb)$ for some $t \in (-1, 1)$.

Proof. Let V_i be a refined filtration of $\mathfrak{g}^*$ as in (3.2). As $\dim_{\mathbb{R}} V_2 = 2$ we have two possibilities for the complex subspace $V_2 \cap JV_2$, either it is non-trivial so $V_2 = JV_2$ or it is zero. If the filtration V_i can be chosen with $V_2 = JV_2$ we will say we are in the complex case, otherwise we are in the real case.

For the complex case, $JV_2 = V_2$ and $V_1 \subseteq V_2 \cap \ker d$, so we may take an orthonormal basis $\{a, Ja\}$ of V_2 with $a \in V_1$. We have $da = 0$ and solvability implies $d(Ja) \in \mathcal{I}(a) \cap \Lambda^2 = \mathbb{R}aJa \oplus a \wedge V_2^{\perp}$. As J is integrable, we must have $d(Ja) \in \Lambda^{1,1}$ too, so $d(Ja) = x_1 aJa$.

In the real case, choose $a \in V_1$ and $b \in V_2 \cap V_1^{\perp}$ of unit length. Then $da = 0$ and the form of $d(Ja)$ follows from the condition $d(Ja) \in \Lambda^{1,1}$. The form of ω follows from $t = g(b, Ja)$ which has absolute value less than 1 by the Cauchy-Schwarz inequality. □

The above equations are necessary but far from sufficient. For integrability it remains to impose $d(b - iJb)^{0,2} = 0$, and to obtain a Lie algebra the Jacobi identity must be satisfied. The latter is equivalent to the condition $d^2 = 0$. Both of these conditions are straightforward to compute. We list the results below. In each case the first line comes from the integrability condition on J, in the last line we provide the SKT condition and the remaining equations are from $d^2 = 0$.

Lemma 3.7. *The structural equations of Lemma 3.6 give an* SKT *structure on a solvable Lie algebra if and only if the following quantities vanish:*

Complex case:

$$\begin{gathered} y_2 - z_2 - u_3 + v_1, \quad y_3 - z_1 + u_2 - v_2, \\ x_1 z_1 - y_3 v_1 - z_2 u_2, \quad (x_1 - y_2 + u_3) z_2 - y_3(z_1 + v_2), \\ y_2 w_1, \quad y_3 w_1, \quad z_1 w_1, \quad z_2 w_1, \\ (x_1 + y_2 - u_3) v_1 - (z_1 + v_2) u_2 + u_1 w_1, \\ x_1 v_2 + y_1 w_1 - y_3 v_1 - z_2 u_2, \\ (x_1 + y_2 + u_3)(y_2 + u_3) + (z_1 - v_2)^2 - u_1 w_1. \end{gathered} \tag{3.8}$$

Real case:

$$\begin{gathered} z_2 - u_3 + v_1, \quad z_3 + u_2 - w_1, \\ x_2 u_2 - x_3(z_2 - v_1) - y_2 u_1, \quad (-x_1 + z_2 + u_3) y_2 + x_2^2 + x_3(x_3 - v_2), \\ x_2 u_3 - x_3(w_1 + z_3) + y_2 z_1, \quad (x_1 + z_2 - u_3) v_1 - (x_3 - v_2) u_1 - u_2 w_1, \\ x_2 v_2 - y_2 w_1, \quad x_3 z_1 + z_3 v_1, \quad y_2 z_1 + z_3 v_2, \quad x_2 z_1 + z_3 w_1, \quad x_2 v_1 - x_3 w_1, \\ x_2 w_1 + x_3 v_1 - y_2 u_1 + z_2 v_2, \quad x_1 w_1 - x_2 u_1 + z_1 v_2 - z_3 v_1, \\ \{(x_1 + z_2 + u_3)(-y_2 + z_2 + u_3) + x_2(x_2 - z_1 + t v_2) \\ \qquad + \big(x_3 - u_1 + t(u_2 - w_1)\big)(x_3 + v_2) + w_1^2\}. \end{gathered} \tag{3.9}$$

In some cases the SKT structure reduces to Kähler. This occurs if and only if the following additional conditions hold:

Complex case:

$$y_1 = 0 = u_1, \quad u_3 = -y_2, \quad v_2 = z_1 \tag{3.10}$$

Real case:

$$\begin{gathered} x_2 - z_1 = t(x_1 + u_3), \quad x_3 - u_1 = -t u_2, \quad y_2 - z_2 - u_3 = t x_2, \\ w_1 = t(x_3 + v_2). \end{gathered} \tag{3.11}$$

3.3 The SKT classification

We are now ready to describe the simply-connected four-dimensional solvable real Lie groups admitting invariant SKT structures. The notation for and distinguishing characteristics of all the solvable real Lie algebras in dimensions up to four are summarised in Section 3.5 following the classification in [ABDO05].

Theorem 3.8. *Let G be a simply-connected four-dimensional solvable real Lie group. Then G admits a left-invariant* SKT *structure if and only if its Lie algebra $\mathfrak{g}$ is listed in Table 3.1. Furthermore the left-invariant* SKT *structures on G may be explicitly determined and the dimension and number of connected components of the moduli space up to homotheties are as in Table 3.1.*

The table also indicates which groups admit invariant Kähler metrics, and gives the dimensions of the Lie algebra cohomology.

$\mathfrak{g}'$	$\mathfrak{g}$	dim	π_0	Kähler	$(b_1 \dots b_4)$
$\{0\}$	$\mathbb{R}^4$	0	1	✓	$(4,6,4,1)$
$\mathbb{R}$	$\mathbb{R}\times\mathfrak{h}_3$	0	1	×	$(3,4,3,1)$
	$\mathbb{R}\times\mathfrak{r}_{3,0}$	1	1	✓	$(3,3,1,0)$
$\mathbb{R}^2$	$\mathbb{R}\times\mathfrak{r}'_{3,0}$	1	1	✓	$(2,2,2,1)$
	$\mathfrak{aff}_{\mathbb{R}}\times\mathfrak{aff}_{\mathbb{R}}$	2	1	✓	$(2,1,0,0)$
$\mathbb{R}^3$	$\mathfrak{r}'_{4,\lambda,0}$ $(\lambda>0)$	1	2	✓	$(1,1,1,0)$
	$\mathfrak{r}_{4,-1/2,-1/2}$	1	1	×	$(1,0,1,1)$
	$\mathfrak{r}'_{4,2\lambda,-\lambda}$ $(\lambda>0)$	1	2	×	$(1,0,1,1)$
$\mathfrak{h}_3$	$\mathfrak{d}_4$	2	1	×	$(1,0,1,1)$
	$\mathfrak{d}_{4,2}$	2	1	✓	$(1,1,1,0)$
	$\mathfrak{d}'_{4,0}$	2	1	×	$(1,0,1,1)$
	$\mathfrak{d}_{4,1/2}$	1	1	✓	$(1,0,0,0)$
	$\mathfrak{d}'_{4,\lambda}$ $(\lambda>0)$	1	1	✓	$(1,0,0,0)$

Table 3.1: The four-dimensional solvable Lie algebras that admit a left-invariant SKT structure. Of these, only $\mathbb{R}^4$ fails to admit an SKT structure that is not Kähler. In the table, dim and π_0 are the dimension and number of components of the SKT moduli space modulo homotheties, b_k denotes $\dim H^k(\mathfrak{g})$.

The proof will occupy the rest of this section. Following Remark 3.2 we analyse the possible solutions to the equations of Section 3.2 case-by-case after the type of $\mathfrak{g}'$. We use the Lie algebra structure of $\mathfrak{g}$ combined with the SKT geometry to determine a canonical choice of basis $\{a, Ja, b, Jb\}$ with $\{a, b\}$ orthonormal, refining the approach of Section 3.2. When talking of the SKT moduli space, we consider only left-invariant structures on the given G. These are determined by (g, J) on $\mathfrak{g}$. Two SKT pairs (g_1, J_1) and (g_2, J_2) on $\mathfrak{g}$ are considered equivalent if there is a Lie algebra automorphism ϕ with $\phi^* g_2 = g_1$

and $\phi \circ J_1 = J_2 \circ \phi$. Equivalent structures have canonical bases with the same structure constants and any remaining parameters in the structure equations are parameters for the SKT moduli space.

3.3.1 Trivial derived algebra

For $\mathfrak{g}' = \{0\}$, $\mathfrak{g} \cong \mathbb{R}^4$ is Abelian, $d \equiv 0$ so all structure constants are zero and each almost Hermitian structure is Kähler. All these Kähler structures are equivalent.

3.3.2 One-dimensional derived algebra

For $\mathfrak{g}' = \mathbb{R}$, we have $\dim W_1 = 3$. It follows that we can choose a, Ja, $b \in W_1$ and are thus in the case $V_2 = JV_2$. The structural equations for $\mathfrak{g}$ in this case are

$$\begin{gathered} da = 0 = d(Ja) = db, \\ d(Jb) = u_1 aJa + u_2(ab + JaJb) + u_3(aJb + bJa) + w_1 bJb, \end{gathered}$$

where the coefficients satisfy $0 = u_2^2 + u_3^2 - u_1 w_1$ and $d(Jb) \neq 0$. Rotating a, Ja in V_2, we may ensure that $u_2 = 0$ and $u_3 \geqslant 0$, so $u_1 w_1 = u_3^2$. Replacing b by $-b$, we obtain $w_1 \geqslant 0$.

If $w_1 = 0$ then $u_3 = 0$ and we may take $u_1 > 0$, after an appropriate choice of b. Thus we have the algebra given by

$$da = 0 = d(Ja) = db, \quad d(Jb) = u_1 aJa. \tag{3.12}$$

Any other orthonormal Hermitian basis $\{a', Ja', b', Jb'\}$ with $a', Ja' \in V_2$, $b' \in W_1$ and $u_1' > 0$ has $b' = b$, $a' = \cos\theta\, a + \sin\theta\, Ja$ and $d(Jb') = u_1'\, a'Ja' = u_1\, aJa$. The parameter $u_1 > 0$ thus describes inequivalent SKT solutions. Scaling of the metric by a homothety, $g \mapsto \lambda^2 g$, $\lambda > 0$, is realised by $a \mapsto \lambda a$, $b \mapsto \lambda b$ and gives $u_1 \mapsto u_1/\lambda$. Thus the resulting SKT metrics are all homothetic to each other. These SKT structures are not Kähler. Moreover we see that $\mathfrak{g}$ is nilpotent and so isomorphic to $\mathbb{R} \times \mathfrak{h}_3$.

If $w_1 > 0$ then $\mathfrak{g}$ is not nilpotent and so isomorphic to $\mathbb{R} \times \mathfrak{r}_{3,0}$. As $u_1 w_1 = u_3^2 \geqslant 0$ we have the structural equations

$$da = 0 = d(Ja) = db, \quad d(Jb) = u_1 aJa + u_3(aJb + bJa) + w_1 bJb,$$

with $u_3 = \sqrt{u_1 w_1}$, $u_1 \geqslant 0$. This is Kähler only if $u_1 = 0$. The non-Kähler solutions have $u_1, u_3, w_1 > 0$ and $u_2 = 0$, which fixes the choice of basis $\{a, Ja, b, Jb\}$. Up to homothety the only parameter is u_1. The moduli space is thus connected.

3.3.3 Two-dimensional derived algebra

For $\mathfrak{g}' = \mathbb{R}^2$, we have $\dim W_1 = 2$, and we shall distinguish between the cases $W_1 = JW_1$ and $W_1 \cap JW_1 = \{0\}$ where $W_1 = \ker d$ is complex or real.

Complex kernel We have $W_1 = JW_1$ and taking $V_2 = W_1$ thus have the structural equations

$$\begin{aligned} da &= 0 = d(Ja), \\ db &= y_1 aJa + y_3 aJb + z_2 JaJb, \\ d(Jb) &= u_1 aJa - y_3 ab + z_2 bJa \end{aligned}$$

with no restrictions on the coefficients other than that db and $d(Jb)$ are linearly independent. Rotating a, Ja we may put $z_2 = 0$, $y_3 > 0$. Rotating b, Jb we can then get $u_1 \geqslant 0$, $y_1 = 0$, reducing the structure to

$$da = 0 = d(Ja), \quad db = y_3 aJb, \quad d(Jb) = u_1 aJa - y_3 ab.$$

The solution is Kähler if and only if $u_1 = 0$. For $u_1 > 0$ the Hermitian basis is unique. The SKT moduli space is connected of dimension 1 modulo homotheties. The Lie algebra $\mathfrak{g}$ is isomorphic to $\mathbb{R} \times \mathfrak{r}'_{3,0}$.

Real kernel Here $W_1 \cap JW_1 = \{0\}$ and we again take $V_2 = W_1$ putting us in the real case and giving the structural equations

$$\begin{aligned} da &= 0 = db, \\ d(Ja) &= x_1 aJa + x_3(aJb + bJa) + y_2 bJb, \\ d(Jb) &= u_1 aJa + u_3(aJb + bJa) + v_2 bJb, \end{aligned}$$

where the last two lines are linearly independent and the coefficients satisfy

$$\begin{gathered} (x_1 - u_3)y_2 = (-v_2 + x_3)x_3, \quad u_1(v_2 - x_3) = u_3(u_3 - x_1), \\ u_3 x_3 = u_1 y_2, \quad (u_1 - x_3)(v_2 + x_3) = (u_3 + x_1)(u_3 - y_2). \end{gathered} \tag{3.13}$$

Lemma 3.9. *We have $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and $\mathfrak{u}(\mathfrak{g}) \cong \mathfrak{r}_{3,-1}$, so $\mathfrak{g} \cong \mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$.*

Proof. We compute the centre via $\mathfrak{z}(\mathfrak{g}) = \{\, X \in \mathfrak{g} : X \lrcorner d\alpha = 0 \text{ for all } \alpha \in \mathfrak{g}^* \,\}$. Writing $X = pA + qB + p'JA + q'JB$, where $\{A, B, JA, JB\}$ is the dual basis to $\{a, b, Ja, Jb\}$, one finds that $X \in \mathfrak{z}(\mathfrak{g})$ implies $(p, q, 0)^T$ and $(0, p, q)^T$ lie in the one-dimensional null space of the rank two matrix

$$Q = \begin{pmatrix} x_1 & x_3 & y_2 \\ u_1 & u_3 & v_2 \end{pmatrix}.$$

We conclude that $p = 0 = q$. The same calculation applies to p' and q', so $X = 0$ and $\mathfrak{z}(\mathfrak{g}) = \{0\}$.

Writing $\boldsymbol{a} = \binom{x_1}{x_3}$, $\boldsymbol{b} = \binom{x_3}{y_2}$, $\boldsymbol{c} = \binom{u_1}{u_3}$, $\boldsymbol{d} = \binom{u_3}{v_2}$, equations (3.13) may be interpreted geometrically as saying that $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{a} - \boldsymbol{d}$ are mutually parallel and that $\boldsymbol{b} - \boldsymbol{c}$ is parallel to $\boldsymbol{a} + \boldsymbol{d}$. Imposing the constraint $\operatorname{rank} Q = 2$, then leads to the fact that $\boldsymbol{a}$ and $\boldsymbol{d}$ are linearly independent.

The map $\chi = \operatorname{Tr}\operatorname{ad}\colon \mathfrak{g} \to \mathbb{R}$ is given by $\chi(A) = -(x_1 + u_3)$, $\chi(B) = -(x_3 + v_2)$, $\chi(JA) = 0 = \chi(JB)$. This is zero only if $\boldsymbol{a} = -\boldsymbol{d}$, which by the above remark, is not possible. Thus $\mathfrak{g}$ is not unimodular. Choosing $a \in \operatorname{Im}\chi^* \leqslant \ker d$, we have $0 = a(B) \propto \chi(B)$ and so $v_2 = -x_3$.

Write $\boldsymbol{a} - \boldsymbol{d} = 2k\boldsymbol{v}$ with $\boldsymbol{v} = \left(\begin{smallmatrix} c \\ s \end{smallmatrix}\right)$, $c^2 + s^2 = 1$. Then (3.13) implies $\boldsymbol{b}, \boldsymbol{c} \in \langle \boldsymbol{v} \rangle$. However $\boldsymbol{a} + \boldsymbol{d} \notin \langle \boldsymbol{v} \rangle$ but is parallel to $\boldsymbol{b} - \boldsymbol{c}$, so we find $\boldsymbol{b} = \boldsymbol{c} = h\boldsymbol{v}$, for some $h \in \mathbb{R}$. This gives $x_3 = ks = hc$, so we may write $k = \ell c$, $h = \ell s$ for some non-zero $\ell \in \mathbb{R}$. Changing the sign of $\boldsymbol{v}$ we may force $\ell > 0$. We get

$$Q = \ell \begin{pmatrix} c^2 + 1 & cs & s^2 \\ cs & s^2 & -cs \end{pmatrix}.$$

The last two columns specify the exterior derivative d on $\mathfrak{u}(\mathfrak{g})^* \cong \mathfrak{g}^* / \operatorname{Im}\chi^*$. One sees that $\mathfrak{u}(\mathfrak{g}) \cong \mathfrak{r}_{3,-1}$ as B acts with eigenvalues $\pm \ell s$. □

To summarise, we get a unique choice of basis $\{a, Ja, b, Jb\}$ with $\{a, b\}$ orthonormal by taking $a \in \operatorname{Im}\chi^*$, $b \in \ker d \cap (\operatorname{Im}\chi^*)^\perp$ with $x_1 > 0$ and $v_2 > 0$.

We may describe the isomorphism of $\mathfrak{g}$ with $\mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$ explicitly by introducing half-angles. Writing $c = \sigma^2 - \tau^2$, $s = 2\sigma\tau$, $\sigma^2 + \tau^2 = 1$, $\sigma > 0$ and using the orthogonal transformation $a' = \sigma a + \tau b$, $b' = -\tau a + \sigma b$, gives the structural equations

$$d(Ja') = 2\ell\sigma\, a' Ja', \qquad d(Jb') = -2\ell\tau\, b' Jb'.$$

We have $\ell, \sigma > 0$ and, replacing b' by $-b'$ if necessary, we may ensure that $\tau < 0$. The SKT moduli space is thus parameterised by $\sigma/\tau \in (-1, 0)$, $\ell > 0$ and the parameter $t = g(b', Ja') \in (-1, 1)$ in the metric. Up to homotheties it is connected of dimension 2. The solutions are Kähler precisely when $t = 0$.

Remark 3.10. If one considers the complex structure on $\mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$ with $de = 0$, $d(Je) = eJe$, $df = 0$, $d(Jf) = fJf$ one sees that a metric with $\omega = eJe + fJf + t(eJf + fJe)$ is SKT (indeed Kähler) only if $t = 0$. Thus for a given complex structure the SKT condition depends on the choice of metric. This is in contrast to the study of SKT structures on six-dimensional nilmanifolds [FPS04]. △

3.3.4 Three-dimensional Abelian derived algebra

For $\mathfrak{g}' = \mathbb{R}^3$, we have $\dim W_1 = 1$, and moreover the assumption that $\mathfrak{g}'$ is Abelian implies that $d(Ja)$, db, $d(Jb) \in \mathcal{I}(a)$. So it is legitimate to assume that $V_2 = JV_2$. The structural equations are thus

$$da = 0, \quad d(Ja) = x_1 aJa,$$
$$db = y_1 aJa + y_2 ab + y_3 aJb, \quad d(Jb) = u_1 aJa - y_3 ab + y_2 aJb,$$

with coefficients satisfying the equation

$$0 = y_2(2y_2 + x_1)$$

and non-degeneracy conditions $x_1 \neq 0$, $y_2^2 + y_3^2 \neq 0$. One may choose a, b so that $x_1 > 0$, $y_1 \geqslant 0$ and $u_1 = 0$. This choice is unique if $y_1 > 0$, for $y_1 = 0$, b is an arbitrary unit vector in $V_2^\perp$. The solutions are then Kähler only if y_1 and y_2 are zero.

If $y_2 = 0$, then $y_3 \neq 0$ and $\mathfrak{g} \cong \mathfrak{r}'_{4,|x_1/y_3|,0}$. Thus on a given $\mathfrak{r}'_{4,\lambda,0}$, $\lambda > 0$, the SKT moduli up to homothety has dimension 1, parameter y_3, with two connected components determined by the sign of y_3, and contains the Kähler solutions as $y_1 = 0$.

For $y_2 \neq 0$, we have $x_1 = -2y_2$. There are two cases. For $y_3 = 0$, we have $\mathfrak{g} \cong \mathfrak{r}_{4,-1/2,-1/2}$ and there is a one-dimensional connected family of solutions up to homothety. For $y_3 \neq 0$, the Lie algebra $\mathfrak{g}$ is $\mathfrak{r}'_{4,2\lambda,-\lambda}$ with $\lambda = |y_2/y_3|$. Again the moduli is of dimension 1 up to homothety and has two connected components.

3.3.5 Three-dimensional non-Abelian derived algebra

For $\mathfrak{g}' = \mathfrak{h}_3$, as above we have $\dim W_1 = 1$. Let d' denote the exterior derivative on $\mathfrak{g}'$. We distinguish between the complex and real cases $V_2 = JV_2$ and $V_2 \cap JV_2 = \{0\}$.

Complex case We have $a \in W_1 = V_1$, and $Ja \in V_2 = JV_2$. Moreover it is possible to take $b \in V_2^\perp$ with $d'b = 0$. The condition $\mathfrak{g}' \cong \mathfrak{h}_3$ then forces $d'(Jb) \in \langle bJa \rangle$, giving the structural equations

$$da = 0, \quad d(Ja) = x_1 aJa,$$
$$db = y_1 aJa + y_2 ab + y_3 aJb, \quad d(Jb) = u_1 aJa + u_2 ab + u_3 aJb + v_1 bJa,$$

with x_1, $y_2^2 + y_3^2$ and v_1 non-zero. Adjusting the choice of a, we may take $x_1 > 0$. The SKT equations are now the vanishing of

$$y_2 - u_3 + v_1, \quad y_3 + u_2, \quad y_3 v_1,$$
$$v_1(x_1 + y_2 - u_3), \quad (y_2 + u_3)(y_2 + u_3 + x_1).$$

We deduce that $y_3 = 0 = u_2$, $v_1 = x_1$ and $u_3 = y_2 + x_1$, leaving the condition $(2y_2 + x_1)(y_2 + x_1) = 0$.

If $y_2 = -x_1$, then the structural equations are

$$da = 0, \quad d(Ja) = x_1 aJa,$$
$$db = y_1 aJa - x_1 ab, \quad d(Jb) = u_1 aJa + x_1 bJa$$

subject only to $x_1 > 0$. We see that $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}')$ is isomorphic to $\mathfrak{r}_{3,-1}$, so $\mathfrak{g}$ itself is isomorphic to $\mathfrak{d}_4$. The only ambiguity in the basis is $b \mapsto -b$, corresponding to $(y_1, u_1) \mapsto (-y_1, -u_1)$. The SKT moduli modulo homotheties is connected and has dimension 2. There are no Kähler solutions.

For $x_1 = -2y_2$, we have the structural equations

$$da = 0, \quad d(Ja) = x_1 aJa,$$
$$db = y_1 aJa - \tfrac{1}{2}x_1 ab, \quad d(Jb) = u_1 aJa + \tfrac{1}{2}x_1 aJb + x_1 bJa,$$

again with $x_1 > 0$. The quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}')$ is isomorphic to $\mathfrak{r}_{3,-1/2}$, and $\mathfrak{g}$ is thus isomorphic to $\mathfrak{d}_{4,2}$. The solutions are Kähler only for $y_1 = 0 = u_1$. There is the same $b \mapsto -b$ ambiguity as above. Again the SKT moduli space up to homotheties is connected of dimension 2.

Real case First note that $\dim W_2 = 3$, so we may choose b to be a unit vector in $W_2 \cap \langle a, Ja\rangle^\perp$. This gives $t = g(b, Ja) = 0$. Now $d'b = 0$, where d' is the differential on $\mathfrak{g}'$, as above. As $\mathfrak{h}_3' = \mathbb{R}$, we have that $d'(Ja)$ and $d'(Jb)$ are linearly dependent, but not both zero. In fact, if $d'(Ja) = 0$, we may take $V_2 = \langle a, Ja\rangle$ and reduce to the complex case described above, so we assume instead $d'(Ja) \neq 0$.

Write $(x_2, x_3, y_2) = m\boldsymbol{p}$, $(w_1, v_1, v_2) = n\boldsymbol{p}$ for some unit vector $\boldsymbol{p} = (p, q, r)$, $m \neq 0$. The structural equations of $\mathfrak{h}_3$, imply $b \wedge d'x = 0$ is zero for all $x \in \mathfrak{g}'$, giving $p = 0$ and $x_2 = 0 = w_1$. Now $q^2 + r^2 = 1$ and one may normalise so that $r \geqslant 0$. Then

$$d'(Ja) = m\, bJc, \quad d'(Jb) = n\, bJc,$$

where

$$c = qa + rb.$$

From this one sees $d'(nJa - mJb) = 0$ and so $(nJa - mJb) \wedge d'x = 0$ is zero too. We conclude that $qJa + rJb$ and $nJa - mJb$ are parallel and write $n = kq$, $m = -kr$, for some $k \neq 0$.

The structural equations are now

$$\begin{gathered} da = 0, \quad d(Ja) = x_1 aJa - kqr(aJb + bJa) - kr^2\, bJb, \\ db = z_1 aJa + z_2 ab + z_3 aJb, \\ d(Jb) = u_1 aJa + u_2 ab + u_3 aJb + kq^2\, bJa + kqr\, bJb, \end{gathered}$$

with $q^2 + r^2 = 1$, $r > 0$, the forms $d(Ja)$, db, $d(Jb)$ non-zero, and subject to

$$\begin{gathered} u_3 = z_2 + kq, \quad u_2 = -z_3, \quad rz_1 = qz_3, \\ kq^3 - qz_2 - ru_1 = 0, \quad 2kq^2 + x_1 - z_2 - u_3 = 0, \\ q(q(x_1 + z_2 - u_3) - 2ru_1) = 0, \quad (x_1 + z_2 + u_3)(z_2 + u_3 + kr^2) = 0. \end{gathered} \tag{3.14}$$

Substituting the first three equations into the remaining four, one sees that the first equation on the last line follows from the two on the middle line. There are thus two cases corresponding to the two factors of the last equation.

The first case is $z_2 = -x_1 - u_3$, which reduces to $x_1 = -kq^2 = -u_3$, $z_2 = 0$, $u_1 = kq^3/r$, giving the structural equations

$$da = 0, \quad d(Ja) = -k\, cJc, \quad db = z_3 r^{-1}\, aJc, \quad d(Jb) = -z_3\, ab + kqr^{-1}\, cJc,$$

with $z_3 \neq 0$. Now $\tilde{\mathfrak{g}}^* = \mathfrak{g} / \mathfrak{z}(\mathfrak{g}')^* \cong \langle a, b, c\rangle$, with $c' = c/r$, has structural equations $\tilde{d}a = 0$, $\tilde{d}b = z_3 ac'$, $\tilde{d}c' = -z_3 ab$ and so is isomorphic to $\mathfrak{r}_{3,0}'$. This gives $\mathfrak{g} \cong \mathfrak{d}_{4,0}'$.

In this case the solutions are never Kähler. The SKT moduli up to homotheties has dimension 2 and is connected. To see this note that a is specified up to sign, which may be fixed by requiring $k > 0$. If $q \neq 0$, replacing b by $\pm b$, we may then ensure $z_3 > 0$. This uniquely specifies b, and the remaining parameter is given by q. For $q = 0$, we may rotate in the b, Jb plan, but this does not change the solution.

The final case is $z_2 = -u_3 - kr^2$. Here one finds $x_1 = -k(1+q^2)$, $z_2 = -k/2$, $u_1 = -kq(2q^2+1)/2r$ giving

$$\begin{aligned} da = 0, \quad d(Ja) = -k(aJa + cJc), \quad db = -\tfrac{1}{2}k\,ab + z_3 r^{-1}\,aJc, \\ d(Jb) = \tfrac{1}{2}kr^{-1}\,a(qJa - rJb) - z_3\,ab + kqr^{-1}\,cJc. \end{aligned} \tag{3.15}$$

This time computing the structural equations for $\tilde{\mathfrak{g}} = \mathfrak{g} / \mathfrak{z}(\mathfrak{g}')$ gives $\tilde{d}a = 0$, $\tilde{d}b = -\frac{1}{2}kab + z_3ac'$, $\tilde{d}c' = -z_3ab - \frac{1}{2}kac'$. If $z_3 \neq 0$, we have $\tilde{\mathfrak{g}} \cong \mathfrak{r}'_{3,\lambda}$ with $\lambda = |k/2z_3|$ giving $\mathfrak{g} \cong \mathfrak{d}'_{4,\lambda}$. The analysis for the choices of a, b is as above. For $z_3 = 0$, we have $\tilde{\mathfrak{g}} \cong \mathfrak{r}_{3,1}$ and $\mathfrak{g} \cong \mathfrak{d}_{4,1/2}$. The basis analysis is similar to above: $k > 0$ fixes a; for $q \neq 0$, b is fixed by $q > 0$; for $q = 0$ we may rotate in the b, Jb plane without changing the solution.

The solutions are Kähler precisely when $q = 0$. The SKT moduli up to homotheties has dimension 1 and is connected both for $\mathfrak{g} = \mathfrak{d}'_{4,\lambda}$ and for $\mathfrak{g} = \mathfrak{d}_{4,1/2}$.

This completes the proof of Theorem 3.8.

3.4 Consequences and concluding remarks

Let us first emphasise Remark 3.10 that for four-dimensional solvable groups the SKT condition depends explicitly on both the metric and the complex structure, in contrast to the situation [FPS04] for six-dimensional nilpotent groups.

Corollary 3.11. *There are four-dimensional solvable complex Lie groups whose family of compatible invariant Hermitian metrics contains both* SKT *and non-*SKT *structures.* □

An alternative approach to our classification of invariant SKT structures in Theorem 3.8 would be to start with results for complex structures on four-dimensional solvable Lie groups (Ovando [Ova00, Ova04], Snow [Sno90]) and then to impose the SKT condition. We have used this approach to cross check our results, but also found that the lists given in [Ova04] for Kähler forms and algebras with complex structures have some errors and omissions. Some of these are corrected in [ABDO05], but we wish to emphasise that the proof given in Section 3.3 is independent of those calculations. In contrast to the compact case we see:

Corollary 3.12. *The four-dimensional solvable Lie algebras $\mathfrak{g}$ that admit invariant complex structures but no compatible invariant* SKT *metric are: $\mathbb{R} \times \mathfrak{r}_{3,1}$, $\mathbb{R} \times \mathfrak{r}'_{3,\lambda>0}$, $\mathfrak{aff}_{\mathbb{C}}$, $\mathfrak{r}_{4,1}$, $\mathfrak{r}_{4,\mu,\lambda}$, ($\mu = \lambda \neq -\frac{1}{2}$ or $\mu \leqslant \lambda = 1$), $\mathfrak{r}'_{4,\mu,\lambda}$ ($\lambda \neq 0, -\mu/2$), $\mathfrak{d}_{4,\lambda}$ ($\lambda \neq \frac{1}{2}, 2$), $\mathfrak{h}_4$.* □

Here the given constraints on the parameters are in addition to the defining constraints for the algebras.

On the other hand if G admits a discrete co-compact subgroup Γ then $M = \Gamma \backslash G$ is a compact manifold (a solvmanifold). By Gauduchon's theorem [Gau84] any complex structure on M admits an SKT metric (indeed one in any compatible conformal class). If G has an invariant complex structure one may then construct a compatible invariant SKT structure on G via pull-back from M (cf. [FG04]). A necessary condition for Γ to exist is that G be unimodular, which is equivalent to $b_4(\mathfrak{g}) = 1$, but in general this is not sufficient. The correct classification of complex solvmanifolds in dimension four has recently been provided by Hasegawa [Has05]. In our notation, one obtains

(i) tori from $\mathfrak{g} = \mathbb{R}^4$,
(ii) primary Kodaira surfaces from $\mathfrak{g} = \mathbb{R} \times \mathfrak{h}_3$,
(iii) hyperelliptic surfaces from $\mathfrak{g} = \mathbb{R} \times \mathfrak{r}'_{3,0}$,
(iv) Inoue surfaces of type S^0 from $\mathfrak{g} = \mathfrak{r}_{4,-\frac{1}{2},-\frac{1}{2}}$ and from $\mathfrak{g} = \mathfrak{r}'_{4,2\lambda,-\lambda}$,
(v) Inoue surfaces of type $S^{\pm}$ from $\mathfrak{g} = \mathfrak{d}_4$ and
(vi) secondary Kodaira surfaces from $\mathfrak{g} = \mathfrak{d}'_{4,0}$.

Comparing this list with our classification we conclude:

Corollary 3.13. *Each unimodular solvable four-dimensional Lie group G with invariant* SKT *structure admits a compact quotient by a lattice.* □

If we have an HKT structure and (g, I) is already SKT then (g, J) and (g, K) are necessarily SKT and the HKT structure is strong. However, the list of HKT structures on solvable Lie groups is known in dimension four from [Bar97].

Corollary 3.14. *The only four-dimensional solvable Lie algebra that is strong* HKT *is $\mathbb{R}^4$, which is hyperKähler. The algebra $\mathfrak{d}_{4,1/2}$ admits both* HKT *and* SKT *structures; these structures are distinct. The remaining* HKT *algebras $\mathfrak{aff}_{\mathbb{C}}$ and $\mathfrak{r}_{4,1,1}$ do not admit invariant* SKT *structures.* □

In the case of $\mathfrak{d}_{4,1/2}$ one may use (3.15) to check that the HKT and SKT metrics are inequivalent.

Finally, let us make the following observation which follows from case-by-case study of the algebras found in our SKT classification Theorem 3.8. The *symmetry rank* of an SKT manifold (M, g, J) is the dimension of the maximal Abelian group of isometries that preserve J, cf. [GS94, Fan04].

Corollary 3.15. *Each invariant* SKT *structure on a four-dimensional solvable Lie group G has symmetry rank at least two.* □

This motivates a future study of SKT structures on Abelian principal bundles over Riemann surfaces. Expectedly multi-moment maps, cf. Chapter 4, will be useful tools since they provide us with one or two natural coordinates in addition to those along the fibers.

3.5 Low-dimensional solvable Lie algebras

The four-dimensional solvable real Lie algebras are classified in [ABDO05]. In this section we summarise the classification and provide the notation for Section 3.3. The quoted results and observations will also be of relevance in our study of $(2,3)$-trivial Lie algebras in Chapter 4.

Our notation for the three-dimensional solvable Lie algebras will be as given in Table 3.2. Note that $\mathfrak{r}_{3,0} \cong \mathbb{R} \times \mathfrak{aff}_{\mathbb{R}}$.

$\mathfrak{aff}_{\mathbb{R}}$	$(0, 21)$	
$\mathfrak{h}_3$	$(0, 0, 21)$	
$\mathfrak{r}_3$	$(0, 21 + 31, 31)$	
$\mathfrak{r}_{3,\lambda}$	$(0, 21, \lambda 31)$	$\lvert\lambda\rvert \leqslant 1$
$\mathfrak{r}'_{3,\lambda}$	$(0, \lambda 21 + 31, -21 + \lambda 31)$	$\lambda \geqslant 0$

Table 3.2: Non-Abelian solvable Lie algebras of dimension at most three that are not of product type.

The four-dimensional solvable Lie algebras are classified as follows.

Theorem 3.16 ([ABDO05])**.** *Let $\mathfrak{g}$ be a four dimensional solvable real Lie algebra. Then $\mathfrak{g}$ is isomorphic to one and only one of the following Lie algebras: $\mathbb{R}^4$, $\mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$, $\mathbb{R} \times \mathfrak{h}_3$, $\mathbb{R} \times \mathfrak{r}_3$, $\mathbb{R} \times \mathfrak{r}_{3,\lambda}$ $(\lvert\lambda\rvert \leqslant 1)$, $\mathbb{R} \times \mathfrak{r}'_{3,\lambda}$ $(\lambda \geqslant 0)$, or one of the algebras in Table 3.3.*

Among these the unimodular algebras are: $\mathbb{R}^4$, $\mathbb{R} \times \mathfrak{h}_3$, $\mathbb{R} \times \mathfrak{r}_{3,-1}$, $\mathbb{R} \times \mathfrak{r}'_{3,0}$, $\mathfrak{n}_4$, $\mathfrak{r}_{4,-1/2}$, $\mathfrak{r}_{4,\mu,-1-\mu}$ $(-1 < \mu \leqslant -\frac{1}{2})$, $\mathfrak{r}'_{4,\mu,-\mu/2}$, $\mathfrak{d}_4$, $\mathfrak{d}'_{4,0}$.

$\mathfrak{n}_4$	$(0, 0, 21, 31)$	
$\mathfrak{aff}_{\mathbb{C}}$	$(0, 0, 31 - 42, 41 + 32)$	
$\mathfrak{r}_4$	$(0, 21 + 31, 31 + 41, 41)$	
$\mathfrak{r}_{4,\lambda}$	$(0, 21, \lambda 31 + 41, \lambda 41)$	
$\mathfrak{r}_{4,\mu,\lambda}$	$(0, 21, \mu 31, \lambda 41)$	$\mu, \lambda \in \mathscr{R}_4$
$\mathfrak{r}'_{4,\mu,\lambda}$	$(0, \mu 21, \lambda 31 + 41, -31 + \lambda 41)$	$\mu > 0$
$\mathfrak{d}_4$	$(0, 21, -31, 32)$	
$\mathfrak{d}_{4,\lambda}$	$(0, \lambda 21, (1 - \lambda) 31, 41 + 32)$	$\lambda \geqslant \frac{1}{2}$
$\mathfrak{d}'_{4,\lambda}$	$(0, \lambda 21 + 31, -21 + \lambda 31, 2\lambda.41 + 32)$	$\lambda \geqslant 0$
$\mathfrak{h}_4$	$(0, 21 + 31, 31, 2.41 + 32)$	

Table 3.3: Four-dimensional solvable Lie algebras not of product type. The set $\mathscr{R}_4$ consists of the $(\mu, \lambda) \in [-1, 1]^2$ with $\lambda \geqslant \mu$ and $\mu, \lambda \neq 0$ and satisfying $\lambda < 0$ if $\mu = -1$.

In the Table 3.4 the four-dimensional solvable real Lie algebras are sorted by their derived algebra $\mathfrak{g}'$. In some cases it is easy to recognise which algebra is at hand using the following observations:

$\mathfrak{g}'$	$\mathfrak{z}(\mathfrak{g})$	$\mathfrak{g}$
$\{0\}$		$\mathbb{R}^4$
$\mathbb{R}$		$\mathbb{R} \times \mathfrak{h}_3,\ \mathbb{R} \times \mathfrak{r}_{3,0}$
$\mathbb{R}^2$	$\{0\}$	$\mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}},\ \mathfrak{aff}_{\mathbb{C}},\ \mathfrak{d}_{4,1}$
	$\mathbb{R}$	$\mathbb{R} \times \mathfrak{r}_3,\ \mathbb{R} \times \mathfrak{r}_{3,\lambda\neq 0},\ \mathbb{R} \times \mathfrak{r}'_{3,\lambda},\ \mathfrak{r}_{4,0},\ \mathfrak{n}_4$
$\mathbb{R}^3$		$\mathfrak{r}_4,\ \mathfrak{r}_{4,\lambda\neq 0},\ \mathfrak{r}_{4,\mu,\lambda},\ \mathfrak{r}'_{4,\mu,\lambda}$
$\mathfrak{h}_3$		$\mathfrak{d}_4,\ \mathfrak{d}_{4,\lambda\neq 1},\ \mathfrak{d}'_{4,\lambda},\ \mathfrak{h}_4$

Table 3.4: The four-dimensional solvable Lie algebras sorted by $\mathfrak{g}'$ and, where necessary, $\mathfrak{z}(\mathfrak{g})$. The conditions on the parameters are in addition to those from Tables 3.2 and 3.3.

$\mathfrak{g}' = \mathbb{R}$: $\mathbb{R} \times \mathfrak{h}_3$ is nilpotent, $\mathbb{R} \times \mathfrak{r}_{3,0}$ is not.

$\mathfrak{g}' = \mathbb{R}^2$, $\mathfrak{z}(\mathfrak{g}) = \{0\}$: $\mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$ and $\mathfrak{d}_{4,1}$ are completely solvable, $\mathfrak{aff}_{\mathbb{C}}$ is not. Moreover these algebras have different unimodular kernels:

$$\mathfrak{u}(\mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}) \cong \mathfrak{r}_{3,-1}, \quad \mathfrak{u}(\mathfrak{d}_{4,1}) \cong \mathfrak{h}_3, \quad \mathfrak{u}(\mathfrak{aff}_{\mathbb{C}}) \cong \mathfrak{r}'_{3,0}\,.$$

$\mathfrak{g}' = \mathfrak{h}_3$: the algebras are distinguished by $\tilde{\mathfrak{g}} = \mathfrak{g} \,/\, \mathfrak{z}(\mathfrak{g}')$ as follows:

$$\tilde{\mathfrak{d}}_4 \cong \mathfrak{r}_{3,-1}, \quad \tilde{\mathfrak{d}}_{4,\lambda\neq 1} \cong \mathfrak{r}_{3,(1-\lambda)/\lambda}, \quad \tilde{\mathfrak{d}}'_{4,\lambda} \cong \mathfrak{r}'_{3,\lambda}, \quad \tilde{\mathfrak{h}}_4 \cong \mathfrak{r}_3\,.$$

Multi-moment maps

Chapter 4

Multi-moment maps for strong geometries

We now pass on from a particular type of three-form geometry to the general notion of strong geometries. These are characterised completely or partly in terms of a closed three-form, and if symmetry is present they often come equipped with a particular type of map, a so-called multi-moment map. While the main source of inspiration is symplectic geometry, the less rigidity of three-forms implies that significantly new ideas are needed.

The chapter is organised as follows. In Section 4.1 we give the fundamental calculations that lead to the definition of multi-moment map and introduce the Lie kernel $\mathcal{P}_{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$. We then consider topological and algebraic criteria for existence and uniqueness of multi-moment maps in Section 4.2. As mentioned in Chapter 1, $(2,3)$-trivial Lie algebras play a natural role and Section 4.3 is devoted to an algebraic study of this class and the description of a number of examples. We then return to strong geometries and their multi-moment maps. The basic example is provided by the total space $\Lambda^2 T^*N$ of the second exterior power of the cotangent bundle of a manifold N. Homogeneous strong geometries with multi-moment maps are closely tied to orbits in the dual $\mathcal{P}_{\mathfrak{g}}^*$ of the Lie kernel and we develop a Kirillov-Kostant-Souriau type theory, pointing out links with nearly Kähler and hypercomplex geometry. In the final section of the chapter we return to the study of $(2,3)$-trivial Lie algebras. A systematic treatment, based on our structural result, Theorem 4.16, enables us to list all algebras of this type in dimensions up to and including five.

4.1 Main definitions

Let (M,c) be a *strong geometry*, meaning that M is a smooth manifold and that c is a closed three-form on M. Note that unlike the symplectic case there is no one canonical form for c, not even pointwise on M. In general, we do not require any non-degeneracy of c. However, when necessary we will use the terminology of [BHR10] that c is 2-*plectic* if $X \lrcorner c = 0$ at $x \in M$ only when $X = 0$ in T_xM.

Remark 4.1. Since c is closed, $\ker c = \{ X \in TM : X \lrcorner c = 0 \}$ is an integrable distribution. Thus if $\ker c$ is of constant rank and has closed leaves, c induces a 2-plectic structure on $M/\ker c$, provided that the quotient is a manifold. △

Remark 4.2. One could consider strongly non-degenerate three-forms c, meaning that $c(X,Y,\cdot) \neq 0$ for all $X \wedge Y \neq 0$. However, by [Mas83] such c exist only in dimensions 3 and 7. The former case is given by a volume form, the latter by a G-structure with $G = G_2$ or its non-compact dual. △

Let G be a *group of symmetries* for (M,c), meaning that G acts on M preserving the three-form c. Thus for each $\mathsf{X} \in \mathfrak{g}$ we have $\mathcal{L}_X c = 0$, where X is the vector field generated by X. As $dc = 0$, this gives

$$0 = \mathcal{L}_X c = d(X \lrcorner c) + X \lrcorner dc = d(X \lrcorner c), \tag{4.1}$$

so the two-form $X \lrcorner c$ is closed. Suppose $\mathsf{Y} \in \mathfrak{g}$ commutes with X. Then we have

$$0 = \mathcal{L}_Y(X \lrcorner c) = d(Y \lrcorner X \lrcorner c) = d((X \wedge Y) \lrcorner c),$$

showing that the one form $(X \wedge Y) \lrcorner c = c(X,Y,\cdot)$ is closed. If for example, $b_1(M) = 0$, we may then write

$$(X \wedge Y) \lrcorner c = d\nu_{\mathsf{X}\wedge\mathsf{Y}}$$

for some smooth function $\nu_{\mathsf{X}\wedge\mathsf{Y}} \colon M \to \mathbb{R}$. This is the basis of the construction of the multi-moment map. However, the set of decomposable elements $\mathsf{X} \wedge \mathsf{Y}$ in $\Lambda^2 \mathfrak{g}$ for which X and Y commute is a complicated variety. It is more natural to consider the following submodule of $\Lambda^2 \mathfrak{g}$.

Definition 4.3. The *Lie kernel* $\mathcal{P}_\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is the $\mathfrak{g}$-module

$$\mathcal{P}_\mathfrak{g} := \ker\left(L \colon \Lambda^2 \mathfrak{g} \to \mathfrak{g}\right),$$

where L is the linear map induced by the Lie bracket.

The previous calculation may now be extended to elements of the Lie kernel. For a bivector $p = \sum_{j=1}^k X_j \wedge Y_j$ we write

$$p \lrcorner c := \sum_{j=1}^k c(X_j, Y_j, \cdot).$$

Lemma 4.4. *Suppose G is a group of symmetries of a strong geometry (M,c). Let $\mathsf{p} = \sum_{j=1}^k \mathsf{X}_j \wedge \mathsf{Y}_j$ be an element of the Lie kernel $\mathcal{P}_\mathfrak{g}$ and let $p = \sum_{j=1}^k X_j \wedge Y_j$ be the corresponding bivector on M. Then*

$$d(p \lrcorner c) = 0. \tag{4.2}$$

Proof. The condition that p lies in $\mathcal{P}_{\mathfrak{g}}$ is that $0 = L(\mathsf{p}) = \sum_{j=1}^{k}[\mathsf{X}_j, \mathsf{Y}_j]$. This together with (4.1) and $dc = 0$ gives

$$\begin{aligned} 0 &= \sum_{j=1}^{k}[Y_j, X_j] \lrcorner c = \sum_{j=1}^{k}\Big([\mathcal{L}_{Y_j}, X_j \lrcorner]c\Big) \\ &= \sum_{j=1}^{k} d(Y_j \lrcorner X_j \lrcorner c) + Y_j \lrcorner d(X_j \lrcorner c) - X_j \lrcorner d(Y_j \lrcorner c) - X_j \lrcorner Y_j \lrcorner dc \\ &= \sum_{j=1}^{k} d(Y_j \lrcorner X_j \lrcorner c) = d(p \lrcorner c), \end{aligned}$$

as required. □

Thus if for example $b_1(M) = 0$, there is a smooth function $\nu_p \colon M \to \mathbb{R}$ with $d\nu_p = p \lrcorner c$ for each $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$.

We are now able to define the main object to be studied in this paper.

Definition 4.5. Let (M, c) be a strong geometry with a symmetry group G. A *multi-moment map* is an equivariant map $\nu \colon M \to \mathcal{P}_{\mathfrak{g}}^*$ satisfying

$$d\langle \nu, \mathsf{p} \rangle = p \lrcorner c \tag{4.3}$$

for each $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$.

Remark 4.6. As the Lie kernel $\mathcal{P}_{\mathfrak{g}}$ is a G-module with respect to the action induced by the adjoint action of G on $\mathfrak{g}$, equivariance of ν may be phrased by the relation

$$\nu(g \cdot m) = \mathrm{Ad}_{g^{-1}}^*(\nu(m)),$$

for all $g \in G$ and $m \in M$. △

Note that for G Abelian $\mathcal{P}_{\mathfrak{g}} = \Lambda^2 \mathfrak{g}$. On the other hand if G is a compact simple Lie group then the Lie kernel is a module familiar from a special class of Einstein manifolds. Indeed Wolf [Wol68, Corollary 10.2] (cf. [Bes08, Proposition 7.49]) showed that in this case $\Lambda^2 \mathfrak{g} = \mathfrak{g} \oplus \mathcal{P}_{\mathfrak{g}}$ as a sum of irreducible modules, so $SO(\dim G)/G$ is an isotropy irreducible space.

4.2 Existence and uniqueness

As mentioned in Chapter 1, one of the principal advantages of multi-moment maps over covariant moment maps is that one can prove that multi-moment maps are guaranteed to exist under a wide range of circumstances.

We start first with topological criteria which follow essentially by the same arguments as in the symplectic setting, see for instance [OR04, Proposition 4.5.19] and [GS84, Addendum to Theorem 26.1].

Theorem 4.7. *Let (M, c) be a strong geometry with a symmetry group G and assume that $b_1(M) = 0$. If either*

(i) *G is compact, or*
(ii) *M is compact and orientable, and G preserves a volume form on M,*
then there exists a multi-moment map $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$.

Proof. Working component by component, we may assume that M is connected. As noted after Lemma 4.4 the condition $b_1(M) = 0$ ensures that there are functions ν_{p} with $d\nu_p = p \lrcorner c$ for each $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$. However, each of these functions may be adjusted by adding a real constant. To build a multi-moment map ν via $\langle \nu, \mathsf{p} \rangle = \nu_p$ we need to ensure equivariance. In the two cases above this may be achieved by either averaging over G or over M. In the second case, one chooses ν_p with mean value 0. In the first case, one chooses a basis (p_i) of $\mathcal{P}_{\mathfrak{g}}$ and puts $\nu(m) = \int_G \sum_i \mathrm{Ad}^*_{g^{-1}}(\nu_{p_i}(g^{-1} \cdot m)) \,\mathrm{vol}_G = \int_G \sum_i \nu_{\mathrm{Ad}_{g^{-1}} \mathsf{p}_i}(g^{-1} \cdot m) \,\mathrm{vol}_G$. In both cases equation (4.3) is satisfied, essentially due to the identity

$$d(\nu_{\mathrm{Ad}_{g^{-1}} \mathsf{p}} \circ g^{-1}) = d\nu_p,$$

which follows since the pull-back $g^*(p)$ is the bivector field corresponding to the element $\mathrm{Ad}_{g^{-1}}(\mathsf{p}) \in \mathcal{P}_{\mathfrak{g}}$. Consequently, ν is multi-moment map. □

As we saw in the above proof, one crucial point is making a canonical choice of function ν_p. The following situation occurs in many examples and provides a differential geometric criterion for a construction of multi-moment maps.

Proposition 4.8. *Suppose G is a group of symmetries of a strong geometry* (M, c) *and that there exists a G-invariant two-form* $b \in \Omega^2(M)$ *such that* $db = c$. *Then* $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$ *given by*

$$\langle \nu, \mathsf{p} \rangle = b(p) \tag{4.4}$$

is a multi-moment map.

Proof. The map ν is equivariant, since b is invariant. We have $\nu_p = b(p)$ with $d(b(p)) = d(p \lrcorner b) = p \lrcorner db = p \lrcorner c$ by the calculation in Lemma 4.4, so equation (4.3) is satisfied, as required. □

Inspired by the symplectic setting [GGK02, Proposition 2.9], we will give an alternative version of Theorem 4.7 which holds if the group acting is Abelian and has compact orbits.

Proposition 4.9. *Let* (M, c) *be a strong geometry with a connected Abelian symmetry group G and assume that* $b_1(M) = 0$. *If G has compact orbits, then there exists a multi-moment map* $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$.

Proof. As in Theorem 4.7, we build a multi-moment map ν via functions $\langle \nu, \mathsf{p} \rangle = \nu_p$ satisfying $d\nu_p = p \lrcorner \nu$. We need to check invariance. From the calculation

$$\mathcal{L}_Y(\mathcal{L}_X \nu_p) = \mathcal{L}_Y(X \lrcorner d\nu_p) = \mathcal{L}_Y(c(p, X)) = 0,$$

we conclude that for each $\mathsf{X} \in \mathfrak{g}$, the function $\mathcal{L}_X(\nu_p)$ is constant along the orbits of G. By compactness, each orbit contains a point where ν_p has a maximum. At this point we must have $\mathcal{L}_X(\nu_p) = 0$ for any $\mathsf{X} \in \mathfrak{g}$. But then $\mathcal{L}_X(\nu_p) = 0$ at all points along the orbit. In conclusion, ν_p is G-invariant, as required. □

Let us now turn to algebraic criteria for multi-moment maps. This involves study of the Lie kernel. The dual of the exact sequence

$$0 \longrightarrow \mathcal{P}_{\mathfrak{g}} \xrightarrow{\iota} \Lambda^2 \mathfrak{g} \xrightarrow{L} \mathfrak{g}$$

is the sequence

$$\mathfrak{g}^* \xrightarrow{d} \Lambda^2 \mathfrak{g}^* \xrightarrow{\pi} \mathcal{P}_{\mathfrak{g}}^* \longrightarrow 0, \tag{4.5}$$

which is also exact. Hence the dual $\mathcal{P}_{\mathfrak{g}}^*$ of the Lie kernel can be identified with the quotient space $\Lambda^2 \mathfrak{g}^* / d(\mathfrak{g}^*)$. As $B^2(\mathfrak{g}) = d(\mathfrak{g}^*)$ is a subspace of $Z^2(\mathfrak{g}) = \ker(d\colon \Lambda^2 \mathfrak{g}^* \to \Lambda^3 \mathfrak{g}^*)$, we have an induced linear map

$$d_{\mathcal{P}}\colon \mathcal{P}_{\mathfrak{g}}^* \to B^3(\mathfrak{g}) \subset Z^3(\mathfrak{g}) \subset \Lambda^3 \mathfrak{g}^* .$$

More concretely, given $\beta \in \mathcal{P}_{\mathfrak{g}}^*$, we choose $\widetilde{\beta} \in \pi^{-1}(\beta)$ and then $d_{\mathcal{P}}\beta = d\widetilde{\beta}$.

Let $b_n(\mathfrak{g})$ denote the dimension of the nth Lie algebra cohomology group, so $b_n(\mathfrak{g}) = \dim H^n(\mathfrak{g}) = \dim Z^n(\mathfrak{g}) - \dim B^n(\mathfrak{g})$. The next result follows directly from the above discussion.

Proposition 4.10. *The linear map $d_{\mathcal{P}}\colon \mathcal{P}_{\mathfrak{g}}^* \to \Lambda^3 \mathfrak{g}^*$ is a $\mathfrak{g}$-morphism with image contained in $Z^3(\mathfrak{g})$. It is injective if and only if $b_2(\mathfrak{g}) = 0$. If this condition holds then $d_{\mathcal{P}}$ is an isomorphism from $\mathcal{P}_{\mathfrak{g}}^*$ onto $Z^3(\mathfrak{g})$ if and only if $b_3(\mathfrak{g}) = 0$.* □

We will see that this distinguishes a class of Lie groups and Lie algebras that play a special role in the theory of multi-moment maps analogous to the role of semi-simple groups in symplectic geometry. We therefore make a definition.

Definition 4.11. A connected Lie group G or its Lie algebra $\mathfrak{g}$ that satisfies $b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$ will be called *(cohomologically)* $(2,3)$*-trivial*.

Theorem 4.12. *Let (M, c) be a strong geometry with connected $(2,3)$-trivial symmetry group G acting nearly effectively. Then there exists a unique multi-moment map $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$.*

More generally, if just $b_2(\mathfrak{g}) = 0$, then multi-moment maps for nearly effective actions of G are unique when they exist.

Proof. The invariant three-form c determines a G-equivariant map $\Psi\colon M \to Z^3(\mathfrak{g})$, given by

$$\langle \Psi, \mathsf{X} \wedge \mathsf{Y} \wedge \mathsf{Z} \rangle = c(X, Y, Z) \tag{4.6}$$

for $\mathsf{X}, \mathsf{Y}, \mathsf{Z} \in \mathfrak{g}$. When $b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$, for each $m \in M$ there is a unique element $\nu(m) \in \mathcal{P}_{\mathfrak{g}}^*$ satisfying $d_{\mathcal{P}}\nu(m) = \Psi(m)$. Since $d_{\mathcal{P}}$ is a G-morphism, it follows that $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$ is also a G-equivariant.

We claim that ν is a multi-moment map. Note that, in general $d_{\mathcal{P}}\colon \mathcal{P}_{\mathfrak{g}}^* \to Z^3(\mathfrak{g}) \cap (\mathfrak{g} \wedge \mathcal{P}_{\mathfrak{g}})^*$. The assumption $b_2(\mathfrak{g}) = 0$, gives that the dual map $d_{\mathcal{P}}^*$ is a surjection $Z^3(\mathfrak{g})^* \cap (\mathfrak{g} \wedge \mathcal{P}_{\mathfrak{g}}) \to \mathcal{P}_{\mathfrak{g}}$. This dual map is given as minus the adjoint action, since

$$\begin{aligned}\langle d_{\mathcal{P}}\alpha, \mathsf{Z} \wedge \mathsf{p}\rangle &= \langle d_{\mathcal{P}}\alpha, \mathsf{Z} \wedge \sum_{i=1}^{k} \mathsf{X}_i \wedge \mathsf{Y}_i\rangle \\ &= -\sum_{i=1}^{k}(\alpha([\mathsf{Z},\mathsf{X}_i],\mathsf{Y}_i) + \alpha([\mathsf{X}_i,\mathsf{Y}_i],\mathsf{Z}) + \alpha([\mathsf{Y}_i,\mathsf{Z}],\mathsf{X}_i)) = -\langle\alpha, \mathrm{ad}_{\mathsf{Z}}(\mathsf{p})\rangle,\end{aligned} \tag{4.7}$$

for $\mathsf{Z} \in \mathfrak{g}$, $\mathsf{p} = \sum_{i=1}^{k} \mathsf{X}_i \wedge \mathsf{Y}_i \in \mathcal{P}_{\mathfrak{g}}$. Hence we may write any $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$ in the form $\mathsf{p} = \sum_{i=1}^{r} \mathrm{ad}_{\mathsf{Z}_i}(\mathsf{q}_i)$, with $\mathsf{Z}_i \in \mathfrak{g}$ and $\mathsf{q}_i \in \mathcal{P}_{\mathfrak{g}}$. Now the function

$$\nu_{\mathsf{p}} = -\sum_{i=1}^{r}\langle\Psi, \mathsf{Z}_i \wedge \mathsf{q}_i\rangle = -\sum_{i=1}^{r} c(Z_i \wedge q_i)$$

satisfies $d\nu_{\mathsf{p}} = -\sum_{i=1}^{r} \mathcal{L}_{Z_i}(q_i \lrcorner c) = p \lrcorner c$, since $d(q_i \lrcorner c) = 0$ by (4.2). Moreover we have that

$$\nu_{\mathsf{p}}(m) = -\sum_{i=1}^{r}\langle d_{\mathcal{P}}\nu(m), \mathsf{Z}_i \wedge \mathsf{q}_i\rangle = \sum_{i=1}^{r}\langle \nu(m), \mathrm{ad}_{\mathsf{Z}_i}(\mathsf{q}_i)\rangle = \langle\nu(m), \mathsf{p}\rangle.$$

Thus ν is a multi-moment map.

For the last part of the theorem, note that a multi-moment map ν defines elements $\nu(m) \in \mathcal{P}_{\mathfrak{g}}^*$ and the above calculations show that $d_{\mathcal{P}}(\nu(m)) = \Psi(m)$. However, $b_2(\mathfrak{g}) = 0$ implies that there is at most one solution $\nu(m)$ to this equation, so ν is then unique. □

Remark 4.13. Note that the calculation (4.7) is a special case of a well-known relation. If we let L denote the dual of the exterior derivative $d\colon \Lambda^k \mathfrak{g}^* \to \Lambda^{k+1} \mathfrak{g}^*$, then one has the relation

$$\mathrm{ad}_{\mathsf{X}}^* \beta = \mathsf{X} \lrcorner d\beta + (\mathsf{X} \lrcorner \beta) \circ L, \tag{4.8}$$

for any $\beta \in \Lambda^k \mathfrak{g}^*$ and any $\mathsf{X} \in \mathfrak{g}$; see also Chapter 5.

When we apply (4.8) to calculate the stabiliser of an element in $\mathcal{P}_{\mathfrak{g}}^*$, we must keep in mind that the dual of the Lie kernel is a quotient space, i.e., we are free to modify representatives by elements $d\alpha$, for $\alpha \in \mathfrak{g}^*$: if $\beta = [\widetilde{\beta}] \in \mathcal{P}_{\mathfrak{g}}^*$, for some lift $\widetilde{\beta} \in \Lambda^2 \mathfrak{g}^*$, then

$$\mathrm{ad}_{\mathsf{X}}^* \beta = [\mathsf{X} \lrcorner d\widetilde{\beta}].$$

For a metric Lie algebra, i.e. a Lie algebra $\mathfrak{g}$ endowed with an ad-invariant inner product $\langle\cdot,\cdot\rangle$, we circumvent this source of confusion by identifying $\mathfrak{g}$ with its dual $\mathfrak{g}^*$; the identification is established via inverse of the map $\mathsf{X} \mapsto \langle\mathsf{X},\cdot\rangle$. △

Note that any semi-simple Lie group G has $b_1(\mathfrak{g}) = 0 = b_2(\mathfrak{g})$. Also any reductive group G with one-dimensional centre still has $b_2(\mathfrak{g}) = 0$; in particular this applies to $G = U(n)$. So when multi-moment maps for these group actions exist, they are unique. However, any simple Lie group G has $b_3(\mathfrak{g}) = 1$, so there can be obstructions to existence.

4.3 (2,3)-trivial Lie algebras

In this section we give a structural description of the $(2,3)$-trivial Lie algebras, list them in low dimensions and show that there are many examples. The classification problem up to and including dimension five is resolved in Section 4.5.

Theorem 4.14. *Any non-trivial finite-dimensional Lie algebra $\mathfrak{g} \neq \mathbb{R}, \mathbb{R}^2$ satisfying $b_3(\mathfrak{g}) = 0$ is solvable and not nilpotent. If in addition we have that $b_2(\mathfrak{g}) = 0$ then $\mathfrak{g}$ cannot be a direct sum of two non-trivial subalgebras, and its derived algebra is a codimension one ideal.*

Proof. To verify the first statement, we consider $\mathfrak{r}$, the solvable radical of $\mathfrak{g}$. This is the maximal solvable ideal of $\mathfrak{g}$ and the quotient $\mathfrak{g} / \mathfrak{r}$ is semi-simple. By [HS53], the cohomology of $\mathfrak{g}$ is given by

$$H^k(\mathfrak{g}) \cong \sum_{i+j=k} H^i(\mathfrak{g} / \mathfrak{r}) \otimes H^j(\mathfrak{r})^{\mathfrak{g}},$$

where $V^{\mathfrak{g}}$ is the set of fixed points of the action $\mathfrak{g}$ on V. We thus have $b_3(\mathfrak{g}) \geqslant b_3(\mathfrak{g} / \mathfrak{r})$. As any non-trivial semi-simple Lie algebra has non-trivial third cohomology group, we deduce that $b_3(\mathfrak{g}) = 0$ implies $\mathfrak{g} = \mathfrak{r}$, so that $\mathfrak{g}$ is solvable. It is necessarily non-nilpotent since it is known [Dix55] that non-Abelian nilpotent Lie algebras are of dimension greater than two and have $b_i \geqslant 2$ for any $0 < i < \dim \mathfrak{g}$, whereas the only non-Abelian three-dimensional nilpotent algebra has $b_3(\mathfrak{g}) = 1$.

For the second statement of the theorem, suppose $\mathfrak{g}$ is a direct sum $\mathfrak{h} \oplus \mathfrak{k}$ of Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$. Using the Künneth formula, we obtain

$$b_2(\mathfrak{g}) = b_2(\mathfrak{h}) + b_2(\mathfrak{k}) + b_1(\mathfrak{h}) b_1(\mathfrak{k}),$$
$$b_3(\mathfrak{g}) = b_3(\mathfrak{h}) + b_3(\mathfrak{k}) + b_2(\mathfrak{h}) b_1(\mathfrak{k}) + b_1(\mathfrak{h}) b_2(\mathfrak{k}).$$

This immediately gives $b_2(\mathfrak{h}) = 0 = b_2(\mathfrak{k})$ and $b_3(\mathfrak{h}) = 0 = b_3(\mathfrak{k})$. It also follows that either $b_1(\mathfrak{h}) = 0$ or $b_1(\mathfrak{k}) = 0$. Reordering the factors, we can assume that $b_1(\mathfrak{h}) = 0$. Thus $\mathfrak{h}$ has $b_1(\mathfrak{h}) = 0 = b_2(\mathfrak{h})$ and so is semi-simple. But now the number of simple factors of $\mathfrak{h}$ is equal to $b_3(\mathfrak{h})$ which is 0. So $\mathfrak{h} = \{0\}$, and $\mathfrak{g}$ is not a non-trivial direct sum.

Now we consider the last assertion of the theorem. Note that $b_1(\mathfrak{g}) = \dim \mathfrak{g} - \dim \mathfrak{g}'$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the derived algebra. As $\mathfrak{g}$ is solvable, we get $b_1(\mathfrak{g}) > 0$. Suppose $b_1(\mathfrak{g}) \geqslant 2$. Then there are two linearly independent elements e_1, e_2 in $Z^1(\mathfrak{g})$. As $e_{12} := e_1 \wedge e_2 \in Z^2(\mathfrak{g})$ and $b_2(\mathfrak{g}) = 0$, we can find an element e_3 with $de_3 = e_{12}$. Note that we have $\dim\langle e_1, e_2, e_3 \rangle = 3$. Inductively, we may find $e_4, \ldots, e_n$ with $de_j = e_{1,j-1}$ such that $e_1, \ldots, e_n$ is a basis for $\mathfrak{g}$. But now $e_{1n} \in Z^2(\mathfrak{g})$ can not be exact, contradicting $b_2(\mathfrak{g}) = 0$. Thus, we must have $b_1(\mathfrak{g}) = 1$. □

We will refine this result later, but it is already sufficient to list the smallest examples of $(2,3)$-trivial Lie algebras. In dimension one, the only Lie algebra is Abelian and is automatically $(2,3)$-trivial. In dimension two a Lie algebra is either Abelian or isomorphic to the $(2,3)$-trivial algebra $(0,21)$. These first two examples are uninteresting from the point of view of multi-moment maps since they have $\mathcal{P}_{\mathfrak{g}} = \{0\}$. However, in dimensions three and four we may use the known classification of solvable Lie algebras [ABDO05] (see also Chapter 3.5) to obtain more interesting examples. Note that for any Lie algebra of dimension n, we have

$$\dim \mathcal{P}_{\mathfrak{g}} = b_1(\mathfrak{g}) + \tfrac{1}{2}n(n-3),$$

since the kernel of leftmost map in (4.5) is $H^1(\mathfrak{g}) = Z^1(\mathfrak{g})$. Thus a $(2,3)$-trivial algebra has $\dim \mathcal{P}_{\mathfrak{g}} = (n-1)(n-2)/2$, which is non-zero for $n \geqslant 3$.

Proposition 4.15. *The inequivalent $(2,3)$-trivial Lie algebras in dimensions three and four are listed in the Tables 4.1 and 4.2.*

$\mathfrak{r}_3$	$(0, 21+31, 31)$	
$\mathfrak{r}_{3,\lambda}$	$(0, 21, \lambda.31)$	$\lambda \in (-1,1] \setminus \{0\}$
$\mathfrak{r}'_{3,\lambda}$	$(0, \lambda.21+31, -21+\lambda.31)$	$\lambda > 0$

Table 4.1: The inequivalent three-dimensional $(2,3)$-trivial Lie algebras.

$\mathfrak{r}_4$	$(0, 21+31, 31+41, 41)$	
$\mathfrak{r}_{4,\lambda}$	$(0, 21, \lambda.31+41, \lambda.41)$	$\lambda \neq -1, -\frac{1}{2}, 0$
$\mathfrak{r}_{4,\mu,\lambda}$	$(0, 21, \mu.31, \lambda.41)$	$(\mu,\lambda) \in \mathscr{R}$
$\mathfrak{r}'_{4,\mu,\lambda}$	$(0, \mu.21, \lambda.31+41, -31+\lambda.41)$	$\mu > 0,\ \lambda \neq -\frac{\mu}{2}, 0$
$\mathfrak{d}_{4,\lambda}$	$(0, \lambda.21, (1-\lambda).31, 41+32)$	$\lambda \geqslant \frac{1}{2},\ \lambda \neq 1, 2$
$\mathfrak{d}'_{4,\lambda}$	$(0, \lambda.21+31, -21+\lambda.31, 2\lambda.41+32)$	$\lambda > 0$
$\mathfrak{h}_4$	$(0, 21+31, 31, 2.41+32)$	

Table 4.2: The inequivalent four-dimensional $(2,3)$-trivial Lie algebras. The set $\mathscr{R}$ consists of the $\mu, \lambda \in (-1,1] \setminus \{0\}$ with $\lambda \geqslant \mu$ and $\mu + \lambda \neq 0, -1$.

The notation follows the conventions of Chapter 3. So considering the example $\mathfrak{h}_4 = (0, 21+31, 31, 2.41+32)$. This means there is a basis $e_1, \dots, e_4$ for $\mathfrak{h}_4^*$ such that $de_1 = 0$, $de_2 = e_{21} + e_{31}$, $de_3 = e_{31}$ and $de_4 = 2e_{41} + e_{32}$.

We will sketch a proof of Proposition 4.15 (see also Section 4.5) that is independent of the classification lists, using the following more detailed structure result.

Theorem 4.16. *A Lie algebra $\mathfrak{g}$ with derived algebra $\mathfrak{k} = \mathfrak{g}'$ is $(2,3)$-trivial if and only if $\mathfrak{g}$ is solvable, $\mathfrak{k}$ is nilpotent of codimension one in $\mathfrak{g}$ and $H^1(\mathfrak{k})^{\mathfrak{g}} = \{0\} = H^2(\mathfrak{k})^{\mathfrak{g}} = H^3(\mathfrak{k})^{\mathfrak{g}}$.*

Proof. The derived algebra $\mathfrak{k} = \mathfrak{g}'$ of a solvable algebra $\mathfrak{g}$ is always nilpotent, so Theorem 4.14 implies that it only remains to check the assertions on the

$\mathfrak{g}$-invariant part of the cohomology of $\mathfrak{k}$. For this, as $\mathfrak{k}$ is an ideal of $\mathfrak{g}$, we may use the spectral sequence of Hochschild and Serre [HS53] that has $E_2^{j,i} \cong H^j(\mathfrak{g}/\mathfrak{k}, H^i(\mathfrak{k}))$. Now the codimension one condition means that we may write $\mathfrak{g}/\mathfrak{k} = \mathbb{R}A$ for some element A. Note that $H^i(\mathfrak{k})$ is a $\mathfrak{g}/\mathfrak{k}$-module. For any $\mathfrak{g}/\mathfrak{k}$-module M, the cohomology groups $H^j(\mathbb{R}A, M)$ are defined from the chain groups $C^j(\mathbb{R}A, M) = \Lambda^j(\mathbb{R}A)^* \otimes M = \operatorname{Hom}(\Lambda^j\mathbb{R}A, M)$. These can only be non-zero for $j = 0, 1$ and in both cases they are isomorphic to M. The chain map is $d_{\mathbb{R}}$ which on C^0 is $(d_{\mathbb{R}}f)(A) = A \cdot f$. Thus $E_2^{0,i} = \ker d_{\mathbb{R}} = M^A$ and $E_2^{1,1} = M/\operatorname{im} d_{\mathbb{R}} \cong \ker d_{\mathbb{R}} = M^A$. We see that the E_2-term of our spectral sequence is

$$E_2^{j,i} \cong \begin{cases} H^i(\mathfrak{k})^{\mathfrak{g}} & \text{for } j = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the spectral sequence degenerates at the E_2-term and we conclude that

$$H^2(\mathfrak{g}) \cong H^2(\mathfrak{k})^{\mathfrak{g}} + H^1(\mathfrak{k})^{\mathfrak{g}}, \quad H^3(\mathfrak{g}) \cong H^3(\mathfrak{k})^{\mathfrak{g}} + H^2(\mathfrak{k})^{\mathfrak{g}},$$

from which the result follows. □

Sketch proof of Proposition 4.15. Let $\mathfrak{g}$ be a $(2,3)$-trivial algebra of dimension three. Then $\mathfrak{k} = \mathfrak{g}'$ is nilpotent and two-dimensional, so $\mathfrak{k} \cong \mathbb{R}^2$. The element A of Theorem 4.16 acts on $\mathbb{R}^2$ invertibly and the induced action on $H^2(\mathbb{R}^2) \cong \Lambda^2\mathbb{R}^2 \cong \mathbb{R}$ is also invertible. So either A is diagonalisable over $\mathbb{C}$ with non-zero eigenvalues whose sum is non-zero, giving cases $\mathfrak{r}_{3,\lambda}$ and $\mathfrak{r}'_{3,\lambda}$, or A acts with Jordan form $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$, $\lambda \neq 0$, giving case $\mathfrak{r}_3$. The particular structure coefficients are obtained by replacing A by a non-zero multiple.

For $\mathfrak{g}$ of dimension four, we have $\mathfrak{k} \cong \mathbb{R}^3$ or the Heisenberg algebra $\mathfrak{h}_3 = (0, 0, 12)$. The former gives the algebras from the $\mathfrak{r}$-series when one enforces that no sum of one, two or three eigenvalues of A is zero. The latter gives the remaining algebras; we have $H^1(\mathfrak{h}_3) \cong \langle e_1, e_2\rangle$, $H^2(\mathfrak{h}_3) \cong \langle e_{13}, e_{23}\rangle$, $H^3(\mathfrak{h}_3) \cong \langle e_{123}\rangle$, A acts invertibly on these spaces and its action in e_3 is determined by its action on e_1 and e_2. □

Theorem 4.16 enables us to generate many examples of $(2,3)$-trivial Lie algebras in higher dimensions. Say that a nilpotent algebra $\mathfrak{k}$ is *positively graded* if there is a vector space direct sum decomposition $\mathfrak{k} = \mathfrak{k}_1 + \cdots + \mathfrak{k}_r$ with $[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j}$ for all i, j.

Corollary 4.17. *Let $\mathfrak{k}$ be any positively graded nilpotent Lie algebra. Then there is a $(2,3)$-trivial Lie algebra whose derived algebra is $\mathfrak{k}$.*

Proof. Let $\mathfrak{g} = \langle A\rangle + \mathfrak{k}$ where ad_A acts as multiplication by i on $\mathfrak{k}_i$. Then $\mathfrak{g}$ is a solvable algebra. Moreover $(\Lambda^s \mathfrak{k})^{\mathfrak{g}} = \{0\}$ for $s \geqslant 1$, so the cohomological condition of Theorem 4.16 is satisfied and $\mathfrak{g}$ is as required. □

The algebras constructed in this way are completely solvable, meaning that each ad_X, for $X \in \mathfrak{g}$, has only real eigenvalues on $\mathfrak{g}$.

Remark 4.18. The Lie kernel has a particularly simple interpretation in the case when $\mathfrak{g} = \langle A \rangle + \mathfrak{k}$ with ad_A acting invertibly on $\mathfrak{k}$; this holds for instance when $\mathfrak{k} = \mathbb{R}^k$ since $H^1(\mathbb{R}^k) \cong \mathbb{R}^k$. Then $\Lambda^2 \mathfrak{k} \cong \mathcal{P}_{\mathfrak{g}}$ as $\mathfrak{k}$-modules. To see this one notes that if $\mathrm{ad}_A \colon \mathfrak{k} \to \mathfrak{k}$ is invertible, then there is an isomorphism $\Phi \colon \Lambda^2 \mathfrak{k} \to \mathcal{P}_{\mathfrak{g}}$ given by

$$\sum_{j=1}^{r} K_1^j \wedge K_2^j \mapsto \sum_{j=1}^{r} \left(K_1^j \wedge K_2^j - A \wedge (\mathrm{ad}_A|_{\mathfrak{k}})^{-1} \circ L(K_1^j \wedge K_2^j) \right).$$

△

Example 4.19. Consider the $(2,3)$-trivial Lie algebra

$$\mathfrak{h}_4 = (0, 21 + 31, 31, 2.41 + 32),$$

and pick a basis A, E_1, E_2, E_3 compatible with these structural equations. Then we have that

$$\mathrm{ad}_A(E_1) = E_1, \quad \mathrm{ad}_A(E_2) = E_1 + E_2, \quad \mathrm{ad}_A(E_3) = 2E_3,$$
$$(\mathrm{ad}_A|_{\mathfrak{k}})^{-1}(E_1) = E_1, \quad (\mathrm{ad}_A|_{\mathfrak{k}})^{-1}(E_2) = -E_1 + E_2, \quad (\mathrm{ad}_A|_{\mathfrak{k}})^{-1}(E_3) = \frac{1}{2}E_3.$$

Using the isomorphism Φ from the above remark, we obtain the following basis for the Lie kernel

$$\mathcal{P}_{\mathfrak{h}_4} = \langle E_1 \wedge E_2 - \tfrac{1}{2} A \wedge E_3, E_1 \wedge E_3, E_2 \wedge E_3 \rangle.$$

♢

Example 4.20. In Section 4.5 we will show that the Lie algebra

$$\mathfrak{p}_5 = (0, 21, 21 + 31, 2.41 + 32, 3.51 + 42)$$

has $H^1(\mathfrak{k})^{\mathfrak{g}} = \{0\} = H^2(\mathfrak{k})^{\mathfrak{g}} = H^3(\mathfrak{k})^{\mathfrak{g}}$. So, by Theorem 4.16, $\mathfrak{p}_5$ is $(2,3)$-trivial. Also note that ad_A acts invertibly on $\mathfrak{k}$. Indeed, let $A, E_1, \dots, E_4$ denote a basis compatible with the specified structural equations, then we have

$$\mathrm{ad}_A(E_1) = E_1 + E_2,\ \mathrm{ad}_A(E_2) = E_2,\ \mathrm{ad}_A(E_3) = 2E_3,\ \mathrm{ad}_A(E_4) = 3E_4,$$
$$(\mathrm{ad}_A|_{\mathfrak{k}})^{-1}(E_1) = E_1 - E_2,\ (\mathrm{ad}_A|_{\mathfrak{k}})^{-1}(E_2) = E_2,\ (\mathrm{ad}_A|_{\mathfrak{k}})^{-1}(E_3) = \frac{1}{2}E_3,$$
$$(\mathrm{ad}_A|_{\mathfrak{k}})^{-1}(E_4) = \frac{1}{3}E_4.$$

We may now use the isomorphism $\Phi \colon \mathfrak{k} \to \mathcal{P}_{\mathfrak{g}}$ from Remark 4.18 to construct a basis for $\mathcal{P}_{\mathfrak{g}}$. If we define elements $p_1^1 := \Phi_1(E_1 \wedge E_2), \dots, p_1^6 := \Phi_1(E_3 \wedge E_4)$, then we have

$$p_1^1 = E_1 \wedge E_2 - \frac{1}{2} A \wedge E_3, \quad p_1^2 = E_1 \wedge E_3 - \frac{1}{3} A \wedge E_4, \quad p_1^3 = E_1 \wedge E_4,$$
$$p_1^4 = E_2 \wedge E_3, \quad p_1^5 = E_2 \wedge E_4, \quad p_1^6 = E_3 \wedge E_4.$$

♢

Example 4.21. It may be checked directly that every nilpotent Lie algebra of dimension at most six can be positively graded. The classification of these nilpotent algebras (see [Sal01]) then gives over 30 different $(2,3)$-trivial algebras in dimension 7, cf. Section 4.5. $\diamondsuit$

Example 4.22. Another class of positively graded algebras is given as follows. Let $\mathrm{Der}(\mathfrak{k})$ be the algebra of derivations of $\mathfrak{k}$. A *maximal torus* $\mathfrak{t}$ for $\mathfrak{k}$ is a maximal Abelian subalgebra of the semi-simple elements of $\mathrm{Der}(\mathfrak{k})$. The nilpotent Lie algebra $\mathfrak{k}$ is said to have *maximal rank* if $\dim \mathfrak{t} = \dim(\mathfrak{k} / \mathfrak{k}')$. Favre [Fav73] showed that there are only finitely many systems of weights for such algebras and following [San82] a number of classification results have been obtained via Kac-Moody techniques, see [FT05] and the references therein. There is a large number (thousands) of families of such algebras. From the general theory, one knows [Fav73, p. 83] that there is a positive grading of each maximal rank nilpotent Lie algebra $\mathfrak{k}$. This grading satisfies $\sum_{i=s+1}^{r} \mathfrak{k}_i = \mathfrak{k}^{(s)} = [\mathfrak{k}, \mathfrak{k}^{(s-1)}]$. Thus each of these distinct nilpotent algebras of maximal rank arises as the derived algebra of non-isomorphic $(2,3)$-trivial Lie algebras. $\diamondsuit$

We note that in the construction of Corollary 4.17, ad_A is a semi-simple derivation of $\mathfrak{k}$. Generally, if $\mathfrak{g}$ is solvable, then $A \in \mathfrak{g} \setminus \mathfrak{g}'$ acts on $\mathfrak{k} = \mathfrak{g}'$ as a derivation. For $\mathfrak{g}$ to be $(2,3)$-trivial, Theorem 4.16 implies that this action is not nilpotent on $H^k(\mathfrak{k})$ for $k = 1, 2, 3$. For $\dim \mathfrak{g} \geqslant 5$, this condition has most force since these three cohomology groups have dimension at least 2 [Dix55].

Now a nilpotent Lie algebra $\mathfrak{k}$ is said to be *characteristically nilpotent* if $\mathrm{Der}(\mathfrak{k})$ acts on $\mathfrak{k}$ by nilpotent endomorphisms. It is known that this is equivalent to $\mathrm{Der}(\mathfrak{k})$ being a nilpotent Lie algebra. For a characteristically nilpotent algebra $\mathfrak{k}$, any solvable extension will act nilpotently on the cohomology of $\mathfrak{k}$. Theorem 4.16 thus gives the following result.

Corollary 4.23. *If $\mathfrak{k}$ is a characteristically nilpotent Lie algebra, then $\mathfrak{k}$ is never the derived algebra of a $(2,3)$-trivial algebra.* $\square$

Example 4.24. The first example of a characteristically nilpotent Lie algebra was constructed by Dixmier and Lister [DL57] in dimension eight. However, there are seven-dimensional examples with the same property and even continuous families [GK96] including:

$$(0, 0, 12, 13, 23, 14 + 25 + \alpha.23, 16 + 25 + 35 + \alpha.24), \qquad \alpha \neq 0.$$

Thus no member of this family of algebras can occur as the derived algebra of any $(2,3)$-trivial Lie algebra. $\diamondsuit$

We recall from Section 3.1.1 that a Lie algebra $\mathfrak{g}$ is called unimodular if the Lie algebra homomorphism $\chi\colon \mathfrak{g} \to \mathbb{R}$ given by $\chi(x) = \mathrm{Tr}(\mathrm{ad}(x))$ has trivial image. As mentioned such Lie algebras are interesting since unimodularity is a necessary condition for the existence of a co-compact discrete subgroup [Mil76].

Corollary 4.25. *The simply-connected $(2,3)$-trivial Lie groups of dimension four or below are not unimodular. In particular they do not admit a compact quotient by a lattice.*

Proof. An n-dimensional Lie algebra $\mathfrak{g}$ is unimodular if and only if $b_n(\mathfrak{g}) = 1$. Moreover, one may show that unimodular algebras satisfy Hodge duality $b_k(\mathfrak{g}) = b_{n-k}(\mathfrak{g})$, cf. Proposition 3.5. For $\mathfrak{g}$ a $(2,3)$-trivial Lie algebra of dimension three, we have $b_3(\mathfrak{g}) = 0$, so $\mathfrak{g}$ is not unimodular. For $\mathfrak{g}$ of dimension four, unimodularity implies $b_1(\mathfrak{g}) = b_3(\mathfrak{g}) = 0$. But $(2,3)$-trivial algebras have $b_1(\mathfrak{g}) = 1$, so they can not be unimodular in dimension four. □

Example 4.26. It can be shown that in dimension five and above there are unimodular $(2,3)$-trivial Lie algebras, see Section 4.5. Moreover one may verify that there are solvmanifolds of the form G/Γ, where G is $(2,3)$-trivial. Indeed using [Boc09, Proposition 7.2.1(i)] one may see that there are $(2,3)$-trivial Lie groups which admit a lattice. One such example has Lie algebra

$$(0, \lambda_1.12, \lambda_2.13, \lambda_3.14, \lambda_4.15),$$

where $\exp(\lambda_i) \approx 0.1277, 0.6297, 2.797, 4.446$ are the four roots of the polynomial $s^4 - 8s^3 + 18s^2 - 10s + 1$. As this Lie algebra is completely solvable it follows from Hattori's Theorem [Hat60] that one has an isomorphism $H^*_{\mathrm{dR}}(G/\Gamma) \cong H^*(\mathfrak{g})$. In particular the five-dimensional solvmanifold constructed in this way has vanishing second and third de Rham cohomology groups. ♢

4.4 Multi-moment maps: examples

As strong geometry has no analogue of the Darboux theorem (see, however, Remark 5.1), the theory of multi-moment maps is in some senses less rigid than that for symplectic moment maps and there is a wider variety of types of examples.

4.4.1 Second exterior power of the cotangent bundle

In symplectic geometry one of the fundamental examples is provided by the cotangent bundle of a manifold, which in mechanics may be interpreted as a phase space. In strong geometry, an analogous example is provided by the second exterior power $M = \Lambda^2 T^*N$ of a base manifold N. This carries a canonical two-form b, given by

$$b_\alpha(W_1, W_2) = \alpha(\pi_* W_1, \pi_* W_2), \qquad W_1, W_2 \in T_\alpha M,$$

where $\pi\colon \Lambda^2 T^*N \to N$ is the bundle projection. From this one defines a closed three-form c on M, via

$$c = db.$$

This form is 2-plectic: in local coordinates $(q^1, \dots, q^n)$ on N we have $\alpha = \sum_{i<j} p_{ij} dq^i \wedge dq^j$ defining local coordinates (q^i, p_{ij}) on $M = \Lambda^2 T^*N$ in which $c = \sum_{i<j} dp_{ij} \wedge dq^i \wedge dq^j$. This is the fundamental example in [BHR10, CCI91].

If G is a group of diffeomorphisms of N, then there is an induced action on $M = \Lambda^2 T^*N$ which preserves b and hence c. As $c = db$, Proposition 4.8 gives that there is a multi-moment map ν determined by (4.4), which here reads

$$\langle \nu(\alpha), \mathsf{p} \rangle = \alpha(p_N)$$

where p_N is the field of bivectors on N determined by $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$. To summarise

Proposition 4.27. *If a Lie group G acts on a smooth manifold N, then the induced action on $M = \Lambda^2 T^*N$ admits a multi-moment map with respect to the canonical 2-plectic structure.* □

Remark 4.28. Suppose N^n carries an H-structure, i.e., a reduction of the structure group of N to $H \leqslant GL(n, \mathbb{R})$. Then at each point of $q \in N$ we have a canonical decomposition $\Lambda^2_q T^*N = \oplus_i V_i(q)$ into isotypical H-modules. If the action of G preserves the H-structure then the induced action on $\Lambda^2 T^*N$ preserves the subbundles V_i. Each bundle V_i carries a strong geometry via the restriction of c on $M = \Lambda^2 T^*N$, and the action of $G \leqslant CO(4)$ again admits a multi-moment map. For example, if N is an oriented four-manifold and G preserves the orientation, then there are multi-moment maps $\nu_\pm$ defined on the 2-plectic seven-manifolds $\Lambda^2_\pm$. The particular case of $SO(4) = Sp(1)_+ Sp(1)_-$ acting on $N = \mathbb{R}^4 = \mathbb{H}$ via $(A, B) \cdot q = Aq\overline{B}$ has multi-moment map on $\Lambda^2_+ N \cong \mathbb{H} + \operatorname{Im} \mathbb{H}$ given by $\langle \nu_+(q, p), a \otimes b \rangle = \frac{1}{2} \operatorname{Re}(paqb\overline{q})$, for $q \in \mathbb{H}$, $p \in \operatorname{Im} \mathbb{H}$, $a \otimes b \in \mathfrak{sp}(1)_+ \otimes \mathfrak{sp}(1)_- = \operatorname{Im} \mathbb{H} \otimes \operatorname{Im} \mathbb{H} \cong \mathcal{P}_{\mathfrak{sp}(1)_+ + \mathfrak{sp}(1)_-}$. △

4.4.2 Homogeneous strong geometries

If G acts transitively on a strong manifold M, then we may define $\Psi \colon M \to Z^3(\mathfrak{g})$ via (4.6), and the image will be a G-orbit in $Z^3(\mathfrak{g})$. Conversely, formula (4.6) can be used to define strong geometries that map to a given orbit in $Z^3(\mathfrak{g})$: given $\Psi \in Z^3(\mathfrak{g})$, let K_Ψ denote the connected subgroup generated by $\ker \Psi = \{ \mathsf{X} \in \mathfrak{g} : \mathsf{X} \lrcorner \Psi = 0 \}$; for any closed group H of G with $H \subset K_\Psi$, equation (4.6) defines a closed three-form c on the homogeneous space G/H and this strong geometry maps to $G \cdot \Psi \subset Z^3(\mathfrak{g})$.

Now suppose that $\Psi = d_{\mathcal{P}} \beta$ for some $\beta \in \mathcal{P}^*_{\mathfrak{g}}$. If the map $d_{\mathcal{P}}$ is injective, then the orbits $G \cdot \Psi$ and $G \cdot \beta$ are identified and the map $\Psi \colon M \to Z^3(\mathfrak{g})$ may now be interpreted as a map $\nu \colon M \to \mathcal{P}^*_{\mathfrak{g}}$. Injectivity of $d_{\mathcal{P}}$ is guaranteed by the condition $b_2(\mathfrak{g}) = 0$. When this holds, the proof of Theorem 4.12 shows that ν is a multi-moment map for the action of G.

Theorem 4.29. *Suppose G is a connected Lie group with $b_2(\mathfrak{g}) = 0$. Let $\mathcal{O} = G \cdot \beta \subset \mathcal{P}^*_{\mathfrak{g}}$ be an orbit of G acting on the dual of the Lie kernel.*

(i) *Then there are homogeneous strong manifolds $(G/H, c)$, with c corresponding to $\Psi = d_{\mathcal{P}} \beta$, such that $\mathcal{O}$ is the image of G/H under the (unique) multi-moment map ν.*

(ii) *The strong geometry may be realised on the orbit $\mathcal{O}$ itself if and only if*

$$\operatorname{stab}_{\mathfrak{g}} \beta = \ker(d_{\mathcal{P}} \beta). \tag{4.9}$$

In this situation, the orbit is 2-plectic and ν is simply the inclusion $\mathcal{O} \hookrightarrow \mathcal{P}^*_{\mathfrak{g}}$.

Proof. It only remains to prove the assertions of the last paragraph of the theorem. We have $\mathcal{O} = G/K$ with $K = \mathrm{stab}_G \beta$, a closed subgroup of G. Now equation (4.9), shows that K has Lie algebra $\ker(d_{\mathcal{P}}\beta)$, so the component of the identity K^0 of K is $K^0 = K_\Psi$ for $\Psi = d_{\mathcal{P}}\beta$. In particular, Ψ vanishes on elements of $\mathfrak{k}$ and induces a well-defined form on $T_\beta\mathcal{O} = \mathfrak{g} / \mathfrak{k}$. The result now follows. □

The rank of the above multi-moment map is clearly equal to $\dim \mathfrak{g} - \dim \mathfrak{k}$. It may be useful to express this number, and more generally the image of the multi-moment map, purely in terms of strong geometric data, meaning data that does not involve the element β.

Corollary 4.30. *Let $(G/H, c)$ be a homogeneous strong manifold as in part (i) of Theorem 4.29. Then the image of the multi-moment map ν is given by G/K, where*

$$\mathfrak{k} = \langle X \in \mathfrak{g} \colon X \lrcorner \Psi \in Z^2(\mathfrak{g}) \rangle. \tag{4.10}$$

Proof. We use the notation of Theorem 4.29. Now consider the linear map $\psi \colon \mathfrak{g} \to \mathcal{P}^*_{\mathfrak{g}}$ given by

$$\psi(X) = \mathrm{ad}^*_X \beta,$$

$X \in \mathfrak{g}$. From (4.8) we see that $X \in \ker \psi$ if and only if $X \lrcorner \Psi$ annihilates $\mathcal{P}_{\mathfrak{g}}$, i.e., $X \lrcorner \Psi \in (\mathcal{P}_{\mathfrak{g}})^\circ = d(\mathfrak{g}^*) = B^2(\mathfrak{g})$. But, as $b_2(\mathfrak{g}) = 0$, we have $B^2(\mathfrak{g}) = Z^2(\mathfrak{g})$, so that

$$\ker \psi = \langle X \in \mathfrak{g} \colon X \lrcorner \Psi \in Z^2(\mathfrak{g}) \rangle,$$

as required. □

Remark 4.31. We obviously have that $\mathfrak{k} \subset \ker \Psi$. So the rank of the strong structure, defined as the codimension of $\ker \Psi$ in $\mathfrak{g}$, is an lower bound for the rank of ν. △

Example 4.32. Suppose G is a $(2,3)$-trivial Lie group. Then, taking $H = \{e\}$, we see that every $\Psi \in Z^3(\mathfrak{g})$ gives rise to a strong geometry on G with multi-moment map whose image is diffeomorphic to the G-orbit of Ψ.

Remark 4.18 shows that this procedure gives a large class of strong geometries with associated multi-moment maps of rank $\geqslant 1$. ♢

Example 4.33. Consider $G = U(2) \cong (S^1 \times SU(2))/\{\pm(1,1)\}$. We have $\mathcal{P}_{\mathfrak{u}(2)} = T \wedge \mathfrak{su}(2)$, where T generates the Lie algebra of S^1. The orbits of $\mathcal{P}_{\mathfrak{u}(2)}$ are thus two-dimensional and can not admit (non-trivial) strong geometries. On the other hand, suppose we write e_1, e_2, e_3 for a standard basis of $\mathfrak{su}(2)^*$ with $de_1 = -e_{23}$. Then the element $\beta = dt \wedge e_1 \in \mathcal{P}^*_{\mathfrak{u}(2)}$, has $d_{\mathcal{P}}\beta = -dt \wedge e_{23}$, defining $\Psi \in Z^3(\mathfrak{u}(2))$. This β does not satisfy condition (4.9) even though $d_{\mathcal{P}}$ identifies the orbits of β and Ψ. However, Ψ defines strong geometries on $U(2)$ and on $U(2)/\operatorname{diag}(e^{i\theta}, e^{-i\theta}) \cong S^1 \times S^2$ with multi-moment map the projection to S^2. Note that $\nu \colon U(2) \to S^2$ is essentially the Hopf fibration. ♢

4.4.2.1 Multi-moment maps and SHKT manifolds

In Example 2.4 we put an invariant SHKT structure on $SU(3)$. Since the multi-moment map for the left action of $SU(3)$ on the corresponding strong geometry $(SU(3), c)$ is trivial, we turn our attention towards the multi-moment maps ν_I, ν_J and ν_K associated with the exact three-forms $d\omega_I$, $d\omega_J$ and $d\omega_K$ on $SU(3)$; an alternative approach, which we will discuss below, would be to enlarge the symmetry group to a maximal subgroup of $SU(3) \times SU(3)$ preserving the SHKT structure.

In the following, we thus consider the triple of multi-moment maps $\nu_{\mathcal{I}}\colon SU(3) \to \mathcal{P}^*_{\mathfrak{su}(3)}$ given by

$$\langle \nu_{\mathcal{I}}, \mathsf{p} \rangle = \omega_{\mathcal{I}}(p), \quad \mathcal{I} = I, J, K, \tag{4.11}$$

and aim to describe their images, up to discrete covers. To this end let us think of $\mathfrak{su}(3)$ as a Lie algebra of complex matrices. Following Example 2.4, we write E_{pq} for the elementary 3×3-matrix with 1 at position (p, q), so that $\mathfrak{su}(3)$ has a basis

$$A_j = i(E_{jj} - E_{j+1,j+1}), \quad B_{k\ell} = E_{k\ell} - E_{\ell k},$$
$$C_{k\ell} = i(E_{k\ell} + E_{\ell k}),$$

for $j, k = 1, 2$, $k < \ell = 2, 3$. Let $a_1, a_2, b_{12}, \dots, c_{23}$ denote the dual basis.

This concrete choice of $\mathfrak{su}(3)$ basis enables us to construct suitable bases for the submodules $\mathcal{P}_{\mathfrak{su}(3)} \subset \Lambda^2 \mathfrak{su}(3)$ and $\mathfrak{su}(3) \subset \Lambda^2 \mathfrak{su}(3)$. While these choices of basis are by no means canonical, they serve the purpose of furnishing $\Lambda^2 \mathfrak{su}(3)$ with a basis that is compatible with the splitting $\Lambda^2 \mathfrak{su}(3) = \mathfrak{su}(3) \oplus \mathcal{P}_{\mathfrak{su}(3)}$. In this way we obtain an explicit realisation of the decomposition of ω_I into its two components at the identity:

$$\begin{aligned}\omega_I = \omega_I^{\mathfrak{su}(3)} + \omega_I^{\mathcal{P}} &= 2\big(b_{12}c_{12} + b_{13}c_{13}\big) \\ &\qquad + \big(-\tfrac{\sqrt{3}}{2}a_1a_2 - (b_{12}c_{12} + b_{23}c_{23} - b_{13}c_{13})\big).\end{aligned}$$

From this decomposition we find the following expression for the multi-moment map ν_I in terms of the chosen $\mathfrak{su}(3)^*$ basis

$$\mathrm{Ad}^*_g \nu_I(g) = -\frac{\sqrt{3}}{2}a_1a_2 - (b_{12}c_{12} + b_{23}c_{23} - b_{13}c_{13}). \tag{4.12}$$

The image of ν_I is the orbit of $\nu_I(e)$ under the action of $SU(3)$. At the algebraic level we have

$$\begin{aligned}\ker(\nu_I)_* = \ker d\nu_I &= \big\{\, A \in \mathfrak{su}(3) : d\nu_I(\mathsf{p}, A) = 0 \text{ for all } \mathsf{p} \in \mathcal{P}_{\mathfrak{su}(3)} \,\big\} \\ &= \big\{\, A \in \mathfrak{su}(3) : c(I\mathsf{p}, IA) = 0 \text{ for all } \mathsf{p} \in \mathcal{P}_{\mathfrak{su}(3)} \,\big\} \\ &= I\big\{\, A \in \mathfrak{su}(3) : g(L(I\mathsf{p}), A) = 0 \text{ for all } \mathsf{p} \in \mathcal{P}_{\mathfrak{su}(3)} \,\big\} \\ &= (L(I\mathcal{P}_{\mathfrak{su}(3)}))^{\perp} = \langle A_1, V \rangle.\end{aligned}$$

The above computation tells us that the Lie algebra of the subgroup stabilising $\nu_I(e)$ is a maximal torus $\mathfrak{t}^2_I \subset \mathfrak{su}(3)$ which is invariant under I. So, up to discrete covers, the orbit of $SU(3)$ acting on $\nu_I(e)$ is a full flag $F_{1,2}(\mathbb{C}^3)$ inside the Lie kernel $\mathcal{P}^*_{\mathfrak{su}(3)}$.

Similarly, we may write

$$\begin{aligned}\omega_J = \omega_J^{\mathfrak{su}(3)} + \omega_J^{\mathcal{P}} &= -\frac{2}{3}\big((2a_1 - a_2)c_{12} + b_{13}b_{23} + c_{13}c_{23}\big)\\ &\quad + \big(\frac{\sqrt{3}}{2}a_2 b_{12} - \frac{1}{3}(\frac{1}{2}(a_2 - 2a_1)c_{12} + b_{13}b_{23} + c_{13}c_{23})\big),\end{aligned}$$

$$\begin{aligned}\omega_K = \omega_K^{\mathfrak{su}(3)} + \omega_K^{\mathcal{P}} &= \frac{2}{3}\big((2a_1 - a_2)b_{12} + b_{13}c_{23} + b_{23}c_{13}\big)\\ &\quad + \big(\frac{\sqrt{3}}{2}a_2 c_{12} + \frac{1}{3}(\frac{1}{2}(a_2 - 2a_1)b_{12} + b_{13}c_{23} + b_{23}c_{13})\big),\end{aligned}$$

so that

$$\mathrm{Ad}^*_g \nu_J(g) = \frac{\sqrt{3}}{2}a_2 b_{12} - \frac{1}{3}(\tfrac{1}{2}(a_2 - 2a_1)c_{12} + b_{13}b_{23} + c_{13}c_{23}),$$

with $\ker(\nu_J)_* = \langle V, B_{12}\rangle$, and

$$\mathrm{Ad}^*_g \nu_K(g) = \frac{\sqrt{3}}{2}a_2 c_{12} + \frac{1}{3}(\frac{1}{2}(a_2 - 2a_1)b_{12} + b_{13}c_{23} + b_{23}c_{13}),$$

with $\ker(\nu_K)_* = \langle V, C_{12}\rangle$.

The three multi-moment maps from above can be put into a single equivariant map $\underline{\nu} = (\nu_I, \nu_J, \nu_K)\colon\ SU(3) \to (\mathcal{P}^*_{\mathfrak{su}(3)})^3$, and from our analysis we see that, up to discrete covers, the image of ν is an Aloff-Wallach space $A_{1,1} = SU(3)/T^1_{1,1}$; $T^1_{1,1}$ has Lie algebra $\mathfrak{t}^1_{1,1} = \langle V\rangle$. The relatively high dimension of this image indicates that multi-moment maps ought to be useful tools in the study homogeneous hyperHermitian structures.

We summarise the above discussion and Example 2.4 as follows.

Proposition 4.34. *The eight-manifold $SU(3)$ carries an invariant* SHKT *metric g, compatible with Joyce's hypercomplex structure (I, J, K). Each of the associated strong geometries $(SU(3), d\omega_{\mathcal{I}})$, for $\mathcal{I} = I, J, K$, admits a multi-moment map $\nu_{\mathcal{I}}\colon\ SU(3) \to \mathcal{P}^*_{\mathfrak{su}(3)}$ given by* (4.11)*. As almost effective spaces, the image of $SU(3)$ under $\nu_{\mathcal{I}}$ is a full flag*

$$SU(3)/T^2_{\mathcal{I}},$$

up to finite covers, and the image of the combined map

$$\underline{\nu} = (\nu_I, \nu_J, \nu_K)\colon\ SU(3) \to (\mathcal{P}^*_{\mathfrak{su}(3)})^3$$

is an Aloff-Wallach space $SU(3)/T^1_{1,1}$. □

Let us indicate, without doing computations, how the above considerations may be generalised to any even-dimensional compact Lie group $T^\ell \times G$ that admits one of Joyce's hypercomplex structures, cf. Section 2.2; it suffices to specify the result in the case when G is simple. Let (g, I, J, K) be a left-invariant SHKT structure on $M = T^\ell \times G$, compatible with the Joyce decomposition of $\mathfrak{g}$. Define multi-moment maps $\nu_{\mathcal{I}}\colon M \to \mathcal{P}^*_{\mathfrak{t}^\ell \oplus \mathfrak{g}}$ by the formula (4.11). Our investigation of the $SU(3)$ case reveals that the images of the multi-moment maps $\nu_{\mathcal{I}}$ and $\underline{\nu}$ can be read of from the hypercomplex data, i.e., from the Joyce decomposition [Joy92, Lemma 4.1, Theorem 4.2] of the Lie algebra $\mathfrak{g}$ of G. Pedersen and Poon [PP99, Section 1] spell out this decomposition for all the compact simple Lie groups, and we will adapt their notation and results.

At hand we have a product manifold $M = T^\ell \times G$, and the tangent space at the identity can be put on the form

$$(2n - r)\,\mathfrak{u}(1) \oplus \mathfrak{g} \cong \mathbb{R}^n \oplus_{j=1}^n \mathfrak{d}_j \oplus \mathfrak{f}_j \quad \text{where} \quad \mathbb{R}^n = \langle e_1, \dots, e_n \rangle,$$

and r denotes the rank of $\mathfrak{g}$. By Joyce's work, we know that there are isomorphisms $\mathbb{H} \cong \langle e_j \rangle \oplus \mathfrak{d}_j$, and $\operatorname{Im}\mathbb{H} \cong \mathfrak{d}_j = \langle x_j, y_j, z_j \rangle$; here $x_j = I(e_j), y_j = J(e_j), z_j = K(e_j)$. One now checks that

$$\ker(\nu_I)_* = \langle e_j, x_j \colon 1 \leqslant j \leqslant n \rangle,$$

and so forth. The combined map $\underline{\nu}_*$ then has $\ker(\underline{\nu})_* = \langle e_j \colon 1 \leqslant j \leqslant n \rangle$.

Finally, we use the following list (see [PP99, Proposition 1])

(i) $\mathfrak{su}(2\ell + 1), r = 2\ell, n = \ell, 2n - r = 0;$
(ii) $\mathfrak{su}(2\ell), r = 2\ell - 1, n = \ell, 2n - r = 1;$
(iii) $\mathfrak{so}(2\ell + 1), r = \ell, n = \ell, 2n - r = \ell;$
(iv) $\mathfrak{sp}(\ell), r = \ell, n = \ell, 2n - r = \ell;$
(v) $\mathfrak{so}(4\ell), r = 2\ell, n = 2\ell, 2n - r = 2\ell;$
(vi) $\mathfrak{so}(4\ell + 2), r = 2\ell + 1, n = 2\ell, 2n - r = 2\ell - 1;$
(vii) $\mathfrak{e}_6, r = 6, n = 4, 2n - r = 2;$
(viii) $\mathfrak{e}_7, r = 7, n = 7, 2n - r = 7;$
(ix) $\mathfrak{e}_8, r = 8, n = 8, 2n - r = 8;$
(x) $\mathfrak{f}_4, r = 4, n = 4, 2n - r = 4;$
(xi) $\mathfrak{g}_2, r = 2, n = 2, 2n - r = 2;$

to derive the Table 4.3 as a generalisation of the result obtained in Proposition 4.34.

Example 4.35. Recently, Gutowski and Papadopoulos studied the geometry of black hole horizons preserving four supersymmetries. In this example we illustrate how the material from [GP10, Section 3.3] fits into the framework of strong geometry and multi-moment maps.

For our purpose, the relevant geometric data of a black hole horizon consist of a horizon section $\mathcal{S}$ which is a holomorphic T^2-fibration over a conformally balanced six-manifold B.

In the context of the above example, we may take $\mathcal{S}$ to be the SKT manifold $(SU(3), g, I)$. Then B will be the full flag $SU(3)/T^2$ realised as the image of the

M	$\nu_{\mathcal{I}}(M)$	$\underline{\nu}(M)$
$SU(2n+1)$	$SU(2n+1)/T^{2n}$	$SU(2n+1)/T^{n}$
$T^1 \times SU(2n)$	$SU(2n)/T^{2n-1}$	$SU(2n)/T^{n-1}$
$T^n \times SO(2n+1)$	$SO(2n+1)/T^{n}$	$SO(2n+1)$
$T^n \times Sp(n)$	$Sp(n)/T^{n}$	$Sp(n)$
$T^{2n} \times SO(4n)$	$SO(4n)/T^{2n}$	$SO(4n)$
$T^{2n-1} \times SO(4n+2)$	$SO(4n+2)/T^{2n+1}$	$SO(4n+2)/T^{1}$
$T^2 \times E_6$	E_6/T^6	E_6/T^2
$T^7 \times E_7$	E_7/T^7	E_7
$T^8 \times E_8$	E_8/T^8	E_8
$T^4 \times F_4$	F_4/T^4	F_4
$T^2 \times G_2$	G_2/T^2	G_2

Table 4.3: The images of $M = T^\ell \times G$ under the multi-moment maps $\nu_{\mathcal{I}}$, $\mathcal{I} = I, J, K$, and $\underline{\nu}$ associated with the Joyce shkt structure (g, I, J, K) on M.

multi-moment map (4.12). We note that this complex six-manifold comes with a compatible left-invariant two-form induced by

$$\omega_B = \omega_I|_{\mathfrak{m}} = b_{12} \wedge c_{12} + b_{13} \wedge c_{13} - b_{23} \wedge c_{23},$$

where $\mathfrak{m} = \langle B_{12}, C_{12}, B_{13}, C_{13}, B_{23}, C_{23} \rangle$ denotes an $\mathrm{ad}_{\mathfrak{t}^2}$-invariant complement of the stabiliser $\mathfrak{t}^2 = \langle A_1, V \rangle$. The corresponding Hermitian metric is

$$g_B = g|_{\mathfrak{m}} = b_{12}^2 + c_{12}^2 + b_{13}^2 + c_{13}^2 + b_{23}^2 + c_{23}^2.$$

Let us now point out the properties ensuring that our that the fibration $\nu_I \colon \mathcal{S} \to B$ is consistent with physical requirements; [GP10, Table 1] summarises the relevant geometric conditions on $\mathcal{S}$ and B imposed by the presence of $\mathcal{N} = 4$ supersymmetry. Firstly, (B, g_B, I_B) is conformally balanced, i.e., the associated Lee one-form $\theta = Id^*\omega_B$ is exact. To verify this we observe that $\theta = -\Lambda_I(d\omega_B)$, where $\Lambda_I \colon \Lambda^3 T_b^* B \to T_b^* B$ denotes the adjoint map of wedging with ω_B. Since $\Lambda_I(d\omega_B) = \langle \cdot \lrcorner\, d\omega_B, \omega_B \rangle$ and

$$d\omega_B = -b_{12} \wedge (b_{13} \wedge c_{23} + b_{23} \wedge c_{13}) - c_{12} \wedge (b_{12} \wedge b_{23} + c_{13} \wedge c_{23}),$$

we see that $\theta(\mathsf{X}) = 0$ for each $\mathsf{X} \in \mathfrak{m}$. Hence $\theta = 0$, so (B, g_B, I_B) is in fact a balanced manifold.

Secondly, let us define two invariant one-forms $k := \sqrt{3}/2a_2$ and $\ell := a_1 - \frac{1}{2}a_2$. In terms of these, the metric of the SKT manifold $(SU(3), g, I)$ takes the form

$$g = k^2 + \ell^2 + g_B.$$

Note that

$$\|k\|^2 = \|\ell\|^2 = \frac{3}{4}.$$

We may think of $\Theta = (\theta_1, \theta_2) := (k, \ell) \in \Omega^2(\mathcal{S}, \mathbb{R}^2)$ as a connection one-form for the principal T^2-fibration. From the calculations

$$d\theta_1 = -\sqrt{3}(b_{13} \wedge c_{13} + b_{23} \wedge c_{23}),$$
$$d\theta_2 = -2b_{12} \wedge c_{12} - b_{13} \wedge c_{13} + b_{23} \wedge c_{23},$$

we see that the principal curvature $d\Theta = \nu_I^*(F)$ has type $(1,1)$ with respect to I_B. In addition, the two components of F satisfy the relations

$$\langle d\theta_1, \omega_B \rangle = 0, \quad \langle d\theta_2, \omega_B \rangle = -8\|k\|^2 = -8\|\ell\|^2,$$

i.e., one component is traceless and the other one traces to a constant determined by the norms $\|k\| = \|\ell\|$.

Finally $\mathcal{N} = 4$ supersymmetry requires that $\mathcal{S}$ is a CYT manifold, meaning the the Bismut connection of (g, I) has holonomy in $SU(3)$. This condition is obviously satisfied, since every HKT structure is CYT, see, e.g., [Gra11]. ♢

If we now define a mathematical notion of *black hole horizon* to be a torus fibration $\mathcal{S} \to B$ of a CYT eight-manifold over a conformally balanced six-manifold, such that the principal curvature satisfies the above conditions, then we may summarise Example 4.35 in the following way.

Proposition 4.36. *Each of the T^2 -fibrations $\nu_{\mathcal{I}}\colon (SU(3), g, \mathcal{I}) \to F_{1,2}(\mathbb{C}^3) \subset \mathcal{P}^*_{\mathfrak{su}(3)}$ from Proposition 4.34 defines a black hole horizon.* □

Example 4.37. Consider $\mathfrak{su}(3)$ as a Lie algebra of complex matrices, and pick a basis $A_1, A_2, B_{12}, \dots, C_{23}$ as in Section 4.4.2.1. Similarly, let $a_1, a_2, b_{12}, \dots, c_{23}$ denote the dual basis. As $p_1 := B_{12} \wedge B_{13} - C_{12} \wedge C_{13} \in \mathcal{P}_{\mathfrak{g}}$, the element

$$\beta_1 = b_{12} \wedge b_{13} - c_{12} \wedge c_{13} \tag{4.13}$$

lies in the Lie kernel $\mathcal{P}^*_{\mathfrak{su}(3)}$. Using the computations (2.8), we find

$$d_{\mathcal{P}}\beta_1 = 3a_1 \wedge (b_{12} \wedge c_{13} - b_{13} \wedge c_{12}).$$

Direct inspection shows that

$$\ker d_{\mathcal{P}}\beta_1 = \langle A_2, B_{23}, C_{23} \rangle = \mathrm{stab}_{\mathfrak{su}(3)}\, \beta_1,$$

cf. Table 4.4. Thus, by Theorem 4.29, the $SU(3)$-orbit $\mathcal{O}_1$ of β_1 is 2-plectic with multi-moment map given by the inclusion in $\mathcal{P}^*_{\mathfrak{su}(3)}$. As the above stabiliser is isomorphic to $\mathfrak{su}(2)$, we see that, up to discrete covers, $\mathcal{O}_1$ is $SU(3)/\,SU(2) = S^5$. Also note that since $\mathrm{stab}_{\mathfrak{su}(3)}\, \beta_1 \subset \ker \beta_1$, we have an induced invariant two-form on the orbit which is determined by the relation

$$b(X \wedge Y) = \langle \beta_1, \mathsf{X} \wedge \mathsf{Y} \rangle \tag{4.14}$$

at $e\, SU(2) \in \mathcal{O}_1$. ♢

Let us summarise the above example.

Proposition 4.38. *Up to finite covers, we may realise $S^5 = SU(3)/SU(2)$ as a 2-plectic orbit inside $\mathcal{P}^*_{\mathfrak{su}(3)}$.* □

4.4.2.2 Strict nearly Kähler six-manifolds

One may obtain $F_{1,2}(\mathbb{C}^3) = SU(3)/T^2$ as a 2-plectic manifold by considering the $SU(3)$-orbit of

$$\beta_2 = b_{12} \wedge c_{12} + c_{13} \wedge b_{13} + b_{23} \wedge c_{23} \in \mathcal{P}^*_{\mathfrak{su}(3)}, \tag{4.15}$$

see Table 4.5 for details. This is in fact an intriguing example, since $F_{1,2}(\mathbb{C}^3)$ is known to carry a nearly Kähler structure. Such a geometry may be specified by a two-form σ and a three-form ψ_+ whose pointwise stabiliser in $GL(6,\mathbb{R})$ is isomorphic to $SU(3)$. The nearly Kähler condition is then $d\sigma = 3\lambda\psi_+$, $d\psi_- = -2\lambda\sigma^2$, where $\psi_+ + i\psi_- \in \Lambda^{3,0}$, cf. [Hit01]. Careful inspection reveals that each homogeneous strict nearly Kähler six-manifold $G/H = F_{1,2}(\mathbb{C}^3)$, $\mathbb{C}P(3)$, $S^3 \times S^3$ and S^6, as classified by Butruille [But05], may be realised as a 2-plectic orbit $G \cdot \beta$ in $\mathcal{P}^*_{\mathfrak{g}}$ for $G = SU(3), Sp(2), SU(2)^3$ and G_2, respectively. Moreover, except for the case $S^3 \times S^3$, this can be done in such a way that $\Psi = d_{\mathcal{P}}\beta$ induces $c = \psi_+$ via (4.6) and β induces σ in a corresponding way.

X	$-\operatorname{ad}^*_X \beta_1$	$\operatorname{ad}_X p_1$
A_1	$3(b_{12}c_{13} - b_{13}c_{12})$	$3(B_{12}C_{13} - B_{13}C_{12})$
B_{12}	$(-2a_1 + a_2)c_{13} - b_{12}b_{23} + c_{12}c_{23}$	$-2A_1C_{13} - B_{12}B_{23} + C_{12}C_{23}$
B_{13}	$(a_1 + a_2)c_{12} - b_{13}b_{23} - c_{13}c_{23}$	$2(A_1 + A_2)C_{12} - B_{13}B_{23} - C_{13}C_{23}$
C_{12}	$(-2a_1 + a_2)b_{13} + b_{12}c_{23} - b_{23}c_{12}$	$-2A_1B_{13} + B_{12}C_{23} - B_{23}C_{12}$
C_{13}	$(a_1 + a_2)b_{12} - b_{13}c_{23} - b_{23}c_{13}$	$2(A_1 + A_2)B_{12} - B_{13}C_{23} - B_{23}C_{13}$

Table 4.4: Specification of the coadjoint action of $\mathfrak{su}(3)$ on the element β_1 from (4.13). Basis elements not on the list, i.e., A_2, B_{23}, C_{23}, act trivially. It is important to think of the above elements as representatives of elements in $\mathcal{P}^*_{\mathfrak{su}(3)} = \Lambda^2 \mathfrak{su}(3)^*/d(\mathfrak{su}(3)^*)$, cf. Remark 4.13. So we are free to modify $\beta \in \mathcal{P}^*_{su(3)}$ by any exact element $d\alpha$, for $\alpha \in \mathfrak{su}(3)^*$. For comparison we also give the adjoint action of $\mathfrak{su}(3)$ on p_1. Note that $\langle A_1, \cdot\rangle = a_1 - \frac{1}{2}a_2$ and that $\langle A_2, \cdot\rangle = a_2 - \frac{1}{2}a_1$.

To obtain such realisations of the homogeneous nearly Kähler six-manifolds, the elements $\beta \in \mathcal{P}^*_{\mathfrak{g}}$ must be chosen with some care. We will now outline a strategy, which is applicable in all cases, except for $S^3 \times S^3$ which will be treated separately. First we pick a basis for $\mathfrak{g}$ and calculate the Lie brackets or, equivalently, the exterior derivative $d\colon \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*$. Then our candidates are elements $\beta \in \mathfrak{g}$ such that $\mathfrak{k} = \operatorname{stab}_{\mathfrak{g}} \beta$ is a codimension six subalgebra and such that (4.9) holds. Finally we must verify that the chosen pair (β, Ψ) determines a nearly Kähler structure on the orbit $\mathcal{O}_\beta = G \cdot \beta$. To this end we first determine an endomorphism $J = J_\Psi\colon V \to V$ via a recipe described by Hitchin in [Hit00]. So consider a six-dimensional real vector space $V \cong \mathfrak{g}/\mathfrak{k}$ and denote by $K_\Psi\colon V \to V \otimes \Lambda^6 V^*$ the linear transformation

$$K_\Psi(X) = A\left((X \lrcorner \Psi) \wedge \Psi\right),$$

where $A\colon \Lambda^5 V \cong V \otimes \Lambda^6 V^*$ is the isomorphism provided by the exterior product pairing $V^* \otimes \Lambda^5 V^* \to \Lambda^6 V^*$. Put $\lambda(\Psi) = \frac{1}{6} \operatorname{Tr} K_\Psi^2 \in (\Lambda^6 V^*)^{\otimes 2}$. Provided that $\lambda(\Psi) < 0$, we may now define J to be

$$J = \frac{1}{\sqrt{-\lambda(\Psi)}} K_\Psi.$$

In order that the pair (β, Ψ) defines a nearly Kähler structure on $\mathcal{O}_\beta = G \cdot \beta$, we must now make sure that the following *characterising properties* are satisfied:

(i) *type* $(1,1)$*:* $\beta \wedge \Psi = 0$;
(ii) *non-degeneracy:* $\beta \wedge \beta \wedge \beta \neq 0$;
(iii) *positive definite:* $\beta(X, JX) > 0$ for all non-zero $X \in V$;
(iv) *differential condition:* $d(J\Psi) = -\kappa \beta \wedge \beta$ for some $\kappa \in \mathbb{R}$.

The first three conditions ensure that the structure group reduces from $GL(6, \mathbb{R})$ to $SU(3)$, which together with the differential condition on $J\Psi = -\Psi(J\cdot, J\cdot, J\cdot)$

and the defining relation $\Psi = d_{\mathcal{P}}\beta$ guarantee that we have nearly Kähler six-manifold; the latter two conditions force the associated Hermitian metric to have weak holonomy $SU(3)$.

The pair (ψ_+, σ) on the orbit is defined via (4.6) and the relation

$$\langle \beta, \mathrm{X} \wedge Y \rangle = \sigma(X, Y) \quad \text{for} \quad \mathrm{X} \wedge \mathrm{Y} \in \mathfrak{m} = \mathfrak{g} / \mathrm{stab}_{\mathfrak{g}}\, \beta,$$

respectively. Note that below we choose forms that fit naturally into our concrete setting. In all three cases, this implies that $\psi_\pm, \sigma$ differ from standard conventions; one could remove this source of confusion by rescaling β and, possibly, change the sign of the complex volume form determined by Ψ.

Let us now discuss the details of the outlined procedure via a case-by-case study based on the orbit types with symmetry $SU(3)$, $Sp(2)$ and G_2, respectively.

Case $F_{1,2}(\mathbb{C}^3)$ First we realise the full flag manifold as a strict nearly Kähler manifold inside $\mathcal{P}^*_{\mathfrak{su}(3)}$. While (4.9) excludes the three copies $F_{1,2}(\mathbb{C}^3) \subset \mathcal{P}^*_{\mathfrak{su}(3)}$ obtained in Proposition 4.34, the full flag obtained from $SU(3)$ acting on the element (4.15) comes with a nearly Kähler structure, which is induced by the forms β_2 and

$$\Psi_2 = d_{\mathcal{P}}\beta_2 = 3(b_{12}(b_{13}c_{23} + b_{23}c_{13}) + c_{12}(b_{13}b_{23} + c_{13}c_{23})).$$

To verify this we first determine J via direct calculations, and find that

$$J(B_{12}) = C_{12}, \quad J(C_{13}) = B_{13}, \quad J(B_{23}) = C_{23}.$$

Note that this gives us

$$J\Psi_2 = -3(c_{12}(c_{13}b_{23} + c_{23}b_{13}) + b_{12}(c_{13}c_{23} + b_{13}b_{23})).$$

We then inspect that the pair (β_2, Ψ_2) satisfies the characterising properties. While the first three of these are easy to check, a few calculations are needed in order to verify the differential condition. We have

$$\begin{aligned}
(dc_{12})c_{13}b_{23} &= 2a_1b_{12}b_{23}c_{13} - a_2b_{12}b_{23}c_{13} + b_{13}b_{23}c_{13}c_{23},\\
-c_{12}(dc_{13})b_{23} &= -a_1b_{13}b_{23}c_{12} - a_2b_{13}b_{23}c_{12} - b_{12}b_{23}c_{12}c_{23},\\
c_{12}c_{13}(db_{23}) &= -a_1c_{12}c_{13}c_{23} + 2a_2c_{12}c_{13}c_{23} + b_{12}b_{13}c_{12}c_{13},\\
(dc_{12})c_{23}b_{13} &= 2a_1b_{12}b_{13}c_{23} - a_2b_{12}b_{13}c_{23} + b_{13}b_{23}c_{13}c_{23},\\
-c_{12}(dc_{23})b_{13} &= -a_1b_{13}b_{23}c_{12} + 2a_2b_{13}b_{23}c_{12} + b_{12}b_{13}c_{12}c_{13},\\
c_{12}c_{23}(db_{13}) &= -a_1c_{12}c_{13}c_{23} - a_2c_{12}c_{13}c_{23} - b_{12}b_{23}c_{12}c_{23},\\
(db_{12})c_{13}c_{23} &= 2a_1c_{12}c_{13}c_{23} - a_2c_{12}c_{13}c_{23} + b_{13}b_{23}c_{13}c_{23},\\
-b_{12}(dc_{13})c_{23} &= -a_1b_{12}b_{13}c_{23} - a_2b_{12}b_{13}c_{23} - b_{12}b_{23}c_{12}c_{23},\\
b_{12}c_{13}(dc_{23}) &= a_1b_{12}c_{13}b_{23} + 2a_2b_{12}b_{23}c_{13} - b_{12}b_{13}c_{12}c_{13},\\
(db_{12})b_{13}b_{23} &= 2a_1b_{13}b_{23}c_{12} - a_2b_{13}b_{23}c_{12} + b_{13}b_{23}c_{12}c_{23},\\
-b_{12}(db_{13})b_{23} &= -a_1b_{12}b_{23}c_{13} - a_2b_{12}b_{23}c_{13} - b_{12}b_{23}c_{12}c_{23},\\
b_{12}b_{13}(db_{23} &= -a_1b_{12}b_{13}c_{23} + 2a_2b_{12}b_{13}c_{23} + b_{12}b_{13}c_{12}c_{13},
\end{aligned}$$

and hence

$$\beta_2 \wedge \beta_2 = 2(b_{12}b_{13}c_{12}c_{13} - b_{12}b_{23}c_{12}c_{23} + b_{13}b_{23}c_{13}c_{23}),$$
$$d(J\Psi_2) = -12(b_{12}b_{13}c_{12}c_{13} - b_{12}b_{23}c_{12}c_{23} + b_{13}b_{23}c_{13}c_{23}),$$

so that $\beta_2 \wedge \beta_2$ and $d(J\Psi_2)$ are proportional, as required.

Case $\mathbb{C}P(3)$ We consider $\mathfrak{sp}(2)$ as a Lie algebra of complex matrices. A basis for $\mathfrak{sp}(2)$ is given by the following 10 complex matrices

$$A_1 = i(E_{11} - E_{33}), \quad A_2 = i(E_{22} - E_{44}),$$
$$Q = E_{12} - E_{21} + E_{34} - E_{43}, \quad R = i(E_{12} + E_{21} - E_{34} - E_{43}),$$
$$B_{k\ell} = E_{k,2+\ell} + E_{\ell,2+k} - E_{2+k,\ell} - E_{2+\ell,k},$$
$$C_{k\ell} = i(E_{k,2+\ell} + E_{\ell,2+k} + E_{2+k,\ell} + E_{2+\ell,k}),$$

for $1 \leqslant k \leqslant \ell \leqslant 2$, and we denote the dual basis by $a_1, a_2, \dots, c_{12}$. Now pick $\beta_3 \in \mathcal{P}^*_{\mathfrak{sp}(2)}$ given by

$$\beta_3 = b_{11} \wedge a_1 + r \wedge b_{12} + q \wedge c_{12}. \tag{4.16}$$

From the commutation relations for the chosen $\mathfrak{sp}(2)$ basis, see Table 4.9, we find that

$$da_1 = -2(4b_{11}c_{11} + b_{12}c_{12} + qr),$$
$$db_{11} = 2a_1c_{11} + b_{12}q - c_{12}r,$$
$$db_{12} = (a_1 + a_2)c_{12} + 2(-b_{11} + b_{22})q - 2(c_{11} + c_{22})r,$$
$$dc_{12} = -(a_1 + a_2)b_{12} + 2(b_{11} + b_{22})r + 2(-c_{11} + c_{22})q,$$
$$dq = (a_1 - a_2)r + 2(b_{11} - b_{22})b_{12} + 2(c_{11} - c_{22})c_{12},$$
$$dr = (-a_1 + a_2)q + 2(c_{11} + c_{22})b_{12} - 2(b_{11} + b_{22})c_{12}.$$

Computations now show that

$$\Psi_3 = d_{\mathcal{P}}\beta_3 = 3(a_1(b_{12}q - c_{12}r) + 2b_{11}(b_{12}c_{12} + qr))$$

Straightforward inspection, cf. Table 4.6, shows that

$$\begin{aligned} \mathrm{stab}_{\mathfrak{sp}(2)}\, \beta_3 &= \langle C_{11}, A_2, \tfrac{1}{2}B_{22}, \tfrac{1}{2}C_{22} \rangle = \mathfrak{u}(1) \oplus \mathfrak{su}(2) \\ &\subset \langle A_1, \tfrac{1}{2}B_{11}, \tfrac{1}{2}C_{11} \rangle \oplus \langle A_2, \tfrac{1}{2}B_{22}, \tfrac{1}{2}C_{22} \rangle = \mathfrak{su}(2) \oplus \mathfrak{su}(2), \end{aligned} \tag{4.17}$$

so that, up to discrete coverings, the $Sp(2)$-orbit $\mathcal{O}_3$ of β_3 is $\mathbb{C}P(3)$. Thus the pair (β_3, Ψ_3) satisfies (4.9). In fact it determine a nearly Kähler structure on $\mathcal{O}_3$. To verify this latter assertion, we will apply Hitchin's description in order to determine the associated almost complex structure J. We find that

$$J(B_{11}) = 2B_{11}, \quad J(R) = B_{12}, \quad J(Q) = C_{12}.$$

Note that this yields

$$J\Psi_3 = 3(2b_{11}(b_{12}c_{12} - qb_{12}) + a_1(rq + c_{12}b_{12})).$$

Let us finally check that the pair (β_3, Ψ_3) satisfies the four characterising properties. The first three of these are rather obvious, and the differential condition follows from the below calculations, where we disregard all terms involving $a_2, b_{22}, c_{11}, c_{22}$, which is legitimate, since we are defining a left-invariant structure on the quotient $Sp(2)/SU(2)$:

$$\begin{gathered} 2(db_{11})b_{12}c_{12} = 0, \quad -2b_{11}(db_{12})c_{12} = 0, \\ 2b_{11}b_{12}(dc_{12}) = 0, \quad -2(db_{11})qb_{12} = 2rb_{12}qc_{12}, \\ 2b_{11}(dq)b_{12} = 2b_{11}a_1rb_{12}, \quad -2b_{11}q(db_{12}) = 2b_{11}a_1qc_{12}, \\ (da_1)rq = 2rb_{12}qc_{12}, \quad -a_1(dr)q = 2b_{11}a_1qc_{12}, \\ a_1r(dq) = 2b_{11}a_1rb_{12}, \quad (da_1)c_{12}b_{12} = 2rb_{12}qc_{12}, \\ -a_1(dc_{12})b_{12} = 2b_{11}a_1rb_{12}, \quad a_1c_{12}(db_{12}) = 2b_{11}a_1qc_{12}. \end{gathered}$$

So we have

$$\begin{aligned} \beta_3 \wedge \beta_3 &= 2(b_{11}a_1rb_{12} + b_{11}a_1qc_{12} + rb_{12}qc_{12}), \\ d(J\Psi_3) &= 18(a_1b_{11}b_{12}r + a_1b_{11}c_{12}q + b_{12}rc_{12}q). \end{aligned}$$

Thus β_3 and $d(J\Psi_3)$ are proportional, as required.

Case S^6 Now let us consider the exceptional Lie algebra $\mathfrak{g}_2$. We choose a basis given by

$$\begin{gathered} A_1 = iH_1, \quad A_2 = iH_2, \\ B_a = X_a - Y_a, \quad C_a = i(X_a + Y_a), \quad 1 \leqslant a \leqslant 6, \end{gathered}$$

where the elements $H_1, \ldots, Y_6$ are defined in [FH91, §22.1] and satisfy and satisfy the commutation relations in Table 4.10. We then have

$$\begin{aligned} db_1 &= (2a_1 - a_2)c_1 + b_3b_2 + c_3c_2 + 2(b_4b_3 + c_4c_3) + b_4b_5 + c_4c_5, \\ dc_1 &= (-2a_1 + a_2)b_1 + c_3b_2 + c_2b_3 + 2(c_4b_3 + c_3b_4) + b_4c_5 + b_5c_4, \\ db_3 &= (-a_1 + a_2)c_3 + b_2b_1 + c_1c_2 + 2(b_1b_4 + c_1c_4) + b_4b_6 + c_4c_6, \\ dc_3 &= (a_1 - a_2)b_3 + c_2b_1 + b_2c_1 + 2(b_1c_4 + b_4c_1) + b_4c_6 + b_6c_4, \\ db_4 &= a_1c_4 + 2(b_3b_1 + c_1c_3) + b_5b_1 + c_5c_1 + c_6c_3 + b_6b_3, \\ dc_4 &= b_4a_1 + 2(c_3b_1 + b_3c_1) + c_5b_1 + c_1b_5 + c_6b_3 + c_3b_6. \end{aligned}$$

In order to obtain S^6 as an orbit in $\mathcal{P}^*_{\mathfrak{g}_2}$, we now consider the G_2-orbit the element

$$\beta_4 = b_1 \wedge c_1 + b_3 \wedge c_3 + c_4 \wedge b_4 \in \mathcal{P}^*_{\mathfrak{g}_2}. \tag{4.18}$$

We have that

$$\operatorname{stab}_{\mathfrak{g}_2} \beta_4 = \langle A_1, A_2, B_2, B_5, B_6, C_2, C_5, C_6 \rangle = \mathfrak{su}(3),$$

cf. Table 4.7. So up to finite covers, $G_2 \cdot \beta_4 = G_2/SU(3) = S^6$. Next, we note that

$$\Psi_4 = d_{\mathcal{P}}\beta_4 = 6(b_1(b_3c_4 - c_3b_4) - c_1(b_3b_4 + c_3c_4)),$$

which follows directly from the following computations

$$\begin{aligned}
(db_1)c_1 &= -b_2b_3c_1 - 2b_3b_4c_1 + b_4b_5c_1 - c_1c_2c_3 - 2c_1c_3c_4 + c_1c_4c_5,\\
-b_1(dc_1) &= b_1b_2c_3 + b_1b_3c_2 + 2b_1b_3c_4 + 2b_1b_4c_3 - b_1b_4c_5 - b_1b_5c_4,\\
(db_3)c_3 &= -b_1b_2c_3 + 2b_1b_4c_3 + b_4b_6c_3 + c_1c_2c_3 - 2c_1c_3c_4 + c_3c_4c_6,\\
-b_3(dc_3) &= -b_1b_3c_2 + 2b_1b_3c_4 + b_2b_3c_1 - b_3b_4c_6 - 2b_3b_4c_1 - b_3b_6c_4,\\
(dc_4)b_4 &= 2b_1b_4c_3 + b_1b_4c_5 - 2b_3b_4c_1 + b_3b_4c_6 - b_4b_5c_1 - b_4b_6c_3,\\
-c_4(db_4) &= 2b_1b_3c_4 + b_1b_5c_4 + b_3b_6c_4 - 2c_1c_3c_4 - c_1c_4c_5 - c_3c_4c_6.
\end{aligned}$$

Clearly, β_4 and Ψ_4 satisfy the necessary condition (4.9). In fact this pair induces a nearly Kähler structure on $\mathcal{O}_4 = G_2 \cdot \beta_4$. The associated almost complex structure is given by

$$J(B_1) = C_1, \quad J(B_3) = C_3, \quad J(C_4) = B_4.$$

From this formula for J we find that

$$J\Psi_4 = 6(c_1(-c_3b_4 + b_3c_4) + b_1(c_3c_4 + b_3b_4)).$$

Finally we observe that the pair (β_4, Ψ_4) satisfies the equations

$$\begin{aligned}
\beta_4 \wedge \beta_4 &= 2(b_1c_1b_3c_3 + b_1c_1c_4b_4 + b_3c_3c_4b_4),\\
d(J\Psi_4) &= -48(b_1c_1b_3c_3 + b_1c_1c_4b_4 + b_3c_3c_4b_4).
\end{aligned}$$

which follow from the calculations

$$\begin{aligned}
-(dc_1)c_3b_4 &= -2c_4b_3c_3b_4, & c_1(dc_3)b_4 &= -2b_1c_1c_4b_4,\\
-c_1c_3(db_4) &= -2b_1c_1b_3c_3, & (dc_1)b_3c_4 &= -2c_4b_3c_3b_4,\\
-c_1(db_3)c_4 &= -2b_1c_1c_4b_4, & c_1b_3(dc_4) &= -2b_1c_1b_3c_3,\\
(db_1)c_3c_4 &= -2c_4b_3c_3b_4, & -b_1(dc_3)c_4 &= -2b_1c_1c_4b_4,\\
b_1c_3(dc_4) &= -2b_1c_1b_3c_3, & (db_1)b_3b_4 &= -2c_4b_3c_3b_4,\\
-b_1(db_3)b_4 &= -2b_1c_1c_4b_4, & b_1b_3(db_4) &= -2b_1c_1b_3c_3,
\end{aligned}$$

where we have ignored terms in $\mathrm{stab}_{\mathfrak{g}_2}\beta_4$. Hence $d(J\Psi_4)$ and $\beta_4 \wedge \beta_4$ are proportional, as required.

Case $S^3 \times S^3$ In order to obtain the homogeneous strict nearly Kähler structure on the group manifold $S^3 \times S^3$, we consider the group $(SU(2))^3$. To be concrete, let us choose standard cyclic bases $\{E_i^1\}, \{E_i^2\}, \{E_i^3\}$ for each copy of $\mathfrak{su}(2)$; so in terms of the dual basis $\{e_i^1\}$, etc., for $\mathfrak{su}(2)^*$ we have that $de_1^i = e_{23}^i$, and so forth. Now consider the element $\beta_5 \in \mathcal{P}^*_{3\,\mathfrak{su}(2)}$ given by

$$\beta_5 = \sum_{i=1}^{3} e_i^1 \wedge e_i^2 + e_i^1 \wedge e_i^3 + e_i^2 \wedge e_i^3. \tag{4.19}$$

We observe that

$$\Psi_5 = d_{\mathcal{P}}\beta_5 = \sum_{i\in\mathbb{Z}/3} e^1_{i+1}\wedge e^1_{i+2}\wedge e^2_i + e^1_{i+1}\wedge e_{i+2}\wedge e^3_i + e^2_{i+1}\wedge e_{i+2}\wedge e^3_i$$
$$- e^1_i\wedge e^2_{i+1}\wedge e^2_{i+2} - e^1_i\wedge e^3_{i+1}\wedge e^3_{i+2} - e^2_i\wedge e^3_{i+1}\wedge e^3_{i+2}.$$

Inspection then shows

$$\mathrm{stab}_{3\,\mathfrak{su}(2)}\,\beta_5 = \langle E^1_i + E^2_i + E^3_i\colon\ 1\leqslant i\leqslant 3\rangle =: \delta\,\mathfrak{su}(2),$$

cf. Table 4.8. To specify the nearly Kähler in this case is somewhat more involved. To keep things simple, we will follow Butruille [But10] and look for nearly Kähler structures invariant under a subgroup $SU(2)^2\subset SU(2)^3$; we emphasise, however, that the strict nearly Kähler structure on $S^3\times S^3$ is invariant under the larger group $SU(2)^3$, cf. [Bär93]. We first choose an $\mathrm{ad}_{\delta\,\mathfrak{su}(2)}$-invariant complement of the stabiliser $\delta_{\mathfrak{su}(2)}$: the subspace $\mathfrak{m} = \mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\{0\}$. Then we have

$$\widehat{\beta}_5 := \beta_5|_{\mathfrak{m}} = e^1_1\wedge e^2_1 + e^1_2\wedge e^2_2 + e^1_3\wedge e^2_3,$$
$$\widehat{\Psi}_5 := \Psi_5|_{\mathfrak{m}} = e^1_{12}e^2_3 + e^1_{23}e^2_1 + e^1_{31}e^2_2 - e^1_1e^2_{23} - e^1_2e^2_{31} - e^1_3e^2_{12}.$$

It is well-known, cf. [But10], that the forms $\widehat{\beta}_5$ and $\widehat{\Psi}_5$ induce a nearly Kähler structure on $S^3\times S^3$. Let us briefly recall Butruille's arguments. First we observe that there is an associated almost complex structure given by

$$J(E^1_i) = (E^1_i + 2E^2_i)/\sqrt{3},\quad i = 1,2,3.$$

From this observation, we see that

$$J\widehat{\Psi}_5 = \frac{1}{\sqrt{3}}(2e^1_{123} - e^1_{12}e^2_3 - e^1_{31}e^2_2$$
$$- e^1_1e^2_{23} - e^1_{23}e^2_1 - e^1_2e^2_{31} - e^1_3e^2_{12} + 2e^2_{123}).$$

Note that the form $\widehat{\beta}_5$ is of type $(1,1)$ and is non-degenerate and positive definite. Finally observe that

$$\widehat{\beta}_5\wedge\widehat{\beta}_5 = 2(e^1_1e^2_1e^1_2e^2_2 + e^1_1e^2_1e^1_3e^2_3 + e^1_2e^2_2e^1_3e^2_3),$$
$$d(J\widehat{\Psi}_5) = \frac{2}{\sqrt{3}}(e^1_1e^2_1e^1_2e^2_2 + e^1_1e^2_1e^1_3e^2_3 + e^1_2e^2_2e^1_3e^2_3).$$

Altogether, the above observations ensure that the pair $(\widehat{\beta}_5,\widehat{\Psi}_5)$ defines a left-invariant nearly Kähler structure on $S^3\times S^3$.

Remark 4.39. The strong structure ψ_5 on $SU(2)^3$ induced by $d_{\mathcal{P}}\beta_5$ via the formula (4.6) has an associated multi-moment map $\nu\colon\ SU(2)^3\to\mathcal{P}^*_{3su(2)}$. The image of ν is the strict nearly Kähler manifold $(S^3\times S^3,\widehat{\psi}_5)$. △

Remark 4.40. The $SU(2) \times SU(2)$-invariant 2-plectic structure $\widehat{\psi}_5$ on the symmetric space $S^3 \times S^3 = (SU(2))^3/\Delta\, SU(2) \subset \mathcal{P}^*_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ also admits a multi-moment map $\widehat{\nu}\colon\ S^3 \times S^3 \to \mathcal{P}^*_{su(2)\oplus\mathfrak{su}(2)}$. The image of this multi-moment map is the $SU(2) \times SU(2)$-orbit of the element

$$\beta_5 = e^1_1 \wedge e^2_1 + e^1_2 \wedge e^2_2 + e^1_3 \wedge e^2_3 \in \mathcal{P}^*_{su(2)\oplus\mathfrak{su}(2)},$$

where $\{e^j_i\}$, $j = 1, 2$, denotes a standard cyclic basis for $\mathfrak{su}(2)^*$ as above. Direct inspection now shows that

$$\mathrm{stab}_{2\,\mathfrak{su}(2)}\, \beta_6 = \langle E^1_1 + E^2_1, E^1_2 + E^2_2, E^1_3 + E^2_3 \rangle,$$

cf. Table 4.8. So up to discrete covers, the image of $\widehat{\nu}$ is the homogeneous space $S^3 = SU(2) \times SU(2)/\Delta\, SU(2)$. △

Note that from Butruille's classification, we know that the above strict nearly Kähler structures are unique up to homothety. In summary we thus have

Proposition 4.41. *As almost effective homogeneous spaces, each strict nearly Kähler six-manifold*

$$F_{1,2}(\mathbb{C}^3), \quad \mathbb{C}\mathrm{P}(3), \quad S^6 \quad \textit{and} \quad S^3 \times S^3$$

may be realised, up to finite covers, as a 2-plectic orbit $\mathcal{O}_\beta = G \cdot \beta$ in $\mathcal{P}^*_{\mathfrak{g}}$ for $G = SU(3), Sp(2), G_2$ and $(SU(2))^3$, respectively. By choosing β as in Table 4.11, the pair $(\beta, d_{\mathcal{P}}\beta)$ determines the nearly Kähler structure on $\mathcal{O}_\beta$. □

4.4.2.3 $\mathcal{P}_{\mathfrak{g}}$-transitive manifolds

Let us now try to analyse the representation theory underlying several of the examples studied in the previous two sections. We consider a compact simple Lie group G and a homogeneous manifold $M = G/K$ carrying a G-invariant two-form $b \in \Omega^2(M)$. Note that b determines a G-morphism $\Phi\colon\ M \to \Lambda^2\,\mathfrak{g}^*$ given by the relation

$$\langle \Phi, \mathsf{X} \wedge \mathsf{Y} \rangle = b(X, Y), \tag{4.20}$$

for $\mathsf{X}, \mathsf{Y} \in \mathfrak{g}$. Put $\beta = \Phi(eK) \in \Lambda^2\,\mathfrak{g}^*$, and note that, as Φ is K-invariant at the point $eK \in M$, we actually have that $\beta \in (\Lambda^2\,\mathfrak{g}^*)^K$. Here

$$(\Lambda^2\,\mathfrak{g}^*)^K = (\mathfrak{g}^* + \mathcal{P}^*_{\mathfrak{g}})^K = (\mathfrak{k}^*)^K + (\mathfrak{m}^*)^K + (\mathcal{P}^*_{\mathfrak{g}})^K,$$

and the first two summands on the rightmost hand side above vanish, e.g., if K is semi-simple and the isotropy action is irreducible. In such cases we will have $\beta \in (\mathcal{P}^*_{\mathfrak{g}})^K \subset \mathcal{P}^*_{\mathfrak{g}}$, and may then use the element $d_{\mathcal{P}}\beta \in Z^3(\mathfrak{g})$ to define a strong geometry (N, c). Moreover, this strong geometry admits a multi-moment map $\nu\colon\ N \to \mathcal{P}^*_{\mathfrak{g}}$ such that $\nu(N) = G \cdot \beta$. Since we have already observed that $\mathcal{P}^*_{\mathfrak{g}}$ is irreducible, Schur's lemma applies, provided that $\mathfrak{m}$ is also irreducible. In particular, we may have that $\nu(N) = G \cdot \beta = M$.

We collect these observations in a slightly more general statement.

X	$\mathrm{ad}^*_X \beta_2$	$-\mathrm{ad}_X p_2$
B_{12}	$(2a_1 - a_2) \wedge b_{12} - 2(b_{13} \wedge c_{23} + b_{23} \wedge c_{13})$	$2A_1 \wedge B_{12} - 2(B_{13} \wedge C_{23} + B_{23} \wedge C_{13})$
C_{12}	$(2a_1 - a_2) \wedge c_{12} - 2(b_{13} \wedge b_{23} + c_{13} \wedge c_{23})$	$2A_1 \wedge C_{12} - 2(B_{13} \wedge B_{23} + C_{13} \wedge C_{23})$
B_{13}	$-(a_1 + a_2) \wedge b_{13} + 2(b_{12} \wedge c_{23} + c_{12} \wedge b_{23})$	$-2(A_1 + A_2) \wedge B_{13} + 2(B_{12} \wedge C_{23} + C_{12} \wedge B_{23})$
C_{13}	$-(a_1 + a_2) \wedge c_{13} - 2(b_{12} \wedge b_{23} + c_{23} \wedge c_{12})$	$-2(A_1 + A_2) \wedge C_{13} - 2(B_{12} \wedge B_{23} + C_{23} \wedge C_{12})$
B_{23}	$(-a_1 + 2a_2) \wedge b_{23} + 2(b_{12} \wedge c_{13} + b_{13} \wedge c_{12})$	$2A_2 \wedge B_{23} + 2(B_{12} \wedge C_{13} + B_{13} \wedge C_{12})$
C_{23}	$(-a_1 + 2a_2) \wedge c_{23} - 2(b_{12} \wedge b_{13} + c_{12} \wedge c_{13})$	$2A_2 \wedge C_{23} - 2(B_{12} \wedge B_{13} + C_{12} \wedge C_{13})$

Table 4.5: Specification of the coadjoint action of $\mathfrak{su}(3)$ action on the element β_2 from (4.15). Basis elements not on the list, i.e., A_1, A_2, act trivially. We also give the adjoint action of $\mathfrak{su}(3)$ on the element $p_2 = B_{12}C_{12} + C_{13}B_{13} + B_{23}C_{23}$.

X	$\mathrm{ad}^*_X \beta_3$	$\mathrm{ad}_X p_3$
A_1	$2(a_1 \wedge c_{11} - b_{12} \wedge q + c_{12} \wedge r)$	$-2(A_1 \wedge C_{11} - B_{12} \wedge Q + C_{12} \wedge R)$
B_{11}	$2(b_{11} \wedge c_{11} - 2b_{12} \wedge c_{12} - 2q \wedge r)$	$-2(B_{11} \wedge C_{11} - 2B_{12} \wedge C_{12} - 2Q \wedge R)$
B_{12}	$(2a_1 + a_2) \wedge q + 2(c_{11} + c_{22}) \wedge b_{12} + 2(2b_{11} - b_{22}) \wedge c_{12}$	$-2(2A_1 + A_2) \wedge Q - (C_{11} + C_{22}) \wedge B_{12} - (2B_{11} - B_{22}) \wedge C_{12}$
R	$(2a_1 - a_2) \wedge c_{12} - 2(2b_{11} + b_{22}) \wedge q + 2(c_{11} + c_{22}) \wedge r$	$-2(2A_1 - A_2) \wedge C_{12} + (2B_{11} + B_{22}) \wedge Q - (C_{11} + C_{22}) \wedge R$
C_{12}	$-(2a_1 + a_2) \wedge r - 2(2b_{11} + b_{22}) \wedge b_{12} + 2(c_{11} - c_{22}) \wedge c_{12}$	$2(2A_1 + A_2) \wedge R + (2B_{11} + B_{22}) \wedge B_{12} - (C_{11} - C_{22}) \wedge C_{12}$
Q	$(-2a_1 + a_2) \wedge b_{12} + 2(2b_{11} - b_{22}) \wedge r + 2(c_{11} - c_{22}) \wedge q$	$-2(-2A_1 + A_2) \wedge B_{12} - (2B_{11} - B_{22}) \wedge R - (C_{11} - C_{22}) \wedge Q$

Table 4.6: Specification of the coadjoint action of $\mathfrak{sp}(2)$ action on the element β_3 from (4.16). Basis elements not on the list, i.e., $A_2, C_{11}, B_{22}, C_{22}$, act trivially. For comparison we also give the adjoint action of the element $p_3 = B_{11} \wedge A_1 + R \wedge B_{12} + Q \wedge C_{12}$. Note that if we take as ad-invariant inner product the mapping $(X, Y) \mapsto -\frac{1}{2}\operatorname{Tr}(XY)$, then $\langle A_i, \cdot \rangle = a_i$, $\langle Q, \cdot \rangle = 2q, \langle R, \cdot \rangle = 2r$, $\langle B_{12}, \cdot \rangle = b_{12}, \langle C_{12}, \cdot \rangle = c_{12}$ and $\langle B_{ii}, \cdot \rangle = 4b_{ii}, \langle C_{ii}, \cdot \rangle = 4c_{ii}$.

X	$-\operatorname{ad}^*_X \beta_4$	$\operatorname{ad}_X p_4$
B_1	$(-2a_1+a_2)b_1+c_3b_2+c_2b_3-4(c_4b_3+c_3b_4)+b_4c_5+b_5c_4$	$-2A_1B_1+3(C_3B_2+C_2B_3)-4(C_4B_3+C_3B_4)$ $+3(B_4C_5+B_5C_4)$
C_1	$(-2a_1+a_2)c_1-b_3b_2-c_3c_2+4(b_4b_3+c_4c_3)-b_4b_5-c_4c_5$	$-2A_1C_1-3(B_3B_2+C_3C_2)+4(B_1C_4+B_4C_1)$ $-3(B_4B_5+C_4C_5)$
B_3	$(a_1-a_2)b_3+c_2b_1+b_2c_1-4(b_1c_4+b_4c_1)+b_4c_6+b_6c_4$	$-2(A_1+3A_2)B_3+3(C_2B_1+B_2C_1)-4(B_1C_4+B_4C_1)$ $+3(B_4C_6+B_6C_4)$
C_3	$(a_1-a_2)c_3-b_2b_1-c_1c_2+4(b_1b_4+c_1c_4)-b_4b_6-c_4c_6$	$-2(A_1+3A_2)C_3-3(B_2B_1+C_1C_2)+4(B_1B_4+C_1C_4)$ $-3(B_4B_6+C_4C_6)$
B_4	$-b_4a_1+4(c_3b_1+b_3c_1)-c_5b_1-c_1b_5-c_6b_3-c_3b_6$	$2(2A_1+3A_2)B_4+4(C_3B_1+B_3C_1)$ $-3(C_5B_1+C_1B_5+C_6B_3+C_3B_6)$
C_4	$a_1c_4-4(b_3b_1+c_1c_3)+b_5b_1+c_5c_1+c_6c_3+b_6b_3$	$2(2A_1+3A_2)C_4-4(B_3B_1+C_1C_3)$ $+3(B_5B_1+C_5C_1+C_6C_3+B_6B_3)$

Table 4.7: Specification of the coadjoint action of $\mathfrak{g}_2$ action on the element β_4 from (4.18). Basis elements not on the list, i.e., $A_1, A_2, B_2, B_5, B_6, C_2, C_5, C_6$, act trivially. We also specify the adjoint action of $\mathfrak{g}_2$ on the element $p_4 = B_1 \wedge C_1 + B_3 \wedge C_3 + C_4 \wedge B_4$.

X	$-\operatorname{ad}^*_X \beta_5$
E_1^1	$e_2^1 \wedge e_3^2 + e_2^1 \wedge e_3^3 - e_3^1 \wedge e_2^2 - e_3^1 \wedge e_2^3$
E_1^2	$-e_2^1 \wedge e_3^2 + e_3^1 \wedge e_2^2 - e_3^2 \wedge e_2^3 + e_2^2 \wedge e_3^3$
E_1^3	$-e_2^1 \wedge e_3^3 + e_3^1 \wedge e_2^3 - e_2^2 \wedge e_3^3 + e_3^2 \wedge e_2^3$
E_2^1	$-e_1^1 \wedge e_3^2 - e_1^1 \wedge e_3^3 + e_3^1 \wedge e_1^2 + e_3^1 \wedge e_1^3$
E_2^2	$e_1^1 \wedge e_3^2 - e_1^2 \wedge e_3^3 - e_3^1 \wedge e_1^2 + e_3^2 \wedge e_1^3$
E_2^3	$e_1^1 \wedge e_3^3 - e_3^1 \wedge e_1^3 + e_1^2 \wedge e_3^3 - e_3^2 \wedge e_1^3$
E_3^1	$e_1^1 \wedge e_2^2 + e_1^1 \wedge e_2^3 - e_2^1 \wedge e_1^2 - e_2^1 \wedge e_1^3$
E_3^2	$-e_1^1 \wedge e_2^2 + e_2^1 \wedge e_1^2 + e_1^2 \wedge e_2^3 - e_2^2 \wedge e_1^3$
E_3^3	$-e_1^1 \wedge e_2^3 + e_2^1 \wedge e_1^3 - e_1^2 \wedge e_2^3 - e_2^2 \wedge e_1^3$

Table 4.8: Specification of the coadjoint action of $3\,\mathfrak{su}(2)$ action on the element β_5 from (4.19). We observe that β_5 is stabilised by the diagonal algebra $\delta\,\mathfrak{su}(2)$ spanned by the elements $E_i^1 + E_i^2 + E_i^3$, $1 \leqslant i \leqslant 3$. Note that we may choose an ad-invariant inner product on $3\,\mathfrak{su}(2)$ such that $\langle E_i^j, \cdot\rangle = e_i^j$. So the adjoint action of $3\,\mathfrak{su}(2)$ in $p_5 = E_1^1 E_1^2 + \dots$ follows immediately from the above calculations.

Theorem 4.42. *Let G be a connected simple Lie group. Assume the homogeneous space $M = G/K$ carries an invariant two-form $b \in \Omega^2(M)$, such that the map Φ, defined via (4.20), satisfies the condition $\beta := \Phi(eK) \in \mathcal{P}_{\mathfrak{g}}^*$. Then there exists a strong geometry (M, c) admitting a unique multi-moment map $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$. The image of ν is $G/\operatorname{stab}_G \beta$.* □

To characterise the homogeneous geometries of Theorem 4.29, we introduce the following terminology.

Definition 4.43. Let G be a group of symmetries of a strong geometry (M, c). We say that the action is *weakly $\mathcal{P}_{\mathfrak{g}}$-transitive* if G acts transitively on M and for each non-zero $X \in T_xM$, there is a $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$ such that $c(X \wedge p)$ is non-zero.

Corollary 4.44. *If G is $(2,3)$-trivial, then the weakly $\mathcal{P}_{\mathfrak{g}}$-transitive 2-plectic geometries with symmetry group G are discrete covers of orbits $\mathcal{O} = G \cdot \beta$ in $\mathcal{P}_{\mathfrak{g}}^*$ satisfying condition (4.9).*

More generally, if G is a Lie group with $b_2(\mathfrak{g}) = 0$, then the orbits $\mathcal{O} = G \cdot \beta \subset \mathcal{P}_{\mathfrak{g}}^*$ satisfying (4.9) are, up to discrete covers, the weakly $\mathcal{P}_{\mathfrak{g}}$-transitive 2-plectic geometries that admit a multi-moment map.

Proof. The differential $\nu_*\colon T_xM \to \mathcal{P}_{\mathfrak{g}}^*$ of the multi-moment map is given by $\langle \nu_*(X), \mathsf{p}\rangle = (X \lrcorner c)(p)$. As G acts weakly $\mathcal{P}_{\mathfrak{g}}$-transitively, we see that $\nu_*(X)$ is non-zero for each non-zero X. Thus ν_* is injective and ν has discrete fibres. Its image is an orbit $G \cdot \beta$ and the proof of Theorem 4.12 shows that the 3-form c on M is induced by $\Psi = d_{\mathcal{P}}\beta$. As ν is a local diffeomorphism and c is 2-plectic it follows that (4.9) is satisfied. Conversely, any orbit $\mathcal{O} = G \cdot \beta$ satisfying (4.9) is 2-plectic with injective multi-moment map ν. Since ν_* is injective, the equation $\langle \nu_*(X), \mathsf{p}\rangle = c(X \wedge p)$ shows that the action is weakly $\mathcal{P}_{\mathfrak{g}}$-transitive. □

	A_2	Q	R	B_{11}	C_{11}	B_{12}	C_{12}	B_{22}	C_{22}
A_1	0	R	$-Q$	$2C_{11}$	$-2B_{11}$	C_{12}	$-B_{12}$	0	0
A_2		$-R$	Q	0	0	C_{12}	$-B_{12}$	$2C_{22}$	$-2B_{22}$
Q			$2(A_1 - A_2)$	$-2B_{12}$	$-2C_{12}$	$B_{11} - B_{22}$	$C_{11} - C_{22}$	$2B_{12}$	$2C_{12}$
R				$2C_{12}$	$-2B_{12}$	$C_{11} + C_{22}$	$-B_{11} - B_{22}$	$2C_{12}$	$-2B_{12}$
B_{11}					$8A_1$	$-2Q$	$2R$	0	0
C_{11}						$-2R$	$-2Q$	0	0
B_{12}							$2(A_1 + A_2)$	$-2Q$	$2R$
C_{12}								$-2R$	$-2Q$
B_{22}									$8A_2$

Table 4.9: Our preferred basis for $\mathfrak{sp}(2)$ satisfies the above commutation relations.

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$
X_1			H_1	X_3	0	$2X_4$	$-3Y_2$	$-3X_5$	$-2Y_3$	0	Y_4	0	0
Y_1				0	$-Y_3$	$3X_2$	$-2Y_4$	$2X_3$	$3Y_5$	$-X_4$	0	0	0
X_2					H_2	0	Y_1	0	0	$-X_6$	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	$-X_5$	0
X_3							$H_1 + 3H_2$	$-3X_6$	$2Y_1$	0	0	0	Y_4
Y_3								$-2X_1$	$3Y_6$	0	0	$-X_4$	0
X_4									$2H_1 + 3H_2$	0	$-Y_1$	0	$-Y_3$
Y_4										X_1	0	X_3	0
X_5											$H_1 + H_2$	0	$-Y_2$
Y_5												X_2	0
X_6													$H_1 + 2H_2$

Table 4.10: Our preferred basis for $\mathfrak{g}_2$ is constructed from elements $H_1, \dots, Y_6$ satisfying the above commutation relations; see further details in [FH91, § 22.1].

G	β	$d_{\mathcal{P}}\beta$	$\mathcal{O} = G \cdot \beta$
$SU(3)$	$b_{12}c_{12}+c_{13}b_{13}+b_{23}c_{23}$	$3(b_{12}(b_{13}c_{23}+b_{23}c_{13})+c_{12}(b_{13}b_{23}+c_{13}c_{23}))$	$F_{1,2}(\mathbb{C}^3)$
$Sp(2)$	$b_{11}a_1+rb_{12}+qc_{12}$	$3(a_1(b_{12}q - c_{12}r) + 2b_{11}(b_{12}c_{12} + qr))$	$\mathbb{C}\mathrm{P}(3)$
G_2	$b_1c_1+b_3c_3+c_4b_4$	$6(b_1(b_3c_4-c_3b_4)-c_1(b_3b_4 + c_3c_4))$	S^6
$SU(2)^3$	$e_1^1e_1^2+e_2^1e_2^2+e_3^1e_3^2 + \cdots$	$e_{12}^1e_3^2 + e_{23}^1e_1^2 + e_{31}^1e_2^2 + \cdots - e_1^1e_{23}^2 - e_2^1e_{31}^2 - e_3^1e_{12}^2 - \cdots$	$S^3 \times S^3$

Table 4.11: Realisations of the homogeneous strict nearly Kähler six-manifolds as orbits in Lie kernels $\mathcal{P}^*_{\mathfrak{g}}$. Note that $\wedge$ signs have been omitted, so that $b_{12}c_{12}$ means $b_{12} \wedge c_{12}$, and so forth.

4.4.3 Compact Lie groups with bi-invariant metric

Let G be a compact semi-simple Lie group. Its Lie algebra $\mathfrak{g}$ admits an inner product $\langle\cdot,\cdot\rangle$ invariant under the adjoint representation, which is proportional to minus the Killing form. The left- and right-invariant Cartan one-forms $\theta^L, \theta^R \in \Omega^1(G,\mathfrak{g})$ are given by $\theta^L(X) = (L_{g^{-1}})_*(X)$, $\theta^R(X) = (R_{g^{-1}})_*(X)$, where $L_g, R_g\colon G \to G$ denote left- and right-multiplication by g. A bi-invariant, and hence closed, three-form is defined on G by

$$c(X,Y,Z) = \langle[\theta^L(X),\theta^L(Y)],\theta^L(Z)\rangle, \qquad \text{for } X,Y,Z \in \Gamma(TG). \tag{4.21}$$

This is 2-plectic but is zero on elements of $\mathcal{P}_{\mathfrak{g}}$ for G acting on the left. Instead for $H,K \leqslant G$, let $H \times K$ act on G by

$$(h,k)\cdot g = L_h \circ R_{k^{-1}}(g) = hgk^{-1}.$$

An element $\mathsf{X} = (\mathsf{X}^H,\mathsf{X}^K) \in \mathfrak{h}\oplus\mathfrak{k}$ induces a vector field X on G given by $X_g = \frac{d}{dt}\exp(t\mathsf{X}^H)g\exp(-t\mathsf{X}^K)|_{t=0} = (R_g)_*\mathsf{X}^H - (L_g)_*\mathsf{X}^K$. For $\mathsf{p} = \sum_{j=1}^k \mathsf{X}_j \wedge \mathsf{Y}_j \in \mathcal{P}_{\mathfrak{h}\oplus\mathfrak{k}}$, we have that $\sum_{j=1}^k [X_j^H, Y_j^H] = 0$ and $\sum_{j=1}^k [X_j^K, Y_j^K] = 0$, and claim that

$$\langle\nu(g),\mathsf{p}\rangle = \sum_{j=1}^k \big(\langle \mathsf{X}_j^H, \mathrm{Ad}_g(\mathsf{Y}_j^K)\rangle - \langle \mathsf{Y}_j^H, \mathrm{Ad}_g(\mathsf{X}_j^K)\rangle\big),$$

defines a multi-moment map $\nu\colon G \to \mathcal{P}^*_{\mathfrak{h}\oplus\mathfrak{k}}$. This follows from the following computation for $A_g = (R_g)_*\mathsf{A}$:

$$\begin{aligned} d\langle\nu,\mathsf{p}\rangle(A)_g &= \frac{d}{dt}\langle\nu(\exp(t\mathsf{A})g),\mathsf{p}\rangle\Big|_{t=0} \\ &= \langle \mathsf{X}_j^H, [\mathsf{A}, \mathrm{Ad}_g(\mathsf{Y}_j^K)]\rangle - \langle \mathsf{Y}_j^H, [\mathsf{A}, \mathrm{Ad}_g(\mathsf{X}_j^K)]\rangle \\ &= -\langle [\mathrm{Ad}_{g^{-1}}\mathsf{X}_j^H, \mathsf{Y}_j^K] + [\mathsf{X}_j^K, \mathrm{Ad}_{g^{-1}}\mathsf{Y}_j^H], \theta^L(A)_g\rangle = (p \lrcorner c)(A)_g, \end{aligned}$$

since $\theta^L(A)_g = \mathrm{Ad}_{g^{-1}}\mathsf{A}$. By considering $\mathsf{p} \in \mathcal{P}_{\mathfrak{h}\oplus\mathfrak{k}}$ of the form $\mathsf{p} = (\mathsf{X}^H, 0) \wedge (0, \mathsf{Y}^K)$ with $\mathsf{X}^H \in \mathfrak{h}$ and $\mathsf{Y}^K \in \mathfrak{k}$ arbitrary, one finds that

$$\ker(\nu_*)_g = (L_g)_*[\mathrm{Ad}_{g^{-1}}\mathfrak{h},\mathfrak{k}]^\perp.$$

In the case when $\mathfrak{h} = \mathfrak{g}$, the set $\ker(\nu_*)_e$ is a subalgebra of $\mathfrak{g}$ and the image of ν is an orbit.

One example is given by $\mathfrak{h} = \mathfrak{g} = \mathfrak{su}(3)$ and $\mathfrak{k} = \mathfrak{u}(1) = \mathrm{diag}(ia,-ia,0)$. Then $\ker(\nu_*)_e = \mathfrak{u}(2)$ and the multi-moment map ν is the projection from $SU(3)$ to $\mathbb{C}P(2) = SU(3)/U(2)$. Now $\mathbb{C}P(2)$ is quaternionic Kähler, and $SU(3)$ carries a hypercomplex structure [Joy92]. The bi-invariant metric on $SU(3)$ realises the hypercomplex structure as a strong HKT manifold whose torsion-three form c is given by (4.21) [GP00]. The symmetry group of this HKT structure is precisely $H \times K = SU(3) \times U(1)$ and the map ν realises $SU(3)$ as a twisted associated bundle over $\mathbb{C}P(2)$ [PPS98].

4.4.4 Strong geometries from symplectic manifolds

Let us show how the theory of multi-moment maps for strong geometries subsumes that of symplectic moment maps. Given a symplectic manifold (N, ω) one has a strong geometry on $M = S^1 \times N$ with $c = \phi \wedge \omega$, where ϕ is the invariant one-form dual to the circle action on S^1. This geometry is 2-plectic. If N comes with a symplectic action of a Lie group H, then $G = S^1 \times H$ is a symmetry group for the strong geometry on M. The corresponding Lie kernel is given by

$$\mathcal{P}_{\mathbb{R}+\mathfrak{h}} \cong \mathcal{P}_{\mathfrak{h}} + \mathbb{R} \otimes \mathfrak{h}\,.$$

Proposition 4.45. *Let (N, ω) be a symplectic manifold with a Hamiltonian action of H, moment map $\mu\colon N \to \mathfrak{h}^*$. Then $M = S^1 \times N$ carries a strong geometry with symmetry group $G = S^1 \times H$ and this has a multi-moment map ν that may be identified with μ.*

Proof. We first claim that $p \lrcorner \omega = 0$, for each $\mathsf{p} \in \mathcal{P}_{\mathfrak{h}} \subset \mathcal{P}_{\mathfrak{g}}$. Writing $\mathsf{p} = \sum_{j=1}^{k} \mathsf{X}_j \wedge \mathsf{Y}_j \in \mathcal{P}_{\mathfrak{h}}$, we have

$$\omega(p) = \sum_{j=1}^{k} \omega(X_j, Y_j) = \sum_{j=1}^{k} Y_j \lrcorner\, d\langle \mu, \mathsf{X}_j \rangle = \sum_{j=1}^{k} \mathcal{L}_{Y_j} \langle \mu, \mathsf{X}_j \rangle.$$

But μ is equivariant, so $\mathcal{L}_Y \langle \mu, \mathsf{X} \rangle = \langle \mu, [X, Y] \rangle$. As $\sum_{j=1}^{k} [\mathsf{X}_j, \mathsf{Y}_j] = 0$ it follows that $\omega(p) = 0$, as required.

Now we may define $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$ by

$$\langle \nu, \mathsf{p} \rangle = 0, \qquad \langle \nu, \mathsf{T} \wedge \mathsf{X} \rangle = \langle \mu, \mathsf{X} \rangle,$$

for $\mathsf{p} \in \mathcal{P}_{\mathfrak{h}}$ and $\mathsf{X} \in \mathfrak{h}$, where T is the generator of the S^1 action on the first factor of $M = S^1 \times G$. Now $d\langle \nu, \mathsf{p} \rangle = 0 = p \lrcorner\, c$ and

$$d\langle \nu, \mathsf{T} \wedge \mathsf{X} \rangle = X \lrcorner\, \mu = (T \wedge X) \lrcorner\, c,$$

so equation (4.3) is satisfied. As the definition of ν is equivariant, we have that ν is a multi-moment map. □

4.4.5 Reduction via multi-moment maps

The Marsden-Weinstein reduction [GS84] is a highly useful tool for obtaining new symplectic manifolds from known examples. Similar roles are played by its cousins in Kähler and hyperKähler geometry [HKLR87]. As these constructions are intimately linked with the theory of moment maps, it seems natural to speculate whether an analogue construction exists in the strong geometric setting. Naively one might hope that if (Y, c) is a strong geometry with symmetry G and multi-moment map ν, and if the quotient space $M = \nu^{-1}(t)/G$ is smooth with projection map $\pi\colon \nu^{-1}(t) \to M$, then M carries a closed three-form. Expectedly the three-form $\widetilde{c}$ on M would be given by the relation $\iota^* c = \pi^* \widetilde{c}$, where ι denotes the inclusion $\nu^{-1}(t) \hookrightarrow Y$.

Unfortunately, this wishful thinking turns out to be nonsense. In contrast to symplectic reduction, it is a subtle task to find strong geometries that are 'strongly reducible'. The above construction fails to hold for the following reason. If $q \in \nu^{-1}(t)$ for some $t \in \nu(M)^G$, then it is generally not true that tangent vectors along the orbit of q are contained in the kernel of $\iota^* c$, that is, we do not have an inclusion

$$T_q(G \cdot q) \subseteq \{X \in T_q\nu^{-1}(t)\colon\ c(\iota_* X, \iota_* \beta) = 0,\ \forall \beta \in \Lambda^2 T_q \nu^{-1}(t)\}.$$

In particular, this means that the form $\iota^* c$ fails to be horizontal. Hence, it cannot be basic and is therefore not the pull-back of a form on M.

While there are no simple criteria telling us when a strong geometry is strongly reducible, it may still be possible to find examples via case-by-case studies. The aim of this section is to find strongly reducible $PSU(3)$-manifolds. The following discussion and results may be regarded as a reinterpretation of parts of Witt's work [Wit08] in the setting of strong geometries and multi-moment maps.

Reducing PSU(3)-manifolds Let us explain the fundamental aspects of $PSU(3)$-geometry following [Wit04]. On $\mathbb{R}^8$ with its standard orientation consider the three-form ρ_0 given by

$$\rho_0 = e_{123} + \frac{1}{2}e_1(e_{47} - e_{56}) + \frac{1}{2}e_2(e_{46} + e_{57}) + \frac{1}{2}e_3(e_{45} - e_{67}) + \frac{\sqrt{3}}{2}e_8(e_{45} + e_{67}), \tag{4.22}$$

where $e_1, \dots, e_8$ is the standard dual basis and wedge signs have been omitted. The stabiliser of ρ_0 is the compact eight-dimensional Lie group

$$PSU(3) = \{g \in GL_+(8, \mathbb{R})\colon\ g^*\rho_0 = \rho_0\} = SU(3)/(\mathbb{Z}/3).$$

This group also preserves the standard metric $g_0 = \sum_{i=1}^8 e_i^2$ on $\mathbb{R}^8$. The associated Hodge $*$-operator gives a five-form $*\rho_0$

$$\begin{aligned} *\rho_0 = e_{45678} &- \frac{1}{2}e_{238}(e_{47} - e_{56}) + \frac{1}{2}e_{138}(e_{46} + e_{57}) - \frac{1}{2}e_{128}(e_{45} - e_{67}) \\ &+ \frac{\sqrt{3}}{2}e_{123}(e_{45} + e_{67}). \end{aligned}$$

A $PSU(3)$-*structure* on an oriented eight-manifold Y is defined by a three-form $\rho \in \Omega^3(Y)$ which is linearly equivalent at each point to ρ_0. It determines a metric g and a four-form $*\rho$. With a slight abuse of terminology we say that a $PSU(3)$-structure is *harmonic* if the forms ρ and $*\rho$ are both closed.

Remark 4.46. The terminology harmonic has its origin in the compact setting, where harmonicity of ρ is equivalent to the closedness of the forms ρ and $*\rho$. Alternatively, one could follow [Puh10] and distinguish $PSU(3)$-structures by their intrinsic torsion. In that nomenclature, we are considering structures of type $\mathcal{W}_6$.

While a harmonic $PSU(3)$-structure need not be parallel, the condition is consistent with the so-called Rarita-Schwinger equations [Hit01]. More precisely this means that any compact harmonic $PSU(3)$-manifold Y carries a spinor valued one-form $\beta \in \Gamma(S^+ \otimes T^*)$ which lies pointwise in $\ker D \cap \ker d^*$, where $D\colon \Gamma(S^+ \otimes T^*) \to \Gamma(S^- \otimes T^*)$ is the Dirac operator with coefficients in the bundle of one-forms, and $d^*\colon \Gamma(S^+ \otimes T^*) \to \Gamma(S^+)$ is the covariant operator on one-forms with coefficients in the spinor bundle. △

Since a harmonic $PSU(3)$-structure comes equipped with a closed three-form, we may study multi-moment maps for such geometries. Let us assume that (Y, ρ) has a two-torus symmetry with a non-constant multi-moment map $\nu\colon Y \to \mathcal{P}^*_{\mathbb{R}^2} \cong \mathbb{R}$. Consider an open neighbourhood $Y_0 \subset Y$ on which T^2 acts freely. Let us then define three two-forms on Y_0 by

$$\omega_0 = (d\nu)^\sharp \lrcorner V \lrcorner U \lrcorner *\rho, \quad \omega_1 = U \lrcorner \rho, \quad \omega_2 = V \lrcorner \rho.$$

To relate these to the $PSU(3)$-structure we introduce two one-forms θ_1, θ_2 and an additional two-form ω_3 as follows. First consider the isomorphism $g^{-1}\colon \Lambda^2 T^*Y_0 \to \Lambda^2 TY_0$ induced by $\sharp\colon T^*Y_0 \to TY_0$. We use this to define $\alpha \in \Omega^1(Y_0)$ given by

$$\alpha = (g^{-1}\omega_0) \lrcorner \rho. \tag{4.23}$$

Also consider the function h defined via the relation $(g_{UU}g_{VV} - g_{UV}^2)h^2 = 1$, where $g_{UU} = g(U, U)$ etc. Now the three forms $\theta_1, \theta_2, \omega_3$ may be expressed as

$$\theta_1 = h^2(g_{VV}U^\flat - g_{UV}V^\flat), \quad \theta_2 = h^2(g_{UU}V^\flat - g_{UV}U^\flat),$$
$$\omega_3 = U \lrcorner V \lrcorner \alpha^\sharp \lrcorner *\rho,$$

where $U^\flat = g(U, \cdot)$ etc. Note that (θ_1, θ_2) is dual to (U, V).

Proposition 4.47. *On Y_0, the three-form ρ and the four-form $*\rho$ are*

$$\rho = d\nu \wedge \theta_1 \wedge \theta_2 + \frac{4}{3}h^4\omega_0 \wedge \alpha + \omega_1 \wedge \theta_1 + \omega_2 \wedge \theta_2 + \frac{4}{3}h^4\omega_3 \wedge d\nu,$$

$$\begin{aligned} *\rho = {}& \frac{8h^2}{9}\omega_0^2 \wedge \alpha + h^4\omega_0 \wedge d\nu \wedge \theta_1 \wedge \theta_2 + \frac{16h^4}{81}\omega_3 \wedge \alpha \wedge \theta_2 \wedge \theta_1 \\ &+ \frac{4}{3}h^4\left(g_{VV}\omega_1 \wedge \theta_2 \wedge \alpha \wedge d\nu - g_{UU}\omega_2 \wedge \theta_1 \wedge \alpha \wedge d\nu\right) \\ &+ \frac{4}{3}h^4 g_{UV}\left(\omega_1 \wedge \theta_2 - \omega_2 \wedge \theta_1\right) \wedge \alpha \wedge d\nu. \end{aligned}$$

Proof. Working locally at a point and using the T^2-action we may write the first two standard basis elements of $\mathbb{R}^8$ as $E_1 = aU = U/g_{UU}^{1/2}$, $E_2 = bU + cV = hg_{UU}^{1/2}(V - g_{UV}g_{UU}^{-1}U)$. We then have $\theta_1 = ae_1 + be_2$ and $\theta_2 = ce_2$. Now using

(4.22) we get $acd\nu = e_3$ and

$$\omega_0 = \frac{\sqrt{3}}{2(ac)^2}(e_{45} + e_{67}), \quad \omega_3 = \frac{1}{ac}\left(e_{12} + \frac{1}{2}(e_{45} - e_{67})\right),$$
$$\omega_1 = \frac{1}{a}\left(e_{23} + \frac{1}{2}(e_{47} - e_{56})\right),$$
$$\omega_2 = \frac{1}{c}\left(-e_{13} + \frac{1}{2}(e_{46} + e_{57})\right) - \frac{b}{ac}\left(e_{23} + \frac{1}{2}(e_{47} - e_{56})\right).$$

From the expression for ω_0, we find that α is given by $(\frac{\sqrt{3}}{2ac})^2 e_8$.

The given expressions now follow. □

Suppose that $t \in \nu(Y_0)$ is a regular value for $\nu\colon Y_0 \to \mathbb{R}$. Then $\nu^{-1}(t)$ is a smooth hypersurface with unit normal $N = h(d\nu)^\sharp$. Assuming that T^2 acts freely on $\nu^{-1}(t)$, the T^2-reduction of Y at level t is defined to be the five-manifold

$$M = \nu^{-1}(t)/T^2.$$

If we let ι denote the inclusion $\nu^{-1}(t) \hookrightarrow Y_0$, then the forms $\iota^*\omega_i$, $i = 0,1,2,3$, and $\iota^*\alpha$ on $\nu^{-1}(t)$ are horizontal and therefore pull-backs of forms σ_i and a on M. We can orthogonalise the triple $(\sigma_1, \sigma_2, \sigma_3)$ to get three forms $\hat\sigma_i$ that satisfy the relations

$$\hat\sigma_i \wedge \hat\sigma_j = \delta_{ij}\hat\sigma_k^2, \quad \hat\sigma_i^2 \wedge a \neq 0,$$
$$X \lrcorner \hat\sigma_1 = Y \lrcorner \hat\sigma_2 \Rightarrow \hat\sigma_3(X,Y) \geqslant 0.$$

The proof of Proposition 4.47 shows that

$$\hat\sigma_1 = \|U\|^{-1}\sigma_1, \quad \hat\sigma_2 = -\frac{\langle U, V\rangle}{\|U\|}h\sigma_1 + \|U\|h\sigma_2, \quad \hat\sigma_3 = h\sigma_3.$$

The quadruple $(a, \hat\sigma_1, \hat\sigma_2, \hat\sigma_3)$ determines an $SU(2)$-reduction of the principal frame bundle of M, cf. [CS07, Proposition 1.1]. In addition, the T^2-reduction carries an induced three-form.

Proposition 4.48. *Let (Y, ρ) be a harmonic $PSU(3)$-manifold with a free T^2-symmetry and admitting a non-constant multi-moment map. Then the T^2-reduction at level t carries an induced three-form $\widetilde{c} = \frac{4}{3}h\sigma_0 \wedge a$. If $(M, \widetilde{c})$ is a strong geometry, then it is 2-plectic.* □

Example 4.49. Starting from a hyperKähler four-manifold $(\mathcal{X} = \mathcal{U} \times \mathbb{R}, k)$ with a circle action, Witt gave a local construction of a harmonic $PSU(3)$-manifold [Wit08]. His starting point is the the Gibbons-Hawking ansatz [GH78]. So let us consider the flat metric $\hat{k} = dx^2 + dy^2 + dz^2$ on an open set $\mathcal{U}$ of $\mathbb{R}^3$. Let $V > 0$ be a harmonic function on $\mathcal{U}$ such that $dV = *_{\hat{k}}\eta$ for a one-form $\eta \in \Omega^1(\mathcal{U})$. Then we have the hyperKähler metric

$$k = V^{-1}(d\tau + \eta)^2 + V(dx^2 + dy^2 + dz^2),$$

which admits circle symmetry generated by $\partial/\partial\tau$. The associated hyperKähler triple of symplectic forms on $\mathcal{U} \times \mathbb{R}$ is given by

$$\sigma_1 = V dy \wedge dz + dx \wedge (d\tau + \eta), \quad \sigma_2 = V dx \wedge dy + dz \wedge (d\tau + \eta),$$
$$\sigma_3 = V dx \wedge dz - dy \wedge (d\tau + \eta).$$

We define a fourth two-form by $\sigma_0 = V dx \wedge dz + dy \wedge (d\tau + \eta)$. This form is closed provided that V is independent of the variable y. For such V we proceed by choosing standard coordinates x_1, x_2, x_3, x_4 on Euclidean space $\mathbb{R}^4$. On the product $\mathcal{X} \times \mathbb{R}^4$ we then obtain an orientation and a metric by declaring

$$e_1 = dx_1, \quad e_2 = dx_2, \quad e_3 = dx_3, \quad e_8 = dx_4, \quad e_4 = V^{1/2} dy,$$
$$e_5 = -V^{-1/2}(d\tau + \eta), \quad e_6 = -V^{1/2} dx, \quad e_7 = V^{1/2} dz,$$

to be an oriented orthonormal coframe.

With these definitions the three-form

$$\rho = e_{123} + \frac{\sqrt{3}}{2} e_8 \wedge \sigma_0 + \frac{1}{2} e_1 \wedge \sigma_1 + \frac{1}{2} e_2 \wedge \sigma_2 + \frac{1}{2} e_3 \wedge \sigma_3 \tag{4.24}$$

defines a harmonic $PSU(3)$-structure on $\mathcal{X} \times \mathbb{R}^4$, and the $PSU(3)$-structure descends to $Y = \mathcal{X} \times T^2 \times \mathbb{R}^2$. On Y there is a natural choice of T^2 acting isometrically by translating the coordinates of the two-torus. In this case a multi-moment map $\nu \colon Y \to \mathbb{R}$ is given by the invariant function $\nu = x_3$.

The T^2 reduction at level $x_3 = t$ is $M \cong \mathcal{X} \times \mathbb{R}$, where the $\mathbb{R}$ factor is parametrised by the coordinate function x_4. The induced three-form $\widetilde{c} = \frac{\sqrt{3}}{2} e_8 \wedge \sigma_0$ is obviously closed, so $(M, \widetilde{c})$ is a 2-plectic, by Proposition 4.48. $\diamondsuit$

In [Wit08] we find further examples of harmonic $PSU(3)$-manifolds that are strongly reducible. For instance let N be the six-dimensional nilpotent Lie group with corresponding Lie algebra $\mathfrak{n} = (0, 0, 0, 0, 0, 23 + 34)$. Then Witt endows the product $T^2 \times N$ with a harmonic $PSU(3)$-structure which is strongly reducible. In that case the one-form $a \in \Omega^1(M)$ is a contact form, meaning that $a \wedge da$ is an orientation form on M.

Other reductions In Chapter 6 we describe a reduction procedure which differs substantially from the above discussion. While the reduced space is still the quotient of a level set of a multi-moment map by a free group action, the resulting manifold is no longer a strong geometry, but rather a particular type of tri-symplectic manifold.

4.5 Classification & further examples of (2,3)-trivial Lie algebras

While Section 4.3 gave a detailed description of the structural aspects of $(2, 3)$-trivial Lie algebras, we now aim to give a more thorough treatment of related classification problems.

4.5.1 Positive gradings of nilpotent algebras

As we have already seen, the relevance of positive gradings in relation to multi-moment maps is that Lie algebras with such a grading generate $(2,3)$-trivial algebras. A grading of an n-dimensional Lie algebra $\mathfrak{k}$ may be specified in terms of n positive integers, referred to as weights, see Example 4.52. One easily verifies:

Proposition 4.50. *Any nilpotent Lie algebra of dimension at most six admits a positive grading. From dimension seven and above, there are nilpotent Lie algebras which do not admit a positive grading.* □

Indeed, the nilpotent Lie algebras of dimension at most six and corresponding primitive positive gradings are listed in Table 4.12, and in Section 4.3 we gave examples of nilpotent algebras $\mathfrak{k}$ of dimension seven that can not arise as the derived algebra of any $(2,3)$-trivial algebra $\mathfrak{g}$. It follows, by Corollary 4.17, that such $\mathfrak{k}$ can not admit a positive grading.

Corollary 4.51. *Each of the* 50 *Lie algebras listed in Table 4.12 is the derived algebra of a completely solvable* $(2,3)$*-trivial Lie algebra.*

Proof. This is of an immediate consequence of Corollary 4.17, but let us give the details for completeness. Let $\mathfrak{g} = \langle A\rangle + \mathfrak{k}$, where $\mathfrak{k}$ is one of the algebras of Table 4.12 and ad_A acts as multiplication by i on $\mathfrak{k}_i$. Then $\mathfrak{g}$ is a solvable algebra. Moreover $(\Lambda^s\,\mathfrak{k})^{\mathfrak{g}} = \{0\}$ for $s \geqslant 1$, so that $H^1(\mathfrak{k})^{\mathfrak{g}} = \{0\} = H^2(\mathfrak{k})^{\mathfrak{g}} = H^3(\mathfrak{k})^{\mathfrak{g}}$. Thus, by Theorem 4.16, $\mathfrak{g}$ is $(2,3)$-trivial. Since ad_X has real eigenvalues for each $X \in \mathfrak{g}$ the Lie algebra is completely solvable. □

Example 4.52. Consider the nilpotent Lie algebra $\mathfrak{k} = (0^2, 12, 13, 14{+}23, 24{+}15) = (0, 0, 12, 13, 14{+}23, 24{+}15)$, which has a basis $E_1, \dots, E_6$ such that the corresponding dual basis $e_1, \dots, e_6$ for $\mathfrak{k}^*$ satisfies

$$de_1 = 0 = de_2, \quad de_3 = e_1 \wedge e_2, \quad \dots \quad , \quad de_6 = e_2 \wedge e_4 + e_1 \wedge e_5.$$

An assignment of weights is deduced from these structural equations, rephrased in terms of the derivative $d\colon\ \mathfrak{k}^* \to \Lambda^2\,\mathfrak{k}^*$. The weight assignment can be formulated, somewhat informally, as

$$e_1 \to a, \quad e_2 \to b, \quad e_3 \to a+b,$$
$$e_4 \to 2a+b,\ e_5 \to 3a+b = a+2b,\ e_6 \to 2(a+b) = 2(a+b),$$

meaning that E_1 gets weight a, E_2 weight b, and so forth. Consistency now requires that $2a = b$, and a grading may be defined by

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \dots \oplus \mathfrak{k}_6, \qquad \text{where} \qquad \mathfrak{k}_i = \langle e_i\rangle.$$

We now choose $a = 1$, and the corresponding weight decomposition is denoted by 123456. Following the proof of Corollary 4.51, we can use this grading to obtain a $(2,3)$-trivial Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}' = \mathfrak{k}$. Explicitly

$$\mathfrak{g} = (0, 12, 2.13, 3.14{+}23, 4.15{+}24, 5.16{+}25{+}34, 6.17{+}24{+}26).$$

♢

Structure	Grading
(0)	1
(0^2)	11
(0^3)	111
$(0^2, 12)$	112
(0^4)	1111
$(0^3, 12)$	1112
$(0^2, 12, 13)$	1123
(0^5)	11111
$(0^4, 12), (0^4, 12{+}34)$	11112
$(0^3, 12, 13)$	11122
$(0^3, 12, 14)$	11123
$(0^3, 12, 13{+}24)$	11223
$(0^2, 12, 13, 23)$	11233
$(0^2, 12, 13, 14)$	11234
$(0^2, 12, 13, 14{+}23)$	12345
(0^6)	111111
$(0^5, 12), (0^5, 12 + 34)$	111112

Structure	Grading
$(0^4, 12, 13), (0^4, 13{+}42, 14{+}23)$, $(0^4, 12, 34), (0^4, 12, 14{+}23)$	111122
$(0^4, 12, 15)$	111123
$(0^3, 12, 13, 23)$	111222
$(0^4, 12, 14{+}25), (0^4, 12, 15{+}34)$, $(0^3, 12, 13, 14), (0^3, 12, 23, 14{\pm}35)$, $(0^3, 12, 13, 24), (0^3, 12, 13, 14{+}35)$	111223
$(0^3, 12, 14, 24)$	111233
$(0^3, 12, 14, 15)$	111234
$(0^3, 12, 13{+}14, 24), (0^3, 12, 13{+}42, 14{+}23)$ $(0^3, 12, 13, 14{+}23), (0^3, 12, 14, 13{+}42)$	112233
$(0^3, 12, 14{-}23, 15{+}34)$	112234
$(0^2, 12, 13, 23, 14{\pm}25), (0^2, 12, 13, 23, 14)$	112334
$(0^2, 12, 13, 14, 15), (0^2, 12, 13, 14, 34{+}52)$	112345
$(0^3, 12, 14, 15{+}23)$	113234
$(0^3, 12, 14, 15{+}24)$	121345
$(0^3, 12, 14, 15{+}23{+}24)$	123345
$(0^2, 12, 13, 14 + 23, 24 + 15)$	123456
$(0^2, 12, 13, 14{+}23, 34{+}52)$	123457
$(0^2, 12, 13, 14, 23{+}15)$	134567

Table 4.12: Positive gradings of nilpotent Lie algebras of dimension $\leqslant 6$. Algebras are ordered according to their dimension and a primitive positive grading.

4.5.2 Classification & families of (2,3)-trivial algebras

While the method of positive gradings provides an effective tool for constructing examples of $(2,3)$-trivial algebras, it is inadequate if one aims for a general understanding of the $(2,3)$-trivial class. Therefore we now turn to give a classification of such algebras in dimensions up to and including five.

As we have already pointed out earlier in this chapter, the only Lie algebra in dimension one is Abelian and is obviously $(2,3)$-trivial. In dimension two a Lie algebra is either Abelian or isomorphic to the $(2,3)$-trivial algebra $(0,21)$. These lowest dimensional examples are uninteresting from the point of view of multi-moment maps since they have $\mathcal{P}_{\mathfrak{g}} = \{0\}$. In next dimensions we have:

Proposition 4.53. *The* $(2,3)$*-trivial Lie algebras in dimensions three, four and five are listed in the tables 4.13 and 4.14.*

We now give a proof of Proposition 4.53; our analysis in the five-dimensional case is greatly facilitated by the work of Mubarakzjanov [Mub63]. Note that we do not fully discuss inequivalence of the obtained algebras; imposing inequivalence would put further restrictions on the parameters, see for instance the tables 4.1 and 4.2.

$\mathfrak{r}_3$	$(0,21{+}31,31)$	
$\mathfrak{r}_{3,\lambda}$	$(0,21,\lambda.31)$	$\lambda \neq -1,0$
$\mathfrak{r}'_{3,\lambda}$	$(0,\lambda.21{+}31,-21{+}\lambda.31)$	$\lambda \neq 0$
$\mathfrak{r}_4$	$(0,21{+}31,31{+}41,41)$	
$\mathfrak{r}_{4,\lambda}$	$(0,21,\lambda.31{+}41,\lambda.41)$	$\lambda \neq -1,-\frac{1}{2},0$
$\mathfrak{r}_{4,\lambda(2)}$	$(0,21,\lambda_1.31,\lambda_2.41)$	$\lambda_i, \lambda_1{+}\lambda_2 \neq -1,0$
$\mathfrak{r}'_{4,\lambda(2)}$	$(0,\lambda_1.21,\lambda_2.31{+}41,-31{+}\lambda_2.41)$	$\lambda_1 \neq 0, \lambda_2 \neq -\frac{\lambda_1}{2},0$
$\mathfrak{d}_{4,\lambda}$	$(0,21,\lambda.31,(1{+}\lambda).41{+}32)$	$\lambda \neq -2,-1,-\frac{1}{2},0$
$\mathfrak{d}'_{4,\lambda}$	$(0,\lambda.21{+}31,-21{+}\lambda.31,2\lambda.41{+}32)$	$\lambda \neq 0$
$\mathfrak{h}_4$	$(0,21{+}31,31,2.41{+}32)$	

Table 4.13: The three- and four-dimensional $(2,3)$-trivial Lie algebras. Note that the above labeling of $\mathfrak{d}_{4,\lambda}$ differs from that in [ABDO05, Theorem 1.5], which we used in Chapter 3 and Table 4.2.

Before going into a detailed case-by-case study, let us give an overview of the overall strategy. Our starting point is Theorem 4.16. Hence we consider a Lie algebra $\mathfrak{g}$ of the form $\mathfrak{g} = \mathbb{R}A + \mathfrak{k}$, where $\mathfrak{k} = \mathfrak{g}'$ is nilpotent. The element A acts on $\mathfrak{k}$ via a linear endomorphism $\mathcal{A}$, and the corresponding action on $\mathfrak{k}^*$ is given by the transpose $\mathcal{A}^*$; concretely note that, in accordance with Remark 4.13, $\mathrm{ad}^*_A(\alpha)(X) = \alpha(\mathrm{ad}_A(X)) = \alpha([A,X]) = -d\alpha(A \wedge X)$, so that $\mathcal{A}^*(\alpha) = -A \lrcorner\, d\alpha$, for $\alpha \in \mathfrak{k}^*$. From $\mathcal{A}^*$ we can calculate the induced action on the cohomology groups $H^i(\mathfrak{k})$, which can be expressed in terms of linear endomorphisms $\mathcal{A}^i \in \mathrm{End}(H^i(\mathfrak{k}))$. We note that the induced action of $\mathfrak{g}$ on

$\mathfrak{r}_5$	$(0, 21{+}31, 31{+}41, 41{+}51, 51)$	
$\mathfrak{r}_{5(1),\lambda}$	$(0, 21, \lambda.31{+}41, \lambda.41{+}51, \lambda.51)$	$\lambda \neq -1, -\frac{1}{2}, 0$
$\mathfrak{r}_{5(2),\lambda}$	$(0, 21{+}31, 31, \lambda.41{+}51, \lambda.51)$	$-\lambda \neq 2, 1, \frac{1}{2}, 0$
$\mathfrak{r}_{5,\lambda(2)}$	$(0, 21, \lambda_1.31, \lambda_2.41{+}51, \lambda_2.51)$	$\lambda_i \neq -1, 0$; $\lambda_1{+}\lambda_2 \neq 0, -1$; $1{+}2\lambda_2, \lambda_1{+}2\lambda_2 \neq 0$
$\mathfrak{r}_{5,\lambda(3)}$	$(0, 21, \lambda_1.31, \lambda_2.41, \lambda_3.51)$	$\lambda_i \neq -1, 0$; $\lambda_1{+}\lambda_2{+}\lambda_3 \neq 0$; $\lambda_i{+}\lambda_j \neq -1, 0\ (i \neq j)$
$\mathfrak{r}'_{5,\lambda(2)}$	$(0, \lambda_1.21{+}31, \lambda_1.31, \lambda_2.41{+}51, -41{+}\lambda_2.51)$	$\lambda_i, \lambda_1{+}2\lambda_2 \neq 0$
$\mathfrak{r}'_{5,\lambda(3)}$	$(0, \lambda_1.21, \lambda_2.31, \lambda_3.41{+}51, -41{+}\lambda_3.51)$	$\lambda_i{\neq}0; \lambda_1{\neq}{-}\lambda_2$; $\lambda_1, \lambda_2{\neq}{-}2\lambda_3$
$\mathfrak{r}''_{5,\lambda}$	$(0, \lambda.21{+}31{+}41, -21{+}\lambda.31{+}51, \lambda.41{+}51, -41{+}\lambda.51)$	$\lambda \neq 0$
$\mathfrak{r}''_{5,\lambda(3)}$	$(0, \lambda_1.21{+}31, -21{+}\lambda_1.31, \lambda_2.41{+}\lambda_3.51, -\lambda_3.41{+}\lambda_2.51)$	$\lambda_i \neq 0$
$\mathfrak{d}_{5(1)}$	$(0, 21, 21{+}31, 31{+}41, 2.51{+}32)$	
$\mathfrak{d}^{\pm}_{5(2)}$	$(0, 21, 21{+}31, 2.41, 2.51 \pm 41{+}32)$	
$\mathfrak{d}_{5(1),\lambda}$	$(0, 21, \lambda.31, (1{+}\lambda).41, (1{+}\lambda).51{+}32{+}41)$	$-\lambda \neq 2, \frac{3}{2}, 1, \frac{2}{3}, \frac{1}{2}, 0$
$\mathfrak{d}_{5(2),\lambda}$	$(0, 21, 21{+}31, \lambda.41, 2.51{+}32)$	$\lambda \neq -3, -1, 0$
$\mathfrak{d}_{5,\lambda(2)}$	$(0, 21, \lambda_1.31, \lambda_2.41, (1{+}\lambda_1).51{+}32)$	$-\lambda_1 \neq 2, \frac{1}{2}, 1, 0$; $\lambda_2 \neq 0, -1$; $\lambda_1{+}\lambda_2 \neq -2, 0$; $\lambda_2{+}2\lambda_1 \neq -1$
$\mathfrak{d}_{5(3),\lambda}$	$(0, \lambda.21, 31, 31{+}41, (1{+}\lambda).51{+}32)$	$-\lambda \neq 3, 2, 1, \frac{1}{2}, 0$
$\mathfrak{d}'^{\pm}_{5,\lambda}$	$(0, \lambda.21{+}31, -21{+}\lambda.31, 2\lambda.41, 2\lambda.51 \pm 41{+}32)$	$\lambda \neq 0$
$\mathfrak{d}'_{5,\lambda(2)}$	$(0, \lambda_1.21{+}31, -21{+}\lambda_1.31, \lambda_2.41, 2\lambda_1.51)$	$\lambda_1, \lambda_2 \neq 0$
$\mathfrak{p}_5$	$(0, 21, 21{+}31, 2.41{+}32, 3.51{+}42)$	
$\mathfrak{p}_{5,\lambda}$	$(0, 21, \lambda.31, (1{+}\lambda).41{+}32, (2{+}\lambda).51{+}42)$	$-\lambda \neq 3, 2, 1, \frac{1}{2}, 0$

Table 4.14: The five-dimensional $(2,3)$-trivial Lie algebras.

$H^i(\mathfrak{k})$ has no non-trivial fixed points, meaning $H^i(\mathfrak{k})^{\mathfrak{g}} = \{0\}$, if and only if $\mathcal{A}^i$ has trivial kernel. In particular, $(2,3)$-triviality of $\mathfrak{g}$ is equivalent to the non-vanishing of $a_i = \det(\mathcal{A}^i)$, for $i = 1,2,3$.

Based on the above observations, we apply the following classification strategy. In dimension $n = 3,4,5$, we list the $(n-1)$-dimensional nilpotent Lie algebras $\mathfrak{k}$. For each of these, we find all possible derivations of $\mathfrak{k}$ expressed in terms of matrices A put on Jordan normal form. This is a simple task when $n \leqslant 4$ or $\mathfrak{k}$ is Abelian, but for the non-Abelian cases $\mathfrak{k}$ with $\dim \mathfrak{k} = 4$ some work is required. We adapt the ideas used in [Mub63]. Each matrix A gives us a solvable Lie algebra. In order to distinguish the $(2,3)$-trivial algebras, we use the transpose of A to calculate $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, and then single out the cases for which the determinants a_1, a_2 and a_3 are non-zero.

Dimension three Let $\mathfrak{g}$ be a $(2,3)$-trivial algebra of dimension three. Then $\mathfrak{k}$ is nilpotent and two-dimensional, so $\mathfrak{k} \cong \mathbb{R}^2$. The transpose A^* acts on $H^1(\mathbb{R}^2) \cong \mathbb{R}^2$ invertibly and the induced action on $H^2(\mathbb{R}^2) \cong \Lambda^2\mathbb{R}^2 \cong \mathbb{R}$ is also invertible. So either A is diagonalisable over $\mathbb{C}$ with non-zero eigenvalues whose sum is non-zero, giving cases $\mathfrak{r}_{3,\lambda\neq-1,0}$ and $\mathfrak{r}'_{3,\lambda\neq0}$, or A acts with Jordan normal form

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \lambda \neq 0,$$

giving case $\mathfrak{r}_3$.

Dimension four For $\mathfrak{g}$ of dimension four we have $\mathfrak{k} \cong \mathbb{R}^3$ or the Heisenberg algebra $\mathfrak{h}_3 = (0^2, 21)$.

Case $\mathfrak{k} \cong \mathbb{R}^3$ In this case we obtain the algebras from the $\mathfrak{r}$- and $\mathfrak{r}'$-series. Any linear endomorphism gives a derivations of $\mathbb{R}^3$, and therefore the relevant list of extensions of $\mathbb{R}^3$ may be obtained by considering invertible 3×3 matrices in normal form:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & -1 & \lambda_2 \end{pmatrix}.$$

The element A_1 gives the family $\mathfrak{r}_{4,\lambda(2)}$, and the induced action on $H^1(\mathbb{R}^3) \cong \mathbb{R}^3$ is, up to sign, given by multiplication by the transposed matrix A_1^*. Using this, we obtain the induced actions on $H^2(\mathbb{R}^3) \cong \mathbb{R}^3$ and $H^3(\mathbb{R}^3) \cong \mathbb{R}$ and then deduce that $(2,3)$-triviality holds if and only if the determinants

$$a_1 = \lambda_1\lambda_2, \quad a_2 = (1+\lambda_1)(1+\lambda_2)(\lambda_1+\lambda_2),$$
$$a_3 = 1 + \lambda_1 + \lambda_2$$

do not vanish.

The matrix A_2 gives us the algebra $\mathfrak{r}_{4,\lambda}$. In this case we have determinants

$$a_1 = \lambda^2, \quad a_2 = 2\lambda(1+\lambda)^2, \quad a_3 = 1+2\lambda,$$

which give the restrictions on parameters displayed in Table 4.13.

The algebra $\mathfrak{r}_4$ corresponds to the choice A_3.

Finally, A_4 occurs when the action has two complex eigenvalues. The corresponding family is $\mathfrak{r}'_{4,\lambda(2)}$, where λ_1, λ_2 are restricted by the condition $a_i \neq 0$ for

$$a_1 = \lambda_1(1+\lambda_2^2), \quad a_2 = 2\lambda_2(1+(\lambda_1+\lambda_2)^2),$$
$$a_3 = \lambda_1 + 2\lambda_2.$$

Case $\mathfrak{k} \cong \mathfrak{h}_3$ The Heisenberg algebra $\mathfrak{h}_3$ has $H^1(\mathfrak{h}_3) \cong \langle e_1, e_2\rangle$, $H^2(\mathfrak{h}_3) \cong \langle e_{13}, e_{23}\rangle$, $H^3(\mathfrak{h}_3) \cong \langle e_{123}\rangle$. The action of A, being a derivation, is represented by a matrix of the form

$$\begin{pmatrix} B & 0 \\ \underline{b} & \operatorname{Tr} B \end{pmatrix}, \quad B \in M(2,\mathbb{R}),\ \underline{b} = (b_1, b_2) \in \mathbb{R}^2.$$

To see this, write $\operatorname{ad}_A(E_i) = \sum_{k=1}^3 b_i^k E_k$, for $i = 1,2,3$, and consider the relation

$$\operatorname{ad}_A(E_3) = \operatorname{ad}_A[E_1, E_2] = [\operatorname{ad}_A(E_1), E_2] + [E_1, \operatorname{ad}_A(E_2)].$$

After the transformation

$$A \mapsto A - b_2 E_1 + b_1 E_2$$

we may furthermore assume $\underline{b} = 0$. Hence the algebras are distinguished by the normal form of B.

The family $\mathfrak{d}_{4,\lambda}$ arises when $B = \operatorname{diag}(1,\lambda)$. The restrictions on λ now follow from the requirement that the determinants

$$a_1 = \lambda, \quad a_2 = (2+\lambda)(1+2\lambda), \quad a_3 = 2(1+\lambda)$$

should be non-zero.

If the matrix B takes the form

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

we have the algebra $\mathfrak{h}_4$.

Finally the action may have complex eigenvalues. Then we have

$$B = \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix},$$

which corresponds to the family $\mathfrak{d}'_{4,\lambda}$. We find determinants

$$a_1 = 1+\lambda^2, \quad a_2 = 1+9\lambda^2, \quad a_3 = 4\lambda,$$

implying the condition $\lambda \neq 0$.

Dimension five A five-dimensional (2,3)-trivial Lie algebra has $\mathfrak{k} \cong \mathbb{R}^4$, $(0^3, 21)$ or $(0^2, 21, 31)$.

Case $\mathfrak{k} \cong \mathbb{R}^4$ In the Abelian case $H^1(\mathbb{R}^4) \cong \mathbb{R}^4$, $H^2(\mathbb{R}^4) \cong \mathbb{R}^6$, $H^3(\mathbb{R}^4) \cong \mathbb{R}^4$. The solvable extensions are found by taking invertible matrices with the normal forms:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$
$$A_5 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & -1 & \lambda_3 \end{pmatrix},$$
$$A_7 = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & -1 & \lambda_2 \end{pmatrix}, \quad A_8 = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ -1 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_3 \\ 0 & 0 & -\lambda_3 & \lambda_2 \end{pmatrix},$$
$$A_9 = \begin{pmatrix} \lambda & 1 & 1 & 0 \\ -1 & \lambda & 0 & 1 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & -1 & \lambda \end{pmatrix}.$$

The matrix A_1 gives the family $\mathfrak{r}_{5,\lambda(3)}$, and restrictions on the parameters λ_i now follow from non-vanishing of the determinants

$$a_1 = \lambda_1\lambda_2\lambda_3, \quad a_2 = \prod_i (1+\lambda_i) \prod_{i<j} (\lambda_i + \lambda_j),$$
$$a_3 = (\lambda_1 + \lambda_2 + \lambda_3) \prod_{i<j} (1 + \lambda_j + \lambda_k).$$

Algebras corresponding to A_2 belong to the family $\mathfrak{r}_{5,\lambda(2)}$. Now the determinants of the actions on cohomology groups are given by

$$a_1 = \lambda_1\lambda_2^2, \quad a_2 = 2\lambda_2(1+\lambda_1)(1+\lambda_2)^2(\lambda_1+\lambda_2)^2,$$
$$a_3 = (1+\lambda_1+\lambda_2)^2(1+2\lambda_2)(\lambda_1+2\lambda_2),$$

and we require that these are non-zero in order to get a (2,3)-trivial Lie algebra.

From A_3 we obtain the family $\mathfrak{r}_{5(2),\lambda}$. The parameter value is now constrained by non-vanishing of

$$a_1 = \lambda^2, \quad a_2 = 4\lambda(1+\lambda)^4, \quad a_3 = (1+2\lambda)^2(2+\lambda)^2.$$

The matrix A_4 gives us the family $\mathfrak{r}_{5(1),\lambda}$. In this case the parameter λ is constrained by $a_i \neq 0$ where

$$a_1 = \lambda^3, \quad a_2 = 8\lambda^3(1+\lambda)^3, \quad a_3 = 3\lambda(1+2\lambda)^3.$$

The algebra $\mathfrak{r}_5$ corresponds to the action of A_5.

Members of the $\mathfrak{r}'$- and $\mathfrak{r}''$-series occur when ad_A has two or four complex eigenvalues, respectively. The algebra $\mathfrak{r}'_{5,\lambda(3)}$ corresponds to A_6. In order to have invertible actions on the first three cohomology groups, the determinants a_1, a_2, a_3 must be non-zero. Here

$$\begin{aligned} a_1 &= \lambda_1\lambda_2(1+\lambda_3^2),\\ a_2 &= 2\lambda_3(\lambda_1+\lambda_2)(1+(\lambda_1+\lambda_3)^2)(1+(\lambda_2+\lambda_3)^2),\\ a_3 &= (\lambda_1+2\lambda_3)(\lambda_2+2\lambda_3)(1+(\lambda_1+\lambda_2+\lambda_3)^2). \end{aligned}$$

The form A_7 gives the family $\mathfrak{r}'_{5,\lambda(2)}$. In order to have invertible induced actions on the first three cohomology groups, λ_1 and λ_2 must be chosen such that the following determinants are not zero:

$$\begin{aligned} a_1 = \lambda_1^2(1+\lambda_2^2), \quad a_2 = 4\lambda_1\lambda_2(1+(\lambda_1+\lambda_2)^2)^2,\\ a_3 = (\lambda_1+2\lambda_2)^2(1+(2\lambda_1+\lambda_2)^2). \end{aligned}$$

The matrix A_8 has $\lambda_3 \neq 0$ and corresponds to the family $\mathfrak{r}''_{5,\lambda(3)}$. Further restrictions on the parameter values follow from requirement that the three determinants

$$\begin{aligned} a_1 &= (1+\lambda_1^2)(\lambda_2^2+\lambda_3^2),\\ a_2 &= 4\lambda_1\lambda_2((\lambda_1+\lambda_2)^2+(1+\lambda_3)^2)((\lambda_1+\lambda_2)^2+(1-\lambda_3)^2),\\ a_3 &= (\lambda_3^2+(2\lambda_1+\lambda_2)^2)(1+(\lambda_1+2\lambda_2)^2) \end{aligned}$$

should be non-zero.

Finally the choice A_9 corresponds to algebras belonging to the family $\mathfrak{r}''_{5,\lambda}$. Here invertibility of the induced action on cohomology requires that

$$a_1 = (1+\lambda^2)^2, \quad a_2 = 64\lambda^4(1+\lambda^2), \quad a_3 = (1+9\lambda^2)^2$$

are non-zero.

Case $\mathfrak{k} \cong (0^3, 21)$ In order to analyse the cases $(0^3,21)$ and $(0^2,21,31)$ we follow and modify arguments given in [Mub63]. We first consider $\mathfrak{k} \cong (0^3,21)$ which has $H^1(\mathfrak{k}) \cong \langle e_1, e_2, e_3\rangle$, $H^2(\mathfrak{k}) \cong \langle e_{13}, e_{14}, e_{23}, e_{24}\rangle$ and $H^3(\mathfrak{k}) \cong \langle e_{124}, e_{134}, e_{234}\rangle$. Write $A(E_i) = \sum_{k=1}^4 a_i^k E_k$ for $i = 1,2,3,4$. From the relations

$$\begin{aligned} \mathrm{ad}_A(E_4) &= [\mathrm{ad}_A(E_1), E_2] + [E_1, \mathrm{ad}_A(E_2)],\\ 0 = \mathrm{ad}_A[E_i, E_3] &= [\mathrm{ad}_A(E_i), E_3] + [E_i, \mathrm{ad}_A(E_3)] \quad i = 1,2, \end{aligned}$$

we deduce that

$$a_4^4 = a_1^1 + a_2^2$$
$$a_4^1 = 0 = a_4^2 = a_4^3 = a_3^2 = a_3^1.$$

After the transformation

$$A \mapsto A - a_2^4 E_1 + a_1^4 E_2$$

we can assume that $a_1^4 = a_2^4 = 0$. The restriction $B = (b_i^k)$ of ad_A to the subspace $\langle e_1, e_2, e_3 \rangle$ has $b_3^1 = 0 = b_3^2$, and may be put Jordan form via the transformations

$$E_1 \to aE_1 + bE_2 + cE_3, \quad E_2 \to pE_1 + qE_2 + rE_3, \quad E_3 \to sE_3,$$

where $aq - bp \neq 0$ and $s \neq 0$. Excluding degenerate matrices, we therefore obtain the following possibilities

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$
$$B_4 = \begin{pmatrix} \lambda_1 & 1 & 0 \\ -1 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad B_5 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Consider first the case

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

If $\lambda_2 \neq 1 + \lambda_1$ we may assume that $a_3^4 = 0$; it may be necessary to make a change of the form

$$E_3 \mapsto E_3 + \alpha E_4.$$

This gives us the family $\mathfrak{d}_{5,\lambda(2)}$. The determinants

$$a_1 = \lambda_1 \lambda_2, \quad a_2 = (1 + \lambda_2)(2 + \lambda_1)(\lambda_1 + \lambda_2)(1 + 2\lambda_1),$$
$$a_3 = 2(1 + \lambda_1)(2 + \lambda_2 + \lambda_1)(1 + 2\lambda_1 + \lambda_2)$$

must be non-zero in order to have a $(2,3)$-trivial algebra.

Turning next to the case $\lambda_2 = 1 + \lambda_1$, let us assume $a_3^4 \neq 0$; if this is not the case, we get a member of the family $\mathfrak{d}_{5,\lambda(2)}$. After rescaling

$$E_i \mapsto |a_3^4|^{1/2} E_i \quad i = 1, 2,$$
$$E_4 \mapsto |a_3^4| E_4,$$

we obtain the families $\mathfrak{d}^{\pm}_{5(1),\lambda}$ given by

$$(0, 21, \lambda.31, (1+\lambda).41, (1+\lambda).51 + 32 \pm 41).$$

If we scale (E_1, E_4, E_5) by the factors $(\lambda, \lambda^{-1}, -1)$ and interchange E_2 and E_3, then we find $\mathfrak{d}^+_{5(1),\lambda} \cong \mathfrak{d}^-_{5(1),1/\lambda}$. So, in fact, there is only one family $\mathfrak{d}_{5(1),\lambda} := \mathfrak{d}^+_{5(1),\lambda}$. In order to have an invertible induced action on first three cohomology groups, the parameter λ must be chosen appropriately, meaning as usual that the following determinants should be non-zero:

$$a_1 = \lambda(1+\lambda), \quad a_2 = (2+\lambda)^2(1+2\lambda)^2,$$
$$a_3 = 2(1+\lambda)(3+2\lambda)(2+3\lambda).$$

Now we turn to the algebra corresponding to the matrix B_2. We can assume that $a^4_3 = 0$, and thus obtain the $(2,3)$-trivial algebra $\mathfrak{d}_5$.

The algebra $\mathfrak{d}_{5(2),\lambda}$ corresponds to choosing the matrix B_3 with $a^4_3 = 0$. The following determinants

$$a_1 = \lambda, \quad a_2 = 9(1+\lambda)^2, \quad a_3 = 4(3+\lambda)^2$$

must be non-zero in order that we have a $(2,3)$-trivial algebra.

For B_3 with $a^4_3 \neq 0$ we obtain the algebra $\mathfrak{d}^\pm_{5(2)}$. To see this, we must rescale as follows

$$E_i \mapsto |a^4_3|^{1/2} E_i \quad i = 1, 2,$$
$$E_4 \mapsto |a^4_3| E_4.$$

From B_4 we obtain $\mathfrak{d}'_{5,\lambda(2)}$ when $a^4_3 = 0$. The $(2,3)$-triviality requirement, i.e., the condition that

$$a_1 = \lambda_2(1+\lambda_1^2),$$
$$a_2 = (1+9\lambda_1^2)(1+(\lambda_1+\lambda_2)^2),$$
$$a_3 = 4\lambda_1(1+(3\lambda_1+\lambda_2)^2)$$

are non-zero, enforces restrictions on the parameters λ_i.

When $a^4_3 \neq 0$ we find, after appropriate rescaling, that B_4 corresponds to the family $\mathfrak{d}'^\pm_{5,\lambda}$. An invertible action on the first three cohomology groups of $\mathfrak{k}$ requires that

$$a_1 = 2\lambda(1+\lambda^2), \quad a_2 = (1+9\lambda^2)^2, \quad a_3 = 4\lambda(1+25\lambda^2)$$

are non-zero.

In the case B_5 we must have $a^4_3 = 0$. Hence we obtain the family $\mathfrak{d}_{5(3),\lambda}$. The allowed values for λ are deduced from non-vanishing of the determinants

$$a_1 = \lambda, \quad a_2 = 2(1+\lambda)(1+2\lambda)(2+\lambda),$$
$$a_3 = 4(1+\lambda)^2(3+\lambda).$$

Case $\mathfrak{k} \cong (0^2, 21, 31)$ In the case $\mathfrak{k} \cong (0^2, 21, 31)$ we have $H^1(\mathfrak{k}) = \langle e_1, e_2 \rangle$, $H^2(\mathfrak{k}) \cong \langle e_{14}, e_{23} \rangle$, $H^3(\mathfrak{k}) \cong \langle e_{134}, e_{234} \rangle$. As above, write $A(E_i) = \sum_{k=1}^4 a_i^k E_k$. Now we consider the relations

$$\begin{aligned} 0 = \mathrm{ad}_A[E_2, E_3] &= [\mathrm{ad}_A(E_2), E_3] + [E_2, \mathrm{ad}_A(E_3)], \\ \mathrm{ad}_A(E_3) &= [\mathrm{ad}_A(E_1), E_2] + [E_1, \mathrm{ad}_A(E_2)], \\ \mathrm{ad}_A(E_4) &= [\mathrm{ad}_A(E_1), E_3] + [E_1, \mathrm{ad}_A(E_3)], \end{aligned}$$

and deduce that

$$a_2^1 = a_3^1 = a_4^1 = a_3^2 = a_4^2 = a_4^3 = 0, \quad a_2^3 = a_3^4,$$
$$a_4^4 = a_1^1 + a_3^3, \quad a_3^3 = a_1^1 + a_2^2.$$

After making the transformation

$$A \mapsto A - a_2^3 E_1 + a_1^3 e E_2 + a_1^4 E_3,$$

we may assume ad_A takes the form $\mathrm{diag}(p, q, p+q, 2p+q) + A'$, where A' only has non-zero entries $a'^2_1 = a^2_1$ and $a'^4_2 = a^4_2$, below the diagonal.

We then obtain $\mathfrak{p}_{5,\lambda}$ and $\mathfrak{p}_5$ as follows. As $\mathfrak{k} = \mathfrak{g}'$ one has $p \neq 0$ and we may rescale ad_A by $1/p$. If $q \neq p$ we make the transformation

$$E_1 \mapsto E_1 + a_1^2 E_2/(p-q).$$

After appropriate transformations,

$$E_1 \mapsto E_1 + aE_4, \quad E_2 \to E_2 + bE_4,$$

we obtain the algebra $\mathfrak{p}_{5,\lambda}$ with $\lambda = q/p$. For this family we calculate the three determinants to be

$$a_1 = \lambda, \quad a_2 = (1+2\lambda)(3+\lambda), \quad a_3 = 6(1+\lambda)(2+\lambda),$$

so that $a_i \neq 0$ enforces λ to be as specified in Table 4.14.

Consider finally the case $q = p$. Note that we may assume $a_1^2 \neq 0$; otherwise we get the algebra $\mathfrak{p}_{5,\lambda}$. After a suitable transformation of the form

$$E_i \mapsto a_1^2 E_i \quad i = 2, 3, 4,$$

followed by

$$E_2 \mapsto E_2 + cE_4,$$

we obtain the algebra $\mathfrak{p}_5$.

This concludes the proof of Proposition 4.53.

Unimodular The lists of $(2,3)$-trivial algebras in dimensions up to and including five reveal that algebraic properties of this class are not fully reflected in low-dimensional examples. In Corollary 4.25 we observed that the $(2,3)$-trivial Lie algebras of dimensions two, three and four are not unimodular. On the other hand there are infinitely many five-dimensional algebras with this property. The structure $\mathfrak{g} = \mathbb{R}A + \mathfrak{k}$ of a $(2,3)$-trivial algebra makes it easy to check unimodularity; it suffices to compute whether the homomorphism $\chi\colon \ \mathfrak{g} \to \mathbb{R}$ evaluated on A is zero. Direct inspection now gives

Corollary 4.54. *The unimodular $(2,3)$-trivial Lie algebras of dimension up to and including five are*

$$\mathbb{R},\ \mathfrak{r}_{5(1),-1/3},\ \mathfrak{r}_{5,\lambda,-(1+\lambda)/2},\ \mathfrak{r}_{5,\lambda,\mu,-(1+\lambda+\mu)},\ \mathfrak{r}'_{5,\lambda,-\lambda},\ \mathfrak{r}''_{5,\lambda,-\lambda,\mu},$$
$$\mathfrak{r}'_{5,\lambda,\mu,-(\lambda+\mu)/2},\ \mathfrak{d}_{5(2),-4},\ \mathfrak{d}_{5,\lambda,-2(1+\lambda)},\ \mathfrak{d}_{5(3),-3/2},\ \mathfrak{d}'_{5,\lambda,-4\lambda},\ \mathfrak{p}_{5,-4/3},$$

where parameters satisfy the conditions in Table 4.14. □

4.5.2.1 Further examples

The quest for higher-dimensional examples is easily met. Indeed, one may construct infinite families of $(2,3)$-trivial Lie algebras following the methods invoked in the proof of Proposition 4.53. In fact all the families appearing in dimension five have higher-dimensional generalisations, and some of these are listed in Table 4.16. Let us now explain how these examples are obtained, and remark that the underlying ideas apply more generally.

Method 1 The members of the $\mathfrak{r}$-series have $\mathfrak{k} \cong \mathbb{R}^{n-1}$ and the linear endomorphism representing ad_A is taken to be one of

$$J(n-1,1), \quad J(k-1,1) \oplus J(n-k,\lambda), \quad \mathrm{diag}(1,\lambda_1,\dots,\lambda_{k-1}) \oplus J(n-k-1,\lambda_k),$$

where $J(m,a)$ is an $m \times m$-Jordan block with a on the diagonal and 1 immediately above the diagonal.

The first matrix, $J(n-1,1)$ corresponds to the algebra $\mathfrak{r}_n$. The second matrix, $J(k-1,1) \oplus J(n-k,\lambda)$ corresponds to the family $\mathfrak{r}_{n(k-1),\lambda}$. Finally, the remaining matrix gives $\mathfrak{r}_{n,\lambda(k)}$. For the last two matrices, the requirement that A acts invertibly on the first three cohomology groups enforces some restrictions on parameters. As A acts on $H^1(\mathbb{R}^{n-1}) \cong \mathbb{R}^{n-1}$ by a lower triangular matrix, these restrictions are easy to find. We must have that the sum of one, two or three diagonal elements is non-zero.

The family $\mathfrak{d}_{n,\lambda(n-3)}$ has $\mathfrak{k} \cong (0^{n-2}, 21)$ and ad_A is

$$\mathrm{diag}(1,\lambda_1,\dots,\lambda_{n-3},1+\lambda_1).$$

Now A acts diagonally on $\mathfrak{k}^*$, and restrictions on parameters may therefore be read off directly from the cohomology groups

$$H^1(\mathfrak{k}) \cong \mathfrak{k}^* \ominus \langle e_{n-1}\rangle, \quad H^2(\mathfrak{k}) \cong \Lambda^2\,\mathfrak{k}^* \ominus \langle e_{12}, e_{i(n-1)}\colon\ i>2\rangle,$$
$$H^3(\mathfrak{k}) \cong \Lambda^3\,\mathfrak{k}^* \ominus \langle e_{12i}, e_{jk(n-1)}\colon\ 2<i<n-1, 2<j<k\rangle.$$

Method 2 An alternative way of constructing infinite families of $(2,3)$-trivial algebras goes via positive gradings of infinite families. We list some examples in Table 4.15.

$\mathfrak{f}_n^1$	$(0, 21, 31, 2.41 + 32, 3.51 + 42, \dots, (n-2).n1 + (n-1)2)$
$\mathfrak{f}_n^2$	$(0, 21, 2.31, 3.41 + 32, 4.51 + 42, 5.61 + 52 + 43, \dots,$ $(n-1).n1 + (n-1)2 + (n-2)3)$
$\mathfrak{f}_n^3$	$(0, 21, 31, 2.41 + 32, \dots, (n-3).(n-1)1 + (n-2)2,$ $(n-2).n1 + (n-1)2 - (n-1)3 + (n-2)3 - \cdots)$

Table 4.15: Infinite families of $(2,3)$-trivial Lie algebras obtained via positive gradings.

Note that

$$(\mathfrak{f}_n^1)' = (0^2, 21, \dots, (n-2)1)$$

has positive grading $1^2 2 \cdots (n-2)$. The derived algebra

$$(\mathfrak{f}_n^2)' = (0^2, 21, 31, 41 + 32, \dots, (n-2)1 + (n-3)2)$$

admits grading $12 \cdots (n-1)$. Finally the nilpotent algebra

$$(\mathfrak{f}_n^3)' = (0^2, 21, 31, \dots, (n-3)1, (n-2)1 - (n-2)2 + (n-3)3 - \dots - (-1)^k (k+1)k), \quad n = 2k+1,$$

has positive grading $1^2 23 \cdots (n-2)$.

Concluding remarks While the above exposition illustrates that $(2,3)$-trivial algebras form a plentiful subclass of the solvable ones, Theorem 4.16 ensures that the general structure of these algebras is fairly well understood. Moreover, we have already argued that $(2,3)$-trivial symmetry groups are particularly interesting objects in the context of strong geometry and multi-moment maps. In summary, $(2,3)$-trivial algebras deserve further attention in future studies of geometries with a closed three-form.

$\mathfrak{r}_n$	$(0, 21{+}31, 31{+}41, \dots, (n{-}1)1{+}n1, n1)$
$\mathfrak{r}_{n(k-1),\lambda}$	$(0, 21{+}31, \dots, (k{-}1)1 + k1, k1, \lambda.(k{+}1)1 + (k{+}2)1, \dots, \lambda.(n{-}1)1 + n1, \lambda.n1)$ with $k > 2$ and $-\lambda \neq 0, 1/2, 1, 2$
$\mathfrak{r}_{n,\lambda(k)}$	$(0, 21, \lambda_1.31, \dots, \lambda_{k-1}.(k{+}1)1, \lambda_k.(k{+}2)1 + (k{+}3)1, \dots, \lambda_k.(n{-}1)1 + n1, \lambda_k.n1)$ with $n > k+2$ and non-zero $\lambda_i, 1{+}\lambda_i, \lambda_i{+}\lambda_j, 1{+}2\lambda_k, \lambda_i{+}2\lambda_k,$ $1{+}\lambda_i{+}\lambda_j (i < j)$ and $\lambda_i{+}\lambda_j{+}\lambda_\ell (i < j < \ell)$
$\mathfrak{d}_{n,\lambda(n-3)}$	$(0, 21, \lambda_1.31, \dots, \lambda_{n-3}.(n{-}1)1, (1{+}\lambda_1).n1{+}32)$ with $\lambda_i \neq 0, -1$ for all i $\lambda_1 \neq -2, -1/2, -\lambda_i, -1/2(1+\lambda_i), -2{-}\lambda_i, -\lambda_i{-}\lambda_j$ for $1 < i, 1 < i < j$ and non-zero $\lambda_i + \lambda_j, 1{+}\lambda_i + \lambda_j (1 < i < j), \lambda_i{+}\lambda_j{+}\lambda_k (1 < i < j < k)$

Table 4.16: A selection of infinite families of $(2,3)$-trivial Lie algebras.

Chapter 5

Multi-moment maps for closed geometries

While Chapter 4 clearly illustrated that most interesting strong geometries carry additional structure (see also Chapter 6), it was still useful for us to focus on the closed three-form as being the essential building block. This leads us to ask whether there is a notion of multi-moment map which is valid for any closed geometry, that is a geometry characterised completely or partly by a closed form. In this chapter we answer this question affirmatively. First we generalise the notion of multi-moment maps to closed geometries in a way that subsumes the concepts in the symplectic and strong settings. We then establish existence results for these maps. Finally, some examples are considered. One of these gives an inverse of the Swann bundle construction in terms of a reduction procedure for multi-moment maps associated with quaternionic four-forms.

5.1 Definitions

Let (M, α) be a *closed geometry*, meaning that M is a smooth manifold and α is a closed $(k+1)$-form on M, for some $k \in \mathbb{N}$. Generally there is not one canonical form for α, neither do we require any non-degeneracy of α, though one could use the terminology of [BHR10] that α is k-plectic if $X \lrcorner \alpha = 0$ at $x \in M$ only when $X = 0$ in T_xM.

Remark 5.1. A k-plectic form $\alpha \in \Omega^{k+1}(M)$ defines pointwise an injective map $\Phi_\alpha \colon T_xM \to \Lambda^k T_x^*M$ given by $\Phi_\alpha(X) = X \lrcorner \alpha$. This map is surjective if and only if $k = 1$, meaning that α is symplectic. Important results in symplectic geometry, in particular the Darboux theorem, rely crucially on bijectivity of Φ_α, injectivity alone is inadequate. One [Mar88] may remedy this problem by considering a restricted class of k-plectic manifolds, see Section 5.3. △

Remark 5.2. For closed geometries with a four-form one could consider a notion of strong non-degeneracy, meaning that $\alpha(X, Y, Z, \cdot) \neq 0$ for all $X \wedge Y \wedge Z \neq 0$. However, such forms only exist in dimension 4 and 8, cf. [Fer86]. The former case is given by a volume form, the latter by a G-structure with $G = Spin(7)$ or its non-compact dual. △

Assume that G is a group of symmetries for (M, α). Thus for each $\mathsf{X} \in \mathfrak{g}$ we have $\mathcal{L}_X\alpha = 0$. We may then use Cartan's formula to show that the k-form $X \lrcorner \alpha$ is closed: $0 = \mathcal{L}_X\alpha = d(X \lrcorner \alpha) + X \lrcorner d\alpha = d(X \lrcorner \alpha)$. Consider now k elements $\mathsf{X}_1, \ldots, \mathsf{X}_k \in \mathfrak{g}$ that satisfy the following generalised commutation relation:

$$\sum_{1 \leqslant i<j \leqslant k} (-1)^{i+j} [\mathsf{X}_i, \mathsf{X}_j] \wedge \mathsf{X}_1 \wedge \ldots \wedge \widehat{\mathsf{X}_i} \wedge \ldots \wedge \widehat{\mathsf{X}_j} \wedge \ldots \wedge \mathsf{X}_k = 0. \tag{5.1}$$

Then the one-form $\beta := X_1 \wedge \ldots \wedge X_k \lrcorner \alpha$ is closed. Assume namely by induction that

$$(-1)^\ell d(X_1 \wedge \ldots \wedge X_\ell \lrcorner \alpha) = \sum_{1 \leqslant i<j \leqslant \ell} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge \widehat{X_j} \wedge \cdots \wedge X_\ell \lrcorner \alpha$$

for $2 \leqslant \ell < k$. The closedness of β then follows from the calculation

$$\begin{aligned}
(-1)^k d\beta &= (-1)^k \mathcal{L}_{X_k}(X_1 \wedge \ldots \wedge X_{k-1} \lrcorner \alpha) - (-1)^k X_k \lrcorner d(X_1 \wedge \ldots \wedge X_{k-1} \lrcorner \alpha) \\
&= \sum_{i=1}^{k-1} (-1)^{i+k} [X_i, X_k] \wedge X_1 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge X_{k-1} \lrcorner \alpha \\
&\quad + X_k \lrcorner \left(\sum_{1 \leqslant i<j \leqslant k-1} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge \widehat{X_j} \wedge \cdots \wedge X_\ell \lrcorner \alpha \right) \\
&= \sum_{1 \leqslant i<j \leqslant k} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge \widehat{X_j} \wedge \ldots \wedge X_k \lrcorner \alpha.
\end{aligned} \tag{5.2}$$

The set of decomposable elements of $\Lambda^k \mathfrak{g}$ that commute in the generalised sense of (5.1) is a complicated variety. We would therefore like to replace this set by an appropriate $\mathfrak{g}$-module. To this end let $L\colon \Lambda^k \mathfrak{g} \to \Lambda^{k-1} \mathfrak{g}$ denote the linear map dual to the exterior derivative $d\colon \Lambda^{k-1} \mathfrak{g}^* \to \Lambda^k \mathfrak{g}^*$. Then the kernel of L obviously includes all decomposable elements $\mathsf{X}_1 \wedge \cdots \wedge \mathsf{X}_k \in \Lambda^k \mathfrak{g}$ satisfying (5.1). This motivates the following notion.

Definition 5.3. The *kth Lie kernel* of a Lie algebra $\mathfrak{g}$ is the $\mathfrak{g}$-module

$$\mathcal{P}_{\mathfrak{g}} := \ker \left(L\colon \Lambda^k \mathfrak{g} \to \Lambda^{k-1} \mathfrak{g} \right).$$

The above calculations extend to elements of the *kth* Lie kernel. For a k-vector $p = \sum_{\ell=1}^r X_\ell^1 \wedge \ldots \wedge X_\ell^k$ we write

$$p \lrcorner \alpha := \sum_{\ell=1}^r \alpha(X_\ell^1, \ldots, X_\ell^k, \cdot).$$

Lemma 5.4. *Suppose G is a group of symmetries of a closed geometry (M,α). Let $\mathsf{p} = \sum_{\ell=1}^r \mathsf{X}_\ell^1 \wedge \ldots \wedge \mathsf{X}_\ell^k$ be an element of the Lie kernel $\mathcal{P}_\mathfrak{g}$ and let $p = \sum_{\ell=1}^r X_\ell^1 \wedge \ldots \wedge X_\ell^k$ be the corresponding k-vector on M. Then*

$$d(p \lrcorner \alpha) = 0. \tag{5.3}$$

Proof. The lemma is a direct consequence of the linearity of the extended fundamental calculation (5.2). If we write $\mathsf{p} = \sum_{\ell=1}^r \mathsf{X}_\ell^1 \wedge \ldots \wedge \mathsf{X}_\ell^k$, then the condition that p lies in $\mathcal{P}_\mathfrak{g}$ is that $0 = L(\mathsf{p}) = \sum_{\ell=1}^r \sum_{1\leqslant i<j\leqslant k} (-1)^{i+j} [\mathsf{X}_\ell^i, \mathsf{X}_\ell^j] \wedge \mathsf{X}_\ell^1 \wedge \ldots \wedge \widehat{\mathsf{X}_\ell^i} \wedge \ldots \wedge \widehat{\mathsf{X}_\ell^j} \wedge \ldots \mathsf{X}_\ell^k$. Now we define r one-forms by $\beta_\ell := X_\ell^1 \wedge \ldots \wedge X_\ell^k \lrcorner \alpha$, and then use the fundamental calculation (5.2) as follows:

$$\begin{aligned}(-1)^k d(p \lrcorner \alpha) &= (-1)^k d\left(\sum_{\ell=1}^r X_\ell^1 \wedge \ldots \wedge X_\ell^k \lrcorner \alpha\right) = (-1)^k \sum_{\ell=1}^r d\beta_\ell \\ &= \sum_{\ell=1}^r \sum_{1\leqslant i<j\leqslant k} (-1)^{i+j} [X_\ell^i, X_\ell^j] \wedge X_\ell^1 \wedge \ldots \wedge \widehat{X_\ell^i} \wedge \ldots \wedge \widehat{X_\ell^j} \wedge \ldots X_\ell^k \lrcorner \alpha.\end{aligned}$$

We see that $\mathsf{p} \in \mathcal{P}_\mathfrak{g}$ implies that $d(p \lrcorner \alpha) = 0$, as required. □

We are now able to define the notion of a multi-moment map for a closed geometry.

Definition 5.5. Let (M,α) be a closed geometry with a symmetry group G. A *multi-moment map* is an equivariant map $\nu\colon M \to \mathcal{P}_\mathfrak{g}^*$ satisfying

$$d\langle \nu, \mathsf{p}\rangle = p \lrcorner \alpha \tag{5.4}$$

for each $\mathsf{p} \in \mathcal{P}_\mathfrak{g}$.

Remark 5.6. For $k = 1,2$ we have that $\mathcal{P}_\mathfrak{g} = \mathfrak{g}$ and $\ker([\cdot,\cdot]\colon \Lambda^2\mathfrak{g} \to \mathfrak{g})$, respectively. Thus Definition 5.5 subsumes the notions of moment maps in the symplectic and strong settings. △

5.2 Existence and uniqueness

While many results from Chapter 4 generalise straightforwardly, it may still be illuminating, and a useful reference, to give precise formulations and proofs of the general existence results. We first address topological existence.

Theorem 5.7. *Let (M,α) be a closed geometry with a symmetry group G and assume that $b_1(M) = 0$. If either*

(i) *G is compact, or*

(ii) *M is compact and orientable, and G preserves a volume form on M,*

then there exists a multi-moment map $\nu\colon M \to \mathcal{P}_\mathfrak{g}^*$.

Proof. Working component by component, we may assume that M is connected. The condition $b_1(M) = 0$ ensures that there are functions ν_{p} with $d\nu_p = p \lrcorner c$ for each $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$. However, each of these functions may be adjusted by adding a real constant. To build a multi-moment map ν via $\langle \nu, \mathsf{p} \rangle = \nu_p$ we need to ensure equivariance. In the two cases above this may be achieved by either averaging over G or over M. In the second case, one chooses ν_p with mean value 0. In the first case, one chooses a basis (p_i) of $\mathcal{P}_{\mathfrak{g}}$ and puts $\nu(m) = \int_G \sum_i \mathrm{Ad}^*_{g^{-1}}(\nu_{p_i}(g^{-1} \cdot m)) \,\mathrm{vol}_G$. In both cases equation (5.4) is satisfied, and ν is a multi-moment map. □

Remark 5.8. Note that certain types of closed geometries, such as symplectic manifolds, come automatically with an invariant volume form vol_M. In such cases a multi-moment map exists provided that M is compact and has $b_1(M) = 0$. △

A geometric existence criterion may be phrased as follows.

Proposition 5.9. *Suppose G is a group of symmetries of a closed geometry (M, α) and that there exists a G-invariant k-form $\beta \in \Omega^k(M)$ such that $d\beta = \alpha$. Then $\nu\colon M \to \mathcal{P}^*_{\mathfrak{g}}$ given by*

$$\langle \nu, \mathsf{p} \rangle = (-1)^k \beta(p) \tag{5.5}$$

is a multi-moment map.

Proof. The map ν is equivariant, since β is invariant. We have $\nu_p = (-1)^k\beta(p)$ with $(-1)^k d(\beta(p)) = (-1)^{2k} p \lrcorner d\beta = p \lrcorner \alpha$ as follows essentially from the arguments that lead to (5.2): we repeat this calculations but replace the $(k+1)$-form $\alpha = d\beta$ with the k-form β:

$$\begin{aligned} &(-1)^k d(X_1 \wedge \ldots \wedge X_k \lrcorner \beta) \\ &= L(X_1 \wedge \ldots \wedge X_k) \lrcorner \beta + (-1)^{2k} X_1 \wedge \ldots \wedge X_k \lrcorner d\beta \\ &= X_1 \wedge \ldots \wedge X_k \lrcorner \alpha. \end{aligned} \tag{5.6}$$

Finally, using linearity as in the proof of Lemma 5.4, we find that equation (5.4) is satisfied, as required. □

In order to discuss algebraic existence, it is useful for us to extend the notation of Section 4.2 in the following way. The dual of the exact sequence

$$0 \longrightarrow \mathcal{P}_{\mathfrak{g}} \xrightarrow{\ \iota\ } \Lambda^k \mathfrak{g} \xrightarrow{\ L\ } \Lambda^{k-1} \mathfrak{g}$$

is the sequence

$$\Lambda^{k-1} \mathfrak{g}^* \xrightarrow{\ d\ } \Lambda^k \mathfrak{g}^* \xrightarrow{\ \pi\ } \mathcal{P}^*_{\mathfrak{g}} \longrightarrow 0.$$

The exterior derivative $d\colon \Lambda^k \mathfrak{g}^* \to \Lambda^{k+1} \mathfrak{g}^*$ induces a linear map $d_{\mathcal{P}}\colon \mathcal{P}^*_{\mathfrak{g}} \to B^{k+1}(\mathfrak{g}) \subset Z^{k+1}(\mathfrak{g}) \subset \Lambda^{k+1} \mathfrak{g}^*$. For $a \in \mathcal{P}^*_{\mathfrak{g}}$, we choose $\widetilde{a} \in \pi^{-1}(a)$ and then $d_{\mathcal{P}} a = d\widetilde{a}$. These observations lead to a generalised version of Proposition 4.10.

Proposition 5.10. *The linear map $d_{\mathcal{P}}\colon \mathcal{P}_{\mathfrak{g}}^* \to \Lambda^{k+1}\mathfrak{g}^*$ is a $\mathfrak{g}$-morphism with image contained in $Z^{k+1}(\mathfrak{g})$. It is injective if and only if $b_k(\mathfrak{g}) = 0$. If this condition holds then $d_{\mathcal{P}}$ is an isomorphism onto $Z^{k+1}(\mathfrak{g})$ if and only if $b_{k+1}(\mathfrak{g}) = 0$.* □

It also turns out useful to generalise the notion of $(2,3)$-triviality to that of $(k, k+1)$-triviality. More generally we introduce the following:

Definition 5.11. A connected Lie group G or its Lie algebra $\mathfrak{g}$ that satisfies $b_{k_1}(\mathfrak{g}) = \cdots = b_{k_\ell}(\mathfrak{g}) = 0$ will be called *(cohomologically) $(k_1, \dots, k_\ell)$-trivial*.

A general algebraic existence criterion, including the known ones from symplectic and strong geometry, may now be phrased as follows:

Theorem 5.12. *Let (M, α) be a closed geometry, $\alpha \in \Omega^{k+1}(M)$. Assume that G is a $(k, k+1)$-trivial symmetry group acting nearly effectively. Then there exists a unique multi-moment map $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$.*

More generally, if just $b_k(\mathfrak{g}) = 0$, then multi-moment maps for nearly effective actions are unique when they exist.

Proof. The invariant $(k+1)$-form α determines a G-equivariant map $\Psi\colon M \to Z^{k+1}(\mathfrak{g})$ given by

$$\langle \Psi, \mathsf{X}_1 \wedge \cdots \wedge \mathsf{X}_{k+1} \rangle = \alpha(X_1, \dots, X_{k+1}) \tag{5.7}$$

for $\mathsf{X}_1, \dots, \mathsf{X}_{k+1} \in \mathfrak{g}$. When $b_k(\mathfrak{g}) = 0 = b_{k+1}(\mathfrak{g})$, for each $m \in M$ there is a unique element $\nu(m) \in \mathcal{P}_{\mathfrak{g}}^*$ satisfying $d_{\mathcal{P}}\nu(m) = \Psi(m)$. Since $d_{\mathcal{P}}$ is a G-morphism, it follows that $\nu\colon M \to \mathcal{P}_{\mathfrak{g}}^*$ is also a G-equivariant.

We claim that ν is a multi-moment map. In general $d_{\mathcal{P}}\colon \mathcal{P}_{\mathfrak{g}}^* \to Z^{k+1}(\mathfrak{g}) \cap (\mathfrak{g} \wedge \mathcal{P}_{\mathfrak{g}})^*$. Consequently, the assumption $b_k(\mathfrak{g}) = 0$, gives that the dual map $d_{\mathcal{P}}^*$ is a surjection $Z^{k+1}(\mathfrak{g})^* \cap (\mathfrak{g} \wedge \mathcal{P}_{\mathfrak{g}}) \to \mathcal{P}_{\mathfrak{g}}$. This dual map may be expressed in terms of the adjoint action, since

$$\begin{aligned} \langle d_{\mathcal{P}}\alpha, \mathsf{Z} \wedge \mathsf{p} \rangle &= \langle d_{\mathcal{P}}\alpha, \mathsf{Z} \wedge \sum_{\ell=1}^{r} \mathsf{X}_\ell^1 \wedge \ldots \wedge \mathsf{X}_\ell^k \rangle \\ &= \sum_{\ell=1}^{r}\sum_{i=1}^{k} (-1)^i \alpha([\mathsf{Z}, \mathsf{X}_\ell^i], \mathsf{X}_\ell^1, \dots, \widehat{\mathsf{X}_\ell^i}, \dots, \mathsf{X}_\ell^k) + \alpha(0, \mathsf{Z}) = -\langle \alpha, \mathrm{ad}_{\mathsf{Z}}(\mathsf{p}) \rangle, \end{aligned} \tag{5.8}$$

for $\mathsf{Z} \in \mathfrak{g}$, $\mathsf{p} = \sum_{\ell=1}^{r} \mathsf{X}_\ell^1 \wedge \ldots \wedge \mathsf{X}_\ell^k \in \mathcal{P}_{\mathfrak{g}}$. Hence we may write any $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$ in the form $\mathsf{p} = \sum_{i=1}^{s} \mathrm{ad}_{\mathsf{Z}_i}(\mathsf{q}_i)$, with $\mathsf{Z}_i \in \mathfrak{g}$ and $\mathsf{q}_i \in \mathcal{P}_{\mathfrak{g}}$. Now the function

$$\nu_{\mathsf{p}} = -\sum_{i=1}^{s} \langle \Psi, \mathsf{Z}_i \wedge \mathsf{q}_i \rangle = -\sum_{i=1}^{s} \alpha(Z_i \wedge q_i)$$

satisfies $d\nu_{\mathsf{p}} = -\sum_{i=1}^{s} \mathcal{L}_{Z_i}(q_i \lrcorner\, \alpha) = p \lrcorner\, \alpha$, since $d(q_i \lrcorner\, \alpha) = 0$ by (5.3). Moreover we have that

$$\nu_{\mathsf{p}}(m) = -\sum_{i=1}^{s} \langle d_{\mathcal{P}}\nu(m), \mathsf{Z}_i \wedge \mathsf{q}_i \rangle = \sum_{i=1}^{s} \langle \nu(m), \mathrm{ad}_{\mathsf{Z}_i}(\mathsf{q}_i) \rangle = \langle \nu(m), \mathsf{p} \rangle.$$

Thus ν is a multi-moment map.

For the last part of the theorem, note that a multi-moment map ν defines elements $\nu(m) \in \mathcal{P}_{\mathfrak{g}}^*$ and the above calculations show that $d_{\mathcal{P}}(\nu(m)) = \Psi(m)$. However, $b_k(\mathfrak{g}) = 0$ implies that there is at most one solution $\nu(m)$ to this equation, so ν is then unique. □

The question remains whether there are interesting closed geometries with a symmetry group which is $(k, k+1)$-trivial for general $k \in \mathbb{N}$. Regarding existence we certainly have an affirmative answer: Hodge duality, Proposition 3.5, tells us that an n-dimensional unimodular Lie algebra is $(k, k+1)$-trivial, for some $k < n+1$, if and only if it is $(n-k-1, n-k)$-trivial, and as illustrated in Section 4.5, there are unimodular $(2,3)$-trivial algebras in dimension five and above. These examples thus provide algebras that are $(3,4)$-trivial, $(4,5)$-trivial, and so forth. Another, perhaps more interesting, class of symmetries consists of the compact simple Lie groups. Based on known results [Sam52, Che52] we obtain:

Proposition 5.13. *Apart from $\mathfrak{su}(n+1)$, $n \geqslant 2$, the compact simple Lie algebras*

$$\mathfrak{su}(2), \quad \mathfrak{so}(2n+1), \quad \mathfrak{sp}(n), \quad \mathfrak{so}(2n),$$
$$\mathfrak{e}_6, \quad \mathfrak{e}_7, \quad \mathfrak{e}_8, \quad \mathfrak{f}_4 \quad \textit{and} \quad \mathfrak{g}_2$$

are all $(1,2,4,5,6)$-trivial.

Proof. One way to keep track of the Betti numbers of an n-dimensional compact simple Lie algebra $\mathfrak{g}$ is in terms of the associated Poincaré polynomial

$$p_{\mathfrak{g}}(t) = \sum_{k=0}^{n} b_k t^k,$$

whose *kth* coefficient is $b_k = \dim H^k(\mathfrak{g})$. In Table 5.1 we list these polynomials based on work of Samelson ([Sam52], the classical algebras) and Chevalley ([Che52], the exceptional algebras). We see that all the compact simple algebras have $b_1 = b_2 = 0$ and $b_3 = 1$. Apart from members of the family $\mathfrak{su}(n+1)$, $n \geqslant 2$, these algebras have next non-zero Betti-number which is one of b_7, b_9 or b_{11}. □

5.2.1 (3,4)-trivial Lie algebras

$(k, k+1)$-trivial Lie algebras play a prominent role as symmetry groups of closed geometries of degree $k+1$. It is therefore natural to strive towards a classification of such algebras. $(1,2)$- and $(2,3)$-trivial Lie algebras are well understood. The next class one may try to describe is that of $(3,4)$-trivial algebras. Our first observation is the following:

Proposition 5.14. *Any non-trivial finite-dimensional Lie algebra $\mathfrak{g} \neq \mathbb{R}, \mathbb{R}^2$ satisfying $b_3(\mathfrak{g}) = 0$ is solvable and not nilpotent. $\mathfrak{g}$ is a direct sum $\mathfrak{h} + \mathfrak{k}$ of non-trivial Lie*

$\mathfrak{g}$	Poincaré polynomial
$\mathfrak{su}(n+1)$	$(1+t^3)(1+t^5)\cdots(1+t^{2n+1})$
$\mathfrak{so}(2n+1)$	$(1+t^3)(1+t^7)\cdots(1+t^{4n-1})$
$\mathfrak{sp}(n)$	$(1+t^3)(1+t^7)\cdots(1+t^{4n-1})$
$\mathfrak{so}(2n)$	$(1+t^3)(1+t^7)\cdots(1+t^{4n-5})(1+t^{2n-1})$
$\mathfrak{e}_6$	$(1+t^3)(1+t^9)(1+t^{11})(1+t^{15})(1+t^{17})(1+t^{23})$
$\mathfrak{e}_7$	$(1+t^3)(1+t^{11})(1+t^{15})(1+t^{19})(1+t^{23})(1+t^{27})(1+t^{35})$
$\mathfrak{e}_8$	$(1+t^3)(1+t^{15})(1+t^{23})(1+t^{27})(1+t^{35})(1+t^{39})(1+t^{47})(1+t^{59})$
$\mathfrak{f}_4$	$(1+t^3)(1+t^{11})(1+t^{15})(1+t^{23})$
$\mathfrak{g}_2$	$(1+t^3)(1+t^{11})$

Table 5.1: Poincaré polynomials for the compact simple Lie algebras, cf. [Sam52, Che52].

algebras if and only if $\mathfrak{h}$ *and* $\mathfrak{k}$ *are* $(2,3)$*-trivial. If in addition* $b_4(\mathfrak{g}) = 0$*, then one can have a direct sum decomposition* $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$ *of non-trivial Lie algebras if and only if* $\mathfrak{h}$ *and* $\mathfrak{k}$ *are* $(2,3,4)$*-trivial.* □

For $j = 1$ or 2, $b_j(\mathfrak{g}) = 0$ implies that $b_1(\mathfrak{g}) < j$. So one may wonder whether the condition $b_3(\mathfrak{g}) = 0$ implies that $b_1(\mathfrak{g}) < 3$. It turns out to be rather difficult to answer this question in general. However, we do have the following elementary result:

Proposition 5.15. *If a Lie algebra* $\mathfrak{g}$ *admits a splitting*

$$0 \to \mathfrak{p} \to \mathfrak{g} \to \mathfrak{q} \to 0 \tag{5.9}$$

then $b_j(\mathfrak{g}) \geqslant b_j(\mathfrak{q})$*. In particular, if* $\mathfrak{g}$ *splits over* $\mathfrak{k} = [\mathfrak{g},\mathfrak{g}]$ *then* $b_j(\mathfrak{g}) \geqslant \binom{b_1(\mathfrak{g})}{j}$*.*

Proof. Assume that we have a splitting (5.9) of $\mathfrak{g}$, i.e., that we may write $\mathfrak{g} = \mathfrak{p} + \mathfrak{q}$ with $\mathfrak{q} \leqslant \mathfrak{g}$ and $[\mathfrak{g},\mathfrak{p}] \subset \mathfrak{p}$. Dually this means that

$$d(\mathfrak{q}^*) \subset \Lambda^2\mathfrak{q}^* \quad \text{and} \quad d(\mathfrak{p}^*) \subset \mathfrak{q}^* \otimes \mathfrak{p}^* + \Lambda^2\mathfrak{p}^*.$$

From these relations we observe that the inclusion $\Lambda^j\mathfrak{q}^* \hookrightarrow \Lambda^j\mathfrak{g}^*$ induces an injection in cohomology $H^j(\mathfrak{q}) \hookrightarrow H^j(\mathfrak{g})$. Hence $b_j(\mathfrak{g}) \geqslant b_j(\mathfrak{q})$, as claimed.

To say that $\mathfrak{g}$ splits over $\mathfrak{k}$ means that we can take $\mathfrak{p} = \mathfrak{k}$ and $\mathfrak{q} = \mathfrak{a} \cong \mathfrak{g}/\mathfrak{k}$ in the above. In that case we have that $d(\mathfrak{a}^*) = 0$. So the inclusion in cohomology tells us that $H^j(\mathfrak{g})$ contains a subspace isomorphic to $\Lambda^j\mathfrak{a}^*$. The last assertion of the proposition now follows since $b_1(\mathfrak{g}) = \dim(\mathfrak{a})$. □

Finally, let us use the Hochschild-Serre spectral sequence to obtain a useful characterisation of the $(3,4)$-trivial Lie algebras $\mathfrak{g}$ that satisfy the condition $b_1(\mathfrak{g}) < 3$. The following result allows us to construct (infinite) families of $(3,4)$-trivial Lie algebras and provide full classifications in low dimensions.

Theorem 5.16. *A Lie algebra* $\mathfrak{g}$ *with derived algebra* $\mathfrak{k} = [\mathfrak{g},\mathfrak{g}]$ *and satisfying the condition that* $b_1(\mathfrak{g}) < 3$ *is* $(3,4)$*-trivial if and only if* $\mathfrak{g}$ *is solvable,* $\mathfrak{k}$ *is nilpotent and either*

(i) $\mathfrak{k}$ *has codimension one and* $H^2(\mathfrak{k})^{\mathfrak{g}} = 0 = H^3(\mathfrak{k})^{\mathfrak{g}} = H^4(\mathfrak{k})^{\mathfrak{g}}$, *or*
(ii) $\mathfrak{k}$ *has codimension two and* $H^i(\mathfrak{k})^{\mathfrak{g}} = 0$ *for* $1 \leqslant i \leqslant 4$.

Proof. The codimension one result is an immediate consequence of the formulae

$$H^k(\mathfrak{g}) = H^k(\mathfrak{h})^X + H^{k-1}(\mathfrak{h})^X,$$

obtained from analysing the Hochschild-Serre spectral sequence relative to a codimension one ideal $\mathfrak{h}$; here one chooses $X \in \mathfrak{g} \setminus \mathfrak{h}$.

Let us now treat the codimension two case. So let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{k}$ be a codimension two ideal containing the derived algebra of $\mathfrak{g}$. Write $\mathfrak{a} = \mathfrak{g} / \mathfrak{k}$. Then the result by Hochschild and Serre tells us that the E_2-page of the spectral sequence converging to $H^*(\mathfrak{g})$ is given by

$$E_2^{p,q} = H^p(\mathfrak{a}, H^q(\mathfrak{k})).$$

Consequently, we need to compute cohomologies of complexes

$$C^0(V^i) = V^i \xrightarrow{d^0} C^1(V^i) = \mathfrak{a}^* \otimes V^i \xrightarrow{d^1} C^2(V^i) = \Lambda^2 \mathfrak{a}^* \otimes V^i,$$

with $V^i = H^i(\mathfrak{k})$,

$$(d^0 f)(A) = A \cdot f, \quad A \in \mathfrak{a},$$

and

$$(d^1(f_1, f_2))(A_1, A_2) = A_1 \cdot f_2 - A_2 \cdot f_1,$$

where A_1, A_2 is a basis for $\mathfrak{a}$ and $\mathfrak{a}^* \otimes V^i \cong 2V^i$.

By assumption, $\mathfrak{k}$ is a codimension two ideal. Therefore the E_1-page of the spectral sequence takes the form:

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ C^0(V^4) & C^1(V^4) & C^2(V^4) \\ C^0(V^3) & C^1(V^3) & C^2(V^3) \\ C^0(V^2) & C^1(V^2) & C^2(V^2) \\ C^0(V^1) & C^1(V^1) & C^2(V^1) \\ \mathbb{R} & 2\mathbb{R} & \mathbb{R} \end{array}$$

Note that $V^0 = \mathbb{R}$ with trivial $\mathfrak{a}$-action. In particular, the d_1-maps on the bottom row are zero, and we have the E_2-page:

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ H^0(V^4) & H^1(V^4) & H^2(V^4) \\ H^0(V^3) & H^1(V^3) & H^2(V^3) \\ H^0(V^2) & H^1(V^2) & H^2(V^2) \\ H^0(V^1) & H^1(V^1) & H^2(V^1) \\ \mathbb{R} & 2\mathbb{R} & \mathbb{R} \end{array}$$

The vanishing of $b_3(\mathfrak{g})$ implies that $H^1(V^2) = 0$, $H^0(V^2)$ surjects onto $H^2(V^1)$, and $H^0(V^3)$ injects into $H^2(V^2)$. Now $H^1(V^2)$ is the middle cohomology of

$$V^2 \to 2V^2 \to V^2.$$

When $H^1(V^2) = 0$, counting dimensions, we find that the first map must be injective and the last surjective, so $H^0(V^2) = 0 = H^2(V^2)$, and therefore the $q = 2$ row of the E_2-page vanishes.

By the same token, the vanishing of $b_4(\mathfrak{g})$ implies that $H^1(V^3) = 0$, $H^0(V^3)$ surjects onto $H^2(V^2)$, and $H^0(V^4)$ injects into $H^2(V^3)$. As above, $H^1(V^3)$ is the middle cohomology of

$$V^3 \to 2V^3 \to V^3,$$

so a dimension count shows that the condition $H^1(V^3) = 0$ yields injectivity of the first map and surjectivity of the last. Hence $H^0(V^3) = 0 = H^2(V^3)$.

Altogether, we now find that the E_2-page takes the form

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ 0 & H^1(V^4) & H^2(V^4) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ H^0(V^1) & H^1(V^1) & 0 \\ \mathbb{R} & 2\mathbb{R} & \mathbb{R} \end{array}.$$

As $H^0(V^i) \cong H^i(\mathfrak{k})^{\mathfrak{g}}$, the condition $H^i(\mathfrak{k})^{\mathfrak{g}} = 0$ is obviously required for $i = 2, 3$ and 4. In order to obtain the condition $H^1(\mathfrak{k})^{\mathfrak{g}} = 0$ a further analysis is needed.

We claim that the vanishing of $H^1(\mathfrak{k})^{\mathfrak{g}}$ is equivalent to the vanishing of $H^2(V^1)$; observe that in order to prove this assertion we might as well change to work over the complex number field. If we can prove the claim, statement (ii) of the theorem will follow. Our problem thus comes down to showing that if we have a two-dimensional Abelian algebra $\mathfrak{a}$ which acts on the finite-dimensional space V, then one has that $V^{\mathfrak{a}} = 0$ if and only if $\mathrm{Im}(\mathfrak{a})(V) = V$, or equivalently, $V^{\mathfrak{a}} \neq 0$ if and only if $\mathrm{Im}(\mathfrak{a})(V) \subsetneq V$; here $\mathrm{Im}(\mathfrak{a})(V) = \mathrm{Im}(A_1) + \mathrm{Im}(A_2)$ with $\mathfrak{a} = \langle A_1, A_2 \rangle$. To prove the latter of these two assertions, we first decompose V in terms of generalised eigenspaces:

$$V = \bigoplus_{\underline{\lambda}_j = (\lambda_j^1, \lambda_j^2)} V_{\underline{\lambda}_j}, \quad V_{\underline{\lambda}_j} = \{v \in V \colon \exists N \in \mathbb{N} \text{ s.t. } (A_i - \lambda_j^i)^N(v) = 0,\ i = 1, 2\}.$$

Now note that for any $\underline{\lambda} \neq 0$ we have that $\mathrm{Im}(\mathfrak{a})(V_{\underline{\lambda}}) = V_{\underline{\lambda}}$, since at least one A_i restricted to $V_{\underline{\lambda}}$ acts invertibly. This also means that the fixed points must be found in $V_{\underline{0}}$. But the restrictions of A_1 and A_2 to $V_{\underline{0}}$ are nilpotent endomorphisms. So, by Engel's theorem, these restrictions are simultaneously upper triangularizable. Hence $V_{\underline{0}} \neq 0$ if and only if $\mathrm{Im}(\mathfrak{a})(V_{\underline{0}}) \subsetneq V_{\underline{0}}$.

In conclusion, the condition $V^{\mathfrak{a}} \neq 0$ holds if and only if $\mathrm{Im}(\mathfrak{a})(V) \subsetneq V$, as required. □

Inspection shows that several of the algebras constructed in Section 4.5.2 satisfy the first of the above restrictions on the invariant cohomology of $\mathfrak{k} = \mathfrak{g}'$. Hence they provide us with examples of $(2,3,4)$-trivial algebras.

Example 5.17. The $(2,3)$-trivial Lie algebra

$$\mathfrak{p}_5 = (0, 21, 21+31, 2.41+32, 3.51+42)$$

also satisfies the condition $b_4(\mathfrak{g}) = 0$. To see this take a basis $A, E_1, \dots, E_4$ as in Example 4.20. Then we find that the induced action of A on $H^4(\mathfrak{k}) \cong \langle E_{1234} \rangle$ is given by multiplication by 7; here E_{1234} denotes $E_1 \wedge E_2 \wedge E_3 \wedge E_4$. So clearly $H^4(\mathfrak{k})^{\mathfrak{g}} = \{0\}$, as required.

Note that the second Lie kernel of $\mathfrak{p}_5$ is non-trivial. Direct calculations show that

$$\mathcal{P}_{\mathfrak{p}_5} = \langle E_{134}, E_{234}, 4E_{123} + A \wedge E_{24}, 5E_{124} + A \wedge E_{34} \rangle. \tag{5.10}$$

◇

In general, we may calculate the dimension of the second Lie kernel of a $(2,3,4)$-trivial Lie algebra via the formula

$$\dim \mathcal{P}_{\mathfrak{g}} = (n-1)(n-2)(n-3)/6.$$

This follows since $\dim \Lambda^3 \mathfrak{g}^* = \dim Z^3(\mathfrak{g}) + \dim B^4(\mathfrak{g})$, and $\dim Z^3(\mathfrak{g}) = (n-1)(n-2)/2$, by $(2,3)$-triviality, while $B^4(\mathfrak{g}) = Z^4(\mathfrak{g}) \cong \mathcal{P}_{\mathfrak{g}}^*$, by $(3,4)$-triviality.

Finally, let us note that there is a systematic way of obtaining the basis (5.10) in Example 5.10. Suppose that we have a $(2,3,4)$-trivial Lie algebra $\mathfrak{g} = \langle A \rangle + \mathfrak{k}$ with ad_A acting invertibly on $\mathfrak{k}$ then we have an injective map $\Phi\colon \Lambda^3 \mathfrak{k} \to \Lambda^3 \mathfrak{g}$ given by

$$\sum_{j=1}^{r} K_1^j \wedge K_2^j \wedge K_3^j \mapsto \sum_{j=1}^{r} \left(K_1^j \wedge K_2^j \wedge K_3^j + A \wedge (\mathrm{ad}_A|_{\mathfrak{k}})^{-1} \circ L(K_1^j \wedge K_2^j \wedge K_3^j) \right).$$

We claim that this map is an isomorphism onto the second Lie kernel $\mathcal{P}_{\mathfrak{g}}$. For dimensional reasons, it obviously suffices to prove that $\Phi(\Lambda^3 \mathfrak{k}) \subset \mathcal{P}_{\mathfrak{g}}$. This assertion follows since

$$\begin{aligned} L(A \wedge (\mathrm{ad}_A|_{\mathfrak{k}})^{-1} \circ (L(K_1 \wedge K_2 \wedge K_3))) &= -\mathrm{ad}_A((\mathrm{ad}_A|_{\mathfrak{k}})^{-1} \circ (L(K_1 \wedge K_2 \wedge K_3))) \\ &\quad - A \wedge L((\mathrm{ad}_A|_{\mathfrak{k}})^{-1} \circ (L(K_1 \wedge K_2 \wedge K_3))) \\ &= -L(K_1 \wedge K_2 \wedge K_3) - A \wedge (\mathrm{ad}_A|_{\mathfrak{k}})^{-1} \circ (L^2(K_1 \wedge K_2 \wedge K_3)) \\ &= -L(K_1 \wedge K_2 \wedge K_3), \end{aligned}$$

where we have used that L is equivariant and squares to zero. Note that Φ commutes with the adjoint action of A: $\Phi \circ \mathrm{ad}_A = \mathrm{ad}_A \circ \Phi$.

5.3 Examples and further discussion

5.3.1 Exterior powers of the cotangent bundle

In Section 4.4.1 a basic example of a 2-plectic geometry was provided by the total space of the second exterior power of the cotangent bundle of a smooth manifold. This example obviously generalises. The *kth* exterior power $M = \Lambda^k T^* N$ of a base manifold N carries a canonical k-form β, given by

$$\beta_a(W_1, \dots, W_k) = a(\pi_* W_1, \dots, \pi_* W_k), \qquad W_1, \dots, W_k \in T_a M,$$

where $\pi\colon \Lambda^k T^* N \to N$ is the bundle projection. From this one defines a closed $(k+1)$-form α on M, via

$$\alpha = d\beta. \tag{5.11}$$

This form is k-plectic: in local coordinates $(q^1, \dots, q^n)$ on N we have $\beta = \sum_{i_1 < \dots < i_k} p_{i_1 \dots i_k} dq^{i_1} \wedge \dots \wedge dq^{i_k}$ defining local coordinates $(q^i, p_{i_1 \dots i_k})$ on $M = \Lambda^k T^* N$ in which $\alpha = \sum_{i_1 < \dots < i_k} dp_{i_1 \dots i_k} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k}$.

If G is a group of diffeomorphisms of N, then there is an induced action on $M = \Lambda^k T^* N$ which preserves β and hence α. As $\alpha = d\beta$, Proposition 5.9 gives that there is a multi-moment map ν determined by (5.11), which here reads

$$\langle \nu(a), \mathsf{p} \rangle = (-1)^k a(p_N)$$

where p_N is the field of k-vectors on N determined by $\mathsf{p} \in \mathcal{P}_{\mathfrak{g}}$. To summarise:

Proposition 5.18. *If a Lie group G acts on a smooth manifold N, then the induced action on $M = \Lambda^k T^* N$ admits a multi-moment map with respect to the canonical k-plectic structure.* □

As a concluding remark, let us note that the k-plectic manifold (M, α) from above appears as a central object in multi-symplectic field theory [CIdL99, Hél11]. Moreover, note that the form α is not only k-plectic, but also determines a unique subbundle $W \subset TM =: V$ that satisfies the following two conditions at each $a \in M$:

(i) $w_1 \wedge w_2 \lrcorner\, \alpha = 0$ for all $w_1, w_2 \in W_a$;

(ii) $\dim W_a = \dim \Lambda^k (V_a / W_a)$ and $\dim W_a \geqslant \dim V_a / W_a$.

In terms of the local coordinates $(q^i, p_{i_1 \dots i_k})$, the subbundle W is spanned by the vector fields $\partial / \partial p_{i_1 \dots i_k}$.

These two additional properties distinguish the restricted class of k-plectic manifolds mentioned in Remark 5.1, that is, the class of closed geometries for which a generalised Darboux theorem [Mar88, Theorem 2.1] is valid.

5.3.2 HyperKähler manifolds with special symmetry

We will now explain how the work of Poon and Swann [Swa91, PS03, PS01, Swa10a] can be rephrased using the notion of multi-moment maps for closed geometries. Recall that a quaternion-Hermitian manifold Q differs from an almost hyperHermitian manifold in that it carries only *locally* defined almost complex structures I, J and K. More precisely, Q is a $4n$-dimensional Riemannian manifold with a rank three subbundle $\mathcal{G} \subset \mathrm{End}(TM)$ which is locally trivialised by anti-commuting almost complex structures I, J and K that satisfy $K = IJ$. In addition, g must be compatible with $\mathcal{G}$, meaning $g(\mathcal{I}X, \mathcal{I}Y)$ for each $X, Y \in T_m M$ and $\mathcal{I} \in \mathcal{G}_m$.

A quaternion-Hermitian manifold carries a non-degenerate four-form Ω. Locally we may write this fundamental form as

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K. \tag{5.12}$$

In dimension eight and above one says that Q is *quaternionic Kähler* if the fundamental form is parallel, $\nabla^{\mathrm{LC}}\Omega = 0$. In dimension four a quaternionic Kähler manifold is defined to be an oriented Riemannian manifold which is Einstein and self-dual.

Swann showed [Swa91] that to any quaternionic Kähler manifold Q^{4n} of positive scalar curvature one may associate a special type of hyperKähler manifold $M^{4n+4} = \mathcal{U}(Q)$ which acts as a hyperKähler generalisation of the twistor space; this is known as the *Swann bundle* and may be written as $\mathcal{U}(Q) = \mathcal{F} \times_{Sp(n)\,Sp(1)} \mathbb{H}^*/\{\pm 1\}$, where $\mathcal{F}$ is the principal $Sp(n)\,Sp(1)$-bundle of frames over Q. Conversely, given a $(4n+4)$-dimensional hyperKähler manifold admitting a certain type of $SU(2)$-action, a version of the Marsden-Weinstein reduction produces a quaternionic-Kähler manifold of positive scalar curvature. Our aim is to explain how this inverse construction may be formulated very naturally via multi-moment maps for the underlying closed geometry.

Suppose that (M, I, J, K) is a hypercomplex manifold, and g is a hyperKähler metric on M. Let a be a real number. A vector field X on M is called a *special homothety of type a* if it satisfies the following conditions:

$$\begin{gathered} \mathcal{L}_X g = ag, \\ \mathcal{L}_{IX} g = 0, \quad \mathcal{L}_{IX} I = 0, \quad \mathcal{L}_{IX} J = -aK, \quad \mathcal{L}_{IX} K = aJ, \\ \mathcal{L}_{JX} g = 0, \quad \mathcal{L}_{JX} I = aK, \quad \mathcal{L}_{JX} J = 0, \quad \mathcal{L}_{JX} K = -aI, \\ \mathcal{L}_{KX} g = 0, \quad \mathcal{L}_{KX} I = -aJ, \quad \mathcal{L}_{KX} J = aI, \quad \mathcal{L}_{KX} K = 0. \end{gathered} \tag{5.13}$$

Remark 5.19. In the more general context of HKT geometry, one considers special homotheties of type (q, r), see e.g. [PS03]. In that terminology we are dealing with homotheties of type $(a, -a)$. Such symmetries are related to superconformal symmetry [dWKV00]; the relevant superalgebra $D(2, 1; -2)$ appears in Kac's classification [Kac77]. △

The equations in (5.13) have a number of consequences. Firstly we observe:

Lemma 5.20. *If (M, g, I, J, K) is a hyperKähler manifold, and X is a special homothety of type a, then*

$$\nabla^{\mathrm{LC}} X^\flat = \frac{1}{2} ag. \tag{5.14}$$

Proof. As ∇^{LC} is metric and torsion-free we have the relation

$$(\mathcal{L}_Y g)(Z, W) = g(\nabla^{\mathrm{LC}}_Z Y, W) + g(Z, \nabla^{\mathrm{LC}}_W Y) = (\nabla^{\mathrm{LC}} Y^\flat)(Z, W) + (\nabla^{\mathrm{LC}} Y^\flat)(W, Z).$$

Combining this observation with the relation $\mathcal{L}_X g = ag$, it follows that we have a decomposition

$$\nabla^{\mathrm{LC}} X^\flat = \frac{1}{2} a g + \alpha,$$

where α is a two-form on M. As $\nabla^{\mathrm{LC}} I = \nabla^{\mathrm{LC}} J = \nabla^{\mathrm{LC}} K = 0$ we may rewrite this expression:

$$g(\nabla^{\mathrm{LC}}(\mathcal{I}X), Z) = \frac{1}{2} a \omega_{\mathcal{I}}(Y, Z) + \alpha(\mathcal{I}Y, Z), \quad \mathcal{I} = I, J, K.$$

Now observe that $0 = (\mathcal{L}_{\mathcal{I}X} g)(Y, Z) = g(\nabla^{\mathrm{LC}}_Y(\mathcal{I}X), Z) + g(\nabla^{\mathrm{LC}}_Z(\mathcal{I}X), Y) = \alpha(\mathcal{I}Y, Z) + \alpha(\mathcal{I}Z, Y)$. Consequently, we have that

$$\begin{aligned}\alpha(IY, Z) &= \alpha(Y, IZ) = \alpha(Y, JKZ) = \alpha(JY, KZ) = \alpha(KJY, Z) \\ &= -\alpha(IY, Z),\end{aligned}$$

which implies $\alpha = 0$, as required. □

A special homothety generates a local action of $\mathbb{H}^*$.

Lemma 5.21. *Let X be a special homothety of type $a \neq 0$. Then the quadruple $\{X, IX, JX, KX\}$ generates a local action of $\mathbb{H}^*$.*

Proof. The statement follows by calculating the commutation relations. We first rewrite (5.14) in the form $\nabla^{\mathrm{LC}} X = \frac{1}{2} a$ and use that $\nabla^{\mathrm{LC}} \mathcal{I} = 0$ to obtain

$$[X, \mathcal{I}X] = \nabla^{\mathrm{LC}}_X(\mathcal{I}X) - \nabla^{\mathrm{LC}}_{\mathcal{I}X} X = \frac{1}{2} a \mathcal{I}X - \frac{1}{2} a \mathcal{I}X = 0, \quad \mathcal{I} = I, J, K.$$

Next we find that

$$[IX, JX] = \nabla^{\mathrm{LC}}_{IX}(JX) - \nabla^{\mathrm{LC}}_{JX}(IX) = \frac{1}{2} a JI(X) - \frac{1}{2} a IJX = -aKX$$

with similar results for cyclic permutations of (I, J, K). □

This lemma implies that a special homothety of type $a \neq 0$ for which IX, JX, KX are complete vector fields generates an isometric action of $SU(2) \cong Sp(1)$.

Lemma 5.22. *Let (M, g, I, J, K) be a hyperKähler manifold, and X a special homothety of type $a \neq 0$. Assume IX, JX, KX are complete vector fields. Then the associated $SU(2)$-action preserves the closed four-form $\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K$.*

Proof. From the defining relations (5.13) we find:

$$\mathcal{L}_{IX} \omega_I = 0, \quad \mathcal{L}_{IX} \omega_J = -a \omega_K, \quad \mathcal{L}_{IX} \omega_K = a \omega_J.$$

It now follows that

$$\begin{aligned}\mathcal{L}_{IX} \Omega &= \mathcal{L}_{IX}(\omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K) \\ &= -2a \omega_J \wedge \omega_K + 2a \omega_J \wedge \omega_K = 0.\end{aligned}$$

Similar calculations show that $\mathcal{L}_{JX} \Omega = 0 = \mathcal{L}_{KX} \Omega$, so that $\mathcal{I}X$ preserves Ω for $\mathcal{I} = I, J, K$, as required. □

In this way we obtain a closed geometry (M, Ω) with symmetry group $SU(2)$. Moreover, there is an associated multi-moment map:

Proposition 5.23. *Let (M, g, I, J, K) be a hyperKähler manifold, and X a special homothety of type $a \neq 0$. Assume IX, JX, KX are complete vector fields. Then the associated closed geometry (M, Ω) with its $SU(2)$-symmetry admits a multi-moment map $\nu\colon M \to \mathbb{R} \cong \mathcal{P}^*_{\mathfrak{su}(2)}$ given by*

$$\nu = -\frac{3}{a}\|X\|^4. \tag{5.15}$$

Moreover, the function $\mu = \sqrt{-\frac{\nu}{3a^3}}$ satisfies

$$dd_{\mathcal{I}}\mu = \omega_{\mathcal{I}}, \quad \mathcal{I} = I, J, K, \tag{5.16}$$

and is thus a hyperKähler potential.

Proof. We first observe that

$$IX \wedge JX \wedge KX \lrcorner (\omega_{\mathcal{I}} \wedge \omega_{\mathcal{I}}) = -2\|X\|^2 X^\flat, \quad \mathcal{I} = I, J, K.$$

Therefore we have $IX \wedge JX \wedge KX \lrcorner \Omega = -6\|X\|^2 X^\flat$.

Using metric compatibility of ∇^{LC} together with the relation $\nabla^{\mathrm{LC}} X = \frac{1}{2}a$ we now find that

$$d(\|X\|^2)(Y) = 2g(\nabla^{\mathrm{LC}}_Y X, X) = aX^\flat(Y),$$

i.e., $d(\|X\|^2) = aX^\flat$. Consequently, we have that

$$d(\|X\|^4) = 2a\|X\|^2 X^\flat = -\frac{a}{3} IX \wedge JX \wedge KX \lrcorner \Omega.$$

Therefore the $SU(2)$-invariant function (5.15) is a multi-moment map for (M, Ω).

In order to prove the last statement of the proposition first note that the function $f = (\frac{1}{a}\|X\|)^2$ satisfies $df = -\frac{1}{a}\mathcal{I}X \lrcorner \omega_{\mathcal{I}}$ for $\mathcal{I} = I, J, K$. Now observe that

$$d_K f(Y) = -df(KY) = \frac{1}{a}(JX \lrcorner \omega_J)(KY) = \frac{1}{a}\omega_I(JX, Y).$$

Consequently

$$dd_K f = d(\frac{1}{a} JX \lrcorner \omega_I) = \frac{1}{a}\mathcal{L}_{JX}\omega_I = \omega_K.$$

Similar computations show that $dd_I f = \omega_I$ and $dd_J f = \omega_J$.

In conclusion, the function $\mu := f = \sqrt{-\frac{\nu}{3a^3}}$, is a hyperKähler potential. □

Remark 5.24. Note that Proposition 5.23 gives an identification between the level sets of ν and those of μ. One has $m \in \nu^{-1}(t)$ if and only if $m \in \mu^{-1}(s)$ with $s = \sqrt{-\frac{t}{3a^3}}$. △

Applying essentially the same arguments as in the proofs of [Swa91, Theorem 5.1] and [PS03, Theorem 4.3], we will now establish a quaternionic four-form analogue of the Marsden-Weinstein reduction which is valid for the data given in the statement of Proposition 5.23. Before we formulate the general result, let us give a conceptual explanation of the construction, which is valid if $\dim M > 12$ and the quotient $Q = \nu^{-1}(t)/SU(2)$ is a manifold.

Pointwise we have a quaternionic splitting $T_mM \cong \mathcal{V}_m \oplus \mathcal{H}_m$, where $\mathcal{V}_m$ is defined as the real span of $\{X_m, IX_m, JX_m, KX_m\}$ and $\mathcal{H}_m$ is the orthogonal complement. We observe that $\mathcal{V}$ contains all vector fields tangent to the $SU(2)$-action while $\mathcal{H}$ is an $SU(2)$-invariant distribution of horizontal vectors for the projection $\pi\colon \nu^{-1}(t) \to Q$. Let ι denote the inclusion $\nu^{-1}(t) \hookrightarrow M$. Then Q carries a four-form $\widetilde{\Omega}$ which is uniquely determined by the relation

$$\iota^*\Omega = \pi^*\widetilde{\Omega}. \tag{5.17}$$

As $\pi^*\widetilde{\Omega}$ is just the restriction of Ω to $\mathcal{H}$, the four-form $\widetilde{\Omega}$ is of the correct algebraic type to determine a quaternionic structure on Q. In addition, the injectivity of π^* combined with the relation (5.17) imply that $d\widetilde{\Omega} = 0$. Consequently, the reduced space Q is quaternionic Kähler.

The general result can be phrased in the following way.

Theorem 5.25. *Let (M^{4n+4}, g, I, J, K) be a hyperKähler manifold, and X a special homothety of type $a \neq 0$. Assume the vector fields IX, JX, KX are complete, and let $SU(2)$ be the corresponding subgroup of $\mathbb{H}^*$. Let ν denote the associated multi-moment map (5.15). Then for any non-zero $t \in \nu(M)$, the group $SU(2)$ acts semi-freely on $\nu^{-1}(t)$, and the quotient $Q^{4n} = \nu^{-1}(t)/SU(2)$ is a quaternionic Kähler orbifold of positive scalar curvature.*

Proof. As $\nu = -\frac{3}{a}\|X\|^4$ and $d\nu = -6\|X\|^2 X^\flat$, each non-zero $t \in \nu(M)$ is a regular value of ν, and X does not vanish on $\nu^{-1}(t)$. The subgroup $SU(2)$ acts semi-freely on

$$\mathcal{X}_t := \nu^{-1}(t),$$

since $\mathcal{I}X$ preserves g and commutes with X, for $\mathcal{I} = I, J, K$. As $SU(2)$ acts isometrically, the quotient $\mathcal{Q} = \mathcal{X}_t/SU(2)$ inherits a Riemannian metric.

Let $\pi\colon \mathcal{X}_t \to Q$ be the projection. In order to define local almost complex structures I_Q, J_Q, K_Q on Q, note that as $\ker \pi_*$ is spanned by IX, JX, KX, the horizontal distribution $\mathcal{H} = (\ker \pi_*)^\perp \subset T\mathcal{X}_t$ is $4n$-dimensional and is preserved by I, J, K. Consequently, each point $x \in \pi^{-1}(q) \subset \mathcal{X}_t$ defines a triple I_Q, J_Q, K_Q of anti-commuting almost complex structures on $T_qQ \cong \mathcal{H}_y$ that satisfy $I_QJ_Q = K_Q$. Moreover, given any other point $x' \in \pi^{-1}(q)$, the corresponding triple I'_Q, J'_Q, K'_Q can be expressed in terms of linear combinations of the triple I_Q, J_Q, K_Q defined by x. We therefore have an almost quaternionic structure $\mathcal{G}_Q$ on Q. As g_Q is compatible with each of the almost complex structures I_Q, J_Q, K_Q, the above arguments show that Q is in fact a quaternion-Hermitian orbifold.

On Q we now define $\nabla^Q\colon \Gamma(TQ) \to \Gamma(TQ \otimes T^*Q)$ via the relation

$$\nabla^Q_Y Z = \pi_*(\nabla^{\mathrm{LC}}_{Y^{\mathcal{H}}} Z^{\mathcal{H}}),$$

where $Y^{\mathcal{H}}, Z^{\mathcal{H}}$ are $SU(2)$-invariant lifts of $Y, Z \in \Gamma(TQ)$ to smooth sections of $\mathcal{H} \subset T\mathcal{X}_t$. One may verify that ∇^Q is in fact the Levi-Civita connection of g_Q. Indeed, fairly straightforward calculations (see, e.g., the proof of [PS03, Theorem 4.3]) show that ∇^Q is metric and torsion-free. Also observe that ∇^Q preserves the almost complex structures: as $\mathcal{I}_Q Y = \pi_*((f_I I + f_J J + f_K K)Y^{\mathcal{H}})$ for some functions f_I, f_J, f_K on $\mathcal{X}_t$, we have that

$$\begin{aligned}(\nabla^Q_Y \mathcal{I}_Q)(Z) &= \pi_* \left(\nabla^{\mathrm{LC}}_{Y^{\mathcal{H}}}((f_I I + f_J J + f_K K)Z^{\mathcal{H}}) - (f_I I + f_J J + f_K K)\nabla^{\mathrm{LC}}_{Y^{\mathcal{H}}} Z^{\mathcal{H}} \right) \\ &= \pi_* \left(((Y^{\mathcal{H}} f_I)I + (Y^{\mathcal{H}} f_J)J + (Y^{\mathcal{H}} f_K)K)Z^{\mathcal{H}} \right),\end{aligned}$$

which is a linear combination of $I_Q Z, J_Q Z, K_Q Z$, as required.

Summarising the above, we have shown that the Levi-Civita connection on Q preserves the rank three vector bundle $\mathcal{G}_Q$. Except in four dimensions, this observation allows us to deduce that Q is a quaternionic Kähler orbifold.

In dimension four one must calculate the curvature, in order to check self-duality and the Einstein condition. The strategy is outlined in the final part of the proof of [Swa91, Theorem 5.1]: first one verifies that the curvature tensor lies pointwise in the complement of $\Lambda^{0,2}_{\mathcal{I}}(Q) \otimes \Lambda^{2,0}_{\mathcal{I}}(Q)$ for $\mathcal{I} = I, J, K$, which implies the self-duality. In order to check the Einstein condition one may apply an immersion computation to the Riemannian submersion $\mathcal{X}_t \to Q$. □

Remark 5.26. We emphasise that the above construction may be generalised to the pseudo-Riemannian setting, cf. [Swa91]. In this way it is possible to produce quaternionic Kähler quotients of negative scalar curvature, see, e.g., the discussion in [PS03]. △

Example 5.27. The fundamental example illustrating Theorem 5.25 comes from taking $M = \mathbb{R}^{4n+4} \setminus \{0\} \cong \mathbb{H}^{n+1} \setminus \{0\}$ with its standard flat metric $g = \sum_{\ell=1}^{4n+4} dx_\ell^2$ and hypercomplex structure I, J, K induced by right multiplication by $-i, -j, -k$, respectively. The dilation vector field $X = \sum_{\ell=1}^{4n+4} x_\ell \partial/\partial x_\ell$ is a special homothety of type 2. The associated multi-moment map and hyperKähler potential are the functions:

$$\nu(x_1, \ldots, x_{4n+4}) = -\frac{3}{2} \left(\sum_{\ell=1}^{4n+4} x_\ell^2 \right)^2 \quad \text{and} \quad \mu(x_1, \ldots, x_{4n+4}) = \frac{1}{4} \sum_{\ell=1}^{4n+4} x_\ell^2.$$

The reduced space $Q = \nu^{-1}(t)/SU(2)$ is the quaternionic projective space $\mathbb{H}P(n) = Sp(n+1)/Sp(n)\,Sp(1)$ which is one of the so-called Wolf spaces [Bes08, Table 14.52]. Apart from the quaternionic projective space, one has the Wolf spaces:

$$Gr(\mathbb{C}^{n+2}) = \frac{U(n+2)}{U(n)\,U(2)}, \quad \widetilde{Gr}_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n)\,SO(4)}, \quad \frac{G_2}{SO(4)},$$
$$\frac{F_4}{Sp(3)\,Sp(1)}, \quad \frac{E_6}{SU(6)\,Sp(1)}, \quad \frac{E_7}{Spin(12)\,Sp(1)}, \quad \frac{E_8}{E_7\,Sp(1)},$$

where the spaces arising from the classical groups have dimension $4n$ while those arising from the exceptional groups have dimensions $4 \times 2 = 8$, $4 \times 7 = 28$, $4 \times 10 = 40$, $4 \times 16 = 64$ and $4 \times 28 = 112$, respectively. All these spaces arise via the reduction procedure described in Theorem 5.25, see [Swa91] or [Swa10a]. ♢

5.3.3 Homogeneous closed geometries

Finally let us generalise the description of the homogeneous strong geometries presented in Section 4.4.2. If G acts transitively on a closed geometry (M, α), $\alpha \in \Omega^{k+1}(M)$, then we may define $\Psi \colon M \to Z^{k+1}(\mathfrak{g})$ via (5.7), and the image will be a G-orbit in $Z^{k+1}(\mathfrak{g})$. Conversely, formula (5.7) can be used to define closed geometries that map to a given orbit in $Z^{k+1}(\mathfrak{g})$: given $\Psi \in Z^{k+1}(\mathfrak{g})$, let K_Ψ denote the connected subgroup generated by $\ker \Psi = \{ X \in \mathfrak{g} : X \lrcorner \Psi = 0 \}$; for any closed group H of G with $H \subset K_\Psi$, equation (5.7) defines a closed $(k+1)$-form α on the homogeneous space G/H and this closed geometry maps to $G \cdot \Psi \subset Z^{k+1}(\mathfrak{g})$.

Now suppose that $\Psi = d_{\mathcal{P}} \beta$ for some $\beta \in \mathcal{P}_{\mathfrak{g}}^*$. If the map $d_{\mathcal{P}}$ is injective, then the orbits $G \cdot \Psi$ and $G \cdot \beta$ are identified and the map $\Psi \colon M \to Z^{k+1}(\mathfrak{g})$ may now be interpreted as a map $\nu \colon M \to \mathcal{P}_{\mathfrak{g}}^*$. Injectivity of $d_{\mathcal{P}}$ is guaranteed by the condition $b_k(\mathfrak{g}) = 0$. When this holds, the proof of Theorem 5.12 shows that ν is a multi-moment map for the action of G.

Theorem 5.28. *Suppose G is a connected Lie group with $b_k(\mathfrak{g}) = 0$. Let $\mathcal{O} = G \cdot \beta \subset \mathcal{P}_{\mathfrak{g}}^*$ be an orbit of G acting on the dual of the kth Lie kernel. Then there are homogeneous closed geometries $(G/H, \alpha)$, with $\alpha \in \Omega^{k+1}(G/H)$ corresponding to $\Psi = d_{\mathcal{P}} \beta$, such that $\mathcal{O}$ is the image of G/H under the (unique) multi-moment map ν.*

The closed geometry may be realised on the orbit $\mathcal{O}$ itself if and only if

$$\operatorname{stab}_{\mathfrak{g}} \beta = \ker(d_{\mathcal{P}} \beta). \tag{5.18}$$

In this situation, the orbit is k-plectic and ν is simply the inclusion $\mathcal{O} \hookrightarrow \mathcal{P}_{\mathfrak{g}}^*$.

Proof. It only remains to prove the assertions of the last paragraph of the theorem. We have $\mathcal{O} = G/K$ with $K = \operatorname{stab}_G \beta$, a closed subgroup of G. Now equation (5.18), shows that K has Lie algebra $\ker(d_{\mathcal{P}} \beta)$, so the component of the identity K^0 of K is $K^0 = K_\Psi$ for $\Psi = d_{\mathcal{P}} \beta$. In particular, Ψ vanishes on elements of $\mathfrak{k}$ and induces a well-defined form on $T_\beta \mathcal{O} = \mathfrak{g} / \mathfrak{k}$. The result now follows. □

Remark 5.29. In the case when $k = 1$, we have

$$\operatorname{ad}_X^* \beta = -X \lrcorner d_{\mathcal{P}} \beta - (X \lrcorner \beta) \circ L = -X \lrcorner d\beta \quad X \in \mathfrak{g},$$

by (4.13), since $L \equiv 0$ and $d_{\mathcal{P}} = d$ for $k = 1$. So (5.18) automatically holds, and the coadjoint orbit in $\mathfrak{g}^*$ is a symplectic manifold endowed with the celebrated Kirillov-Kostant-Sourieau symplectic structure. △

Example 5.30. Suppose G is a $(k, k+1)$-trivial Lie group. Then, taking $H = \{e\}$, we see that every $\Psi \in Z^{k+1}(\mathfrak{g})$ gives rise to a closed geometry on G with multi-moment map whose image is diffeomorphic to the G-orbit of Ψ. ♢

Chapter 6

Exceptional holonomy metrics and torus symmetry

METRICS OF EXCEPTIONAL HOLONOMY have received much attention from both mathematicians and physicists over the years. The mathematical motivation for studying exceptional holonomy metrics was initiated with Berger's classification of possible holonomy groups for irreducible non-symmetric Riemannian manifolds [Ber55], though their existence was first shown much later in Bryant's paper [Bry87]. Significant results then followed, in particular it is worth mentioning the complete exceptional holonomy metrics discovered by Bryant and Salamon [BS89] and Joyce's constructions [Joy96b, Joy96a, Joy00] of compact Riemannian manifolds with holonomy G_2 and $Spin(7)$. The ideas of Bryant-Salamon and Joyce greatly influenced later developments. While some authors have studied metrics of cohomogeneity-one [CS02b, DW04, Rei10b], others have extended and refined Joyce's methods [Nor08, KN10, Cla10]. From the physical perspective one motivation for studying exceptional holonomy metrics comes from superstring theories [AW02, AG04, CGLP02b, CGLP03a, GS02, SS09].

In this chapter we study how to reduce toric torsion-free G_2- and $Spin(7)$-manifolds to tri-symplectic four-manifolds. We also explain how to obtain all torsion-free G_2- and $Spin(7)$-manifolds with free T^2- or T^3-symmetry, respectively, starting from tri-symplectic four-manifolds. In this way we obtain a local classification result, which is similar to the Gibbons-Hawking ansatz for hyper-Kähler surfaces with circle symmetry. Finally, we present several examples that illustrate our reduction and reconstruction procedures. Some of the examples complement previous ones that have appeared in the context of domain wall problems in supergravity theories [GLPS02, MM05, GS07].

6.1 Reduction of torsion-free G_2-manifolds

Let us recall the fundamental aspects of G_2-geometry from [Bry87]. On $\mathbb{R}^7$ we consider the three-form ϕ_0 given by

$$\phi_0 = e_{123} + e_1(e_{45} + e_{67}) + e_2(e_{46} - e_{57}) - e_3(e_{47} + e_{56}), \tag{6.1}$$

where $e_1, \dots, e_7$ is the standard dual basis and $\wedge$ signs have been omitted. The stabiliser of ϕ_0 is the compact 14-dimensional Lie group

$$G_2 = \{\, g \in GL(7, \mathbb{R}) : g^*\phi_0 = \phi_0 \,\}.$$

This group preserves the standard metric on $g_0 = \sum_{i=1}^{7} {e_i}^2$ on $\mathbb{R}^7$ and the volume form $\mathrm{vol}_0 = e_{1234567}$. These tensors are uniquely determined by ϕ_0 via the relation $6g_0(X,Y)\,\mathrm{vol}_0 = (X \lrcorner \phi_0) \wedge (Y \lrcorner \phi_0) \wedge \phi_0$. The Hodge $*$-operator gives a four-form

$$*\phi_0 = e_{4567} + e_{23}(e_{67} + e_{45}) + e_{13}(e_{57} - e_{46}) - e_{12}(e_{56} + e_{47}).$$

A G_2-structure on a seven-manifold Y is given by a three-form $\phi \in \Omega^3(Y)$ which is linearly equivalent at each point to ϕ_0. It determines a metric g, a volume form vol and a four-form $*\phi$ on Y. The G_2-structure is called *torsion-free* if both of the forms ϕ and $*\phi$ are closed. This happens precisely when $\nabla^{\mathrm{LC}}\phi = 0$ [FG82]. One then calls (Y, ϕ) a torsion-free G_2-manifold. In this situation the metric g has holonomy contained in G_2 and is Ricci-flat. This implies real-analyticity of g in harmonic coordinates.

Since a torsion-free G_2-geometry comes equipped with a closed three-form, we may study multi-moment maps for such manifolds. Let us assume that (Y, ϕ) has a two-torus symmetry with a non-constant multi-moment map $\nu\colon Y \to \mathcal{P}^*_{\mathbb{R}^2} \cong \mathbb{R}$. Choosing generating vector fields U and V for the T^2-action, we have $d\nu = \phi(U, V, \cdot)$. The latter is non-zero if and only if U and V are linearly independent. So T^2 acts locally freely on some open set $Y_0 \subset Y$.

Remark 6.1. On a Ricci-flat manifold the metric dual one-form of an infinitesimal isometry lies in the kernel of the Laplacian [Kob72, Theorem 2.3]. This implies, by elliptic regularity, that the generating vector fields U, V are real-analytic. △

We may define three two-forms on Y_0 by

$$\omega_0 = V \lrcorner U \lrcorner *\phi, \quad \omega_1 = U \lrcorner \phi \quad \text{and} \quad \omega_2 = V \lrcorner \phi.$$

To relate these to the G_2-structure, consider the positive function h and one-forms θ_i given by

$$(g_{UU}g_{VV} - g_{UV}^2)\,h^2 = 1$$
$$\theta_1 = h^2(g_{VV}U^\flat - g_{UV}V^\flat), \quad \theta_2 = h^2(g_{UU}V^\flat - g_{UV}U^\flat),$$

where $U^\flat = g(U, \cdot)$ and $g_{UU} = g(U,U)$, etc. Note that h is well-defined on Y_0, and that (θ_1, θ_2) is dual to (U, V).

Proposition 6.2. *On Y_0, the three-form ϕ and the four-form $*\phi$ are*

$$\begin{aligned} \phi &= h^2\omega_0 \wedge d\nu + \omega_1 \wedge \theta_1 + \omega_2 \wedge \theta_2 + d\nu \wedge \theta_2 \wedge \theta_1, \\ *\phi &= \omega_0 \wedge \theta_1 \wedge \theta_2 + h^2\big(g_{VV}\omega_1 \wedge \theta_2 \wedge d\nu - g_{UU}\omega_2 \wedge \theta_1 \wedge d\nu \\ &\qquad + g_{UV}(\omega_1 \wedge \theta_1 - \omega_2 \wedge \theta_2) \wedge d\nu + \tfrac{1}{2}\omega_0 \wedge \omega_0\big). \end{aligned}$$

Proof. Working locally at a point and using the T^2-action we may write the first two standard basis elements of $\mathbb{R}^7$ as $E_1 = aU = U/g_{UU}^{1/2}$, $E_2 = bU + cV = hg_{UU}^{1/2}(V - g_{UV}g_{UU}^{-1}U)$. We then have $\theta_1 = ae_1 + be_2$ and $\theta_2 = ce_2$. Now using (6.1) we get $ac\,d\nu = e_3$, $ac\,\omega_0 = -(e_{56} + e_{47})$, $a\,\omega_1 = e_{23} + e_{45} + e_{67}$ and

$$ac\,\omega_2 = -a(e_{13} - e_{46} + e_{57}) - b(e_{23} + e_{45} + e_{67}).$$

The given expressions now follow. □

Now suppose that $t \in \nu(Y_0) \subset \mathbb{R}$ is a regular value for $\nu\colon Y_0 \to \mathbb{R}$. Then $\mathcal{X}_t = \nu^{-1}(t)$ is a real-analytic hypersurface with unit normal $N = h(d\nu)^\sharp$. This inherits an $SU(3)$-structure $(\sigma, \psi_\pm)$ given by

$$\begin{gathered}\sigma = N \lrcorner \phi = h\omega_0 + h^{-1}\theta_1 \wedge \theta_2, \quad \psi_+ = \iota^*\phi = \iota^*\omega_1 \wedge \theta_1 + \iota^*\omega_2 \wedge \theta_2, \\ \psi_- = -N \lrcorner *\phi = h\big(g_{VV}\iota^*\omega_1 \wedge \theta_2 - g_{UU}\iota^*\omega_2 \wedge \theta_1 \\ + g_{UV}(\iota^*\omega_1 \wedge \theta_1 - \iota^*\omega_2 \wedge \theta_2)\big),\end{gathered} \tag{6.2}$$

where $\iota\colon \mathcal{X}_t \to Y_0$ is the inclusion. As shown in [CS02a], oriented hypersurfaces in torsion-free G_2-manifolds are *half-flat*, meaning that

$$\sigma \wedge d\sigma = 0 \quad \text{and} \quad d\psi_+ = 0. \tag{6.3}$$

Suppose T^2 acts freely on $\mathcal{X}_t = \nu^{-1}(t)$.

Definition 6.3. The T^2*-reduction* of Y at level t is the four-manifold

$$M = \nu^{-1}(t)/T^2 = \mathcal{X}_t/T^2.$$

Proposition 6.4. *The T^2-reduction M carries three pointwise linearly independent symplectic forms defining the same orientation.*

Proof. Consider the two-forms ω_0, ω_1, ω_2 on Y_0. These forms are T^2-invariant and closed, since $d\omega_0 = \mathcal{L}_V(U \lrcorner *\phi) = 0$ and $d\omega_1 = \mathcal{L}_U\phi = 0$, cf. (4.1). Furthermore, as $V \lrcorner \omega_1 = d\nu$, their pull-backs to $\mathcal{X}_t = \nu^{-1}(t)$ are basic. Thus they descend to three closed forms σ_0, σ_1 and σ_2 on M. The proof of Proposition 6.2 shows that at a point $h\sigma_0 = -(e_{56} + e_{47})$, $h\sigma_1 = c(e_{45} + e_{67})$ and $h\sigma_2 = a(e_{46} + e_{75}) - b(e_{45} + e_{67})$, with $ac = h \neq 0$. Thus σ_0, σ_1 and σ_2 are non-degenerate symplectic forms defining the same orientation. □

The expressions for the forms in this proof show that they satisfy the following relations on M:

$$\begin{gathered}h^2\,{\sigma_0}^2 = g_{UU}^{-1}\,{\sigma_1}^2 = g_{VV}^{-1}\,{\sigma_2}^2 = 2\,\mathrm{vol}_M, \\ \sigma_0 \wedge \sigma_1 = 0 = \sigma_0 \wedge \sigma_2, \quad \sigma_1 \wedge \sigma_2 = 2g_{UV}\,\mathrm{vol}_M\,.\end{gathered} \tag{6.4}$$

Here vol_M is induced by the element e_{4567} on in Y, which is the volume element on directions orthogonal to the T^2-action on $\mathcal{X}_t$. Note that (θ_1, θ_2) is a connection one-form for $\mathcal{X}_t \to M$ regarded as a principal T^2-bundle.

Inversion via a flow We now consider how this construction may be inverted, producing the G_2-geometry of Y from a triple of symplectic forms on a four-manifold M. Note that the relations (6.4) show that the symplectic forms σ_i define the same orientation on M and are pointwise linearly independent. Indeed the intersection matrix $\widetilde{Q} = (q_{ij})$ with $\sigma_i \wedge \sigma_j = q_{ij}\sigma_0^2$, for $i, j = 1, 2, 3$, is positive definite. As in [DK90], the positive three-dimensional subbundle $\Lambda^+ = \langle \sigma_0, \sigma_1, \sigma_2 \rangle \subset \Lambda^2 T^*M$ corresponds to a unique oriented conformal structure on M.

Definition 6.5. A *coherent symplectic triple* $\mathscr{C}$ on a four-manifold M consists of three symplectic forms σ_0, σ_1, σ_2 that pointwise span a maximal positive subspace of $\Lambda^2 T^*M$ and satisfy $\sigma_0 \wedge \sigma_i = 0$ for $i = 1, 2$.

Let $Q = (q_{ij})_{i,j=1,2}$ be the lower-right 2×2 submatrix of $\widetilde{Q}$. Since $\det Q$ is positive, we may write $h = \sqrt{\det Q} \in C^\infty(M)$.

Proposition 6.6. *Let $(M, \mathscr{C})$ be a coherently tri-symplectic four-manifold. Suppose $\mathcal{X}$ is a principal T^2-bundle over M with connection one-form $\Theta = (\theta_1, \theta_2)$. Then the forms σ, $\psi_\pm$ given by*

$$\begin{aligned} \sigma = h\sigma_0 + h^{-1}\theta_1 \wedge \theta_2, \quad \psi_+ = \sigma_1 \wedge \theta_1 + \sigma_2 \wedge \theta_2, \\ \psi_- = h^{-1}(q_{22}\sigma_1 \wedge \theta_2 - q_{11}\sigma_2 \wedge \theta_1 + q_{12}(\sigma_1 \wedge \theta_1 - \sigma_2 \wedge \theta_2)) \end{aligned} \tag{6.5}$$

define an $SU(3)$-structure on $\mathcal{X}$. This structure is half-flat if and only if $d\Theta^+ = (\sigma_1, \sigma_2)A$ with $\langle A, Q \rangle = \operatorname{Tr}(AQ) = 0$.

Proof. Choose a conformal basis $e_4, \dots, e_7$ of T_x^*M so that $h\sigma_i$ are as in the proof of Proposition 6.4 with $c^2 = q_{11}$, $bc = -q_{12}$ and $a^2 = q_{22} - b^2$. This is consistent with the equation $ac = h$. Now inspired by the proof of Proposition 6.2 we write $\theta_1 = ae_1 + be_2$ and $\theta_2 = ce_2$. The basis $e_1, e_2, e_7, e_4, e_6, e_5$ is then an $SU(3)$-basis for $T^*\mathcal{X}$, with defining forms given via (6.2) for $g_{UU} = q_{11}/h^2$, $g_{UV} = q_{12}/h^2$ and $g_{VV} = q_{22}/h^2$.

For the final assertion we need to study the equations (6.3). Firstly, $\sigma \wedge d\sigma = \sigma_0 \wedge d\theta_1 \wedge \theta_2 + \sigma_0 \wedge d\theta_2 \wedge \theta_1$, which vanishes only if $d\Theta^+$ is orthogonal to σ_0. This implies that $d\Theta^+$ is a linear combination $(\sigma_1, \sigma_2)A$ of σ_1 and σ_2. Now $d\psi_+ = \sigma_1 \wedge d\theta_1 + \sigma_2 \wedge d\theta_2$, and the vanishing of $d\psi_+$ gives the constraint $\operatorname{Tr}(AQ) = 0$. □

Remark 6.7. The $SU(3)$-structures found here are more general than those studied in [GP04] since the connection one-forms are not orthonormal. △

Remark 6.8. Existence of two-torus bundles over a coherent tri-symplectic four-manifold $(M, \mathscr{C})$ is related to Chern-Weil theory. One finds that for any closed two-form F with integral periods, $F \in \Omega^2_{\mathbb{Z}}(M, \mathbb{R}^2)$, there exists a T^2-bundle $\pi_M \colon \mathcal{X} \to M$ with connection one-form Θ that satisfies $\pi_M^*(d\Theta) = F$. If such a two-form has self-dual part F_+ satisfying the orthogonality condition $\langle F_+, Q \rangle = 0$ of Proposition 6.6, then we will say that F is *orthogonal*. △

Studying a certain Hamiltonian flow, Hitchin [Hit01] developed a relationship between torsion-free G_2-metrics and half-flat $SU(3)$-manifolds, see also [CLSSH11]. In particular, he derived evolution equations that describe the one-dimensional flow of a half-flat $SU(3)$-manifold along its unit normal in a torsion-free G_2-manifold. When the flow equations have a solution, this determines a torsion-free G_2-metric from a half-flat $SU(3)$-manifold. In inverting our construction, one could use Hitchin's flow on the half-flat structure of Proposition 6.6. However, Hitchin's flow does not preserve the level sets of the multi-moment map: the unit normal is $h(d\nu)^\sharp$, but $\partial/\partial\nu = h^2(d\nu)^\sharp$. It is thus more natural for us to determine the flow equations associated to the latter vector field.

Proposition 6.9. *Suppose T^2 acts freely on a connected seven-manifold Y preserving a torsion-free G_2-structure ϕ and admitting a multi-moment map ν. Let M be the topological reduction $\nu^{-1}(t)/T^2$ for any t in the image of ν. Then M is equipped with a t-dependent coherent symplectic triple $\sigma_0, \sigma_1, \sigma_2$ and $\mathcal{X}_t = \nu^{-1}(t)$ carries the half-flat $SU(3)$-structure $(\sigma, \psi_\pm)$ of Proposition 6.6. The forms on $\mathcal{X}_t$ satisfy the following system of differential equations:*

$$\begin{aligned} \psi_+' &= d(h\sigma) \\ (\tfrac{1}{2}\sigma^2)' &= -d(h\psi_-), \end{aligned} \tag{6.6}$$

where $'$ denotes differentiation with respect to t.

Conversely, given a real-analytic half-flat $SU(3)$-structure of the form (6.5) on a six-manifold $\mathcal{X}_0$. Then the system (6.6) admits a unique solution on some neighbourhood of $\mathcal{X}_0 \times \{0\} \subset \mathcal{X}_0 \times \mathbb{R}$ and that solution determines a torsion-free G_2-structure.

Proof. We have

$$\phi = \sigma \wedge h d\nu + \psi_+ \quad \text{and} \quad *\phi = \psi_- \wedge h d\nu + \tfrac{1}{2}\sigma^2.$$

These have derivatives

$$\begin{aligned} d\phi &= (hd\sigma + dh \wedge \sigma) \wedge d\nu + d\psi_+, \\ d{*\phi} &= (hd\psi_- + dh \wedge \psi_-) \wedge d\nu + \sigma \wedge d\sigma \end{aligned}$$

Half-flatness of $(\sigma, \psi_\pm)$ gives $d\phi = 0 = d{*\phi}$ if and only if

$$0 = \frac{\partial}{\partial\nu} \lrcorner\, d\phi = -d(h\sigma) + \psi_+' \quad \text{and} \quad 0 = \frac{\partial}{\partial\nu} \lrcorner\, d{*\phi} = d(h\psi_-) + \sigma \wedge \sigma'.$$

Hence we have a torsion-free G_2-structure if and only if the evolution equations (6.6) are satisfied.

Given real-analytic initial data, the Cauchy-Kovalevskaya theorem (see, e.g., [BCG$^+$91, Theorem 2.1] or [Spi75, Chapter 10.4]) applies and provides us with a unique solutions of the evolution equations on an open neighbourhood of $\mathcal{X}_0 \times \{0\} \subset \mathcal{X}_0 \times \mathbb{R}$.

We now rewrite the evolution equations as a set of first order differential equations for the data on M. Firstly, the derivatives of σ_0, σ_1, σ_2 and h with respect to $\partial/\partial v$ are:

$$\begin{gathered}\sigma_0' = 0, \quad \sigma_1' = -d\theta_2, \quad \sigma_2' = d\theta_1, \\ hh'\sigma_0^2 = (q_{11}\sigma_2 - q_{12}\sigma_1) \wedge d\theta_1 + (q_{12}\sigma_2 - q_{22}\sigma_1) \wedge d\theta_2.\end{gathered} \tag{6.7}$$

Using (6.7) and the definition of Q, we obtain the following equations:

$$q_{11}'\sigma_0^2 = -2\sigma_1 \wedge d\theta_2, \quad q_{22}'\sigma_0^2 = 2\sigma_2 \wedge d\theta_1, \quad q_{12}'\sigma_0^2 = \sigma_1 \wedge d\theta_1 - \sigma_2 \wedge d\theta_2. \tag{6.8}$$

If we combining (6.5) and (6.6), we get the following relations for the derivatives of the connection one-form

$$\sigma_0 \wedge \theta_1' = dq_{12} \wedge \sigma_2 - dq_{22} \wedge \sigma_1, \quad \sigma_0 \wedge \theta_2' = dq_{11} \wedge \sigma_2 - dq_{12} \wedge \sigma_1. \tag{6.9}$$

Finally let us verify that these equations together with an initial half-flat $SU(3)$-structure on $\mathcal{X}_0$ of the form (6.5) already ensure that the family consists of half-flat structures. Firstly we note that the flow equations (6.6) ensure that the conditions $\sigma \wedge d\sigma = 0$ and $d\psi_+ = 0$ are preserved for all times. Next we observe that the normalisation

$$\sigma^3 = \psi_+ \wedge \psi_-$$

automatically holds, by construction of the defining forms $(\sigma, \psi_\pm)$ and the functions q_{ij} and h. Hence, in order to have a family of half-flat structures, we must verify that the condition

$$\sigma \wedge \psi_+ = 0$$

is preserved for all times. To show this we note, by inspection, that (6.7) implies that if $d\sigma_i(0) = 0$, $i = 0, 1, 2$, then $d\sigma_i = 0$ for all times. Combining this with (6.9) enable us to conclude that that

$$(\sigma_0 \wedge d\theta_1)' = 0 = (\sigma_0 \wedge d\theta_2)'.$$

As $d\theta_1(0), d\theta_2(0) \in \langle\sigma_0\rangle^\perp$, we thus have $\sigma_0 \wedge d\theta_i = 0$ for all times. We may use this to deduce that

$$(\sigma_0 \wedge \sigma_1)' = 0 = (\sigma_0 \wedge \sigma_2)'.$$

Since we start out with a coherent triple, we deduce that

$$\sigma_0 \wedge \sigma_1 = 0 = \sigma_0 \wedge \sigma_2$$

for all times. Hence, σ_0 lies pointwise in $\langle\sigma_1, \sigma_2\rangle^\perp$, which clearly ensures that $\sigma \wedge \psi_+ = 0$, as required. □

Remark 6.10. By solving the flow equations we obtain a holonomy G_2-metric with T^2-symmetry. Indeed, if g_M is the time-dependent metric in the conformal class on M with volume form $\frac{1}{2}h^2\sigma_0^2$, then the G_2-metric is explicitly

$$h^2dt^2 + g_M + h^{-2}(q_{11}\theta_1^2 + q_{22}\theta_2^2 + q_{12}(\theta_1\theta_2 + \theta_2\theta_1)).$$

Note that Bryant's study of the Hitchin flow [Bry10] shows that non-analytic initial data can lead to an ill-posed Hitchin system that has no solution. △

Summarising the results of this section we have:

Theorem 6.11. *Let (Y^7, ϕ) be a torsion-free G_2-structure with a free T^2-symmetry and admitting a multi-moment map. Then the reduction M at a level t is a real-analytic coherently tri-symplectic four-manifold and the level set $\mathcal{X}_t$ is the total space of a T^2-bundle over M satisfying the orthogonality condition on $F_+ = d\Theta^+$ of Proposition 6.6.*

Conversely for real-analytic data, a coherently tri-symplectic four-manifold together with an orthogonal $F \in \Omega^2_{\mathbb{Z}}(M, \mathbb{R}^2)$ define a torsion-free G_2-metric with T^2-symmetry. □

6.1.1 Examples

Let us now study some examples that illustrate the analysis of the previous section. First we show that even in the flat case $\mathbb{R}^7$, with isometric action given by maximal torus $T^2 \subset SU(3)$ acting via diagonal matrices, the geometry of the reduction procedure is quite complicated. Thereafter we study multi-moment maps associated with some of the known examples of torsion-free cohomogeneity-one G_2-structures. Finally we investigate hyperKähler four-manifolds, complementing previous examples that have appeared in the context of domain-wall problems in supergravity [GLPS02, MM05, GS07].

Example 6.12. Consider $Y = \mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3$ endowed with the usual three-form and the action of the standard diagonal maximal torus $T^2 \subset SU(3)$. Concretely, ϕ is given by

$$\phi = \tfrac{i}{2}dx \wedge (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3) + \operatorname{Re}(dz_1 \wedge dz_2 \wedge dz_3),$$

and T^2 acts by $(e^{i\theta}, e^{i\varphi}) \cdot (x, z_1, z_2, z_3) = (x, e^{i\theta}z_1, e^{i\varphi}z_2, e^{-i(\theta+\varphi)}z_3)$. The action is generated by the vector fields $U = \operatorname{Re}\{i(z_1\frac{\partial}{\partial z_1} - z_3\frac{\partial}{\partial z_3})\}$ and $V = \operatorname{Re}\{i(z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3})\}$. It follows that the multi-moment map $\nu\colon Y \to \mathbb{R}$ is given by

$$\nu(x, z_1, z_2, z_3) = -\tfrac{1}{4}\operatorname{Re}(z_1z_2z_3).$$

By definition, the T^2-reduction of Y at level t is the quotient space $M_t = \nu^{-1}(t)/T^2$. In this case M_0 is singular, whereas M_t is a smooth manifold for each $t \neq 0$. Indeed considering $\Phi_t\colon M_t \to \mathbb{R}^4$ given by

$$\begin{aligned}\Phi_t(x, z_1, z_2, z_3) &= \big(x, \tfrac{1}{2}(\|z_1\|^2 - \|z_3\|^2), \tfrac{1}{2}(\|z_2\|^2 - \|z_3\|^2), \operatorname{Im}(z_1z_2z_3)\big)\\ &=: (x, u, v, w)\end{aligned}$$

we have global smooth coordinates on M_t for $t \neq 0$.

In this smooth case, writing $4\eta_u = h^2(g_{VV}du - g_{UV}dv)$ and $4\eta_v = h^2(g_{UU}dv - g_{UV}du)$, the two-forms σ_0, σ_1, σ_2 are given by

$$4\sigma_0 = dx \wedge dw + dv \wedge du, \quad 2\sigma_1 = dx \wedge du + dw \wedge \eta_v,$$
$$2\sigma_2 = dx \wedge dv + \eta_u \wedge dw.$$

These forms depend (implicitly) on t via the relations $4g_{UU} = \|z_1\|^2 + \|z_3\|^2$, $4g_{VV} = \|z_2\|^2 + \|z_3\|^2$, $4g_{UV} = \|z_3\|^2$ and $z_1z_2z_3 = -4t + iw$. In particular, g_{UV} is a non-constant function, so the coherent triple does not specify a hyperKähler a structure. The (oriented) conformal class has representative metric

$$dx^2 + \frac{h^2}{16}dw^2 + 4g_{UU}\eta_u^2 + 4g_{VV}\eta_v^2 + 4g_{UV}(\eta_u\eta_v + \eta_v\eta_u).$$

The curvature of the principal bundle $\nu^{-1}(t) \to M_t$ is given by

$$4d\theta_1 = th^4 dw \wedge ((2g_{VV} - g_{UV})\eta_u + (g_{VV} - 2g_{UV})\eta_v)$$
$$4d\theta_2 = th^4 dw \wedge ((g_{UU} - 2g_{UV})\eta_u + (2g_{UU} - g_{UV})\eta_v).$$

In the singular case $t = 0$, the two-torus collapses in two ways: to a point along the real axis $\mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{C}^3$ and to a circle away from $\mathbb{R} \times \{0\}$ along $z_1 = z_2 = 0$, $z_1 = z_3 = 0$ or $z_2 = z_3 = 0$. The collapsing happens when $w = 0$ and u, v satisfy one of the following three constraints: $(u = v \leqslant 0)$, $(u = 0,\ v \geqslant 0)$ or $(u \geqslant 0,\ v = 0)$. ◇

Example 6.13. A Lie group G acts on (Y, ϕ) with cohomogeneity 1 if G preserves ϕ and the largest G-orbits are of dimension six. Cohomogeneity-one G_2-structures have been studied by a number of authors [CS02b, CGLP02a, DW04]. This class is particularly interesting, since it includes the complete holonomy G_2-metrics discovered by Bryant and Salamon [BS89]. As almost effective spaces, the principal orbits for a cohomogeneity-one G_2-structure are of the form G/K with K acting on the isotropy representation as a subgroup of $SU(3)$. A case-by-case study [CS02b, Theorem 3.1] (see also [Rei10a, Theorem 1 & Remark 5.3]) gives the following list of possibilities, up to finite covers:

$$S^6 = \frac{G_2}{SU(3)}, \quad \mathbb{C}P(3) = \frac{Sp(2)}{SU(2)\,U(1)}, \quad F_{1,2}(\mathbb{C}^3) = \frac{SU(3)}{T^2},$$
$$S^3 \times S^3 = \frac{SU(2)^3}{SU(2)} = \frac{SU(2)^2T^1}{T^1} = SU(2)^2, \quad V_2(\mathbb{R}^4) \times T^1 = \frac{SO(4)}{SO(2)} \times T^1,$$
$$S^5 \times S^1 = \frac{SU(3)T^1}{SU(2)}, \quad S^3 \times (S^1)^3 = SU(2)T^3, \quad (S^1)^6 = T^6.$$

Case $F_{1,2}(\mathbb{C}^3)$ Let us consider cohomogeneity-one G_2-structures with $SU(3)$-symmetry. The principal orbits are thus $F_{1,2}(\mathbb{C}^3) = SU(3)/T^2_R$, and the principal isotropy group $K = T^2_R = S^1_U \times S^1_V$ acts on the standard representation

$\Lambda^{1,0} \cong \mathbb{C}^3$ as $L_U + L_V + \bar{L}_U\bar{L}_V$, where $L_U, L_V \cong \mathbb{C}$ are the standard representations of $S^1_U, S^1_V \cong U(1)$. From the isomorphism $\mathfrak{su}(3) \otimes \mathbb{C} \cong \Lambda^{1,1}_0$ we see that the isotropy representation is $[\![L_1\bar{L}_2]\!] + [\![L_1L_2^2]\!] + [\![L_1^2L_2]\!]$, where $[\![L_1\bar{L}_2]\!]$ denotes the real vector space underlying $L_1\bar{L}_2$. A careful analysis [CS02b] now shows that any $SU(3)$-invariant torsion-free G_2-structure on $Y = I \times SU(3)/T^2_R$ can be put on the form

$$\begin{aligned}\phi = 4(&f_3^2 b_{12}c_{12} - f_2^2 b_{13}c_{13} + f_1^2 b_{23}c_{23})ds \\ &+ 8\epsilon f_1 f_2 f_3(b_{12}b_{13}c_{23} + b_{12}b_{23}c_{13} + b_{13}b_{23}c_{12} + c_{12}c_{13}c_{23}),\end{aligned} \tag{6.10}$$

at the point $(s, eT^2_R) \in Y$. In the above, $b_{12}, \dots, c_{23}$ denote elements from our usual basis in $\mathfrak{su}(3)^*$, cf. Example 2.4. The parameter ϵ is a fixed number ± 1, and f_1, f_2, f_3 are non-vanishing real functions on $I \subset \mathbb{R}$. These quantities must satisfy the following set of differential equations in the parameter s on I:

$$(f_1^2 f_2^2)' = (f_2^2 f_3^2)' = (f_3^2 f_1^2)' = 2\epsilon f_1 f_2 f_3, \quad (\epsilon f_1 f_2 f_3)' = \frac{1}{2}(f_1^2 + f_2^2 + f_3^2). \tag{6.11}$$

The system (6.11) ensures that the G_2-form ϕ closed and co-closed [CS02b]. By integration, we obtain three functions $F_1, F_2, F_3 \colon I \to \mathbb{R}$ satisfying the equations

$$\begin{gathered}F_1' = \frac{1}{2}(f_2^2 + f_3^2),\ F_2' = \frac{1}{2}(f_3^2 + f_1^2),\ F_3' = \frac{1}{2}(f_1^2 + f_2^2), \\ 3\epsilon f_1 f_2 f_3 - F_1 - F_2 - F_3 = \epsilon f_1 f_2 f_3.\end{gathered} \tag{6.12}$$

Define a two-form β on Y given by

$$\begin{aligned}\beta = 8((&\epsilon f_1 f_2 f_3 - F_3)b_{12}c_{12} - (\epsilon f_1 f_2 f_3 - F_2)b_{13}c_{13} \\ &+ (\epsilon f_1 f_2 f_3 - F_1)b_{23}c_{23}),\end{aligned} \tag{6.13}$$

at the point (s, eT^2_R). The vector fields

$$U_{(s,gT^2_R)} = (R_g)_*(A_1) \quad \text{and} \quad V_{(s,gT^2_R)} = (R_g)_*(A_2).$$

are infinitesimal generators of a left action of $T^2_L \subset SU(3)$ on Y, and β is clearly invariant under this action. Since

$$\begin{aligned}d(b_{12}c_{12}) &= d(c_{13}b_{13}) = d(b_{23}c_{23}) \\ &= b_{12}b_{13}c_{23} + b_{12}b_{23}c_{13} + b_{13}b_{23}c_{12} + c_{12}c_{13}c_{23},\end{aligned}$$

direct calculation shows that $\phi = d\beta$. By Theorem 4.8, the strong geometry (Y, ϕ) therefore admits a multi-moment map $\nu \colon Y \to \mathbb{R}$ given by

$$(s, gT^2_R) \mapsto \beta_{(s,gT^2_R)}(U \wedge V). \tag{6.14}$$

We can write the map (6.14) more explicitly. If we think of a point in $F_{1,2}(\mathbb{C}^3)$ as an element $g = \left(\begin{smallmatrix} A & B & C \\ D & E & F \\ G & H & K \end{smallmatrix}\right) \in SU(3) \subset M(3, \mathbb{C})$ modulo the right T^2_R-action, then we claim that

$$\nu(y) = -24\epsilon f_1 f_2 f_3 \operatorname{Im}(\overline{A}BD\overline{E}) \tag{6.15}$$

at $y = (s, gT^2_R) \in Y$. In order to verify this formula, let us spell out the quantities involved in determining the multi-moment map. Firstly, note that if we, momentarily, ignore the T^2_R-action, then

$$\begin{aligned}
\underline{b}_{12}(U_g) &= b_{12}((L_{g^{-1}})_*A_1(R_g)_*) = \tfrac{i}{2}(\overline{A}B - \overline{B}A + D\overline{E} - \overline{D}E),\\
\underline{b}_{12}(V_g) &= b_{12}((L_{g^{-1}})_*A_2(R_g)_*) = \tfrac{i}{2}(\overline{D}E - D\overline{E} + G\overline{H} - \overline{G}H),\\
\underline{c}_{12}(U_g) &= \tfrac{1}{2}(\overline{A}B + A\overline{B} - D\overline{E} - \overline{D}E),\\
\underline{c}_{12}(V_g) &= \tfrac{1}{2}(\overline{D}E + D\overline{E} - G\overline{H} - \overline{G}H),\\
\underline{b}_{13}(U_g) &= \tfrac{i}{2}(\overline{A}C - A\overline{C} + D\overline{F} - \overline{D}F),\\
\underline{b}_{13}(V_g) &= \tfrac{i}{2}(\overline{D}F - D\overline{F} + G\overline{K} - \overline{G}K),\\
\underline{c}_{13}(U_g) &= \tfrac{1}{2}(\overline{A}C + A\overline{C} - D\overline{F} - \overline{D}F),\\
\underline{c}_{13}(V_g) &= \tfrac{1}{2}(\overline{D}F + D\overline{F} + G\overline{K} - \overline{G}K),\\
\underline{b}_{23}(U_g) &= \tfrac{i}{2}(\overline{B}C - B\overline{C} + E\overline{F} - \overline{E}F),\\
\underline{b}_{23}(V_g) &= \tfrac{i}{2}(\overline{E}F - E\overline{F} + H\overline{K} - \overline{H}K),\\
\underline{c}_{23}(U_g) &= \tfrac{1}{2}(\overline{B}C + B\overline{C} - E\overline{F} - \overline{E}F),\\
\underline{c}_{23}(V_g) &= \tfrac{1}{2}(\overline{E}F + E\overline{F} - H\overline{K} - \overline{H}K),
\end{aligned}$$

where $\underline{b}_{12}$ denotes the left-translate of b_{12}, and so forth. We then have

$$\begin{aligned}
\underline{b}_{12}\underline{c}_{12}(U_g \wedge V_g) &= \tfrac{i}{2}(\overline{A}BD\overline{E} - \overline{A}BG\overline{H} - A\overline{B}\overline{D}E + A\overline{B}\overline{G}H + \overline{D}EG\overline{H} - D\overline{E}\overline{G}H)\\
&= -3\operatorname{Im}(\overline{A}BD\overline{E}),
\end{aligned}$$

where the last equality uses relations derived from the identities $g^{-1} = g^*$ and $gg^{-1} = 1 = g^{-1}g$; specifically the $(2,1)$-entry of g^*g tells us that $A\overline{B} + D\overline{E} + G\overline{H} = 0$. Similarly, we find that

$$\begin{aligned}
-\underline{b}_{13}\underline{c}_{13}(U_g \wedge V_g) &= \tfrac{i}{2}(A\overline{D}\overline{C}F - \overline{A}DC\overline{F} + \overline{A}CG\overline{K} - A\overline{C}\overline{G}K + D\overline{F}\overline{G}K - \overline{D}FG\overline{K})\\
&= -3\operatorname{Im}(\overline{A}BD\overline{E}),
\end{aligned}$$

$$\begin{aligned}
\underline{b}_{23}\underline{c}_{23}(U_g \wedge V_g) &= \tfrac{i}{2}(E\overline{F}\overline{B}C - \overline{E}FB\overline{C} + \overline{E}FH\overline{K} - E\overline{F}\overline{H}K + B\overline{C}\overline{H}K - \overline{B}CH\overline{K})\\
&= -3\operatorname{Im}(\overline{A}BD\overline{E}).
\end{aligned}$$

From these calculations and the last equality in (6.12), one readily obtains the expression (6.15) for the multi-moment map.

Remark 6.14. It is worth emphasising that the above considerations include the complete Bryant-Salamon metric on the total space $\Lambda^2_-(\mathbb{C}P(2))$ of the bundle of anti-self-dual two-forms over $\mathbb{C}P(2)$ [BS89]. In that case we can simplify the formula (6.15) slightly, since the analysis in [CS02b] enables us to perform a suitable parameter change. Indeed, if we define a positive function r on I by the relation $r^2 = f_1^2 f_3^2$, then we have

$$f_1 f_2 f_3 = r(r^2 + \vartheta^2)^{1/4},$$

where ϑ is a positive constant. △

Remark 6.15. In the case when $f_1 = f_2 = f_3 =: f$, we may write (6.13) in the form $b = \frac{8}{3}\epsilon(f(s))^3(b_{12}c_{12} + c_{13}b_{13} + b_{23}c_{23}) \in \mathcal{P}^*_{\mathfrak{su}(3)}$, for each fixed $s \in I$. From Example 4.4.2.2, we know that the pair $(b, d_{\mathcal{P}}b)$ determines a strict nearly Kähler structure on $F_{1,2}(\mathbb{C}^3) \subset \mathcal{P}^*_{\mathfrak{su}(3)}$. This link between nearly Kähler six-manifolds and metrics with holonomy G_2 is well-known [Bär93, Sal03]. △

Case $S^3 \times S^3$ Let us now turn the attention from the full flag to the homogeneous space $S^3 \times S^3$. Cohomogeneity-one G_2-structures with such principal orbits have been studied by a number of authors [Hit01, CGLP03b, Bra02]. We will adapt the notation used by Brandhuber. To express ϕ we thus introduce two copies of $\mathfrak{su}(2)^*$; each copy is endowed with a cyclic basis (cf. Example 4.4.2.2), say $\{e_i\}$ and $\{f_i\}$, respectively. At a point $(s, e) \in Y = I \times S^3 \times S^3$, we now write ϕ in the form

$$\phi = \psi + d\beta,$$
$$\psi = s_0^3(pe_1 \wedge e_2 \wedge e_3 + qf_1 \wedge f_2 \wedge f_3) \text{ and } \beta = a(s)\sum_{i=1}^{3} e_i \wedge f_i, \tag{6.16}$$

where p, q are integers, and a is an appropriate real function on $I \subset \mathbb{R}$. Torsion-free G_2-structures of this type include the complete Bryant-Salamon metric on the spin bundle of S^3; this corresponds to picking $(p, q) = (-1, 0)$ and $a(s) = \frac{4}{3}(s^3 - s_0^3)$, cf. [Bra02].

In order to obtain a multi-moment map for (Y, ϕ) we consider an isometric left action of $T^2 \subset SU(2) \times SU(2)$ on (Y, ϕ). Concretely, we pick the action generated by the vector fields

$$U_{(s,g)} = (R_g)_*(E_1) \quad \text{and} \quad V_{(s,g)} = (R_g)_*(F_1),$$

where

$$E_1 = \frac{i}{2}\begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad \text{and} \quad F_1 = \frac{i}{2}\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{pmatrix}.$$

Note that

$$\underline{e}_1(V_g) = e_1((L_{g^{-1}})_*F_1(R_g)_*) = 0 = \underline{e}_2(V_g) = \underline{e}_3(V_g),$$
$$\underline{f}_1(U_g) = f_1((L_{g^{-1}})_*E_1(R_g)_*) = 0 = \underline{f}_2(U_g) = \underline{f}_3(U_g),$$

where $\underline{e}_1$ denotes the left-translate of e_1, and so forth. We therefore have that $U \wedge Y \lrcorner \psi = 0$, which combined with (6.16) implies that the strong geometry (Y, ϕ) admits a multi-moment map $\nu\colon Y \to \mathbb{R}$ of the form

$$(s, g) \mapsto \beta_{(s,g)}(U \wedge V).$$

To be explicit, pick an element $g = \left(\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}, \begin{pmatrix} C & -\overline{D} \\ D & \overline{C} \end{pmatrix}\right) \in S^3 \times S^3$. We then have

$$\nu(y) = a(s)\left((|A|^2 - |B|^2)(|C|^2 - |D|^2) + 4\,\mathrm{Re}(AB\overline{CD})\right), \tag{6.17}$$

at the point $y = (s, g) \in Y$. To verify this formula, first note that

$$\underline{e}_1(U_g) = e_1((L_{g^{-1}})_* E_1(R_g)_*) = |A|^2 - |B|^2,\ \underline{f}_1(V_g) = |C|^2 - |D|^2,$$
$$\underline{e}_2(U_g) = -2\,\mathrm{Im}(AB),\ \underline{f}_2(V_g) = -2\,\mathrm{Im}(CD),$$
$$\underline{e}_3(U_g) = -2\,\mathrm{Re}(AB),\ \underline{f}_3(V_g) = -2\,\mathrm{Re}(CD),$$

We thus have

$$\underline{e}_1\underline{f}_1(U_g \wedge V_g) = (|A|^2 - |B|^2)(|C|^2 - |D|^2),$$
$$\underline{e}_2\underline{f}_2(U_g \wedge V_g) = 4\,\mathrm{Im}(AB)\,\mathrm{Im}(CD),$$
$$\underline{e}_2\underline{f}_2(U_g \wedge V_g) = 4\,\mathrm{Re}(AB)\,\mathrm{Re}(CD).$$

From these calculations one easily derives the formula (6.17).

◇

Example 6.16. Let M be a hyperKähler four-manifold. Then M comes equipped with three symplectic forms σ_0, σ_1, σ_2 that satisfy the relations $\sigma_i \wedge \sigma_j = \delta_{ij}\sigma_0^2$. In particular, $(\sigma_0, \sigma_1, \sigma_2)$ forms a coherent symplectic triple, and Q is the identity matrix: $h^2 = q_{11}^2 = q_{22}^2 = 1$ and $q_{12} = 0$. If the two-forms σ_1, σ_2 have integral periods, we may construct a T^2-bundle over M with connection one-form Θ that satisfies $d\Theta = (\sigma_1, \sigma_2)\left(\begin{smallmatrix} \alpha & a \\ b & -\alpha \end{smallmatrix}\right)$ for integers $\alpha, a, b \in \mathbb{Z}$. The total space $\mathcal{X}_0$ of this bundle carries a half-flat $SU(3)$-structure given by (6.5), and the associated metric is complete if the hyperKähler base manifold is complete.

We shall now illustrate how one may solve the flow equations, starting from the above data at initial time $t = 0$. As an a priori simplifying assumption, we consider the case when $(d\Theta)' = 0$, i.e., the principal curvatures are t-independent. Then the differential equations for the symplectic triple simplify considerably:

$$\sigma_0' = 0, \quad \sigma_1' = -a\Omega_1 + \alpha\Omega_2, \quad \sigma_2' = \alpha\Omega_1 + b\Omega_2,$$

where $\Omega_1 = \sigma_1(0)$, $\Omega_2 = \sigma_2(0)$. Integrating these equations, we find that

$$\sigma_0(t) = \sigma_0, \quad \sigma_1(t) = (1 - at)\Omega_1 + \alpha t\Omega_2, \quad \sigma_2(t) = \alpha t\Omega_1 + (1 + bt)\Omega_2.$$

Using this observation, we may rewrite the equations for q_{ij}' as follows:

$$q_{11}' = 2(\alpha^2 + a^2)t - 2a, \quad q_{22}' = 2(\alpha^2 + b^2)t + 2b, \quad q_{12}' = 2\alpha((b - a)t + 1),$$

and from this we see that $Q(t) = (1 + tB)^2$, where $B = \left(\begin{smallmatrix} \alpha & a \\ b & -\alpha \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. As a consequence we have that $dq_{ij}(t) = 0$. Hence, from (6.9), $\Theta' = 0$ so that $\Theta(t) = \Theta$. Moreover, one may check that the function $h(t) = \det(B)t^2 + \mathrm{Tr}(B)t + 1$ evolves in accordance with the equation $hh'\sigma_0^2 = (q_{11}\sigma_2 - q_{12}\sigma_1) \wedge d\theta_1 + (q_{12}\sigma_2 - q_{22}\sigma_1) \wedge d\theta_2$.

The above solution is defined on $\mathcal{X}_0 \times I$, where the interval $I \subset \mathbb{R}$ is determined by non-degeneracy of the matrix $1 + tB$ and $0 \in I$. By uniqueness of the solution on $\mathcal{X}_0 \times I$, we deduce that the property $(d\Theta)' = 0$ is already implied by the initial data, i.e., it is not a simplifying assumption.

The associated torsion-free G_2-structure is determined by the three-form

$$\phi = h(t)^2\sigma_0 \wedge dt + \theta_1 \wedge \theta_2 \wedge dt + \sigma_1(t) \wedge \theta_1 + \sigma_2(t) \wedge \theta_2,$$

and the corresponding holonomy G_2-metric is given by

$$g = h(t)^2 dt^2 + h(t) g_0 + h(t)^{-2}(q_{11}(t)\theta_1^2 + q_{22}(t)\theta_2^2 + q_{12}(t)(\theta_1\theta_2 + \theta_2\theta_1)),$$

where g_0 is the initial hyperKähler metric on M.

If the initial hyperKähler four-manifold is complete, then we may describe completeness properties of g in terms of the matrix B. Provided g remains finite and non-degenerate, completeness corresponds to completeness of $h(t)^2dt^2$ on I, cf. [BO69]. We find that the metric is half-complete, in the terminology of [AS04], precisely when $\det B \geqslant 0$; completeness is obtained only for $B = 0$. $\diamondsuit$

6.2 Reduction of torsion-free Spin(7)-manifolds

We now turn to eight-manifolds with holonomy contained in $Spin(7)$. First let us recall some fundamental aspects of $Spin(7)$-geometry, again following [Bry87]. On $\mathbb{R}^8$ we consider the four-form Φ_0 given by

$$\begin{aligned}\Phi_0 = e_{1234} + (e_{12} + e_{34})(e_{56} + e_{78}) + (e_{13} - e_{24})(e_{57} - e_{68}) \\ - (e_{14} + e_{23})(e_{58} + e_{67}) + e_{5678},\end{aligned} \tag{6.18}$$

where $e_1, \dots, e_8$ is the standard dual basis and $\wedge$ signs have been omitted. The stabiliser of Φ_0 is the compact 21-dimensional Lie group

$$Spin(7) = \{ g \in GL(8, \mathbb{R}) : g^*\Phi_0 = \Phi_0 \}.$$

This group preserves the standard metric $g_0 = \sum_{i=1}^8 e_i{}^2$ on $\mathbb{R}^8$ and the volume form $\mathrm{vol}_0 = e_{12345678}$. These tensors are uniquely determined by Φ_0 via the relations $14\,\mathrm{vol}_0 = \Phi_0^2$ and $(Y \lrcorner X \lrcorner \Phi_0) \wedge (Y \lrcorner X \lrcorner \Phi_0) \wedge \Phi_0 = 6\|X \wedge Y\|^2 \mathrm{vol}_0$, cf. [Kar05]. The form Φ_0 is self-dual, meaning $*\Phi_0 = \Phi_0$.

A $Spin(7)$-structure on an eight-manifold Y is given by a four-form $\Phi \in \Omega^4(Y)$ which is linearly equivalent at each point to Φ_0. It determines a metric g and a volume form vol. The $Spin(7)$-structure is called *torsion-free* if the form Φ is parallel with respect to the Levi-Civita connection, meaning $\nabla^{\mathrm{LC}}\Phi = 0$. This happens precisely when Φ is closed. One then calls (Y, Φ) a torsion-free $Spin(7)$-manifold. In this situation the metric g has holonomy contained in $Spin(7)$ and is Ricci-flat. In particular, g is real-analytic in harmonic coordinates.

Since a torsion-free $Spin(7)$-manifold comes equipped with a closed four-form, we may study multi-moment maps for such manifolds. Assume that (Y, Φ) has a three-torus symmetry, generated by vector fields U_i, necessarily real-analytic [Kob72, Theorem 2.3], and that there is a non-constant multi-moment map ν. Then $d\nu = \Phi(U_1, U_2, U_3, \cdot)$ is non-zero if and only if U_1, U_2 and U_3 are linearly independent, cf. [Fer86]. So T^3 acts locally freely on some open set $Y_0 \subset Y$.

Let us define three two-forms on Y_0 by

$$\omega_1 = U_2 \lrcorner U_3 \lrcorner \Phi, \quad \omega_2 = U_3 \lrcorner U_1 \lrcorner \Phi, \quad \omega_3 = U_1 \lrcorner U_2 \lrcorner \Phi.$$

To relate these to the $Spin(7)$-structure we introduce two $\mathbb{R}^3$-valued one-forms $\theta = (\theta_1, \theta_2, \theta_3)$ and $\Theta = (\Theta_1, \Theta_2, \Theta_3)$. The one-form θ is defined by the formula $\theta = U^\flat G^{-1}$, where $U^\flat$ has entries $U_i^\flat = g(U_i, \cdot)$, and $G^{-1} = (g^{ij})$ denotes the inverse of the matrix $G = (g_{ij})$ that has entries $g_{ij} = g(U_i, U_j)$. Note that $\theta_i(U_j) = \delta_{ij}$. The second $\mathbb{R}^3$-valued one-form is given by the formula $\Theta = h^2 U^\flat$, where h is the positive real-analytic function $h = \sqrt{\det(G^{-1})}$; componentwise we have $\Theta_i = h^2 \sum_{j=1}^3 g_{ij}\theta_j$.

Proposition 6.17. *On Y_0, the four-form Φ is*

$$\begin{aligned}\Phi = dv \wedge (2\,\theta_2 \wedge \theta_3 \wedge \theta_1 + \Theta_1 \wedge \omega_1 + \Theta_2 \wedge \omega_2 + \Theta_3 \wedge \omega_3) \\ + \theta_3 \wedge \theta_2 \wedge \omega_1 + \theta_1 \wedge \theta_3 \wedge \omega_2 + \theta_2 \wedge \theta_1 \wedge \omega_3 + *(dv \wedge \theta_3 \wedge \theta_2 \wedge \theta_1).\end{aligned} \tag{6.19}$$

Proof. Working locally at a point and using the T^3-action we may write the first three standard basis elements of $\mathbb{R}^8$ as $E_1 = k_1 U_1$, $E_2 = k_2 U_1 + \ell_2 U_2$, $E_3 = k_3 U_1 + \ell_3 U_2 + m_3 U_3$ for appropriate functions $k_1, \dots, m_3$. Now, using (6.18), we get $k_1 \ell_2\, \omega_3 = -e_{34} - e_{56} - e_{78}$, $k_1 m_3\, \omega_2 - k_1 \ell_3\, \omega_3 = -e_{24} + e_{57} - e_{68}$ and $-\ell_2 m_3\, \omega_1 + k_2 m_3\, \omega_2 + (\ell_2 k_3 - k_2 \ell_3)\, \omega_3 = e_{14} - e_{58} - e_{67}$. We therefore have

$$\begin{aligned}\ell_2 m_3\, \omega_1 &= -e_{14} + e_{58} + e_{67} - \tfrac{k_2}{k_1}(e_{24} - e_{57} + e_{68}) - \tfrac{k_3}{k_1}(e_{34} + e_{56} + e_{78}) \\ k_1 m_3\, \omega_2 &= -e_{24} + e_{57} - e_{68} - \tfrac{\ell_3}{\ell_2}(e_{34} + e_{56} + e_{78}) \\ k_1 \ell_2\, \omega_3 &= -e_{34} - e_{56} - e_{78}.\end{aligned}$$

Next, we write $\theta_1 = k_1 e_1 + k_2 e_2 + k_3 e_3$, $\theta_2 = \ell_2 e_2 + \ell_3 e_3$ and $\theta_3 = m_3 e_3$. Also note that $h\, dv = e_4$. We then find

$$\begin{aligned}e_{1234} = dv \wedge \theta_3 \wedge \theta_2 \wedge \theta_1, \quad & e_{5678} = *(dv \wedge \theta_3 \wedge \theta_2 \wedge \theta_1), \\ \theta_3 \wedge \theta_2 \wedge \omega_1 &= e_{1234} - e_{23}(e_{58} + e_{67}) - \tfrac{k_2}{k_1} e_{23}(e_{57} - e_{68}) + \tfrac{k_3}{k_1} e_{23}(e_{56} + e_{78}), \\ \theta_1 \wedge \theta_3 \wedge \omega_2 &= e_{1234} + e_{13}(e_{57} - e_{68}) - \tfrac{\ell_3}{\ell_2} e_{13}(e_{56} + e_{78}) \\ &\quad + \tfrac{k_2}{k_1} e_{23}(e_{57} - e_{68}) - \tfrac{k_2 \ell_3}{k_1 \ell_2} e_{23}(e_{56} + e_{78}), \\ \theta_2 \wedge \theta_1 \wedge \omega_3 &= e_{1234} + e_{12}(e_{56} + e_{78}) - \tfrac{k_3}{k_1} e_{23}(e_{56} + e_{78}) \\ &\quad + \tfrac{k_2 \ell_3}{k_1 \ell_2} e_{23}(e_{56} + e_{78}) + \tfrac{\ell_3}{\ell_2} e_{13}(e_{56} + e_{78}), \\ dv \wedge (\Theta_1 \wedge \omega_1 + \Theta_2 \wedge \omega_2 + \Theta_3 \wedge \omega_3) &= -e_{14}(e_{58} + e_{67}) - e_{24}(e_{57} - e_{68}) \\ &\quad + e_{34}(e_{56} + e_{78}),\end{aligned}$$

and the given expression for Φ follows. □

Remark 6.18. The functions $k_1,\dots,m_3$ from the proof of Proposition 6.17 are related to G in the following way

$$G = \begin{pmatrix} \frac{1}{k_1^2} & -\frac{k_2}{k_1^2\ell_2} & \frac{k_2\ell_3-k_3\ell_2}{k_1^2\ell_2 m_3} \\ -\frac{k_2}{k_1^2\ell_2} & \frac{k_2^2}{k_1^2\ell_2^2}+\frac{1}{\ell_2^2} & \frac{k_2(k_3\ell_2-k_2\ell_3)}{k_1^2\ell_2^2 m_3}-\frac{\ell_3}{\ell_2^2 m_3} \\ \frac{k_2\ell_3-k_3\ell_2}{k_1^2\ell_2 m_3} & \frac{k_2(k_3\ell_2-k_2\ell_3)}{k_1^2\ell_2^2 m_3}-\frac{\ell_3}{\ell_2^2 m_3} & \frac{(k_2\ell_3-k_3\ell_2)^2}{(k_1\ell_2 m_3)^2}+\frac{\ell_3^2}{\ell_2^2 m_3^2}+\frac{1}{m_3^2} \end{pmatrix},$$

and for $G^{-1} = (g^{ij})$ we have

$$G^{-1} = \begin{pmatrix} k_1^2+k_2^2+k_3^2 & k_2\ell_2+k_3\ell_3 & k_3m_3 \\ k_2\ell_2+k_3\ell_3 & \ell_2^2+\ell_3^2 & \ell_3 m_3 \\ k_3m_3 & \ell_3m_3 & m_3^2 \end{pmatrix}. \tag{6.20}$$

△

Now suppose that $t \in \nu(Y_0)$ is a regular value for $\nu\colon Y_0 \to \mathbb{R}$. Then $\mathcal{X}_t = \nu^{-1}(t)$ is a real-analytic hypersurface and has unit normal $N = h(d\nu)^\sharp$. We shall denote by ι the inclusion $\mathcal{X}_t \hookrightarrow Y_0$.

Definition 6.19. The T^3-*reduction* of Y_0 at level t is the four-manifold

$$M = \nu^{-1}(t)/T^3 = \mathcal{X}_t/T^3.$$

This quotient space is a tri-symplectic manifold.

Proposition 6.20. *The T^3-reduction M carries three pointwise linearly independent symplectic forms defining the same orientation.*

Proof. Consider the real-analytic two-forms ω_1, ω_2 and ω_3 on Y_0. These forms are T^3-invariant and closed since for instance $\mathcal{L}_{U_i}\omega_1 = \mathcal{L}_{U_i}(U_2 \lrcorner U_3 \lrcorner \Phi) = 0$ and $d\omega_1 = d(U_2 \lrcorner U_3 \lrcorner \Phi) = \mathcal{L}_{U_2}(U_3 \lrcorner \Phi) = 0$, respectively. Furthermore, as $U_1 \lrcorner \omega_1 = -d\nu$, etc., their pull-backs to $\mathcal{X}_t = \nu^{-1}(t)$ are basic. Thus they descend to three closed forms σ_1, σ_2 and σ_3 on M.

The proof of Proposition 6.17 shows that at a point $k_1\ell_2m_3\,\sigma_1 = k_1(e_{58} + e_{67}) + k_2(e_{57} - e_{68}) - k_3(e_{56} + e_{78})$, $k_1\ell_2m_3\,\sigma_2 = \ell_2(e_{57} - e_{68}) - \ell_3(e_{56} + e_{78})$ and $k_1\ell_2m_3\sigma_3 = -m_3(e_{56} + e_{78})$. Consequently, σ_1, σ_2 and σ_3 are non-degenerate symplectic forms defining the same orientation. □

The symplectic triple $(\sigma_1, \sigma_2, \sigma_3)$ on M defines a matrix $Q = (q_{ij})$ given by $\sigma_i \wedge \sigma_j = 2q_{ij}\,\mathrm{vol}_M$, where vol_M is the induced volume form on M.

Proposition 6.21. *The matrices G and Q are related via $G^{-1} = h^2 Q$. In particular, $\mathrm{vol}_M = \frac{h^2}{6}\sum_{i,j=1}^{3} g_{ij}\sigma_i \wedge \sigma_j$. Moreover, for any positive smooth function λ on M, the redefinitions $\widetilde{Q} = \lambda^2 Q$, $\widetilde{G} = \lambda G$, $\widetilde{h}^2 = \det(\widetilde{G}^{-1})$ retain the relation $\widetilde{G}^{-1} = \widetilde{h}^2\widetilde{Q}$.*

Proof. Working locally at a point and using the T^3-action, as in the proof of Proposition 6.17, we have

$$\sigma_1 \wedge \sigma_2 = 2\tfrac{k_2\ell_2+k_3\ell_3}{h^2}\,\mathrm{vol}_M, \quad \sigma_1 \wedge \sigma_3 = 2\tfrac{k_3m_3}{h^2}\,\mathrm{vol}_M, \quad \sigma_2 \wedge \sigma_3 = 2\tfrac{\ell_3m_3}{h^2}\,\mathrm{vol}_M,$$
$$\tfrac{h^2}{(k_1^2+k_2^2+k_3^2)}\sigma_1^2 = \tfrac{h^2}{(\ell_2^2+\ell_3^2)}\sigma_2^2 = \tfrac{h^2}{m_3^2}\sigma_3^2 = 2\,\mathrm{vol}\,.$$

where $\mathrm{vol}_M = e_{5678}$ is induced volume form on M. The relation between Q and G^{-1} now follows directly from the expression (6.20), and it immediately implies the last two assertions of the proposition. □

As we shall see below, the above behavior of G and Q with respect to rescaling plays a subtle role in the description of induced geometry on the hypersurface $\mathcal{X}_t$.

It is well-known, cf. [MC97], that any orientable hypersurface in a $Spin(7)$-manifold carries an induced G_2-structure. To express the G_2-structure $\phi = N \lrcorner \Phi$ on $\mathcal{X}_t$ it is useful to rewrite Φ in a way that abuses notation slightly, namely using the forms defined on M.

$$\begin{aligned}\Phi = dv \wedge (\theta_3 \wedge \theta_2 \wedge \theta_1 + \Theta_1 \wedge \sigma_1 + \Theta_2 \wedge \sigma_2 + \Theta_3 \wedge \sigma_3) \\ + \theta_3 \wedge \theta_2 \wedge \sigma_1 + \theta_1 \wedge \theta_3 \wedge \sigma_2 + \theta_2 \wedge \theta_1 \wedge \sigma_3 + \mathrm{vol}_M .\end{aligned} \tag{6.21}$$

From (6.21) we see that

$$h\phi = \theta_3 \wedge \theta_2 \wedge \theta_1 + \Theta_1 \wedge \sigma_1 + \Theta_2 \wedge \sigma_2 + \Theta_3 \wedge \sigma_3. \tag{6.22}$$

Alternatively we may, up to orientation, specify the G_2-structure by the four-form $\psi = \iota^*\Phi\,(= *\phi)$:

$$\psi = \theta_3 \wedge \theta_2 \wedge \sigma_1 + \theta_1 \wedge \theta_3 \wedge \sigma_2 + \theta_2 \wedge \theta_1 \wedge \sigma_3 + \mathrm{vol}_M .$$

As the $Spin(7)$-structure is torsion-free, the induced real-analytic G_2-structure on $\mathcal{X}_t$ is *cosymplectic*, meaning $d\psi = 0$.

It turns out that there is a family of smooth cosymplectic G_2-structures on $\mathcal{X}_t$ obtained by scaling of the volume form on M:

Proposition 6.22. *Let (ϕ, ψ) be the G_2-structure on $\mathcal{X}_t$ described above. For any positive smooth function λ on M, the changes $\lambda^2 Q =: \widetilde{Q}$ and $\lambda G =: \widetilde{G}$ of Q and G, respectively, give a new cosymplectic G_2-structure $(\widetilde{\phi}, \widetilde{\psi})$ on $\mathcal{X}_t$:*

$$\widetilde{h}\widetilde{\phi} = \theta_3 \wedge \theta_2 \wedge \theta_1 + \widetilde{\Theta}_1 \wedge \sigma_1 + \widetilde{\Theta}_2 \wedge \sigma_2 + \widetilde{\Theta}_3 \wedge \sigma_3, \tag{6.23}$$

$$\widetilde{\psi} = \theta_3 \wedge \theta_2 \wedge \sigma_1 + \theta_1 \wedge \theta_3 \wedge \sigma_2 + \theta_2 \wedge \theta_1 \wedge \sigma_3 + \widetilde{\mathrm{vol}}_M, \tag{6.24}$$

where $\widetilde{h} = \det(\widetilde{Q})^{-\frac{1}{4}} = \lambda^{-\frac{3}{2}} h$, $\widetilde{\Theta}_i = \sum_{j=1}^{3} \tilde{q}^{ij}\theta_j = \lambda^{-2}\Theta_i$, $\widetilde{\mathrm{vol}}_M = \frac{1}{6}\sum_{i,j=1}^{3} \tilde{q}^{ij}\sigma_i \wedge \sigma_j$ $= \lambda^{-2}\,\mathrm{vol}_M$.

Proof. Working locally at a point, as in the proof of Proposition 6.17, we have the basis $(e_1, \dots, \widehat{e_4}, \dots, e_8)$ for $T^*\mathcal{X}_t$. We now define a new basis $(f_1, \dots, \widehat{f_4}, \dots, f_8)$ for $T^*\mathcal{X}_t$ by letting $f_i := \sqrt{\lambda} e_i$, for $i = 1, 2, 3$, and $f_i := \frac{1}{\sqrt{\lambda}} e_i$, for $i = 5, \dots, 8$. Writing $\widetilde{\phi}$ and $\widetilde{\psi}$ in terms of f_i we have that

$$\begin{aligned}\widetilde{\phi} &= -f_{123} - f_3(f_{56} + f_{78}) + f_2(f_{57} - f_{68}) + f_1(f_{58} + f_{67}), \\ \widetilde{\psi} &= f_{12}(f_{56} + f_{78}) + f_{13}(f_{57} - f_{68}) - f_{23}(f_{58} + f_{67}) + f_{5678},\end{aligned}$$

which shows that $\widetilde{\phi}$ and $\widetilde{\psi}$ define a G_2-structure with volume form $\widetilde{\mathrm{vol}}_{\mathcal{X}} = \frac{1}{\sqrt{\lambda}} \mathrm{vol}_{\mathcal{X}}$. Clearly, $\widetilde{\psi}$ is closed. Hence the new G_2-structure is also cosymplectic. □

Inversion via a flow We now consider how the reduction procedure from the previous section may be inverted, constructing a $Spin(7)$-metric starting from a triple of symplectic forms on a four-manifold M. First we need a weakening of the notion of coherent symplectic triple [MS10, Definition 6.4].

Definition 6.23. A *weakly coherent symplectic triple* $\mathscr{C}$ on a four-manifold M consists of three symplectic forms σ_1, σ_2, σ_3 that pointwise span a maximal positive subspace of $\Lambda^2 T^*M$.

As in [DK90], the positive three-dimensional subbundle $\Lambda^+ = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \subset \Lambda^2 T^*M$ corresponds to a unique oriented conformal structure on M. Fix a volume form vol_M on M compatible with the orientation and define a 3×3-matrix $Q = (q_{ij})$ by $\sigma_i \wedge \sigma_j = 2q_{ij}\,\mathrm{vol}_M$, for $i, j = 1,2,3$. Subsequently, denote by h the positive smooth function satisfying $h^{-4} = \det(Q)$. We now consider a T^3-bundle $\pi_M \colon \mathcal{X} \to M$ endowed with connection one-form $\theta = (\theta_1, \theta_2, \theta_3) \in \Omega^1(\mathcal{X}, \mathbb{R}^3)$. We define three one-forms Θ_i, for $i = 1,2,3$, by the formula $\Theta_i = \sum_{j=1}^3 q^{ij}\theta_j$. Finally, denote the curvature by $F = \pi_M^*(d\theta) \in \Omega^2(M, \mathbb{R}^3)$. With these definitions in mind we have:

Proposition 6.24. *Let $(M, \mathscr{C})$ be a weakly coherent tri-symplectic four-manifold. Suppose that $\mathcal{X}$ is a principal T^3-bundle over M with connection one-form $\theta = (\theta_1, \theta_2, \theta_3)$ and curvature F. Define a three-form ϕ and a four-form ψ by*

$$\begin{aligned} h\phi &= \theta_3 \wedge \theta_2 \wedge \theta_1 + \Theta_1 \wedge \sigma_1 + \Theta_2 \wedge \sigma_2 + \Theta_3 \wedge \sigma_3, \\ \psi &= \theta_3 \wedge \theta_2 \wedge \sigma_1 + \theta_1 \wedge \theta_3 \wedge \sigma_2 + \theta_2 \wedge \theta_1 \wedge \sigma_3 + \mathrm{vol}_M . \end{aligned} \tag{6.25}$$

Then ϕ determines a G_2-structure on $\mathcal{X}$ satisfying $*\phi = \psi$.

Let $A = (a_{ij})$ be the 3×3-matrix defined pointwise by the projection $F_+ = (\sigma_1, \sigma_2, \sigma_3)A$. Then the G_2-structure ϕ is cosymplectic if and only if the matrix QA is symmetric:

$$QA = A^t Q \tag{6.26}$$

Proof. Write the entries of $G^{-1} := h^2 Q$ as in (6.20) and then express the functions $k_1, \dots, m_3$ in terms of the entries g^{ij} of $G^{-1} = h^2 Q$. Next, choose a conformal basis e_5, e_6, e_7, e_8 of T^*M so that $h\sigma_i$ are as in the proof of Proposition 6.17 and then write $\theta_1 = k_1 e_1 + k_2 e_2 + k_3 e_3$, $\theta_2 = \ell_2 e_2 + \ell_3 e_3$, $\theta_3 = m_3 e_3$. It now follows, using Proposition 6.22, that the basis $(e_1, \dots, \widehat{e_4}, \dots, e_8)$ is a G_2-basis for $T^*\mathcal{X}$ with defining form ϕ given via (6.25).

For the final assertion we need to study the condition $d\psi = 0$. The equation $d\psi = 0$ holds if and only if one has

$$d\theta_1 \wedge \sigma_2 - d\theta_2 \wedge \sigma_1 = d\theta_3 \wedge \sigma_1 - d\theta_1 \wedge \sigma_3 = d\theta_2 \wedge \sigma_3 - d\theta_3 \wedge \sigma_2 = 0.$$

A calculation shows that these relations correspond to the three equations

$$\begin{aligned} -a_{13}q_{12} + a_{12}q_{13} - a_{23}q_{22} + (a_{22} - a_{33})q_{23} + a_{32}q_{33} &= 0, \\ a_{13}q_{11} + a_{23}q_{12} + (a_{33} - a_{11})q_{13} - a_{21}q_{23} - a_{31}q_{33} &= 0, \\ -a_{12}q_{11} + (a_{11} - a_{22})q_{12} - a_{32}q_{13} + a_{21}q_{22} + a_{31}q_{23} &= 0, \end{aligned} \tag{6.27}$$

and these are equivalent to the condition (6.26). □

Remark 6.25. Condition (6.26) on F is independent of the choice of orientation compatible volume form on M. Though the bilinear form on $\Lambda^2 T^*M$, given by wedging, is only well-defined after choosing a representative volume form, self-adjointness of the projection $F_+ \in \Lambda^+ \subset \Lambda^2 T^*M$ does not depend on the specific choice.

Provided the assumptions of Proposition 6.24 hold, we therefore obtain a family of cosymplectic G_2-manifolds. This is a consequence of Proposition 6.22, and contrasts with the corresponding analysis of $SU(3)$-structures on T^2-bundles over coherently tri-symplectic four-manifolds (Proposition 6.6). In that situation we made a particular choice of volume form to obtain a half-flat structure. △

Remark 6.26. Existence of three-torus bundles over a weakly coherent tri-symplectic four-manifold $(M, \mathscr{C})$ is related to Chern-Weil theory. One finds that for any closed two-form F with integral periods, $F \in \Omega^2_{\mathbb{Z}}(M, \mathbb{R}^3)$, there exists a T^3-bundle $\pi_M \colon \mathcal{X} \to M$ with connection one-form θ that satisfies $\pi_M^*(d\theta) = F$. △

Studying a certain Hamiltonian flow, Hitchin [Hit01] developed a relationship between torsion-free $Spin(7)$-metrics and cosymplectic G_2-manifolds. In particular, he derived evolution equations that describe the one-dimensional flow of a cosymplectic G_2-manifold along its unit normal in a torsion-free $Spin(7)$-manifold. In inverting our construction, one could use Hitchin's flow on the cosymplectic structure of Proposition 6.24. However, Hitchin's flow does not preserve the level sets of the multi-moment map: the unit normal is $h(d\nu)^\sharp$, but $\partial/\partial\nu = h^2(d\nu)^\sharp$. It is therefore more natural for us to determine the flow equations associated to the latter vector field.

Proposition 6.27. *Suppose T^3 acts freely on a connected eight-manifold Y preserving the torsion-free $Spin(7)$-structure Φ and admitting a multi-moment map ν. Let M be the topological reduction $\nu^{-1}(t)/T^3$ for any t in the image of ν. Then M is equipped with a t-dependent weakly coherent real-analytic symplectic triple σ_1, σ_2, σ_3 and the seven-manifold $\mathcal{X}_t = \nu^{-1}(t)$ carries a cosymplectic real-analytic G_2-structure of the form (6.25). On $\mathcal{X}_t$ the following evolution equation holds:*

$$\psi' = d(h\phi), \tag{6.28}$$

where $'$ denotes differentiation with respect to t.

Conversely, given a cosymplectic real-analytic G_2-structure of the form (6.25) defined on a seven-manifold $\mathcal{X}_0$. Then the flow equation (6.28) admits a unique solution on some open neighbourhood of $\mathcal{X}_0 \times \{0\} \subset \mathcal{X}_0 \times \mathbb{R}$, and that solution determines a torsion-free $Spin(7)$-structure.

Proof. We have

$$\Phi = hd\nu \wedge \phi + \psi.$$

This has derivative

$$d\Phi = d\nu \wedge (-dh \wedge \phi - hd\phi) + d\psi.$$

By assumption, the G_2-structure is cosymplectic, i.e., $d\psi = 0$ on each level set. We therefore find that $d\Phi = 0$ if and only if

$$0 = \frac{\partial}{\partial v} \lrcorner\, d\Phi = -d(h\phi) + \psi'.$$

Hence we have a torsion-free $Spin(7)$-structure if and only if the evolution equation (6.28) is satisfied.

Observe that equation (6.28) together with an initial cosymplectic G_2-structure on $\mathcal{X}_0$ already ensure that the family consists of cosymplectic structures; the time derivative of $d\psi$ vanishes according to (6.28).

We note that given real-analytic initial data, the Cauchy-Kovalevskaya theorem applies. Therefore we obtain existence and uniqueness of a solution defined on some open neighbourhood of $\mathcal{X}_0 \times \{0\} \subset \mathcal{X}_0 \times \mathbb{R}$.

For later use, we shall rewrite the evolution equation as a set of first order differential equations for the quantities defined by data on M. First we note that

$$\begin{aligned}\psi' = \sigma_1' \wedge \theta_3 \wedge \theta_2 + \sigma_2' \wedge \theta_1 \wedge \theta_3 + \sigma_3' \wedge \theta_2 \wedge \theta_1 + (\theta_2' \wedge \sigma_3 - \theta_3' \wedge \sigma_2) \wedge \theta_1 \\ + (\theta_3' \wedge \sigma_1 - \theta_1' \wedge \sigma_3) \wedge \theta_2 + (\theta_1' \wedge \sigma_2 - \theta_2' \wedge \sigma_1) \wedge \theta_3 + \mathrm{vol}_M',\end{aligned}$$

$$\begin{aligned}d(h\phi) = d\theta_1 \wedge \theta_3 \wedge \theta_2 + d\theta_2 \wedge \theta_1 \wedge \theta_3 + d\theta_3 \wedge \theta_2 \wedge \theta_1 \\ + \sigma_1 \wedge d\Theta_1 + \sigma_2 \wedge d\Theta_2 + \sigma_3 \wedge d\Theta_3,\end{aligned}$$

where

$$\sum_{i=1}^{3} \sigma_i \wedge d\Theta_i = \sum_{i,j=1}^{3} \sigma_i \wedge \left(d(q^{ij}) \wedge \theta_j + q^{ij} d\theta_j \right).$$

From these equations we get the t-derivatives for $\sigma_1, \sigma_2, \sigma_3$:

$$\sigma_i' = d\theta_i, \quad \text{for} \quad i = 1, 2, 3. \tag{6.29}$$

The t-derivative of the connection one-form $\theta = (\theta_1, \theta_2, \theta_3)$ is given by

$$\theta_i' \wedge \sigma_j - \theta_j' \wedge \sigma_i = \sum_{\ell=1}^{3} \sigma_\ell \wedge dq^{\ell k}, \quad \text{for} \quad sgn(ijk) = +1. \tag{6.30}$$

The volume form vol_M evolves via

$$\mathrm{vol}_M' = \sum_{i,j=1}^{3} q^{ij} \sigma_i \wedge d\theta_j. \tag{6.31}$$

Finally the t-derivatives of entries q_{ij} of Q may be expressed via

$$2q_{ij}' \,\mathrm{vol}_M = d\theta_i \wedge \sigma_j + \sigma_i \wedge d\theta_j - 2q_{ij} \sum_{k,\ell=1}^{3} q^{k\ell} \sigma_k \wedge d\theta_\ell, \quad \text{for} \quad i,j = 1,2,3. \tag{6.32}$$

Note that the equations for the entries q_{ij} now determine the evolution of h and G via the relations $h^{-4} = \det(Q)$ and $G^{-1} = h^2 Q$, respectively. □

Remark 6.28. By solving the flow equations we obtain a holonomy $Spin(7)$-metric with three-torus symmetry. Indeed, if g_M is the time-dependent metric in the conformal class on M with volume form vol_M, then the $Spin(7)$-metric is explicitly

$$h^2 dt^2 + g_M + g_{11}\theta_1^2 + g_{22}\theta_2^2 + g_{33}\theta_3^2 + g_{12}\theta_1\theta_2 + g_{13}\theta_1\theta_3 + g_{23}\theta_2\theta_3, \qquad (6.33)$$

where $G = (g_{ij}) = h^{-2}Q^{-1}$.

Real-analyticity of the cosymplectic G_2-structures is a subtle matter. Bryant's study of the Hitchin flow [Bry10] shows that non-analytic cosymplectic G_2-structures can lead to an ill-posed Hitchin system that has no solution. △

Remark 6.29. Though the torsion-free G_2-manifolds studied in [MS10] fiber over (weakly) coherently tri-symplectic four-manifolds, they do not fit naturally into the above framework. The constructed G_2-flow does not preserve the $Spin(7)$-data. △

Summarising the results discussed so far we have:

Theorem 6.30. *Let (Y^8, Φ) be a torsion-free $Spin(7)$-manifold with a free T^3-symmetry and admitting a multi-moment map. Then the reduction M at level t carries a weakly coherent real-analytic symplectic triple and the level set $\mathcal{X}_t$ is the total space of a T^3-bundle over M satisfying condition* (6.26) *on the curvature.*

Conversely, let $(M, \mathscr{C})$ be a weakly coherent tri-symplectic four-manifold with a closed two-form $F \in \Omega^2_{\mathbb{Z}}(M, \mathbb{R}^3)$ and a choice of orientation compatible volume form. Assume F satisfies condition (6.26)*. When these data are real-analytic, they define a torsion-free $Spin(7)$-metric with T^3-symmetry.* □

6.2.1 Examples

Let us now turn to some examples that illustrate the analysis of the previous two sections. First we show that even in the flat case $\mathbb{R}^8$, with isometric action given by maximal torus $T^3 \subset SU(4)$ acting via diagonal matrices, the geometry of the reduction procedure is somewhat complicated. In the final example we study hyperKähler four-manifolds, complementing previous examples that have appeared in the physics literature [GLPS02, GS07].

Example 6.31. Consider $Y = \mathbb{R}^8 = \mathbb{C}^4$ endowed with the usual four-form and the action of the standard diagonal maximal torus $T^3 \subset SU(4)$. Concretely, Φ is given by

$$\begin{aligned}\Phi = \tfrac{1}{2}\left(\tfrac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3 + dz_4 \wedge d\bar{z}_4)\right)^2 \\ + \mathrm{Re}(dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4),\end{aligned}$$

and T^3 acts by $(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot (z_1, z_2, z_3, z_4) = (e^{i\theta_1}z_1, e^{i\theta_2}z_2, e^{i\theta_3}z_3, e^{-i(\theta_1+\theta_2+\theta_3)}z_4)$. The action is generated by the vector fields $U_j = \mathrm{Re}\left\{i(z_j\frac{\partial}{\partial z_j} - z_4\frac{\partial}{\partial z_4})\right\}$, for $j = 1, 2, 3$. It follows that a multi-moment map $\nu\colon Y \to \mathbb{R}$ is given by

$$\nu(z_1, z_2, z_3, z_4) = \tfrac{1}{8}\,\mathrm{Im}(z_1 z_2 z_3 z_4).$$

By definition, the T^3-reduction of Y at level t is the quotient space $M_t = \nu^{-1}(t)/T^3$. In this case M_0 is singular, whereas M_t is a smooth manifold for each $t \neq 0$. Indeed, considering $\Xi_t \colon M_t \to \mathbb{R}^4$ given by

$$\begin{aligned}\Xi_t(z_1, z_2, z_3, z_4) &= \Big(\tfrac{\|z_1\|^2-\|z_4\|^2}{2}, \tfrac{\|z_2\|^2-\|z_4\|^2}{2}, \tfrac{\|z_3\|^2-\|z_4\|^2}{2}, \mathrm{Re}(z_1z_2z_3z_4)\Big)\\ &=: (v_1, v_2, v_3, w),\end{aligned}$$

we have global smooth coordinates on M_t for $t \neq 0$.

In this smooth case, writing $(\eta_1, \eta_2, \eta_3) = (dv_1, dv_2, dv_3)G^{-1}$, the two-forms σ_1, σ_2, σ_3 are given by

$$16\sigma_1 = \eta_1 \wedge dw + 4dv_2 \wedge dv_3, \quad 16\sigma_2 = \eta_2 \wedge dw + 4dv_3 \wedge dv_1,$$
$$16\sigma_3 = \eta_3 \wedge dw + 4dv_1 \wedge dv_2.$$

These forms depend (implicitly) on t via the relations $4g_{ij} = \delta_{ij}\|z_i\|^2 + \|z_4\|^2$, for $i, j = 1, 2, 3$, and $z_1z_2z_3z_4 = w + 8it$. In particular g_{ij} is a non-constant positive function f, for $i \neq j$. Thus the weakly coherent triple does not specify a coherent triple, in particular it is not a hyperKähler structure.

The (oriented) conformal class has representative metric

$$\frac{h^2}{16}dw^2 + g_{11}\eta_1^2 + g_{22}\eta_2^2 + g_{33}\eta_3^2 + f(\eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3),$$

where $h^2 = \det(G^{-1})$.

The curvature $F = (F_1, F_2, F_3)$ of the principal T^3-bundle $\nu^{-1}(t) \to M_t$ is given by

$$\begin{aligned}F_1 &= 2th^2\eta_w \wedge ((2g_{22}g_{33} - f(g_{22} + g_{33}))\eta_1\\ &\qquad + (g_{33} - f)(g_{22} - 2f)\eta_2 + (g_{22} - f)(g_{33} - 2f)\eta_3),\\ F_2 &= 2th^2\eta_w \wedge ((2g_{11}g_{33} - f(g_{11} + g_{33}))\eta_2\\ &\qquad + (g_{11} - f)(g_{33} - 2f)\eta_3 + (g_{33} - f)(g_{11} - 2f)\eta_1),\\ F_3 &= 2th^2\eta_w \wedge ((2g_{11}g_{22} - f(g_{11} + g_{22}))\eta_3\\ &\qquad + (g_{22} - f)(g_{11} - 2f)\eta_1 + (g_{11} - f)(g_{22} - 2f)\eta_2).\end{aligned}$$

where $\eta_w = g_{ww}^{-1}dw$ satisfies $\eta_w((dw)^\sharp) = 1$ and $\eta_w((dv_i)^\sharp) = 0$, for $i = 1, 2, 3$. Note that $F \neq F_+$.

In the singular case $t = 0$, the three-torus collapses in three different ways: to a point, a circle or a two-torus. At the origin $(z_1, z_2, z_3, z_4) = 0$ the three-torus collapses to a point. Next, if $z_i = z_j = z_k = 0$ for exactly three different indices, then the torus collapses to a circle. In terms of the quadruple (v_1, v_2, v_3, w) this collapsing happens for $w = 0$ when v_1, v_2, v_3 satisfy one of the following constraints: $(v_1 = v_2 = v_3 \leqslant 0)$, $(v_1 = v_2 = 0,\ v_3 \geqslant 0)$, $(v_1 = v_3 = 0,\ v_2 \geqslant 0)$ or $(v_2 = v_3 = 0,\ v_1 \geqslant 0)$. Finally, if $z_i = z_j = 0$ for exactly two different indices, the T^3 collapses to a two-torus. This happens for $w = 0$ when v_1, v_2, v_3 satisfy one of: $(v_1 = v_2 \leqslant 0)$, $(v_1 = v_3 \leqslant 0)$, $(v_1 = 0,\ v_2, v_3 \geqslant 0)$, $(v_2 = v_3 \leqslant 0)$, $(v_2 = 0,\ v_1, v_3 \geqslant 0)$ or $(v_3 = 0,\ v_1, v_2 \geqslant 0)$. ◇

Example 6.32. As in the G_2-case, we may investigate T^3-reductions associated with some of the known cohomogeneity-one $Spin(7)$-structures. Examples come from spaces with principal orbits

$$Q^{1,1,1} = \frac{SU(2)^3}{U(1)^2_{1,1,1}} \quad \text{or} \quad M^{1,1,0} = \frac{SU(3) \times SU(2)}{(SU(2) \times U(1))_{1,1,0}},$$

where the indices refer to the embeddings of $U(1)^2$ into $SU(2)^3$ and of the Abelian factor of $SU(2) \times U(1)$ into $SU(3) \times SU(2)$. Reidegeld has constructed examples of this type [Rei10b] with holonomy $SU(4)$. In the first case, $Q^{1,1,1}$, we can choose an isometric left action of $T^3 \subset SU(2)^3$. In the latter case we may pick the three-torus $T^3 \subset SU(3) \times SU(2)$, also acting on the left. Calculations may now be carried out along the lines of Example 6.13. But the concrete expressions become somewhat unwieldy in the $Spin(7)$-case. ♢

Example 6.33. Let M be a hyperKähler four-manifold. Then M comes equipped with three symplectic forms σ_1, σ_2, σ_3 that satisfy the relations $\sigma_i \wedge \sigma_j = \delta_{ij}\sigma_k^2$ for $i, j, k = 1, 2, 3$. Choosing the volume form $\mathrm{vol}_M^0 = \frac{1}{2}\sigma_1^2$, we have that $Q = \mathrm{diag}(1, 1, 1)$. The compatible hyperKähler metric is denoted by g_M^0.

Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ denote the hyperKähler triple and assume there is a constant matrix $A = (a_{ij})$ such that $\sigma A \in \Omega^2_{\mathbb{Z}}(M, \mathbb{R}^3)$. Then we may construct a T^3-bundle over M with connection one-form θ that has curvature $F = \sigma A$. The total space $\mathcal{X}_0$ of this bundle carries the G_2-structure of Proposition 6.24, which is now cosymplectic if and only if A is symmetric. The associated metric on $\mathcal{X}_0$ is complete if the hyperKähler base manifold is complete.

We shall illustrate how one may solve the flow equations (6.29)–(6.32) starting from the above data at $t = 0$. As an a priori simplifying assumption we consider the case when $F' = 0$, i.e., the curvature is t-independent. Then the differential equations for the symplectic triple simply read $\sigma' = \Omega A$, where $\Omega = \sigma(0)$. Integrating, we find that $\sigma(t) = \Omega(1 + tA)$.

We next solve the equations (6.31) and (6.32). First we observe that the volume develops according to the equation $\mathrm{vol}_M' = v\,\mathrm{vol}_M^0$, where $v = 2\,\mathrm{Tr}(Q^{-1}(1 + tA)A)$. We may therefore write $\mathrm{vol}_M(t) = V(t)\,\mathrm{vol}_M^0$, where $V' = v$ and $V(0) = 1$. The equation for Q' now takes the form $VQ' = 2(1 + At)A - vQ$. It follows that we must find the unique solution of the differential equation $\ln(V)' = 2Tr((1 + tA)^{-1}A)$, $V(0) = 1$. We find that $V(t) = \det(1 + tA)^2$. Consequently, vol_M and Q take the form $\mathrm{vol}_M(t) = \det(1 + tA)^2\,\mathrm{vol}_M^0$ and $\det(1 + tA)^2 Q(t) = (1 + tA)^2$. Note also that $h(t) = \det(1 + tA)$ and that $dq_{ij}(t) = 0$. The latter observation implies, by (6.30), that the connection one-form is t-independent, $\theta(t) = \theta$.

The above solution is defined on $\mathcal{X}_0 \times I$, where the interval $I \subset \mathbb{R}$ is determined by non-degeneracy of the matrix $1 + tA$ and $0 \in I$. By uniqueness of the solution on $\mathcal{X}_0 \times I$, we deduce that the condition $F' = 0$ already follows from the initial data, i.e., it is not a simplifying assumption.

The torsion-free $Spin(7)$-structure corresponding to the above solution has associated metric g given by

$$h^2(t)dt^2 + h(t)g_M^0 + h(t)^{-2}\left(\sum_{i=1}^{3} q^{ii}(t)\theta_i^2 + \sum_{1\leqslant i<j\leqslant 3} q^{ij}(t)\theta_i\theta_j\right). \tag{6.34}$$

If the initial hyperKähler four-manifold is complete, we may describe completeness properties of g in terms of the matrix A. Provided g remains finite and non-degenerate, completeness corresponds to completeness of $h(t)^2dt^2$ on I, cf. [BO69]. We now find that g is half-complete, cf. [AS04], if and only if A does not have two eigenvalues of the opposite sign; the metric is complete only for $A = 0$. ♢

Future research

Chapter 7

Kähler like aspects of HKT geometry

GIVEN A HYPERCOMPLEX MANIFOLD (M, I, J, K), we can always find a compatible Riemannian metric g, meaning that each of the pairs (g, I), and so forth, forms a Hermitian structure on M. Indeed, given a metric g', then the metric $g = \frac{1}{4}(g' + g'(I\cdot, I\cdot) + g'(J\cdot, J\cdot) + g'(K\cdot, K\cdot))$ is hyperHermitian. However, existence becomes a non-trivial matter if we want g to satisfy additional requirements. The best known example is a hyperKähler metric, i.e., a hyperHermitian metric which has closed fundamental two-forms $\omega_I = g(I\cdot, \cdot)$, etc. It is highly non-trivial to construct examples of such metrics.

A more flexible notion than being hyperKähler is that of an HKT manifold, introduced in Chapter 2. HKT metrics seem to be good quaternionic analogues of Kähler metrics, for instance many hypercomplex manifolds, but not all [FG04, Swa10b], admit a compatible HKT metric. There is also a good potential theory [BS04] ensuring that HKT metrics admit locally a potential. Moreover, a version of Hodge theory [Ver02] applies to HKT manifolds with special Obata holonomy. Another intriguing Kähler like aspect is expressed in some work towards an HKT version of the Calabi-Yau result [AV10, Mad09]. It is this particular problem we now turn to discuss.

7.1 A Calabi-Yau problem for HKT manifolds

We first discuss some known results from HKT geometry and introduce the notions required so that we can formulate the HKT Calabi-Yau problem.

7.1.1 The DD_I-operator

To make the analogy between Kähler and HKT geometry transparent, one [BS04] introduces the differential complex studied by Salamon in [Sal86]. We thus consider the following complex defined on any hypercomplex manifold (M^{4n}, I, J, K):

$$0 \xrightarrow{D} \mathcal{A}^0 \xrightarrow{D=d} \mathcal{A}^1 \xrightarrow{D} \cdots \xrightarrow{D} \mathcal{A}^{2n} \longrightarrow 0, \qquad (7.1)$$

where $\mathcal{A}^k = \Gamma(A^k)$ and $A^k \subset \Lambda^k T^*M =: \Lambda^k$ is the subbundle

$$A^k = \sum_{\mathcal{I}\in S^2} \left(\Lambda^{k,0}_{\mathcal{I}} \oplus \Lambda^{0,k}_{\mathcal{I}}\right),$$
$$S^2 = \{aI + bJ + cK\colon\ a^2+b^2+c^2 = 1\}.$$

In the above expression for A^k, the sum denotes fibrewise finite linear combinations, and $\Lambda^{k,0}_{\mathcal{I}}$ is the space of $(k,0)$-forms relative to $\mathcal{I} \in S^2$. The differential $D = \pi \circ d$ in (7.1) is given by exterior differentiation d followed by projection $\pi\colon \Lambda^k \to A^k$.

The kernel of π is denoted by B^k, and Salamon observed [Sal86, Proposition 4.2] that one has

$$B^k = \bigcap_{\mathcal{I}\in S^2} \left(\Lambda^{k-1,1}_{\mathcal{I}} \oplus \Lambda^{k-2,2}_{\mathcal{I}} \oplus \cdots \oplus \Lambda^{1,k-1}_{\mathcal{I}}\right),$$

with the intersection interpreted fibrewise.

Remark 7.1. While Salamon's approach [Sal86] was quite general, in the sense that he studied quaternionic manifolds, Verbitsky gave a reinterpretation of the Salamon complex in a purely hypercomplex setting [Ver07] and included a discussion of some Dolbeault like properties. Such aspects have also been studied in [Wid02]. △

Remark 7.2. Since all the manifolds considered in this chapter are hypercomplex, they come endowed with the *Obata connection* [Oba56]: ∇^{Ob} is the unique torsion-free connection preserving I, J and K. △

We are mainly interested in the first four terms of the Salamon complex. While one obviously has $\mathcal{A}^0 = C^\infty(M)$ and $\mathcal{A}^1 = \Omega^1(M)$, an explicit description of $\mathcal{A}^k$ and $\mathcal{B}^k$, for $k = 2, 3$, requires a few calculations. To this end it is useful to adapt the notation of [MCS08]. For $\chi \in \Omega^\ell(M)$, we thus write

$$\mathcal{I}_p\chi = -\chi(X_1,\dots,\mathcal{I}X_p,\dots,X_\ell), \quad \mathcal{I}_{pq\dots r} = \mathcal{I}_p\mathcal{I}_q\dots\mathcal{I}_r,$$
$$\text{and} \quad \mathcal{I}\chi(X_1,\dots,X_\ell) = (-1)^\ell\chi(\mathcal{I}X_1,\dots,\mathcal{I}X_\ell),$$

and if we have operators $\mathcal{P}_{\mathcal{I}}$, for $\mathcal{I} = I, J, K$, acting on ℓ-forms, then we use $\mathcal{P}$ to denote their quaternionic average defined by

$$\mathcal{P} = \mathcal{P}_I + \mathcal{P}_J + \mathcal{P}_K.$$

Proposition 7.3. *Let (M, I, J, K) be a hypercomplex manifold. Then any sections $\omega \in \Omega^2(M)$ and $\eta \in \Omega^3(M)$ decompose in the following way:*

$$\omega = \tfrac{1}{4}(3-\mathcal{P})(\omega) + \tfrac{1}{4}(1+\mathcal{P})(\omega) \quad \textit{where} \quad \mathcal{P}_{\mathcal{I}} = \mathcal{I}, \tag{7.2}$$
$$\eta = \tfrac{1}{6}(3-\mathcal{P})(\eta) + \tfrac{1}{6}(3+\mathcal{P})(\eta) \quad \textit{where} \quad \mathcal{P}_{\mathcal{I}} = \mathcal{I}_{12}+\mathcal{I}_{13}+\mathcal{I}_{23}. \tag{7.3}$$

In particular, the Salamon differentials of sections $\theta \in \mathcal{A}^1$ and $\xi \in \mathcal{A}^2$ are given by

$$D\theta = (d\theta)^{2,0} + (d\theta)^{0,2} + \tfrac{1}{2}(1-J)(d\theta)^{1,1} \quad \textit{and}$$
$$D\xi = (d\xi)^{3,0} + (d\xi)^{0,3} + \tfrac{1}{6}(3-\mathcal{P})\left((d\xi)^{2,1} + (d\xi)^{1,2}\right),$$

where the type decomposition is with respect to the complex structure I.

Proof. We observe that the operators $\mathcal{P} = \mathcal{P}_I + \mathcal{P}_J + \mathcal{P}_K$ defined in (7.2) and (7.3) act on $\Omega^k(M)$, for $k = 2$ and 3, respectively, with two distinct eigenvalues. The corresponding eigenspaces are the modules $\mathcal{A}^k$ and $\mathcal{B}^k$.

For $k = 2$, we find that $\mathcal{A}^2$ is the -1-eigenspace of $\mathcal{P}$, while $\mathcal{B}^2$ is the $+3$-eigenspace. It now follows that

$$\pi(\omega) = \frac{1}{4}(3 - \mathcal{P})(\omega) \quad \text{and} \quad (1 - \pi)(\omega) = \frac{1}{4}(1 + \mathcal{P})(\omega).$$

If we decompose into types with respect to I, these expressions become

$$\pi(\omega) = \omega^{2,0} + \omega^{0,2} + \frac{1}{2}(1 - J)\omega^{1,1} \quad \text{and} \quad (1 - \pi)(\omega) = \frac{1}{2}(1 + J)\omega^{1,1}.$$

The first of these two formulae gives the stated expression for $D\theta$.

For $k = 3$, the operator $\mathcal{P}$ has eigenvalues ± 3. $\mathcal{A}^3$ is the -3-eigenspace, while $\mathcal{B}^3$ is the $+3$-eigenspace. Consequently, we may write

$$\pi(\eta) = \frac{1}{6}(3 - \mathcal{P})(\eta) \quad \text{and} \quad (1 - \pi)(\eta) = \frac{1}{6}(3 + \mathcal{P})(\eta).$$

Decomposing into types with respect to I, we find

$$\pi(\eta) = \eta^{3,0} + \eta^{0,3} + \frac{1}{6}(3 - \mathcal{P})(\eta^{2,1} + \eta^{1,2}) \quad \text{and}$$
$$(1 - \pi)(\eta) = \frac{1}{6}(3 + \mathcal{P})(\eta^{2,1} + \eta^{1,2}).$$

□

The significance of the Salamon complex is that we obtain a DD_I-operator, which serves as an analogue of the usual dd_I-operator in Kähler geometry; here $D_I\phi(X) = -D\phi(IX)$ for $\phi \in C^\infty(M)$. For instance the following result introduces the notion of an HKT potential, which is a function ϕ, such that $DD_I\phi(\cdot, I\cdot)$ is an HKT metric. The result is well-known, but we will give a proof for completeness, since this allows us to correct a minor misprint in [BS04, Remark, p. 3132].

Proposition 7.4. *Let (M, I, J, K) be a hypercomplex manifold, and ϕ a smooth real function on M. If the symmetric tensor*

$$k_\phi = \frac{1}{2}(1 + I + J + K)(\nabla^{\mathrm{Ob}})^2(\phi)$$

is positive definite, then k_ϕ is an HKT *metric. The associated fundamental two-forms $F_{\mathcal{I}} = k_\phi(\mathcal{I}\cdot, \cdot)$ are given by*

$$F_I = \frac{1}{2}(dd_I + d_J d_K)(\phi), \quad F_J = \frac{1}{2}(dd_J + d_K d_I)(\phi), \quad F_K = \frac{1}{2}(dd_K + d_I d_J)(\phi),$$

where $d_{\mathcal{I}}\chi = (-1)^\ell \mathcal{I} d(\mathcal{I}\chi)$, for $\chi \in \Omega^\ell(M)$.

Proof. First we note that k_ϕ is obviously compatible with I, J, K, and therefore defines a hyperHermitian metric if it is positive definite. When this holds, k_ϕ is HKT if and only if $IdF_I = JdF_J = KdF_K$.

Let us now assume that the fundamental two-forms are given by the formulae $F_I = \frac{1}{2}(dd_I + d_J d_K)(\phi)$, etc. Then a calculations shows that

$$IdF_I = d_I F_I = \frac{1}{2} d_I d_J d_K(\phi) = \frac{1}{2} d_J d_K d_I(\phi) = JdF_J, \quad \text{etc.},$$

where we have used that $\mathcal{I}F_\mathcal{I} = F_\mathcal{I}$ and that $d_I d_J = -d_J d_I$, etc. The HKT condition is thus satisfied.

It now remains to verify the expressions $F_I = (dd_I + d_J d_K)(\phi)$, and so forth. Using the properties of the Obata connection, direct calculations show that

$$\begin{aligned} dd_I\phi(X,Y) &= -(\nabla^{\mathrm{Ob}}_X d\phi)(IY) + (\nabla^{\mathrm{Ob}}_Y d\phi)(IX), \\ d_J d_K\phi(X,Y) &= (\nabla^{\mathrm{Ob}}_{JX} d\phi)(KY) - (\nabla^{\mathrm{Ob}}_{JY} d\phi)(KX). \end{aligned}$$

From this we obtain that

$$\begin{aligned} (dd_I + d_J d_K)(\phi)(X, IY) &= (\nabla^{\mathrm{Ob}}_X d\phi)(Y) + (\nabla^{\mathrm{Ob}}_{IX} d\phi)(IY) \\ &\qquad + (\nabla^{\mathrm{Ob}}_{JX} d\phi)(JY) + (\nabla^{\mathrm{Ob}}_{KX} d\phi)(KY) \\ &= (1 + I + J + K)(\nabla^{\mathrm{Ob}})^2(\phi), \end{aligned}$$

as required. □

7.1.2 An HKT Calabi-Yau problem

Using the results from the previous section, we obtain the following.

Theorem 7.5. *Let (M^{4n}, I, J, K) be a connected compact hypercomplex manifold, and g an* HKT *metric on M. Let $A \in \mathbb{R}$ and $f \in C^\infty(M)$. Assume a smooth real function ϕ satisfies the equation*

$$(\omega_I + DD_I\phi)^{2n} = Ae^f \omega_I^{2n}. \tag{7.4}$$

Then $g_\phi = g + \frac{1}{2}(1 + I + J + K)(\nabla^{\mathrm{Ob}})^2\phi$ is an HKT *metric. Moreover, if ψ is another solution of* (7.4) *then $\phi - \psi$ is a constant.*

Proof. By Proposition 7.4, the symmetric tensor g_ϕ defines an HKT metric if and only if it is pointwise positive definite. To verify the definiteness we consider a point $m \in M$ where ϕ achieves its minimum. At m we have $dd_I\phi(X, IX) \geqslant 0$, for any $X \in T_m M$, and hence

$$\begin{aligned} \frac{1}{2}(1 + I + J + K)(\nabla^{\mathrm{Ob}})^2(\phi)(X,X) &= \frac{1}{2}(dd_I + d_J d_K)(\phi)(X, IX) \\ &= \frac{1}{2}(dd_I(\phi)(X, IX) + dd_I(\phi)(JX, IJX)) \geqslant 0. \end{aligned}$$

Consequently, g_ϕ is positive definite at m. As M is connected and the right hand side of (7.4) is a volume form, g_ϕ must be positive definite at all points of M, as required.

In order to prove the last statement of the theorem, let us assume we have two smooth real functions ϕ and ψ each satisfying (7.4). Writing $\omega_\phi = g_\phi(I\cdot,\cdot)$ and $\omega_\psi = g_\psi(I\cdot,\cdot)$ we then have

$$0 = \omega_\phi^{2n} - \omega_\psi^{2n} = \gamma \wedge DD_I(\phi - \psi),$$

where $\gamma = \sum_{j=0}^{2n-1} \omega_\phi^j \wedge \omega_\psi^{2n-1-j}$. Since γ is a positive linear combination of positive forms, the operator

$$\varphi \mapsto P(\varphi) = \gamma \wedge DD_I(\varphi)$$

is a second order (overdetermined) elliptic operator without constant terms, cf. Proposition 7.9 below. Therefore, by the maximum principle, P has kernel equal to the constant functions on M. Consequently, the difference $\phi - \psi$ is a constant. □

Remark 7.6. If we add the condition that $\int_M \phi \operatorname{vol}_g = 0$ to equation (7.4), then the last assertion of Theorem 7.5 may be interpreted as a uniqueness statement of solutions in ϕ. This uniqueness result clearly holds under weaker regularity assumptions. Indeed, it suffices to take $f \in C^1(M)$ and $\phi \in C^3(M)$. △

The striking resemblance between (7.4) regarded as an equation in ϕ and the complex Monge-Ampère equation studied by Yau in [Yau78], leads us to formulate the following HKT version of the Calabi-Yau problem.

Question 7.7. Let (M^{4n}, I, J, K) be a connected compact hypercomplex manifold that admits an HKT metric g. Given any $f \in C^\infty(M)$ do there exist a unique smooth real function ϕ and a unique $A \in \mathbb{R}$ such that the equations

$$\int_M \phi \operatorname{vol}_g = 0 \quad \text{and} \quad (\omega_I + DD_I\phi)^{2n} = Ae^f \omega_I^{2n} \tag{7.5}$$

hold? ▽

Remark 7.8. The positive constant A is uniquely determined by the relation

$$\operatorname{vol}_{g_\phi}(M) = A \int_M e^f \operatorname{vol}_g,$$

where vol_g and $\operatorname{vol}_{g_\phi}$ denote the Riemannian volume forms associated with g and g_ϕ, respectively. In the Kähler version of the Calabi-Yau problem one considers fundamental two-forms belonging to the same de Rham cohomology class. Hence they have the same total volume. Contrasting with this we will generally have that $\operatorname{vol}_{g_\phi}(M) \neq \operatorname{vol}_g(M)$, and therefore the constant A will generally not satisfy the simple relation $A = \operatorname{vol}_g(M)/\int_M e^f \operatorname{vol}_g$. △

From a regularity theoretical viewpoint the differential operator DD_I is of the right type in order to make the continuity method tractable.

Proposition 7.9. *The linear second order differential operator $DD_I\colon \mathcal{A}^0 = C^\infty(M) \to \mathcal{A}^2$ is (overdetermined) elliptic.*

Proof. To verify that DD_I is (overdetermined) elliptic we must show that the associated symbol $(m,v) \mapsto \sigma_{DD_I}(m,v)$ is injective, as a map $\underline{\mathbb{R}}_m \to \Lambda^2 T_m^* M$, for all $m \in M$ and $v \in T_m^* M \setminus \{0\}$; here $\underline{\mathbb{R}}_m$ denotes the trivial bundle over M.

To this end we rewrite DD_I as a composite of two zeroth order and two first order differential operators. Specifically we have $DD_I = \frac{1}{2}(1-J) \circ d \circ I \circ d$. Using this expression, we find

$$\begin{aligned} \sigma_{DD_I}(m,v)\alpha &= \frac{1}{2}(1-J) \circ \sigma_d(m,v) \circ I \circ \sigma_d(m,v)\alpha \\ &= \frac{\alpha}{2}(v \wedge Iv + Jv \wedge Kv), \end{aligned} \tag{7.6}$$

for $m \in M$, $v \in T_m^* M \setminus \{0\}$ and $\alpha \in \underline{\mathbb{R}}_m$. Clearly (7.6) is zero if and only if $\alpha = 0$, as required. □

Proposition 7.9 implies that equation (7.4) is a non-linear elliptic second order partial differential equation in ϕ. This follows from the form of the linearisation LP of $\phi \mapsto P(\phi) = (\omega_I + DD_I\phi)^{2n}$:

$$LP_\phi(\psi) = 2n(\omega_I + DD_I\phi)^{2n-1} \wedge DD_I\psi.$$

7.1.3 Cohomology interpretations

While the analytical resemblance between Question 7.7 and the Calabi-Yau problem seems convincing, we still need to address the geometric significance of the problem at hand. Again our aim is to find an analogue of the interpretation in the Kähler setting. We thus recall that Calabi's original conjecture was a statement about representatives of certain cohomology classes. Indeed, the Calabi-Yau theorem tells us that on a compact connected Kähler manifold each representative of the first Chern class is realised as the Ricci-curvature of a unique Kähler metric in each Kähler class. To obtain a similar interpretation of (7.4) a further study of (modified) Salamon cohomology is required.

First we address the notion of an HKT class. Banos and Swann defined [BS04] the HKT class of an HKT metric g to be the Salamon class $[\omega_I] \in H^2_{Sal}(M)$. In general, however, this need not be the best definition. More precisely, their definition is appropriate if the global DD_I-lemma holds [BS04, Section 2.3]. For a general hypercomplex manifold it seems more natural to define the HKT class of g to be the class $[\omega_I] \in H_{\text{HKT}}(M)$ in the Bott-Chern like cohomology group defined by the complex

$$\mathcal{A}^0 \xrightarrow{DD_I} \mathcal{A}^{1,1} \xrightarrow{D} \mathcal{A}^3, \tag{7.7}$$

where $\mathcal{A}^{1,1} = \Gamma(A^2 \cap \Lambda_I^{1,1})$.

Proposition 7.10. *For a hypercomplex manifold (M, I, J, K) the associated complex (7.7) is elliptic. In particular, $H_{\text{HKT}}(M)$ is finite-dimensional when M is compact.*

Sketch of proof. To prove ellipticity of the complex complex (7.7), let us pick $m \in M$ and $v \in T^*_m M \setminus \{0\}$. We then consider the symbol sequence

$$\underline{\mathbb{R}}_m \xrightarrow{\sigma_{DD_I}(m,v)} A^{1,1}_m \xrightarrow{\sigma_D(m,v)} A^3_m. \tag{7.8}$$

We must verify that the sequence (7.8) is exact. Direct calculations lead to the following expressions

$$\sigma_{DD_I}(m,v)\alpha = \frac{\alpha}{2}(v \wedge Iv + Jv \wedge Kv), \quad \alpha \in \underline{\mathbb{R}}_m,$$
$$\sigma_D(m,v)\beta = \frac{1}{6}(3 - \mathcal{P})(v \wedge \beta), \quad \beta \in A^{1,1}_m.$$

Exactness may thus be characterised in the following way: for any $\beta \in A^{1,1}_m$, the element $v \wedge \beta$ lies in $\mathcal{B}^3_m$ if and only if β lies in the image of $\sigma_{DD_I}(m,v)$. This assertion is readily verified. Firstly, we have

$$\mathcal{P}(v \wedge Iv + Jv \wedge Kv) = 3(v \wedge Iv + Jv \wedge Kv),$$

which shows that the image of $\sigma_{DD_I}(m,v)$ is contained in $\mathcal{B}^3_m$. Secondly, we extend $v \wedge Iv + Jv \wedge Kv$ to a basis $\{v \wedge Iv + Jv \wedge Kv, X_1 \wedge IX_1 + JX_1 \wedge KX_1, \dots\}$ for $A^{1,1}_m$. Using such a basis, we observe that the $+3$-eigenspace of $\mathcal{P}$ is indeed spanned by $v \wedge Iv + Jv \wedge Kv$, as required.

If M is compact, finite-dimensionality of $H_{\text{HKT}}(M)$ follows from ellipticity of (7.7) combined with Hodge theory, cf. [Wel80, Chapter IV]. □

Remark 7.11. The failure of the global DD_I-lemma to hold is measured by the kernel of the natural surjection

$$\Phi\colon H_{\text{HKT}}(M) \to H^2_{Sal}(M).$$

Verbitsky's arguments in [Ver09, Remark 4.5] imply that $\ker \Phi$ is trivial if $\mathrm{Hol}(\nabla^{\mathrm{Ob}}) \subset SL(n, \mathbb{H})$. As a consequence, the global DD_I-lemma holds on any hypercomplex manifold with special Obata holonomy. △

On a hypercomplex manifold (M, I, J, K) any locally dd_I-exact two-form $\rho = dd_I\varphi$ determines a D-closed form

$$\hat{\rho} = \frac{1}{2}(1 - J)\rho \in \mathcal{A}^{1,1}.$$

In particular, given any HKT metric g the first Chern form ρ^C_I of the Chern connection of ω_I determines a class $\widehat{c_I} = [\widehat{\rho^C_I}] \in H_{\text{HKT}}(M)$.

In order to obtain a cohomological interpretation of the quaternionic Calabi-Yau problem, we now relate the projected Chern forms of two HKT metrics g and $g' = g_\phi$ that satisfy (7.4). We find that

$$\widehat{\rho'^C_I} = \widehat{\rho^C_I} - DD_I f.$$

Based on this observation, we obtain the following reformulation of Question 7.7.

Question 7.12. Let (M^{4n}, I, J, K) be compact connected hypercomplex manifold, and g an HKT metric on M. Is it then possible to realise each representative of $\widehat{c_I} \in H_{\text{HKT}}(M)$ as the projected Chern form $\widehat{\rho'^{C}_I}$ of an HKT metric g' such that $[\omega'_I] = [\omega_I] \in H_{\text{HKT}}(M)$? ▽

Remark 7.13. Note that, by the proof of Proposition 7.4, the local expression

$$\widehat{\rho_I^C} = -DD_I(\log\det g)$$

corresponds to the symmetric tensor

$$\frac{1}{2}(1 + I + J + K)(\nabla^{\text{Ob}})^2(\log\det g),$$

which is manifestly a quaternionic object. △

A useful notion associated with Kähler classes is that of a Kähler cone. The analogous construction in the HKT setting is the set

$$\mathcal{C} = \{\omega \in \mathcal{A}^{1,1} \colon \omega(\cdot, I\cdot) > 0,\, D\omega = 0\}$$

of positive D-closed Salamon $(1,1)$-forms.

While $\mathcal{C}$ fits naturally into the cohomological framework, the associated quaternionic object is a subset $\mathcal{H}$ of the hyperHermitian metrics on (M, I, J, K). This set $\mathcal{H}$ is obtained via the correspondence $\Psi\colon \mathcal{C} \to \mathcal{H}$ given by $\Psi(\omega) = \omega(\cdot, I\cdot)$.

Proposition 7.14. *Let (M, I, J, K) be a compact hypercomplex manifold. Then $\mathcal{C}$ is an open convex cone in the linear space $\{\omega \in \mathcal{A}^{1,1} \colon D\omega = 0\}$, and the convex subcones*

$$\mathcal{C}_b = \{\omega \in \mathcal{C} \colon d\omega^{2n-1}\} \quad \text{and} \quad \mathcal{C}_s = \{\omega \in \mathcal{C} \colon d(Id\omega) = 0\}$$

of balanced and strong HKT metrics, respectively, are finite-dimensional. Moreover, the intersection $\mathcal{C}_b \cap \mathcal{C}_s$ corresponds via Ψ to the set of hyperKähler metrics on (M, I, J, K).

Sketch of proof. The first statement is proved in the same way as in the Kähler setting, cf. [Huy05, Corollary 3.1.8]. We give a brief outline for completeness. Firstly, observe that the condition $\Psi(\omega) = \omega(\cdot, I\cdot) > 0$ is an open property on the set of forms $\omega \in \mathcal{A}^{1,1}$, and that the differential condition $D\omega = 0$ ensures that $\Psi(\omega)$ is an HKT metric. Secondly, note that for $\lambda \in \mathbb{R}_{>0}$ and $\omega, \omega' \in \mathcal{C}$ we obviously have $\lambda\omega, \omega + \omega' \in \mathcal{C}$. Thus $\mathcal{C}$ is a convex cone.

As the subsets $\mathcal{C}_b, \mathcal{C}_s \subset \mathcal{C}$ are defined via one extra linear constraint, they are clearly convex subcones. To show that $\mathcal{C}_b$ and $\mathcal{C}_s$ are finite-dimensional, it suffices, by Proposition 7.10, to argue that each HKT class contains finitely many metrics of the respective type. Finite-dimensionality of $\mathcal{C}_s$ follows from the work of Verbitsky [Ver09, Remark 4.12]. The essential ingredient in his argument is the uniqueness statement in Theorem 7.5.

Turning to the cone $\mathcal{C}_s$, let consider two SHKT metrics g and g_ϕ belonging to the same HKT class. We must have that $dIdDD_I(\phi) = 0$. As the fourth order linear differential operator $\psi \mapsto P(\psi) = d \circ I \circ d \circ DD_I(\psi)$ has symbol given by

$$\sigma_P(m, v) = v \wedge Iv \wedge Jv \wedge Kv,$$

it is (overdetermined) elliptic. Consequently, we have that $\ker(P) \subset C^\infty(M)$ is finite-dimensional, cf. [Bes08, Appendix I, Corollary 32], as required.

In order to verify the final statement of the proposition, note that an element ω_I in the intersection $\mathcal{C}_b \cap \mathcal{C}_s$ has corresponding HKT metric $\Psi(\omega_I)$ that is both balanced and strong. Such a metric is clearly hyperKähler in dimension four, since $d\omega_I = 0$, by definition, which implies $d\omega_J = d\omega_K = 0$, since $Id\omega_I = Jd\omega_J = Kd\omega_K$. In higher dimensions, the result is implied by [FPS04, Proposition 1.4] or, equivalently, by [AI01, Remark 1]. The core of the argument is a calculation, which shows that on a balanced manifold, $\|d\omega_I\|^2$ is proportional to the inner product of the forms $dd_I\omega_I$ and ω_I^2. Hence $dId\omega_I = 0$ implies that $d\omega_I = 0$, and consequently $\Psi(\omega_I)$ is Kähler. □

Remark 7.15. On a compact balanced HKT-manifold (M^{4n}, g, I, J, K), $n \geqslant 3$, the existence of an SHKT metric $g' \in \mathcal{C}_s$, such that $[g'] = [g] \in H_{\text{HKT}}(M)$, forces g to be hyperKähler, cf. [Ver09], and in fact $g = g'$. For an $SL(n, \mathbb{H})$-manifold an affirmative answer to Question 7.7 implies the existence of a unique balanced metric in each HKT class. Due to these observations we do not expect to find (non-hyperKähler) SHKT metrics on $SL(n, \mathbb{H})$-manifolds. △

7.1.4 Comparison with the Alesker-Verbitsky conjecture

Before turning to discuss some technical details, let us remark that Alesker and Verbitsky recently studied a quaternionic Monge-Ampère equation [AV10]. In their setting, the Salamon complex is replaced by Verbitsky's quaternionic Dolbeault complex [Ver07]. In particular, ω_I and DD_I are replaced by $\Omega_I = \omega_J + i\omega_K$ and $\partial\partial_J$, respectively; here ∂ denotes the ∂-derivative with respect to I, and ∂_J is $\bar\partial$ appropriately 'twisted' by J. It turns out that their Calabi-Yau problem is closely related to Question 7.7. This observation is important since it greatly facilitates the work required in order to obtain our first a priori estimate. To compare the two problems first observe the following.

Proposition 7.16. *Let (M^{4n}, I, J, K) be a hypercomplex manifold endowed with hyperHermitian metric g, and denote by Ω_I the $(2,0)$-form for I given by $\omega_J + i\omega_K$. Then the following relation holds:*

$$\omega_I^{2n} = \lambda_n \Omega_I^n \wedge \overline{\Omega}_I^n, \quad \textit{where} \quad \lambda_n = \tfrac{4(n!)^2}{(2n)!}. \tag{7.9}$$

Proof. Let $p \in M$ be any point. Choose an orthonormal basis for T_p^*M of the form $\{e_j, Ie_j, Je_j, Ke_j \colon 1 \leqslant j \leqslant n\}$. We may now write

$$\omega_I = \sum_{j=1}^n e_j \wedge Ie_j + Je_j \wedge Ke_j,\ \omega_J = \sum_{j=1}^n e_j \wedge Je_j - Ie_j \wedge Ke_j,$$

$$\omega_K = \sum_{j=1}^n e_j \wedge Ke_j + Ie_j \wedge Je_j,$$

$$\Omega_I = \omega_J + i\omega_K,\ \overline{\Omega}_I = \omega_J - i\omega_K,$$

$$\mathrm{vol}_M = e_1 \wedge Ie_1 \wedge Je_1 \wedge Ke_1 \wedge \ldots \wedge Ke_n.$$

Calculations show that

$$\omega_I^{2n} = (2n)!\,\mathrm{vol}_M \quad \text{and} \quad \Omega_I \wedge \overline{\Omega}_I = \left(\prod_{j=1}^{n} 2j\right)^2 \mathrm{vol}_M,$$

so that relation (7.9) follows. □

We may now bridge the gap between Question 7.7 and the problem studied by Alesker and Verbitsky.

Proposition 7.17. *Let (M^{4n}, I, J, K) be a connected compact hypercomplex manifold, and g an* HKT *metric. Let $A \in \mathbb{R}$ and $f \in C^\infty(M)$. A smooth real function ϕ satisfies* (7.4) *if and only if it satisfies the equation*

$$(\Omega_I + \partial\partial_J\phi)^n = Be^{f/2}\Omega_I^n \tag{7.10}$$

for $B \in \mathbb{R}$, such that $B^2 = A$.

Proof. By Theorem 7.5 and [AV10, Lemma 4.9] we know that if either of the equations (7.4) or (7.10) is satisfied, then there is a corresponding HKT metric g_ϕ that has $\omega_I^\phi = g_\phi(I\cdot,\cdot) = \omega_I + DD_I\phi$ and $\Omega_I^\phi = \omega_J^\phi + i\omega_K^\phi = \Omega_I + \partial\partial_J\phi$.

Now assume that (7.10) holds. Then, by Proposition 7.16, we find

$$\begin{aligned}(\omega_I^\phi)^{2n} &= \lambda_n\,(\Omega_I^\phi)^n \wedge (\overline{\Omega}_I^\phi)^n = \lambda_n(Be^{f/2}\Omega_I^n) \wedge (Be^{f/2}\overline{\Omega}_I^n)\\ &= \lambda_n(Be^{f/2})^2(\lambda_n^{-1}\omega_I^{2n}) = Ae^f\omega_I^2,\end{aligned}$$

as required.

Conversely, assume that (7.4) holds. Let us write $(\Omega_I^\phi)^n = h\Omega_I^n$, for some function h. Note that h must be real, since we have $\overline{(\Omega_I^\phi)^n} = \bar{h}\overline{\Omega}_I^n$ and $\overline{(\Omega_I^\phi)^n} = J((\Omega_I^\phi)^n) = hJ(\Omega_I)^n = h\overline{\Omega}_I^n$. Using this observation, we find that

$$\begin{aligned}(A^{1/2}e^{f/2})^2\Omega_I^n \wedge \overline{\Omega}_I^n &= \lambda_n^{-1}(A^{1/2}e^{f/2})^2\omega_I^{2n} = \lambda_n^{-1}(\omega_I^\phi)^{2n}\\ &= (\Omega_I^\phi)^n \wedge \overline{(\Omega_I^\phi)}^n = h^2\Omega_I^n \wedge \overline{\Omega}_I^n\end{aligned} \tag{7.11}$$

It follows that, up to a sign, $h = A^{1/2}e^{f/2}$, and thus equation (7.10) holds, as required. □

7.2 Solution strategy: the continuity method

The continuity method has proven to be a successful approach for solving the complex Monge-Ampère equation, not only for Kähler manifolds but also, more recently, in the general Hermitian setting [TW10]. It is therefore reasonable to expect that a similar approach might be applied in order to answer Question 7.7 affirmatively.

In this section, we will follow, and when required modify, the arguments applied by Joyce in [Joy00, Chapter 5]. We thus consider a one-parameter family of equations:

$$(\omega_I + DD_I\phi_t)^{2n} = A_t e^{tf} \omega_I^{2n}, \quad t \in [0,1], \tag{7.12}$$

where A_t are positive real numbers. Equation (7.12) is obviously satisfied when $t = 0$; we put $\phi_0 = 0$ and $A_0 = 1$. The aim is now to show that the set S of parameter values $t \in [0,1]$ for which the corresponding equation (7.12) has a solution is both open and closed. This will imply that $S = [0,1]$, and we may then solve (7.12) for $t = 1$, as required.

While the openness of S follows from Theorem 7.19 below, the closedness is technically much more difficult to show. Vaguely speaking, the idea is to take a sequence $\{t_j\}_{j\in\mathbb{N}} \subset S$ that converges to a number t'. As each t_j lies in S, there is a corresponding sequence $\{\phi_{t_j}\}_{j\in\mathbb{N}}$ of solutions to (7.12). The task is then to establish so-called a priori bounds on all solutions ϕ_t in some appropriate Banach space and to show that they lie in a compact subset. In that case the sequence $\{\phi_{t_j}\}_{j\in\mathbb{N}}$ contains a convergent subsequence, and provided we can show that the corresponding limit $\phi_{t'}$ is a solution of (7.12), we obtain that $t' \in S$. Consequently, we will have that S is closed.

7.2.1 Technical results

As part of the continuity method we will now prove two technical results. First we obtain an a priori bound on ϕ, under certain assumptions on the underlying hypercomplex manifold (M, I, J, K); we will assume that the Obata connection has holonomy in $SL(n, \mathbb{H})$. Thereafter we prove openness of the set S. To formulate these results we need some tools from analysis. In particular, we have to choose appropriate Banach spaces.

Our conventions are those of [Joy00, Chapter 1]. For a compact Riemannian manifold (M, g) we denote by $L^q_k = L^q_k(M)$ the Sobolev space consisting of functions $f \in L^q(M)$ that are k times weakly differentiable and have $|\nabla^j f| \in L^q(M)$; here $q \geqslant 1$ and k is a nonnegative integer. The associated Banach norm is given by

$$\|f\|^q_{L^q_k} = \sum_{j=0}^{k} \int_M |\nabla^j f|^q \operatorname{vol}_g .$$

By $C^{k,\alpha} = C^{k,\alpha}(M)$ we denote the Hölder spaces; here $k \geqslant 0$ is an integer and $\alpha \in (0,1)$. These are Banach spaces consisting of functions $f \in C^k(M)$ for which $\nabla^k f$ is Hölder continuous with exponent α. The norm on such a space is given by

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + [\nabla^k f]_\alpha,$$

where $\|f\|_{C^k} = \sum_{j=0}^{k} \sup_M |\nabla^j f|$ and

$$[f]_\alpha = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{\operatorname{dist}(x,y)^\alpha}.$$

Zeroth order a priori estimate In order to prove the closedness of S we must establish a priori bounds. A first step in this direction is the following theorem.

Theorem 7.18. *Let (M^{4n}, I, J, K) be a connected compact $SL(n,\mathbb{H})$-manifold, and g an HKT metric on M. Let $Q_1 \geqslant 0$. Then there exists $Q_2 \geqslant 0$ depending only on M, g, I, J, K and Q_1 such that the following holds.*

Suppose $f \in C^3(M)$, $\phi \in C^5(M)$ and $A > 0$ satisfy the following equations

$$\|\log A + f\|_{C^3} \leqslant Q_1, \quad \int_M \phi \, \mathrm{vol}_g = 0, \quad \text{and} \quad (\omega_I + DD_I\phi)^{2n} = Ae^f \omega_I^{2n}.$$

Then $\|\phi\|_{C^0} \leqslant Q_2$.

The above theorem is a direct consequence of Proposition 7.17 combined with [AV10, Corollary 5.7]. For completeness, and in order to specify the precise estimate, we will give an overview of the five main steps going into the proof of this result. We emphasise that apart from Step 1, which applies [AV10, Proposition 5.3 & Lemma 5.2], the arguments are essentially identical those in [Joy00, pp. 108-111]; in fact we deliberately use Joyce's notation, since this might be helpful if one aims to modify his higher order a priori estimates to the HKT setting. The proof outline will occupy the rest of this section.

Step 1 First we argue that for any $p \geqslant 2$, a solution ϕ of (7.5) satisfies

$$\|\nabla(|\phi|^{p/2})\|^2_{L^2} \leqslant \frac{1}{16n}\frac{p^2}{p-1}\int_M (1 - Be^{f/2})\phi|\phi|^{p-2}F\omega_I^{2n}, \tag{7.13}$$

where $B^2 = A$, and $F > 0$ is a smooth real function that depends only on M, I, J, K and g.

To see this we introduce a (positive) non-vanishing I-holomorphic $(2n,0)$-form $\Theta \in \Gamma(\Lambda_I^{2n,0})$; this is possible since we are on an $SL(n,\mathbb{H})$-manifold, i.e., $\mathrm{Hol}(\nabla^{\mathrm{Ob}}) \subset SL(n,\mathbb{H})$. Direct calculations, following the proof of [AV10, Proposition 5.3], now show that

$$\begin{aligned}
&\int_M (1 - Be^{f/2})\phi|\phi|^{p-2}\Omega_I^n \wedge \overline{\Theta} = \\
&\qquad (p-1)\int_M |\phi|^{p-2}\partial\phi \wedge \partial_J\phi \wedge \left(\sum_{j=0}^{n-1}\Omega_I^j \wedge (\Omega_I^\phi)^{n-j-1}\right) \wedge \overline{\Theta} \\
&\qquad \geqslant (p-1)\int_M |\phi|^{p-2}\partial\phi \wedge \partial_J\phi \wedge \Omega_I^{n-1} \wedge \overline{\Theta} \\
&\qquad = (p-1)\frac{4}{p^2}\int_M \partial(|\phi|^{p/2}) \wedge \partial_J(|\phi|^{p/2}) \wedge \Omega_I^{n-1} \wedge \overline{\Theta}.
\end{aligned}$$

In addition to straightforward computations, the first equality uses Stokes' theorem and the inequality uses positivity of the forms Ω_I, Ω_I^ϕ and Θ.

In order to obtain the estimate (7.13), we now apply the pointwise equality

$$4n\partial(|\phi|^{p/2}) \wedge \partial_J(|\phi|^{p/2}) \wedge \Omega_I^{n-1} = |\nabla(|\phi|^{p/2})|_g^2\,\Omega_I^n,$$

cf. [AV10, Lemma 5.2], combined with the observation that $\Omega_I^n \wedge \overline{\Theta} = F\omega_I^{2n}$, where the positive function F depends only on M, g and the hypercomplex structure I, J, K.

For the remainder of the proof we put $\varepsilon = \frac{2n}{2n-1}$.

Step 2 We next obtain constants C_1 and C_2 depending only on M and g such that if $\psi \in L_1^2$ then

$$\|\psi\|_{L^{2\varepsilon}}^2 \leqslant C_1(\|\nabla\psi\|_{L^2}^2 + \|\psi\|_{L^2}^2), \tag{7.14}$$

and if in addition $\int_M \psi \operatorname{vol}_g = 0$ then

$$\|\psi\|_{L^2} \leqslant C_2\|\nabla\psi\|_{L^2}. \tag{7.15}$$

The inequality (7.14) follows, since L_1^2 is continuously embedded in $L^{2\varepsilon}$, by Sobolev's embedding theorem. If $\int_M \psi \operatorname{vol}_g = 0$, the inequality (7.15) follows, since the non-zero eigenvalues of the Laplacian on (M, g) are positive and form a discrete spectrum. In fact, cf. [Joy00, Proposition 5.4.2], we can take $C_2 = \lambda_1^{-1/2}$, where λ_1 is the smallest non-zero eigenvalue of the Laplacian.

Step 3 We go on to find a priori estimates of $\|\phi\|_{L^p}$, for $p \geqslant 2$. First we consider the case when $2 \leqslant p \leqslant 2\varepsilon$. Our aim is to find a constant C_3 depending on M, g, I, J, K and Q_1 such that if $2 \leqslant p \leqslant 2\varepsilon$ then

$$\|\phi\|_{L^p} \leqslant C_3.$$

To this end, we define a positive constant Q depending only on Q_1. Concretely, we may take $Q := \log(1 + e^{Q_1/2})$. Then we have that $|1 - Be^{f/2}| \leqslant e^Q$. From (7.13), with $p = 2$, we thus get

$$\|\nabla\phi\|_{L^2}^2 \leqslant \kappa e^Q \|\phi\|_{L^1},$$

where $\kappa = \frac{(2n)!}{4n}\|F\|_{C^0}$. Combining this estimate with (7.15) and the estimate $\|\phi\|_{L^1} \leqslant \operatorname{vol}_g(M)^{1/2}\|\phi\|_{L^2}$, obtained via Hölder's inequality, we find that

$$\|\nabla\phi\|_{L^2}^2 \leqslant \kappa e^Q C_2 \operatorname{vol}_g(M)^{1/2}\|\nabla\phi\|_{L^2},$$

and therefore $\|\nabla\phi\|_{L^2} \leqslant \kappa e^Q C_2 \operatorname{vol}_g(M)^{1/2} =: c$.

Now put $C_3 := \max\left\{cC_2, cC_1^{1/2}(1 + C_2^2)^{1/2}\right\}$. Then, by (7.14) and (7.15), we have that $\|\phi\|_{L^2}, \|\phi\|_{L^{2\varepsilon}} \leqslant C_3$. So, by Hölder's inequality, $\|\phi\|_{L^p} \leqslant C_3$ for $2 \leqslant p \in\leqslant 2\varepsilon$, as required.

Step 4 We then find constants Q_2, C_4 depending on M, g, I, J, K and Q_1 such that for each $p \geqslant 2$, we have

$$\|\phi\|_{L^p} \leqslant Q_4(C_4 p)^{-2n/p}. \tag{7.16}$$

Define a positive constant $C_4 := C_1\varepsilon^{2n-1}(\kappa e^Q + \frac{1}{2})$, and choose a positive number Q_2 such that

$$Q_2 \geqslant C_3(C_4p)^{2n/p}, \quad \text{for } 2 \leqslant p \leqslant 2\varepsilon,$$
$$Q_2 \geqslant (C_4p)^{2n/p}, \quad \text{for } 2 \leqslant p < \infty.$$

We will prove the estimate (7.16) by induction on p. We already know that for $2 \leqslant p \leqslant 2\varepsilon$ one has that

$$\|\phi\|_{L^p} \leqslant C_3 \leqslant Q_2(C_4p)^{-2n/p}.$$

In order to verify the inductive step, let us assume that $\|\phi\|_{L^p} \leqslant Q_2(C_4p)^{-2n/p}$ holds for all $2 \leqslant p \leqslant k$, where $k \geqslant 2\varepsilon$. We now argue that the estimate

$$\|\phi\|_{L^q} \leqslant Q_2(C_4q)^{-2n/q}$$

holds for all $2 \leqslant q \leqslant \varepsilon k$. Then, by induction, the inequality (7.16) holds for all $p \geqslant 2$.

Let $2 \leqslant p \leqslant k$. By (7.13), we have that

$$\|\nabla(|\phi|^{p/2})\|^2_{L^2} \leqslant p\kappa\|\phi\|^{p-1}_{L^{p-1}}.$$

If we combine this estimate with the inequality

$$\|\phi\|^p_{L^{\varepsilon p}} \leqslant C_1\left(\|\nabla(|\phi|^{p/2})\|^2_{L^2} + \|\phi\|^p_{L^p}\right),$$

which follows from (7.14) applied to $|\phi|^{p/2}$, we get

$$\|\phi\|^p_{L^{\varepsilon p}} \leqslant C_1\left(p\kappa\|\phi\|^{p-1}_{L^{p-1}} + \|\phi\|^p_{L^p}\right).$$

Put $q = \varepsilon p$. As $2 \leqslant p \leqslant k$, we have $\|\phi\|_{L^p} \leqslant Q_2(C_4p)^{-2n/p}$ as well as $1 \leqslant Q_2(C_4p)^{-2n/p}$. Combining these observations with the inequality $\|\phi\|_{L^{p-1}} \leqslant \|\phi\|_{L^p}$, we get

$$\|\phi\|^p_{L^q} \leqslant Q_2^p(C_4p)^{-2n}C_1(p\kappa + 1).$$

As $p \geqslant 2$, the definition of C_4 ensures that the inequality $C_1(p\kappa+1) \leqslant C_4p\varepsilon^{1-2n}$ holds. But as $Q_2^p(C_4p\varepsilon)^{1-2n} = (Q_2(C_4q)^{-2n/q})^p$, these observations allow us to conclude that

$$\|\phi\|_{L^q} \leqslant Q_2(C_4q)^{-2n/q},$$

for all $2\varepsilon \leqslant q \leqslant \varepsilon k$. This completes the inductive step.

Step 5 Finally, we are able to verify the statement of Theorem 7.18. By construction, Q_2 depends only on M, g, I, J, K and Q_1, and if we combine continuity of ϕ and compactness of M with (7.16), we get

$$\|\phi\|_{C^0} = \lim_{p\to\infty}\|\phi\|_{L^p} \leqslant \lim_{p\to\infty} Q_2(C_4p)^{-2n/p} = Q_2,$$

as required.

This completes our sketch of the proof of Theorem 7.18.

Openness The following theorem implies that S is open, and is the HKT analogue of [Joy00, Theorem C3].

Theorem 7.19. *Let (M^{4n}, I, J, K) be a connected compact hypercomplex manifold, and g an* HKT *metric on M. Fix $\alpha \in (0,1)$ and suppose that $f' \in C^{3,\alpha}(M)$, $\phi' \in C^{5,\alpha}(M)$ and $A' > 0$ satisfy the equations*

$$\int_M \phi' \operatorname{vol}_g = 0 \quad \textit{and} \quad (\omega_I + DD_I\phi')^{2n} = A' e^{f'} \omega_I^{2n}. \tag{7.17}$$

Then whenever $f \in C^{3,\alpha}(M)$ and $\|f - f'\|_{C^{3,\alpha}}$ is sufficiently small, there exist $\phi \in C^{5,\alpha}$ and $A > 0$ such that

$$\int_M \phi \operatorname{vol}_g = 0 \quad \textit{and} \quad (\omega_I + DD_I\phi)^{2n} = A e^{f} \omega_I^{2n}. \tag{7.18}$$

Proof. Let X be the vector subspace consisting of functions $\phi \in C^{5,\alpha}$ for which $\int_M \phi \operatorname{vol}_g = 0$. Then the subset $U \subset X$ for which $\omega_I^{\phi} = g_{\phi}(I\cdot, \cdot)$ is a positive $(1,1)$-form on M is open in X.

Suppose that $\phi \in U$ and that a is a real number. Then $(\omega_I^{\phi})^{2n}$ is a positive multiple of ω_I^{2n}. In particular, there exists a unique function $f \in C^{3,\alpha}$ such that

$$(\omega_I^{\phi})^{2n} = e^{a+f} \omega_I^{2n}. \tag{7.19}$$

Now define a map $\Phi \colon U \times \mathbb{R} \to C^{3,\alpha}$ by $\Phi(\phi, a) = f$ with f satisfying (7.19). Φ is a well-defined smooth map between Banach spaces. If we choose f', ϕ' and A' as in the statement of the theorem and let $a' = \log A'$, then, by (7.17), one has that $\phi' \in U$ and $\Phi(\phi', a') = f'$. Let us evaluate the derivative of Φ at the point $p = (\phi', a')$. A calculation shows that

$$(\omega_I + DD_I(\phi' + \varepsilon\psi))^{2n} = e^{a'+f'} \omega_I^{2n} + \varepsilon C \Delta'^{c}(\psi) \omega_I^{2n} + O(\varepsilon^2), \tag{7.20}$$

where Δ'^{c} is the complex Laplacian with respect to $g' = g_{\phi}$ and C is a positive $C^{3,\alpha}$ function. Now let $f_\varepsilon := f' - \varepsilon b + \varepsilon C \Delta'^{c}(\psi) + O(\varepsilon^2)$ and observe that

$$(\omega_I + DD_I(\phi' + \varepsilon\psi))^{2n} = e^{a' + \varepsilon b + f_\varepsilon} \omega_I^{2n},$$

so that $\Phi(\phi' + \varepsilon\psi, a' + \varepsilon b) = f_\varepsilon$. Consequently, the derivative $d\Phi_p \colon X \times \mathbb{R} \to C^{3,\alpha}$ is given by

$$d\Phi_p(\psi, b) = -b + C\Delta'^{c}(\psi). \tag{7.21}$$

The linear differential operator $P := C\,\Delta'^{c} \colon C^{5,\alpha} \to C^{3,\alpha}$ is a second order elliptic operator without constant term, so its kernel is the set of constant functions on M; this follows from the maximum principle. In addition P has vanishing index, since it is the composite of two Fredholm operators each of index zero. These observations imply that $\ker P^*$ is one-dimensional, say spanned by the function φ; the adjoint is taken with respect to the inner product induced by g. From the theory of elliptic operators, we then know that $\operatorname{Im} P$ consists of elements $\vartheta \in C^{3,\alpha}$ orthogonal to ϕ and that the restriction of P to X

is injective. A straightforward argument now shows that $d\Phi_p\colon X \times \mathbb{R} \to C^{3,\alpha}$ is an invertible linear map, cf. [Mad09, Theorem 5.5]. Since it is continuous and has continuous inverse, the inverse mapping theorem for Banach spaces applies. Hence, there is an open neighbourhood $V \subset U \times \mathbb{R}$ of $p \in X \times \mathbb{R}$ and an open neighbourhood $W \subset C^{3,\alpha}$ of $f' \in C^{3,\alpha}$ such that $\Phi\colon V \to W$ is a homeomorphism.

In conclusion we have that whenever $f \in C^{3,\alpha}$ and $\|f - f'\|_{C^{3,\alpha}}$ is sufficiently small then $f \in W$, and there is a unique point $(\phi, a) \in V$ such that $\Phi(\phi, a) = f$. As $\phi \in X$ the first equation of (7.18) holds and the second equality follows if we take $A = e^a$ so that $\Phi(\phi, a) = f$. □

In order to see that this result implies openness of the set $S \subset [0,1]$, we have to be more specific regarding the relevant topologies on the function spaces that are involved: in (7.12) we take $f \in C^{3,\alpha}$ and $\phi \in C^{5,\alpha}$. Now pick $t' \in S$. Then, by definition of S, there is a function $\phi' \in C^{5,\alpha}$ with $\int_M \phi' \operatorname{vol}_g = 0$ and $A' > 0$ such that

$$(\omega_I + DD_I\phi')^{2n} = A' e^{t'f} \omega_I^{2n}.$$

Applying Theorem 7.19 with $t'f$ in place of f' and tf in place of f, for $t \in [0,1]$, shows that whenever $|t - t'|\|f\|_{C^{3,\alpha}}$ is sufficiently small, then there exist $\phi \in C^{5,\alpha}$ and a positive real number A such that

$$\int_M \phi \operatorname{vol}_g = 0 \quad \text{and} \quad (\omega_I + DD_I\phi)^{2n} = A e^{tf} \omega_I^{2n}.$$

Hence, $t \in S$ whenever chosen sufficiently close to t'. So S is open, as claimed.

7.3 Concluding remarks

While Verbitsky [Ver09] argues that balanced HKT metrics are good quaternionic analogues of Calabi-Yau metrics, SHKT metrics are distinguished from the strong geometric point of view. The above results therefore suggest that future studies of HKT geometry should be twofold. On compact hypercomplex manifolds with Obata holonomy in $SL(n, \mathbb{H})$, e.g., on hypercomplex nilmanifolds [BDV09], we expect to find to find balanced HKT metrics. On the other hand hypercomplex manifolds with $\mathrm{Hol}(\nabla^{\mathrm{Ob}}) \subsetneq SL(n, \mathbb{H})$, e.g., compact Lie groups [Sol11] or their product with a torus, are more likely to carry SHKT metrics.

In summary, we should strive to prove the HKT Calabi-Yau problem for $SL(n, \mathbb{H})$-manifolds, but at the same time put a separate effort into the construction and understanding of SHKT metrics.

Appendix A

Published work

A substantial part of this thesis is based on published or submitted papers [MS11b, MS10, MS11a, Mad11]; the first three items are joint work with Andrew Swann. Below I explain to which extent the material in the individual chapters already appeared in these four references.

Chapter 1 Some of the motivational material is based on the introductions to the papers [MS11b] and [MS10].

Chapter 2 Section 2.1 on SKT geometry is based on [MS11b], and Example 2.4 appeared in [MS11a]. Example 2.6 on generalized hyperKähler structures has been added.

Chapter 3 Apart from the two supplementary results on unimodular Lie algebras, Remark 3.4 and Proposition 3.5, this chapter is based on [MS11b].

Chapter 4 While the first part of the chapter, the Sections 4.1–4.4, consists of material mainly from [MS10], Section 4.5 is based on the paper [MS11a]. Most of Section 4.4.2 differs significantly from the material appearing in the aforementioned papers; I have rearranged, clarified and expanded the exposition. I have also added Proposition 4.9, the Examples 4.19–4.20, and Section 4.4.5.

Chapter 5 This chapter consists of unpublished material.

Chapter 6 Apart from a few supplementary remarks, clarifications, and the addition of Example 6.13, Section 6.1 is based on the last part of the paper [MS10]. Section 6.2 is based on [Mad11].

Chapter 7 This chapter describes a *future research project*. Some of the main ideas were conceived in [Mad09].

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