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Octonions and Supergravity

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Abstract

This thesis makes manifest the roles of the normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} in various supergravity theories. Of particular importance are the octonions \mathbb{O} , which frequently occur in connection with maximal supersymmetry, and hence also in the context of string and M-theory. Studying the symmetries of M-theory is perhaps the most straightforward route towards understanding its nature, and the division algebras provide useful tools for such study via their deep relationship with Lie groups.

After reviews of supergravity and the definitions and properties of $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} , a division-algebraic formulation of pure super Yang-Mills theories is developed. In any spacetime dimension a Yang-Mills theory with \mathcal{Q} real supercharge components is written over the division algebra with dimension $\mathcal{Q}/2$. In particular then, maximal $\mathcal{Q} = 16$ super Yang-Mills theories are written over the octonions, since \mathbb{O} is eight-dimensional. In such maximally supersymmetric theories, the failure of the supersymmetry algebra to close off-shell (using the conventional auxiliary field formalism) is shown to correspond to the non-associativity of the octonions.

Making contact with the idea of ‘gravity as the square of gauge theory’, these division-algebraic Yang-Mills multiplets are then tensored together in each spacetime dimension to produce a pyramid of supergravity theories, with the Type II theories at the apex in ten dimensions. The supergravities at the base of the pyramid have global symmetry groups that fill out the famous Freudenthal-Rosenfeld-Tits magic square. This magic square algebra is generalised to a ‘magic pyramid algebra’, which describes the global symmetries of each Yang-Mills-squared theory in the pyramid.

Finally, a formulation of eleven-dimensional supergravity over the octonions is presented. Toroidally compactifying this version of the theory to four or three spacetime dimensions leads to an interpretation of the dilaton vectors (which organise the coupling of the seven or eight dilatons to the other bosonic fields) as the octavian integers – the octonionic analogue of the integers.

Declarations

Unless otherwise referenced, all research presented in this thesis is the present author's own work, carried out alone or in collaboration with Alexandros Anastasiou, Leron Borsten, Michael Duff and Silvia Nagy. The material is based on the following research papers:

- L. Borsten, M. J. Duff, L. J. Hughes, S. Nagy, *A Magic Square from Yang-Mills Squared*, Physical Review Letters 112, 131601
- A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, S. Nagy, *Super Yang-Mills, Division Algebras and Triality*, JHEP 1408 (2014) 080
- A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, S. Nagy, *A Magic Pyramid of Supergravities*, JHEP 1404 (2014) 178
- A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, S. Nagy, *An Octonionic Formulation of the M-Theory Algebra*, JHEP 1411 (2014) 022
- A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, S. Nagy, *Yang-Mills Origin of Gravitational Symmetries*, Physical Review Letters 113, 231606
- A. Anastasiou, M. J. Hughes, *Octonionic $D = 11$ Supergravity and 'Octavian Integers' as Dilaton Vectors*, arXiv:1502.02578 [hep-th]
- A. Anastasiou, L. Borsten, M. J. Hughes, S. Nagy, *Global Symmetries of Yang-Mills Squared in Various Dimensions*, arXiv:1502.05359 [hep-th].

Note that the author's first initials were previously L.J. (rather than the current M.J.) until January 2015.

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1. Introduction

1.1. Supergravity, Superstrings and M-Theory

The goal of high energy physics is arguably to determine the fundamental microscopic constituents of the universe and the basic rules that govern their interactions. For most theorists, the hope is that there are simple, logical organising principles underlying the structure of these elementary building blocks, which might allow us to paint an essentially complete picture of their properties and behaviour in the form of one or two short equations – a ‘theory of everything’.

Quantum field theory and the Standard Model of particle physics have given us a powerful description of this microscopic world, bringing into focus the realm of quarks, leptons and gauge bosons, the ingredients required to explain all non-gravitational phenomena observed so far. To incorporate gravity into this quantum picture requires a marriage between the Standard Model and Einstein’s theory of general relativity, which stipulates that at the macroscopic scale gravity is the manifestation of the smooth curvature of spacetime. However, the basic frameworks of quantum field theory and general relativity are infamously incompatible; unlike the field theories describing electromagnetism and the weak and strong nuclear forces, general relativity is not renormalisable, and hence any naive attempt to calculate graviton scattering amplitudes (beyond one loop [2, 3]) is thwarted by untameable divergences.

In the early 1980s many physicists believed that a candidate ‘theory of everything’ was Cremmer and Julia’s theory of *eleven-dimensional supergravity* [4, 5]. The vital ingredient in this theory, and a protagonist in this thesis, is *supersymmetry*, a symmetry which unites fermions and bosons. Intriguingly, supersymmetry requires the existence of Einstein’s gravity [6], as well as placing an upper limit [7] of *eleven* on the number of spacetime dimensions D . The ultraviolet (UV) behaviour of supersymmetric quantum field theories also tends to be good [8], due to cancellations between divergences coming from bosons and divergences coming (with the opposite sign) from fermions, so it was hoped that this might render $D = 11$ supergravity renormalisable.

Far from being a disadvantage, the seven extra dimensions of the theory are a potentially great asset, thanks to the techniques of Kaluza-Klein reduction [9, 10, 11, 12, 13]. Supposing that the extra dimensions take the form of some compact seven-manifold \mathcal{M}_7 of length scale L , then at scales much larger than L the low-energy effective theory is not only four-dimensional, but also turns out to have a gauge symmetry: the isometry group [14] of \mathcal{M}_7 . In other words, the existence of extra dimensions might reveal the origin of the Standard Model's mysterious $SU(3) \times SU(2) \times U(1)$ gauge symmetry, which serves as its (slightly ad hoc) organising principle; in the low-energy four-dimensional physics this symmetry would be a remnant of the underlying higher-dimensional gravitational theory's invariance under general coordinate transformations.

One seemingly attractive feature of $D = 11$ supergravity came from Witten's discovery [14] that seven extra dimensions are the *minimum* required to produce the Standard Model's gauge group via Kaluza-Klein reduction. This means that a world with eleven spacetime dimensions is singled out both from above *and* below, by supersymmetry and the Kaluza-Klein theory of the Standard Model, respectively. Is this a profound clue about nature, or just a strange coincidence? Or both? The answer remains to be seen.

For a while, interest in eleven-dimensional supergravity faded as it appeared to be non-renormalisable after all, and there were difficulties obtaining the chiral fermions of the Standard Model by Kaluza-Klein reduction [14]. Moreover, by 1984 ten-dimensional *superstring theory* was gaining popularity [15, 16, 17, 18] and seemed promising as a unified and apparently UV-finite theory of all particles and interactions, including quantum gravity. Its central idea is that at the Planck scale any individual particle is a tiny one-dimensional (supersymmetric) string, and that the particular quantum-mechanical vibrational mode of this string dictates its corresponding particle type. It is in this sense that string theory's description of particle physics is unifying; each particle type is a different manifestation of the same underlying object. In particular, there is a vibrational mode with the properties of a massless spin-2 particle, which may be identified with the graviton. Better yet, quantum consistency of superstring theory demands that its low-energy field theory approximation is equivalent to general relativity – or rather to its supersymmetric generalisation in ten spacetime dimensions.

Unlike the unique $D = 11$ supergravity theory, there are several (anomaly-free) $D = 10$ supergravities, each of which is the low-energy description of one of *five* distinct consistent string theories known as Type I, Heterotic $SO(32)$, Heterotic $E_8 \times E_8$, Type IIA and Type IIB [15, 16, 17, 18]. The five string theories are

distinguished by fundamental assumptions regarding basic features of their strings, such as whether the strings are oriented or unoriented, or whether they may be open line segments or only closed loops. Remarkably in the 1990s it was discovered that the five theories are connected to one another via a web of *dualities*, that is, they are equivalent to one another upon taking various limits [19, 20, 21, 22]. For example the Type I theory in the limit of strong coupling is equivalent to the Heterotic SO(32) theory with weak coupling [19, 21], and vice versa. Because of this web of dualities, the five string theories are now understood to be limiting cases of a single theory, dubbed *M-theory*, which is yet to be properly defined.

At its fundamental level M-theory is thought to include extended vibrating objects of various dimensionalities called *branes* [22, 23]. Just as the $D = 10$ supergravity field theories correspond to the low-energy physics of strings, it is believed that the low-energy physics of M-theory is described by none-other than $D = 11$ supergravity. For this reason, there is as much interest in supergravity in ten and eleven dimensions today – and in supergravity more generally – as there was in the 1970s and 1980s. These supersymmetric generalisations of Einstein’s theory are perhaps the best handle we have on working towards an understanding of M-theory, as well as its hoped eventual reduction to the Standard Model.

1.2. Octonions and Supergravity

Complex numbers, ubiquitous in modern physics, generalise the real line to the complex plane. It is natural to ask whether this can be taken further; are there higher-dimensional number systems whose mathematics may have some utility or significance in physical theories? In fact, there is a set of precisely four such number systems, the so-called *normed division algebras* – the real numbers \mathbb{R} , the complex numbers \mathbb{C} , the quaternions \mathbb{H} and the octonions \mathbb{O} – and they all have roles to play in physics¹.

A recurring theme in the study of supersymmetry and string theory is the connection to these four special algebras. See, for example, references [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. The octonions, titular in this thesis, are perhaps the most intriguing and the most mysterious of these. Not only do they hold an exceptional status as the largest division algebra, but they may be used to describe representations of the Lorentz group in spacetime dimensions $D = 10, 11$ – dimensions that are

¹In fact, there is actually another useful series of algebras that generalise \mathbb{R} , \mathbb{C} and \mathbb{H} : the Clifford algebras, $\text{Cl}(N)$. They begin as $\text{Cl}(0) \cong \mathbb{R}$, $\text{Cl}(1) \cong \mathbb{C}$, $\text{Cl}(2) \cong \mathbb{H}$, but they depart from the division algebras at $\text{Cl}(3) \cong \mathbb{H} \oplus \mathbb{H}$ and continue from there on to $N \rightarrow \infty$ in terms of matrices over \mathbb{R} , \mathbb{C} and \mathbb{H} . These will be introduced in Chapter 4.

themselves distinguished as those preferred by string and M-theory, as well as being the maximum dimensions typically allowable for rigid and local supersymmetry, respectively.

The aim of this thesis is to make manifest the roles of the four normed division algebras \mathbb{R} , \mathbb{C} , \mathbb{H} and \mathbb{O} in various supergravity theories. The octonions will frequently appear in connection with maximal supersymmetry, and hence also in the context of string and M-theory. Studying the symmetries of these theories is perhaps the most straightforward route towards understanding their nature, and the division algebras provide natural building blocks for such study via their deep relationship with Lie groups. In particular, as will be demonstrated in the chapters to come, exceptional groups such as E_8 are inherently octonionic in structure and often arise as the symmetries of supergravities [4, 35], strings [36] and M-theory [37, 38].

After a brief review of supersymmetry and supergravity in Chapter 2, Chapter 3 will provide a detailed introduction to the normed division algebras and their relationship with triality, a pivotal cog in the machinery of string theory [36, 39, 16]. Chapter 4 will build on a well-established description of minimally supersymmetric Yang-Mills theories in dimensions $D = 3, 4, 6$ and 10 in terms of \mathbb{R} , \mathbb{C} , \mathbb{H} and \mathbb{O} [24, 31], and by dimensional reduction extend this description to incorporate additional supersymmetries and to include any spacetime dimension $3 \leq D \leq 10$. Maximally supersymmetric Yang-Mills theories, which are descended from $D = 10$, are all written over the octonions in this formalism.

Chapter 5 will introduce the idea of ‘gravity as the square of gauge theory’, and in this spirit the division-algebraic Yang-Mills multiplets will be ‘multiplied’ together to produce supergravity theories valued over tensor products of division algebras. In $D = 3$ this leads to a set of supergravities whose global symmetry groups correspond to a famous set of Lie algebras called the Freudenthal-Rosenfeld-Tits *magic square*², which will also be introduced in Chapter 5.

In $D = 3$ there are four super Yang-Mills theories ($\mathcal{N} = 1, 2, 4, 8$), while in $D = 10$ there is just one ($\mathcal{N} = 1$); hence ‘squaring Yang-Mills’ in the range $3 \leq D \leq 10$ leads to a *pyramid* of supergravity theories, which will be the subject of Chapter 6. Type II supergravity sits at the apex in $D = 10$, while the magic square supergravities populate the 4×4 base of the pyramid in $D = 3$. The magic square algebra of Freudenthal-Rosenfeld-Tits will be generalised to a ‘magic pyramid algebra’, which describes the global symmetries of each Yang-Mills-squared theory.

Finally, in Chapter 7, a formulation of eleven-dimensional supergravity over the

²Note that the magic square has appeared before in supergravity in an almost entirely different context [25, 26], with each of its four rows in a different dimension $D = 3, 4, 5, 6$ (and with different real forms of its Lie algebras).

octonions will be presented. Toroidally compactifying this version of the theory to four or three spacetime dimensions leads to an effective Lagrangian written in terms of the octavian integers [40, 41] – the octonionic analogue of the integers.

1.3. Conventions and Units

All physics in this thesis will be expressed in natural units, with $c = \hbar = G = 1$. The metric will have ‘mostly plus’ signature $(-, +, +, \dots, +)$ in any number of spacetime dimensions D .

When dealing with decompositions of Lie algebras, the direct sum symbol \oplus will be reserved to sit between commuting Lie subalgebras, while $+$ will denote the direct sum for subspaces that do *not* commute with these subalgebras – for example,

$$\mathfrak{so}(m+n) \cong \mathfrak{so}(m) \oplus \mathfrak{so}(n) + \mathbb{R}^m \otimes \mathbb{R}^n. \quad (1.1)$$

Similarly, the symbol \ominus between two commuting Lie algebras will denote ‘direct subtraction’, e.g.

$$\mathfrak{u}(n) \ominus \mathfrak{u}(1) \cong \mathfrak{su}(n), \quad (1.2)$$

while the symbol $-$ will denote the removal of a subspace that does not commute with the rest of the algebra, as in

$$\mathfrak{so}(m+n) - \mathbb{R}^m \otimes \mathbb{R}^n \cong \mathfrak{so}(m) \oplus \mathfrak{so}(n). \quad (1.3)$$

Also note that direct sums and direct products of vector spaces will be used interchangeably wherever such operations are essentially indistinguishable – for example $\mathbb{R}^2 \cong \mathbb{R} \oplus \mathbb{R}$.

2. A Review of Supersymmetry and Supergravity

This chapter reviews the basic ideas and features of supersymmetric theories, mostly at the classical level. Although the quantum properties of supersymmetric field theories make up a rich subject, they will mostly be beyond the scope of this thesis, whose focus is at the level of Lagrangians and their symmetries.

The first section is an illustration of the basic features of a theory with a global supersymmetry by means of a simple ‘toy’ model, roughly following the exposition in [42]. Then comes a brief introduction to super Yang-Mills theory, followed by a review of supergravity and the all-important eleven-dimensional theory. The final two sections will demonstrate the techniques of Kaluza-Klein reduction on a torus T^d and examine the symmetries of the resulting lower-dimensional theories.

2.1. The Basics of Rigid Supersymmetry

Consider the simple theory of a free scalar field ϕ and Dirac spinor field ψ , both of mass m :

$$S = \int d^4x \left(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - i \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{1}{2} m^2 \phi^2 + im \bar{\psi} \psi \right). \quad (2.1)$$

The Lagrangian is invariant – up to a total derivative – under the following transformations [8]:

$$\delta \phi = i \bar{\epsilon} \psi, \quad \delta \psi = \frac{1}{2} (\gamma^\mu \partial_\mu + m) \phi \epsilon, \quad (2.2)$$

where ϵ is some constant spinor parameter. This is called a supersymmetry. The transformations ‘rotate’ a bosonic field into a fermionic one, and vice versa. In general the set of fields rotated into one another by supersymmetry transformations is called a *supermultiplet* – in this case this is just the pair $\{\phi, \psi\}$. Thanks to Noether’s theorem, this symmetry implies the existence of a conserved current

$$\mathcal{J}^\mu = (\gamma^\nu \partial_\nu \phi - m \phi) \gamma^\mu \psi. \quad (2.3)$$

Note that this *supercurrent* is a vector-spinor quantity. There is also a corresponding conserved spinor *supercharge*, which can be expressed in the usual form:

$$Q = \int d^3x \mathcal{J}^0(\mathbf{x}, t). \quad (2.4)$$

The total number of supercharges – equal to the number of independent supersymmetries – is usually denoted by \mathcal{N} . The theory above has $\mathcal{N} = 1$ since there is just one set of transformations and one corresponding spinor supercharge. The supercharge generates the supersymmetry transformations via Poisson brackets,

$$\delta\phi = [\bar{\epsilon}Q, \phi]_{\text{PB}} = i\bar{\epsilon}\psi, \quad \delta\psi = [\bar{\epsilon}Q, \psi]_{\text{PB}} = \frac{1}{2}(\gamma^\mu\partial_\mu + m)\phi\epsilon, \quad (2.5)$$

where the Poisson bracket between any two functions F and G of the fields and their conjugate momenta is defined as

$$[F, G]_{\text{PB}} := \int d^{D-1}\vec{x} \left(\frac{\delta F}{\delta\Phi^i(\vec{x})} \frac{\delta G}{\delta\Pi_i(\vec{x})} - \frac{\delta F}{\delta\Pi_i(\vec{x})} \frac{\delta G}{\delta\Phi^i(\vec{x})} \right), \quad (2.6)$$

where the sum i runs over all fields Φ^i in a given theory and their momenta Π_i .

Of course, to see any interesting physics one must consider an interacting field theory, such as that introduced in the following section.

2.2. Super Yang-Mills Theory

Super Yang-Mills theory (SYM) is, as the name suggests, the supersymmetric generalisation of pure Yang-Mills theory. As such, the four-dimensional $\mathcal{N} = 1$ theory contains a massless gauge boson A_μ^A , where the index $A = 1, \dots, \dim[G]$ labels the adjoint representation of some simple, compact, non-Abelian gauge group G . Its ‘superpartner’ is a massless Majorana spinor λ^A sometimes called a ‘gaugino’. The action is

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}^A F^{A\mu\nu} - \frac{i}{2}\bar{\lambda}^A \gamma^\mu D_\mu \lambda^A \right), \quad (2.7)$$

with the covariant derivative and field strength given by the usual expressions

$$\begin{aligned} D_\mu \lambda^A &= \partial_\mu \lambda^A + gf_{BC}^A A_\mu^B \lambda^C, \\ F_{\mu\nu}^A &= \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + gf_{BC}^A A_\mu^B A_\nu^C. \end{aligned} \quad (2.8)$$

Again up to a total derivative, the Lagrangian is invariant under the supersymmetry transformations

$$\delta A_\mu^A = -\frac{i}{2}\bar{\epsilon}\gamma_\mu\lambda^A, \quad \delta\lambda^A = \frac{1}{4}F_{\mu\nu}^A\gamma^{\mu\nu}\epsilon, \quad (2.9)$$

where in general the matrix $\gamma^{\mu_1\mu_2\dots\mu_r}$ is defined as the antisymmetrised product of r gamma matrices

$$\gamma^{\mu_1\mu_2\dots\mu_r} := \gamma^{[\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_r]} \quad (2.10)$$

and is called the rank- r Clifford algebra element. A very important term of the form $f_{BC}{}^A (\bar{\lambda}^B \gamma_\nu \lambda^C) (\bar{\epsilon} \gamma^\nu \lambda^A)$ arises in the variation of the Lagrangian, which vanishes by virtue of a special spinor identity [15] that holds *only* in dimensions $D = 3, 4, 6, 10$. A proof of this will be given in Chapter 4.

The fact that a super Yang-Mills theory with supermultiplet $\{A_\mu, \lambda\}$ exists only in these particular dimensions can be understood by counting the on-shell degrees of freedom of the fields; it is a basic requirement for a non-trivial supersymmetric theory that the numbers of bosonic and fermionic on-shell degrees of freedom are equal – see Section 2.4. In D dimensions a vector field carries $D - 2$ on-shell degrees of freedom, which matches that of a single spinor only when $D - 2 = 1, 2, 4, 8$, since the number of degrees of freedom for a spinor is always a power of 2.

This is a first hint that the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ might be related to supersymmetry; the number of bosonic (fermionic) on-shell degrees of freedom in SYM can be either 1, 2, 4 or 8 – precisely the dimensions of $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} . This is indeed no coincidence, as will be demonstrated in Chapter 4.

2.3. Supersymmetry Algebras

It is instructive to consider the commutator of two successive supersymmetry transformations with independent parameters ϵ_1 and ϵ_2 . In the case of the SYM example above, a straightforward calculation using the supersymmetry transformations gives

$$\begin{aligned} \delta_1(\delta_2 A_\mu^A) - \delta_2(\delta_1 A_\mu^A) &= -\frac{i}{2}(\bar{\epsilon}_1 \gamma^\nu \epsilon_2) F_{\nu\mu}^A, \\ \delta_1(\delta_2 \lambda^A) - \delta_2(\delta_1 \lambda^A) &= -\frac{i}{2}(\bar{\epsilon}_1 \gamma^\mu \epsilon_2) D_\mu \lambda^A + \left(\frac{i}{2}(\bar{\epsilon}_2 \gamma^\mu D_\mu \lambda^A) \epsilon_1 - (1 \leftrightarrow 2)\right), \end{aligned} \quad (2.11)$$

which, on substitution of the fermionic equation of motion $\gamma^\mu D_\mu \lambda^A = 0$, becomes simply:

$$\begin{aligned} [\delta_1, \delta_2] A_\mu^A &= a^\nu F_{\nu\mu}^A, \\ [\delta_1, \delta_2] \lambda^A &= a^\mu D_\mu \lambda^A, \quad \text{with } a^\mu := -\frac{i}{2} \bar{\epsilon}_1 \gamma^\mu \epsilon_2. \end{aligned} \quad (2.12)$$

The commutator of the two supersymmetry transformations is thus a *translation* – in this case a gauge-covariant one. That a supersymmetry transformation is the ‘square root of a translation’ demonstrates the crucial point that supersymmetry is to be regarded as a spacetime symmetry rather than an internal one. This can also

be seen from the fact is that the associated conserved charge is a spinor, and hence transforms non-trivially under the Lorentz group, meaning supersymmetry does not commute with Lorentz transformations.

Using equation (2.5) one can write the transformations in terms of Poisson brackets with the supercharge Q :

$$\begin{aligned} [\delta_1, \delta_2]A_\mu^A &= [\bar{\epsilon}_1 Q, [\bar{\epsilon}_2 Q, A_\mu^A]_{\text{PB}}]_{\text{PB}} - (1 \leftrightarrow 2) \\ &= [[\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q]_{\text{PB}}, A_\mu^A]_{\text{PB}}. \end{aligned} \quad (2.13)$$

However, this gives a translation by a^μ , which can be written as

$$[[\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q]_{\text{PB}}, A_\mu^A]_{\text{PB}} = -\frac{i}{2}\bar{\epsilon}_1 \gamma^\nu \epsilon_2 [P_\nu, A_\mu^A]_{\text{PB}}, \quad (2.14)$$

where the momentum P_μ is the conserved charge associated with translational symmetry, obtained from the energy-momentum tensor $T_{\mu\nu}$:

$$P_\mu = \int d^3x T_{\mu 0}. \quad (2.15)$$

In fact, equation (2.14) holds not just for A_μ^A , but for any field in a supersymmetric theory; thus in general

$$[\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q]_{\text{PB}} = -\frac{i}{2}\bar{\epsilon}_1 \gamma^\nu \epsilon_2 P_\nu. \quad (2.16)$$

Then, since ϵ_1 and ϵ_2 are arbitrary, one can strip them from this equation (making use of the anti-commutativity of fermionic quantities) and conclude that:

$$\{Q_\alpha, \bar{Q}^\beta\}_{\text{PB}} = -\frac{i}{2}(\gamma^\mu)_{\alpha}{}^{\beta} P_\mu, \quad (2.17)$$

where α, β are spinor indices and curly braces denote the Poisson bracket of two fermionic quantities. Equation (2.17) expresses the relationship between supersymmetry and translations – the former as the ‘square root’ of the latter – at the level of classical conserved charges. Similarly one can show that

$$\{Q_\alpha, Q_\beta\}_{\text{PB}} = 0, \quad [Q_\alpha, P_\mu]_{\text{PB}} = 0, \quad (2.18)$$

and since Q_α transforms as a spinor under Lorentz transformations, its Poisson bracket with the associated conserved charge $J_{\mu\nu}$, the angular momentum tensor, is given by

$$[Q_\alpha, J_{\mu\nu}]_{\text{PB}} = -\frac{i}{2}(\gamma_{\mu\nu})_{\alpha}{}^{\beta} Q_\beta. \quad (2.19)$$

Along with the usual brackets between P_μ and $J_{\mu\nu}$, equations (2.17), (2.18) and

(2.19) make up the classical $\mathcal{N} = 1$ super-Poincaré algebra in terms of conserved charges. When the theory is quantised, the fields – and hence also the conserved charges – become operators acting on a Fock space. The charge operators act as generators for the symmetries to which they each correspond via Noether’s theorem, and their Poisson brackets usually become algebraic commutators: $[\cdot, \cdot]_{\text{PB}} \rightarrow -i[\cdot, \cdot]$. However, because the supersymmetry generators Q_α are anti-commuting spinor quantities, the $[Q, Q]$ Poisson brackets become *anti-commutators* in the quantum theory:

$$\{Q_\alpha, \bar{Q}^\beta\} = \frac{1}{2}(\gamma^\mu)_\alpha{}^\beta P_\mu, \quad \{Q_\alpha, Q_\beta\} = 0. \quad (2.20)$$

For theories with extended supersymmetry, meaning $\mathcal{N} > 1$, there are \mathcal{N} supercharges Q_α^I with $I = 1, \dots, \mathcal{N}$. These obey the algebra

$$\{Q_\alpha^I, \bar{Q}_J^\beta\} = \frac{1}{2}(\gamma^\mu)_\alpha{}^\beta P_\mu \delta_J^I, \quad (2.21)$$

while for the $\{Q_\alpha^I, Q_\beta^J\}$ bracket the index structure suggests the new possibility of including antisymmetric ‘central charges’ $Z^{IJ} = -Z^{JI}$ appearing on the right-hand side:

$$\{Q_\alpha^I, Q_\beta^J\} = C_{\alpha\beta} Z^{IJ}, \quad (2.22)$$

where $C_{\alpha\beta}$ is the charge conjugation matrix, which raises and lowers Majorana spinor indices. The central charges are ‘central’ in the group-theoretic sense in that they commute with all the generators of the super-Poincaré group¹.

In general there is a group R of linear transformations that preserve the superalgebra – its automorphisms. This automorphism group (excluding the Poincaré group itself) is called the R-symmetry. Its transformations simply act like rotations on the set of supercharges and commute with the Poincaré generators P_μ and $J_{\mu\nu}$. To see the R-symmetry in $D = 4$ it is useful to choose the Weyl representation representation for the gamma matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad (2.23)$$

where σ^μ are the set of Pauli matrices σ^i , $i = 1, 2, 3$, with the identity included for the zero-component:

$$\begin{aligned} \bar{\sigma}_\mu &= \sigma^\mu = (+\mathbf{1}, \sigma^i), \\ \sigma_\mu &= \bar{\sigma}^\mu = (-\mathbf{1}, \sigma^i). \end{aligned} \quad (2.24)$$

¹In general the *center* of a group G is the set of elements $Z(G)$ that commute with every element of the group: $Z(G) = \{z \in G \mid \forall g \in G, zg = gz\}$.

Then one finds that since the supersymmetry generator Q_α^I is a Majorana spinor it has the form

$$Q_\alpha^I = \left(Q_1^I, Q_2^I, Q_{I2}^\dagger, -Q_{I1}^\dagger \right). \quad (2.25)$$

Thus, since the second two components are Hermitian conjugates of the first two (still considering the supercharges as quantum operators), any complex linear transformation of the generators will need to transform the second two components in a conjugate representation to that of the first two. Then for any Q^I satisfying the (anti-)commutators (2.21) and (2.22) above, the algebra is invariant under

$$\left(Q_1^I, Q_2^I \right) \rightarrow U^I{}_J \left(Q_1^J, Q_2^J \right), \quad (2.26)$$

where U is a unitary matrix $UU^\dagger = U^\dagger U = \mathbf{1}$. Thus the algebra has a chiral $U(\mathcal{N})$ R-symmetry when $D = 4$.

In the case of $\mathcal{N} = 1$ supersymmetry in $D = 4$, although there is only one Q there is still a $U(1)$ R-symmetry. Here equation (2.26) can be written in terms of the full Majorana spinor Q

$$Q \rightarrow e^{i\theta\gamma_*} Q, \quad \theta \in \mathbb{R}, \quad (2.27)$$

where γ_* (sometimes called γ_5) is

$$\gamma_* = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \quad (2.28)$$

This $U(1)$ symmetry is realised in the $\mathcal{N} = 1$ super Yang-Mills theory by the invariance of the Lagrangian (2.7) and transformation rules (2.9) under

$$\lambda^A \rightarrow e^{i\theta\gamma_*} \lambda^A. \quad (2.29)$$

Note that θ is a constant so this is a global symmetry. In most supersymmetric theories in this thesis the R-symmetry will appear as a global symmetry of the Lagrangian.

2.4. Supermultiplets

A natural question to ask when constructing theories is what kind of supermultiplets exist in four spacetime dimensions when there are \mathcal{N} supersymmetries. Only massless multiplets will appear in this thesis. The simplest way to derive the possible supermultiplets is to consider quantum particle states transforming under the supersymmetry generators. These states must form a representation of the su-

persymmetry algebra, i.e. transform into one another under the action of the Q^I operators, so the goal is to determine a basis for such a set of states transforming irreducibly under supersymmetry.

Since the supercharges Q^I commute with translations P_μ , supersymmetry transformations must preserve the momentum of on-shell states. As a result one may simply set $p^\mu = (|E|, 0, 0, E)$ for any state for the remainder of this discussion without loss of generality. Then, since all particle states are assumed to have momentum p^μ of this form, a one-particle state $|h\rangle$ is entirely specified by its helicity $h = \pm s$, where s is the particle's spin. Helicity can be defined as the eigenvalue under the operator $J^3 = J_{12}$, which generates the little group $\text{SO}(2)$ – the subgroup of $\text{SO}(1, 3)$ that preserves the form of p^μ above².

Using the Weyl basis (2.23) and the corresponding form of Q^I in (2.25), the $\{Q_\alpha^I, \bar{Q}_J^\beta\}$ anti-commutator becomes

$$\{Q_1^I, Q_{J1}^\dagger\} = 0, \quad \{Q_2^I, Q_{J2}^\dagger\} = \delta_J^I |E|, \quad (2.30)$$

for the components Q_1^I and Q_2^I . The Q_1^I operator produces unphysical states of zero norm, since for any state $|\psi\rangle$,

$$\langle\psi|\{Q_1^I, Q_{I1}^\dagger\}|\psi\rangle = 2|Q_{I1}^\dagger|\psi\rangle|^2 = 0, \quad (\text{no sum on } I). \quad (2.31)$$

Hence Q_1^I and Q_{I1}^\dagger can be ignored. As for the Q_2^I operator, it is useful to define a normalised version $\alpha^I := |E|^{-1/2} Q_2^I$ so that

$$\{\alpha^I, \alpha_J^\dagger\} = \delta_J^I, \quad \{\alpha^I, \alpha^J\} = \{\alpha_I^\dagger, \alpha_J^\dagger\} = 0. \quad (2.32)$$

Then, using the super-Poincaré algebra it is easy to show that

$$[J^3, \alpha_I^\dagger] = -\frac{1}{2}\alpha_I^\dagger, \quad [J^3, \alpha^I] = \frac{1}{2}\alpha^I, \quad (2.33)$$

i.e. one may view α^I as having helicity $1/2$ and α_I^\dagger as having helicity $-1/2$. Thus each supersymmetry generator Q_α^I corresponds to an anti-commuting lowering (raising) operator α_I^\dagger (α^I), which lowers (raises) the helicity of a state by $1/2$. Define the ‘vacuum’ state $|h_0\rangle$, with h_0 any positive or negative integer or half-integer, to be such that

$$\alpha^I|h_0\rangle = 0, \quad J^3|h_0\rangle = h_0|h_0\rangle, \quad \forall I = 1, \dots, \mathcal{N}. \quad (2.34)$$

²Note that the little group in D dimensions is actually $\text{ISO}(D-2)$, but the translation generators are neglected since they annihilate physical states, leaving only $\text{SO}(D-2)$ with a non-trivial action.

The representation will then have a basis built up by acting with successive α_I^\dagger operators on $|h_0\rangle$:

$$|h_0\rangle, \quad \alpha_I^\dagger|h_0\rangle, \quad \alpha_I^\dagger\alpha_J^\dagger|h_0\rangle, \quad \alpha_I^\dagger\alpha_J^\dagger\alpha_K^\dagger|h_0\rangle, \quad \dots, \quad \alpha_{I_1}^\dagger\alpha_{I_2}^\dagger\cdots\alpha_{I_N}^\dagger|h_0\rangle. \quad (2.35)$$

A state with m lowering operators has helicity $h_0 - \frac{1}{2}m$. The lowering operators anti-commute with one another, so each level comes with its I, J, K, \dots indices antisymmetrised. Thus the multiplicity of states at level m is the binomial coefficient $\binom{\mathcal{N}}{m}$, and there are $2^{\mathcal{N}}$ states in total. Furthermore, since the α_I^\dagger operators transform under R-symmetry $U(\mathcal{N}) \cong SU(\mathcal{N}) \times U(1)$ as

$$\alpha_I^\dagger \rightarrow e^{i\theta} U_I^J \alpha_J^\dagger, \quad U_I^J \in SU(\mathcal{N}), \quad \theta \in \mathbb{R}, \quad (2.36)$$

the states at level m transform as the m -index antisymmetric tensor representation of $SU(\mathcal{N})$ with charge m under the $U(1)$ factor:

$$\alpha_{I_1}^\dagger \alpha_{I_2}^\dagger \cdots \alpha_{I_m}^\dagger |h_0\rangle \rightarrow e^{im\theta} U_{I_1}^{J_1} U_{I_2}^{J_2} \cdots U_{I_m}^{J_m} \alpha_{J_1}^\dagger \alpha_{J_2}^\dagger \cdots \alpha_{J_m}^\dagger |h_0\rangle \quad (2.37)$$

For every supermultiplet the total number of bosonic and fermionic states must be equal; if h_0 is an integer then states with odd numbers of α_I^\dagger operators have half-integer helicity and there are $2^{\mathcal{N}-1}$ of these in total, while states with even numbers of α_I^\dagger operators have integer helicity and again there are $2^{\mathcal{N}-1}$ of these. The argument is the same with (odd and even reversed) when h_0 is a half-integer.

As an example, consider $h_0 = 1$ with $\mathcal{N} = 2$ – so $I = 1, 2$ and the R-symmetry group is $U(2)$. There is one state $|1\rangle$ with helicity 1, two states $|1/2\rangle_I = \alpha_I^\dagger|1\rangle$ with helicity 1/2 and one state $|0\rangle_{IJ} = \alpha_I^\dagger\alpha_J^\dagger|1\rangle$ with helicity 0 (note that the \mathcal{N} -index antisymmetric tensor is a singlet under $SU(\mathcal{N})$ since it is proportional to the invariant epsilon tensor $\varepsilon_{I_1 I_2 \dots I_N}$). Group theoretically these transform as the representations $\mathbf{1}_{1,0}$, $\mathbf{2}_{\frac{1}{2},1}$ and $\mathbf{1}_{0,2}$, where the first subscript indicates the helicity and the second is the charge under the $U(1)$ factor of the R-symmetry $U(2) \cong SU(2) \times U(1)$.

There is slight a subtlety in the above example: since any physical space of states must include *both* helicities $h = \pm s$, one must add to the irreducible multiplet above its CPT-conjugate, giving a *reducible* representation of the supersymmetry algebra. The CPT-conjugate multiplet is obtained by starting with $| -h_0\rangle$ as the vacuum and acting successively with α^I operators, raising the helicity of the states. Thus the CPT-conjugate representations for $h_0 = 1$ and $\mathcal{N} = 2$ are $\mathbf{1}_{-1,0}$, $\mathbf{2}_{-\frac{1}{2},-1}$ and $\mathbf{1}_{0,-2}$. In total this means that the corresponding theory's field content should be a spin-

1 vector A_μ , two spin-1/2 fermions λ_I and a complex scalar field ϕ , transforming on-shell as the representations

$$A_\mu : \mathbf{1}_{1,0} + \mathbf{1}_{-1,0}, \quad \lambda_I : \mathbf{2}_{\frac{1}{2},1} + \mathbf{2}_{-\frac{1}{2},-1}, \quad \phi : \mathbf{1}_{0,2}, \quad \phi^* : \mathbf{1}_{0,-2} \quad (2.38)$$

of $U(1)_{\text{ST}} \times U(2)$, where $U(1)_{\text{ST}} \cong \text{SO}(2)$ is the spacetime little group. This is the field content of $\mathcal{N} = 2$ super Yang-Mills theory, which will appear several times in the coming chapters.

For typical supermultiplets the CPT-conjugate must be added as above. However, in the special cases when $\mathcal{N} = 4|h_0|$, the initial multiplet in (2.35) is already self-conjugate. For example, consider the special case of $\mathcal{N} = 4$ and $h_0 = 1$ with multiplet content

$$A_\mu : \mathbf{1}_{1,0} + \mathbf{1}_{-1,4}, \quad \lambda_I : \mathbf{4}_{\frac{1}{2},1} + \bar{\mathbf{4}}_{-\frac{1}{2},3}, \quad \phi_{[IJ]} : \mathbf{6}_{0,2}. \quad (2.39)$$

These are the fields of $\mathcal{N} = 4$ super Yang-Mills. Note that it is the *maximal* theory containing spins $s = |h|$ no greater than 1; starting from $h_0 = 1$, if \mathcal{N} were any greater then the helicities would be lowered beyond -1 . Again there is a subtlety, which is that the $U(1)$ R-symmetry charges do not match up for the positive and negative helicity states. This means that this group is not compatible with the spacetime little group $U(1)_{\text{ST}}$ and hence cannot be a physical symmetry. The charges on the right may simply be discarded. In general then for multiplets with $\mathcal{N} = 4|h_0|$, the physical R-symmetry group is $SU(\mathcal{N})$ rather than $U(\mathcal{N})$.

The other particular case of interest is $\mathcal{N} = 8$, $h_0 = 2$ with R-symmetry $SU(8)$. Here the multiplet content is

$$\begin{aligned} g_{\mu\nu} &: \mathbf{1}_2 + \mathbf{1}_{-2}, \\ \Psi_{\mu I} &: \mathbf{8}_{\frac{3}{2}} + \bar{\mathbf{8}}_{-\frac{3}{2}}, \\ A_{\mu[IJ]} &: \mathbf{28}_1 + \bar{\mathbf{28}}_{-1}, \\ \lambda_{[JK]} &: \mathbf{56}_{\frac{1}{2}} + \bar{\mathbf{56}}_{-\frac{1}{2}}, \\ \phi_{[JKLM]} &: \mathbf{70}_0. \end{aligned} \quad (2.40)$$

Since the highest spin in this multiplet is 2, it contains a graviton and hence corresponds to a *supergravity* theory. Also note the presence of eight spin- $\frac{3}{2}$ fields. These represent fermionic vector-spinors called ‘gravitinos’ (or sometimes ‘gravitini’ is used for the plural), which will be discussed in the following section. Any multiplet with $h_0 = 2$ must contain \mathcal{N} gravitino fields transforming in the fundamental rep of the R-symmetry $U(\mathcal{N})$ (or $SU(\mathcal{N})$), since they correspond to the states $|3/2\rangle_I = \alpha_I^\dagger|2\rangle$.

By the same argument used for $\mathcal{N} = 4$ super Yang-Mills above, the $\mathcal{N} = 8$ multiplet is the largest containing spins $s = |h|$ no greater than 2. Because consistent interacting theories with spins greater than 2 are very difficult – if not impossible – to write down, the $\mathcal{N} = 8$ theory is considered the *maximally supersymmetric* field theory.

Before moving onto supergravity proper there is a little terminology to define. A supermultiplet is often referred to by the name of its field with the highest spin. A *gravity multiplet* has highest spin 2; a *vector multiplet* has highest spin 1; a *chiral multiplet* (or Wess-Zumino multiplet) has highest spin 1/2 (this particular name is reserved only for $\mathcal{N} = 1$); a *hyper-multiplet* has highest spin 1/2 (this time for $\mathcal{N} = 2$ only). Analogous multiplets also appear in spacetime dimensions $D \neq 4$ – see [43] for the original and complete survey.

2.5. Local Supersymmetry: Supergravity

In the field theories discussed above, the supersymmetry parameter ϵ is a constant spinor; the symmetry is global. However, since any realistic field theory must ultimately contain gravity, the problem arises that on a non-trivial spacetime manifold a ‘constant’ spinor is no longer well-defined in general³. This is easy to see, since if $\epsilon(x)$ is constant in the whole of some coordinate patch \mathcal{U}_1 with coordinates x , then on the overlap with another coordinate patch \mathcal{U}_2 with coordinates x' , in general, $\epsilon(x')$ will not be constant. Therefore, if gravity is to be included in a supersymmetric field theory one must replace $\epsilon \rightarrow \epsilon(x)$, making supersymmetry *local*.

Conversely, by the following argument, local supersymmetry requires gravity. The commutator of two supersymmetry transformations with parameters ϵ_1 and ϵ_2 yields a spacetime translation by $a^\mu = -\frac{i}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2$. Thus letting ϵ_1 and ϵ_2 depend on the spacetime coordinates x results in a *local translation*, i.e. an infinitesimal *general coordinate transformation*⁴. Therefore local supersymmetry implies general coordinate invariance and hence requires gravity. Supergravity is this marriage of supersymmetry and gravity.

A local symmetry needs a gauge field, which acts like a connection, allowing one to define covariant derivatives. In the case of local supersymmetry this gauge field is the fermionic vector-spinor *gravitino* Ψ_μ . Also known as a Rarita-Schwinger field,

³The closest thing to a ‘constant spinor’ is a *covariantly constant* spinor, but this will still in general depend on the manifold’s coordinates.

⁴For any tensor field T , the change due to an infinitesimal local translation along a vector field ξ is a Lie derivative $\delta T = \mathcal{L}_\xi T$, but this is just the change under an infinitesimal diffeomorphism. Therefore infinitesimal local translations and diffeomorphisms have the same effect.

in four dimensions the gravitino is a spin- $\frac{3}{2}$ field and is sometimes referred to as the ‘superpartner’ of the spin-2 graviton. As demonstrated in the previous section, any supersymmetric multiplet that contains the graviton – and has spins ≤ 2 – must also contain at least one gravitino. More precisely, a theory with extended local supersymmetry $\mathcal{N} > 1$ must have \mathcal{N} gravitino fields – one to gauge each of the \mathcal{N} independent supersymmetries.

The action for a free massless Rarita-Schwinger field in D -dimensional Minkowski spacetime is [42]

$$S = - \int d^D x \, i \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho, \quad (2.41)$$

where $\gamma^{\mu\nu\rho}$ is the rank-3 Clifford algebra element

$$\gamma^{\mu\nu\rho} = \gamma^{[\mu} \gamma^\nu \gamma^{\rho]}. \quad (2.42)$$

Note that (2.41) is the most obvious action to write down that is Lorentz-invariant, quadratic in Ψ_μ , Hermitian and first order in derivatives; this is everything one might reasonably expect from the action of a relativistic fermion, given the features of the Dirac action for an ordinary massless spin- $\frac{1}{2}$ field. In this thesis all Rarita-Schwinger fields will be massless, but in general a mass term $-m \bar{\Psi}_\mu \gamma^{\mu\nu} \Psi_\nu$ may also be added.

Just as the Maxwell action has a gauge symmetry under the replacement $A_\mu \rightarrow A_\mu + \partial_\mu \theta$, the Rarita-Schwinger action (2.41) is invariant up to a total derivative under

$$\Psi_\mu \rightarrow \Psi_\mu + \partial_\mu \epsilon, \quad (2.43)$$

where $\epsilon(x)$ is an arbitrary spinor parameter. This gauge symmetry makes the gravitino a suitable gauge field for local supersymmetry.

For a supergravity theory the gravitino and any other fermions must be described in curved spacetime. In general in a gravitational theory, since a spinor is defined to transform in the double cover of the Lorentz group $\text{Spin}(1, D-1)$, it must be described by its components in a local Lorentz frame. This requires the vielbein field $e_\mu^a(x)$, which relates the curved-space metric $g_{\mu\nu}(x)$ to the ordinary Minkowski metric $\eta_{ab} = \text{diag}(-1, 1, \dots, 1)$:

$$g_{\mu\nu}(x) =: e_\mu^a(x) e_\nu^b(x) \eta_{ab}. \quad (2.44)$$

This equation is invariant under local Lorentz transformations $e_\mu^a(x) \rightarrow \Lambda^a_b(x) e_\mu^b(x)$. Thus the indices a, b are referred to as local Lorentz indices and are raised and lowered using η^{ab} and η_{ab} , respectively.

In the vielbein formalism for general relativity, e_μ^a becomes the central dynamical field in place of $g_{\mu\nu}$ and carries the same information. As a 4×4 matrix e_μ^a has *a priori* D^2 components in D dimensions, but since it is only defined up to a local Lorentz transformation one must subtract $\dim[\text{O}(1, D-1)] = \frac{1}{2}D(D-1)$ components, leaving a total of $\frac{1}{2}D(D+1)$ – precisely the same number as the usual symmetric matrix $g_{\mu\nu}$.

The analogue of the affine connection $\Gamma_{\nu\rho}^\mu$ in this language is the spin connection ω_μ^{ab} , which is defined to be the gauge field associated with local Lorentz transformations, and thus transforms as

$$\omega_\mu^a{}_b \rightarrow \Lambda^{-1a}{}_c \partial_\mu \Lambda^c{}_b + \Lambda^{-1a}{}_c \omega_\mu^c{}_d \Lambda^d{}_b, \quad (2.45)$$

i.e. as a Yang-Mills field with gauge group $\text{O}(1, D-1)$. The covariant derivatives of Lorentz vectors and tensors may then be defined in the usual way; for example,

$$D_\mu V^a := \partial_\mu V^a + \omega_\mu^a{}_b V^b. \quad (2.46)$$

For a metric-compatible connection, the spin connection is related to the connection coefficients $\Gamma_{\nu\rho}^\mu$ by

$$\omega_\mu^a{}_b = e_\nu^a (\partial_\mu e_b^\nu + \Gamma_{\mu\rho}^\nu e_b^\rho) = e_\nu^a \nabla_\mu e_b^\nu, \quad (2.47)$$

where e_a^μ is the inverse of the vielbein.

Since the generators of Lorentz transformations for spinors are $\frac{1}{2}\gamma_{ab}$, the Lorentz-covariant derivative for a spinor field $\psi(x)$ is defined as

$$D_\mu \psi := \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \right) \psi. \quad (2.48)$$

With the above definitions in place, it is simple to write down the action for $D = 4$, $\mathcal{N} = 1$ supergravity with the single gravity multiplet $\{g_{\mu\nu}, \Psi_\mu\}$ or $\{e_\mu^a, \Psi_\mu\}$:

$$S = \int d^4x \sqrt{-g} \left(R - i \bar{\Psi}_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho \right), \quad (2.49)$$

where R is the Ricci scalar. The supersymmetry transformations are

$$\delta e_\mu^a = \frac{i}{2} \bar{\epsilon} \gamma^a \Psi_\mu, \quad \delta \Psi_\mu = D_\mu \epsilon. \quad (2.50)$$

Note that the action (2.49) assumes a first order formalism in which e_μ^a and ω_μ^{ab} are treated as separate independent variables, with $R = R(\omega)$ treated as a function of ω rather than e . To obtain a second order action one must vary first with respect to

the spin connection to obtain its equation of motion, which may then be substituted back into (2.49).

2.6. Eleven-Dimensional Supergravity

A spinor of the Clifford algebra in D dimensions has $2^{\lfloor D/2 \rfloor}$ components, where $\lfloor D/2 \rfloor$ is defined to be the integer part of $D/2$, i.e. equal to $D/2$ when D is even and $(D-1)/2$ when D is odd. In Section 2.4 it was demonstrated that with $D = 4$ the maximal supergravity has $\mathcal{N} = 8$. Since a Majorana spinor in $D = 4$ has 4 components this means that the 8 supercharges of the $\mathcal{N} = 8$ theory make up 32 components in total. This matches the dimension of a single spinor supercharge in $D = 11$, which means that $D = 11$, $\mathcal{N} = 1$ supergravity must have the maximum dimension possible for a supersymmetric theory. This is equivalent to the requirement that there are no spins greater than 2 in the supersymmetry multiplet.

Having established that $D = 11$ is the maximum dimension for supersymmetry with a unique $\mathcal{N} = 1$ theory, it is natural to ask what this theory's field content and Lagrangian look like. As an $\mathcal{N} = 1$ supergravity theory it must contain a massless graviton $g_{\mu\nu}$ and a gravitino Ψ_μ . The little group $\text{SO}(D-2)$ for $D = 11$ is $\text{SO}(9)$ and the graviton's on-shell degrees of freedom must transform as the symmetric traceless representation **44**, while the gravitino's transform as the vector-spinor **128**. This means there is mismatch of 84 degrees of freedom between the bosonic and fermionic sectors. Fortunately there is a representation of precisely this dimension, the 3-form $C_{\mu\nu\rho}$, whose on-shell degrees of freedom transform as the irreducible **84** representation of $\text{SO}(9)$. Thus one expects the supermultiplet in eleven dimensions to have field content

$$\{g_{\mu\nu}, C_{\mu\nu\rho}, \Psi_\mu\}. \quad (2.51)$$

Cremmer and Julia derived the Lagrangian of the corresponding theory. Their method was simply to write down the kinetic terms

$$\mathcal{L} = \sqrt{-g} \left(R - i\bar{\Psi}_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho - \frac{1}{24} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \right), \quad (2.52)$$

where $F_{\mu\nu\rho\sigma} = 4\partial_{[\mu} C_{\nu\rho\sigma]}$ is the field strength of the 3-form field. Then, a sensible ansatz for the supersymmetry transformations is

$$\begin{aligned} \delta g_{\mu\nu} &= c_1 (i\bar{\epsilon} \gamma_{(\mu} \Psi_{\nu)}), \\ \delta C_{\mu\nu\rho} &= c_2 (i\bar{\epsilon} \gamma_{[\mu\nu} \Psi_{\rho]}), \\ \delta \Psi_\mu &= c_3 D_\mu \epsilon + \Gamma^{\nu\rho\sigma\tau}{}_\mu F_{\nu\rho\sigma\tau} \epsilon, \end{aligned} \quad (2.53)$$

where c_1, c_2, c_3 are constants to be fixed and $\Gamma^{\nu\rho\sigma\tau}{}_{\mu}$ is a linear combination of products of gamma matrices to be determined. Varying the Lagrangian with respect to these transformations one finds that various corrections and interactions must be added and eventually, after many calculations, the full supersymmetric theory is obtained. The interaction terms are not given here but will be presented in an octonionic formalism in Chapter 7.

As a theory with the *maximal* number of dimensions for *maximal* supersymmetry, eleven-dimensional supergravity is certainly singled out as special. It should also be the low-energy limit of M-theory, the speculated eleven-dimensional master theory that contains in various limits the five ten-dimensional string theories. By the process of dimensional reduction introduced in the following section, the $D = 11$ supergravity also leads to maximal supergravity theories for all $D < 11$ – including the $\mathcal{N} = 8$ theory in $D = 4$ – as well as truncations to a myriad of non-maximal supergravities.

2.7. Kaluza-Klein Theory

String theory predicts that the world has ten spacetime dimensions, while its conjectured parent M-theory goes precisely one step further, predicting eleven. One way to reconcile this with the apparent four-dimensional universe is to use Kaluza-Klein theory. In general this is a technique for recovering a low-energy effective D -dimensional theory from a higher-dimensional $(D + d)$ -dimensional theory. The most remarkable aspect of the idea is its scope for unification; gauge symmetries in D dimensions are interpreted simply as gravitational symmetries in $D + d$ dimensions. The key is to assume that the $(D + d)$ -dimensional spacetime \mathcal{M}_{D+d} locally takes the form of a product manifold

$$\mathcal{M}_{D+d} \sim \mathcal{M}_D \times \mathcal{M}_d, \quad (2.54)$$

where \mathcal{M}_d is a compact d -manifold whose size (with respect to the metric on \mathcal{M}_{D+d}) is very small compared to the energy scales of interest, such as those probed by current particle accelerators. The resulting effective theory appears to have spacetime \mathcal{M}_D , with only D observable dimensions.

As a simple example, consider the theory of a massless real scalar field Φ in $D + 1$ flat dimensions satisfying the Klein-Gordon equation

$$\square_{(D+1)}\Phi = 0, \quad (2.55)$$

where $\square_{(D+1)} = \partial_M \partial^M$ with $M = 0, 1, \dots, D$. Suppose that the spacetime manifold is a product of D -dimensional Minkowski space $\mathbb{R}^{1,D-1}$ and a circle S^1 ,

$$\mathcal{M}_{D+1} = \mathbb{R}^{1,D-1} \times S^1. \quad (2.56)$$

One may parameterise such that the first D coordinates x^μ , $\mu = 0, \dots, (D-1)$, refer to $\mathbb{R}^{1,D-1}$ while the $(D+1)$ th coordinate $x^D =: z$ is the coordinate on S^1 . This means that z is periodic,

$$z \sim z + 2\pi nR, \quad n \in \mathbb{Z}, \quad R = \text{constant}, \quad (2.57)$$

where R is the radius of the circle. One may Fourier expand Φ in the circular dimension, so that

$$\Phi(x^\mu, z) = \sum_{n \in \mathbb{Z}} \Phi_n(x^\mu) e^{inz/R} \quad (2.58)$$

for Fourier modes $\Phi_n(x^\mu)$, which depend only on the D extended dimensions. Substituting this into the Klein-Gordon equation (2.55) gives

$$\left(\square_{(D)} - \frac{n^2}{R} \right) \Phi_n(x^\mu) = 0, \quad (2.59)$$

where $\square_{(D)} = \partial_\mu \partial^\mu$. This is the equation of motion for a scalar field of mass $|n|/R$; in the D -dimensional theory the massless $(D+1)$ -dimensional field $\Phi(x^M)$ appears as an infinite tower of massive scalars $\Phi_n(x^\mu)$. Since the extra dimension is intended to be unobservable, R is taken to be very small compared to the physical scale of interest, which means that the mass $|n|/R$ is very large for $n \neq 0$. These modes are not excited at low energies. Thus in the low-energy physics only the massless zero mode $\Phi_0(x^\mu)$ is detectable and one may truncate all $\Phi_n(x^\mu)$ with $n \neq 0$ from the theory by setting them to zero. By equation (2.58) this is equivalent to setting $\partial_z \Phi = 0$, i.e. demanding that Φ is independent of the compact coordinate z . This is the typical procedure of Kaluza-Klein reduction: a compactification and a truncation to the massless sector⁵.

Since any free dynamical field satisfies the Klein-Gordon equation, the same argument must also apply to spinors, vectors, Rarita-Schwinger fields and tensor fields on any $(D+d)$ -dimensional background of the form $\mathbb{R}^{1,D-1} \times T^d$, where $T^d = S^1 \times S^1 \cdots \times S^1$ is the d -torus. The set of higher-dimensional coordinates

⁵A subtlety is that these truncations must be consistent. That is, one must check in the field equations that the massive modes are not sourced by the massless modes, as this would preclude setting the massive modes to zero. However, for Kaluza-Klein reduction on a circle the truncation is always guaranteed to be consistent.

$\{x^M\}$ may be split into $\{x^\mu, z^m\}$, where x^μ are coordinates on $\mathbb{R}^{1,D-1}$ and z^m are coordinates on T^d ,

$$\mu = 0, 1, \dots, (D-1), \quad m = 1, \dots, d, \quad (2.60)$$

and the low-energy physics depends only on field configurations that are independent of the z^m directions. As a result one may formally just write $\partial_m \simeq 0$ to obtain the D -dimensional theory. However, for fields carrying non-trivial Lorentz representations one must also decompose their components into irreducible representations of the lower-dimensional Lorentz group.

For example, a massless Maxwell 1-form field A_M becomes a D -dimensional 1-form A_μ and d scalar fields A_m . With $\partial_m A_M = 0$ the Lagrangian simply becomes

$$-\frac{1}{4}F_{MN}F^{MN} = -\frac{1}{4}(F_{\mu\nu}F^{\mu\nu} + 2F_{\mu m}F^{\mu m}) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu A_m \partial^\mu A^m, \quad (2.61)$$

as expected for the Lagrangian of d massless scalars and a Maxwell field in D dimensions. Note that the scalar fields A_m have an internal global $\text{SO}(d)$ symmetry under rotations $A^m \rightarrow O^m_n A^n$ with $O \in \text{SO}(d)$. This is just the residual symmetry from breaking the higher-dimensional Lorentz group $\text{SO}(1, D+d-1)$ into $\text{SO}(1, D-1)$:

$$\text{SO}(1, D-1) \times \text{SO}(d) \subset \text{SO}(1, D+d-1). \quad (2.62)$$

The same logic applies to any p -form field $A_{M_1 M_2 \dots M_p}$ in $D+d$ dimensions; in the D -dimensional theory this becomes a p -form $A_{\mu_1 \mu_2 \dots \mu_p}$, d distinct $(p-1)$ -forms $A_{\mu_1 \mu_2 \dots \mu_{(p-1)} m}$, $\binom{d}{2}$ distinct $(p-2)$ -forms $A_{\mu_1 \mu_2 \dots \mu_{(p-2)} mn}$, \dots and so on.

For spinor fields a complete and general treatment requires more detail on Clifford algebras than will be presented here. However, the key is that a spinor $\lambda_{\bar{\alpha}}$, where $\bar{\alpha}$ is a spinor index of $\text{SO}(1, D+d-1)$, may be written as $\lambda_{\alpha a}$, where α is a spinor index of $\text{SO}(1, D-1)$ and a is an *internal* spinor index of $\text{SO}(d)$.

For a Rarita-Schwinger field $\Psi_{M\bar{\alpha}}$ the vector index must also be decomposed into μ and m . A particularly relevant example is $D=4$ and $d=7$: the reduction of a gravitino in eleven dimensions down to $D=4$. In this case $\bar{\alpha} = 1, \dots, 32$. Then one may write $\bar{\alpha}$ as a composite index $\bar{\alpha} = \alpha a$, where $\alpha = 1, \dots, 4$ is the Majorana spinor index of $\text{SO}(1, 3)$ and $a = 1, \dots, 8$ (since $4 \times 8 = 32$). Thus the gravitino $\Psi_{M\bar{\alpha}} = \Psi_{M\alpha a}$ in $D+d=11$ becomes *eight* $D=4$ Majorana gravitinos $\Psi_{\mu\alpha a}$ (with $\mu = 0, 1, 2, 3$) and $7 \times 8 = 56$ Majorana fermions $\Psi_{m\alpha a}$ (with $m = 1, \dots, 7$).

A more involved but more interesting torus reduction is that of gravity. Consider pure Einstein gravity in $D+d$ dimensions with a spacetime that is locally a (warped) product $\mathcal{M}_D \times T^d$, where \mathcal{M}_D is a Lorentzian D -manifold. The full metric G_{MN}

decomposes into a D -dimensional metric $G_{\mu\nu}$, d vector fields $G_{\mu m} = G_{m\mu}$ and $\frac{1}{2}d^2(d+1)$ scalar fields packaged in a symmetric matrix $G_{mn} = G_{nm}$. As argued above, in the low-energy physics each of these fields is independent of the torus coordinates z^m .

The following particular choice of metric ansatz turns out to give a straightforward Lagrangian for the lower-dimensional fields:

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & G_{\mu n} \\ G_{m\nu} & G_{mn} \end{pmatrix} = \begin{pmatrix} e^{2\alpha\varphi} g_{\mu\nu} + e^{2\beta\varphi} \mathcal{A}_\mu^m \mathcal{A}_\nu^n \mathcal{M}_{mn} & e^{2\beta\varphi} \mathcal{M}_{mn} \mathcal{A}_\mu^m \\ e^{2\beta\varphi} \mathcal{M}_{mn} \mathcal{A}_\nu^n & e^{2\beta\varphi} \mathcal{M}_{mn} \end{pmatrix}, \quad (2.63)$$

where $\mathcal{M}_{mn} = \mathcal{M}_{nm}$ is taken to be in $\text{SL}(d, \mathbb{R})$; its determinant has been factorised to give the exponential factor $e^{2\beta\varphi}$, so that $(e^{2\beta\varphi})^d$ is the volume of the d -torus as a function on \mathcal{M}_D . After a suitable choice of the constants α and β in terms of D and d , the Einstein-Hilbert Lagrangian becomes

$$\sqrt{-G}R_G = \sqrt{-g} \left(R_g - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{4} \partial_\mu \mathcal{M}_{mn} \partial^\mu \mathcal{M}^{nm} - \frac{1}{4} e^{2c\varphi} \mathcal{F}_{\mu\nu}^m \mathcal{M}_{mn} \mathcal{F}^{n\mu\nu} \right), \quad (2.64)$$

where $\mathcal{F}_{\mu\nu}^m = \partial_\mu \mathcal{A}_\nu^m - \partial_\nu \mathcal{A}_\mu^m$, the matrix \mathcal{M}^{mn} is the inverse of \mathcal{M}_{mn} and c is a constant depending on D and d . This is precisely the Lagrangian one would hope for: the D -dimensional Einstein theory coupled to scalars and Maxwell-like vectors. Note however that the scalars couple non-canonically to the kinetic term of the vector fields. This is not a problem in general, but was actually what ruled out Kaluza and Klein's original theory as a unification of gravity and electromagnetism (the special case with $D = 4$ and $d = 1$).

The $(D+d)$ -dimensional gravity theory is of course invariant under general coordinate transformations (GCTs). For a small change in the coordinates

$$\delta x^M = -\xi^M(x), \quad (2.65)$$

the metric transforms via a Lie derivative

$$\delta G_{MN} = \xi^P \partial_P G_{MN} + G_{PN} \partial_M \xi^P + G_{MP} \partial_N \xi^P. \quad (2.66)$$

In general such a transformation does not preserve the metric ansatz (2.63) and so will not be a symmetry of the D -dimensional theory. However, if ξ is chosen to be independent of the z^m coordinates, $\partial_m \xi^M = 0$, then the metric ansatz (2.63) will be preserved. In this case, splitting equation (2.66) into its lower-dimensional

components according to (2.63) gives

$$\begin{aligned}
\delta g_{\mu\nu} &= \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho, \\
\delta \mathcal{A}_\mu^m &= \xi^\rho \partial_\rho \mathcal{A}_\mu^m + \mathcal{A}_\nu^m \partial_\mu \xi^\nu + \partial_\mu \xi^m, \\
\delta \mathcal{M}_{mn} &= \xi^\rho \partial_\rho \mathcal{M}_{mn}, \\
\delta \varphi &= \xi^\rho \partial_\rho \varphi.
\end{aligned} \tag{2.67}$$

Thus the higher-dimensional GCT leads to a lower-dimensional GCT as well as a *gauge transformation* for each of the Maxwell fields, $\delta \mathcal{A}_\mu^m = \partial_\mu \theta^m$ with $\theta^m = \xi^m$. In this framework the $U(1) \cong S^1$ gauge symmetry of each vector field simply corresponds to the residual general coordinate invariance of each circular extra dimension. This demonstrates the powerful potential for unification offered by Kaluza-Klein theory; gauge symmetries are simply special cases of higher-dimensional gravitational symmetries.

There are in fact more $(D+d)$ -dimensional GCTs that preserve the metric ansatz (2.63). It is easy to check that one may allow the diffeomorphism parameters ξ^m to depend linearly on the torus coordinates z^m :

$$\xi^m = S^m_n z^n + \theta^m, \tag{2.68}$$

where S^m_n is a constant matrix belonging to the Lie algebra $\mathfrak{sl}(d, \mathbb{R})$, i.e. real and traceless ($S^m_m = 0$), and $\theta^m(x^\mu)$ depends only on the x^μ coordinates. Again, taking lower-dimensional components of equation (2.66) using (2.63) shows that this results in

$$\begin{aligned}
\delta \mathcal{A}_\mu^m &= S^m_n \mathcal{A}_\mu^n, \\
\delta \mathcal{M}_{mn} &= S^p_m \mathcal{M}_{pn} + S^p_n \mathcal{M}_{mp},
\end{aligned} \tag{2.69}$$

while $g_{\mu\nu}$ and φ remain invariant (here θ^m has been set to zero). Thus the Lagrangian (2.64) is invariant under the $SL(d, \mathbb{R})$ transformations (2.69). Furthermore, the full symmetry is $GL(d, \mathbb{R}) \cong SL(d, \mathbb{R}) \times \mathbb{R}$, since the Lagrangian is also invariant under constant shifts of φ combined with scaling \mathcal{A}_μ^m :

$$\begin{aligned}
\varphi(x) &\rightarrow \varphi(x) + k, \quad k \in \mathbb{R}, \\
\mathcal{A}_\mu^m(x) &\rightarrow e^{-ck} \mathcal{A}_\mu^m.
\end{aligned} \tag{2.70}$$

Because $(e^{2\beta\varphi})^d$ represents the volume of the d -torus, this shift corresponds to the symmetry under scaling or ‘dilating’ its overall size⁶. The φ field is usually called a

⁶More precisely, this comes from combining the symmetry of the higher-dimensional Einstein

dilaton due to this fact.

To gain a better understanding of the scalar kinetic term $\frac{1}{4}\partial_\mu\mathcal{M}_{mn}\partial^\mu\mathcal{M}^{nm}$ in (2.64) it is useful to consider the higher-dimensional vielbein E_M^A , satisfying

$$\eta_{AB}E_M^AE_N^B = G_{MN}, \quad (2.71)$$

so that the A, B indices label local Lorentz frames. These indices then split into $a = 0, 1, \dots, (D-1)$ and $i = 1, \dots, d$. Due to the local Lorentz-invariance of equation (2.71) under $E_M^A \rightarrow \Lambda^A{}_B(x)E_M^B$ with $\Lambda^C{}_A\eta_{CD}\Lambda^D{}_B = \eta_{AB}$, it is possible to choose a gauge such that the vielbein corresponding to G_{MN} above is upper-triangular:

$$E_M^A = \begin{pmatrix} e^{\alpha\varphi}e_\mu^a & e^{\beta\varphi}\mathcal{V}_m^i\mathcal{A}_\mu^m \\ 0 & e^{\beta\varphi}\mathcal{V}_m^i \end{pmatrix}, \quad (2.72)$$

where $\mathcal{V}_m^i \in \text{SL}(d, \mathbb{R})$ is like the vielbein for \mathcal{M}_{mn} ,

$$\mathcal{V}_m^i\mathcal{V}_n^i = \mathcal{M}_{mn}, \quad (2.73)$$

and e_μ^a is the D -dimensional vielbein for the metric $g_{\mu\nu}$. Equation (2.73) is invariant under

$$\mathcal{V}_m^i \rightarrow \mathcal{O}^i{}_j(x)\mathcal{V}_m^j, \quad (2.74)$$

where $\mathcal{O}^i{}_j$ is a $d \times d$ unimodular orthogonal matrix. This local $\text{SO}(d)$ symmetry is hence also a symmetry of the Lagrangian (2.64), and simply represents the redundancy in using a unimodular $d \times d$ matrix \mathcal{V}_m^i to package fewer than $d^2 - 1$ scalars ($\frac{1}{2}d^2(d+1) - 1$ scalars, to be precise). The higher-dimensional origin of this symmetry is that the upper-triangular gauge in (2.72) is preserved under block-diagonal local Lorentz transformations $E_M^A \rightarrow \Lambda^A{}_B(x)E_M^B$ of the form

$$\Lambda^A{}_B(x) = \begin{pmatrix} \Lambda^a{}_b(x) & 0 \\ 0 & \mathcal{O}^i{}_j(x) \end{pmatrix} \in \text{SO}(1, D-1) \times \text{SO}(d), \quad (2.75)$$

where $\Lambda^a{}_b$ is a Lorentz transformation in D dimensions and $\mathcal{O}^i{}_j$ is orthogonal.

The scalar fields in the kinetic term $\frac{1}{4}\partial_\mu\mathcal{M}_{mn}\partial^\mu\mathcal{M}^{nm}$ describe a *non-linear sigma model*. This is a scalar field theory where the scalars are smooth maps from space-time to another manifold, called the target space or ‘scalar manifold’. For example, consider $N+1$ scalar fields ϕ_i , $i = 1, \dots, (N+1)$, described by the Lagrangian

equations under scalings of the metric by a positive constant $G_{MN} \rightarrow e^{k_1}G_{MN}$, $k_1 \in \mathbb{R}$, with uniform scaling of the torus $\delta z^m = k_2 z^m$. The constant shift k of φ is then a linear combination of k_1 and k_2 . See [44] for more detail on this.

$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i$ subject to the constraint that $\phi_i\phi_i = 1$. These scalars are maps from spacetime to the unit sphere S^N . Since \mathcal{V}_m^i only appears in the Lagrangian via \mathcal{M}_{mn} , which is invariant under (2.74), \mathcal{V}_m^i and $\mathcal{O}^i_j(x)\mathcal{V}_m^j$ are physically indistinguishable, and $\mathcal{V}_m^i \in \text{SL}(d, \mathbb{R})$ is essentially only defined up to an $\text{SO}(d)$ transformation. Thus the scalar fields in \mathcal{V}_m^i parameterise the *non-compact* coset manifold $\text{SL}(d, \mathbb{R})/\text{SO}(d)$. Note that the coset $\text{SL}(d, \mathbb{R})/\text{SO}(d)$ has dimension $\frac{1}{2}d^2(d+1) - 1$, as required, since this is the number of scalar fields described by the original matrix \mathcal{M}_{mn} .

To construct more traditional kinetic terms of the form $-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \dots$, a gauge must be chosen for the coset representative \mathcal{V}_m^i , with a particular labelling of its components in terms of the individual scalar fields it parameterises. This can be achieved by working directly with the Lie algebra $\mathfrak{sl}(d, \mathbb{R})$. As a specific example, consider the $d = 2$ case. The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ has basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (2.76)$$

with commutators

$$[H, E_+] = 2E_+, \quad [H, E_-] = -2E_-, \quad [E_+, E_-] = H, \quad (2.77)$$

so E_+ is positive-root generator and E_- is a negative root generator under the Cartan subalgebra, which in this case consists of the only diagonal generator H . A convenient choice of upper-triangular 2×2 coset representative turns out to be

$$\mathcal{V} = e^{\frac{1}{2}\phi(x)H}e^{\chi(x)E_+} = \begin{pmatrix} e^{\frac{1}{2}\phi} & 0 \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix} \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\frac{1}{2}\phi} & \chi e^{\frac{1}{2}\phi} \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix}, \quad (2.78)$$

which gives

$$\mathcal{M}_{mn} = \mathcal{V}_m^i \mathcal{V}_n^i = \begin{pmatrix} e^\phi & \chi e^\phi \\ \chi e^\phi & \chi^2 e^\phi + e^{-\phi} \end{pmatrix}. \quad (2.79)$$

Substituting \mathcal{M}_{mn} into the kinetic term gives

$$\frac{1}{4}\partial_\mu\mathcal{M}_{mn}\partial^\mu\mathcal{M}^{nm} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{2\phi}\partial_\mu\chi\partial^\mu\chi. \quad (2.80)$$

In this form the physical content is easier to see, but the $\text{SL}(2, \mathbb{R})$ symmetry that acted linearly on the matrix \mathcal{M}_{mn} is no longer manifest. Its action on the individual

fields ϕ and χ is non-linear and rather complicated:

$$\begin{aligned} e^\phi &\rightarrow (C\chi + D)^2 e^\phi + C^2 e^{-\phi}, \\ \chi e^\phi &\rightarrow (A\chi + B)(C\chi + D)e^\phi + ACe^{-\phi}, \end{aligned} \quad (2.81)$$

where A, B, C, D are the parameters of the finite linear transformation that acts on \mathcal{M}_{mn} :

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (e^S)^m_n \quad \Rightarrow \quad AD - BC = 1. \quad (2.82)$$

In the general case, with Lie algebra $\mathfrak{sl}(d, \mathbb{R})$, constructing the scalar coset Lagrangian works analogously. The Cartan subalgebra consists of the basis of traceless $d \times d$ diagonal matrices \vec{H} , written here all together as a $(d-1)$ -component vector. The positive-root generators are matrices E_{ij} , $i < j$, which have components $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, i.e. the ij^{th} element of E_{ij} is equal to 1 and every other element is zero. The commutators can be written as

$$[\vec{H}, E_{ij}] = \vec{b}_{ij} E_{ij}, \quad i < j \text{ (no sum)}, \quad (2.83)$$

where $\{\vec{b}_{ij}\}$ with $i < j$ is the set of positive root vectors.

In analogy with equation (2.78), a useful parameterisation of \mathcal{V} is the upper-triangular matrix

$$\mathcal{V} = e^{\frac{1}{2}\vec{\phi}(x)\cdot\vec{H}} \prod_{i < j} e^{\chi_{ij}(x)E_{ij}}. \quad (2.84)$$

Then one may form $\mathcal{M} = \mathcal{V}^T \mathcal{V}$ and the scalar Lagrangian can be written in terms of the individual fields as

$$\frac{1}{4} \partial_\mu \mathcal{M}_{mn} \partial^\mu \mathcal{M}^{nm} = -\frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{1}{2} \sum_{i < j} e^{\vec{b}_{ij}\cdot\vec{\phi}} \partial_\mu \chi_{ij} \partial^\mu \chi_{ij}. \quad (2.85)$$

The root vectors \vec{b}_{ij} in this context are often referred to as ‘dilaton vectors’, since each of the ϕ fields is a dilaton representing the radius of one of the circles of the d -torus (overall φ and $\vec{\phi}$ are just the diagonal components of the original metric on the torus). In Chapter 7 they will be connected with the octonions in a surprising way.

It is also important to note that since all fields in a gravitational theory with matter couple to the metric (via G^{MN} factors which contract their kinetic-term indices, as well as the determinant factor $\sqrt{-G}$), one must use the ansatz (2.63) to carefully determine the couplings of the various lower-dimensional fields. In particular, after some field redefinitions, the lower-dimensional theory may be parameterised such

that every bosonic field (except for the metric) has an exponential coupling $e^{\vec{b}\cdot\vec{\phi}}$ as a factor appearing in its kinetic term. Thus, each bosonic field has its own unique dilaton vector.

2.8. U-Dualities

Having discussed the Kaluza-Klein theory of pure gravity, the next step is to turn to eleven-dimensional supergravity and compactify on a d -torus to obtain maximal supergravities in dimensions $D = 11 - d$. This will not be carried out explicitly here, but the general features of the resulting theories will be discussed. For a detailed and pedagogical account see [44].

One particularly interesting aspect of these reductions is that scalars descended from the 3-form field⁷ C_{MNP} conspire with those descended from the metric g_{MN} to give an enlarged scalar coset sigma model with a greatly enlarged ‘*hidden*’ *symmetry group*. For example, consider the scalars of the $D = 4$ theory obtained from reduction on the 7-torus. In the previous section it was demonstrated that the Kaluza-Klein reduction of pure gravity on T^7 gives rise to an $\text{SL}(7, \mathbb{R})/\text{SO}(7)$ scalar coset Lagrangian and the total global symmetry group was in fact $\text{GL}(7, \mathbb{R})$. Thus there are

$$\dim[\text{GL}(7, \mathbb{R})] - \dim[\text{SO}(7)] = 49 - 21 = 28 \quad (2.86)$$

scalar fields descended from the eleven-dimensional metric. There are also $\binom{7}{3} = 35$ scalars given by the 3-form’s internal components C_{ijk} , where $i, j, k = 1, \dots, 7$. Finally there are an additional seven scalar fields coming from the 2-forms $C_{\mu\nu i}$ by the process of *dualisation*; the 3-form field strengths $F_{\mu\nu\rho i} = 3\partial_{[\mu}C_{\nu\rho]i}$ of these 2-form fields may be defined as the Hodge duals of seven 1-form field strengths

$$\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi_i = \frac{1}{3!}\varepsilon^{\mu\nu\rho\sigma}e^{\vec{a}_i\cdot\vec{\phi}}F_{\nu\rho\sigma i} \quad (2.87)$$

for seven scalar fields Φ_i , where \vec{a}_i is the dilaton vector found in the 2-form kinetic term

$$-\frac{1}{2\cdot 3!}\sqrt{-g}e^{\vec{a}_i\cdot\vec{\phi}}F_{\mu\nu\rho i}F^{\mu\nu\rho i}. \quad (2.88)$$

Differentiating (2.87) shows that the Bianchi identity for the 2-forms gives an equation of motion for the scalar fields,

$$\partial_\mu\left(e^{-\vec{a}_i\cdot\vec{\phi}}\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi_i\right) = \partial_\mu\left(\frac{1}{3!}\varepsilon^{\mu\nu\rho\sigma}F_{\nu\rho\sigma i}\right) = \partial_\mu\left(\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_{[\nu}C_{\rho\sigma]i}\right) \equiv 0, \quad (2.89)$$

⁷Now labelled with capital M, N, \dots indices in preparation for decomposing into those of the lower-dimensional spacetime μ, ν, \dots and internal indices i, j, \dots .

while the Bianchi identity for the scalar fields gives the equation of motion for the 2-forms:

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu\Phi_i = \partial_\mu\left(\sqrt{-g}g^{\mu\nu}g^{\rho\lambda}g^{\sigma\tau}e^{\vec{a}_i\cdot\vec{\phi}}F_{\nu\lambda\tau i}\right) = 0. \quad (2.90)$$

This means that the scalar kinetic term

$$-\frac{1}{2}\sqrt{-g}e^{-\vec{a}_i\cdot\vec{\phi}}\partial_\mu\Phi_i\partial^\mu\Phi^i \quad (2.91)$$

will give equivalent overall equations of motion and Bianchi identities, and thus the seven 2-forms may be replaced with the seven scalars Φ_i . In total this gives $7 + 35 + 28 = 70$ scalar fields, just as required for the $\mathcal{N} = 8$ supermultiplet derived in Section 2.4. There it was shown that the 70 scalars transformed as the irreducible **70** representation of the R-symmetry group $SU(8)$. Since $SU(8)$ is a compact group, the maximal compact subgroup $SO(7) \subset GL(7, \mathbb{R})$ that is a symmetry of the naive Lagrangian ought to be a subgroup of the $SU(8)$ R-symmetry. Indeed there is a subgroup $SO(7) \subset SU(8)$ such that the **70** decomposes as

$$\mathbf{70} \rightarrow \mathbf{1} + \mathbf{27} + \mathbf{35} + \mathbf{7}, \quad (2.92)$$

corresponding precisely to the scalars introduced above: φ , \mathcal{M}_{mn} , C_{ijk} and Φ_i , respectively.

The various other fields obtained from the dimensional reduction of $D = 11$ to $D = 4$ also assemble into $SU(8)$ representations. This is easy to see at the on-shell level where the $D = 11$ fields furnish representations of the little group $SO(9)_{ST}$; reduction to $D = 4$ simply corresponds to splitting $SO(9) \supset SO(2) \times SO(7)$ to give the $D = 4$ little group $SO(2) \cong U(1)_{ST}$ and the internal $SO(7)$ symmetry. The resulting representations then assemble into precisely those of $SU(8) \times U(1)_{ST}$ given in (2.40) – see Table 2.1.

Returning to the 70 scalars, one might guess that they belong to a larger coset $G/SU(8)$, where G is a non-compact group whose maximal compact subgroup is $SU(8)$ – just as the 28 scalars descended from the metric belonged to the coset $GL(7, \mathbb{R})/SO(7)$, where $SO(7)$ is the maximal compact subgroup of $GL(7, \mathbb{R})$. This would mean that

$$\dim[G] - \dim[SU(8)] = \dim[G] - 63 = 70, \quad (2.93)$$

so G must be a non-compact group of dimension 133. There is only one simple Lie group of this dimension whose maximal compact subgroup is $SU(8)$: the exceptional group $E_{7(7)}$. Indeed it can be shown that the scalars of $\mathcal{N} = 8$ supergravity

| | | $g_{\mu\nu}$ | Ψ_μ | A_μ | λ | ϕ |
|-----------|------------|----------------------------------|--|------------------------------------|--|--------------------------------|
| | | $\mathbf{1}_2 + \mathbf{1}_{-2}$ | $\mathbf{8}_{\frac{3}{2}} + \mathbf{8}_{-\frac{3}{2}}$ | $\mathbf{28}_1 + \mathbf{28}_{-1}$ | $\mathbf{56}_{\frac{1}{2}} + \mathbf{56}_{-\frac{1}{2}}$ | $\mathbf{70}_0$ |
| g_{MN} | 44 | $\mathbf{1}_2 + \mathbf{1}_{-2}$ | — | $\mathbf{7}_1 + \mathbf{7}_{-1}$ | — | $\mathbf{27}_0 + \mathbf{1}_0$ |
| C_{MNP} | 84 | — | — | $\mathbf{21}_1 + \mathbf{21}_{-1}$ | — | $\mathbf{35}_0 + \mathbf{7}_0$ |
| Ψ_M | 128 | — | $\mathbf{8}_{\frac{3}{2}} + \mathbf{8}_{-\frac{3}{2}}$ | — | $\mathbf{48}_{\frac{1}{2}} + \mathbf{48}_{-\frac{1}{2}}$ $+ \mathbf{8}_{\frac{1}{2}} + \mathbf{8}_{-\frac{1}{2}}$ | — |

Table 2.1.: Little group representations of $D = 4$, $\mathcal{N} = 8$ supergravity. The rows show the decomposition of the $D = 11$ little group $\text{SO}(9)_{\text{ST}}$ representations into those of $\text{SO}(7) \times \text{U}(1)_{\text{ST}}$, corresponding to dimensional reduction to $D = 4$, with little group $\text{U}(1)_{\text{ST}}$. The columns show the decomposition of the same $\mathcal{N} = 8$ supergravity degrees of freedom as representations of $\text{SU}(8) \times \text{U}(1)_{\text{ST}}$, as given in (2.40).

parameterise the 70-dimensional coset $E_{7(7)}/\text{SU}(8)$ and thus their kinetic terms have a global non-linearly realised $E_{7(7)}$ symmetry. In the context of M-theory and its brane solutions, this symmetry is seen as a discrete subgroup $E_{7(7)}(\mathbb{Z})$ called the *U-duality* group. However, throughout this thesis the term ‘U-duality’ will be used mainly to refer to the continuous global symmetry groups of classical supergravity theories.

In general the U-duality group is not a symmetry of the whole Lagrangian, only of the equations of motion and the Bianchi identities. For example, there are 28 vector fields in $\mathcal{N} = 8$ supergravity – \mathcal{A}_μ^i coming from the metric and $C_{\mu ij}$ coming from the 3-form – but the smallest representation of $E_{7(7)}$ is the **56**. The $E_{7(7)}$ transformation actually rotates the 28 vector fields’ equations of motion into their 28 Bianchi identities as the **56**, generalising electromagnetic duality.

| D | G | H |
|-----|--|------------------------------------|
| 10 | $\text{SO}(1, 1)$ | $\mathbb{1}$ |
| 9 | $\text{SL}(2, \mathbb{R})$ | $\text{SO}(2)$ |
| 8 | $\text{SL}(3, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ | $\text{SO}(3) \times \text{SO}(2)$ |
| 7 | $\text{SL}(5, \mathbb{R})$ | $\text{SO}(5)$ |
| 6 | $\text{SO}(5, 5)$ | $\text{SO}(5) \times \text{SO}(5)$ |
| 5 | $E_{6(6)}$ | $\text{Sp}(4)$ |
| 4 | $E_{7(7)}$ | $\text{SU}(8)$ |
| 3 | $E_{8(8)}$ | $\text{SO}(16)$ |

Table 2.2.: U-duality groups G for the maximal supergravities obtained by torus reductions of the $D = 11$ theory. The scalars in each case parameterise the coset manifolds G/H , where H is the maximal compact subgroup of G .

The maximal $D = 4$, $\mathcal{N} = 8$ theory is not the only maximal supergravity where

such symmetry enhancements occur when reducing down from eleven dimensions. The first case is $D = 8$ where there are six scalars descended from the metric and one scalar C_{123} from the 3-form: seven scalars in total. Instead of the naive $GL(3, \mathbb{R})/SO(3)$ coset expected from pure gravity, the seven scalars turn out to parameterise the coset $(SL(3, \mathbb{R}) \times SL(2, \mathbb{R})) / (SO(3) \times SO(2))$. Similar enhancements occur for $D \leq 8$ all the way down to $D = 3$ where the coset is $E_{8(8)}/SO(16)$ – see Table 2.2 for the complete list of cosets G/H , where G is a (maximally) non-compact group and H its maximal compact subgroup. Note that in each case the maximal compact subgroup H is also the R-symmetry group.

3. The Normed Division Algebras

*The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete. The quaternions, being non-commutative, are the eccentric cousin who is shunned at important family gatherings. But the octonions are the crazy old uncle nobody lets out of the attic: they are **non-associative**.*

– John Baez, *The Octonions*

This purely mathematical chapter is dedicated to the normed division algebras and their deep relationship with group theory. It loosely follows the treatment in the excellent review [1] by Baez, but often uses – in that author’s words – the ‘index-ridden’ notation favoured by most physicists.

Section 3.1 contains the definition of the division algebras and a detailed introduction to their basic properties, as well as a proof that they are indeed normed division algebras. Next, Section 3.2 describes the close connection between the division algebras and the orthogonal groups that act naturally upon them, which in Section 3.3 is discussed in relation to the notion of triality. The final two Sections, 3.4 and 3.5, review the broader relationship between division algebras and classical Lie groups and their Lie algebras.

3.1. Definitions and Basic Properties

An algebra \mathbb{A} is a vector space equipped with a bilinear multiplication rule and a unit element. We say \mathbb{A} is a division algebra if, given $x, y \in \mathbb{A}$ with $xy = 0$, then either $x = 0$ or $y = 0$. A *normed division algebra* is an algebra \mathbb{A} equipped with a positive-definite norm satisfying the condition that the norm of a product xy is the product of the respective norms of x and y ,

$$\|xy\| = \|x\| \|y\|. \tag{3.1}$$

One may see from setting $\|xy\| = 0$ that (3.1) also implies that \mathbb{A} is a division algebra, since the norm of any element is zero if and only if that element is itself

zero. From now on in this thesis, the terms ‘division algebra’ and ‘*normed* division algebra’ shall be used interchangeably, since the division algebras that arise will always be normed unless stated otherwise.

Although condition (3.1) might not seem so stringent, there is a remarkable theorem due to Hurwitz [45], which states that there are only four normed division algebras: the real numbers \mathbb{R} , the complex numbers \mathbb{C} , the quaternions \mathbb{H} and the octonions \mathbb{O} – with dimensions $n = 1, 2, 4$ and 8 , respectively.

Searching for a generalisation of the complex numbers, Hamilton discovered the quaternions \mathbb{H} in 1853. A quaternion is a linear combination of the basis vectors $\{1, i, j, k\}$, where the three ‘imaginary’ units i, j and k are all square roots of -1 ,

$$i^2 = -1, \quad j^2 = -1, \quad k^2 = -1, \quad (3.2)$$

and they satisfy the associative, non-commutative multiplication rule

$$ij = -ji = k. \quad (3.3)$$

It follows that cyclic permutations $i \rightarrow j \rightarrow k \rightarrow \dots$ of the above equation also hold. The rule may be neatly summarised using the notation $e_1 = i, e_2 = j, e_3 = k$:

$$e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k, \quad i = 1, 2, 3, \quad (3.4)$$

where ε_{ijk} is the permutation symbol with $\varepsilon_{123} = 1$. This is reminiscent of the algebra of the three-dimensional vector cross product, or (up to a factor of i) equivalent to that of the Pauli spin matrices σ_i :

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \varepsilon_{ijk} \sigma_k, \quad i = 1, 2, 3, \quad (3.5)$$

where as usual

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.6)$$

Evidently the Pauli matrices give a *complex* representation of the quaternion algebra with $e_i \rightarrow -i\sigma_i$. It makes sense that \mathbb{H} can be represented by matrices, since it is an associative algebra.

Inspired by Hamilton’s quaternions, later in 1853 Graves discovered the octonions \mathbb{O} with basis $\{1, e_1, e_2, \dots, e_7\}$, equipped with the multiplication rule

$$e_i e_j = -\delta_{ij} + C_{ijk} e_k, \quad i = 1, \dots, 7, \quad (3.7)$$

| | | | | | | | |
|-------|--------|--------|--------|--------|--------|--------|--------|
| | e_1 | e_2 | e_3 | e_4 | e_5 | e_6 | e_7 |
| e_1 | -1 | e_4 | e_7 | $-e_2$ | e_6 | $-e_5$ | $-e_3$ |
| e_2 | $-e_4$ | -1 | e_5 | e_1 | $-e_3$ | e_7 | $-e_6$ |
| e_3 | $-e_7$ | $-e_5$ | -1 | e_6 | e_2 | $-e_4$ | e_1 |
| e_4 | e_2 | $-e_1$ | $-e_6$ | -1 | e_7 | e_3 | $-e_5$ |
| e_5 | $-e_6$ | e_3 | $-e_2$ | $-e_7$ | -1 | e_1 | e_4 |
| e_6 | e_5 | $-e_7$ | e_4 | $-e_3$ | $-e_1$ | -1 | e_2 |
| e_7 | e_3 | e_6 | $-e_1$ | e_5 | $-e_4$ | $-e_2$ | -1 |

Table 3.1.: The multiplication table of the seven imaginary basis octonions.

A brief inspection of the Fano plane reveals that the octonions are non-associative, but they do exhibit a similar, weaker property called *alternativity*. An algebra \mathbb{A} is alternative if and only if for all $x, y \in \mathbb{A}$ the following relations are satisfied:

$$(xx)y = x(xy), \quad (xy)x = x(yx), \quad (yx)x = y(xx) \quad (3.10)$$

(note that in fact any one of these conditions may be derived from the remaining two [1]). This property is trivially satisfied by the three associative division algebras \mathbb{R} , \mathbb{C} and \mathbb{H} , and therefore all division algebras are alternative. This turns out to be crucial for supersymmetry in $D = 3, 4, 6, 10$, as will be demonstrated in Chapter 4. The three conditions (3.10) can be neatly summed up by defining a trilinear map called the associator given by

$$[x, y, z] := (xy)z - x(yz), \quad x, y, z \in \mathbb{A}, \quad (3.11)$$

which measures the failure of associativity. An algebra \mathbb{A} is then alternative if and only if the associator is an antisymmetric function of its three arguments, since setting any two of the arguments to be equal then yields one of the three conditions in (3.10).

Sometimes it will be useful to denote the division algebra of dimension n by \mathbb{A}_n . From the above discussion, it is clear that the division algebra \mathbb{A}_n contains $\mathbb{A}_{n/2}$ as a subalgebra (excluding the $n = 1$ case). This is no coincidence. The four division algebras can be constructed, one-by-one, using the so-called ‘Cayley-Dickson doubling’ method, starting with \mathbb{R} ; the complex numbers are pairs of real numbers equipped with a particular multiplication rule, quaternions are pairs of complex numbers and octonions are pairs of quaternions. At the level of vector

spaces,

$$\begin{aligned} \mathbb{C} &\cong \mathbb{R}^2, \\ \mathbb{H} &\cong \mathbb{C}^2 \cong \mathbb{R}^4, \\ \mathbb{O} &\cong \mathbb{H}^2 \cong \mathbb{C}^4 \cong \mathbb{R}^8. \end{aligned} \tag{3.12}$$

The real numbers are ordered, commutative and associative, but with each doubling one such property is lost: \mathbb{C} is commutative and associative, \mathbb{H} is associative, \mathbb{O} is *non-associative*. The Cayley-Dickson procedure yields an infinite sequence of algebras, but in doubling the octonions to obtain the 16-dimensional ‘sedenions’ both alternativity and the division algebra property are lost.

It will be useful to introduce some general notation here for working with all four division algebras. A division algebra element $x \in \mathbb{A}_n$ is written as a linear combination of the n basis elements with real coefficients: $x = x_a e_a$, $x_a \in \mathbb{R}$ and $a = 0, \dots, (n-1)$. The first basis element $e_0 = 1$ is ‘real’¹, while the remaining $(n-1)$ basis elements e_i are ‘imaginary’:

$$e_0^2 = 1, \quad e_i^2 = -1, \tag{3.13}$$

where $i = 1, \dots, (n-1)$. By analogy with the complex case, we define a conjugation operation indicated by $*$, which is a linear involution that changes the sign of the imaginary basis elements:

$$e_0^* = e_0, \quad e_i^* = -e_i. \tag{3.14}$$

It is natural then to define the linear projections

$$\operatorname{Re}(x) := \frac{1}{2}(x + x^*) = x_0, \quad \operatorname{Im}(x) := \frac{1}{2}(x - x^*) = x_i e_i, \tag{3.15}$$

for $x \in \mathbb{A}$. Note that this differs slightly with the convention typically used for the complex numbers (since $\operatorname{Im}(x_0 + x_1 e_1) = x_1 e_1$ rather than just x_1). A division algebra element is said to be ‘real’ if it belongs to the subspace $\operatorname{Re}(\mathbb{A}) \cong \mathbb{R}$ or ‘imaginary’ if it belongs to $\operatorname{Im}(\mathbb{A})$.

The multiplication rule for the imaginary basis elements of a general division

¹Following the tradition favoured for complex numbers, authors often take the liberty of not distinguishing between the basis element e_0 , which is a vector, and the number 1, which is of course an element of \mathbb{R} , the field over which the vectors are defined. In practice this is always legitimate, since e_0 is the multiplicative identity.

algebra is:

$$e_i e_j = -\delta_{ij} + A_{ijk} e_k. \quad (3.16)$$

where the tensor A_{ijk} is totally antisymmetric, reflecting anti-commutativity. The antisymmetry of A_{ijk} means all of its components are identically zero for $\mathbb{A} = \mathbb{R}, \mathbb{C}$. For the quaternions $A_{ijk} = \varepsilon_{ijk}$, while for the octonions $A_{ijk} = C_{ijk}$:

$$A_{ijk}(\mathbb{A}) = \begin{cases} 0 & \text{for } \mathbb{A} = \mathbb{R}, \mathbb{C}, \\ \varepsilon_{ijk} & \text{for } \mathbb{A} = \mathbb{H}, \\ C_{ijk} & \text{for } \mathbb{A} = \mathbb{O}. \end{cases} \quad (3.17)$$

It is clear from (3.16) that the commutator of two imaginary basis elements is

$$[e_i, e_j] = 2A_{ijk} e_k, \quad (3.18)$$

which shows that when \mathbb{A} is a non-commutative algebra, i.e. $\mathbb{A} = \mathbb{H}, \mathbb{O}$, the subspace $\text{Im}(\mathbb{A})$ forms a closed algebra under commutation. In fact, (3.18) shows that $\text{Im}(\mathbb{H})$ under the commutator is just the Lie algebra $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, since $A_{ijk}(\mathbb{H}) = \varepsilon_{ijk}$. This is exactly what one would expect given that the imaginary quaternions can be represented by the Pauli matrices, which are of course the generators of $\mathfrak{su}(2)$. Note also that the ordinary vector cross product in $\mathbb{R}^3 \cong \text{Im}(\mathbb{H})$ is just $e_i \times e_j = \frac{1}{2}[e_i, e_j]$.

Including $e_0 = 1$, the multiplication rule can be summarised as

$$e_a e_b = (+\delta_{a0}\delta_{bc} + \delta_{0b}\delta_{ac} - \delta_{ab}\delta_{0c} + A_{abc}) e_c =: \Gamma_{bc}^a e_c, \quad (3.19)$$

where the definition of A_{ijk} is extended to A_{abc} , which is totally antisymmetric with $A_{0ab} = 0$, and the structure constants Γ_{bc}^a are defined² as

$$\Gamma_{bc}^a := \delta_{a0}\delta_{bc} + \delta_{b0}\delta_{ac} - \delta_{ab}\delta_{c0} + A_{abc}. \quad (3.20)$$

These will have an important role to play in the next section. Taking the $ab = ij$ components of (3.19) returns the simpler pure-imaginary version of the rule, given in (3.16). Note that

$$\begin{aligned} e_a^* e_b &= (+\delta_{a0}\delta_{bc} - \delta_{0b}\delta_{ac} + \delta_{ab}\delta_{0c} - A_{abc}) e_c =: \bar{\Gamma}_{bc}^a e_c, \\ e_a e_b^* &= (-\delta_{a0}\delta_{bc} + \delta_{0b}\delta_{ac} + \delta_{ab}\delta_{0c} - A_{abc}) e_c = \bar{\Gamma}_{ab}^c e_c, \end{aligned} \quad (3.21)$$

²The choice of index structure is for later convenience – see equations (3.96) and (3.102).

where

$$\bar{\Gamma}_{cb}^a := \Gamma_{bc}^a. \quad (3.22)$$

Using (3.19) one can easily verify that conjugation of a product of $x, y \in \mathbb{A}$ gives

$$(xy)^* = y^*x^*. \quad (3.23)$$

Again, just as for the complex numbers, the norm $\|x\|$ of a division algebra element x may be defined as:

$$\|x\|^2 = xx^* = x^*x = x_a x_a. \quad (3.24)$$

This norm provides the notion of ‘division’ that gives these algebras their name; each element x has a multiplicative inverse

$$x^{-1} := \frac{x^*}{x^*x} \quad \Rightarrow \quad x^{-1}x = xx^{-1} = 1. \quad (3.25)$$

In general, for any normed vector space, the polarisation identity gives a natural inner product constructed from the norm [1]:

$$\begin{aligned} \langle x|y \rangle &:= \frac{1}{2} (\|x+y\|^2 - \|x\|^2 - \|y\|^2) \\ &= \frac{1}{2} (xy^* + yx^*) = \frac{1}{2} (x^*y + y^*x) = x_a y_a, \quad \text{i.e.} \quad \langle e_a|e_b \rangle = \delta_{ab}. \end{aligned} \quad (3.26)$$

This is just the canonical inner product on \mathbb{R}^n , which is preserved by the orthogonal group $O(n)$. This group and its Lie algebra thus have a natural action on the division algebra elements that will be explored in detail in the next section.

The proof that the normed division algebra condition (3.1) is satisfied contains several identities that will be used throughout this work and so will be demonstrated here. The calculation requires careful consideration for the non-associativity of the octonions, so some discussion of the practicalities of this is beneficial. For a product such as $(xy)z$, with $x, y, z \in \mathbb{O}$, the brackets may be moved at the cost of adding an associator term,

$$(xy)z = x(yz) + [x, y, z]. \quad (3.27)$$

The properties of the associator often render this term benign. It clearly vanishes if one of its arguments is real,

$$[x_0, y, z] = 0 \quad \text{for } x_0 \in \text{Re}(\mathbb{O}) \cong \mathbb{R}, \quad (3.28)$$

so the associator actually only takes into account the imaginary parts of its argu-

ments:

$$[x, y, z] = [\text{Im}(x), \text{Im}(y), \text{Im}(z)]. \quad (3.29)$$

Thus conjugation of any one of its arguments has the effect of changing the sign,

$$[x^*, y, z] = [x, y^*, z] = [x, y, z^*] = -[x, y, z], \quad (3.30)$$

which, when combined with alternativity, shows that the associator is always pure-imaginary:

$$\begin{aligned} [x, y, z]^* &= ((xy)z)^* - (x(yz))^* \\ &= z^*(y^*x^*) - (z^*y^*)x^* \\ &= -[z^*, y^*, x^*] \\ &= +[z, y, x] \\ &= -[x, y, z] \quad \Leftrightarrow \quad \text{Re}([x, y, z]) = 0. \end{aligned} \quad (3.31)$$

This also implies that the associator of an octonion x and its conjugate x^* vanishes:

$$[x, x^*, z] = -[x, x, z] \equiv 0. \quad (3.32)$$

Now, returning to the proof that the norm satisfies (3.1), consider $x, y \in \mathbb{A}$, where $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Their product $xy \in \mathbb{A}$ has norm-squared $\|xy\|^2$ given by

$$(xy)(xy)^* = (xy)(y^*x^*), \quad (3.33)$$

using $(xy)^* = y^*x^*$. Since $(xy)(y^*x^*)$ is a real number, it is equal to its real part,

$$(xy)(y^*x^*) = \text{Re}((xy)(y^*x^*)). \quad (3.34)$$

The brackets can be moved using the associator:

$$\begin{aligned} \text{Re}((xy)(y^*x^*)) &= \text{Re}(x(y(y^*x^*))) + \text{Re}([x, y, (y^*x^*)]) \\ &= \text{Re}(x(y(y^*x^*))), \end{aligned} \quad (3.35)$$

since the associator is imaginary. Then, by (3.32), $y(y^*x^*) = (yy^*)x^*$, but since (yy^*) is real it may be moved through any brackets. Putting all of this together,

$$\begin{aligned} (xy)(xy)^* &= \text{Re}(x(y(y^*x^*))) = \text{Re}(x((yy^*)x^*)) \\ &= \text{Re}(xx^*)(yy^*) \\ &= (xx^*)(yy^*). \end{aligned} \quad (3.36)$$

Thus the product of the norms is the norm of the product, as required. Note that this proof relies on the alternativity of the division algebras, and thus would fail for the 16-dimensional sedenions, which are non-alternative.

For the discussion of the relationships between the division algebras and various Lie groups, it will be helpful to present a few more definitions and identities for working with octonions and their components.

The complement of a line in the Fano plane is called a *quadrangle*. Thus the Fano plane has seven points, seven lines and seven quadrangles. Just as multiplication of the octonionic basis elements is encoded in the oriented lines of the Fano plane, the associator of three octonionic basis elements is encoded in its seven oriented quadrangles:

$$[e_a, e_b, e_c] = 2Q_{abcd}e_d, \quad (3.37)$$

where the tensor Q_{abcd} is totally antisymmetric with $Q_{0abc} = 0$, and the non-trivial Q_{ijkl} are given by:

$$Q_{ijkl} = Q_{[ijkl]} = \begin{cases} 1 & \text{if } ijkl \in \mathbf{Q} = \{3567, 4671, 5712, 6123, 7234, 1345, 2456\}, \\ 0 & \text{if } \sigma(ijkl) \notin \mathbf{Q} \text{ for any permutation } \sigma. \end{cases} \quad (3.38)$$

Since a quadrangle is the complement of a line in the Fano plane, by definition, the tensors Q_{ijkl} and C_{ijk} are dual to one another in seven dimensions:

$$Q_{ijkl} = -\frac{1}{3!}\varepsilon_{ijklmnp}C_{mnp}, \quad (3.39)$$

with $\varepsilon_{1234567} = 1$. The tensors Q_{ijkl} and C_{ijk} are also related by various identities that can be derived by carefully considering the Fano plane [46] (or by brute-force computation, according to taste):

$$\begin{aligned} C_{ijm}C_{klm} &= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + Q_{ijkl}, \\ C_{ijn}Q_{klmn} &= 3(C_{i[kl}\delta_{m]j} - C_{j[kl}\delta_{m]i}), \\ Q^{ijkl}Q_{mnp} &= 6\delta_m^{[i}\delta_n^j\delta_p^{k]} - C^{ijk}C_{mnp} + 9Q^{[ij}{}_{[mn}\delta_p^{k]}. \end{aligned} \quad (3.40)$$

Note that, by convention, division-algebraic indices are raised and lowered with the Kronecker delta, so the upper index placement in the final formula is only a matter of notational convenience, i.e. $Q^{ijkl} = Q_{ijkl}$.

3.1.1. Exponentials and Polar Form

One can define the exponential of $x \in \mathbb{A}$ using the power series definition,

$$e^x := \sum_{m=0}^{\infty} \frac{x^m}{m!}. \quad (3.41)$$

In the octonionic case non-associativity does not stop one from defining x^m unambiguously, since all products of m copies of x are equivalent by alternativity. Now consider any non-zero $x \in \mathbb{A} = \mathbb{H}, \mathbb{O}$ and write it as a sum of its real and imaginary parts, $x = \text{Re}(x) + \text{Im}(x)$. Its norm is then

$$\|x\|^2 = \text{Re}(x)^2 + \|\text{Im}(x)\|^2, \quad (3.42)$$

One may parameterise these two terms using an angle $\theta \in [0, 2\pi]$, defined by

$$\theta := \tan^{-1} \left(\frac{\|\text{Im}(x)\|}{\text{Re}(x)} \right), \quad (3.43)$$

which is of course chosen such that

$$\text{Re}(x) = \|x\| \cos \theta, \quad \|\text{Im}(x)\| = \|x\| \sin \theta. \quad (3.44)$$

Now, as a vector, the imaginary part $\text{Im}(x)$ can easily be rewritten

$$\text{Im}(x) = \text{Im}(x) \frac{\|x\|}{\|\text{Im}(x)\|} \frac{\|\text{Im}(x)\|}{\|x\|} = \|x\| u \sin \theta, \quad u := \frac{\text{Im}(x)}{\|\text{Im}(x)\|}, \quad (3.45)$$

where u manifestly has unit norm. Putting this together, the division algebra element x is just

$$x = \|x\| (\cos \theta + u \sin \theta). \quad (3.46)$$

This looks very similar to Euler's formula for complex numbers, but with the complex unit i replaced by u . However, just like i , any unit-norm imaginary division algebra element u also squares to -1 :

$$u^2 = u_i e_i u_j e_j = u_i u_j (-\delta_{ij} + A_{ijk} e_k) = -u_i u_i = -1, \quad (3.47)$$

since $u_i u_j$ is symmetric under the interchange $i \leftrightarrow j$ while A_{ijk} is antisymmetric. Thus Euler's formula applies as usual and x may be written in polar form

$$x = \|x\| e^{u\theta}. \quad (3.48)$$

This shows that a non-zero division algebra element may be expressed by a positive real number $\|x\|$, an imaginary axis u and an angle θ . There is a helpful geometrical interpretation of this parameterisation: the real axis and the u axis are orthogonal to one another, and since $u^2 = -1$, they define a complex plane $\mathbb{C} \subset \mathbb{A}$ on which $x = \|x\|e^{u\theta}$ is just the usual complex polar form; on this plane $\|x\|$ is the modulus while θ is the angle x makes with the real axis.

Equation (3.48) shows that a division algebra element has unit norm if and only if it may be written as e^α for some $\alpha \in \text{Im}(\mathbb{A})$ (excluding the real case). Thus the identity (3.48) can also be thought of simply as the statement that any non-zero $x \in \mathbb{A}_n$ can be written as a product of its norm $\|x\| \in \mathbb{R}^+$ and a unit element $e^{u\theta} \in S^{n-1}$, where S^{n-1} is the unit sphere in \mathbb{A}_n . More symbolically,

$$\mathbb{A}_n - \{0\} \cong S^{n-1} \times \mathbb{R}^+. \quad (3.49)$$

Since $\text{Im}(\mathbb{H}) \cong \mathfrak{su}(2)$, a simple corollary of the above is the well-known fact that the group $\text{SU}(2)$ must be isomorphic to the 3-sphere,

$$\text{SU}(2) \cong e^{\mathfrak{su}(2)} \cong e^{\text{Im}(\mathbb{H})} \cong S^3, \quad (3.50)$$

as is readily checked from the definition of $\text{SU}(2)$ as 2×2 unitary matrices with unit determinant.

3.2. Orthogonal Groups and Clifford Algebras

3.2.1. Symmetries of the Norm

As mentioned in the previous section, the norm of a division algebra element

$$x^*x = xx^* = x_a x_a, \quad x \in \mathbb{A}_n, \quad (3.51)$$

is manifestly invariant under

$$x_a \rightarrow O_a^b x_b, \quad (3.52)$$

where $O_a^b \in O(n)$ is an orthogonal matrix. However, it is both enlightening and useful to express these rotations in terms of division-algebraic operations such as the multiplication rule defined on \mathbb{A}_n .

Beginning with the trivial case, for the reals the $O(1) \cong \mathbb{Z}_2$ symmetry just corresponds to the symmetry under multiplication by the unit-norm elements $\{1, -1\}$, i.e. $x \rightarrow \pm x$.

For any of the division algebras with $n > 1$, the full rotation group $O(n)$ has two connected components: the group of orientation-preserving rotations $SO(n)$ (i.e. determinant $+1$) and the component consisting of orientation-reversing *reflections* (i.e. determinant -1). In terms of division-algebraic operations, conjugation $x \rightarrow x^*$ is clearly a discrete orientation-reversing symmetry of the norm (3.51). Thus any *reflection* in $O(n)$ is a product of a rotation in $SO(n)$ and conjugation $x \rightarrow x^*$. The remaining question then is how to write a general element of $SO(n)$ purely in terms of the multiplication rule of \mathbb{A}_n .

Of course for the complex numbers $SO(2) \cong U(1)$ acts again as multiplication by arbitrary elements of \mathbb{C} with unit norm:

$$x \rightarrow e^{i\theta}x, \quad \theta \in \mathbb{R}. \quad (3.53)$$

Or at the Lie algebra level, i.e. for small θ ,

$$\delta x = i\theta x. \quad (3.54)$$

It is clear that this preserves the squared norm (3.51).

The quaternionic analogue with $SO(4)$ acting on \mathbb{H} follows closely. It is easy to see that the norm is preserved under multiplication of x from both the left and right by arbitrary unit-norm quaternions, which may be parameterised as $e^{-\theta_+}$ and e^{θ_-} with θ_{\pm} imaginary:

$$x \rightarrow e^{-\theta_+}x e^{\theta_-}, \quad \theta_+, \theta_- \in \text{Im}(\mathbb{H}). \quad (3.55)$$

This means that at the Lie algebra level

$$\delta x = -\theta_+x + x\theta_-. \quad (3.56)$$

In fact, *any* $SO(4)$ transformation may be written in this way. As can be verified using the quaternionic multiplication rule, an arbitrary $SO(4)$ rotation with parameters $\theta^{ab} = -\theta^{ba}$ is enacted by choosing

$$\theta_+ = \frac{1}{2} (\theta^{0i} + \frac{1}{2}\varepsilon_{ijk}\theta^{jk}) e_i, \quad \theta_- = \frac{1}{2} (-\theta^{0i} + \frac{1}{2}\varepsilon_{ijk}\theta^{jk}) e_i. \quad (3.57)$$

It is instructive to examine these transformations in a little more detail. Equation (3.56) shows that the Lie algebra $\mathfrak{so}(4)$ is isomorphic to two copies of $\text{Im}(\mathbb{H})$: one for left-multiplication by $-\theta_+$ and one for right-multiplication by θ_- . Furthermore, they commute because left- and right-multiplication commute in \mathbb{H} , i.e. \mathbb{H} is associative.

Thus

$$\mathfrak{so}(4) \cong \text{Im}(\mathbb{H}) \oplus \text{Im}(\mathbb{H}) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2). \quad (3.58)$$

This is a reflection of the fact that any antisymmetric 4×4 matrix $\theta^{ab} \in \mathfrak{so}(4)$ can be split into its self-dual and anti-self-dual parts θ_+^{ab} and θ_-^{ab} :

$$\theta_{\pm}^{ab} := \frac{1}{2}(\theta^{ab} \pm \frac{1}{2}\varepsilon^{abcd}\theta^{cd}) \quad \Rightarrow \quad \frac{1}{2}\varepsilon^{abcd}\theta_{\pm}^{cd} = \pm\theta_{\pm}^{ab}, \quad (3.59)$$

which each have only three independent components (rather than six) since they obey

$$\theta_{\pm}^{0i} = \pm\frac{1}{2}\varepsilon^{ijk}\theta_{\pm}^{jk} = \frac{1}{2}(\theta^{0i} \pm \frac{1}{2}\varepsilon_{ijk}\theta^{jk}). \quad (3.60)$$

It is easy to check that the sets of self-dual and anti-self-dual matrices not only commute with one another, respectively, but each forms a closed subalgebra isomorphic to $\mathfrak{su}(2)$. With this in mind the particular expressions for θ_+ and θ_- in (3.56) make sense: $\theta_+ = \theta_+^{0i}e_i$ corresponds to the $\mathfrak{su}(2)$ generated by the self-dual part, while $\theta_- = \theta_-^{0i}e_i$ corresponds to the $\mathfrak{su}(2)$ generated by the anti-self-dual part. Returning to the transformation at the finite group level, the $\text{SO}(4)$ action $x \rightarrow e^{-\theta_+}xe^{\theta_-}$ is specified by a pair of unit quaternions $(e^{-\theta_+}, e^{\theta_-}) \in S^3 \times S^3 \cong \text{SU}(2)^2$, but since the pairs $(e^{-\theta_+}, e^{\theta_-})$ and $(-e^{-\theta_+}, -e^{\theta_-})$ induce the same $\text{SO}(4)$ transformation, $\text{SU}(2)^2$ must be the double cover of $\text{SO}(4)$, i.e. $\text{SU}(2)^2 \cong \text{Spin}(4)$.

Due to non-associativity the octonionic case is a little more complicated, but it leads to some interesting considerations. In any normed division algebra \mathbb{A} , by the defining condition $\|xy\| = \|x\| \|y\|$, multiplying x from the left and right by any unit-norm elements e^α and e^β clearly preserves the norm

$$\|e^\alpha(xe^\beta)\| = \|(e^\alpha x)e^\beta\| = \|e^\alpha\| \|x\| \|e^\beta\| = \|x\|. \quad (3.61)$$

Along with conjugation $x \rightarrow x^*$ this exhausted the symmetries of the norm for $\mathbb{R}, \mathbb{C}, \mathbb{H}$. However, for \mathbb{O} left- and right-multiplication by unit elements e^α and e^β gives a set of symmetries isomorphic (up to a sign) to a pair of 7-spheres, $S^7 \times S^7$, but these do not close to form a group; for the octonions there are symmetries of the norm that arise from non-associativity and hence cannot be written in such a simple form. Since $S^7 \times S^7$ is 14-dimensional and the total unimodular norm-preserving group is $\text{SO}(8)$, with dimension 28, the remaining symmetries must also be 14-dimensional. These are the *automorphisms* of the octonions, forming the exceptional group G_2 , which will be introduced in the following.

3.2.2. Automorphisms and Derivations

The automorphisms $\text{Aut}(\mathbb{A})$ of an algebra \mathbb{A} are the group of linear transformations $g : \mathbb{A} \rightarrow \mathbb{A}$ that preserve multiplication in \mathbb{A} :

$$g(xy) = g(x)g(y), \quad x, y \in \mathbb{A}. \quad (3.62)$$

For the normed division algebra \mathbb{A}_n , the automorphisms leave real elements x_0 invariant, since by linearity

$$g(x_0y) = x_0g(y), \quad x_0 \in \text{Re}(\mathbb{A}_n) \cong \mathbb{R}, \quad (3.63)$$

but by definition

$$g(x_0y) = g(x_0)g(y), \quad (3.64)$$

so that multiplying on the right by $g(y)^{-1}$ gives

$$g(x_0) = x_0. \quad (3.65)$$

This means that for the real numbers $\text{Aut}(\mathbb{R}) \cong \mathbb{1}$ is the trivial group. Furthermore this shows that in a general division algebra automorphisms preserve the norm since the norm is real:

$$g(x^*x) = g(x^*)g(x) = g(x)^*g(x) = x^*x \quad \Leftrightarrow \quad \|g(x)\| = \|x\|. \quad (3.66)$$

In other words $\text{Aut}(\mathbb{A}_n)$ is a subgroup of $O(n)$. More specifically, since the real part of a division algebra element is invariant under automorphisms, $\text{Aut}(\mathbb{A}_n)$ is a subgroup of the $O(n-1)$ group that acts only on the imaginary subspace. This means that in terms of the basis elements e_i the automorphisms take the form $g(e_i) = O_{ij}e_j$ with $O_{ij} \in O(n-1)$, i.e. $O_{ik}O_{jk} = \delta_{ij}$. An automorphism is then just an orthogonal rotation of the basis elements $e_i \rightarrow g(e_i) = O_{ij}e_j$ such that the new set of basis elements $g(e_i)$ satisfy the same multiplication rule (3.16) as the original set:

$$g(e_i)g(e_j) = g(e_ie_j), \quad \Leftrightarrow \quad O_{ik}e_k O_{jl}e_l = -\delta_{ij} + A_{ijk} O_{kl}e_l. \quad (3.67)$$

For the complex numbers there is only one imaginary basis element, $e_1 \sim i$, so $A_{ijk} \equiv 0$ and $\text{Aut}(\mathbb{C}) \cong O(1) \cong \mathbb{Z}_2$ corresponding to the replacement $i \rightarrow -i$; of course $-i$ squares to -1 and hence satisfies all that is required to do the job of i .

When $A_{ijk} \neq 0$ (i.e. when $\mathbb{A}_n = \mathbb{H}, \mathbb{O}$) multiplying (3.67) by two inverse rotations

shows that it is satisfied if and only if

$$O_{i'i}O_{j'j}O_{k'k}A_{i'j'k'} = A_{ijk}. \quad (3.68)$$

For the quaternions $A_{ijk} = \varepsilon_{ijk}$ so the left-hand side is proportional to the determinant of O :

$$O_{i'i}O_{j'j}O_{k'k}\varepsilon_{i'j'k'} = \det(O)\varepsilon_{ijk} = \varepsilon_{ijk}, \quad (3.69)$$

which is satisfied only for $\det(O) = +1$. Thus $\text{Aut}(\mathbb{H}) \cong \text{SO}(3)$. To see how these transformations act on an arbitrary quaternion x , note that the subgroup of $\text{SO}(4)$ matrices that leave the $\text{Re}(\mathbb{H})$ subspace invariant is just those with Lie algebra parameters $\theta^{0i} = 0$. Equations (3.55), (3.56) and (3.57) then give

$$x \rightarrow e^{-\theta}xe^{\theta} \quad \Leftrightarrow \quad \delta x = [x, \theta], \quad \theta \in \text{Im}(\mathbb{H}), \quad (3.70)$$

since $\theta^{0i} = 0$ implies $\theta_- = \theta_+ =: \theta$ in (3.57). This clearly transforms the imaginary components x_i as the adjoint of $\text{SO}(3)$, i.e. as an ordinary spatial 3-vector. Because these transformations are just a special case of the left- and right-multiplications by unit elements discussed above, $\text{Aut}(\mathbb{H})$ gives no new contribution to the symmetries of the quaternionic norm.

For the octonions the structure tensor A_{ijk} becomes C_{ijk} and it is less clear which matrices will satisfy (3.68). Unlike those of the quaternions, the automorphisms of the octonions $\text{Aut}(\mathbb{O}) =: G_2$ give new norm-preserving transformations that cannot be written as simple left- and right-multiplications. But what do these G_2 transformations look like and how does one perform them? This question is easier to answer at the infinitesimal level, where automorphisms become the Lie algebra $\mathfrak{der}(\mathbb{A}) := \mathfrak{aut}(\mathbb{A})$ of *derivations*.

A derivation of an algebra \mathbb{A} is a linear transformation $d : \mathbb{A} \rightarrow \mathbb{A}$ such that

$$d(xy) = xd(y) + d(x)y, \quad x, y \in \mathbb{A}. \quad (3.71)$$

Exponentiating this equation with $g = e^d$ recovers the finite version (3.62). For a division algebra \mathbb{A}_n , writing $O_{ij} = (e^{\theta})_{ij}$ with $\theta_{ij} = -\theta_{ji}$ an arbitrary $\mathfrak{so}(n-1)$ matrix, equation (3.68) becomes

$$\delta A_{ijk} = \theta_{il}A_{ljk} + \theta_{jl}A_{ilk} + \theta_{kl}A_{ijl} = 0 \quad (3.72)$$

for small θ . Thus $\mathfrak{der}(\mathbb{A}_n)$ is the subalgebra of $\mathfrak{so}(n-1)$ consisting of antisymmetric matrices θ_{ij} satisfying (3.72).

For the octonions, $A_{ijk} = C_{ijk}$, so $\mathfrak{der}(\mathbb{O}) \cong \mathfrak{g}_2 \subset \mathfrak{so}(7)$ is the Lie algebra of antisymmetric matrices θ^{ij} such that

$$\delta C_{ijk} = \theta^{il} C_{ljk} + \theta^{jl} C_{ilk} + \theta^{kl} C_{ijl} = 0. \quad (3.73)$$

Contracting this equation with C_{ijm} and using the first identity in (3.40) gives

$$\frac{2}{3} (\delta_{i[k} \delta_{l]j} - \frac{1}{4} Q_{ijkl}) \theta^{kl} = \theta^{ij}. \quad (3.74)$$

Viewing θ^{ij} as a 21-component vector (as an antisymmetric 7×7 matrix it has $\binom{7}{2} = 21$ independent components), this can be interpreted as a projection equation

$$P_{ijkl}^{14} \theta^{kl} = \theta^{ij}, \quad \text{where} \quad P_{ijkl}^{14} := \frac{2}{3} (\delta_{i[k} \delta_{l]j} - \frac{1}{4} Q_{ijkl}), \quad (3.75)$$

where P^{14} is a rank-14 projection operator on the 21-dimensional vector space of antisymmetric 7×7 matrices, satisfying

$$(P^{14})^2 = P^{14} \quad \Leftrightarrow \quad P_{ijkl}^{14} P_{klmn}^{14} = P_{ijmn}^{14}. \quad (3.76)$$

That P^{14} is rank-14 may be seen as follows. As a 21×21 matrix, P^{14} is real and symmetric – $P_{ijkl}^{14} = P_{klij}^{14}$ – and therefore may be diagonalised by an $\text{SO}(21)$ transformation. Because it is a projector its diagonalised entries will each be equal to either 1 or 0. Thus its trace is equal to its rank:

$$\text{rank}(P^{14}) = \text{Tr}(P^{14}) = \delta_{i[k} \delta_{l]j} P_{ijkl}^{14} = \frac{2}{3} \times 21 = 14. \quad (3.77)$$

In other words θ^{ij} *a priori* has 21 components, but P^{14} projects out 7 of these, leaving $14 = \dim[G_2]$.

It is now easy to write down the generators of G_2 in terms of those of $\text{SO}(7)$. The $\text{SO}(7)$ generators $J_{[ij]}$ are labelled³ by the antisymmetric pair of indices $[ij]$ and an $\text{SO}(7)$ transformation with parameters θ^{ij} is enacted by the linear combination $\frac{1}{2} \theta^{ij} J_{[ij]}$. In the defining **7** representation of $\text{SO}(7)$ the generators $J_{[ij]}$ are the 7×7 matrices

$$(J_{[ij]})_{kl} = 2\delta_{i[k} \delta_{l]j}, \quad (3.78)$$

since a 7-vector x_i transforms as

$$\delta x_i = \frac{1}{2} \theta^{kl} (J_{[kl]})_{ij} x_j = \theta^{kl} \delta_{i[k} \delta_{l]j} x_j = \theta^{ij} x_j. \quad (3.79)$$

³I adopt the notation of writing antisymmetric generator labels in square brackets to distinguish them from the generator components.

If the parameters θ^{ij} satisfy equation (3.75) then this is a G_2 transformation. In that case

$$\frac{1}{2}\theta^{ij}J_{[ij]} = \frac{1}{2}\theta^{ij}P_{ijkl}^{\mathbf{14}}J_{[kl]} = \frac{1}{2}\theta^{ij}P_{ijkl}^{\mathbf{14}}P_{klmn}^{\mathbf{14}}J_{[mn]}, \quad (3.80)$$

from which one can see that G_2 has generators $(G_{[ij]})_{kl}$ given by the linear combinations

$$(G_{[ij]})_{kl} := P_{ijmn}^{\mathbf{14}}(J_{[mn]})_{kl} = \frac{4}{3}(\delta_{i[k}\delta_{l]j} - \frac{1}{4}Q_{ijkl}). \quad (3.81)$$

Thus under $G_2 \subset \text{SO}(7)$, the vector transforms as

$$\delta x_i = \frac{1}{2}\theta^{kl}(G_{[kl]})_{ij}x_j = \frac{2}{3}\theta^{kl}(\delta_{i[k}\delta_{l]j} - \frac{1}{4}Q_{ijkl})x_j. \quad (3.82)$$

This is the defining **7** representation of G_2 .

Finally, this can be used to find the explicit form of the derivations of the octonions in terms of the multiplication rule in \mathbb{O} . This makes use of the commutator and the associator, since they satisfy the identities

$$[e_i, e_j] = 2C_{ijk}e_k \quad \text{and} \quad [e_i, e_j, e_k] = 2Q_{ijkl}e_l. \quad (3.83)$$

It is then easy to verify that the following particular combination gives the correct action of the G_2 generators $G_{[ij]}$:

$$\widehat{G}_{[ij]}x := \frac{1}{2}[e_i, e_j, x] - \frac{1}{6}[[e_i, e_j], x] = (G_{[ij]})_{kl}x_l e_k. \quad (3.84)$$

where in general a ‘hat’ denotes a linear operator on \mathbb{O} . This means that any pair of imaginary octonions $y_1, y_2 \in \text{Im}(\mathbb{O})$ generates a unique derivation, since one can simply contract the free indices in equation (3.84) with arbitrary components y_1^i, y_2^j :

$$\widehat{d}_{y_1, y_2}x := \frac{1}{2}[y_1, y_2, x] - \frac{1}{6}[[y_1, y_2], x]. \quad (3.85)$$

Furthermore, since any derivation may be written as a linear combination of the generators $G_{[ij]}$, any derivation must take the form (3.85) for some $y_1, y_2 \in \text{Im}(\mathbb{O})$. This gives another way to see the dimension of G_2 ; it takes $7 + 7 = 14$ parameters to specify the two imaginary octonions required to define a derivation.

Returning to the symmetries of the norm, the full $\mathfrak{so}(8)$ can now be realised in terms of octonions using derivations, combined with left- and right-multiplication by imaginary octonions $\alpha, \beta \in \text{Im}(\mathbb{O})$. This corresponds to the decomposition $\text{SO}(8) \supset G_2$:

$$\mathbf{28} \rightarrow \mathbf{14} + \mathbf{7} + \mathbf{7}, \quad (3.86)$$

with the **14** corresponding to derivations and the two **7**s corresponding to α and β . For an SO(8) transformation parameterised by the antisymmetric matrix $\theta^{ab} \in \mathfrak{so}(8)$:

$$\delta x = \widehat{d}x + \alpha x + x\beta, \quad \Leftrightarrow \quad \delta x_a = \theta^{ab}x_b \quad (3.87)$$

where

$$\widehat{d} = \frac{1}{2}\theta^{ij}\widehat{G}_{[ij]}, \quad \alpha = \left(-\frac{1}{2}\theta^{0i} - \frac{1}{12}C_{ijk}\theta^{jk}\right)e_i, \quad \beta = \left(-\frac{1}{2}\theta^{0i} + \frac{1}{12}C_{ijk}\theta^{jk}\right)e_i. \quad (3.88)$$

Due to non-associativity, exponentiating (3.87) to obtain a finite SO(8) transformation must be approached carefully. One fool-proof way to achieve this is to define operators that multiply from the left and right,

$$\widehat{L}_\alpha x := \alpha x \quad \widehat{R}_\beta x := x\beta, \quad (3.89)$$

so that δx may be written

$$\delta x = (\widehat{d} + \widehat{L}_\alpha + \widehat{R}_\beta)x. \quad (3.90)$$

Now \widehat{L}_α , \widehat{R}_β and \widehat{d} are just linear transformations of $\mathbb{R}^8 \cong \mathbb{O}$, so their composition is by definition associative. This means they may be exponentiated as usual and the finite SO(8) transformation is

$$x \rightarrow e^{\widehat{d} + \widehat{L}_\alpha + \widehat{R}_\beta}x. \quad (3.91)$$

3.2.3. Spin(n) Spinors and Clifford Algebras

So far in the above discussion on SO(n), the group of symmetries of the norm of \mathbb{A}_n , the division algebra elements have transformed in the defining – or ‘vector’ – representation of the group. However, the division algebras have an equally close relationship with the spinor representations of Spin(n) and their associated Clifford algebras. To see this, consider an arbitrary division algebra element $\psi \in \mathbb{A}_n$, and multiply it by the basis elements as follows (for $n > 1$):

$$e_a^*(e_b\psi) + e_b^*(e_a\psi) = (\Gamma_{cd}^a\bar{\Gamma}_{de}^b + \Gamma_{cd}^b\bar{\Gamma}_{de}^a)\psi_e e_c, \quad (3.92)$$

where the structure constants Γ_{bc}^a and $\bar{\Gamma}_{bc}^a$ are defined by the division-algebraic multiplication rules

$$e_a e_b = \Gamma_{bc}^a e_c, \quad e_a^* e_b = \bar{\Gamma}_{bc}^a e_c. \quad (3.93)$$

Moving both sets of brackets on the left-hand side of (3.92) costs a pair of associators, but they cancel with one another due to alternativity:

$$e_a^*(e_b\psi) + e_b^*(e_a\psi) = (e_a^*e_b)\psi + (e_b^*e_a)\psi. \quad (3.94)$$

The final line here can be factorised and then gives a Kronecker delta, since it is proportional to the inner product of the basis elements:

$$(e_a^*e_b + e_b^*e_a)\psi = 2\langle e_a|e_b\rangle\psi = 2\delta_{ab}\psi. \quad (3.95)$$

Comparing (3.92) and (3.95) shows – perhaps surprisingly – that Γ_{bc}^a and $\bar{\Gamma}_{bc}^a$ are actually $n \times n$ ‘gamma matrices’ satisfying the familiar relations

$$\begin{aligned} \Gamma^a\bar{\Gamma}^b + \Gamma^b\bar{\Gamma}^a &= 2\delta^{ab}\mathbf{1}, \\ \bar{\Gamma}^a\Gamma^b + \bar{\Gamma}^b\Gamma^a &= 2\delta^{ab}\mathbf{1} \end{aligned} \quad (3.96)$$

(where the second line comes from the other version of the inner product with $2\langle e_a|e_b\rangle = e_ae_b^* + e_b e_a^*$). This means that the $2n \times 2n$ matrices

$$\gamma^a = \begin{pmatrix} 0 & \Gamma^a \\ \bar{\Gamma}^a & 0 \end{pmatrix} \quad (3.97)$$

satisfy the $\text{Spin}(n)$ Clifford algebra,

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta^{ab}\mathbf{1}, \quad (3.98)$$

and so may be used to form $2n \times 2n$ generators of $\text{Spin}(n)$ by the usual antisymmetrised product

$$\frac{1}{2}\gamma^{[a}\gamma^{b]} = \begin{pmatrix} \frac{1}{2}\Gamma^{[a}\bar{\Gamma}^{b]} & 0 \\ 0 & \frac{1}{2}\bar{\Gamma}^{[a}\Gamma^{b]} \end{pmatrix}. \quad (3.99)$$

However, as these matrices are block-diagonal, this is a reducible representation of $\text{Spin}(n)$; it decomposes into the $n \times n$ irreducible spinor and conjugate spinor representations, with generators [46]

$$\begin{aligned} \Sigma^{[ab]} &:= \frac{1}{2}\Gamma^{[a}\bar{\Gamma}^{b]}, \\ \bar{\Sigma}^{[ab]} &:= \frac{1}{2}\bar{\Gamma}^{[a}\Gamma^{b]}, \end{aligned} \quad (3.100)$$

whose components can be found by explicit computation using the identities (3.40):

$$\begin{aligned}\Sigma_{cd}^{[ab]} &= \delta_{c[a}\delta_{b]d} - \delta_{0[a}A_{b]cd} + \delta_{0[c}A_{d]ab} - \frac{1}{2}Q_{abcd} + 4\delta_{0[c}\delta_{d][a}\delta_{b]0}, \\ \bar{\Sigma}_{cd}^{[ab]} &= \delta_{c[a}\delta_{b]d} + \delta_{0[a}A_{b]cd} + \delta_{0[c}A_{d]ab} - \frac{1}{2}Q_{abcd},\end{aligned}\tag{3.101}$$

where the Q terms vanish for $n \neq 8$.

The above discussion demonstrates that multiplying a division algebra element ψ by the basis element e_a has the effect of multiplying ψ 's components by the gamma matrix $\bar{\Gamma}^a$:

$$e_a\psi = e_a e_b \psi_b = \Gamma_{bc}^a e_c \psi_b = e_c \bar{\Gamma}_{cb}^a \psi_b,\tag{3.102}$$

which means one can represent an irreducible spinor ψ_a and conjugate spinor χ_a of $\text{Spin}(n)$ using division algebra elements ψ and χ , transforming as

$$\begin{aligned}\delta\psi &= \frac{1}{4}\theta^{ab}e_a^*(e_b\psi) = \frac{1}{2}\theta^{ab}\Sigma_{cd}^{[ab]}\psi_d e_c, \\ \delta\chi &= \frac{1}{4}\theta^{ab}e_a(e_b^*\chi) = \frac{1}{2}\theta^{ab}\bar{\Sigma}_{cd}^{[ab]}\chi_d e_c.\end{aligned}\tag{3.103}$$

These representations will be important for the fermions in the supersymmetric theories presented in later chapters.

Looking at equations (3.103) more closely for the division algebras \mathbb{A}_n with $n > 1$:

- For \mathbb{C} , there is only one independent parameter $\theta := \theta^{01} = -\theta^{10}$, so the transformations become simply (writing e_1 as the usual complex unit i):

$$\begin{aligned}\delta\psi &= +\frac{1}{2}i\theta\psi, \\ \delta\chi &= -\frac{1}{2}i\theta\chi.\end{aligned}\tag{3.104}$$

- For \mathbb{H} , associativity means that

$$\begin{aligned}\delta\psi &= \frac{1}{4}\theta^{ab}(e_a^*e_b)\psi = \frac{1}{2}\left(+\theta^{0i} - \frac{1}{2}\varepsilon_{ijk}\theta^{jk}\right)e_i\psi = -\theta_-\psi, \\ \delta\chi &= \frac{1}{4}\theta^{ab}(e_a e_b^*)\chi = \frac{1}{2}\left(-\theta^{0i} - \frac{1}{2}\varepsilon_{ijk}\theta^{jk}\right)e_i\chi = -\theta_+\chi,\end{aligned}\tag{3.105}$$

where θ_+ and θ_- are defined as in (3.57). This shows that the spinor and conjugate spinor transform under the anti-self-dual and self-dual parts of θ^{ab} , respectively. The imaginary quaternions θ_+ and θ_- each generate an $\text{SU}(2)$ factor of the double cover of the norm-preserving group $\text{Spin}(4) \cong \text{SU}(2)^2$.

- For \mathbb{O} , the transformations can once again be written in terms of derivations

and multiplication from the left and right:

$$\begin{aligned}\delta\psi &= \frac{1}{4}\theta^{ab}e_a^*(e_b\psi) = \widehat{d}\psi - \beta\psi + \psi(\alpha - \beta), \\ \delta\chi &= \frac{1}{4}\theta^{ab}e_a(e_b^*\chi) = \widehat{d}\chi + \alpha\chi + \chi(\alpha - \beta),\end{aligned}\tag{3.106}$$

where \widehat{d} , α and β are defined in equations (3.88). The derivations term is common to the transformations of ψ , χ and x in (3.87); it is the terms involving left- and right-multiplication by various combinations of α and β that distinguish the three representations. Another way to say this is that under the subgroup $G_2 \subset \text{Spin}(8)$, the vector $\mathbf{8}_v$, spinor $\mathbf{8}_s$ and conjugate spinor $\mathbf{8}_c$ all decompose in the same way, i.e.

$$\mathbf{8}_v \rightarrow \mathbf{7} + \mathbf{1}, \quad \mathbf{8}_s \rightarrow \mathbf{7} + \mathbf{1}, \quad \mathbf{8}_c \rightarrow \mathbf{7} + \mathbf{1}.\tag{3.107}$$

Note that the singlet $\mathbf{1}$ in each of these decompositions corresponds to the real part, which is invariant under G_2 (so $\widehat{d}x_0 = 0$ for any $x_0 \in \text{Re}(\mathbf{O})$); as far as G_2 is concerned, the $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ are all simply octonions on equal footing.

3.3. Triality

Having seen how to transform the vector, spinor and conjugate-spinor representations of $\text{Spin}(n)$ using division algebras, the obvious next step is to see how the three representations are related to one another. Consider the following octonion a , using the multiplication rule (3.19):

$$a := \chi\psi^* = \chi^b \bar{\Gamma}_{bc}^a \psi^c e_a,\tag{3.108}$$

where ψ is an octonionic spinor and χ is a conjugate-spinor. As a gamma matrix sandwiched between two spinors, this ought to transform as a vector under $\text{Spin}(8)$ (cf. the familiar spinor bilinear $\bar{\chi}\gamma^\mu\psi$ for Lorentz spinors χ, ψ). Indeed one may verify that this is the case using the transformations (3.106). This highlights the most elegant aspect of using division-algebraic representations of $\text{Spin}(n)$: the relationships between the three fundamental representations may be expressed *without* gamma matrices, simply using the multiplication rule. Assuming that ψ and χ have unit norm (since this section is only concerned with their transformation properties),

(3.108) may be written in three equivalent forms,

$$\begin{aligned}
a &= \chi\psi^* = \chi^b\psi^c\bar{\Gamma}_{bc}^a e_a = (\chi^\top\bar{\Gamma}^a\psi)e_a, \\
\Leftrightarrow \psi &= a^*\chi = a^a\chi^b\bar{\Gamma}_{bc}^a e_c = (\chi^\top\phi)_a e_a, \\
\Leftrightarrow \chi &= a\psi = a^a\psi^c\bar{\Gamma}_{bc}^a e_b = (\phi\psi)_a e_a,
\end{aligned} \tag{3.109}$$

where $\phi := \bar{\Gamma}^b a_b$. This further demonstrates the intimate relationship between division-algebraic multiplication and $\text{Spin}(n)$.

As well as transforming covariantly under $\text{Spin}(n)$, for the associative division algebras, equations (3.109) are also invariant under (right-)multiplication of the spinors by an arbitrary unit-norm element $u \in \mathbb{A}_n$:

$$\psi \rightarrow \psi u, \quad \chi \rightarrow \chi u, \quad u^* u = 1. \tag{3.110}$$

Writing $u = e^{\theta_I}$ in the complex and quaternionic cases this becomes

$$\delta\psi = \psi\theta_I, \quad \delta\chi = \chi\theta_I, \quad \theta_I \in \text{Im}(\mathbb{A}_n), \quad \mathbb{A}_n = \mathbb{C}, \mathbb{H}, \tag{3.111}$$

for small θ_I . At the Lie algebra level this means that equations (3.109) have an extra $\mathfrak{u}(1)$ and $\mathfrak{su}(2)$ symmetry for $\mathbb{A}_n = \mathbb{C}, \mathbb{H}$, respectively, which can be labelled as $\mathfrak{r}(\mathbb{A}_n) = \emptyset, \mathfrak{u}(1), \mathfrak{su}(2), \emptyset$, for $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. The total Lie algebra $\mathfrak{so}(n) \oplus \mathfrak{r}(\mathbb{A}_n)$ combining $\mathfrak{so}(n)$ with these extra symmetries is called the *trality algebra* $\mathfrak{tri}(\mathbb{A}_n)$:

$$\begin{aligned}
\mathfrak{tri}(\mathbb{R}) &= \emptyset, \\
\mathfrak{tri}(\mathbb{C}) &= \mathfrak{u}(1) \oplus \mathfrak{u}(1), \\
\mathfrak{tri}(\mathbb{H}) &= \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2), \\
\mathfrak{tri}(\mathbb{O}) &= \mathfrak{so}(8).
\end{aligned} \tag{3.112}$$

Intuitively speaking, it makes sense that there is no extra symmetry in the octonionic case, since a , ψ and χ each have 8 real components and are already transforming as the irreducible vector $\mathbf{8}_v$, spinor $\mathbf{8}_s$ and conjugate-spinor $\mathbf{8}_c$ representations of $\mathfrak{so}(8)$; there is ‘no room’ left for them to transform under any further group action. In contrast, for the quaternions, the vector a transforms as the $(\mathbf{2}, \mathbf{2})$ representation of $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, while the quaternionic spinors ψ and χ in (3.105) each only transform as a $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$. Since ψ and χ have 4 real components each, there is room for them to transform also as a $\mathbf{2}$ under the third $\mathfrak{su}(2)$ generated by θ_I . Thus the three representations transform under $\mathfrak{tri}(\mathbb{H}) = \mathfrak{so}(4) \oplus \mathfrak{su}(2)$, isomorphic

to $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, as

$$a \sim (\mathbf{2}, \mathbf{2}, \mathbf{1}), \quad \psi \sim (\mathbf{2}, \mathbf{1}, \mathbf{2}), \quad \chi \sim (\mathbf{1}, \mathbf{2}, \mathbf{2}), \quad (3.113)$$

where the three slots in order correspond to θ_+ , θ_- and θ_I .

In the mathematics literature, the triality algebras are usually introduced via a more elegant definition, designed so as to generalise the algebra of derivations. This is given as follows: $\mathfrak{tri}(\mathbb{A}_n)$ is the algebra of triples $(\widehat{A}, \widehat{B}, \widehat{C}) \in \mathfrak{so}(n) \oplus \mathfrak{so}(n) \oplus \mathfrak{so}(n)$ such that

$$\widehat{A}(xy) = x(\widehat{B}y) + (\widehat{C}x)y, \quad x, y \in \mathbb{A}_n. \quad (3.114)$$

The analogy with derivations is very clear – compare (3.114) with (3.71). Note that \widehat{A} , \widehat{B} and \widehat{C} each belong to a *different* copy of $\mathfrak{so}(n)$, and that each transformation is taken *a priori* to be in the vector representation of each of the three copies. Hence, in terms of three sets of antisymmetric parameters θ_A^{ab} , θ_B^{ab} and θ_C^{ab} , equation (3.114) becomes

$$\theta_A^{ab}(xy)_b e_a = x \theta_B^{ab} y_b e_a + \theta_C^{ab} x_b e_a y. \quad (3.115)$$

Using the multiplication rule and factoring out the arbitrary coefficients x_a and y_a gives

$$\theta_A^{ad} \bar{\Gamma}_{bc}^d + \theta_B^{bd} \bar{\Gamma}_{dc}^a + \theta_C^{cd} \bar{\Gamma}_{bd}^a = 0. \quad (3.116)$$

Thus the triality algebras represent the overall symmetries of the structure constants $\bar{\Gamma}_{bc}^a$ under independent rotations of each of their indices, just as derivations are the orthogonal transformations preserving C_{ijk} – see (3.72). Equation (3.116) relates the parameters θ_A^{ab} , θ_B^{ab} and θ_C^{ab} to one another, breaking $\mathfrak{so}(n) \oplus \mathfrak{so}(n) \oplus \mathfrak{so}(n)$ to its subalgebra $\mathfrak{tri}(\mathbb{A}_n)$, as given in (3.112).

To see how this works in terms of division-algebraic multiplication, consider the octonionic case. Using (3.87) gives

$$\begin{aligned} \widehat{A}(xy) &= \widehat{d}_A(xy) + \alpha_A(xy) + (xy)\beta_A, \\ \widehat{B}y &= \widehat{d}_B y + \alpha_B y + y\beta_B, \\ \widehat{C}x &= \widehat{d}_C x + \alpha_C x + x\beta_C, \end{aligned} \quad (3.117)$$

for three independent derivations d_A , d_B and d_C and six independent imaginary parameters $\alpha_A, \alpha_B, \alpha_C, \beta_A, \beta_B, \beta_C \in \text{Im}(\mathbb{O})$. Temporarily setting all but the derivations to zero (3.114) becomes

$$\widehat{d}_A(xy) = x(\widehat{d}_B y) + (\widehat{d}_C x)y, \quad (3.118)$$

but by definition any derivation must satisfy the Leibniz rule (3.71) and one must identify $\widehat{d}_A = \widehat{d}_B = \widehat{d}_C$; the subscript can then be dropped and all three of them simply written as \widehat{d} . Temporarily setting $\widehat{d} = 0$ and substituting the transformations (3.117) into (3.114) gives

$$\alpha_A(xy) + (xy)\beta_A = x(\alpha_B y + y\beta_B) + (\alpha_C x + x\beta_C)y, \quad (3.119)$$

but by alternativity,

$$\begin{aligned} \alpha(xy) &= (\alpha x)y + (x\alpha)y - x(\alpha y), \\ (xy)\beta &= x(y\beta) + x(\beta y) - (x\beta)y, \end{aligned} \quad (3.120)$$

for any $\alpha, \beta, x, y \in \mathbb{O}$. This means $\alpha_B, \alpha_C, \beta_B, \beta_C$ may be eliminated, leaving only α_A and β_A :

$$\begin{aligned} \alpha_B &= -(\alpha_A - \beta_A), & \beta_B &= \beta_A, \\ \alpha_C &= \alpha_A, & \beta_C &= \alpha_A - \beta_A. \end{aligned} \quad (3.121)$$

Dropping the A subscripts, the triple of transformations then becomes

$$\begin{aligned} \widehat{A}(xy) &= \widehat{d}(xy) + \alpha(xy) + (xy)\beta, \\ \widehat{B}y &= \widehat{d}y - (\alpha - \beta)y + y\beta, \\ \widehat{C}x &= \widehat{d}x + \alpha x + x(\alpha - \beta). \end{aligned} \quad (3.122)$$

In fact, the argument above works for *any* of the division algebras, since orthogonal transformations can always be written in the form (3.117). Thus the triality triple $(\widehat{A}, \widehat{B}, \widehat{C})$ is determined entirely by a derivation \widehat{d} and two imaginary parameters α and β as in (3.122), i.e.

$$\mathbf{tri}(\mathbb{A}_n) \cong \mathfrak{der}(\mathbb{A}_n) + \text{Im}(\mathbb{A}_n) + \text{Im}(\mathbb{A}_n). \quad (3.123)$$

It is easy to see that this leads to the algebras listed in (3.112) (note that derivations are trivial for \mathbb{R} and \mathbb{C}).

To see how this relates to the discussion of $\text{Spin}(n)$ representations above, simply relabel $x = \chi$ and $y = \psi^*$ in (3.114) so that that $xy = \chi\psi^* =: a -$ at this point these

are just labels. Then the transformations (3.122) become

$$\begin{aligned}\delta a &= \widehat{A}a = \widehat{d}a + \alpha a + a\beta, \\ \delta\psi^* &= \widehat{B}\psi^* = \left(\widehat{d}\psi - \beta\psi + \psi(\alpha - \beta)\right)^*, \\ \delta\chi &= \widehat{C}\chi = \widehat{d}\chi + \alpha\chi + \chi(\alpha - \beta).\end{aligned}\tag{3.124}$$

In the octonionic case these are exactly the Spin(8) transformations of the vector a , spinor ψ and conjugate-spinor χ representations, as seen in equations (3.87) and (3.106). The statement of triality (3.114) then becomes an identity:

$$\delta a = \delta(\chi\psi^*) = \delta\chi\psi^* + \chi\delta\psi^*.\tag{3.125}$$

In component form, this is equivalent to saying that (3.116) is solved by

$$\theta_A^{ab} = \theta^{ab}, \quad \theta_B^{ab} = \frac{1}{2}\theta^{cd}\Sigma_{ab}^{[cd]}, \quad \theta_C^{ab} = \frac{1}{2}\theta^{cd}\bar{\Sigma}_{ab}^{[cd]},\tag{3.126}$$

where Σ and $\bar{\Sigma}$ are the spinor and conjugate-spinor generators defined in (3.101). This picks out a single diagonal $\mathfrak{so}(8)$ subalgebra in $\mathfrak{so}(8) \oplus \mathfrak{so}(8) \oplus \mathfrak{so}(8)$, relying on the fact that the Lie algebra elements transforming the vector a_a , spinor ψ_a and conjugate-spinor χ_a are all just 8×8 (real) antisymmetric matrices.

Similarly for the quaternions any derivation is generated by a commutator with an imaginary parameter $\theta \in \text{Im}(\mathbb{H})$, so

$$\begin{aligned}\widehat{A}a &= a\theta - \theta a + \alpha a + a\beta, \\ \widehat{B}\psi^* &= (\psi\theta - \theta\psi - \beta\psi + \psi(\alpha - \beta))^*, \\ \widehat{C}\chi &= \chi\theta - \theta\chi + \alpha\chi + \chi(\alpha - \beta).\end{aligned}\tag{3.127}$$

Identifying

$$\theta - \alpha = \theta_+, \quad \theta + \beta = \theta_-, \quad \theta + \alpha - \beta = \theta_1\tag{3.128}$$

recovers the usual transformations of a , ψ and χ preserving $a = \chi\psi^*$, as outlined in the beginning of this section. A similar argument works for the complex numbers, while the triality algebra of the reals is empty.

The triality algebras are so named because they each have a discrete symmetry – called triality – isomorphic to the group S_3 of permutations on three objects. For $\mathbb{A}_n = \mathbb{H}, \mathbb{O}$, the first hint of these outer automorphisms⁴ is in the manifest symmetry

⁴Group automorphisms are linear mappings A of a group G onto itself such that the multiplication rule is preserved: $A(g_1g_2) = A(g_1)A(g_2)$, $g_1, g_2 \in G$. Outer automorphisms are those that are non-trivial in the sense that $A(g) \neq hgh^{-1}$ for any $h \in G$. The same concept applies also at the Lie algebra level.

of the $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{so}(8)$ Dynkin diagrams in Fig. 3.2. Furthermore, for any \mathbb{A}_n the relationship $a = \chi\psi^*$ is invariant under the following representation of S_3 :

$$\begin{aligned} & \begin{pmatrix} a & \psi & \chi^* \\ a & \psi & \chi^* \end{pmatrix}, \quad \begin{pmatrix} a & \psi & \chi^* \\ \psi & \chi^* & a \end{pmatrix}, \quad \begin{pmatrix} a & \psi & \chi^* \\ \chi^* & a & \psi \end{pmatrix}, \\ & \begin{pmatrix} a & \psi & \chi^* \\ \chi & \psi^* & a^* \end{pmatrix}, \quad \begin{pmatrix} a & \psi & \chi^* \\ \psi^* & a^* & \chi \end{pmatrix}, \quad \begin{pmatrix} a & \psi & \chi^* \\ a^* & \chi & \psi^* \end{pmatrix}, \end{aligned} \quad (3.129)$$

since these permutations simply result in six equivalent versions of the same relation

$$\begin{aligned} & a = (\chi^*)^* \psi^*, \quad \psi = a^* (\chi^*)^*, \quad \chi^* = \psi^* a^* \\ \Leftrightarrow & a^* = \psi \chi^*, \quad \psi^* = \chi^* a, \quad \chi = a \psi. \end{aligned} \quad (3.130)$$

Note that the cyclic permutations involve a , ψ and χ^* (rather than a , ψ and χ) and that odd permutations must be accompanied by conjugating all three of these division algebra elements.

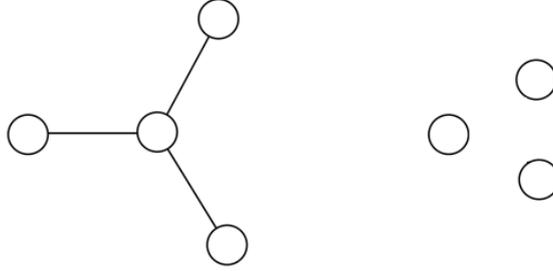


Figure 3.2.: The Dynkin diagrams for the Lie algebras $\mathfrak{so}(8)$ (left) and $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ (right).

Clearly the three representations a , ψ and χ^* of $\mathfrak{tri}(\mathbb{A}_n)$ are isomorphic, since their transformations

$$\begin{aligned} \delta a &= \widehat{d}a + \alpha a + a\beta, \\ \delta \psi &= \widehat{d}\psi - \beta\psi + \psi(\alpha - \beta), \\ \delta \chi^* &= \widehat{d}\chi^* - (\alpha - \beta)\chi^* - \chi^*\alpha, \end{aligned} \quad (3.131)$$

each involve a derivation, a left-multiplication by an element of $\text{Im}(\mathbb{A}_n)$ and a right-multiplication by an element of $\text{Im}(\mathbb{A}_n)$. Thus exchanging the arbitrary parameters

$$\alpha \rightarrow -\beta, \quad \beta \rightarrow \alpha - \beta \quad \Rightarrow \quad \alpha - \beta \rightarrow -\alpha, \quad (3.132)$$

results in

$$\begin{aligned}
\delta a &= \widehat{d}a - \beta a + a(\alpha - \beta), \\
\delta \psi &= \widehat{d}\psi - (\alpha - \beta)\psi - \psi\alpha, \\
\delta \chi^* &= \widehat{d}\chi^* + \alpha\chi^* + \chi^*\beta,
\end{aligned} \tag{3.133}$$

so that from this perspective the *vector* can be seen as a *spinor*, while the spinor looks like a conjugate-spinor and the conjugate-spinor like a vector. The exchange of parameters (3.132) simply corresponds to the order-two permutation

$$\vartheta := \begin{pmatrix} a & \psi & \chi^* \\ \chi^* & a & \psi \end{pmatrix}, \tag{3.134}$$

from (3.129). The other permutations in (3.129) can be obtained by similar parameter redefinitions, resulting in the full S_3 group of outer automorphisms. Explicitly, letting

$$\alpha \rightarrow \alpha, \quad \beta \rightarrow \alpha - \beta \quad \Rightarrow \quad \alpha - \beta \rightarrow \beta \tag{3.135}$$

results in the order-one *odd* permutation

$$\varrho := \begin{pmatrix} a & \psi & \chi^* \\ \chi & \psi^* & a^* \end{pmatrix} \tag{3.136}$$

and the six elements of S_3 are then $\{\mathbf{1}, \vartheta, \vartheta^2, \varrho, \varrho\vartheta, \varrho\vartheta^2\}$.

As an aside, when constructing the exceptional groups it will be useful to define the action of the particular elements ϑ and ϱ above in terms of the triality transformation triple $(\widehat{A}, \widehat{B}, \widehat{C})$ in (3.122). The transformations (3.131) can be written

$$\delta a = \widehat{A}a, \quad \delta \psi^* = \widehat{B}\psi^*, \quad \delta \chi = \widehat{C}\chi, \tag{3.137}$$

whereas the ϑ -permuted transformations (3.133) are

$$\delta a = (\widehat{B}a^*)^*, \quad \delta \psi^* = \widehat{C}\psi^*, \quad \delta \chi = (\widehat{A}\chi^*)^*. \tag{3.138}$$

Thus the action of ϑ is equivalent to

$$\vartheta : (\widehat{A}, \widehat{B}, \widehat{C}) \mapsto (\widehat{B}^*, \widehat{C}, \widehat{A}^*), \tag{3.139}$$

where the \star of an operator is defined as

$$\widehat{O}^*x := (\widehat{O}x^*)^*, \quad \widehat{O} \in \mathfrak{so}(n), \quad x \in \mathbb{A}. \tag{3.140}$$

By similar logic, one may also easily derive that the action of ϱ is equivalent to

$$\varrho : (\widehat{A}, \widehat{B}, \widehat{C}) \mapsto (\widehat{C}, \widehat{B}^*, \widehat{A}). \quad (3.141)$$

Both ϑ and ϱ will be put to use in Chapter 5.

The triality algebras find utility in supergravity mainly through their relationship with the exceptional groups and the magic square. Most importantly, the discrete triality symmetry associated with the octonions is one of the fundamental ingredients that allows the exceptional groups F_4 , E_6 , E_7 and E_8 to exist, as will be shown in Chapter 5. The triality algebras are also the symmetries of $\mathcal{N} = 1$ super Yang-Mills theories (on-shell) in $D = 3, 4, 6, 10$. The various Lie algebras associated with the division algebras that have appeared so far are summarised in Table 3.2.

| \mathbb{A}_n | $\mathfrak{so}(n)$ | $\mathfrak{der}(\mathbb{A}_n)$ | $\mathfrak{tri}(\mathbb{A}_n)$ | $\mathfrak{er}(\mathbb{A}_n) :=$ $\mathfrak{tri}(\mathbb{A}_n) \ominus \mathfrak{so}(n)$ |
|----------------|--------------------|--------------------------------|--|---|
| \mathbb{R} | — | — | — | — |
| \mathbb{C} | $\mathfrak{so}(2)$ | — | $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$ | $\mathfrak{u}(1)$ |
| \mathbb{H} | $\mathfrak{so}(4)$ | $\mathfrak{so}(3)$ | $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ | $\mathfrak{su}(2)$ |
| \mathbb{O} | $\mathfrak{so}(8)$ | \mathfrak{g}_2 | $\mathfrak{so}(8)$ | — |

Table 3.2.: Various Lie algebras associated with the division algebras. The ‘extra’ algebras $\mathfrak{er}(\mathbb{A}_n) := \mathfrak{tri}(\mathbb{A}_n) \ominus \mathfrak{so}(n)$ are the subalgebras of $\mathfrak{tri}(\mathbb{A}_n)$ that commute with $\mathfrak{so}(n)$ (for $\mathbb{A}_n = \mathbb{H}, \mathbb{O}$ there are of course three ways to embed the $\mathfrak{so}(n)$ into $\mathfrak{tri}(\mathbb{A}_n)$, but they are equivalent up to discrete triality transformations).

3.4. Division Algebras and Simple Lie Algebras

Above and beyond the action of the norm-preserving and triality algebras on $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and \mathbb{O} , the division algebras have a close relationship with *all* simple Lie groups and their Lie algebras. Consider the Lie group $A(N, \mathbb{A})$ consisting of $N \times N$ ‘unitary’ matrices with entries in $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$:

$$A(N, \mathbb{A}) := \{X \in \mathbb{A}[N] \mid X^\dagger X = X X^\dagger = \mathbf{1}\}, \quad (3.142)$$

where $\mathbb{A}[N]$ denotes the set of $N \times N$ matrices with entries in \mathbb{A} and x^\dagger is the conjugate-transpose $(x^\top)^*$. Its Lie algebra is made up of $N \times N$ anti-Hermitian

matrices with entries in \mathbb{A} :

$$\mathfrak{a}(N, \mathbb{A}) := \{x \in \mathbb{A}[N] \mid x^\dagger = -x\}, \quad (3.143)$$

The Lie brackets are taken using ordinary matrix commutation, $[x, y] = xy - yx$, and despite the non-commutativity of the quaternions, the commutators satisfy the required properties: bilinearity, antisymmetry and the Jacobi identity. In fact, for the three different division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, this definition encompasses all of the classical (compact) Lie algebras [47]

$$\mathfrak{a}(N, \mathbb{R}) \cong \mathfrak{so}(N), \quad \mathfrak{a}(N, \mathbb{C}) \cong \mathfrak{u}(N), \quad \mathfrak{a}(N, \mathbb{H}) \cong \mathfrak{sp}(N). \quad (3.144)$$

Aside from these infinite classical families, there are only five other simple Lie algebras in existence (up to different real forms): those of the exceptional groups G_2, F_4, E_6, E_7 and E_8 . The first of these G_2 was introduced earlier as the automorphism group of \mathbb{O} , so one might guess that the other four exceptional groups are also somehow related to the octonions. Indeed this turns out to be the case; just as the octonions themselves hold an exceptional status as the largest and only non-associative division algebra, their existence gives rise to the five exceptional groups. These will be introduced in Chapter 5.

3.5. Complex and Quaternionic Structures

An important concept for understanding the symmetries of super Yang-Mills theories and the magic pyramid of supergravities is the idea of a complex structure, and its generalisation to a quaternionic structure.

A complex structure defines an embedding of \mathbb{C}^N into \mathbb{R}^{2N} , where N is any natural number. This defines a corresponding embedding of $\mathbb{C}[N]$ into $\mathbb{R}[2N]$. Consider a vector $v \in \mathbb{R}^{2N}$. Writing $v = (x, y)$, with $x, y \in \mathbb{R}^N$, one can identify x and y with the real and imaginary parts of a complex vector $x + iy \in \mathbb{C}^N$. Multiplying v by the real $2N \times 2N$ matrix

$$E := \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \in \mathbb{R}[2N], \quad (3.145)$$

which clearly satisfies $E^2 = -\mathbf{1}$, is equivalent to multiplying $x + iy$ by i :

$$\begin{aligned} (x + iy) &\rightarrow i(x + iy) = -y + ix \\ \Leftrightarrow v &\rightarrow Ev = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}. \end{aligned} \quad (3.146)$$

For an arbitrary real $2N \times 2N$ matrix $M \in \mathbb{R}[2N]$ acting on \mathbb{R}^{2N} to behave like a complex $N \times N$ matrix acting on \mathbb{C}^N , it must commute with the complex structure matrix E . This is the purely real version of the statement that any complex matrix commutes with multiplication by i . A quick calculation shows that if M commutes with E then it must be of the form

$$M = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \quad A, B \in \mathbb{R}[N]. \quad (3.147)$$

In this case, acting on v with M is equivalent to acting on $x + iy$ with the matrix $A + iB \in \mathbb{C}[N]$

$$\begin{aligned} (x + iy) &\rightarrow (A + iB)(x + iy) = (Ax - By) + i(Bx + Ay) \\ \Leftrightarrow v \rightarrow Mv &= \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax - By \\ Bx + Ay \end{pmatrix}. \end{aligned} \quad (3.148)$$

Thus in general there is an algebra isomorphism

$$\mathbb{C}[N] \cong \left\{ M \in \mathbb{R}[2N] \mid [M, E] = 0, E^2 = -\mathbf{1} \right\}. \quad (3.149)$$

In particular, for the Lie algebra of real antisymmetric $2N \times 2N$ matrices $\mathfrak{so}(2N)$, this restricts to

$$\mathfrak{u}(N) \cong \left\{ u \in \mathfrak{so}(2N) \mid [u, E] = 0, E^2 = -\mathbf{1}, E \in \mathfrak{so}(2N) \right\}, \quad (3.150)$$

where the commuting $\mathfrak{u}(1)$ part of $\mathfrak{u}(N) \cong \mathfrak{su}(N) \oplus \mathfrak{u}(1)$ is generated by the complex structure E itself.

It is straightforward to generalise this notion to that of a quaternionic structure, which embeds $\mathbb{H}[N]$ into $\mathbb{R}[4N]$. This is simply a pair of $4N \times 4N$ complex structures E_1 and E_2 that anti-commute with one another:

$$E_1^2 = E_2^2 = -\mathbf{1}, \quad E_1 E_2 = -E_2 E_1 =: E_3. \quad (3.151)$$

The three matrices E_1 , E_2 and E_3 then mimic the algebra of the quaternionic basis elements i , j and k , and it is easy to prove that there is an algebra isomorphism

$$\mathbb{H}[N] \cong \left\{ M \in \mathbb{R}[4N] \mid [M, E_1] = [M, E_2] = 0 \right\}. \quad (3.152)$$

This time the requirement that M commutes with E_1 and E_2 (and hence also E_3) is the real version of the statement that left- and right-multiplication in \mathbb{H} commute;

E_1, E_2 and E_3 represent right-multiplication of a quaternionic vector in \mathbb{H}^N by i, j and k , which commutes with left-multiplying the vector by a quaternionic matrix⁵.

Similarly to the complex case, for the Lie algebra of real antisymmetric $4N \times 4N$ matrices $\mathfrak{so}(4N)$, the isomorphism (3.152) restricts to

$$\mathfrak{sp}(N) \cong \left\{ u \in \mathfrak{so}(4N) \mid [u, E_1] = [u, E_2] = 0, E_1, E_2 \in \mathfrak{so}(4N) \right\}. \quad (3.153)$$

The three quaternionic structure matrices E_1, E_2 and E_3 themselves generate a copy of $\mathfrak{sp}(1)$ that commutes with the $\mathfrak{sp}(N)$ above, corresponding to right-multiplication by an imaginary quaternion.

There is no such straightforward generalisation to an ‘octonionic structure’, since ordinary associative matrix multiplication can only represent the associative division algebras.

⁵Note that E_1, E_2 and E_3 may alternatively correspond to left-multiplication if the quaternionic matrix is thought of as quaternionic-multiplying on the right, e.g. if the matrix $\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}$ with

$q_1, q_2, q_3, q_4 \in \mathbb{H}$ is taken to act like the operator $\begin{pmatrix} \widehat{R}_{q_1} & \widehat{R}_{q_2} \\ \widehat{R}_{q_3} & \widehat{R}_{q_4} \end{pmatrix}$.

4. Super Yang-Mills, Division Algebras and Triality

Over the years the relationship between supersymmetry and the division algebras has been a recurring theme. See, for example, [24, 27, 28, 31, 32, 48]. In particular, owing to the Lie algebra isomorphism

$$\mathfrak{so}(1, 1 + n) \cong \mathfrak{sl}(2, \mathbb{A}_n), \quad n = \dim \mathbb{A}_n = 1, 2, 4, 8, \quad (4.1)$$

the normed division algebras $\mathbb{A}_n = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ provide a natural framework for describing relativistic physics in $D = n + 2 = 3, 4, 6$ and 10 spacetime dimensions, which will be introduced in Section 4.1. Furthermore, the classical Green-Schwarz superstring and $\mathcal{N} = 1$ super Yang-Mills (SYM) theories of a single vector and spinor can exist *only* in these dimensions [15]. In Section 4.2 this division-algebraic language is used to give a unified formulation of these $\mathcal{N} = 1$ SYM theories. A version using division-algebraic auxiliary fields is also given so that the supersymmetry algebra closes off-shell¹ in $D = 3, 4, 6$, while for $D = 10$ the failure to close is explicitly demonstrated to be a result of the non-associativity of the octonions, as hinted at in [49].

Finally in Section 4.3 these theories are dimensionally reduced to give a unified division-algebraic description of SYM theories with $(D = 3, \mathcal{N} = 1, 2, 4, 8)$, $(D = 4, \mathcal{N} = 1, 2, 4)$, $(D = 5, \mathcal{N} = 1, 2)$, $(D = 6, (\mathcal{N}_+ \mathcal{N}_-) = (1, 0), (1, 1))$ and $(D = 10, 9, 8, 7, \mathcal{N} = 1)$. In particular the maximally supersymmetric theories in each dimension, descended from $D = 10$, are formulated over the octonions and thus the failure of the supersymmetry algebra to close in these cases is attributed to the non-associativity of \mathbb{O} . Consistent truncation to theories with fewer supersymmetries then corresponds simply to finding \mathbb{H}, \mathbb{C} and \mathbb{R} subalgebras of \mathbb{O} . There is a single ‘master Lagrangian’ for all of these theories with a single unified set of supersymmetry transformation rules.

¹Recall from Chapter 2 that the $\{Q, \bar{Q}\}$ anti-commutator only ‘closes’ to produce a translation after the fermionic equation of motion $\not{D}\lambda = 0$ has been applied, i.e. on-shell. Using unphysical auxiliary fields the algebra can be made to close *off-shell*.

The important role of triality algebras as the symmetries of the on-shell degrees of freedom of $\mathcal{N} = 1$ SYM theories will also be discussed in Section 4.3. By defining a new algebra \mathfrak{int} in terms of the usual triality algebras, the on-shell symmetries for super Yang-Mills with any (D, \mathcal{N}) as above are summarised in a single formula.

4.1. Spacetime Fields in $D = n + 2$

The isomorphism (4.1) holds for the same reasoning commonly used in $D = 4$ for $\mathfrak{so}(1, 3) \cong \mathfrak{sl}(2, \mathbb{C})$. In dimension $D = n + 2$ with $n = 1, 2, 4, 8$, a vector X^μ in Minkowski spacetime can be represented by the components:

$$X^\mu = (X^0, X^1, \dots, X^n, X^{n+1}) =: (t, X^{a+1}, z), \quad a = 0, 1, \dots, (n-1). \quad (4.2)$$

However, the vector can just as well be parametrised by $X \in \mathfrak{h}_2(\mathbb{A}_n)$, where $\mathfrak{h}_2(\mathbb{A}_n)$ is the set of 2×2 Hermitian matrices [24, 50, 51, 48, 31] with entries in the division algebra \mathbb{A}_n :

$$X = \begin{pmatrix} t + z & x^* \\ x & t - z \end{pmatrix} \quad \text{where } t, z \in \mathbb{R} \text{ and } x = X^{a+1}e_a \in \mathbb{A}_n. \quad (4.3)$$

Then the determinant of the matrix is the Minkowski metric for D -dimensional spacetime:

$$-\det X = -t^2 + z^2 + |x|^2. \quad (4.4)$$

The group of determinant-preserving transformations $\text{SL}(2, \mathbb{A}_n)$ must then be equivalent to the group of Lorentz transformations, although care is needed to define elements of $\text{SL}(2, \mathbb{A}_n)$ and its Lie algebra, due to the general non-commutativity and non-associativity of \mathbb{A}_n .

In $D = 4$ the Pauli matrices $\{\bar{\sigma}_\mu\}$ are used as a basis for Hermitian matrices, so that $X = X^\mu \bar{\sigma}_\mu$. This suggests a generalised set of Pauli matrices for $\mu = 0, 1, \dots, (n+1)$. The straightforward generalisation of the usual Pauli matrices to all four normed division algebras is the linearly-independent basis [24, 51, 29]

$$\begin{aligned} \bar{\sigma}_\mu &= \sigma^\mu = (+\mathbb{1}, \sigma_{a+1}, \sigma_{n+1}), \\ \text{or } \sigma_\mu &= \bar{\sigma}^\mu = (-\mathbb{1}, \sigma_{a+1}, \sigma_{n+1}), \end{aligned} \quad (4.5)$$

where

$$\sigma_{a+1} := \begin{pmatrix} 0 & e_a^* \\ e_a & 0 \end{pmatrix}, \quad \sigma_{n+1} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.6)$$

The notation is chosen so that in $D = 4$ (where $n = 2$ and $\mathbb{A}_2 = \mathbb{C}$) the matrices reduce to the usual Pauli basis:

$$\sigma_1 = \begin{pmatrix} 0 & e_0^* \\ e_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & e_1^* \\ e_1 & 0 \end{pmatrix} =: \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.7)$$

It is easy to check that the generalised Pauli matrices satisfy

$$\begin{aligned} \sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu &= 2\eta^{\mu\nu} \mathbf{1}, \\ \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu &= 2\eta^{\mu\nu} \mathbf{1}. \end{aligned} \quad (4.8)$$

This means they can be used to generate Lorentz transformations. To see this, consider a vector field A_μ transforming under an infinitesimal Lorentz transformation as

$$\delta A_\mu = \lambda_\mu{}^\nu A_\nu, \quad (4.9)$$

with $\lambda_\mu{}^\rho \eta_{\rho\nu} = \lambda_{\mu\nu} = -\lambda_{\nu\mu}$. The vector may be contracted with the Pauli matrices σ^μ or $\bar{\sigma}^\mu$ to give Hermitian matrices

$$\begin{aligned} A &= A_\mu \sigma^\mu = \begin{pmatrix} -A^0 + A^{n+1} & A^{a+1} e_a^* \\ A^{a+1} e_a & -A^0 - A^{n+1} \end{pmatrix}, \\ \bar{A} &= A_\mu \bar{\sigma}^\mu = \begin{pmatrix} +A^0 + A^{n+1} & A^{a+1} e_a^* \\ A^{a+1} e_a & +A^0 - A^{n+1} \end{pmatrix} = A - (\text{Tr } A) \mathbf{1}. \end{aligned} \quad (4.10)$$

Then A transforms under Lorentz transformations as

$$\delta A = \frac{1}{4} \lambda^{\mu\nu} \left(\sigma_\mu (\bar{\sigma}_\nu A) - A (\bar{\sigma}_\mu \sigma_\nu) \right). \quad (4.11)$$

Of course the positioning of the brackets above is only important for the octonions. To see that equations (4.9) and (4.11) are equivalent, one may add and subtract a term in (4.11):

$$\begin{aligned} \delta A &= \frac{1}{4} \lambda^{\mu\nu} \left(\sigma_\mu (\bar{\sigma}_\nu A) - A (\bar{\sigma}_\mu \sigma_\nu) + \sigma_\mu (\bar{A} \sigma_\nu) - \sigma_\mu (\bar{A} \sigma_\nu) \right) \\ &= \frac{1}{4} \lambda^{\mu\nu} \left(\sigma_\mu (\bar{\sigma}_\nu A + \bar{A} \sigma_\nu) - (A \bar{\sigma}_\mu + \sigma_\mu \bar{A}) \sigma_\nu \right), \end{aligned} \quad (4.12)$$

where the second line follows from alternativity. Then since

$$\begin{aligned} \sigma^\mu \bar{A} + A \bar{\sigma}^\mu &= A_\nu (\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu) = 2\eta^{\mu\nu} A_\nu \mathbf{1}, \\ \bar{\sigma}^\mu A + \bar{A} \sigma_\mu &= A_\nu (\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu) = 2\eta^{\mu\nu} A_\nu \mathbf{1}. \end{aligned} \quad (4.13)$$

it follows that

$$\begin{aligned}\delta A &= \frac{1}{2}\lambda^{\mu\nu}(\sigma_\mu A_\nu - \sigma_\nu A_\mu) \\ &= \sigma^\mu \lambda_\mu{}^\nu A_\nu,\end{aligned}\tag{4.14}$$

and hence by the linear independence of the sigma matrices one recovers (4.9). A parallel argument also leads to

$$\delta \bar{A} = \frac{1}{4}\lambda^{\mu\nu}(\bar{\sigma}_\mu(\sigma_\nu \bar{A}) - \bar{A}(\sigma_\mu \bar{\sigma}_\nu)).\tag{4.15}$$

Note the similarity with the familiar $SL(2, \mathbb{C})$ formalism using the Pauli matrices.

For the spinor representations, consider constructing gamma matrices in $D = n+2$ modeled on the Weyl basis (2.23) using the generalised Pauli matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \mu = 0, \dots, (n+1).\tag{4.16}$$

Left-multiplication by these matrices generates the Clifford algebra

$$\gamma^\mu(\gamma^\nu \zeta) + \gamma^\nu(\gamma^\mu \zeta) = 2\eta^{\mu\nu} \zeta \quad \forall \quad \zeta \in (\mathbb{A}_n)^4,\tag{4.17}$$

and thus multiplying any 4-component ‘column-vector’ ζ valued in \mathbb{A}_n by γ^μ has the effect of multiplying ζ ’s $4n$ real components by an ordinary real $4n \times 4n$ gamma matrix. This means that ζ may be seen as a Dirac spinor, and the spinor representation of $\text{Spin}(1, n+1)$ will be given infinitesimally by

$$\delta \zeta = \frac{1}{4}\lambda_{\mu\nu}\gamma^\mu(\gamma^\nu \zeta).\tag{4.18}$$

However, this representation is reducible; writing $\zeta = (\Psi, \mathcal{X})$ where $\Psi, \mathcal{X} \in (\mathbb{A}_n)^2$ are 2-component spinors,

$$\delta \zeta = \delta \begin{pmatrix} \Psi \\ \mathcal{X} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\lambda^{\mu\nu}\sigma_\mu(\bar{\sigma}_\nu \Psi) \\ \frac{1}{4}\lambda^{\mu\nu}\bar{\sigma}_\mu(\sigma_\nu \mathcal{X}) \end{pmatrix}.\tag{4.19}$$

Thus Ψ and \mathcal{X} transform respectively as the irreducible left-handed spinor and right-handed spinor representations of $\text{Spin}(1, n+1)$:

$$\begin{aligned}\delta \Psi &= \frac{1}{4}\lambda^{\mu\nu}\sigma_\mu(\bar{\sigma}_\nu \Psi), \\ \delta \mathcal{X} &= \frac{1}{4}\lambda^{\mu\nu}\bar{\sigma}_\mu(\sigma_\nu \mathcal{X}).\end{aligned}\tag{4.20}$$

Again, the bracket positioning only matters in the octonionic case. Making the

incorrect choice $(\sigma_{[\mu}\bar{\sigma}_{\nu]})\Psi$, one finds that the octonionic matrices $\sigma_{[\mu}\bar{\sigma}_{\nu]}$ have only 31 independent components, rather than the required $45 = \dim[\text{SO}(1,9)]$. The 14 missing generators are precisely those of G_2 , the automorphism group of the octonions; this G_2 subgroup is thus encoded in the non-associativity of the octonions.

Note that in $D = 3$, the 2×2 matrices

$$\Gamma^\mu := \sigma^\mu \varepsilon \quad \text{with} \quad \varepsilon := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (4.21)$$

already satisfy the Clifford algebra

$$\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2\eta^{\mu\nu} \mathbf{1}, \quad (4.22)$$

so using the two sets of matrices σ^μ and $\bar{\sigma}^\mu$ is not actually necessary, i.e. the spinor and conjugate-spinor representations are isomorphic. However, it is convenient to work with the sigmas for consistency with $D = 4, 6, 10$.

The framework presented above offers a nice picture of spinors in $D = 3, 4, 6, 10$; in each case the minimal spinor is just a 2-component \mathbb{A}_n -valued column [31]:

- $\mathbb{A}_n = \mathbb{R}$: Ψ is a Majorana spinor in $D = 3$
- $\mathbb{A}_n = \mathbb{C}$: Ψ, \mathcal{X} are left- and right-handed Weyl spinors² in $D = 4$
- $\mathbb{A}_n = \mathbb{H}$: Ψ, \mathcal{X} are left- and right-handed Symplectic-Weyl spinors in $D = 6$
- $\mathbb{A}_n = \mathbb{O}$: Ψ, \mathcal{X} are left- and right-handed Majorana-Weyl spinors in $D = 10$.

Finally, before moving on to describe super Yang-Mills theories it will be useful to define octonionic operators:

$$\begin{aligned} \hat{\sigma}_{\mu\nu} &= \frac{1}{2} \left[\sigma_\mu(\bar{\sigma}_\nu \cdot) - \sigma_\nu(\bar{\sigma}_\mu \cdot) \right], \\ \hat{\bar{\sigma}}_{\mu\nu} &= \frac{1}{2} \left[\bar{\sigma}_\mu(\sigma_\nu \cdot) - \bar{\sigma}_\nu(\sigma_\mu \cdot) \right], \end{aligned} \quad (4.23)$$

where the dots represent slots for left- and right-handed spinors, respectively (continuing with the convention from the previous chapter that octonionic operators are written with hats). This means that $\frac{1}{2}\hat{\sigma}_{\mu\nu}$ and $\frac{1}{2}\hat{\bar{\sigma}}_{\mu\nu}$ are generators of $\text{Spin}(1, n+1)$

²The left- and right-handed spinors in $D = 4$ are related by complex conjugation, i.e. if Ψ is a left-handed spinor then $\tilde{\Psi} := \varepsilon \Psi^*$ (with ε given above) transforms like a right-handed spinor. This is *not* true in $D = 6$ and $D = 10$ due to non-commutativity. Thus the ‘dotted’ and ‘un-dotted’ spinor index notation is not useful in the general case.

and necessarily satisfy the Lorentz algebra

$$\begin{aligned}\frac{1}{4}[\hat{\sigma}^{\mu\nu}, \hat{\sigma}^{\rho\sigma}] &= \frac{1}{2}(\eta^{\sigma\mu}\hat{\sigma}^{\rho\nu} + \eta^{\nu\sigma}\hat{\sigma}^{\mu\rho} - \eta^{\rho\mu}\hat{\sigma}^{\sigma\nu} - \eta^{\nu\rho}\hat{\sigma}^{\mu\sigma}), \\ \frac{1}{4}[\hat{\hat{\sigma}}^{\mu\nu}, \hat{\hat{\sigma}}^{\rho\sigma}] &= \frac{1}{2}(\eta^{\sigma\mu}\hat{\hat{\sigma}}^{\rho\nu} + \eta^{\nu\sigma}\hat{\hat{\sigma}}^{\mu\rho} - \eta^{\rho\mu}\hat{\hat{\sigma}}^{\sigma\nu} - \eta^{\nu\rho}\hat{\hat{\sigma}}^{\mu\sigma}).\end{aligned}\tag{4.24}$$

These can then be used to represent spacetime 2-forms such as a field strength $F_{\mu\nu}$:

$$\begin{aligned}\hat{F} &:= \frac{1}{2}F_{\mu\nu}\hat{\sigma}^{\mu\nu}, \\ \hat{\hat{F}} &:= \frac{1}{2}F_{\mu\nu}\hat{\hat{\sigma}}^{\mu\nu},\end{aligned}\tag{4.25}$$

so that \hat{F} and $\hat{\hat{F}}$ transform under commutation using the Lorentz algebra (4.24):

$$\delta\hat{F} = \frac{1}{4}\lambda^{\mu\nu}[\hat{F}, \hat{\sigma}_{\mu\nu}], \quad \delta\hat{\hat{F}} = \frac{1}{4}\lambda^{\mu\nu}[\hat{\hat{F}}, \hat{\hat{\sigma}}_{\mu\nu}].\tag{4.26}$$

For the other division algebras 2-forms can be written similarly, but the positioning of the brackets is no longer important and the hats can be dropped.

4.2. Super Yang-Mills Theories in $D = n + 2$

4.2.1. $\mathcal{N} = 1$ Lagrangian and Transformation Rules

Writing a spinor kinetic term in $D = n + 2$ will require a real, Lorentz-scalar spinor bilinear. For *commuting* spinors this is given by

$$\text{Re}(\Psi^\dagger \mathcal{X}) := \frac{1}{2}(\Psi^\dagger \mathcal{X} + (\Psi^\dagger \mathcal{X})^\dagger) = \frac{1}{2}(\Psi^\dagger \mathcal{X} + \mathcal{X}^\dagger \Psi),\tag{4.27}$$

where Ψ is a left-handed spinor, \mathcal{X} is a right-handed spinor and the dagger denotes the conjugate-transpose. It is simple to verify that this is Lorentz-invariant using the transformations (4.20). Writing $\Psi = \Psi_a e_a$ and $\mathcal{X} = \mathcal{X}_a e_a$ in terms of their division-algebraic components this becomes

$$\frac{1}{2}(\Psi_a^\top \mathcal{X}_b e_a^* e_b + \mathcal{X}_b^\top \Psi_a e_b^* e_a) = \frac{1}{2}\Psi_a^\top \mathcal{X}_b (e_a^* e_b + e_b^* e_a) = \Psi_a^\top \mathcal{X}_a,\tag{4.28}$$

which is clearly a real number. However, the first equality here relies on the components Ψ_a and \mathcal{X}_b being ordinary commuting real numbers; for physical spinor fields, the components Ψ_a and \mathcal{X}_b are anti-commuting Grassmann numbers and the scalar product above must be replaced by

$$\text{Re}(i\Psi^\dagger \mathcal{X}) = \frac{1}{2}(i\Psi^\dagger \mathcal{X} + (i\Psi^\dagger \mathcal{X})^\dagger) = \frac{i}{2}(\Psi^\dagger \mathcal{X} - \mathcal{X}^\dagger \Psi),\tag{4.29}$$

where the complex unit i in the above equation is independent of (and commutes with) the division algebra \mathbb{A}_n – the \dagger operation is defined to conjugate the factor of i as well as the elements of \mathbb{A}_n . Note that for anti-commuting spinors

$$\operatorname{Re}(i\Psi^\dagger \mathcal{K}) = -\operatorname{Re}(i\mathcal{K}^\dagger \Psi). \quad (4.30)$$

The Lagrangian for a left-handed spinor Ψ is then obtained by applying (4.29) to Ψ and the right-handed spinor $\bar{\sigma}^\mu \partial_\mu \Psi$:

$$-\operatorname{Re}(i\Psi^\dagger \bar{\sigma}^\mu \partial_\mu \Psi) = -\frac{i}{2}\Psi^\dagger (\bar{\sigma}^\mu \partial_\mu \Psi) - \frac{i}{2}(\Psi^\dagger \bar{\sigma}^\mu) \partial_\mu \Psi + \text{total derivative}, \quad (4.31)$$

where the association brackets on the left-hand side have been omitted since the associator is pure-imaginary:

$$\operatorname{Re}(xyz) := \operatorname{Re}((xy)z) = \operatorname{Re}(x(yz)), \quad \forall x, y, z \in \mathbb{A}_n. \quad (4.32)$$

The overall sign in (4.31) ensures energy-positivity and agreement with the usual $D = 4$ expression $-i\Psi^\dagger \bar{\sigma}^\mu \partial_\mu \Psi$.

The action for $(n + 2)$ -dimensional $\mathcal{N} = 1$ SYM with gauge group G over the division algebra \mathbb{A}_n is then

$$S = \int d^{n+2}x \left(-\frac{1}{4} F_{\mu\nu}^A F^{A\mu\nu} - \operatorname{Re}(i\Psi^\dagger \bar{\sigma}^\mu D_\mu \Psi^A) \right), \quad \Psi \in \mathbb{A}_n^2, \quad (4.33)$$

where $A = 0, \dots, \dim[G]$ and the covariant derivative and field strength are given by the usual expressions

$$\begin{aligned} D_\mu \Psi^A &= \partial_\mu \Psi^A + g f_{BC}^A A_\mu^B \Psi^C, \\ F_{\mu\nu}^A &= \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g f_{BC}^A A_\mu^B A_\nu^C. \end{aligned} \quad (4.34)$$

The supersymmetry transformations are

$$\delta A_\mu^A = \operatorname{Re}(i\Psi^{A\dagger} \bar{\sigma}_\mu \epsilon), \quad \delta \Psi^A = \frac{1}{2} \hat{F}^A \epsilon. \quad (4.35)$$

For the associative division algebras $\mathbb{A}_n = \mathbb{R}, \mathbb{C}, \mathbb{H}$ the action and transformation rules are invariant under right-multiplication of the spinors by a unit-modulus division algebra element $u \in \mathbb{A}_n$

$$\Psi^A \rightarrow \Psi^A u, \quad \epsilon \rightarrow \epsilon u, \quad u^* u = 1. \quad (4.36)$$

For $\mathbb{A}_n = \mathbb{C}, \mathbb{H}$ the unit u may be written as $u = e^{\theta_1}$ with $\theta_1 \in \text{Im}(\mathbb{A}_n)$, so that for small θ_1

$$\delta\Psi = \Psi\theta_1, \quad \delta\epsilon = \epsilon\theta_1, \quad \theta_1 \in \text{Im}(\mathbb{A}_n), \quad \mathbb{A}_n = \mathbb{C}, \mathbb{H}. \quad (4.37)$$

Thus there is an internal $U(1)$ and $\text{Sp}(1) \cong \text{SU}(2)$ symmetry for $\mathbb{A}_n = \mathbb{C}, \mathbb{H}$, respectively. This is none other than the R-symmetry of $\mathcal{N} = 1$ SYM in $D = 4, 6$. Note the similarity with the triality algebras in Section 3.3 where there was an extra algebra $\mathfrak{rx}(\mathbb{A}_n) = \mathcal{O}, \mathfrak{u}(1), \mathfrak{su}(2), \mathcal{O}$ for $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathcal{O}$, respectively.

An advantage of using division-algebraic spinors is that one may write the vector's transformation without sigma matrices simply by taking the outer product:

$$\delta A^A = \delta A_\mu^A \sigma^\mu = i(\Psi^A \epsilon^\dagger - \epsilon \Psi^{\dagger A}). \quad (4.38)$$

Using (4.10), reversing the trace gives this in terms of $A^A = A_\mu \bar{\sigma}^\mu$:

$$\begin{aligned} \delta \bar{A}^A &= \delta A_\mu^A \bar{\sigma}^\mu = i(\Psi^A \epsilon^\dagger - \epsilon \Psi^{\dagger A}) - (\text{trace}) \\ &= i(\Psi^A \epsilon^\dagger - \epsilon \Psi^{\dagger A}) + i(\epsilon^\dagger \Psi^A - \Psi^{\dagger A} \epsilon) \mathbb{1}, \end{aligned} \quad (4.39)$$

where the trace term was computed as follows. Since $i(\Psi^A \epsilon^\dagger - \epsilon \Psi^{\dagger A})$ is a Hermitian matrix its trace is real:

$$\text{Tr } i(\Psi^A \epsilon^\dagger - \epsilon \Psi^{\dagger A}) = \text{Re } \text{Tr } i(\Psi^A \epsilon^\dagger - \epsilon \Psi^{\dagger A}). \quad (4.40)$$

Then by the cyclicity property of the real part of a trace,

$$\text{Re } \text{Tr } i(\Psi^A \epsilon^\dagger - \epsilon \Psi^{\dagger A}) = -\text{Re } \text{Tr } i(\epsilon^\dagger \Psi^A - \Psi^{\dagger A} \epsilon) = -i(\epsilon^\dagger \Psi^A - \Psi^{\dagger A} \epsilon), \quad (4.41)$$

where the minus sign in the second equality follows from taking into account the Grassmann nature of the spinors.

4.2.2. Proof of Supersymmetry

The following proof that the action (4.33) is supersymmetric follows the method found in the literature [52, 31]. It turns out that the variation vanishes by virtue of the alternativity of the division algebras. Varying the action gives

$$\delta S = \int d^{n+2}x \left(\delta A_\nu^A D_\mu F^{A\mu\nu} - \text{Re}(igf_{BC}{}^A \Psi^{\dagger A} \delta \bar{A}^B \Psi^C + 2i\Psi^{\dagger A} \bar{\sigma}^\mu D_\mu \delta \Psi^A) \right). \quad (4.42)$$

Crucially, the ‘ 3Ψ ’ term $\text{Re}(igf_{BC}{}^A \Psi^{\dagger A} \delta \bar{A}^B \Psi^C)$ vanishes. First define the trace-reversed Hermitian outer product of any two anti-commuting spinors Ψ_1 and Ψ_2

by

$$\begin{aligned}
\Psi_1 \cdot \Psi_2 &= i(\Psi_1 \Psi_2^\dagger - \Psi_2 \Psi_1^\dagger) - (\text{trace}) \\
&= i(\Psi_1 \Psi_2^\dagger - \Psi_2 \Psi_1^\dagger) + i(\Psi_2^\dagger \Psi_1 - \Psi_1^\dagger \Psi_2) \mathbf{1} \\
&= \text{Re}(i \Psi_1^\dagger \bar{\sigma}_\mu \Psi_2) \bar{\sigma}^\mu.
\end{aligned} \tag{4.43}$$

Note that the trace $-i(\Psi_2^\dagger \Psi_1 - \Psi_1^\dagger \Psi_2)$ is a real number. Then acting on a third spinor Ψ_3 and adding cyclic permutations gives zero

$$(\Psi_1 \cdot \Psi_2) \Psi_3 + (\Psi_2 \cdot \Psi_3) \Psi_1 + (\Psi_3 \cdot \Psi_1) \Psi_2 \equiv 0. \tag{4.44}$$

To see this, note that

$$\begin{aligned}
(\Psi_1 \cdot \Psi_2) \Psi_3 &= i(\Psi_1 \Psi_2^\dagger - \Psi_2 \Psi_1^\dagger) \Psi_3 + i(\Psi_2^\dagger \Psi_1 - \Psi_1^\dagger \Psi_2) \Psi_3 \\
&= i(\Psi_1 \Psi_2^\dagger - \Psi_2 \Psi_1^\dagger) \Psi_3 + i \Psi_3 (\Psi_2^\dagger \Psi_1 - \Psi_1^\dagger \Psi_2),
\end{aligned} \tag{4.45}$$

so that adding cyclic permutations (c.p.) just results in a sum of associators:

$$\begin{aligned}
(\Psi_1 \cdot \Psi_2) \Psi_3 + \text{c.p.} &= i \left(+[\Psi_1, \Psi_2^\dagger, \Psi_3] + [\Psi_2, \Psi_3^\dagger, \Psi_1] + [\Psi_3, \Psi_1^\dagger, \Psi_2] \right. \\
&\quad \left. - [\Psi_1, \Psi_3^\dagger, \Psi_2] - [\Psi_2, \Psi_1^\dagger, \Psi_3] - [\Psi_3, \Psi_2^\dagger, \Psi_1] \right).
\end{aligned} \tag{4.46}$$

The six associators then cancel in pairs by alternativity. For example,

$$\begin{aligned}
[\Psi_1, \Psi_2^\dagger, \Psi_3] - [\Psi_1, \Psi_3^\dagger, \Psi_2] &= \Psi_{1i} \Psi_{2j}^\dagger \Psi_{3k} [e_i, e_j^*, e_k] - \Psi_{1i} \Psi_{3j}^\dagger \Psi_{2k} [e_i, e_j^*, e_k] \\
&= -\Psi_{1i} \Psi_{2j}^\dagger \Psi_{3k} [e_i, e_j, e_k] + \Psi_{1i} \Psi_{3j}^\dagger \Psi_{2k} [e_i, e_j, e_k] \\
&= \Psi_{1i} \Psi_{3j}^\dagger \Psi_{2k} ([e_i, e_k, e_j] + [e_i, e_j, e_k]) \\
&= 0.
\end{aligned} \tag{4.47}$$

With a little more work, using the variation (4.39), the 3Ψ term can be rewritten

$$gf_{BC}^A \text{Re} (i \Psi^{\dagger A} \delta \bar{A}^B \Psi^C) = gf_{BC}^A \text{Re} (i \epsilon^\dagger (\Psi^C \cdot \Psi^A) \Psi^B), \tag{4.48}$$

but since $\Psi^A \cdot \Psi^B = -\Psi^B \cdot \Psi^A$, this is just

$$\frac{1}{3} gf_{BC}^A \text{Re} (i \epsilon^\dagger [(\Psi^C \cdot \Psi^A) \Psi^B + (\Psi^A \cdot \Psi^B) \Psi^C + (\Psi^B \cdot \Psi^C) \Psi^A]) = 0. \tag{4.49}$$

Thus the 3Ψ term is zero by virtue of the alternativity of the division algebras.

As for the remaining terms in δS , substituting in the variations of A and Ψ gives

$$\delta S = \int d^{n+2}x \left(\text{Re}(i\Psi^{A\dagger}\bar{\sigma}_\nu\epsilon)D_\mu F^{A\mu\nu} - \text{Re}(i\Psi^{\dagger A}\bar{\sigma}^\mu D_\mu(\hat{F}^A\epsilon)) \right), \quad (4.50)$$

so by the Leibniz rule

$$\begin{aligned} \delta S = \int d^{n+2}x & \left(\text{Re}(i\Psi^{A\dagger}\bar{\sigma}_\nu\epsilon)D_\mu F^{A\mu\nu} - \frac{1}{2}\text{Re}(i\Psi^{\dagger A}\bar{\sigma}^\mu(\sigma^{[\nu}(\bar{\sigma}^{\rho]}\epsilon))D_\mu F_{\nu\rho}^A \right. \\ & \left. - \frac{1}{2}\text{Re}(i\Psi^{\dagger A}\bar{\sigma}^\mu(\sigma^{[\nu}(\bar{\sigma}^{\rho]}\partial_\mu\epsilon))F_{\nu\rho}^A) \right), \end{aligned} \quad (4.51)$$

where the $\partial_\mu\epsilon$ term has been retained (despite the fact ϵ is of course constant) since this can be used to read off the supercurrent – see below. Then, invoking the identity

$$\bar{\sigma}^\mu(\sigma^{[\nu}(\bar{\sigma}^{\rho]}\Psi)) = \bar{\sigma}^{[\mu}(\sigma^{\nu]}(\bar{\sigma}^{\rho]}\Psi) + 2\eta^{\mu[\nu}(\bar{\sigma}^{\rho]}\Psi) \quad \forall \Psi \in (\mathbb{A}_n)^2, \quad (4.52)$$

and applying it to the second term results in

$$\delta S = \int d^{n+2}x \left(-\frac{1}{2}\text{Re}(i\Psi^{\dagger A}\bar{\sigma}^{[\mu}(\sigma^{\nu]}(\bar{\sigma}^{\rho]}\epsilon))D_{[\mu}F_{\nu\rho]}^A - \frac{1}{2}\text{Re}(i\Psi^{\dagger A}\bar{\sigma}^\mu(\sigma^{[\nu}(\bar{\sigma}^{\rho]}\partial_\mu\epsilon))F_{\nu\rho}^A) \right). \quad (4.53)$$

The first term then vanishes by the gauge Bianchi identity $D_{[\mu}F_{\nu\rho]}^A \equiv 0$ and the second (which contains the supercurrent) because ϵ is constant. The supercurrent term can be rewritten (by taking the dagger and repeatedly applying (4.32)) as

$$-\frac{1}{2}\text{Re}(i\Psi^{\dagger A}\bar{\sigma}^\mu(\sigma^{[\nu}(\bar{\sigma}^{\rho]}\partial_\mu\epsilon))F_{\nu\rho}^A = \frac{1}{2}\text{Re}(i\partial_\mu\epsilon^\dagger\bar{\sigma}^{[\rho}(\sigma^{\nu]}(\bar{\sigma}^\mu\Psi^A)))F_{\nu\rho}^A, \quad (4.54)$$

from which one may recover the supercurrent in this formalism,

$$\mathcal{J}^\mu = \hat{F}^A(\bar{\sigma}^\mu\Psi^A), \quad (4.55)$$

and hence the supercharge

$$Q = - \int d^{n+1}x \hat{F}^A\Psi^A \quad (4.56)$$

(since $\bar{\sigma}^0 = -\mathbb{1}$). The supercharge is valued in $(\mathbb{A}_n)^2$ and hence has $\mathcal{Q} = 2n$ real components.

4.2.3. Supersymmetry Algebra and Off-Shell Formulation

Taking commutators of the supersymmetry transformations given in equation (4.35) results in

$$\begin{aligned}
[\delta_1, \delta_2]A_\mu^A &= \text{Re}(i\epsilon_2^\dagger \bar{\sigma}^\nu \epsilon_1) F_{\nu\mu}^A, \\
[\delta_1, \delta_2]\Psi^A &= \text{Re}(i\epsilon_2^\dagger \bar{\sigma}^\mu \epsilon_1) D_\mu \Psi^A \\
&\quad + \left(\frac{i}{2} [\epsilon_1, \epsilon_2^\dagger, (\bar{\sigma}^\mu D_\mu \Psi^A)] + \frac{1}{2} \epsilon_1 \text{Im}(i\epsilon_2^\dagger (\bar{\sigma}^\mu D_\mu \Psi^A)) - (1 \leftrightarrow 2) \right).
\end{aligned} \tag{4.57}$$

As usual, the commutator of two supersymmetry transformations is a gauge-covariant translation, but the algebra fails to close by terms proportional to the fermionic equation of motion $\bar{\sigma}^\mu D_\mu \Psi^A = 0$. The difference between the number of fermionic and bosonic degrees of freedom is

$$2n - (n + 1) = n - 1, \tag{4.58}$$

so if the algebra is to close off-shell, the counting suggests using an auxiliary $\text{Im}(\mathbb{A})$ -valued scalar field $D^A = D_i^A e_i$. This idea was explored in [53, 29, 54]. Add to the action (4.33) the term

$$S_D = \int d^{m+2}x \left(\frac{1}{2} D^{A*} D^A \right), \tag{4.59}$$

and modify the supersymmetry transformations to

$$\begin{aligned}
\delta A_\mu^A &= \text{Re}(i\Psi^{A\dagger} \bar{\sigma}_\mu \epsilon), \\
\delta \Psi^A &= \frac{1}{2} (\hat{F}^A \epsilon + \epsilon D^A), \\
\delta D^A &= \text{Im}((iD_\mu \Psi^{A\dagger} \bar{\sigma}_\mu) \epsilon)
\end{aligned} \tag{4.60}$$

(note that $\text{Im}(iz) = i\text{Re}(z)$ for some division algebra element z). It is straightforward to check that the action $S + S_D$ is invariant under these new transformations. In the $D = 4, 6$ cases $D^A \in \text{Im}(\mathbb{A}_n)$ appears in $\delta\Psi^A$ just like a local R-symmetry transformation (4.37) with $\theta_I \rightarrow D^A$, so the auxiliary fields must transform in the adjoint of the R-symmetry group, i.e. the singlet for $U(1)$ and the $\mathbf{3}$ for $\text{Sp}(1)$.

However, in the $D = 10$ case multiplying the octonionic objects ϵ and D^A in the transformations actually *breaks* Lorentz symmetry. Consider attempting to Lorentz-transform $\delta\Psi^A$. Left- and right-multiplication do not commute in \mathbb{O} , i.e. \mathbb{O} is non-associative, so the term ϵD^A only transforms like a spinor if D^A itself transforms under Lorentz transformations. The Lorentz group $\text{SO}(1, 9)$ can be broken into $\text{SO}(1, 2) \times \text{SO}(7)$, where the $\text{SO}(1, 2) \cong \text{SL}(2, \mathbb{R})$ is generated by real matrices and the $\text{SO}(7)$ is octonionic, generated by $\hat{\sigma}^{(i+1)(j+1)}$. The only way to make sense of the

ϵD^A term is to break $\text{SO}(1, 9) \supset \text{SO}(1, 2) \times G_2$ and allow D^A to transform as the **7** of G_2 . In conclusion, in the $D = 10$ octonionic theory the imaginary auxiliary field is *not* a scalar at all but a G_2 vector. It is interesting to consider the formalism regardless.

The commutators of the supersymmetry transformations (4.60) are as follows:

$$\begin{aligned} [\delta_1, \delta_2] A_\mu^A &= \text{Re}(i\epsilon_2^\dagger \bar{\sigma}^\nu \epsilon_1) F_{\nu\mu}^A - \frac{1}{2} \text{Re} \left(i[\epsilon_2^\dagger, \bar{\sigma}_\mu, \epsilon_1] D^A \right), \\ [\delta_1, \delta_2] \Psi^A &= \text{Re}(i\epsilon_2^\dagger \bar{\sigma}^\mu \epsilon_1) D_\mu \Psi^A + \left(\frac{i}{2} [\epsilon_1, \epsilon_2^\dagger, (\bar{\sigma}^\mu D_\mu \Psi^A)] - (1 \leftrightarrow 2) \right), \\ [\delta_1, \delta_2] D^A &= \text{Re}(i\epsilon_2^\dagger \bar{\sigma}^\mu \epsilon_1) D_\mu D^A - i[\epsilon_2^\dagger, \bar{\sigma}^\nu, \epsilon_1] D^\nu F_{\mu\nu}^A + \frac{i}{2} \text{Re} \left([\epsilon_2^\dagger, \bar{\sigma}^\mu, \epsilon_1] D_\mu D^A \right). \end{aligned} \quad (4.61)$$

Thus the algebra is closed for the associative algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, corresponding to $D = 3, 4, 6$, but fails to close by associators for $D = 10$ over \mathbb{O} . Interestingly, in the octonionic case all of these associators vanish if 7 of the 16 real components of the supersymmetry parameters are set to zero by constraining one of the two octonionic components of ϵ to be real:

$$\epsilon = \begin{pmatrix} \epsilon_s \\ \epsilon_c \end{pmatrix} \rightarrow \begin{pmatrix} \text{Re}(\epsilon_s) \\ \epsilon_c \end{pmatrix} \quad (4.62)$$

(the subscripts s and c are chosen to reflect the little group representations – see the following section). This is in agreement with [53, 29], where an imaginary octonionic auxiliary field was used to close the algebra for 9 out of 16 supersymmetries.

4.3. Dimensional Reductions

4.3.1. Little Group Representations and Triality

Having established the Lagrangian and transformation rules for $\mathcal{N} = 1$ super Yang-Mills in $D = 3, 4, 6, 10$, it is interesting to move on to dimensional reduction. This will result in every pure Yang-Mills theory with extended supersymmetry $\mathcal{N} \geq 1$ in all dimensions $3 \leq D \leq 10$. As an example, consider dimensional reduction from $D = 10$ to $D = 6$. The Lorentz group in $D = 6$ is $\text{SL}(2, \mathbb{H})$, which is clearly a subgroup of $\text{SL}(2, \mathbb{O})$ acting on the \mathbb{H} subalgebra of \mathbb{O} . Using the Cayley-Dickson doubling procedure described in Chapter 3 an octonion can be constructed as a pair of quaternions, and thus reducing from $D = 10$ to $D = 6$ is a case of ‘Cayley-Dickson halving’ to split \mathbb{O} into a pair of \mathbb{H} constituents. Similar arguments apply for each other case, although to obtain SYM for $D \neq 3, 4, 6, 10$ will require some additional background on Clifford algebras.

To perform these dimensional reductions it will be simpler to consider only the on-shell degrees of freedom transforming under the little group $\text{SO}(n)_{\text{ST}}$, but the process may equally well be carried out using the full off-shell fields. The on-shell states also have the advantage that they make the connection between super Yang-Mills and triality manifest, and this is the focus of this subsection.

The little group is the subgroup of $\text{SO}(1, 1+n)$ generated by sums of Lie algebra elements with parameters $\lambda^{0\mu} = \lambda^{n+1,\mu} = 0$. For notational convenience define

$$\theta^{ab} := \lambda^{a+1,b+1}. \quad (4.63)$$

Then, writing the spinor Ψ 's two components as $\psi, \chi \in \mathbb{A}_n$ and setting $\lambda^{0\mu} = \lambda^{n+1,\mu} = 0$, Ψ transforms as

$$\delta\Psi = \frac{1}{4}\lambda^{\mu\nu}\hat{\sigma}_{\mu\nu}\Psi \Rightarrow \delta\begin{pmatrix} \psi \\ \chi \end{pmatrix} = \frac{1}{4}\theta^{ab}\begin{pmatrix} e_a^*(e_b\psi) \\ e_a(e_b^*\chi) \end{pmatrix}. \quad (4.64)$$

These are just the transformations of the spinor ψ and conjugate-spinor χ representations of $\text{SO}(n)$ from Chapter 3. For a solution of the free momentum-space equation of motion $\bar{\sigma}^\mu p_\mu\Psi = 0$ (with gauge indices suppressed) with momentum $p_\mu = (E, 0, \dots, 0, E)$, the form of the sigma matrices gives

$$p_\mu\bar{\sigma}^\mu\Psi = 2E\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix} = 0, \quad (4.65)$$

i.e. $\chi = 0$, so the on-shell fermionic content is just the single division algebra element ψ .

Similarly, defining $a := A^{a+1}e_a$, with $\lambda^{0\mu} = \lambda^{n+1,\mu} = 0$ the vector A transforms as:

$$\delta\begin{pmatrix} A^0 + A^{n+1} & a^* \\ a & A^0 - A^{n+1} \end{pmatrix} = \frac{1}{4}\theta^{ab}\begin{pmatrix} 0 & e_a^*(e_b a^*) - a^*(e_a e_b^*) \\ e_a(e_b^* a) - a(e_a^* e_b) & 0 \end{pmatrix}, \quad (4.66)$$

which gives the transformation of the vector a of $\text{SO}(n)_{\text{ST}}$:

$$\delta a = \frac{1}{4}\theta^{ab}\left(e_a(e_b^* a) - a(e_a^* e_b)\right). \quad (4.67)$$

After applying the radiation gauge and imposing the vector's equation of motion, the bosonic content also amounts only to a single division algebra element a .

Then, writing the supersymmetry parameter as

$$\epsilon = \begin{pmatrix} \epsilon_s \\ \epsilon_c \end{pmatrix}, \quad (4.68)$$

where $\epsilon_s, \epsilon_c \in \mathbb{A}_n$ transform respectively as the spinor and conjugate-spinor under $\text{SO}(n)_{\text{ST}}$, only supersymmetry transformations with parameter $\epsilon_s = 0$ preserve the solutions with $p_\mu = (E, 0, \dots, 0, E)$. The supersymmetry transformations (4.35) then reduce to

$$\delta a = -i\epsilon_c \psi^*, \quad \delta \psi = iE a^* \epsilon_c. \quad (4.69)$$

These are of exactly the same form as the triality relations from equations (3.109), which were preserved by the triality algebra $\mathfrak{tri}(\mathbb{A}_n)$. This means that the overall symmetry algebra of the on-shell supersymmetry transformations (4.69) should be $\mathfrak{tri}(\mathbb{A}_n)$, and this *is* precisely the combination of the spacetime little group's Lie algebra $\mathfrak{so}(n)_{\text{ST}}$ with the R-symmetry $\mathfrak{r}(\mathbb{A}_n)$. In conclusion, the total on-shell symmetry of super Yang-Mills in $D = n + 2$ is the triality algebra $\mathfrak{tri}(\mathbb{A}_n) \cong \mathfrak{so}(n)_{\text{ST}} \oplus \mathfrak{r}(\mathbb{A}_n)$, whose triples $T = (\widehat{A}, \widehat{B}, \widehat{C})$ act on a, ψ, ϵ (dropping the subscript c on ϵ) via equation (3.124):

$$\begin{aligned} \delta a &= \widehat{A}a = \frac{1}{4}\theta^{ab}(e_a(e_b^*a) - a(e_a^*e_b)), \\ \delta \psi &= \widehat{B}^*\psi = \frac{1}{4}\theta^{ab}e_a^*(e_b\psi) + \psi\theta_1, \\ \delta \epsilon &= \widehat{C}\epsilon = \frac{1}{4}\theta^{ab}e_a(e_b^*\epsilon) + \epsilon\theta_1, \end{aligned} \quad (4.70)$$

where $\theta^{ab} \in \mathfrak{so}(n)_{\text{ST}}$ and $\theta_1 \in \mathfrak{r}(\mathbb{A}_n)$. Expanding $\theta^{ab} \rightarrow \theta^{0i}, \theta^{ij}$, the spacetime little group part of the these transformations may be written

$$\begin{aligned} \delta a &= \widehat{A}_{\text{ST}}a = -\frac{1}{2}\theta^{0i}(e_i a + a e_i) - \frac{1}{4}\theta^{ij}(e_i(e_j a) - a(e_i e_j)), \\ \delta \psi &= \widehat{B}_{\text{ST}}^*\psi = +\frac{1}{2}\theta^{0i}e_i\psi - \frac{1}{4}\theta^{ij}e_i(e_j\psi) \\ \delta \epsilon &= \widehat{C}_{\text{ST}}\epsilon = -\frac{1}{2}\theta^{0i}e_i\epsilon - \frac{1}{4}\theta^{ij}e_i(e_j\epsilon), \end{aligned} \quad (4.71)$$

which it is useful to express as

$$\delta(a, \psi^*, \epsilon) = -\frac{1}{2}\theta^{0i}T_i(a, \psi^*, \epsilon) - \frac{1}{4}\theta^{ij}T_i T_j(a, \psi^*, \epsilon), \quad (4.72)$$

where

$$T_i = (\widehat{A}_i, \widehat{B}_i, \widehat{C}_i) := (\widehat{L}_{e_i} + \widehat{R}_{e_i}, \widehat{R}_{e_i}, \widehat{L}_{e_i}) \in \mathfrak{tri}(\mathbb{A}_n) \quad (4.73)$$

and

$$T(x, y, z) := (\widehat{A}x, \widehat{B}y, \widehat{C}z) \quad (4.74)$$

for any $T = (\widehat{A}, \widehat{B}, \widehat{C}) \in \mathfrak{tri}(\mathbb{A}_n)$ and $(x, y, z) \in 3\mathbb{A}_n$. Better still, defining $T_{ab} = -T_{ba}$ with $T_{0i} := -T_i$ and $T_{ij} := -T_{[i}T_{j]}$ the $\mathfrak{so}(n)_{\text{ST}}$ transformation becomes

$$\delta(a, \psi^*, \epsilon) = T_{\text{ST}}(a, \psi^*, \epsilon) = \frac{1}{4}\theta^{ab}T_{ab}(a, \psi^*, \epsilon). \quad (4.75)$$

Note that for $n = 1$ (i.e. $D = 3$) any sum of spacetime little group generators gives $T_{\text{ST}} = 0$, corresponding to the trivial little group $\text{SO}(1)_{\text{ST}} \cong \mathbf{1}$. It will be convenient in the later chapters to write the $\text{SO}(n)$ generators T_{ab} as follows:

$$T_{ab} := (\widehat{S}_{ab}, \widehat{R}_{e_b}\widehat{R}_{e_a^*} - \widehat{R}_{e_a}\widehat{R}_{e_b^*}, \widehat{L}_{e_b}\widehat{L}_{e_a^*} - \widehat{L}_{e_a}\widehat{L}_{e_b^*}), \quad (4.76)$$

where

$$\widehat{S}_{ab}x = \langle e_a|x\rangle e_b - \langle e_b|x\rangle e_a, = -\frac{1}{2}(e_a(e_b^*x) + x(e_a^*e_b)), \quad (4.77)$$

(which appears in (4.67) as the vector transformation) and \widehat{L} and \widehat{R} denote left- and right-multiplication.

The appearance of $\mathfrak{tri}(\mathbb{A}_n)$ raises the question of the role the discrete triality symmetry S_3 plays in these theories. Here the spinor ψ and conjugate spinor ϵ have anti-commuting components and hence cannot be exchanged with the vector a as in (3.129) (and of course it would make no sense to exchange the constant parameter ϵ with the spacetime-dependent solutions ψ and a regardless), but group theoretically the transformations of the three representations are on completely equal footing. For example, in the quaternionic case the symmetry is $\mathfrak{tri}(\mathbb{H}) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, with two of the $\mathfrak{su}(2)$ parts being the spacetime symmetry algebra $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and the third being the R-symmetry $\mathfrak{cr}(\mathbb{H})$; the three representations transform as

$$\begin{aligned} \delta a &= -\theta_+ a + a\theta_- \sim (\mathbf{2}, \mathbf{2}, \mathbf{1}), \\ \delta \psi &= -\theta_- \psi + \psi\theta_+ \sim (\mathbf{2}, \mathbf{1}, \mathbf{2}), \\ \delta \epsilon &= -\theta_+ \epsilon + \epsilon\theta_- \sim (\mathbf{1}, \mathbf{2}, \mathbf{2}), \end{aligned} \quad (4.78)$$

so the three $\mathfrak{su}(2)$ s act democratically.

The triality algebra formalism is also suited to describing the various Lie algebras that emerge in dimensional reduction. First consider the bosonic sector. Group theoretically, on-shell dimensional reduction from $D = n + 2$ to $D = N + 2$ (with $1 \leq N \leq n$) amounts to restricting the little group $\mathfrak{so}(n)_{\text{ST}}$ so that it preserves an \mathbb{R}^N subspace of \mathbb{A}_n , corresponding to breaking the higher-dimensional vector $a' \in \mathbb{A}_n$ into a lower-dimensional vector $a \in \mathbb{R}^N$ and scalars $\phi \in \mathbb{R}^{n-N}$, where \mathbb{R}^{n-N}

is the orthogonal complement of \mathbb{R}^N in \mathbb{A}_n . Symbolically,

$$\begin{aligned}\mathbb{A}_n &= \mathbb{R}^N \oplus \mathbb{R}^{n-N}, \\ a' &= a + \phi = a_{\underline{a}}e_{\underline{a}} + \phi_{\bar{i}}e_{\bar{i}},\end{aligned}\tag{4.79}$$

where the index $a = 0, 1, \dots, (n-1)$ labelling the division-algebraic basis elements has been split into

$$\begin{aligned}\underline{a} &= 0, 1, \dots, (N-1), \\ \bar{i} &= N, (N+1), \dots, (n-1),\end{aligned}\tag{4.80}$$

so that $\text{span}\{e_{\underline{a}}\} = \mathbb{R}^N$, while $\text{span}\{e_{\bar{i}}\} = \mathbb{R}^{n-N}$. This decomposition simply breaks $\mathfrak{so}(n)_{\text{ST}}$ into the subalgebra

$$\mathfrak{so}(N)_{\text{ST}} \oplus \mathfrak{so}(n-N) \subseteq \mathfrak{so}(n)_{\text{ST}},\tag{4.81}$$

with equality only for the trivial case of *no* dimensional reduction: $n = N$. This subalgebra is obtained by decomposing the $\mathfrak{so}(n)_{\text{ST}}$ parameters $\theta^{ab} \rightarrow \theta^{\underline{a}\underline{b}}, \theta^{\underline{a}\bar{i}}, \theta^{\bar{i}\bar{j}}$ and then setting $\theta^{\underline{a}\bar{i}} = 0$, in order to preserve the decomposition in (4.79). The parameters $\theta^{\underline{a}\underline{b}}$ are those of the lower-dimensional spacetime little algebra $\mathfrak{so}(N)_{\text{ST}}$, which acts on the N -dimensional vector a and leaves the $(n-N)$ scalars ϕ invariant; the parameters $\theta^{\bar{i}\bar{j}}$ are those of an internal $\mathfrak{so}(n-N)$ symmetry, which acts on ϕ and leaves a invariant. Explicitly, the transformation of the higher-dimensional vector with $\theta^{\underline{a}\bar{i}} = 0$ reduces to:

$$\begin{aligned}\delta a' &= \frac{1}{4}\theta^{\underline{a}\underline{b}}(e_{\underline{a}}(e_{\underline{b}}^*a') - a'(e_{\underline{a}}^*e_{\underline{b}})) \\ &= \frac{1}{4}\theta^{\underline{a}\underline{b}}(e_{\underline{a}}(e_{\underline{b}}^*a) - a(e_{\underline{a}}^*e_{\underline{b}})) + \frac{1}{4}\theta^{\underline{a}\underline{b}}(e_{\underline{a}}(e_{\underline{b}}^*\phi) - \phi(e_{\underline{a}}^*e_{\underline{b}})) \\ &\quad + \frac{1}{4}\theta^{\bar{i}\bar{j}}(e_{\bar{i}}(e_{\bar{j}}^*a) - a(e_{\bar{i}}^*e_{\bar{j}})) + \frac{1}{4}\theta^{\bar{i}\bar{j}}(e_{\bar{i}}(e_{\bar{j}}^*\phi) - \phi(e_{\bar{i}}^*e_{\bar{j}})) \\ &= e_{\underline{a}}\theta^{\underline{a}\underline{b}}a_{\underline{b}} + e_{\bar{i}}\theta^{\bar{i}\bar{j}}\phi_{\bar{j}},\end{aligned}\tag{4.82}$$

so that a transforms as a vector under $\mathfrak{so}(N)_{\text{ST}}$, while ϕ transforms as a vector under $\mathfrak{so}(n-N)$.

Hence, taking into account the $\mathfrak{tr}(\mathbb{A}_n)$ symmetry, the total internal symmetry of such a theory in $D = N + 2$ is given by

$$\begin{aligned}\text{int}_N(\mathbb{A}_n) &:= \{(\widehat{A}, \widehat{B}, \widehat{C}) \in \text{tri}(\mathbb{A}_n) \mid \widehat{A}(\mathbb{R}^N) = 0, \mathbb{R}^N \subseteq \mathbb{A}_n\} \\ &= \mathfrak{so}(n-N) \oplus \mathfrak{tr}(\mathbb{A}_n).\end{aligned}\tag{4.83}$$

while the total on-shell symmetry is

$$\mathfrak{so}(N)_{\text{ST}} \oplus \mathfrak{int}_N(\mathbb{A}_n) \subseteq \mathfrak{tri}(\mathbb{A}_n) \quad (4.84)$$

(equal to $\mathfrak{tri}(\mathbb{A}_n)$ only for $n = N$). The extra internal symmetry algebra $\mathfrak{e}\mathfrak{t}(\mathbb{A}_n)$ living in $\mathfrak{int}_N(\mathbb{A}_n)$ acts only on the fermionic sector, whose field content may be understood using the theory of Clifford algebras presented in the following subsection.

4.3.2. Clifford Algebras

For a real vector space V with a bilinear inner product $\langle \cdot | \cdot \rangle$, the Clifford algebra $\text{Cl}(V)$ is the (associative) algebra of tensors $T(V)$ modulo the relation

$$vw + wv = -2\langle v | w \rangle, \quad \forall v, w \in V. \quad (4.85)$$

In other words the product in the algebra $\text{Cl}(V)$ is just the ordinary tensor product, subject to the rule that the symmetric product of two vectors gives their inner product (multiplied by -2). When $V = \mathbb{R}^N$, equipped with its canonical Euclidean inner product, $\text{Cl}(V)$ is usually denoted $\text{Cl}(N)$. For an orthonormal basis $\{e_m\}$ of \mathbb{R}^N with $m = 1, \dots, N$, the defining relation (4.85) becomes

$$e_m e_n + e_n e_m = -2\delta_{mn}, \quad (4.86)$$

and so $\text{Cl}(N)$ is the unital associative algebra consisting of linear combinations of products of N distinct anti-commuting square roots of -1 : the ‘generating’ basis elements e_m . It is easy to see that the first three Clifford algebras of this type are the associative division algebras³

$$\text{Cl}(0) \cong \mathbb{R}, \quad \text{Cl}(1) \cong \mathbb{C}, \quad \text{Cl}(2) \cong \mathbb{H}, \quad (4.87)$$

while \mathbb{O} cannot be a Clifford algebra, since it is non-associative. The fourth Clifford algebra is actually $\text{Cl}(3) \cong \mathbb{H} \oplus \mathbb{H}$, which is isomorphic to \mathbb{O} as a vector space, but as an algebra consists of diagonal quaternionic matrices under associative matrix multiplication. The first eight Clifford algebras are listed in Table 4.1 (as in the previous chapter, $\mathbb{A}[N]$ denotes the set of $N \times N$ matrices with entries in \mathbb{A}).

Remarkably, the continuation of Table 4.1 for higher values of N obeys the fol-

³Note that for \mathbb{H} there are two independent anti-commuting square roots of -1 , namely i and j , while the third imaginary basis element $k \equiv ij$ is simply a product of the first two; hence $\text{Cl}(2) = \mathbb{H}$.

| N | $\text{Cl}(N)$ | Generators $\{e_m\}$ | \mathcal{P}_N | $p_N := \dim_{\mathbb{R}}[\mathcal{P}_N]$ |
|-----|--------------------------------------|--|------------------------------------|---|
| 0 | \mathbb{R} | \emptyset | \mathbb{R} | 1 |
| 1 | \mathbb{C} | $\{i\}$ | \mathbb{C} | 2 |
| 2 | \mathbb{H} | $\{i, j\}$ | \mathbb{H} | 4 |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | $\{i\sigma^3, j\sigma^3, k\sigma^3\}$ | $\mathbb{H} \oplus \mathbb{H}$ | 4+4 |
| 4 | $\mathbb{H}[2]$ | $\{i\sigma^3, j\sigma^3, k\sigma^3, \varepsilon\}$ | \mathbb{H}^2 | 8 |
| 5 | $\mathbb{C}[4]$ | $\{i\sigma^3 \otimes \sigma^1, \sigma^3 \otimes \varepsilon, i\sigma^3 \otimes \sigma^3, \varepsilon \otimes \mathbf{1}, i\sigma^1 \otimes \mathbf{1}\}$ | \mathbb{C}^4 | 8 |
| 6 | $\mathbb{R}[8]$ | $\{\sigma^3 \otimes \sigma^1 \otimes \varepsilon, \sigma^3 \otimes \varepsilon \otimes \mathbf{1}, \sigma^3 \otimes \sigma^3 \otimes \varepsilon, \varepsilon \otimes \mathbf{1} \otimes \mathbf{1}, \sigma^1 \otimes \mathbf{1} \otimes \varepsilon, \sigma^1 \otimes \varepsilon \otimes \sigma^1\}$ | \mathbb{R}^8 | 8 |
| 7 | $\mathbb{R}[8] \oplus \mathbb{R}[8]$ | $\{\sigma^3 \otimes \sigma^1 \otimes \varepsilon \otimes \sigma^3, \sigma^3 \otimes \varepsilon \otimes \mathbf{1} \otimes \sigma^3, \sigma^3 \otimes \sigma^3 \otimes \varepsilon \otimes \sigma^3, \varepsilon \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma^3, \sigma^1 \otimes \mathbf{1} \otimes \varepsilon \otimes \sigma^3, \sigma^1 \otimes \varepsilon \otimes \sigma^1 \otimes \sigma^3, \sigma^1 \otimes \varepsilon \otimes \sigma^3 \otimes \sigma^3\}$ | $\mathbb{R}^8 \oplus \mathbb{R}^8$ | 8+8 |

Table 4.1.: The Clifford (sub)algebras $\text{Cl}(N) \cong \text{Cl}_0(N+1)$ as matrix algebras, their generators and (s)pinor representations \mathcal{P}_N for the first Bott period, $0 \leq N \leq 7$. Here $\sigma^1, \sigma^2 =: i\varepsilon$ and σ^3 are the usual Pauli matrices (3.6). The choice of generators is unique up to $O(N)$ orthogonal transformations $e_m \rightarrow O_m^n e_n$, which are the automorphisms of $\text{Cl}(N)$.

lowing rule, known as *Bott periodicity*:

$$\text{Cl}(N+8) \cong \text{Cl}(N) \otimes \mathbb{R}[16], \quad (4.88)$$

i.e. the algebra $\text{Cl}(N+8)$ is made up of 16×16 matrices with entries in $\text{Cl}(N)$. Thus every Clifford algebra $\text{Cl}(N)$ is isomorphic to a matrix algebra over the algebra \mathbb{D}_N , with

$$\mathbb{D}_N = \begin{cases} \mathbb{R}, & N = 0, 6 \pmod{8} \\ \mathbb{C}, & N = 1, 5 \pmod{8} \\ \mathbb{H}, & N = 2, 4 \pmod{8} \\ \mathbb{H} \oplus \mathbb{H}, & N = 3 \pmod{8} \\ \mathbb{R} \oplus \mathbb{R}, & N = 7 \pmod{8} \end{cases} \quad (4.89)$$

where (for the cases $N = 3, 7 \pmod{8}$) a matrix algebra ‘over’ a direct sum of two division algebras is defined to be the direct sum of the two corresponding matrix algebras; for example,

$$\text{Cl}(7) = \mathbb{D}_7[8] = (\mathbb{R} \oplus \mathbb{R})[8] := \mathbb{R}[8] \oplus \mathbb{R}[8]. \quad (4.90)$$

Any Clifford algebra $\text{Cl}(N)$ hence has a unique irreducible representation (or two irreducible representations in the cases with $N = 3, 7 \pmod{8}$) – that is, the vector space these matrices naturally act upon. This is called the space of *pinors* \mathcal{P}_N , listed in Table 4.1. Like the Clifford algebras that act upon them, the pinor representations exhibit Bott periodicity: $\mathcal{P}_{N+8} \cong \mathcal{P}_N \otimes \mathbb{R}^{16}$.

The dimension of $\text{Cl}(N)$ is 2^N , since the whole Clifford algebra has basis

$$\{1, e_m, e_m e_n, \dots, e_{m_1} e_{m_2} \cdots e_{m_N}\}, \quad (4.91)$$

with $\binom{N}{r}$ basis elements at level r , i.e. elements of the form $e_{m_1} e_{m_2} \cdots e_{m_r}$. Under the involution $e_m \rightarrow -e_m$, which corresponds to a space inversion of \mathbb{R}^N , the Clifford algebra splits into two eigenspaces of dimensions 2^{N-1} , called the even part $\text{Cl}_0(N)$ and the odd part $\text{Cl}'_0(N)$:

$$\begin{aligned} \text{Cl}_0(N) &:= \{x \in \text{Cl}(N) \mid x \rightarrow +x \text{ for } e_m \rightarrow -e_m\}, \\ \text{Cl}'_0(N) &:= \{x \in \text{Cl}(N) \mid x \rightarrow -x \text{ for } e_m \rightarrow -e_m\}. \end{aligned} \quad (4.92)$$

The even part clearly forms a closed subalgebra, consisting of sums of products of even numbers of generators e_m . Splitting the N generators $\{e_m\}$ into $\{e_m, e_N\}$ with $m = 1, \dots, (N-1)$, it is easy to see that any element of $\text{Cl}_0(N)$ is a sum of products of elements $\gamma_m := e_m e_N$, which satisfy

$$\gamma_m \gamma_n + \gamma_n \gamma_m = -2\delta_{mn}, \quad (4.93)$$

and thus $\text{Cl}_0(N)$ is in fact a Clifford algebra itself, isomorphic to $\text{Cl}(N-1)$,

$$\text{Cl}_0(N) \cong \text{Cl}(N-1), \quad (4.94)$$

and hence may be represented by \mathbb{D}_{N-1} -valued matrices.

The group $\text{Spin}(N) \subset \text{Cl}_0(N)$ consists of all elements of $\text{Cl}(N)$ that are a product of an even number of unit vectors in \mathbb{R}^N . The irreducible representations of $\text{Spin}(N)$, known of course as spinors, are then the irreducible representations of $\text{Cl}_0(N) \cong \text{Cl}(N-1)$. In other words, a spinor in N dimensions is a pinor in $N-1$ dimensions⁴,

$$\mathcal{P}_{N-1} \cong \mathcal{S}_N, \quad (4.95)$$

⁴The group $\text{Pin}(N)$ consists of all elements of $\text{Cl}(N)$ that are a product of unit vectors in \mathbb{R}^N , and is the double cover of $\text{O}(N)$, while its subgroup $\text{Spin}(N)$ is the double cover of $\text{SO}(N)$. The irreducible representations of $\text{Pin}(N)$, are the irreducible representations of $\text{Cl}(N)$ – hence the name *pinors* [1].

where \mathcal{S}_N is the space of spinors. It will be useful to define:

$$p_N := \dim_{\mathbb{R}}[\mathcal{P}_N], \quad s_N := \dim_{\mathbb{R}}[\mathcal{S}_N]. \quad (4.96)$$

In the context of on-shell supersymmetric Yang-Mills multiplets, usually there are \mathcal{N} spinors $\psi_I \in \mathcal{S}_N$, $I = 1, 2, \dots, \mathcal{N}$, which may be packaged as a reducible representation $(\mathcal{S}_N)^{\mathcal{N}}$ of the Clifford algebra $\text{Cl}_0(N)$. The action of the generators e_m on these spinors is simply left matrix multiplication: $e_m \psi_I$ (where the e_m here are \mathbb{D}_{N-1} -valued matrices acting on the \mathbb{D}_{N-1} -valued spinors ψ_I). The largest algebra of linear transformations M^J_I of $(\mathcal{S}_N)^{\mathcal{N}}$ that commutes with the action of the Clifford algebra $\text{Cl}_0(N)$ is the matrix algebra $\mathbb{D}_{N-1}[\mathcal{N}]$, which, because of the potential non-commutativity of \mathbb{D}_{N-1} , acts via *right* matrix multiplication: $\psi_J M^J_I$. It will be of vital importance in this chapter and in Chapter 6 to consider how this looks in terms of *real* matrices acting on $\mathbb{R}^{\mathcal{N}s_N} \cong (\mathcal{S}_N)^{\mathcal{N}}$.

If a real vector space V has dimension $\dim[V] = \mathcal{N}s_N$ for some natural numbers N and \mathcal{N} , then of course there exists a set of $(N-1)$ matrices $E_m \in \mathbb{R}[\mathcal{N}s_N]$ satisfying

$$E_m E_n + E_n E_m = -2\delta_{mn} \mathbf{1}, \quad (4.97)$$

which generates a reducible representation of \mathcal{N} copies of the 2^{N-1} -dimensional Clifford algebra $\text{Cl}(N)$. The Clifford algebra generated by the E_m acts on a real vector $v \in V$ as if it were \mathcal{N} spinors ψ_I valued in \mathcal{S}_N :

$$V \cong \mathbb{R}^{\mathcal{N}s_N} \cong (\mathcal{S}_N)^{\mathcal{N}}. \quad (4.98)$$

For such a real vector space V with $\dim[V] = \mathcal{N}s_N$, one finds that

$$\mathbb{D}_{N-1}[\mathcal{N}] \cong \left\{ M \in \mathbb{R}[\mathcal{N}s_N] \mid [M, E_m] = 0 \right\}, \quad (4.99)$$

for $E_m \in \mathbb{R}[\mathcal{N}s_N]$ satisfying (4.97). In other words, just as stated above, the largest algebra of linear transformations of $V \cong (\mathcal{S}_N)^{\mathcal{N}}$ that commutes with the action of the Clifford algebra $\text{Cl}(N)$ is isomorphic to the matrix algebra $\mathbb{D}_{N-1}[\mathcal{N}]$, which takes \mathbb{D}_{N-1} -linear combinations of the \mathcal{N} different spinors.

Equation (4.99) generalises equations (3.149) and (3.152) for complex and quaternionic structures in Section 3.5, since the complex numbers $\mathbb{C} = \text{Cl}(1)$ and quaternions $\mathbb{H} = \text{Cl}(2)$ are just Clifford algebras; both the complex and quaternionic structures are simply matrices that obey the defining Clifford algebra relation (4.86) for $\text{Cl}(1)$ and $\text{Cl}(2)$, respectively. Just as for complex and quaternionic structures,

equation (4.99) restricts to

$$\mathfrak{a}(\mathcal{N}, \mathbb{D}_{N-1}) \cong \left\{ M \in \mathfrak{so}(\mathcal{N}_{S_N}) \mid [M, E_m] = 0 \right\}, \quad (4.100)$$

where $\mathfrak{a}(\mathcal{N}, \mathbb{D}_{N-1})$ is defined in (3.143).

For Chapter 6 it will be useful to generalise this to $\mathbb{D}_{N-1}[\mathcal{N}_L, \mathcal{N}_R]$, the set of rectangular $\mathcal{N}_L \times \mathcal{N}_R$ matrices with entries in \mathbb{D}_{N-1} , which maps $(\mathcal{S}_N)^{\mathcal{N}_L}$ and $(\mathcal{S}_N)^{\mathcal{N}_R}$ into one another. In this case one requires both a left and right set of Clifford algebra generators, $E_m \in \mathbb{R}[\mathcal{N}_L S_N]$ and $\tilde{E}_m \in \mathbb{R}[\mathcal{N}_R S_N]$, each satisfying (4.97). Then one finds that

$$\mathbb{D}_{N-1}[\mathcal{N}_L, \mathcal{N}_R] \cong \left\{ M \in \mathbb{R}[\mathcal{N}_L S_N, \mathcal{N}_R S_N] \mid E_m M - M \tilde{E}_m = 0 \right\}. \quad (4.101)$$

Equation (4.100) has an obvious application in supersymmetry. The R-symmetry group – that is, the group of automorphisms of the \mathcal{N} -extended supersymmetry algebra – is the group of norm-preserving⁵ linear transformations of the spinor supercharges, which must commute with the Clifford algebra $\text{Cl}_0(D-2) \cong \text{Cl}(D-3) \supset \text{Spin}(D-2)$ corresponding to the spacetime little group $\text{Spin}(D-2)$. Thus the R-symmetry for \mathcal{N} supercharges in D dimensions has Lie algebra [43]

$$\mathfrak{r}(\mathcal{N}, D) = \mathfrak{a}(\mathcal{N}, \mathbb{D}), \quad (4.102)$$

where $\mathbb{D} := \mathbb{D}_{D-3}$ is the algebra associated with each spacetime dimension D , over which $\text{Cl}(D-3)$ may be formulated – see (4.89). The Clifford algebras for dimensions $3 \leq D \leq 10$, along with their corresponding \mathbb{D} algebras, spinor representations and R-symmetry groups are presented in Table 4.2. Note that when \mathbb{D} is a direct sum, i.e. in $D = 6, 10$, there exist chiral spinors. In this case the number of supersymmetries N becomes an ordered pair $(\mathcal{N}^+, \mathcal{N}^-)$, where \mathcal{N}^+ is the number of left-handed chiral spinor supercharges and \mathcal{N}^- is that of the right-handed⁶. The R-symmetry group preserves chirality in these dimensions, and so is given by $\mathfrak{so}(\mathcal{N}^+) \oplus \mathfrak{so}(\mathcal{N}^-)$ in $D = 10$ and $\mathfrak{sp}(\mathcal{N}^+) \oplus \mathfrak{sp}(\mathcal{N}^-)$ in $D = 6$.

With the above machinery in hand, the question of dimensionally reducing on-shell super Yang-Mills from $D = n + 2$ to $D = N + 2$ is fairly straightforward. The higher-dimensional quantities $(a', \psi'^*, \epsilon') \in 3\mathbb{A}_n$ transform under the lower-

⁵By definition, R-symmetry transformations must preserve the δ_J^I on the right-hand side of the supersymmetry algebra's anti-commutator $\{Q_\alpha^I, \bar{Q}_J^\beta\} = P_\mu(\gamma^\mu)_{\alpha\beta} \delta_J^I$.

⁶Thus, strictly speaking, what was called $\mathcal{N} = 1$ super Yang-Mills in $D = 6, 10$ sometimes in this chapter should have been called $(\mathcal{N}^+, \mathcal{N}^-) = (1, 0)$.

| D | $\text{Cl}_0(D-2)$ $\cong \text{Cl}(D-3)$ | \mathbb{D} | $D-2$ spinor rep $\cong D-3$ pinor rep | R-symmetry $\cong \mathfrak{a}(\mathcal{N}, \mathbb{D})$ |
|-----|--|--------------------------------|---|--|
| 10 | $\mathbb{R}[8] \oplus \mathbb{R}[8]$ | $\mathbb{R} \oplus \mathbb{R}$ | $\mathbb{R}^8 \oplus \mathbb{R}^8$ | $\mathfrak{so}(\mathcal{N}^+) \oplus \mathfrak{so}(\mathcal{N}^-)$ |
| 9 | $\mathbb{R}[8]$ | \mathbb{R} | \mathbb{R}^8 | $\mathfrak{so}(\mathcal{N})$ |
| 8 | $\mathbb{C}[4]$ | \mathbb{C} | \mathbb{C}^4 | $\mathfrak{u}(\mathcal{N})$ |
| 7 | $\mathbb{H}[2]$ | \mathbb{H} | \mathbb{H}^2 | $\mathfrak{sp}(\mathcal{N})$ |
| 6 | $\mathbb{H} \oplus \mathbb{H}$ | $\mathbb{H} \oplus \mathbb{H}$ | $\mathbb{H} \oplus \mathbb{H}$ | $\mathfrak{sp}(\mathcal{N}^+) \oplus \mathfrak{sp}(\mathcal{N}^-)$ |
| 5 | \mathbb{H} | \mathbb{H} | \mathbb{H} | $\mathfrak{sp}(\mathcal{N})$ |
| 4 | \mathbb{C} | \mathbb{C} | \mathbb{C} | $\mathfrak{u}(\mathcal{N})$ |
| 3 | \mathbb{R} | \mathbb{R} | \mathbb{R} | $\mathfrak{so}(\mathcal{N})$ |

Table 4.2.: The Clifford (sub)algebras, \mathbb{D} , spinor representation and R-symmetry algebra for dimensions $D = 3, \dots, 10$.

dimensional little algebra $\mathfrak{so}(N)_{\text{ST}} \subset \mathfrak{tri}(\mathbb{A}_n)$ as

$$\begin{aligned} \delta(a', \psi'^*, \epsilon') &= \frac{1}{4} \theta^{ab} T_{ab}(a', \psi'^*, \epsilon') \\ &= -\frac{1}{2} \theta^{0i} T_{\underline{i}}(a', \psi'^*, \epsilon') - \frac{1}{4} \theta^{ij} T_{\underline{i}} T_{\underline{j}}(a', \psi'^*, \epsilon'), \end{aligned} \quad (4.103)$$

where $\underline{i} = 1, 2, \dots, (N-1)$. For the fermionic quantities this reads

$$\begin{aligned} \delta\psi' &= \frac{1}{4} \theta^{ab} e_a^*(e_b \psi') = -\frac{1}{2} \theta^{0i} \widehat{B}_{\underline{i}}^* \psi' - \frac{1}{4} \theta^{ij} \widehat{B}_{\underline{i}}^* \widehat{B}_{\underline{j}}^* \psi', \\ \delta\epsilon' &= \frac{1}{4} \theta^{ab} e_a(e_b^* \epsilon') = -\frac{1}{2} \theta^{0i} \widehat{C}_{\underline{i}} \epsilon' - \frac{1}{4} \theta^{ij} \widehat{C}_{\underline{i}} \widehat{C}_{\underline{j}} \epsilon', \end{aligned} \quad (4.104)$$

since

$$T_{\underline{i}} = (\widehat{A}_{\underline{i}}, \widehat{B}_{\underline{i}}, \widehat{C}_{\underline{i}}) := (\widehat{L}_{e_{\underline{i}}} + \widehat{R}_{e_{\underline{i}}}, \widehat{R}_{e_{\underline{i}}}, \widehat{L}_{e_{\underline{i}}}) \in \mathfrak{tri}(\mathbb{A}_n). \quad (4.105)$$

These spacetime Clifford triality generators $T_{\underline{i}}$ give rise to a representation of $\text{Cl}(N-1)$ on the higher-dimensional fermion ψ' and supersymmetry parameter ϵ' :

$$\begin{aligned} (\widehat{B}_{\underline{i}} \widehat{B}_{\underline{j}} + \widehat{B}_{\underline{j}} \widehat{B}_{\underline{i}}) \psi'^* &= -2\delta_{\underline{ij}} \psi'^*, \\ (\widehat{C}_{\underline{i}} \widehat{C}_{\underline{j}} + \widehat{C}_{\underline{j}} \widehat{C}_{\underline{i}}) \epsilon' &= -2\delta_{\underline{ij}} \epsilon', \quad \forall \psi', \epsilon' \in \mathbb{A}_n. \end{aligned} \quad (4.106)$$

Thus each of $\widehat{B}_{\underline{i}}$ and $\widehat{C}_{\underline{i}}$ determines an isomorphism,

$$\mathbb{A}_n \cong (\mathcal{S}_N)^{\mathcal{N}}, \quad (4.107)$$

where $\mathcal{N} = n/s_N$, just as in equation (4.98). In this way, the higher-dimensional fermion $\psi' \in \mathbb{A}_n$ breaks into \mathcal{N} lower-dimensional spinors valued in \mathcal{S}_N – and similar for the supersymmetry parameter ϵ' – so the lower-dimensional theory has $\mathcal{N} = n/s_N$ supersymmetries. For example, for $D = N + 2 = 4$ and $\mathcal{N} = 4$, there is just one

Clifford generator \widehat{B}_1 (or \widehat{C}_1), which generates $\text{Cl}_0(N) = \text{Cl}_0(2) \cong \mathbb{C}$, acting on the reducible representation $\mathbb{A}_n = \mathbb{O}$. In this case \widehat{B}_1 (\widehat{C}_1) acts as a complex structure on \mathbb{O} , allowing one to reinterpret $\psi' \in \mathbb{O}$ ($\epsilon' \in \mathbb{O}$) as $\mathcal{N} = 4$ complex spinors: $\mathbb{O} \cong \mathbb{C}^4$ (note that $\mathcal{S}_2 = \mathbb{C}$, so $s_2 = 2$).

Table 4.3 shows how the fermion fields in each $D = N + 2$ fit into the division algebra \mathbb{A}_n over which the theory may be written, and Appendix A.1 contains explicit demonstrations of this. As outlined above, this is achieved via dimensional reduction from $D = n + 2$. The examples given are the maximal theories valued over $\mathbb{A}_n = \mathbb{O}$ and so are obtained from $D = 10$. The theories with fewer supersymmetries can then be obtained simply by truncation to division subalgebras of \mathbb{O} .

| N | $\text{Cl}_0(N)$ | $\mathbb{O} \cong (\mathcal{S}_N)^{\mathcal{N}}$ $\mathcal{Q} = 16$ | $\mathbb{H} \cong (\mathcal{S}_N)^{\mathcal{N}}$ $\mathcal{Q} = 8$ | $\mathbb{C} \cong (\mathcal{S}_N)^{\mathcal{N}}$ $\mathcal{Q} = 4$ | $\mathbb{R} \cong (\mathcal{S}_N)^{\mathcal{N}}$ $\mathcal{Q} = 2$ |
|---|--------------------------------------|--|---|--|--|
| 8 | $\mathbb{R}[8] \oplus \mathbb{R}[8]$ | $\mathbb{O} \cong \mathcal{S}_8^+ \cong \mathbb{R}^8$ $\Rightarrow \mathcal{N} = (1, 0)$ | — | — | — |
| 7 | $\mathbb{R}[8]$ | $\mathbb{O} \cong \mathcal{S}_7 \cong \mathbb{R}^8$ $\Rightarrow \mathcal{N} = 1$ | — | — | — |
| 6 | $\mathbb{C}[4]$ | $\mathbb{O} \cong \mathcal{S}_6 \cong \mathbb{C}^4$ $\Rightarrow \mathcal{N} = 1$ | — | — | — |
| 5 | $\mathbb{H}[2]$ | $\mathbb{O} \cong \mathcal{S}_5 \cong \mathbb{H}^2$ $\Rightarrow \mathcal{N} = 1$ | — | — | — |
| 4 | $\mathbb{H} \oplus \mathbb{H}$ | $\mathbb{O} \cong \mathcal{S}_4^+ \oplus \mathcal{S}_4^- \cong \mathbb{H} \oplus \mathbb{H}$ $\Rightarrow \mathcal{N} = (1, 1)$ | $\mathbb{H} \cong \mathcal{S}_4^+ \cong \mathbb{H}$ $\Rightarrow \mathcal{N} = (1, 0)$ | — | — |
| 3 | \mathbb{H} | $\mathbb{O} \cong (\mathcal{S}_3)^2 \cong \mathbb{H}^2$ $\Rightarrow \mathcal{N} = 2$ | $\mathbb{H} \cong \mathcal{S}_3 \cong \mathbb{H}$ $\Rightarrow \mathcal{N} = 1$ | — | — |
| 2 | \mathbb{C} | $\mathbb{O} \cong (\mathcal{S}_2)^4 \cong \mathbb{C}^4$ $\Rightarrow \mathcal{N} = 4$ | $\mathbb{H} \cong (\mathcal{S}_2)^2 \cong \mathbb{C}^2$ $\Rightarrow \mathcal{N} = 2$ | $\mathbb{C} \cong \mathcal{S}_2 \cong \mathbb{C}$ $\Rightarrow \mathcal{N} = 1$ | — |
| 1 | \mathbb{R} | $\mathbb{O} \cong (\mathcal{S}_1)^8 \cong \mathbb{R}^8$ $\Rightarrow \mathcal{N} = 8$ | $\mathbb{H} \cong (\mathcal{S}_1)^4 \cong \mathbb{R}^4$ $\Rightarrow \mathcal{N} = 4$ | $\mathbb{C} \cong (\mathcal{S}_1)^2 \cong \mathbb{R}^2$ $\Rightarrow \mathcal{N} = 2$ | $\mathbb{R} \cong \mathcal{S}_1 \cong \mathbb{R}$ $\Rightarrow \mathcal{N} = 1$ |

Table 4.3.: Division algebras \mathbb{A}_n decomposed into spinor representations of $\text{Cl}_0(N)$.

For $N > 1$ ($D > 3$) the internal symmetry $\mathbf{int}_N(\mathbb{A}_n)$ is just the subalgebra of $\mathbf{tri}(\mathbb{A}_n)$ that commutes with $\mathfrak{so}(N)_{\text{ST}}$:

$$\begin{aligned} \mathbf{int}_N(\mathbb{A}_n) &= \left\{ T \in \mathbf{tri}(\mathbb{A}_n) - \mathfrak{so}(N)_{\text{ST}} \mid [T, \mathfrak{so}(N)_{\text{ST}}] = 0 \right\}, \quad N > 1 \\ &= \mathfrak{so}(n - N) \oplus \mathfrak{cr}(\mathbb{A}_n), \end{aligned} \quad (4.108)$$

which acts on the a', ψ', ϵ' as

$$\begin{aligned}\delta a' &= -\frac{1}{4}\theta^{\bar{i}j}(e_{\bar{i}}(e_j a') - a'(e_{\bar{i}}e_j)), \\ \delta \psi' &= -\frac{1}{4}\theta^{\bar{i}j}e_{\bar{i}}(e_j \psi') + \psi' \theta_{\mathbf{I}} \\ \delta \epsilon' &= -\frac{1}{4}\theta^{\bar{i}j}e_{\bar{i}}(e_j \epsilon') + \epsilon' \theta_{\mathbf{I}}.\end{aligned}\tag{4.109}$$

There is more structure in the internal symmetry algebra $\mathbf{int}_N(\mathbb{A}_n)$ than this might at first suggest. Specifically, $\mathbf{int}_N(\mathbb{A}_n)$ is a direct sum of Lie algebras, one of which is $\mathfrak{sa}(\mathcal{N}, \mathbb{D})$, as expected from the R-symmetry (4.102). This can be seen as follows. From (4.103) it is clear that $[T, \mathfrak{so}(N)_{\text{ST}}] = 0$ if and only if $[T, T_i] = 0$ for every i . Writing $T = (\widehat{A}, \widehat{B}, \widehat{C})$, for the spinors this condition translates to

$$[\widehat{B}, \widehat{B}_i] = 0, \quad [\widehat{C}, \widehat{C}_i] = 0,\tag{4.110}$$

where $\widehat{B}, \widehat{C} \in \mathfrak{so}(n)$. This means the internal symmetry is generated by the subset of $\mathfrak{so}(n)$ matrices that commutes with the Clifford algebra generators, which according to (4.100) means that the internal symmetry contains a factor of $\mathfrak{sa}(\mathcal{N}, \mathbb{D})$. Indeed one finds that for $N > 1$

$$\mathbf{int}_N(\mathbb{A}_n) = \mathfrak{sa}(\mathcal{N}, \mathbb{D}) \oplus \mathfrak{er}_N(\mathbb{A}_n),\tag{4.111}$$

where $\mathfrak{er}_N(\mathbb{A}_n)$ is a possible commuting $\mathfrak{u}(1)$:

$$\mathfrak{er}_N(\mathbb{A}_n) := \mathbf{int}_N(\mathbb{A}_n) \ominus \mathfrak{sa}(\mathcal{N}, \mathbb{D}) = \begin{cases} \mathfrak{u}(1), & (N, n) = (2, 2), (2, 4), (6, 8) \\ \emptyset, & \text{otherwise.} \end{cases}\tag{4.112}$$

For $N = 2$ (i.e. $D = 4$) these additional factors correspond to the inclusion of the CPT conjugate. It is particularly convenient that the definition of $\mathbf{int}_N(\mathbb{A}_n)$ is such that in $D = 4$ the internal symmetry works out to be the R-symmetry $\mathfrak{u}(1), \mathfrak{u}(2), \mathfrak{su}(4)$ for $\mathcal{N} = 1, 2, 4$ super Yang-Mills; the missing $\mathfrak{u}(1)$ in the CPT-self-conjugate $\mathcal{N} = 4$ theory is taken into account automatically:

$$\mathbf{int}_N(\mathbb{A}_n) = \mathfrak{a}(\mathcal{N}, \mathbb{D}) \ominus \delta_{4,D} \delta_{4,\mathcal{N}} \mathfrak{u}(1), \quad D = N + 2 > 3.\tag{4.113}$$

Also intriguing is the fact that the two equations (4.83) and (4.111) are compatible with one another only because of the existence of the so-called ‘accidental’ Lie

algebra isomorphisms

$$\begin{aligned}
\mathfrak{so}(2) &\cong \mathfrak{u}(1), & \mathfrak{so}(3) &\cong \mathfrak{su}(2) \cong \mathfrak{sp}(1), & \mathfrak{so}(4) &\cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1), \\
\mathfrak{so}(5) &\cong \mathfrak{sp}(2), & \mathfrak{so}(6) &\cong \mathfrak{su}(4). & &
\end{aligned} \tag{4.114}$$

Moreover, these constitute *all* the low-dimensional (compact) Lie algebra isomorphisms, *every* one of which is relied upon for the consistency of equations (4.83) and (4.111).

| $\text{Cl}_0(N) \setminus \mathbb{A}_n$ | \mathbb{O} $\mathcal{Q} = 16$ | \mathbb{H} $\mathcal{Q} = 8$ | \mathbb{C} $\mathcal{Q} = 4$ | \mathbb{R} $\mathcal{Q} = 2$ |
|---|--|---|---|---|
| $\mathbb{R}[8] \oplus \mathbb{R}[8]$ | $\mathfrak{so}(8)_{\text{ST}}$ $\cong \mathfrak{tri}(\mathbb{O})$ | - | - | - |
| $\mathbb{R}[8]$ | $\mathfrak{so}(7)_{\text{ST}}$ | - | - | - |
| $\mathbb{C}[4]$ | $\mathfrak{so}(6)_{\text{ST}} \oplus \mathfrak{so}(2)$ $\cong \mathfrak{su}(4)_{\text{ST}} \oplus \mathfrak{u}(1)$ | - | - | - |
| $\mathbb{H}[2]$ | $\mathfrak{so}(5)_{\text{ST}} \oplus \mathfrak{so}(3)$ $\cong \mathfrak{sp}(2)_{\text{ST}} \oplus \mathfrak{sp}(1)$ | - | - | - |
| $\mathbb{H} \oplus \mathbb{H}$ | $\mathfrak{so}(4)_{\text{ST}} \oplus \mathfrak{so}(4)$ $\cong 2\mathfrak{sp}(1)_{\text{ST}} \oplus 2\mathfrak{sp}(1)$ | $\mathfrak{so}(4)_{\text{ST}} \oplus \mathfrak{sp}(1)$ $\cong 2\mathfrak{sp}(1)_{\text{ST}} \oplus \mathfrak{sp}(1)$ $\cong \mathfrak{tri}(\mathbb{H})$ | - | - |
| \mathbb{H} | $\mathfrak{so}(3)_{\text{ST}} \oplus \mathfrak{so}(5)$ $\cong \mathfrak{sp}(1)_{\text{ST}} \oplus \mathfrak{sp}(2)$ | $\mathfrak{so}(3)_{\text{ST}} \oplus \mathfrak{sp}(1)$ $\mathfrak{sp}(1)_{\text{ST}} \oplus \mathfrak{sp}(1)$ | - | - |
| \mathbb{C} | $\mathfrak{so}(2)_{\text{ST}} \oplus \mathfrak{so}(6)$ $\cong \mathfrak{u}(1)_{\text{ST}} \oplus \mathfrak{su}(4)$ | $\mathfrak{so}(2)_{\text{ST}} \oplus \mathfrak{so}(2) \oplus \mathfrak{sp}(1)$ $\cong \mathfrak{u}(1)_{\text{ST}} \oplus \mathfrak{u}(2)$ | $\mathfrak{so}(2)_{\text{ST}} \oplus \mathfrak{u}(1)$ $\cong \mathfrak{u}(1)_{\text{ST}} \oplus \mathfrak{u}(1)$ $\cong \mathfrak{tri}(\mathbb{C})$ | - |
| \mathbb{R} | $\mathfrak{so}(7)$ | $\mathfrak{so}(3) \oplus \mathfrak{sp}(1)$ $\cong \mathfrak{so}(4)$ | $\mathfrak{u}(1)$ $\cong \mathfrak{so}(2)$ | \emptyset $\cong \mathfrak{tri}(\mathbb{R})$ |

Table 4.4.: The symmetry algebras of pure super Yang-Mills on-shell: $\mathfrak{so}(N)_{\text{ST}} \oplus \mathfrak{int}_N(\mathbb{A}_n)$. Each slot corresponds to the SYM theory in $D = N + 2$ dimensions with $\mathcal{Q} = 2n$ real supercharge components. Note that *every* pure SYM theory is included in this table.

Table 4.4 shows the overall (spacetime plus internal) symmetry of each pure super Yang-Mills in each dimension $D = N + 2$. In each dimension $D > 3$ the internal symmetry $\mathfrak{int}_N(\mathbb{A}_n)$ is just the R-symmetry. Contrastingly, in $D = 3$ the algebra $\mathfrak{int}_N(\mathbb{A}_n)$ coincides with the R-symmetry only for the $\mathcal{N} = 1, 2, 4$ theories; the maximal $\mathcal{N} = 8$ theory is an exceptional case. The R-symmetry for $D = 3$ (that is, the automorphisms of the supersymmetry algebra) for a theory with \mathcal{N} supersymmetries

in $D = 3$ is $\mathfrak{so}(\mathcal{N})$, which simply rotates the \mathcal{N} supercharges into one another in the vector representation. For the maximal $D = 3$, $\mathcal{N} = 8$ theory the R-symmetry is then $\mathfrak{so}(8)$ rather than the $\mathfrak{so}(7)$ given in Table 4.4. The full R-symmetry $\mathfrak{so}(8)$ is only realised when the vector is dualised to a scalar, which can only be carried out in the free Yang-Mills theory with coupling constant $g = 0$ (see the Lagrangian in the next subsection)⁷.

4.3.3. The Master Lagrangian

Dimensionally reducing the minimally supersymmetric Lagrangian (4.33) from $D = n + 2$ to $D = N + 2$ using the techniques described in the previous subsections results in a ‘master Lagrangian’, whose input is the dimension D and the division algebra \mathbb{A}_n , and whose output is the Yang-Mills theory in $D = N + 2$ with \mathcal{N} supersymmetries (recall that $n = \mathcal{N}S_N = \mathcal{Q}/2$). The higher-dimensional vector $A' \in \mathfrak{h}_2(\mathbb{A}_n)$ decomposes as

$$A' = \begin{pmatrix} -A^0 + A^{n+1} & A^{a+1}e_a^* \\ A^{a+1}e_a & -A^0 - A^{n+1} \end{pmatrix} + \begin{pmatrix} 0 & (\phi_{\bar{i}}e_{\bar{i}})^* \\ \phi_{\bar{i}}e_{\bar{i}} & 0 \end{pmatrix} = A + \phi\varepsilon. \quad (4.115)$$

The spinor $\Psi \in (\mathbb{A}_n)^2$ is left just as it is, on the understanding that there are actually \mathcal{N} spinors in $D = N + 2$ dimensions, each valued in $(\mathcal{S}_N)^2$. The resulting action is

$$\begin{aligned} S_D(\mathbb{A}_n) = \int d^D x \Big(& -\frac{1}{4}F_{\mu\nu}^A F^{A\mu\nu} - \frac{1}{2}D_\mu\phi^{A*} D^\mu\phi^A - \text{Re}(i\Psi^{\dagger A}\bar{\sigma}^\mu D_\mu\Psi^A) \\ & - \frac{1}{16}g^2 f_{BC}{}^A f_{DE}{}^A (\phi^{B*}\phi^D + \phi^{D*}\phi^B)(\phi^{C*}\phi^E + \phi^{E*}\phi^C) \\ & - g f_{BC}{}^A \text{Re}(i\Psi^{\dagger A}\varepsilon\phi^B\Psi^C) \Big) \end{aligned} \quad (4.116)$$

where the set $\{\bar{\sigma}^\mu\}$ is a D -dimensional basis for matrices of the form that A takes in (4.115), i.e. \mathbb{A}_n -valued Hermitian matrices. The supersymmetry transformations are

$$\begin{aligned} \delta A^A &= i(\Psi^A\epsilon^\dagger - \epsilon\Psi^{\dagger A})_{\mathbb{R}^N}, \\ \delta\phi^A &= -\frac{i}{2}\text{Tr}(\varepsilon(\Psi^A\epsilon^\dagger - \epsilon\Psi^{\dagger A})_{\mathbb{R}^{N-n}}), \\ \delta\Psi^A &= \frac{1}{2}\hat{F}^A\epsilon + \frac{1}{2}\sigma^\mu\varepsilon(D_\mu\phi^A\epsilon) + \frac{1}{4}f_{BC}{}^A\phi^C(\phi^B\epsilon), \end{aligned} \quad (4.117)$$

⁷Incidentally, this is the reason for the insistence on using the generic term ‘internal symmetry’ throughout this chapter, rather than just writing ‘R-symmetry’.

where the subscripts \mathbb{R}^N and \mathbb{R}^{n-N} refer to the respective projections onto these subspaces of \mathbb{A}_n .

To obtain the conventional actions, one can always multiply out the division algebra basis elements as appropriate to the theory of interest. For example, the $D = 4$, $\mathcal{N} = 4$ theory has $\mathbb{A}_n = \mathbb{A}_8 = \mathbb{O}$ and $\mathcal{S}_N = \mathcal{S}_2 = \mathbb{C}$, so the fermions form an octonion, written to look like a four complex numbers – see equation (A.14) in Appendix A.1.2. Multiplying out the basis elements $e_{\hat{a}}$ returns the conventional action for $D = 4$, $\mathcal{N} = 4$, in terms of four complex fermions $\Psi_{\hat{a}}$ and six real scalars $\phi_{\hat{i}}$:

$$\begin{aligned}
S(\mathbb{C}, \mathbb{O}) = \int d^4x \left(& -\frac{1}{4} F_{\mu\nu}^A F^{A\mu\nu} - \frac{1}{2} D_\mu \phi_{\hat{i}}^A D^\mu \phi_{\hat{i}}^A - \text{Re}(i \Psi_{\hat{a}}^{\dagger A} \bar{\sigma}^\mu D_\mu \Psi_{\hat{a}}^A) \right. \\
& - \frac{1}{4} g^2 f_{BC}^A f_{DE}^A \phi_{\hat{i}}^B \phi_{\hat{i}}^D \phi_{\hat{j}}^C \phi_{\hat{j}}^E \\
& \left. - \frac{i}{2} g f_{BC}^A \phi_{\hat{i}}^B \left(\Psi_{\hat{a}}^{\dagger A} \varepsilon^{\hat{i}\hat{a}\hat{b}} \Psi_{\hat{b}}^C + \Psi_{\hat{a}}^{\dagger A} \varepsilon^{\hat{i}\hat{a}\hat{b}} \Psi_{\hat{b}}^{C*} \right) \right), \quad (4.118)
\end{aligned}$$

where the complex matrices $\Upsilon_{\hat{a}\hat{b}}^{\hat{i}}$ are defined in Appendix A.1.2.

Since the master Lagrangian comes from dimensional reduction of the fundamental $\mathcal{N} = 1$ Lagrangians in $D = 3, 4, 6, 10$, it is guaranteed to be supersymmetric by the proof given in the previous section. To close the supersymmetry algebra off-shell it is clear that the appropriate auxiliary field is valued in $\text{Im}(\mathbb{A}_n)$; otherwise, the form of the terms in the supersymmetry transformations (4.60) is unchanged. Interestingly, the transformations in the $D = 3$ octonionic case ($N = 1$, $\mathcal{N} = 8$) are Lorentz-covariant with symmetry $\text{SO}(1, 2) \times G_2$. However, this must be broken to $\text{SO}(1, 1) \times G_2$ to close the algebra [29], since one must still impose the constraint of equation (4.62).

4.4. Summary

This chapter gave a demonstration of how any super Yang-Mills theory in $D = N + 2$ may be written using a pair of algebras: $\text{Cl}(N - 1) \cong \text{Cl}_0(N)$ and \mathbb{A}_n . The internal symmetry $\mathbf{int}_N(\mathbb{A}_n)$ is the subalgebra of $\mathbf{tri}(\mathbb{A}_n)$ for which one element of the triality triple $(\hat{A}, \hat{B}, \hat{C})$ – say \hat{A} – annihilates the subspace $\mathbb{R}^N \subset \mathbb{A}_n$. Imaginary \mathbb{A}_n -valued auxiliary fields may be used to close the non-maximal supersymmetry algebra off-shell, while the failure to close for maximally supersymmetric theories is attributed directly to the non-associativity of the octonions.

Pure super Yang-Mills in D dimensions with \mathcal{N} supersymmetries can always be thought of as the descendant of a $D = n + 2$ theory with $\mathcal{N} = 1$ – its ‘oxidation endpoint’ – written over \mathbb{A}_n with $\mathcal{Q} = 2n$ real supercharge components. However,

if one would rather not think in terms of dimensional reduction (or oxidation), then the number of real supercharge components $\mathcal{Q} = 2n$ is sufficient to associate a SYM theory in D dimensions with the n -dimensional division algebra \mathbb{A}_n .

Of course the master Lagrangian (4.116) would be cumbersome to work with; the key point of the dimensional reductions is to highlight how the triality algebra breaks up into pieces in each of the dimensions, as this will be important for understanding supergravity symmetries in the coming chapters. In $D = 3$ however the $\mathcal{N} = 1, 2, 4, 8$ theories have a particularly simple form written over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, and these theories will be used in the next chapter to construct a magic square of supergravities.

5. A Magic Square from Yang-Mills Squared

The octonions provide an intuitive basis for defining the five exceptional groups G_2 , F_4 , E_6 , E_7 and E_8 . For example, the smallest of these G_2 is the group of automorphisms of \mathbb{O} . Efforts to understand the remaining exceptional groups geometrically in terms of octonions resulted in the Freudenthal-Rosenfeld-Tits *magic square* [55, 56, 57, 58, 59, 60] presented in Table 5.1. Each slot gives the Lie algebra of the isometries of the ‘projective plane’¹ over a tensor product of division algebras $\mathbb{A}_L \otimes \mathbb{A}_R$, with exceptional groups appearing whenever one of the two algebras is \mathbb{O} . A detailed construction of the magic square will be given in Section 5.2.

| $\mathbb{A}_L \backslash \mathbb{A}_R$ | \mathbb{R} | \mathbb{C} | \mathbb{H} | \mathbb{O} |
|--|--------------------|--|---------------------|------------------|
| \mathbb{R} | $\mathfrak{so}(3)$ | $\mathfrak{su}(3)$ | $\mathfrak{sp}(3)$ | \mathfrak{f}_4 |
| \mathbb{C} | $\mathfrak{su}(3)$ | $\mathfrak{su}(3) \times \mathfrak{su}(3)$ | $\mathfrak{su}(6)$ | \mathfrak{e}_6 |
| \mathbb{H} | $\mathfrak{sp}(3)$ | $\mathfrak{su}(6)$ | $\mathfrak{so}(12)$ | \mathfrak{e}_7 |
| \mathbb{O} | \mathfrak{f}_4 | \mathfrak{e}_6 | \mathfrak{e}_7 | \mathfrak{e}_8 |

Table 5.1.: The magic square $\mathcal{L}_3(\mathbb{A}_L, \mathbb{A}_R)$ of compact real forms.

In apparently completely different developments, a popular thread in attempts to understand the quantum theory of gravity is the idea of ‘gravity as the square of Yang-Mills’. The idea in its most basic form is that a symmetric tensor $g_{\mu\nu}$ can be built from the symmetric tensor product of two vector fields A_μ and \tilde{A}_μ as $g_{\mu\nu} \sim A_{(\mu}\tilde{A}_{\nu)}$ (or rather any symmetric tensor may be written as a sum of such tensor products). It is easy to see that this can incorporate supersymmetry; tensoring a minimal Yang-Mills multiplet $\{A_\mu, \lambda\}$ (in $D = 4$ say) with a vector field \tilde{A}_μ results in a minimal supergravity multiplet $\{g_{\mu\nu}, \Psi_\mu\} \sim \{A_{(\mu}\tilde{A}_{\nu)}, \lambda\tilde{A}_\mu\}$.

This idea of tensoring Yang-Mills multiplets appears in many different guises, which often overlap: KLT relations in string theory [61], $D = 10$ Type IIA and

¹Note that the manifolds associated with the $(\mathbb{H} \otimes \mathbb{O})$ and $(\mathbb{O} \otimes \mathbb{O})$ cases are not strictly speaking projective spaces, but nevertheless constitute geometries which are often referred to as projective planes.

IIB supergravity multiplets from $D = 10$ super Yang-Mills multiplets [15], asymmetric orbifold constructions [62], gravity anomalies from gauge anomalies [63], (super)gravity scattering amplitudes from those of (super) Yang-Mills [64, 65, 66] in various dimensions and even classical general relativity solutions from classical gauge theory solutions [67, 68]. While it would seem there is now a growing web of relations connecting gravity to ‘gauge \times gauge’, it is as yet not clear to what extent gravity may really be regarded as the square of Yang-Mills. In an attempt to address this question, Section 5.1 will contain some further discussion of what it means to tensor Yang-Mills multiplets at the classical linearised level.

Bringing all of this together, the culmination of this chapter in Section 5.3 is the following result: tensoring pairs of $D = 3$ SYM multiplets with $\mathcal{N}_L = 1, 2, 4, 8$ and $\mathcal{N}_R = 1, 2, 4, 8$ yields a magic square of $D = 3$ supergravity theories with $\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R = 2, 3, 4, 5, 6, 8, 9, 10, 12, 16$, as presented in Table 5.6. For $\mathcal{N} > 8$ the resulting multiplets are those of pure supergravity; for $\mathcal{N} \leq 8$ pure supergravity is coupled to vector multiplets. In both cases the field content is such that the U-dualities exactly match the groups of the magic square, Table 5.3. This will be interpreted using the division algebra description of $D = 3$ Yang-Mills with $\mathcal{N} = 1, 2, 4, 8$ over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, as introduced in the previous chapter.

5.1. Gravity as the Square of Yang-Mills

In recent years gauge and gravitational scattering amplitudes have been subject to something of a renaissance [69], resulting not only in dramatic computational advances but also important conceptual insights. One such development, straddling both the technical and conceptual, is the *colour-kinematic duality* of gauge amplitudes introduced by Bern, Carrasco and Johansson [64]. Exploiting this duality it has been shown that gravitational amplitudes may be reconstructed using a double-copy of gauge amplitudes, suggesting a possible interpretation of perturbative gravity as ‘the square of Yang-Mills’ [65, 70]. This perspective has proven itself remarkably effective, rendering possible previously intractable gravitational scattering amplitude calculations [71]; it is both conceptually suggestive and technically advantageous. Yet, the idea of gravity as the square of Yang-Mills is not specific to amplitudes, having appeared previously in a number of different, but often related, contexts [61, 15, 62, 63, 72, 67, 68].

These many-faceted relations have furthered our understanding of (super)gravity itself. For example, the Bern-Carrasco-Johansson (BCJ) color-kinematic duality [64, 65] has facilitated the computation of higher-loop $D = 4, \mathcal{N} = 8$ supergravity

amplitudes previously regarded as beyond reach. See, for example, [71] and the references therein. This promises to answer the long-standing questions [73] of when and how perturbative $\mathcal{N} = 8$ supergravity diverges – if indeed it diverges at all [74].

In spite of these remarkable developments, it is still not entirely clear what precisely it means to say that gravity is the square of Yang-Mills. This thesis will address this question mainly at the group-theoretic level, where the idea is simply to take the symmetric tensor product of a pair of momentum-space Yang-Mills vector fields $A_\mu(k)$ and $\tilde{A}_\nu(k)$ in order to construct a symmetric traceless matrix $h_{\mu\nu}(k) = A_{(\mu}\tilde{A}_{\nu)} - \frac{1}{D}A^\rho\tilde{A}_\rho\eta_{\mu\nu}$, which could be interpreted as the momentum-space graviton field, i.e. the Fourier transform $h_{\mu\nu}(k)$ of a small deviation $h_{\mu\nu}(x)$ of the metric $g_{\mu\nu}(x)$ from Minkowski space, $g_{\mu\nu}(x) \simeq \eta_{\mu\nu} + h_{\mu\nu}(x)$.

This can be generalised to supersymmetric theories. A pure \mathcal{N} -extended super Yang-Mills theory in D spacetime dimensions has field content $\{A_\mu, \lambda^I, \phi^{\bar{i}}\}$, where the (defining representation) R-symmetry index I labels the \mathcal{N} fermions and \bar{i} labels the $(\mathcal{Q}/2 + 2 - D)$ scalars, with \mathcal{Q} the total number of real supercharge components (so for example $\mathcal{N} = 4$ SYM in $D = 4$ has $\mathcal{Q} = 16$ and hence contains $16/2 + 2 - 4 = 6$ scalar fields). Temporarily ignoring gauge indices, tensoring a (momentum-space) SYM multiplet $\{A_\mu, \lambda^I, \phi^{\bar{i}}\}$ with another $\{\tilde{A}_\mu, \tilde{\lambda}^{I'}, \tilde{\phi}^{\bar{i}'}\}$ produces the field content of a supergravity theory:

$$\begin{array}{c|ccc}
\otimes & \tilde{A}_\nu & \tilde{\lambda}^{I'} & \tilde{\phi}^{\bar{i}'} \\
\hline
A_\mu & h_{\mu\nu} + B_{\mu\nu} + \varphi & \Psi_\mu^{I'} + \chi^{I'} & A_\mu^{\bar{i}'} \\
\lambda^I & \Psi_\nu^I + \chi^I & \varphi_{\text{RR}}^{II'} + \dots & \chi^{I\bar{i}'} \\
\phi^{\bar{i}} & A_\nu^{\bar{i}} & \chi^{I'\bar{i}} & \varphi^{\bar{i}\bar{i}'}
\end{array} \tag{5.1}$$

where the symbol \otimes denotes the tensor product of Lorentz representations, which are carried out as follows. The vector-vector tensor product $A_\mu\tilde{A}_\nu$ gives

$$h_{\mu\nu} = A_{(\mu}\tilde{A}_{\nu)} - \frac{1}{D}A^\rho\tilde{A}_\rho\eta_{\mu\nu}, \quad B_{\mu\nu} = A_{[\mu}\tilde{A}_{\nu]}, \quad \varphi = A^\mu\tilde{A}_\mu, \tag{5.2}$$

while the fermion-vector tensor products $A_\mu\tilde{\lambda}^{I'}$ and $\lambda^I\tilde{A}_\mu$ give

$$\begin{aligned}
\Psi_\mu^I &= \frac{1}{2}\gamma^\nu\gamma_\mu\lambda^I\tilde{A}_\nu, & \Psi_\mu^{I'} &= \frac{1}{2}\gamma^\nu\gamma_\mu A_\nu\tilde{\lambda}^{I'}, \\
\chi^I &= \frac{1}{2}\gamma^\mu\lambda^I\tilde{A}_\mu, & \chi^{I'} &= \frac{1}{2}\gamma^\mu A_\mu\tilde{\lambda}^{I'}.
\end{aligned} \tag{5.3}$$

The tensor product of any field with the scalars is trivial, giving

$$\begin{aligned} A_{\mu}^{\bar{i}} &= \phi^{\bar{i}} \tilde{A}_{\mu}, & A_{\mu}^{\bar{i}'} &= A_{\mu} \tilde{\phi}^{\bar{i}'}, \\ \chi^{I\bar{i}'} &= \lambda^I \tilde{\phi}^{\bar{i}'}, & \chi^{I'\bar{i}} &= \phi^{\bar{i}} \tilde{\lambda}^{I'}, \end{aligned} \quad (5.4)$$

and

$$\varphi^{\bar{i}\bar{i}'} = \phi^{\bar{i}} \tilde{\phi}^{\bar{i}'}. \quad (5.5)$$

Finally, the fermion-fermion tensor product must be decomposed using Fierz expansion, which gives a different result in each dimension, depending on the symmetry of the gamma matrices. In string theory parlance this is known as the Ramond-Ramond sector. Although the particular set of Ramond-Ramond p -forms $\phi_{\text{RR}}^{II'} + \dots$ is dimension-dependent, there is always a set of $\mathcal{N}_L \mathcal{N}_R$ scalar fields $\phi_{\text{RR}}^{II'}$, which are related to the Yang-Mills fermions as

$$i\bar{\lambda}^I \gamma_{\mu} \tilde{\lambda}^{I'} = \partial_{\mu} \phi_{\text{RR}}^{II'}. \quad (5.6)$$

In general it is the $(p+1)$ -form field strengths F of Ramond-Ramond p -forms that are produced from the fermion-fermion products:

$$i\bar{\lambda}^I \gamma_{\mu_1 \dots \mu_{p+1}} \tilde{\lambda}^{I'} = F_{\mu_1 \dots \mu_{p+1}}. \quad (5.7)$$

This can be seen from dimensional analysis, which will be discussed below.

Since the Yang-Mills R-symmetry (fundamental representation) indices take values $I = 1, \dots, \mathcal{N}_L$ and $I' = 1, \dots, \mathcal{N}_R$, there are $\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R$ gravitini Ψ_{μ}^I and $\Psi_{\mu}^{I'}$ in the resulting Yang-Mills-squared multiplet. Indeed the field content produced for each D and $\mathcal{N}_L, \mathcal{N}_R$ constitutes that of a supergravity theory with $\mathcal{N}_L + \mathcal{N}_R$ supersymmetries:

$$[\mathcal{N}_L \text{ SYM}] \otimes [\mathcal{N}_R \text{ SYM}] \rightarrow [\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R \text{ Supergravity}]. \quad (5.8)$$

Attempting to interpret gravity as the ‘square of’ gauge theory, the tensor products above can be considered from one of two viewpoints. The first and more conservative of these is that this is a purely (super-)group-theoretic dictionary, which applies only to supermultiplets as representations of the super-Poincaré group. The second viewpoint is that equations (5.2)–(5.7) represent a genuine ansatz describing gravity fields in terms of Yang-Mills fields at the *linearised* level. This is a fairly new idea, explored in [75].

If equations (5.2)–(5.7) are to be taken seriously as a gravity-Yang-Mills dictio-

nary, the first question is how the two (in general distinct) Yang-Mills gauge groups G_L and G_R are to be accommodated. The Yang-Mills multiplets $\{A_\mu^A, \lambda^{IA}, \phi^{\bar{I}A}\}$ and $\{\tilde{A}_\mu^{A'}, \tilde{\lambda}^{I'A'}, \tilde{\phi}^{\bar{I}'A'}\}$ carry adjoint indices $A = 1, \dots, \dim[G_L]$ and $A' = 1, \dots, \dim[G_R]$, which may be taken into account by introducing a ‘spectator’ scalar function $\Phi_{AA'}(k)$ valued in the bi-adjoint of $G_L \times G_R$. The candidate graviton field $h_{\mu\nu}$ may then be written as

$$h_{\mu\nu} = \Phi_{AA'} \left(A_{(\mu}^A \tilde{A}_{\nu)}^{A'} - \frac{1}{D} A^{\rho A} \tilde{A}_\rho^{A'} \eta_{\mu\nu} \right). \quad (5.9)$$

The appearance of this scalar $\Phi_{AA'}$ seems to be roughly consistent with the observation [76, 77, 78] that at tree-level the product of two SYM amplitudes (or, to be precise, their integrands) produces a gravitational amplitude multiplied by an additional factor that happens to be precisely the appropriate amplitude for a bi-adjoint scalar field with a *cubic* Lagrangian. From this perspective the schematic relation ‘Yang-Mills \times Yang-Mills = gravity’ is replaced by the even more peculiar statement that ‘Yang-Mills \times Yang-Mills = gravity $\times \phi^3$,’ which turns out to lead to the BCJ color-kinematic duality [77].

The relation ‘Yang-Mills \times Yang-Mills = gravity $\times \phi^3$ ’ refers of course to scattering amplitudes calculated and multiplied in momentum space, and the gravity-Yang-Mills dictionary presented above is also for fields in momentum space. The Fourier transform of a product $f(k)g(k)$ of two functions of momentum $f(k)$ and $g(k)$ is a *convolution* in position space $(f \star g)(x)$, defined by

$$(f \star g)(x) = \int d^D y f(y) g(x - y), \quad (5.10)$$

where $f(x)$ and $g(x)$ are the respective Fourier transforms of $f(k)$ and $g(k)$. The position-space version of (5.9) is then

$$h_{\mu\nu} = \Phi_{AA'} \star \left(A_{(\mu}^A \star \tilde{A}_{\nu)}^{A'} - \frac{1}{D} A^{\rho A} \star \tilde{A}_\rho^{A'} \eta_{\mu\nu} \right). \quad (5.11)$$

The equations (5.2)–(5.7) define a complete supergravity-Yang-Mills dictionary, provided that one adopts the convention that juxtaposition of position-space Yang-Mills fields with and without tildes \sim denotes a convoluted contraction with the gauge-bi-adjoint spectator scalar $\Phi_{AA'}(x)$,

$$f(x)\tilde{g}(x) := (\Phi_{AA'} \star f^A \star \tilde{g}^{A'})(x), \quad \forall f(x), \tilde{g}(x), \quad (5.12)$$

while for momentum-space Yang-Mills fields juxtaposition denotes an ordinary product contraction with $\Phi_{AA'}(k)$. The physical meaning of such a description at the field theory level is somewhat mysterious, due to the non-local nature of convolutions.

A remarkable feature of this proposed gravity-Yang-Mills dictionary is that if the two gauge supermultiplets are taken to describe linearised Yang-Mills theories then their respective gauge transformations result in gravitational gauge transformations. A gauge field A_μ^A of linearised Yang-Mills transforms as

$$\delta A_\mu^A(x) = \partial_\mu \sigma^A(x) + f^A{}_{BC} A_\mu^B(x) \theta^C, \quad (5.13)$$

where the $\sigma^A(x)$ are local gauge parameters and the θ^A are global parameters. Transforming the two Yang-Mills fields in (5.11) by (5.13), while transforming the spectator field in the bi-adjoint as

$$\delta \Phi_{AA'} = -f^C{}_{AB} \Phi_{CA'} \theta^B + -\tilde{f}^{C'}{}_{A'B'} \Phi_{AC'} \tilde{\theta}^{B'} \quad (5.14)$$

results in a linearised general coordinate transformation, or spin-2 gauge transformation,

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad (5.15)$$

where, using the shorthand notation of (5.12),

$$\xi_\mu := \frac{1}{2}(\sigma \tilde{A}_\mu + A_\mu \tilde{\sigma}). \quad (5.16)$$

This relies upon the following vital property of convolutions under a derivative:

$$\partial_\mu (f \star g) = (\partial_\mu f) \star g = f \star (\partial_\mu g). \quad (5.17)$$

In fact, the local symmetries of linearised super Yang-Mills give rise via ‘squaring’ to *all* of the local symmetry transformations of the resulting linearised supergravities: spin-2 gauge transformations, local Lorentz transformations, gravitino gauge transformations and p -form gauge transformations [75]. For example, the gravitini $\Psi_\mu^I, \Psi_\mu^{I'}$ constructed as in (5.3) transform under Yang-Mills gauge transformations as

$$\delta \Psi_\mu^I = \partial_\mu \epsilon^I, \quad \delta \Psi_\mu^{I'} = \partial_\mu \epsilon^{I'}, \quad (5.18)$$

where

$$\epsilon^I := \lambda^I \tilde{\sigma}, \quad \epsilon^{I'} := \sigma \tilde{\lambda}^{I'}, \quad (5.19)$$

which is exactly as required for gravitino gauge transformations (see (2.43)).

The dimensions of the gravitational fields on the respective left-hand sides of the ansätze (5.2)–(5.7) must match those of the corresponding Yang-Mills expressions on the right-hand sides. For example, the fields $h_{\mu\nu}$, A_μ^A and $\tilde{A}_\mu^{A'}$ each have mass

dimension $\frac{1}{2}(D - 2)$, so the equation (5.11) is only consistent if $\Phi_{AA'}$ has mass dimension $-\frac{1}{2}(D - 2)$. This is the exact *negative* of the dimension expected for a scalar field. That $\Phi_{AA'}$ has mass dimension $-\frac{1}{2}(D - 2)$ is also consistent with the remaining equations in (5.2)–(5.7) (taking into account the notation defined in (5.12)), since all the fermions have mass dimension $\frac{1}{2}(D - 1)$ and all the bosons have $\frac{1}{2}(D - 2)$. One obvious possible explanation for the negative dimension of $\Phi_{AA'}$ is that it could be related to a bi-adjoint scalar field $\phi_{AA'}$ with canonical kinetic term and cubic interaction term by

$$\Phi_{AA'} = \frac{\phi_{AA'}}{\phi^2}, \quad (5.20)$$

where $\phi^2 = \phi_{AA'}\phi^{AA'}$. Although it is admittedly speculative, (5.20) gives $\Phi_{AA'}$ the correct dimension and is also superficially consistent with the relations between spin s scattering amplitudes $M(s)$ of the kind discussed in the double-copy literature [76, 77, 78], which take the form

$$\begin{aligned} M(2) &= M(1)M^{-1}(0)M(1), \\ M\left(\frac{3}{2}\right) &= M\left(\frac{1}{2}\right)M^{-1}(0)M(1). \end{aligned} \quad (5.21)$$

The main focus of this and the following chapter is on the Yang-Mills origin of the *global* symmetries of supergravities. Supergravities are characterized by non-compact global symmetries G (these are the so-called U-dualities – see Section 2.8) with local compact subgroups H , for example $G = E_{7(7)}$ and $H = \text{SU}(8)$ for $\mathcal{N} = 8$ supergravity in $D = 4$; whereas the initial Yang-Mills theories have global R-symmetries, for example $R = \text{SU}(4)$ for $\mathcal{N} = 4$ in $D = 4$ (see [79] for an approach linking $\text{SU}(4)$ to $\text{SU}(8)$ based on scattering amplitudes). In the following section, the $D = 3$ super Yang-Mills theories from Chapter 4 will be tensored with one another to reveal a magic square of $D = 3$ supergravity theories. Hence, looking through the prism of ‘gravity = gauge \times gauge’ uncovers novel structural features of the symmetries in $D = 3$ supergravity. Understanding supergravity and its symmetries is essential in the context of string/M-theory, since it constitutes their low-energy effective field theory limit. In particular, supergravity has been central in exposing the non-perturbative aspects of string theory. Here, symmetries, especially U-duality, have played a crucial role – for example in constructing black hole solutions – and this highlights their significance.

Three-dimensional supergravity is rather special, since the metric and gravitino carry no dynamical degrees of freedom, while a vector field may be dualised – see Section 2.8 – to a scalar. Thus the dynamical bosonic degrees of freedom are unified as scalar fields of a G/H coset. This throws light on higher-dimensional theories

that produce $D = 3$ supergravities upon dimensional reduction, and this is very much the case in the next chapter when the magic square will be generalised to $D = 3, 4, \dots, 9, 10$ [80, 81]. Moreover, $D = 3$ is intrinsically interesting for a number of reasons [82, 83, 84, 66], one important example being the surprising observation that pure three-dimensional quantum gravity is actually solvable [85, 86].

5.2. Mathematical Interlude: The Magic Square

5.2.1. F_4 and the Exceptional Jordan Algebra

After G_2 , the second smallest exceptional group is F_4 . It has triality intricately built into its structure and is a useful a prototype for constructing the remaining exceptional groups E_6 , E_7 and E_8 (and is a subgroup of each). Arguably the simplest interpretation of F_4 comes via its role as the automorphism group of the *exceptional Jordan algebra*. A Jordan algebra \mathfrak{J} with product \circ is a commutative but non-associative algebra (over a field) satisfying the Jordan identity

$$(X \circ Y) \circ (X \circ X) = X \circ (Y \circ (X \circ X)), \quad X, Y \in \mathfrak{J}. \quad (5.22)$$

Hermitian $N \times N$ matrices $\mathfrak{h}_N(\mathbb{A})$ with $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $N \geq 2$ form a Jordan algebra $\mathfrak{J}_N(\mathbb{A})$ under the product

$$H_1 \circ H_2 := \frac{1}{2}(H_1 H_2 + H_2 H_1), \quad H_1, H_2 \in \mathfrak{h}_N(\mathbb{A}). \quad (5.23)$$

It turns out that Hermitian 3×3 matrices $\mathfrak{h}_3(\mathbb{O})$ also form a Jordan algebra under this product: the exceptional Jordan algebra $\mathfrak{J}_3(\mathbb{O})$. Simple counting shows that this has dimension $(3 \times 1) + (3 \times 8) = 27$, since a matrix of $\mathfrak{h}_3(\mathbb{O})$ has three real elements on its diagonal and three independent octonionic components on its off-diagonal.

Consider the automorphisms $\text{Aut}(\mathfrak{J}_N(\mathbb{A}_n))$ of the Jordan algebra $\mathfrak{J}_N(\mathbb{A}_n)$ – see Subsection 3.2.2 for the definition of automorphisms. It is clear that for the associative division algebras, $\mathbb{A}_n = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the Jordan product (5.23) is preserved by the ‘unitary’ transformation

$$H \rightarrow U H U^\dagger, \quad U^\dagger U = U U^\dagger = \mathbb{1} \quad (5.24)$$

for all $H \in \mathfrak{J}_N(\mathbb{A}_n)$. Writing $U = e^T$, at the Lie algebra level (5.24) becomes

$$\delta H = [T, H], \quad T^\dagger = -T, \quad (5.25)$$

which is by definition a derivation of $\mathfrak{J}_N(\mathbb{A}_n)$, since (5.24) is an automorphism.

In fact, any derivation of $\mathfrak{J}_N(\mathbb{A}_n)$, $\mathbb{A}_n = \mathbb{R}, \mathbb{C}, \mathbb{H}$, may be written in the form (5.25). The Lie algebra of such anti-Hermitian matrices T is $\mathfrak{a}(N, \mathbb{A}_n)$, as defined in (3.143), which is just $\mathfrak{so}(N)$, $\mathfrak{u}(N)$ and $\mathfrak{sp}(N)$ for $\mathbb{A}_n = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively. However, it is not true that $\mathfrak{der}(\mathfrak{J}_N(\mathbb{A}_n))$ is just $\mathfrak{a}(N, \mathbb{A}_n)$, since there is a slight subtlety involving the trace of such anti-Hermitian matrices. Any $N \times N$ matrix M with entries in $\mathbb{A}_n = \mathbb{R}, \mathbb{C}, \mathbb{H}$ may be decomposed into a traceless part M' and a trace part proportional to the identity:

$$M = M' + \frac{1}{N} \text{Tr}(M)\mathbf{1}, \quad (5.26)$$

where

$$M' := M - \frac{1}{N} \text{Tr}(M)\mathbf{1}. \quad (5.27)$$

Doing this for $T \in \mathfrak{a}(N, \mathbb{A}_n)$ gives

$$T = T' + \alpha\mathbf{1}, \quad (5.28)$$

where $\alpha = \frac{1}{N} \text{Tr}(T)$ is pure-imaginary, since T is anti-Hermitian. Equation (5.25) then becomes

$$\delta H = [T', H] + [\alpha, H]. \quad (5.29)$$

For $\mathbb{A}_n = \mathbb{R}, \mathbb{C}$ the second term actually does not contribute – for \mathbb{R} there is no imaginary subspace, so $\alpha = 0$, while \mathbb{C} is commutative, so $\alpha \in \text{Im}(\mathbb{C})$ commutes with any H . Contrastingly, in the quaternionic case, since \mathbb{H} is non-commutative, the second term generates a derivation of \mathbb{H} itself, which acts component-wise on H . It then makes sense to define

$$\mathfrak{sa}(N, \mathbb{A}_n) := \mathfrak{a}'(N, \mathbb{A}_n) + \mathfrak{der}(\mathbb{A}_n), \quad (5.30)$$

where $\mathfrak{a}'(N, \mathbb{A}_n)$ is the space of traceless anti-Hermitian matrices,

$$\mathfrak{a}'(N, \mathbb{A}_n) := \{T' \in \mathbb{A}[N] \mid T'^{\dagger} = -T', \text{Tr}(T') = 0\}, \quad (5.31)$$

in which case the algebra of derivations $\mathfrak{der}(\mathfrak{J}_N(\mathbb{A}_n))$ is just $\mathfrak{sa}(N, \mathbb{A}_n)$ for the three smallest division algebras. The $\mathfrak{der}(\mathbb{A}_n)$ term in (5.30) corresponds to the α commutator in (5.29), which is trivial for $\mathbb{A}_n = \mathbb{R}, \mathbb{C}$, while $\mathfrak{der}(\mathbb{H}) = \mathfrak{so}(3)$. It is easy to see that the algebra $\mathfrak{sa}(N, \mathbb{A}_n)$ is just $\mathfrak{so}(N)$, $\mathfrak{su}(N)$ and $\mathfrak{sp}(N)$ for $\mathbb{A}_n = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively. It will be useful below to note that for these division algebras the

definition (5.30) could alternatively be replaced by

$$\mathfrak{sa}(N, \mathbb{A}_n) := \mathfrak{a}'(N, \mathbb{A}_n) + \mathfrak{so}(n-1), \quad (5.32)$$

since for $\mathbb{A}_n = \mathbb{R}, \mathbb{C}, \mathbb{H}$ the derivations $\mathfrak{der}(\mathbb{A}_n)$ are the same as $\mathfrak{so}(n-1)$ rotations of the imaginary subspace (which are of course trivial for \mathbb{R}, \mathbb{C}).

In fact $\mathfrak{sa}(N, \mathbb{A}_n)$ can also be defined analogously for $\mathbb{A}_n = \mathbb{O}$, $N = 2, 3$, but there is a subtlety: for $\mathbb{A}_n = \mathbb{O}$ the two definitions (5.30) and (5.32) are inequivalent, since $\mathfrak{der}(\mathbb{O}) = \mathfrak{g}_2$, while rotations of the imaginary subspace form the larger group $\mathfrak{so}(7) \supset \mathfrak{g}_2$. For $N = 3$ the first definition (5.30) works well for the octonions. However, it turns out that for $N = 2$ only the second definition (5.32) gives a set closed under the Lie bracket (taking repeated commutators of the set of matrices in the first definition just generates the remaining seven generators of $\mathfrak{so}(7) - \mathfrak{g}_2$). As a result, $\mathfrak{sa}(N, \mathbb{O})$ is defined (in this thesis) as

$$\mathfrak{sa}(N, \mathbb{O}) := \begin{cases} \mathfrak{a}'(2, \mathbb{O}) + \mathfrak{so}(7), & N = 2, \\ \mathfrak{a}'(3, \mathbb{O}) + \mathfrak{der}(\mathbb{O}), & N = 3. \end{cases} \quad (5.33)$$

Then, by construction, for any Hermitian Jordan algebra $\mathfrak{J}_N(\mathbb{A}_n)$ (where N is understood to be ≥ 2 for $\mathbb{A}_n = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and equal to 3 if $\mathbb{A}_n = \mathbb{O}$) the derivations are given by

$$\mathfrak{der}(\mathfrak{J}_N(\mathbb{A}_n)) \cong \mathfrak{sa}(N, \mathbb{A}_n). \quad (5.34)$$

Although 2×2 octonionic Hermitian matrices $\mathfrak{h}_2(\mathbb{O})$ do not strictly form a Jordan algebra, $\mathfrak{sa}(2, \mathbb{A}_n)$ still naturally acts on $\mathfrak{h}_2(\mathbb{A}_n)$ for any division algebra \mathbb{A}_n . Recall that $\mathfrak{h}_2(\mathbb{A}_n)$ is just the space of matrices of the form in (4.10) used to represent $(n+2)$ -dimensional spacetime vectors in the previous chapter. In fact, $\mathfrak{sa}(2, \mathbb{A}_n)$ is none other than $\mathfrak{so}(n+1)$, the ‘spatial’ subalgebra of the Lorentz algebra $\mathfrak{sl}(2, \mathbb{A}_n) \cong \mathfrak{so}(1, n+1)$ leaving the trace (or the ‘time’ component) of $H \in \mathfrak{h}_2(\mathbb{A}_n)$ invariant – see equations (4.5) and (4.6). An infinitesimal Lorentz transformation as in equation (4.11) with parameters $\lambda^{\mu\nu} = -\lambda^{\nu\mu}$ may be split up into boosts λ^{0r} , $r = 1, \dots, (n+1)$, and compact $\mathfrak{so}(n+1)$ rotations λ^{rs} . Then by (4.11) $H \in \mathfrak{h}_2(\mathbb{A}_n)$ transforms under $\mathfrak{sl}(2, \mathbb{A}_n)$ as

$$\begin{aligned} \delta H &= \frac{1}{4} \lambda^{\mu\nu} (\sigma_\mu (\bar{\sigma}_\nu H) - H (\bar{\sigma}_\mu \sigma_\nu)) \\ &= \frac{1}{4} \lambda^{rs} (\sigma_r (\sigma_s H) - H (\sigma_r \sigma_s)) + \frac{1}{2} \lambda^{0r} (\sigma_r H + H \sigma_r), \end{aligned} \quad (5.35)$$

i.e. by combining an $\mathfrak{sa}(2, \mathbb{A}_n) \cong \mathfrak{so}(n+1)$ rotation with parameters λ^{rs} and a boost given by an anti-commutator with $\lambda^{0r} \sigma_r \in \mathfrak{h}'_2(\mathbb{A}_n)$. Incidentally, this highlights that

$\mathfrak{sl}(2, \mathbb{A}_n) \cong \mathfrak{sa}(2, \mathbb{A}_n) + \mathfrak{h}'_2(\mathbb{A}_n)$, just as one might expect². The compact part of the transformation with $\lambda^{0r} = 0$ may be rewritten as

$$\delta H = \widehat{O}H + [T, H], \quad \widehat{O} \in \mathfrak{so}(n-1), \quad T \in \mathfrak{a}'(2, \mathbb{A}_n), \quad (5.36)$$

demonstrating that $\mathfrak{so}(n+1)$ is indeed $\mathfrak{sa}(2, \mathbb{A}_n)$, as defined in (5.32)³. Note that the trace (time component) is invariant under this transformation – the trace part commutes with any T , and, since it is real, is annihilated by any $\widehat{O} \in \mathfrak{so}(n-1)$.

For $N > 2$ (restricted to $N = 3$ in the octonionic case), $\mathfrak{sa}(N, \mathbb{A}_n)$ as given in (5.30) acts on $H \in \mathfrak{h}_N(\mathbb{A}_n)$ as

$$\delta H = \widehat{d}H + [T, H], \quad \widehat{d} \in \mathfrak{der}(\mathbb{A}_n), \quad T \in \mathfrak{a}'(N, \mathbb{A}_n). \quad (5.37)$$

Any $H \in \mathfrak{h}_N(\mathbb{A}_n)$ may be written as a sum of a traceless part and a real trace part

$$H = H' + \frac{1}{N} \text{Tr}(H)\mathbf{1}, \quad (5.38)$$

and only the traceless part H' transforms non-trivially under $\mathfrak{sa}(N, \mathbb{A}_n)$ – the trace part commutes with any T , and, since it is real, is annihilated by any $\widehat{d} \in \mathfrak{der}(\mathbb{A}_n)$.

For $N = 3$, $\mathfrak{sa}(3, \mathbb{A}_n)$ is $\mathfrak{so}(3)$, $\mathfrak{su}(3)$ and $\mathfrak{sp}(3)$ for \mathbb{R} , \mathbb{C} and \mathbb{H} , respectively, while $\mathfrak{sa}(3, \mathbb{O})$ may be taken as the definition of \mathfrak{f}_4 , the Lie algebra of F_4 . The space of traceless anti-Hermitian 3×3 octonionic matrices $\mathfrak{a}'(3, \mathbb{O})$ has dimension $(2 \times 7) + (3 \times 8) = 38$, while $\mathfrak{der}(\mathbb{O}) = \mathfrak{g}_2$ has dimension 14, so by (5.30), the dimension of F_4 must be $38 + 14 = 52$. The 26-dimensional space of Hermitian traceless 3×3 octonionic matrices $\mathfrak{h}'_3(\mathbb{O})$ transforms irreducibly under F_4 and defines its smallest non-trivial representation, the **26**.

It is illuminating to consider the Lie algebra $\mathfrak{f}_4 \cong \mathfrak{sa}(3, \mathbb{O})$ in terms of its 2×2 subalgebra $\mathfrak{so}(9) \cong \mathfrak{sa}(2, \mathbb{O})$, which will emerge as follows. Consider the particular diagonal matrix $P'_0 \in \mathfrak{h}'_3(\mathbb{O})$ defined as

$$P'_0 := \frac{1}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (5.39)$$

(P'_0 is just the traceless part of the matrix $P_0 := \text{diag}(1, 0, 0)$, which will be used

²In general for the associative division algebras $\mathfrak{sl}(N, \mathbb{A}_n) \cong \mathfrak{sa}(N, \mathbb{A}_n) + \mathfrak{h}'_N(\mathbb{A}_n)$. For example $\mathfrak{sl}(N, \mathbb{R})$ is the algebra of all anti-symmetric and symmetric traceless $N \times N$ real matrices.

³Specifically, $\widehat{O}H = \frac{1}{4}\lambda^{i+1, j+1}(e_i(e_j H) - H(e_i e_j))$ and $T = \begin{pmatrix} \alpha & -x^* \\ x & -\alpha \end{pmatrix}$ with $\alpha = \frac{1}{2}\lambda^{1, i+1}e_i$ and $x = \frac{1}{2}(\lambda^{1, n+1} - \lambda^{i+1, n+1}e_i)$.

later in the chapter in Section 5.4). From (5.37), the subalgebra of $\mathfrak{sa}(3, \mathbb{O})$ such that $\delta P'_0 = 0$ acts via derivations and commutation with block matrices of the form

$$T = \begin{pmatrix} -\text{Tr}(t) & 0 \\ 0 & t \end{pmatrix}, \quad (5.40)$$

where t is an anti-Hermitian (but not in general traceless) 2×2 matrix, i.e. an element of $\mathfrak{a}(2, \mathbb{O})$. To see how the transformation (5.37) with the block-diagonal T of (5.40) acts upon a general $H \in \mathfrak{h}'_3(\mathbb{O})$, one may write it in terms of the lower 2×2 block subspace,

$$H = \begin{pmatrix} -\text{Tr}(A) & \Psi^\dagger \\ \Psi & A \end{pmatrix}, \quad A \in \mathfrak{h}_2(\mathbb{O}), \quad \Psi \in \mathbb{O}^2. \quad (5.41)$$

It is easy to check using (5.37) that δH may be written as an $\mathfrak{so}(9)$ transformation:

$$\delta H = \frac{1}{4} \lambda^{rs} \begin{pmatrix} 0 & (\Psi^\dagger \sigma_s) \sigma_r \\ \sigma_r (\sigma_s \Psi) & \sigma_r (\sigma_s A) - A (\sigma_r \sigma_s) \end{pmatrix}. \quad (5.42)$$

Thus in terms of the individual pieces

$$\begin{aligned} \delta \Psi &= \frac{1}{4} \theta^{rs} \sigma_r (\sigma_s \Psi) + \Psi \theta_1, \\ \delta A &= \frac{1}{4} \theta^{rs} (\sigma_r (\sigma_s A) - A (\sigma_r \sigma_s)) \end{aligned} \quad (5.43)$$

(and $\delta(\text{Tr}(A)) = 0$). For $\mathbb{A}_n = \mathbb{O}$, these are just the Spin(9) transformations of a spinor Ψ and vector A as obtained from (4.20) and (4.11), demonstrating the decomposition

$$\mathbf{26} \rightarrow \mathbf{1} + \mathbf{9} + \mathbf{16}. \quad (5.44)$$

The adjoint of F_4 itself decomposes as

$$\mathbf{52} \rightarrow \mathbf{36} + \mathbf{16}, \quad (5.45)$$

so F_4 can be seen as the group resulting from combining the adjoint of Spin(9) with its spinor representation. This is the interpretation of F_4 that will generalise to include E_6 , E_7 and E_8 .

In fact, for any division algebra one may consider breaking $\mathfrak{sa}(3, \mathbb{A}_n)$ into the stabiliser of P'_0 . Since this is just derivations of \mathbb{A}_n and 2×2 matrices $t \in \mathfrak{sa}(2, \mathbb{A}_n)$ as in (5.40), this subalgebra is just $\mathfrak{der}(\mathbb{A}_n) + \mathfrak{a}(2, \mathbb{A}_n) \subset \mathfrak{sa}(3, \mathbb{A}_n)$. It is straightforward to check that this gives $\mathfrak{so}(2)$, $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$, $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ and $\mathfrak{so}(9)$ for $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$,

respectively⁴. In other words, in general these algebras are just the direct sum of two pieces: $\mathfrak{so}(n+1) \cong \mathfrak{sa}(2, \mathbb{A}_n)$ and an extra $\mathfrak{u}(1)$ for $\mathbb{A}_n = \mathbb{C}$ and $\mathfrak{sp}(1)$ for $\mathbb{A}_n = \mathbb{H}$ – precisely the same extra pieces that appear in the triality algebras: $\mathfrak{cr}(\mathbb{A}_n) = \emptyset, \mathfrak{u}(1), \mathfrak{sp}(1), \emptyset$ for $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (see Table 3.2). In each case one may view $\mathfrak{sa}(3, \mathbb{A}_n)$ as the algebra resulting from combining $\mathfrak{sa}(2, \mathbb{A}_n) \oplus \mathfrak{cr}(\mathbb{A}_n)$ with its spinor representation: $(\mathbb{A}_n)^2$.

5.2.2. The Reduced Magic Square

Just as the algebras $\mathfrak{sa}(3, \mathbb{A}_n)$ may be built up from $\mathfrak{so}(n+1)$ subalgebras, the magic square in Table 5.1 may be built up from Lie algebras of orthogonal groups acting on tensor products of division algebras $\mathbb{A}_L \otimes \mathbb{A}_R$. The tensor product $\mathbb{A}_L \otimes \mathbb{A}_R$ is the algebra with basis elements $e_a \tilde{e}_{a'} = \tilde{e}_{a'} e_a$, where e_a and $\tilde{e}_{a'}$ are those of \mathbb{A}_L and \mathbb{A}_R , respectively, with $a = 0, 1, \dots, (n_L - 1)$ and $a' = 0, 1, \dots, (n_R - 1)$. An element of $\mathbb{A}_L \otimes \mathbb{A}_R$ is a linear combination $x_{aa'} e_a \tilde{e}_{a'}$, with $x_{aa'} \in \mathbb{R}$, and the multiplication rule is simply inherited from those of \mathbb{A}_L and \mathbb{A}_R .

Consider the set of 2×2 matrices $\{\sigma_A\}$, $A = 1, \dots, (n_L + n_R)$ defined as

$$\{\sigma_A\} = \{\sigma_{a+1}, \sigma_{a'+1}\varepsilon\} := \left\{ \begin{pmatrix} 0 & e_a^* \\ e_a & 0 \end{pmatrix}, \begin{pmatrix} \tilde{e}_{a'}^* & 0 \\ 0 & -\tilde{e}_{a'} \end{pmatrix} \right\}. \quad (5.46)$$

Note that for the special case $\mathbb{A}_R = \mathbb{R}$ there is only one \mathbb{A}_R basis element $\tilde{e}_{a'} = 1$ and these matrices become the spatial $\text{Spin}(n_L + 1)$ Pauli matrices σ_r from the previous subsection. Defining $\bar{\sigma}_A := \sigma_A^\dagger$, it is easy to check that for any $\Psi, \mathcal{X} \in (\mathbb{A}_L \otimes \mathbb{A}_R)^2$,

$$\begin{aligned} \sigma_A(\bar{\sigma}_B \Psi) + \sigma_B(\bar{\sigma}_A \Psi) &= 2\delta_{AB} \Psi, \\ \bar{\sigma}_A(\sigma_B \Psi) + \bar{\sigma}_B(\sigma_A \Psi) &= 2\delta_{AB} \mathcal{X}. \end{aligned} \quad (5.47)$$

Then, by the usual reasoning, this means that Ψ and \mathcal{X} transform as left- and right-handed Weyl spinors of $\mathfrak{so}(n_L + n_R)$ as

$$\begin{aligned} \delta \Psi &= \frac{1}{4} \theta^{AB} \sigma_A(\bar{\sigma}_B \Psi), \\ \delta \mathcal{X} &= \frac{1}{4} \theta^{AB} \bar{\sigma}_A(\sigma_B \mathcal{X}), \end{aligned} \quad (5.48)$$

where $\theta^{AB} = -\theta^{BA}$ are arbitrary parameters. Writing $\Psi = (\Psi_1, \Psi_2)$ with $\Psi_1, \Psi_2 \in$

⁴In the octonionic case note that $\mathfrak{so}(7) \cong \mathfrak{der}(\mathbb{O}) + \text{Im}(\mathbb{O})$ so that $\mathfrak{a}(2, \mathbb{O}) + \mathfrak{der}(\mathbb{O})$ becomes $\mathfrak{a}'(2, \mathbb{O}) + \mathfrak{so}(7)$, which is just $\mathfrak{so}(9) \cong \mathfrak{sa}(2, \mathbb{O})$.

$\mathbb{A}_L \otimes \mathbb{A}_R$ and summing over θ^{AB} , the Ψ transformation becomes

$$\begin{aligned} \frac{1}{4}\theta^{AB}\sigma_A(\bar{\sigma}_B\Psi) &= \frac{1}{4}\theta^{ab}\begin{pmatrix} e_a^*(e_b\Psi_1) \\ e_a(e_b^*\Psi_2) \end{pmatrix} + \frac{1}{4}\theta^{a'b'}\begin{pmatrix} \tilde{e}_{a'}^*(\tilde{e}_{b'}\Psi_1) \\ \tilde{e}_{a'}(\tilde{e}_{b'}^*\Psi_2) \end{pmatrix} \\ &+ \frac{1}{2}\theta^{aa'}\begin{pmatrix} 0 & -e_a^*\tilde{e}_{a'} \\ e_a\tilde{e}_{a'} & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \end{aligned} \quad (5.49)$$

and similar for \mathcal{X} . The first two terms are just the spinor transformations of $\mathfrak{so}(n_L)$ and $\mathfrak{so}(n_R)$ from (3.103), which demonstrates the decomposition

$$\mathfrak{so}(n_L + n_R) \cong \mathfrak{so}(n_L) \oplus \mathfrak{so}(n_R) + \mathbb{A}_L \otimes \mathbb{A}_R. \quad (5.50)$$

However, in general Ψ and \mathcal{X} are natural irreducible representations of a larger algebra,

$$\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R) := \mathfrak{so}(n_L + n_R) \oplus \mathfrak{rx}(\mathbb{A}_L) \oplus \mathfrak{rx}(\mathbb{A}_R), \quad (5.51)$$

rather than just $\mathfrak{so}(n_L + n_R)$, since when $\mathbb{A}_L, \mathbb{A}_R = \mathbb{C}, \mathbb{H}$ the transformations (5.48) commute with (right-)multiplication of the spinors Ψ and \mathcal{X} by imaginary division algebra elements $\theta_1 \in \text{Im}(\mathbb{A}_L)$ and $\tilde{\theta}_1 \in \text{Im}(\mathbb{A}_R)$:

$$\begin{aligned} \delta\Psi &= \frac{1}{4}\theta^{AB}\sigma_A(\bar{\sigma}_B\Psi) + \Psi\theta_1 + \Psi\tilde{\theta}_1, \\ \delta\mathcal{X} &= \frac{1}{4}\theta^{AB}\bar{\sigma}_A(\sigma_B\mathcal{X}) + \mathcal{X}\theta_1 + \mathcal{X}\tilde{\theta}_1. \end{aligned} \quad (5.52)$$

This generates an extra $\mathfrak{u}(1)$ when one of the two division algebras is \mathbb{C} and an extra $\mathfrak{sp}(1)$ when one of the algebras is \mathbb{H} – see Table 5.2. This is of course reminiscent of the triality algebras in Chapters 3 and 4, and indeed (5.50) means that $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$ may be decomposed into

$$\begin{aligned} \mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R) &\cong \mathfrak{so}(n_L) \oplus \mathfrak{rx}(\mathbb{A}_L) \oplus \mathfrak{so}(n_R) \oplus \mathfrak{rx}(\mathbb{A}_R) + \mathbb{A}_L \otimes \mathbb{A}_R \\ &\cong \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + \mathbb{A}_L \otimes \mathbb{A}_R. \end{aligned} \quad (5.53)$$

The array of Lie algebras given by $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$ is called the *reduced magic square* and is presented in Table 5.2. For the magic square of supergravities these Lie algebras will be shown to be those of the maximal compact subgroups H of the non-compact U-duality groups G . Note that $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{R})$ gives the algebras satisfying $\delta P'_0 = 0$ from the previous subsection.

| | \mathbb{R} | \mathbb{C} | \mathbb{H} | \mathbb{O} |
|--------------|--|---|--|---|
| \mathbb{R} | $\mathfrak{so}(2)$ | $\mathfrak{so}(3) \oplus \mathfrak{u}(1)$ | $\mathfrak{so}(5) \oplus \mathfrak{sp}(1)$ | $\mathfrak{so}(9)$ |
| \mathbb{C} | $\mathfrak{so}(3) \oplus \mathfrak{u}(1)$ | $\mathfrak{so}(4) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ | $\mathfrak{so}(6) \oplus \mathfrak{sp}(1) \oplus \mathfrak{u}(1)$ | $\mathfrak{so}(10) \oplus \mathfrak{u}(1)$ |
| \mathbb{H} | $\mathfrak{so}(5) \oplus \mathfrak{sp}(1)$ | $\mathfrak{so}(6) \oplus \mathfrak{sp}(1) \oplus \mathfrak{u}(1)$ | $\mathfrak{so}(8) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ | $\mathfrak{so}(12) \oplus \mathfrak{sp}(1)$ |
| \mathbb{O} | $\mathfrak{so}(9)$ | $\mathfrak{so}(10) \oplus \mathfrak{u}(1)$ | $\mathfrak{so}(12) \oplus \mathfrak{sp}(1)$ | $\mathfrak{so}(16)$ |

Table 5.2.: The reduced magic square $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$. The Lie algebras are those of the maximal compact subgroups of the groups from the Lorentzian magic square of Table 5.3.

5.2.3. The Magic Square Construction

Equation (5.45) demonstrated that F_4 is the group resulting from combining the adjoint $\mathbf{36}$ of Spin(9) with additional generators transforming as its spinor representation $\mathbf{16}$. This unusual result can be seen as a consequence of Spin(8) triality. Decomposing F_4 into Spin(9) and then further into Spin(8) this becomes clear:

$$\begin{aligned} \mathbf{52} &\rightarrow \mathbf{36} + \mathbf{16} \\ &\rightarrow \mathbf{28} + \mathbf{8}_v + \mathbf{8}_s + \mathbf{8}_c. \end{aligned} \tag{5.54}$$

Thus the action of F_4 on its adjoint rotates the three representations $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ into one another, which is a possibility granted by the discrete triality symmetry of $\mathfrak{tri}(\mathbb{O}) = \mathfrak{so}(8)$. In more octonionic language (5.54) becomes

$$\mathfrak{f}_4 \cong \mathfrak{tri}(\mathbb{O}) + 3\mathbb{O}. \tag{5.55}$$

The commutators of this Lie algebra can be calculated fairly straightforwardly by acting successively on $\mathfrak{h}_3(\mathbb{O})$ using (5.37). First, write the Jordan algebra element $H \in \mathfrak{h}_3(\mathbb{O})$ as

$$H = \begin{pmatrix} h_1 & \psi^* & \chi^* \\ \psi & h_2 & a^* \\ \chi & a & h_3 \end{pmatrix}, \quad a, \psi, \chi \in \mathbb{O}, \quad h_1, h_2, h_3 \in \mathbb{R}. \tag{5.56}$$

Similarly, an element T of $\mathfrak{a}'(3, \mathbb{O})$ may be parameterised as

$$T = \begin{pmatrix} -(\alpha - \beta) & -x_s^* & x_c^* \\ x_s & -\beta & -x_v^* \\ -x_c & x_v & \alpha \end{pmatrix}, \quad x_v, x_s, x_c \in \mathbb{O}, \quad \alpha, \beta \in \text{Im}(\mathbb{O}). \tag{5.57}$$

Temporarily setting $x_v, x_s, x_c = 0$, one finds using (5.37) that α, β and the derivation \widehat{d} combine just such that the three octonionic matrix elements a, ψ, χ of H transform

by none other than the standard $\mathfrak{so}(8)$ triality transformations in (3.124):

$$\delta H = \widehat{d}H + [T, H] = \begin{pmatrix} 0 & \widehat{B}\psi^* & (\widehat{C}\chi)^* \\ \widehat{B}^*\psi & 0 & (\widehat{A}a)^* \\ \widehat{C}\chi & \widehat{A}a & 0 \end{pmatrix}, \quad (5.58)$$

with $(\widehat{A}, \widehat{B}, \widehat{C}) \in \mathfrak{tri}(\mathbb{O})$ defined in (3.124) (and $\widehat{B}^*\psi := (\widehat{B}\psi^*)^*$). This is how $\mathfrak{tri}(\mathbb{O}) \subset \mathfrak{f}_4$ acts on $H \in \mathfrak{h}_3(\mathbb{O})$. The $[\mathfrak{tri}(\mathbb{O}), \mathfrak{tri}(\mathbb{O})]$ commutators are then given by the natural commutators of $\mathfrak{so}(8)$. Now reinstating $(x_v, x_s^*, x_c) \in 3\mathbb{O}$ and setting $(\widehat{A}, \widehat{B}, \widehat{C}) = 0$ gives δH equal to

$$[T, H] = \begin{pmatrix} 2(\langle x_c | \chi \rangle - \langle x_s | \psi \rangle) & \delta\psi^* & \delta\chi^* \\ \delta\psi & 2(\langle x_s | \psi \rangle - \langle x_v | a \rangle) & \delta a^* \\ \delta\chi & \delta a & 2(\langle x_v | a \rangle - \langle x_c | \chi \rangle) \end{pmatrix} \quad (5.59)$$

with

$$\begin{aligned} \delta a &= (h_2 - h_3)x_v + \chi x_s^* - x_c \psi^*, \\ \delta\psi &= (h_1 - h_2)x_s + a^* x_c - x_v^* \chi, \\ \delta\chi &= (h_3 - h_1)x_c + x_v \psi + a x_s, \end{aligned} \quad (5.60)$$

which manifestly makes use of the triality relations (3.109) – this shows that these transformations are covariant under $\mathfrak{so}(8)$. Now, to shorten the notation for the $[3\mathbb{O}, 3\mathbb{O}]$ commutators write

$$(x_v, x_s^*, x_c) \sim \begin{pmatrix} 0 & -x_s^* & x_c^* \\ x_s & 0 & -x_v^* \\ -x_c & x_v & 0 \end{pmatrix} \in 3\mathbb{O} \quad (5.61)$$

(the conjugation on the $\mathfrak{so}(8)$ spinor x_s here is just for convenience; it corresponds to the fact that under $\mathfrak{so}(8)$ it is the octonionic conjugate of the spinor that transforms with the operator \widehat{B} – see equations (3.124)). Then applying the transformations (5.60) twice one finds that

$$\begin{aligned} [(0, 0, x_c), (0, x_s^*, 0)] &= (x_c x_s^*, 0, 0), \\ [(x_v, 0, 0), (0, 0, x_c)] &= (0, x_v^* x_c, 0), \\ [(0, x_s^*, 0), (x_v, 0, 0)] &= (0, 0, x_v x_s), \end{aligned} \quad (5.62)$$

which again uses the triality relations (3.109). Next, applying (5.60) twice for two matrices of the same type shows that

$$\begin{aligned} [(x_v, 0, 0), (y_v, 0, 0)] &= T_{x_v, y_v}, \\ [(0, x_s^*, 0), (0, y_s^*, 0)] &= \vartheta^2 T_{x_s, y_s}, \\ [(0, 0, x_c), (0, 0, y_c)] &= \vartheta T_{x_c, y_c}, \end{aligned} \tag{5.63}$$

where ϑ is the order-three Lie algebra automorphism

$$\vartheta : (\widehat{A}, \widehat{B}, \widehat{C}) \mapsto (\widehat{B}^*, \widehat{C}, \widehat{A}^*), \tag{5.64}$$

introduced in Section 3.3, and $T_{x,y} \in \mathfrak{tri}(\mathbb{O})$ is defined by

$$T_{x,y} := (\widehat{S}_{x,y}, \widehat{R}_y \widehat{R}_{x^*} - \widehat{R}_x \widehat{R}_{y^*}, \widehat{L}_y \widehat{L}_{x^*} - \widehat{L}_x \widehat{L}_{y^*}), \tag{5.65}$$

where

$$\widehat{S}_{x,y} z = \langle x|z \rangle y - \langle y|z \rangle x, \quad \widehat{L}_x y = xy, \quad \widehat{R}_x y = yx. \tag{5.66}$$

This can also be written in terms of the $\mathfrak{so}(8)$ generators $T_{ab} = -T_{ba}$ used in (4.75) and given in (4.76):

$$T_{x,y} := x^a y^b T_{ab}. \tag{5.67}$$

Finally, acting on (5.59) with the transformations of (5.58) and invoking the defining triality relations (3.114), the $[\mathfrak{tri}(\mathbb{O}), 3\mathbb{O}]$ commutators are just given by the natural action of $\mathfrak{tri}(\mathbb{O})$ on $3\mathbb{O}$:

$$\left[(\widehat{A}, \widehat{B}, \widehat{C}), (x_v, x_s^*, x_c) \right] = (\widehat{A}x_v, \widehat{B}x_s^*, \widehat{C}x_c). \tag{5.68}$$

This completes the specification of the commutators of $\mathfrak{f}_4 \cong \mathfrak{tri}(\mathbb{O}) + 3\mathbb{O}$. In fact, the entire discussion in this subsection also works for any other division algebra \mathbb{A}_n since

$$\mathfrak{sa}(3, \mathbb{A}_n) \cong \mathfrak{tri}(\mathbb{A}_n) + 3\mathbb{A}_n, \tag{5.69}$$

with the commutators given as above. This can also be seen as adjoining the subalgebra $\mathfrak{L}_2(\mathbb{A}_n, \mathbb{R}) = \mathfrak{so}(n+1) \oplus \mathfrak{tr}(\mathbb{A}_n)$ with its spinor representation.

By analogy with this, making use of the triality principle also allows the Lie algebra $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R) = \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + \mathbb{A}_L \otimes \mathbb{A}_R$ to be adjoined to its spinor representation $\Psi \in (\mathbb{A}_L \otimes \mathbb{A}_R)^2$ in order to construct a much bigger algebra [87]:

the magic square algebra,

$$\mathfrak{L}_3(\mathbb{A}_L, \mathbb{A}_R) = \mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R) + (\mathbb{A}_L \otimes \mathbb{A}_R)^2, \quad (5.70)$$

which may be written as

$$\mathfrak{L}_3(\mathbb{A}_L, \mathbb{A}_R) = \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + 3(\mathbb{A}_L \otimes \mathbb{A}_R). \quad (5.71)$$

The commutators are defined as follows. First of all, $\mathfrak{tri}(\mathbb{A}_L)$ and $\mathfrak{tri}(\mathbb{A}_R)$ are Lie subalgebras. For elements $T_L = (\widehat{A}_L, \widehat{B}_L, \widehat{C}_L) \in \mathfrak{tri}(\mathbb{A}_L)$ and elements⁵

$$(X_v, X_s^*, X_c) = (x_v \tilde{x}_v, x_s^* \tilde{x}_s^*, x_c \tilde{x}_c) \in 3(\mathbb{A}_L \otimes \mathbb{A}_R),$$

the commutators are given by the natural action of $\mathfrak{tri}(\mathbb{A}_L)$,

$$[T_L, (X_v, X_s^*, X_c)] = (\widehat{A}_L X_v, \widehat{B}_L X_s^*, \widehat{C}_L X_c). \quad (5.72)$$

Similarly for $T_R = (\widehat{A}_R, \widehat{B}_R, \widehat{C}_R) \in \mathfrak{tri}(\mathbb{A}_R)$,

$$[T_R, (X_v, X_s^*, X_c)] = (\widehat{A}_R X_v, \widehat{B}_R X_s^*, \widehat{C}_R X_c). \quad (5.73)$$

The commutators between the three copies of $\mathbb{A}_L \otimes \mathbb{A}_R$ are as follows:

$$\begin{aligned} [(0, 0, x_s \tilde{x}_s), (0, x_c^* \tilde{x}_c^*, 0)] &= \zeta((x_c x_s^*)(\tilde{x}_c \tilde{x}_s^*), 0, 0), \\ [(x_v \tilde{x}_v, 0, 0), (0, 0, x_s \tilde{x}_s)] &= (0, (x_v x_s)(\tilde{x}_v \tilde{x}_s), 0), \\ [(0, x_c^* \tilde{x}_c^*, 0), (x_v \tilde{x}_v, 0, 0)] &= (0, 0, (x_v^* x_c)(\tilde{x}_v^* \tilde{x}_c)), \end{aligned} \quad (5.74)$$

where $\zeta = \pm 1$ is a constant to be chosen, whose two possible values result in different real forms of $\mathfrak{L}_3(\mathbb{A}_L, \mathbb{A}_R)$; selecting $\zeta = 1$ gives the original magic square of compact real forms in Table 5.1, whereas selecting $\zeta = -1$ gives the *Lorentzian* magic square $\mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R)$ [88] of non-compact real forms given in Table 5.3.

Finally, for two elements belonging to the same $(\mathbb{A}_L \otimes \mathbb{A}_R)$ summand in (5.71) the commutator is defined using the natural map

$$\Lambda^2(\mathbb{A}_L \otimes \mathbb{A}_R)_i \rightarrow \Lambda^2(\mathbb{A}_L) \oplus \Lambda^2(\mathbb{A}_R) \rightarrow \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R), \quad (5.75)$$

where $\Lambda^2(\mathbb{A})$ for any algebra \mathbb{A} denotes antisymmetric linear maps on \mathbb{A} . The first

⁵Although a typical element $X \in \mathbb{A}_L \otimes \mathbb{A}_R$ does not factorise into a single tensor product $x\tilde{x}$ with $x \in \mathbb{A}_L$ and $\tilde{x} \in \mathbb{A}_R$, it will simplify the notation in this section to write the commutators only for such factorisable elements. Since a general $X \in \mathbb{A}_L \otimes \mathbb{A}_R$ may always be written as a sum of such elements, it is easy to find the general commutators from those given in this section.

arrow uses the norm on \mathbb{A}_L and \mathbb{A}_R and the second is the inclusion of $\Lambda^2(\mathbb{A}_{L,R})$ inside $\mathfrak{tri}(\mathbb{A}_{L,R})$. More explicitly,

$$\begin{aligned}
[(x_v \tilde{x}_v, 0, 0), (y_v \tilde{y}_v, 0, 0)] &= \langle x_v | y_v \rangle T_{\tilde{x}_v, \tilde{y}_v}^R + \langle \tilde{x}_v | \tilde{y}_v \rangle T_{x_v, y_v}^L, \\
[(0, x_c^* \tilde{x}_c^*, 0), (0, y_c^* \tilde{y}_c^*, 0)] &= \zeta \langle x_c^* | y_c^* \rangle \vartheta T_{\tilde{x}_c^*, \tilde{y}_c^*}^R + \zeta \langle \tilde{x}_c^* | \tilde{y}_c^* \rangle \vartheta T_{x_c^*, y_c^*}^L, \\
[(0, 0, x_s \tilde{x}_s), (0, 0, y_s \tilde{y}_s)] &= \zeta \langle x_s | y_s \rangle \vartheta^2 T_{\tilde{x}_s, \tilde{y}_s}^R + \zeta \langle \tilde{x}_s | \tilde{y}_s \rangle \vartheta^2 T_{x_s, y_s}^L.
\end{aligned} \tag{5.76}$$

With these commutators the magic square formula (5.71) describes the Lie algebras of the groups presented in Table 5.1 and Table 5.3. This construction finally reveals \mathfrak{e}_6 , \mathfrak{e}_7 and \mathfrak{e}_8 as the analogues of \mathfrak{f}_4 for octonionic tensor product algebras $\mathbb{C} \otimes \mathbb{O}$, $\mathbb{H} \otimes \mathbb{O}$ and $\mathbb{O} \otimes \mathbb{O}$. By counting $\dim[\mathfrak{tri}(\mathbb{A}_L)] + \dim[\mathfrak{tri}(\mathbb{A}_R)] + 3n_L n_R$ one finds that these algebras have dimensions 78, 133 and 248, respectively.

The $\zeta = -1$ case, $\mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R)$, gives the magic square algebra relevant to interpreting $D = 3$ supergravity as the square of Yang-Mills. It is straightforward to check the signatures (real forms) appearing in Table 5.3 as follows. A non-compact real form \mathfrak{g}_{nc} of a complex semi-simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ admits a symmetric decomposition $\mathfrak{g}_{nc} = \mathfrak{h} + \mathfrak{p}$,

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}, \tag{5.77}$$

where \mathfrak{h} is the maximal compact subalgebra. If a compact real form \mathfrak{g}_c shares with some non-compact real form \mathfrak{g}_{nc} a common subalgebra, $\mathfrak{g}_{nc} = \mathfrak{h} + \mathfrak{p}$ and $\mathfrak{g}_c = \mathfrak{h} + \mathfrak{p}'$, and the brackets in $[\mathfrak{h}, \mathfrak{p}]$ are the same as those in $[\mathfrak{h}, \mathfrak{p}']$, but equivalent brackets in $[\mathfrak{p}, \mathfrak{p}]$ and $[\mathfrak{p}', \mathfrak{p}']$ differ by a sign, then \mathfrak{h} is the maximal compact subalgebra of \mathfrak{g}_{nc} . This observation is sufficient to confirm that the $\zeta = -1$ construction yields the real forms in Table 5.3 and that $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$ in each case is the maximal compact subalgebra – see Table 5.2.

| $\mathbb{A}_L \backslash \mathbb{A}_R$ | \mathbb{R} | \mathbb{C} | \mathbb{H} | \mathbb{O} |
|--|-------------------------|--|------------------------|-------------------------|
| \mathbb{R} | $\mathfrak{so}(2, 1)$ | $\mathfrak{su}(2, 1)$ | $\mathfrak{sp}(2, 1)$ | $\mathfrak{f}_{4(-20)}$ |
| \mathbb{C} | $\mathfrak{su}(2, 1)$ | $\mathfrak{su}(2, 1) \times \mathfrak{su}(2, 1)$ | $\mathfrak{su}(4, 2)$ | $\mathfrak{e}_{6(-14)}$ |
| \mathbb{H} | $\mathfrak{sp}(2, 1)$ | $\mathfrak{su}(4, 2)$ | $\mathfrak{so}(8, 4)$ | $\mathfrak{e}_{7(-5)}$ |
| \mathbb{O} | $\mathfrak{f}_{4(-20)}$ | $\mathfrak{e}_{6(-14)}$ | $\mathfrak{e}_{7(-5)}$ | $\mathfrak{e}_{8(8)}$ |

Table 5.3.: The Lorentzian magic square $\mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R)$ of real forms required in $D = 3$ supergravity. The subscripts in parentheses on the exceptional groups are the numbers of non-compact generators minus the number of compact generators. For example, $\mathfrak{e}_{8(8)}$ has 128 non-compact generators from $(\mathbb{O} \otimes \mathbb{O})^2$ and 120 compact generators from $\mathfrak{L}_2(\mathbb{O}, \mathbb{O}) = \mathfrak{so}(16)$, which is why the number in brackets on the subscript is $128 - 120 = 8$.

5.3. The Magic Square of $D = 3$ Supergravities

From Chapter 4, the $D = 3$, $\mathcal{N} = 1, 2, 4, 8$ SYM Lagrangian over $\mathbb{A}_{\mathcal{N}} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ is⁶

$$\begin{aligned} \mathcal{L}(\mathbb{A}_{\mathcal{N}}) = & -\frac{1}{4}F_{\mu\nu}^A F^{A\mu\nu} - \frac{1}{2}D_{\mu}\phi^{*A}D^{\mu}\phi^A - i\bar{\lambda}^A\gamma^{\mu}D_{\mu}\lambda^A \\ & - \frac{1}{16}g^2 f_{BC}{}^A f_{DE}{}^A (\phi^{B*}\phi^D + \phi^{D*}\phi^B)(\phi^{C*}\phi^E + \phi^{E*}\phi^C) \\ & + \frac{i}{2}g f_{BC}{}^A ((\bar{\lambda}^A\phi^B)\lambda^C - \bar{\lambda}^A(\phi^{*B}\lambda^C)), \end{aligned} \quad (5.78)$$

where $\phi = \phi_i e_i$ is an $\text{Im}(\mathbb{A}_{\mathcal{N}})$ -valued scalar field, $\lambda = \lambda_a e_a$ is an $\mathbb{A}_{\mathcal{N}}$ -valued two-component Majorana spinor and $\bar{\lambda} := \lambda_a^{\top} \varepsilon e_a^*$. Note that λ_a is anti-commuting; these are the division algebras defined over Grassmann numbers. In $D = 3$ there is no need to add the Hermitian conjugate of the fermion kinetic term as its imaginary part is a total derivative.

The supersymmetry transformations in this language are given by

$$\begin{aligned} \delta\lambda^A &= \frac{1}{4}(F^{A\mu\nu} + \varepsilon^{\mu\nu\rho}D_{\rho}\phi^A)\gamma_{\mu\nu}\epsilon - \frac{1}{4}g f_{BC}{}^A\phi^B(\phi^C\epsilon), \\ \delta A_{\mu}^A &= \frac{i}{2}(\bar{\lambda}^A\gamma_{\mu}\epsilon - \bar{\epsilon}\gamma_{\mu}\lambda^A), \\ \delta\phi^A &= \frac{i}{2}\text{Tr}(\epsilon\bar{\lambda}^A - \lambda^A\bar{\epsilon}), \end{aligned} \quad (5.79)$$

where ϵ is an $\mathbb{A}_{\mathcal{N}}$ -valued two-component spinor, $\frac{1}{2}\gamma_{\mu\nu} = \frac{1}{2}\gamma_{[\mu}\gamma_{\nu]}$ are the generators of the Lorentz algebra $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(1, 2)$ and the trace on the third line is over the suppressed spinor indices.

The Lagrangian (5.78) and transformation rules (5.79) exhibit a symmetry known in the division-algebraic literature as the ‘intermediate algebra’ [87, 89],

$$\text{int}_1(\mathbb{A}_{\mathcal{N}}) := \{(\widehat{A}, \widehat{B}, \widehat{C}) \in \text{tri}(\mathbb{A}_{\mathcal{N}}) \mid \widehat{A}\mathbb{1} = 0\} \cong \mathfrak{so}(\mathcal{N} - 1) \oplus \mathfrak{rt}(\mathbb{A}_{\mathcal{N}}), \quad (5.80)$$

which is $\text{int}_1(\mathbb{A}_{\mathcal{N}}) = \emptyset, \mathfrak{so}(2), \mathfrak{so}(4), \mathfrak{so}(7)$ for $\mathbb{A}_{\mathcal{N}} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. The condition $\widehat{A}\mathbb{1} = 0$ in (5.80) corresponds to the fact that in the bosonic sector the symmetry leaves the gauge field invariant, while the $\mathfrak{so}(\mathcal{N} - 1)$ piece transforms the $\text{Im}(\mathbb{A}_{\mathcal{N}})$ -valued scalar field in the fundamental representation:

$$\begin{aligned} \delta A_{\mu}^A &= 0, \\ \delta\phi^A &= -\frac{1}{4}\theta^{ij}(e_i(e_j\phi^A) - \phi^A(e_i e_j)). \end{aligned} \quad (5.81)$$

where $\theta^{ij} = -\theta^{ji} \in \mathfrak{so}(\mathcal{N} - 1)$ and $\theta_i \in \mathfrak{rt}(\mathbb{A}_{\mathcal{N}})$. The fermions and supersymmetry

⁶Comparing to the ‘master Lagrangian’ (4.116), the gamma matrices γ_{μ} here can be obtained as $\gamma_{\mu} = -\varepsilon\bar{\sigma}_{\mu}$, with the $\bar{\sigma}_{\mu}$ matrices defined in (4.5) and (4.6).

parameters transform under this symmetry as:

$$\begin{aligned}\delta\lambda^A &= -\frac{1}{4}\theta^{ij}e_i(e_j\lambda^A) + \lambda^A\theta_1, \\ \delta\epsilon &= -\frac{1}{4}\theta^{ij}e_i(e_j\epsilon) + \epsilon\theta_1.\end{aligned}\tag{5.82}$$

Also, note that $\mathbf{int}_1(\mathbb{A}_{\mathcal{N}})$ can also be written as $\mathbf{int}_1(\mathbb{A}_{\mathcal{N}}) \cong \mathbf{der}(\mathbb{A}_{\mathcal{N}}) + \text{Im}(\mathbb{A}_{\mathcal{N}})$.

The form of the first term in the λ^A supersymmetry transformation highlights the vector's status as the 'missing' real part of the $\text{Im}(\mathbb{A}_{\mathcal{N}})$ -valued scalar field. Indeed, in the free $g = 0$ theory one may dualise the vector A_μ to a scalar ϕ_0 to obtain a full $\mathbb{A}_{\mathcal{N}}$ -valued scalar field $\phi^A \rightarrow \phi_0^A + \phi_i^A e_i$:

$$\frac{1}{2}\varepsilon_{\mu\sigma\tau}\eta^{\sigma\nu}\eta^{\tau\rho}F_{\nu\rho}^A = \partial_\mu\phi_0^A.\tag{5.83}$$

In this case the following Lagrangian gives equations of motion and Bianchi identities equivalent to those of (5.78) with $g = 0$,

$$\mathcal{L}(\mathbb{A}_{\mathcal{N}}) = -\frac{1}{2}\partial_\mu\phi^{A*}\partial^\mu\phi^A - i\bar{\lambda}^A\gamma^\mu\partial_\mu\lambda^A,\tag{5.84}$$

where ϕ and λ each take values in $\mathbb{A}_{\mathcal{N}}$. The supersymmetry transformations become:

$$\delta\phi^A = i\text{Tr}\epsilon\bar{\lambda}^A, \quad \delta\lambda^A = -\frac{1}{2}\partial_\mu\phi^{A*}\gamma^\mu\epsilon.\tag{5.85}$$

Equations (5.84) and (5.85) enjoy a global internal symmetry whose Lie algebra is $\mathbf{tri}(\mathbb{A}_{\mathcal{N}}) \cong \emptyset, \mathfrak{u}(1) \oplus \mathfrak{u}(1), \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathfrak{so}(8)$ for $\mathbb{A}_{\mathcal{N}} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively:

$$\begin{aligned}\delta\phi^A &= \frac{1}{4}\theta^{ab}(e_a(e_b^*\phi^A) - \phi^A(e_a^*e_b)), \\ \delta\lambda^A &= \frac{1}{4}\theta^{ab}\sigma^{a+1}(\sigma^{b+1}\lambda^A) + \lambda^A\theta_1, \\ \delta\epsilon &= \frac{1}{4}\theta^{ab}\sigma^{a+1}(\sigma^{b+1}\epsilon) + \epsilon\theta_1.\end{aligned}\tag{5.86}$$

This sum of the R-symmetry $\mathfrak{so}(\mathcal{N})$ and the extra algebras $\mathfrak{tr}(\mathbb{A}_{\mathcal{N}})$ is inherited from the $D = 4, 6, 10$ minimally supersymmetric theories, which dimensionally reduce to give $\mathcal{N} = 2, 4, 8$ in $D = 3$. See Table 5.4 for a clarification of the various $D = 3$ symmetry algebras.

After applying the equations of motion and fixing the gauge of A_μ , the $g = 0$ Yang-Mills theory and its dualised counterpart are equivalent. For a single plane wave solution to the momentum-space equations of motion with momentum $p_\mu = (E, 0, E)$, the gauge field can be written as $A_\mu = (0, A_1, 0)$ (after choosing the radiation gauge), in which case (5.83) has only one non-zero component, equivalent

| $D = 3$ | R-symmetry: $\mathfrak{so}(\mathcal{N})$ | $g \neq 0$ Theory: $\mathfrak{int}_1(\mathbb{A}_{\mathcal{N}})$ | $g = 0$ Theory: $\mathfrak{tri}(\mathbb{A}_{\mathcal{N}})$ |
|-------------------|---|--|--|
| $\mathcal{N} = 1$ | — | — | — |
| $\mathcal{N} = 2$ | $\mathfrak{so}(2)$ | $\mathfrak{so}(2)$ | $\mathfrak{u}(1) \oplus \mathfrak{u}(1)$ |
| $\mathcal{N} = 4$ | $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ | $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ | $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ |
| $\mathcal{N} = 8$ | $\mathfrak{so}(8)$ | $\mathfrak{so}(7)$ | $\mathfrak{so}(8)$ |

Table 5.4.: Symmetries in $D = 3$ SYM theories. The symmetries of the $g = 0$ theories are the triality algebras of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, while the symmetries of the $g \neq 0$ theories are known in the division algebras literature as ‘intermediate algebras’ (these are just the subgroups of the triality algebras such that $\hat{A}1 = 0$ in equation (3.114)).

to the relation $A_1 = \phi_0$. Hence the single dynamical degree of freedom carried by A_μ is of course the exact same degree of freedom carried by the corresponding dual scalar ϕ_0 .

The fermionic equation of motion with $p_\mu = (E, 0, E)$ has the same form as (4.65), which removes one of the $\mathbb{A}_{\mathcal{N}}$ -valued components of the two-component spinor. Thus the overall on-shell degrees of freedom for the $D = 3$ free theory are

$$\phi^A = \phi_0^A + \phi_i^A e_i \in \mathbb{A}_{\mathcal{N}}, \quad \psi^A \in \mathbb{A}_{\mathcal{N}} \quad (5.87)$$

(whether the vector is dualised or not). These transform under supersymmetry as

$$\delta\phi = -i\epsilon\psi^*, \quad \delta\psi = iE\phi^*\epsilon, \quad \epsilon \in \mathbb{A}_{\mathcal{N}}, \quad (5.88)$$

where ϵ has anti-commuting components ϵ_a (as does ψ). It is convenient to take advantage of triality and redefine the parameters θ^{ab}, θ_I such that under $\mathfrak{tri}(\mathbb{A}_{\mathcal{N}})$:

$$\begin{aligned} \delta\phi &= \frac{1}{4}\theta^{ab}e_a(e_b^*\phi) + \phi\theta_I, \\ \delta\psi^* &= \frac{1}{4}\theta^{ab}e_a^*(e_b\psi^*) + \psi^*\theta_I, \\ \delta\epsilon &= \frac{1}{4}\theta^{ab}(e_a(e_b^*\epsilon) - \epsilon(e_a^*e_b)); \end{aligned} \quad (5.89)$$

with respect to the $\mathfrak{so}(\mathcal{N})$ generated by the new θ^{ab} , it is the supersymmetry parameter ϵ that transforms as a vector, while ψ^* is a spinor and ϕ is a conjugate-spinor (this is the triality transformation ϱ defined at the end of Section 3.3). Hence θ^{ab} corresponds to the $\mathfrak{so}(\mathcal{N})$ R-symmetry, which rotates the \mathcal{N} supercharges or \mathcal{N} supersymmetry parameters into one another in the vector representation, while θ_I generates the additional internal global symmetry $\mathfrak{tr}(\mathbb{A}_{\mathcal{N}})$.

Taking an ‘un-dualised’ left SYM multiplet $\{A_\mu \in \text{Re}(\mathbb{A}_L), \phi \in \text{Im}(\mathbb{A}_L), \lambda \in \mathbb{A}_L\}$ and tensoring with a right multiplet $\{\tilde{A}_\mu \in \text{Re}(\mathbb{A}_R), \tilde{\phi} \in \text{Im}(\mathbb{A}_R), \tilde{\lambda} \in \mathbb{A}_R\}$, as

described in Section 5.1, results in the field content of a supergravity theory valued in $\mathbb{A}_L \otimes \mathbb{A}_R$. See Table 5.5. Note that the left/right SYM global symmetries act on each slot of the $\mathbb{A}_L, \mathbb{A}_R$ tensor products.

| $\mathbb{A}_L \backslash \mathbb{A}_R$ | $\tilde{A}_\mu \in \text{Re}(\mathbb{A}_R)$ | $\tilde{\phi} \in \text{Im}(\mathbb{A}_R)$ | $\tilde{\lambda} \in \mathbb{A}_R$ |
|--|---|---|---|
| $A_\mu \in \text{Re}(\mathbb{A}_L)$ | $g_{\mu\nu}, \varphi \in \text{Re}(\mathbb{A}_L) \otimes \text{Re}(\mathbb{A}_R)$ | $\varphi \in \text{Re}(\mathbb{A}_L) \otimes \text{Im}(\mathbb{A}_R)$ | $\Psi_\mu, \chi \in \text{Re}(\mathbb{A}_L) \otimes \mathbb{A}_R$ |
| $\phi \in \text{Im}(\mathbb{A}_L)$ | $\varphi \in \text{Im}(\mathbb{A}_L) \otimes \text{Re}(\mathbb{A}_R)$ | $\varphi \in \text{Im}(\mathbb{A}_L) \otimes \text{Im}(\mathbb{A}_R)$ | $\chi \in \text{Im}(\mathbb{A}_L) \otimes \mathbb{A}_R$ |
| $\lambda \in \mathbb{A}_L$ | $\Psi_\mu, \chi \in \mathbb{A}_L \otimes \text{Re}(\mathbb{A}_R)$ | $\chi \in \mathbb{A}_L \otimes \text{Im}(\mathbb{A}_R)$ | $\varphi \in \mathbb{A}_L \otimes \mathbb{A}_R$ |

Table 5.5.: Tensor product of left/right ($\mathbb{A}_L/\mathbb{A}_R$) SYM multiplets, using $\text{SO}(1,2)$ spacetime representations and dualising all vectors to scalars (and the 2-form $B_{\mu\nu} := A_{[\mu}\tilde{A}_{\nu]}$ absent from the top-left slot is dualised to nothing, since a 3-form field strength is dual to a ‘0-form field strength’, which cannot correspond to any physical field).

Gathering together spacetime fields of the same type, one finds the following overall field content:

$$g_{\mu\nu} \in \mathbb{R}, \quad \Psi_\mu \in \mathbb{A}_L \oplus \mathbb{A}_R, \quad \varphi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}, \quad \chi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}. \quad (5.90)$$

Dynamically speaking, the \mathbb{R} -valued graviton and $(\mathbb{A}_L \oplus \mathbb{A}_R)$ -valued gravitino carry no degrees of freedom, while the $2(\mathbb{A}_L \otimes \mathbb{A}_R)$ -valued scalar φ and Majorana spinor χ each have $2(\dim \mathbb{A}_L \times \dim \mathbb{A}_R) = 2\mathcal{N}_L\mathcal{N}_R$ degrees of freedom. The fact that there are $\dim[\mathbb{A}_L \oplus \mathbb{A}_R] = \mathcal{N}_L + \mathcal{N}_R$ real gravitinos shows that there are $\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R$ supersymmetries.

The dynamical degrees of freedom of the supergravity theories are contained in the doublets $\varphi, \chi \in 2(\mathbb{A}_L \otimes \mathbb{A}_R)$. Such doublets were demonstrated in Subsection 5.2.2 to be the irreducible representation spaces of the reduced magic square algebra

$$\begin{aligned} \mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R) &\cong \mathfrak{so}(\mathcal{N}_L + \mathcal{N}_R) \oplus \mathfrak{cr}(\mathbb{A}_L) \oplus \mathfrak{cr}(\mathbb{A}_R) \\ &\cong \mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + \mathbb{A}_L \otimes \mathbb{A}_R. \end{aligned} \quad (5.91)$$

which by (5.52) transforms them as spinor and conjugate-spinor representations

$$\begin{aligned} \delta\varphi &= \frac{1}{4}\theta^{AB}\sigma_A(\bar{\sigma}_B\varphi) + \varphi\theta_{\mathbf{I}} + \varphi\tilde{\theta}_{\mathbf{I}}, \\ \delta\chi &= \frac{1}{4}\theta^{AB}\bar{\sigma}_A(\sigma_B\chi) + \chi\theta_{\mathbf{I}} + \chi\tilde{\theta}_{\mathbf{I}}, \end{aligned} \quad (5.92)$$

where $A = 1, 2, \dots, (\mathcal{N}_L + \mathcal{N}_R)$, and $\theta_{\mathbf{I}} \in \mathfrak{cr}(\mathbb{A}_L) \cong \mathcal{O}, \text{Im}(\mathbb{C}), \text{Im}(\mathbb{H}), \mathcal{O}$ and $\tilde{\theta}_{\mathbf{I}} \in \mathfrak{cr}(\mathbb{A}_R)$. The scalar degrees of freedom in these supergravity theories belong to non-linear sigma models whose target spaces are coset manifolds G/H , where G is the

| | R | C | H | O |
|---|--|---|---|--|
| R | $\mathcal{N} = 2, \ell = 4$ $G = \text{SL}(2, \mathbb{R})$ $H = \text{SO}(2)$ | $\mathcal{N} = 3, \ell = 8$ $G = \text{SU}(2, 1)$ $H = \text{SU}(2) \times \text{SO}(2)$ | $\mathcal{N} = 5, \ell = 16$ $G = \text{USp}(4, 2)$ $H = \text{USp}(4) \times \text{USp}(2)$ | $\mathcal{N} = 9, \ell = 32$ $G = F_{4(-20)}$ $H = \text{SO}(9)$ |
| C | $\mathcal{N} = 3, \ell = 8$ $G = \text{SU}(2, 1)$ $H = \text{SU}(2) \times \text{SO}(2)$ | $\mathcal{N} = 4, \ell = 16$ $G = \text{SU}(2, 1)^2$ $H = \text{SU}(2)^2 \times \text{SO}(2)^2$ | $\mathcal{N} = 6, \ell = 32$ $G = \text{SU}(4, 2)$ $H = \text{SU}(4) \times \text{SU}(2) \times \text{SO}(2)$ | $\mathcal{N} = 10, \ell = 64$ $G = E_{6(-14)}$ $H = \text{SO}(10) \times \text{SO}(2)$ |
| H | $\mathcal{N} = 5, \ell = 16$ $G = \text{USp}(4, 2)$ $H = \text{USp}(4) \times \text{USp}(2)$ | $\mathcal{N} = 6, \ell = 32$ $G = \text{SU}(4, 2)$ $H = \text{SU}(4) \times \text{SU}(2) \times \text{SO}(2)$ | $\mathcal{N} = 8, \ell = 64$ $G = \text{SO}(8, 4)$ $H = \text{SO}(8) \times \text{SO}(4)$ | $\mathcal{N} = 12, \ell = 128$ $G = E_{7(-5)}$ $H = \text{SO}(12) \times \text{SO}(3)$ |
| O | $\mathcal{N} = 9, \ell = 32$ $G = F_{4(-20)}$ $H = \text{SO}(9)$ | $\mathcal{N} = 10, \ell = 64$ $G = E_{6(-14)}$ $H = \text{SO}(10) \times \text{SO}(2)$ | $\mathcal{N} = 12, \ell = 128$ $G = E_{7(-5)}$ $H = \text{SO}(12) \times \text{SO}(3)$ | $\mathcal{N} = 16, \ell = 256$ $G = E_{8(8)}$ $H = \text{SO}(16)$ |

Table 5.6.: Magic square of $D = 3$ supergravity theories. The first row of each entry indicates the amount of supersymmetry \mathcal{N} and the total number of degrees of freedom ℓ . The second (third) row indicates the U-duality group G (the maximal compact subgroup $H \subset G$) and its dimension. The scalar fields in each case parametrise the coset G/H , where $\dim_{\mathbb{R}}(G/H) = \ell/2$.

non-compact U-duality group, which acts non-linearly on the scalars, and H is the maximal compact subgroup of G . The largest linearly realised global symmetry of these theories is H , so one should expect its Lie algebra \mathfrak{h} to be $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$, and this indeed turns out to be the case. All the fields in (5.90) carry linear representations of H : the graviton $h_{\mu\nu}$ is a singlet; $\Psi_\mu \in \mathbb{A}_L \oplus \mathbb{A}_R$ transforms as the vector of the supergravity R-symmetry $\mathfrak{so}(\mathcal{N}_L + \mathcal{N}_R) \subseteq \mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$ and is inert under $\mathfrak{r}(\mathbb{A}_L) \oplus \mathfrak{r}(\mathbb{A}_R)$; and φ and χ transform as the spinor and conjugate-spinor of $\mathfrak{so}(\mathcal{N}_L + \mathcal{N}_R)$ and by right-multiplication under $\mathfrak{r}(\mathbb{A}_L) \oplus \mathfrak{r}(\mathbb{A}_R)$, as shown in (5.92). For example, in the maximal case of $\mathbb{A}_L, \mathbb{A}_R = \mathbb{O}$, these representations are the **1**, **16**, **128** and **128'** of $\mathfrak{so}(16)$.

The full non-linear U-duality groups G are fixed by the field content and H symmetries, as described in the literature [83, 84]. The groups are, of course, those with Lie algebras $\mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R)$, as presented in the magic square in Table 5.3. The $D = 3$ magic square theories are summarised in Table 5.6. The $\mathcal{N} > 8$ supergravities in $D = 3$ are unique, all fields belonging to the gravity multiplet, while those with $\mathcal{N} \leq 8$ may be coupled to k additional matter multiplets [83, 84]. The real beauty is that tensoring left and right SYM multiplets yields the field content of $\mathcal{N} = 2, 3, 4, 5, 6, 8$ supergravity with $k = 1, 1, 2, 1, 2, 4$: just the right matter content to produce the U-duality groups appearing in Table 5.3.

The compact symmetries $\mathfrak{h} = \mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$ can be traced directly back to Yang-Mills origins. The dynamical fields φ and χ can be defined as in the linearised

($g = 0$) dictionary of Section 5.1 as follows:

$$\partial_\mu \varphi = \begin{pmatrix} \partial_\mu \varphi_1 \\ \partial_\mu \varphi_2 \end{pmatrix}, \quad \chi = \frac{1}{2} \gamma^\mu \begin{pmatrix} \lambda^* \tilde{A}_\mu \\ A_\mu \tilde{\lambda}^* \end{pmatrix} + \begin{pmatrix} \lambda^* \tilde{\phi} \\ \phi \tilde{\lambda}^* \end{pmatrix}, \quad (5.93)$$

with

$$\begin{aligned} \partial_\mu \varphi_1 &:= i \bar{\lambda} \gamma_\mu \tilde{\lambda}^*, \\ \partial_\mu \varphi_2 &= \partial_\mu (A_\nu \tilde{A}^\nu + \phi \tilde{\phi}) + \frac{1}{2} \varepsilon_{\mu\sigma\tau} \eta^{\sigma\nu} \eta^{\tau\rho} (F_{\nu\rho} \tilde{\phi} + \phi \tilde{F}_{\nu\rho}) \end{aligned} \quad (5.94)$$

The division-algebraic conjugations $*$ are a matter of convention, chosen for later convenience. The on-shell doublets then become

$$\varphi = \begin{pmatrix} \frac{1}{E} \psi^* \tilde{\psi}^* \\ \phi \tilde{\phi} \end{pmatrix}, \quad \chi = \begin{pmatrix} \psi^* \tilde{\phi} \\ \phi \tilde{\psi}^* \end{pmatrix}. \quad (5.95)$$

The $\mathfrak{tri}(\mathbb{A}_L)$ and $\mathfrak{tri}(\mathbb{A}_R)$ symmetries of (5.89) act on the φ doublet as

$$\delta \varphi = \frac{1}{4} \theta^{ab} \begin{pmatrix} \frac{1}{E} e_a^* (e_b \psi^* \tilde{\psi}^*) \\ e_a (e_b^* \phi \tilde{\phi}) \end{pmatrix} + \frac{1}{4} \theta^{a'b'} \begin{pmatrix} \frac{1}{E} \tilde{e}_{a'}^* (\tilde{e}_{b'} \psi^* \tilde{\psi}^*) \\ \tilde{e}_{a'} (\tilde{e}_{b'}^* \phi \tilde{\phi}) \end{pmatrix} + \varphi \theta_1 + \varphi \tilde{\theta}_1, \quad (5.96)$$

while a new possibility in the ‘squared’ theory is to consider acting on φ with off-diagonal rotations of the form

$$\begin{pmatrix} 0 & -e_a^* \tilde{e}_{a'}^* \\ e_a \tilde{e}_{a'} & 0 \end{pmatrix}. \quad (5.97)$$

Including the action of these off-diagonal matrices, the total algebra of linear transformations is given by $\mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R) + \mathbb{A}_L \otimes \mathbb{A}_R$, i.e. the reduced magic square algebra $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$, as required – compare the above with equations (5.49) and (5.52). The analogous reasoning also works for χ .

It is interesting to note that the off-diagonal transformations of (5.97) take Yang-Mills fermions into Yang-Mills bosons, and vice versa, but are bosonic generators in the supergravity theory. It is tempting to guess that left-multiplication by (5.97) is related to the action of the operator $Q \otimes \tilde{Q}$, where Q, \tilde{Q} are the respective supersymmetry generators of the left and right Yang-Mills theories, and indeed this is the case. Consider the variation of φ under the simultaneous supersymmetry variation of the left and right theories,

$$\delta_L \delta_R \varphi = \begin{pmatrix} \frac{1}{E} \delta \psi^* \delta \tilde{\psi}^* \\ \delta \phi \delta \tilde{\phi} \end{pmatrix} = \begin{pmatrix} -E \epsilon^* \phi \tilde{\epsilon}^* \tilde{\phi} \\ -\epsilon \psi^* \tilde{\epsilon} \tilde{\psi}^* \end{pmatrix}. \quad (5.98)$$

Taking into account the anti-commutativity of the components of $\tilde{\epsilon}$ and ψ^* , this may be written as

$$\delta_L \delta_R \varphi = E \epsilon_a \tilde{\epsilon}_{a'} \begin{pmatrix} 0 & -e_a^* \tilde{\epsilon}_{a'} \\ e_a \tilde{\epsilon}_{a'} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{E} \psi^* \tilde{\psi}^* \\ \phi \tilde{\phi} \end{pmatrix}. \quad (5.99)$$

Similarly, for the analogous variation of χ

$$\delta_L \delta_R \chi = E \epsilon_a \tilde{\epsilon}_{a'} \begin{pmatrix} 0 & -e_a^* \tilde{\epsilon}_{a'} \\ e_a \tilde{\epsilon}_{a'} & 0 \end{pmatrix} \begin{pmatrix} \psi^* \tilde{\phi} \\ \phi \tilde{\psi}^* \end{pmatrix}. \quad (5.100)$$

Hence *all* of the compact symmetries $\mathfrak{h} = \mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$ come from those of the original Yang-Mills theories.

To further understand the non-linear U-duality groups G in terms of Yang-Mills squared will require a brief introduction to division-algebraic projective spaces, since it turns out that the symmetric spaces G/H , which the scalars parameterise, are in fact *projective planes* defined over $\mathbb{A}_L \otimes \mathbb{A}_R$. This is given in the following section.

5.4. Division-Algebraic Projective Planes

Real projective space $\mathbb{R}\mathbb{P}^{N-1}$ is the space of undirected lines through the origin in \mathbb{R}^N . Each line passes through a unique pair of antipodal points v and $-v$ on the unit sphere S^{N-1} and hence $\mathbb{R}\mathbb{P}^{N-1}$ is the quotient of S^{N-1} by the antipodal map, i.e. the points v and $-v \in S^{N-1}$ are identified:

$$\mathbb{R}\mathbb{P}^{N-1} \cong S^{N-1} / \mathbb{Z}_2. \quad (5.101)$$

with $\mathbb{Z}_2 = \{+1, -1\}$.

This can easily be generalised to \mathbb{C} and \mathbb{H} , defining $\mathbb{A}\mathbb{P}^{N-1}$ for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ as

$$\mathbb{A}\mathbb{P}^{N-1} \cong \{v \in \mathbb{A}^N \mid v^\dagger v = 1\} / \sim, \quad (5.102)$$

with the equivalence relation \sim defined by $v_1 \sim v_2 \Leftrightarrow v_1 = v_2 u$ where $u \in \mathbb{A}$ has unit modulus: $u^* u = 1$. In other words $\mathbb{A}\mathbb{P}^{N-1}$ is the quotient of the unit sphere in \mathbb{A}^N by the unit sphere in \mathbb{A} :

$$\mathbb{A}\mathbb{P}^{N-1} \cong S^{Nn-1} / S^{n-1}, \quad (5.103)$$

where $\mathbb{A} = \mathbb{A}_n$. In the real $n = 1$ case the unit ‘sphere’ is $S^0 \cong \mathbb{Z}_2 \cong \{+1, -1\}$.

The unit sphere in \mathbb{A}^N comes with an induced metric

$$ds^2 = dv^\dagger dv \quad (5.104)$$

whose isometry group G is inherited from the natural action of $N \times N$ unitary matrices $U \in A(N, \mathbb{A})$ on \mathbb{A}^N : $v \rightarrow Uv$. This metric and its isometries are then passed onto $\mathbb{A}\mathbb{P}^{N-1}$, so

$$G = \text{Isom}(\mathbb{A}\mathbb{P}^{N-1}) \cong A(N, \mathbb{A}). \quad (5.105)$$

The isometry group G acts transitively on the unit sphere and hence also on $\mathbb{A}\mathbb{P}^{N-1}$. Thus $\mathbb{A}\mathbb{P}^{N-1}$ is a homogeneous space and may be identified with the coset G/H , where H is the subgroup of G that fixes any given point, called the isotropy group. To elaborate on this, any unit vector $v \in \mathbb{A}^N$ can be written $g_v v_0$ where v_0 is some reference vector, say

$$v_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (5.106)$$

and g_v is a transformation in G . Hence one may parameterise points of $\mathbb{A}\mathbb{P}^{N-1}$ by the particular g_v required to reach them from v_0 . However, there is a redundancy in this description in that v_0 is invariant under a subgroup H , meaning these transformations should be excluded from the parameterisation. Thus as a manifold $\mathbb{A}\mathbb{P}^{N-1}$ is diffeomorphic to G/H .

The group H fixing a point may be seen by considering transformations that leave the reference vector v_0 above invariant. Clearly it is invariant under multiplication by block-diagonal elements of G of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}, \quad (5.107)$$

where h is $(N-1) \times (N-1)$ and hence an element of $A(N-1, \mathbb{A})$. In $\mathbb{A}\mathbb{P}^{N-1}$ the vector v_0 gets identified with $v_0 u$, so left-multiplication of v_0 by

$$\begin{pmatrix} u & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad (5.108)$$

which in this case is equivalent to right-multiplication by u , also leaves the corresponding point of $\mathbb{A}\mathbb{P}^{N-1}$ invariant. Thus (ignoring subtleties involving orientation-

reversing isometries in the real case)

$$\begin{aligned}\mathbb{R}\mathbb{P}^{N-1} &\cong \frac{\mathrm{SO}(N)}{\mathrm{SO}(N-1) \times \mathrm{O}(1)}, \\ \mathbb{C}\mathbb{P}^{N-1} &\cong \frac{\mathrm{U}(N)}{\mathrm{U}(N-1) \times \mathrm{U}(1)} \cong \frac{\mathrm{SU}(N)}{\mathrm{SU}(N-1) \times \mathrm{U}(1)}, \\ \mathbb{H}\mathbb{P}^{N-1} &\cong \frac{\mathrm{Sp}(N)}{\mathrm{Sp}(N-1) \times \mathrm{Sp}(1)}.\end{aligned}\tag{5.109}$$

Having seen that *all* of the classical compact groups appear as isometries of projective spaces over $\mathbb{R}, \mathbb{C}, \mathbb{H}$, it is obvious to guess that the exceptional groups are isometries of projective spaces over the octonions. Up to one or two subtleties this idea turns out to be correct. The definition above does not generalise to \mathbb{O} due to non-associativity, but there is an equivalent definition that will work for $\mathbb{O}\mathbb{P}^1$ and $\mathbb{O}\mathbb{P}^2$. Consider the matrix

$$P := vv^\dagger, \quad v \in \mathbb{A}^N, \quad v^\dagger v = 1,\tag{5.110}$$

for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. This is clearly a projector ($P^2 = P$) projecting onto the line defined by v . It is manifestly Hermitian and has unit trace:

$$\mathrm{Tr}(P) = \mathrm{Re} \mathrm{Tr}(P) = \mathrm{Re} \mathrm{Tr}(vv^\dagger) = v^\dagger v = 1,\tag{5.111}$$

where the cyclicity property of the real trace was used in the penultimate equality. Evidently P is invariant under right-multiplication of v by a unit element $u \in \mathbb{A}$,

$$v \rightarrow vu \quad \Rightarrow \quad P \rightarrow P,\tag{5.112}$$

so v and vu map to the same P . Moreover, any Hermitian rank-1 projector P may be written as in (5.110) for some v , meaning there is an isomorphism

$$\mathbb{A}\mathbb{P}^{N-1} \cong \{P \in \mathfrak{h}_N(\mathbb{A}) \mid P^2 = P, \mathrm{Tr}(P) = 1\}.\tag{5.113}$$

This may also be taken as the *definition* of $\mathbb{A}\mathbb{P}^{N-1}$, *including* in the octonionic case for $N = 2, 3$. In fact for all four division algebras P may be written as vv^\dagger with $v \in \mathbb{A}^N$ satisfying $v^\dagger v = 1$, but in the $N = 3$ octonionic case there is an additional constraint that the associator between the three components of v must vanish, i.e.

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{with } [x, y, z] = 0.\tag{5.114}$$

This ensures that the components of P belong to an associative subalgebra of \mathbb{O} , and hence P squares to itself:

$$P^2 = (vv^\dagger)(vv^\dagger) = v(v^\dagger v)v^\dagger = P. \quad (5.115)$$

The $N = 3$ case gives the *projective plane* $\mathbb{A}\mathbb{P}^2$, which is of particular interest since its isometry group has Lie algebra $\mathfrak{sa}(3, \mathbb{A})$, i.e. the first row (or column) of the compact magic square Table 5.1. The isotropy group is the subgroup that preserves a reference projector, say

$$P_0 := v_0 v_0^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (5.116)$$

This matrix can be written as a sum of its traceless part P'_0 and trace part,

$$P_0 = (P_0 - \frac{1}{3}\mathbb{1}) + \frac{1}{3}\mathbb{1} =: P'_0 + \frac{1}{3}\mathbb{1}, \quad (5.117)$$

with P'_0 given as in (5.39). Hence, by the discussion in Subsection 5.2.1 the isotropy groups must have Lie algebras $\mathfrak{sa}(2, \mathbb{A}) \oplus \mathfrak{r}(\mathbb{A})$, which agrees with (5.109) for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$; for $\mathbb{A} = \mathbb{O}$ this implies that

$$\mathbb{O}\mathbb{P}^2 \cong \frac{F_4}{\text{Spin}(9)}. \quad (5.118)$$

This manifold is sometimes referred to as the *Cayley plane*.

Next it is natural to suppose that there exist projective planes

$$(\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2 \cong \frac{L_3(\mathbb{A}_L, \mathbb{A}_R)}{L_2(\mathbb{A}_L, \mathbb{A}_R)}, \quad (5.119)$$

where $L_3(\mathbb{A}_L, \mathbb{A}_R)$ is the group resulting from exponentiation of the Lie algebra $\mathfrak{L}_3(\mathbb{A}_L, \mathbb{A}_R)$ and similar for $L_2(\mathbb{A}_L, \mathbb{A}_R)$ with Lie algebra $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$. At first sight this appears to work, since $(\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2$ should have tangent space $(\mathbb{A}_L \otimes \mathbb{A}_R)^2 = 2(\mathbb{A}_L \otimes \mathbb{A}_R)$, which is exactly the Lie-algebra subspace $\mathfrak{L}_3(\mathbb{A}_L, \mathbb{A}_R) - \mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$. However, in general these manifolds (5.119) are not strictly speaking projective spaces because the tensor product $\mathbb{A}_L \otimes \mathbb{A}_R$ is in general not a division algebra, and hence has zero divisors. One must settle then for taking $(\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2$ to be *defined* by equation (5.119) [1, 57, 90]. For more detailed and elegant treatments of magic square projective geometry see [91, 1, 90, 92] and the references therein.

For the magic square of supergravities the scalars parametrise the symmetric

spaces G/H , where $G = L_{1,2}(\mathbb{A}_L, \mathbb{A}_R)$ is the U-duality group with Lie algebra $\mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R)$ and $H = L_2(\mathbb{A}_L, \mathbb{A}_R)$ is its maximal compact subgroup with Lie algebra $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$. Thus, the scalar fields may be regarded as points in non-compact Lorentzian division-algebraic ‘projective planes’⁷

$$(\mathbb{A}_L \otimes \mathbb{A}_R)\tilde{\mathbb{P}}^2 \cong \frac{L_{1,2}(\mathbb{A}_L, \mathbb{A}_R)}{L_2(\mathbb{A}_L, \mathbb{A}_R)}. \quad (5.120)$$

Taking $\mathbb{A}_R = \mathbb{R}$, i.e. $\mathcal{N}_R = 1$, the scalar manifolds can be understood using Lorentzian Jordan algebras [1, 88]:

$$\mathfrak{h}_{1,2}(\mathbb{A}) := \{H \in \mathbb{A}[3] \mid H = \eta H^\dagger \eta\}, \quad (5.121)$$

where $\eta = \text{diag}(-1, 1, 1)$. Then the non-compact Lorentzian plane is just the Lorentzian analogue of (5.113):

$$\mathbb{A}\tilde{\mathbb{P}}^2 \cong \{P \in \mathfrak{h}_{1,2}(\mathbb{A}) \mid P^2 = P, \text{Tr}(P) = 1\}. \quad (5.122)$$

Rank-1 projectors P of this form may always be written as

$$P = -vv^\dagger\eta, \quad (5.123)$$

where $v \in \mathbb{A}^3$ is a ‘time-like’ unit vector, $v^\dagger\eta v = -1$, whose three components associate, as in (5.114). The metric on this space is then inherited from the Lorentzian metric η on \mathbb{A}^3 :

$$ds^2 = -dv^\dagger\eta dv = \frac{1}{2} \text{Tr}(dP dP). \quad (5.124)$$

Thus the $\mathcal{N} = \mathcal{N}_L + 1 = 2, 3, 5, 9$ supergravities resulting from Yang-Mills theories with $\mathcal{N}_R = 1$ have a scalar sector Lagrangian given by

$$\mathcal{L} = \frac{1}{4}\sqrt{-g} \text{Tr}(\partial_\mu P \partial^\mu P), \quad (5.125)$$

with

$$P \in \mathbb{A}_L\tilde{\mathbb{P}}^2 \cong G/H = \begin{cases} \text{SO}(1, 2)/(\text{SO}(2) \times \text{O}(1)), & \mathbb{A}_L = \mathbb{R} \\ \text{SU}(1, 2)/(\text{SU}(2) \times \text{U}(1)), & \mathbb{A}_L = \mathbb{C} \\ \text{Sp}(1, 2)/(\text{Sp}(2) \times \text{Sp}(1)), & \mathbb{A}_L = \mathbb{H} \\ F_{4(-20)}/\text{Spin}(9), & \mathbb{A}_L = \mathbb{O}. \end{cases} \quad (5.126)$$

⁷For example $\mathbb{R}\mathbb{P}^2 \cong \text{SO}(3)/(\text{SO}(2) \times \text{O}(1))$ vs. $\mathbb{R}\tilde{\mathbb{P}}^2 \cong \text{SL}(2, \mathbb{R})/(\text{SO}(2) \times \text{O}(1))$.

In order to see the form of the Lagrangian (5.125) in terms of individual scalar fields (of which there are $2\mathcal{N}_L$ in total), one may take the coset representative $\mathcal{V} \in G/H$ in more conventional language and then write it as an \mathbb{A}_L -valued 3×3 matrix in the representation of $\text{SA}(3, \mathbb{A}_L)$ described in Subsection 5.2.1. Then simply write

$$v = \mathcal{V}^\dagger v_0 \quad (5.127)$$

with v_0 defined in (5.106). Then defining P as in (5.123) gives an explicit Lagrangian. For example, the $\mathbb{A}_L = \mathbb{A}_R = \mathbb{R}$ case has scalar coset $\text{SO}(1, 2)/(\text{SO}(2) \times \text{O}(1))$, which is equivalent to the $\text{SL}(2, \mathbb{R})/\text{O}(2)$ sigma model presented in equations (2.78)–(2.82). The coset representative \mathcal{V} as defined in (2.78) becomes

$$\mathcal{V} = \begin{pmatrix} \cosh \varphi_1 + \frac{1}{2}e^{\varphi_1}(\varphi_2)^2 & e^{\varphi_1}\varphi_2 & \sinh \varphi_1 - \frac{1}{2}e^{\varphi_1}(\varphi_2)^2 \\ \varphi_2 & 1 & -\varphi_2 \\ \sinh \varphi_1 + \frac{1}{2}e^{\varphi_1}(\varphi_2)^2 & e^{\varphi_1}\varphi_2 & \cosh \varphi_1 + \frac{1}{2}e^{\varphi_1}(\varphi_2)^2 \end{pmatrix} \quad (5.128)$$

in the defining 3×3 representation of $\text{SO}(1, 2)$. Then

$$v = \mathcal{V}^\top v_0 = \begin{pmatrix} \cosh \varphi_1 + \frac{1}{2}e^{\varphi_1}(\varphi_2)^2 \\ e^{\varphi_1}\varphi_2 \\ \sinh \varphi_1 - \frac{1}{2}e^{\varphi_1}(\varphi_2)^2 \end{pmatrix} \quad (5.129)$$

(with $\dagger \rightarrow \top$ since \mathcal{V} has real entries), which leads to the Lagrangian

$$\mathcal{L} = \frac{1}{4}\sqrt{-g} \text{Tr}(\partial_\mu P \partial^\mu P) = \sqrt{-g}(-\frac{1}{2}\partial_\mu \varphi_1 \partial^\mu \varphi_1 - \frac{1}{2}e^{2\varphi_1} \partial_\mu \varphi_2 \partial^\mu \varphi_2), \quad (5.130)$$

in agreement with equation (2.80).

The Lagrangian (5.125) is an elegant formulation of the scalar sigma model for $\mathcal{N}_R = 1$, but unfortunately it does not easily generalise to include $\mathcal{N}_R > 1$, by the lack of a projective interpretation of the manifolds $(\mathbb{A}_L \otimes \mathbb{A}_R)\tilde{\mathbb{P}}^2$ defined in (5.120). Such a version of the supergravity Lagrangian is desirable, since writing the theory over $\mathbb{A}_L \otimes \mathbb{A}_R$ makes manifest its Yang-Mills-squared origins.

5.5. Summary

In this chapter the idea of gravity as the square of gauge theory was combined with the division-algebraic formulation of Yang-Mills to uncover a magic square of $D = 3$ supergravities. Each theory is obtained by tensoring an \mathbb{A}_L -valued SYM multiplet with an \mathbb{A}_R -valued SYM multiplet and has a U-duality group G whose Lie algebra

is the magic square algebra $\mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R)$. This U-duality acts non-linearly on the scalar sector, which describes a sigma model whose target space is the Lorentzian ‘projective plane’ $(\mathbb{A}_L \otimes \mathbb{A}_R)\tilde{\mathbb{P}}^2$.

The next step is to lift this process into higher dimensions, generalising to all cases in the range $3 \leq D \leq 10$. Since there are fewer and fewer available super Yang-Mills theories to ‘square’ as D increases, this results in a *magic pyramid* of supergravity theories, whose base in $D = 3$ is the magic square. This is the subject of Chapter 6.

6. A Magic Pyramid of Supergravities

Having established in the previous chapter that taking tensor products of $D = 3$ super Yang-Mills multiplets leads to a magic square of supergravities, it makes sense to generalise this construction to higher dimensions. The goal of this chapter is to give a unified presentation of the field content and global symmetries of ‘SYM-squared’ supergravities in the range $3 \leq D \leq 10$.

In terms of Lorentz representations, the field content of a general SYM-squared supergravity theory is presented in Section 5.1, or in further detail in Table A.1 and Table A.2 in the Appendix. For each D , \mathcal{N}_L and \mathcal{N}_R this field content is sufficient to determine the U-duality groups G of the resulting supergravities, as well as their maximal compact subgroups H . There are two possibilities:

1. If one of the Yang-Mills theories has maximal supersymmetry then the tensor product results in the unique gravity multiplet with $\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R$ supersymmetries, whose U-duality group in each case is familiar from the literature [42]. For example, in $D = 4$, tensoring $\mathcal{N}_L = 4$ SYM and $\mathcal{N}_R = 2$ SYM results in the unique $\mathcal{N} = 6$ supergravity theory with $G = \text{SO}^*(12)$ and $H = \text{U}(6)$.
2. If neither SYM theory is maximal then the tensor product gives a supergravity with $\mathcal{N} = \mathcal{N}_L + \mathcal{N}_R$, consisting of the gravity multiplet coupled to additional matter multiplets. Again, the U-dualities of such theories are well-known. For example, in $D = 4$, tensoring $\mathcal{N}_L = 2$ SYM and $\mathcal{N}_R = 2$ SYM results in $\mathcal{N} = 4$ supergravity coupled to two vector multiplets¹ [93] with $G = \text{SO}(6, 2) \times \text{SL}(2, \mathbb{R})$ and $H = \text{SO}(6) \times \text{SO}(2)^2 \sim \text{U}(4) \times \text{U}(1)$.

The resulting G and H groups are given in Figure 6.1 and Figure 6.2, as well as Tables 6.3–6.6.

Just as the magic square was described by a single Lie-algebraic formula, one may define a *magic pyramid formula*, which takes as its arguments three algebras:

¹In general, $\mathcal{N} = 4$ supergravity coupled to n vector multiplets has $G = \text{SO}(6, n) \times \text{SL}(2, \mathbb{R})$ and $H = \text{SO}(6) \times \text{SO}(n) \times \text{SO}(2)$. The factor $\text{SO}(6) \times \text{SO}(2) \sim \text{U}(4)$ is the R-symmetry, while the factor $\text{SO}(n)$ rotates the n vector multiplets into one another [93].

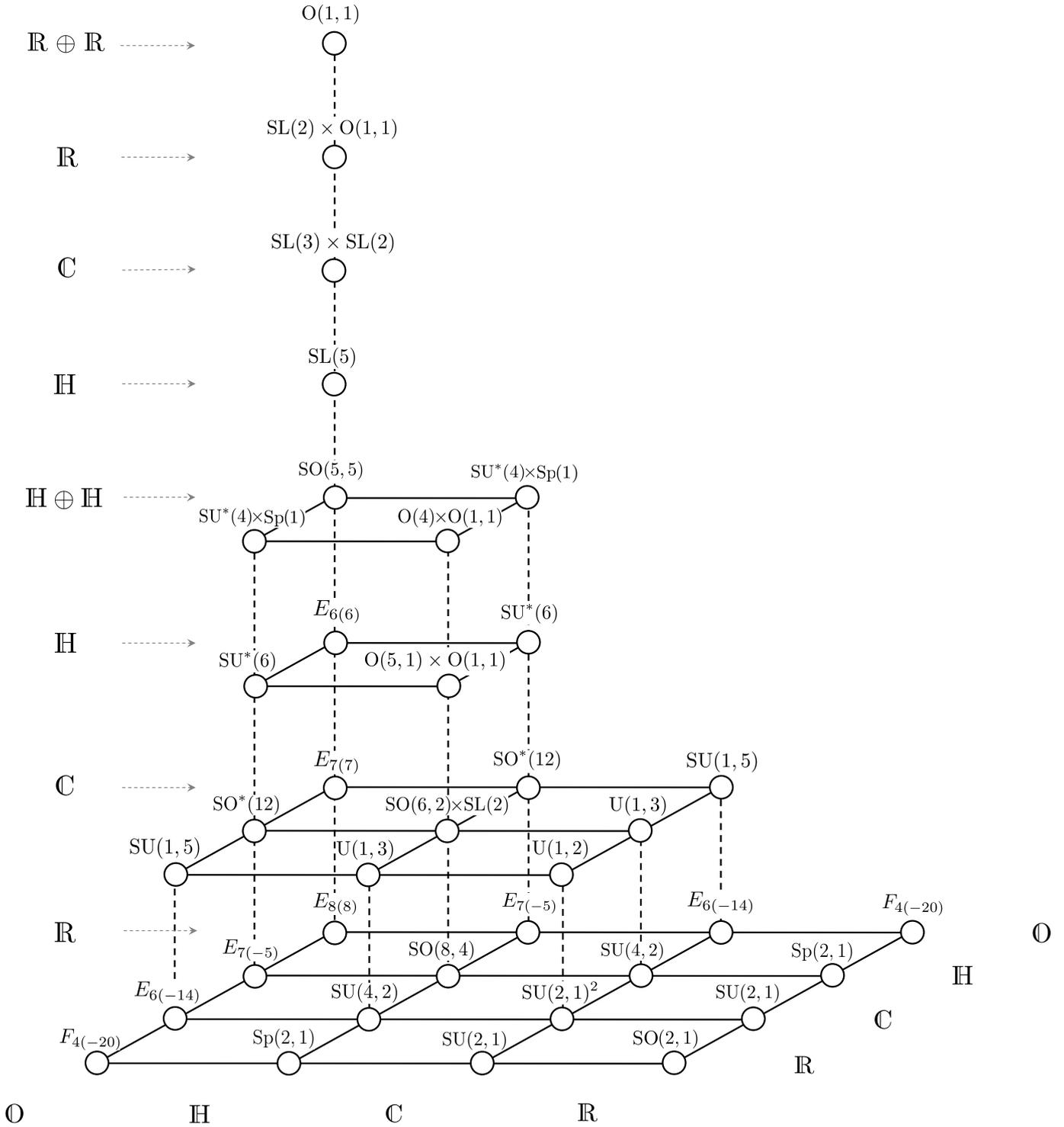


Figure 6.1.: Pyramid of U-duality groups G . Each layer of the pyramid corresponds to a different spacetime dimension, with $D = 3$ at the base and $D = 10$ at the summit. The spacetime dimensions are labelled by the (direct sum of) division algebra(s) \mathbb{D} on the vertical axis as given in Table 4.2. The horizontal axes label the number of supersymmetries of the left and right Yang-Mills theories: \mathbb{O} means maximal supersymmetry, \mathbb{H} means half-maximal, and so on.

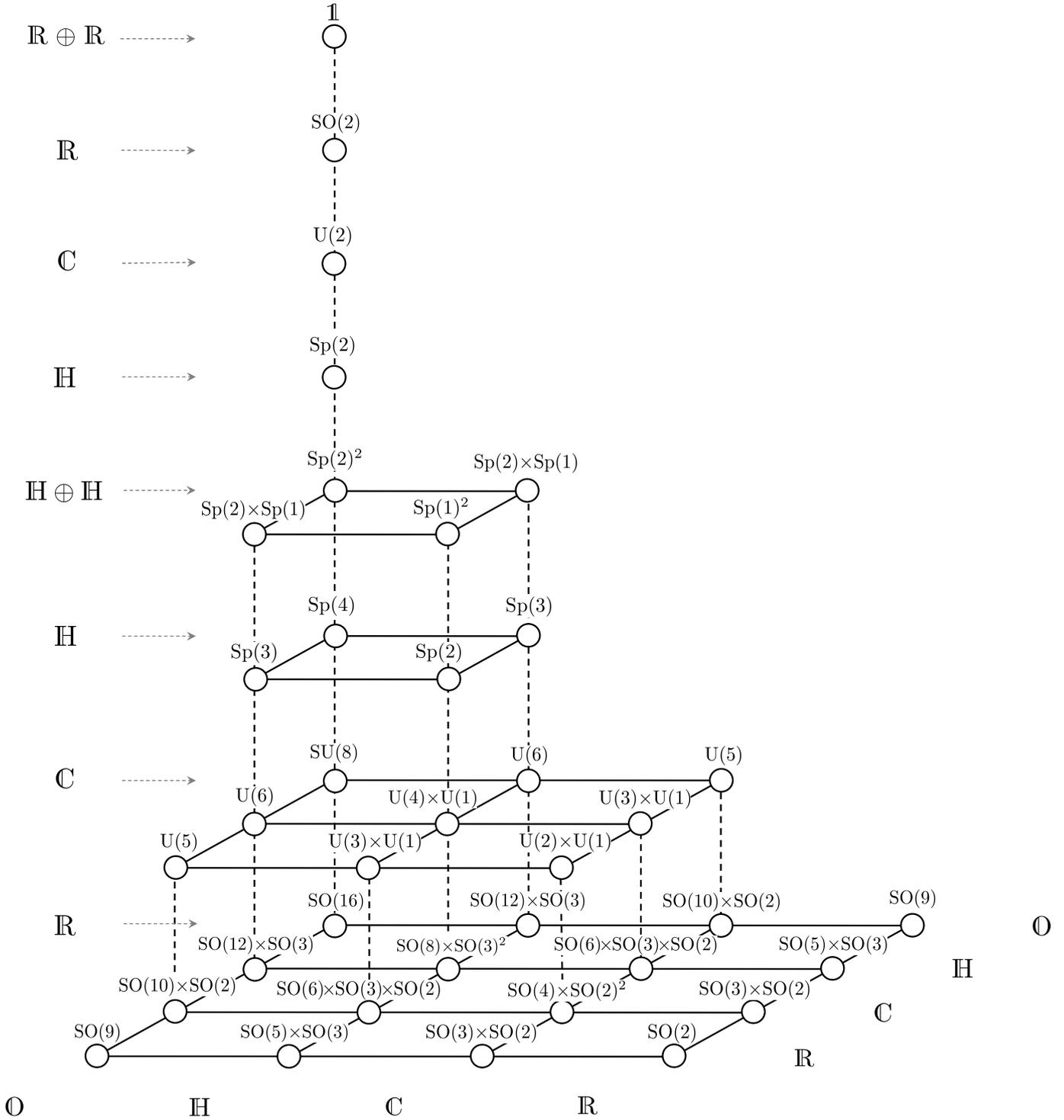


Figure 6.2.: Pyramid of compact subgroups $H \subset G$. Each layer corresponds to a different spacetime dimension, with $D = 3$ at the base and $D = 10$ at the summit. The spacetime dimensions are labelled by the (direct sum of) division algebra(s) \mathbb{D} on the vertical axis as given in Table 4.2. The horizontal axes label the number of supersymmetries of the left and right Yang-Mills theories: \mathbb{O} means maximal supersymmetry, \mathbb{H} means half-maximal, and so on.

division algebras \mathbb{A}_L and \mathbb{A}_R , and the Clifford algebra $\text{Cl}_0(N)$, $1 \leq N \leq n_L, n_R$, where $N = D - 2$ and $n_L = \dim[\mathbb{A}_L]$ and $n_R = \dim[\mathbb{A}_R]$. Each SYM-squared supergravity's U-duality group G then has Lie algebra \mathfrak{g} given by the magic pyramid formula, and a maximal compact subalgebra \mathfrak{h} given by a corresponding *reduced* magic pyramid algebra.

Section 6.1 contains a brief summary of one of the main the results of Chapter 4: how a general pure SYM theory may be written over a division algebra \mathbb{A}_n . Then, Section 6.2 describes how this gives $(\mathbb{A}_L \otimes \mathbb{A}_R)$ -valued SYM-squared supergravities. Finally, Section 6.3 presents the mathematical definition of the magic pyramid algebra, and a demonstration of how it describes the Lie algebras of the groups presented in Figure 6.1, 6.2 and Tables 6.3–6.6 in terms of the symmetries of the original super Yang-Mills theories.

6.1. Division-Algebraic Super Yang-Mills

Every pure super Yang-Mills theory can be obtained by dimensional reduction of the minimally supersymmetric $D = 3, 4, 6, 10$ theories presented in Chapter 4, whose on-shell symmetries are given by the triality algebras $\mathfrak{tri}(\mathbb{A}_n)$. There it was demonstrated that a super Yang-Mills theory in $D = N + 2$ dimensions with \mathcal{N} supersymmetries has field content $\{A_\mu, \phi_{\bar{i}}, \lambda^I\}$, or on-shell $\{a, \phi_{\bar{i}}, \psi^I\}$, where:

- $\psi^I \in \mathcal{S}_N$, with $I = 1, \dots, \mathcal{N}$, are the \mathcal{N} fermions, each valued in \mathcal{S}_N , i.e. the space of spinors of $\text{Cl}_0(N)$. These can be packaged as an element f of the division algebra $\mathbb{A}_n \cong (\mathcal{S}_N)^{\mathcal{N}}$, where $n = \mathcal{N}s_N$ with $s_N := \dim[\mathcal{S}_N]$;
- $a \in \mathbb{R}^N$ is the on-shell gauge field, which may be embedded in a subspace $\mathbb{R}^N \subseteq \mathbb{A}_n$;
- $\phi_{\bar{i}}$ are the $(n - N)$ scalar fields, which may be embedded in a subspace $\mathbb{R}^{n-N} \subseteq \mathbb{A}_n$, complementary to that of the vector a .

Hence, the on-shell content may be organised into a bosonic sector b and a fermionic sector f , each of which takes values in \mathbb{A}_n :

$$b = a + \phi = a_{\underline{a}}e_{\underline{a}} + \phi_{\bar{i}}e_{\bar{i}} \in \mathbb{A}_n, \quad f \in \mathbb{A}_n \cong (\mathcal{S}_N)^{\mathcal{N}} \quad (6.1)$$

where, for the bosons, the usual index $a = 0, 1, \dots, (n - 1)$ labelling the division-algebraic basis elements is decomposed as

$$\begin{aligned} a &= 0, 1, \dots, (N - 1), \\ \bar{i} &= N, (N + 1), \dots, (n - 1), \end{aligned} \tag{6.2}$$

so that $\text{span}\{e_a\} = \mathbb{R}^N \subseteq \mathbb{A}_n$, while $\text{span}\{e_{\bar{i}}\} = \mathbb{R}^{n-N} \subset \mathbb{A}_n$. For solutions of the free Fourier-space equations of motion with momentum $p_\mu = (E, 0 \dots, 0, E)$, the on-shell supersymmetry transformations are

$$\delta b = -i\epsilon f^*, \quad \delta f = iE b^* \epsilon, \quad b, f, \epsilon \in \mathbb{A}_n, \tag{6.3}$$

where the components f_a and ϵ_a are Grassmann numbers. The overall symmetry is $\mathfrak{so}(N)_{\text{ST}} \oplus \mathfrak{int}_N(\mathbb{A}_n) \subseteq \mathfrak{tri}(\mathbb{A}_n)$ – the spacetime little group algebra $\mathfrak{so}(N)_{\text{ST}}$ plus the internal symmetry $\mathfrak{int}_N(\mathbb{A}_n)$. The spacetime part $\mathfrak{so}(N)_{\text{ST}}$ acts on the three different representations b, f, ϵ via equation (4.103):

$$\delta(b, f^*, \epsilon) = T_{\text{ST}}(b, f^*, \epsilon) = \frac{1}{4}\theta^{ab} T_{ab}(b, f^*, \epsilon), \tag{6.4}$$

where $\theta^{ab} \in \mathfrak{so}(N)_{\text{ST}}$, while $\mathfrak{int}_N(\mathbb{A}_n) \cong \mathfrak{so}(n - N) \oplus \mathfrak{cr}(\mathbb{A}_n)$ acts as

$$\delta(b, f^*, \epsilon) = T_{\text{INT}}(b, f^*, \epsilon) = \frac{1}{4}\theta^{\bar{i}\bar{j}} T_{\bar{i}\bar{j}}(b, f^*, \epsilon) + T_{\text{EX}}(b, f^*, \epsilon), \tag{6.5}$$

where

$$T_{\text{EX}} = (0, -\widehat{L}_{\theta_I}, \widehat{R}_{\theta_I}) \in \mathfrak{tri}(\mathbb{A}_n), \quad \theta_I \in \mathfrak{cr}(\mathbb{A}_n). \tag{6.6}$$

It is convenient to define a new algebra $\mathfrak{sym}_N(\mathbb{A}_n)$, made up of those elements of $\mathfrak{tri}(\mathbb{A}_n)$ that commute with its $\mathfrak{so}(N)_{\text{ST}}$ subalgebra:

$$\mathfrak{sym}_N(\mathbb{A}_n) := \left\{ T \in \mathfrak{tri}(\mathbb{A}_n) - \mathfrak{so}(N)_{\text{ST}} \mid [T, \mathfrak{so}(N)_{\text{ST}}] = 0 \right\}. \tag{6.7}$$

For $N > 1$ this is clearly just the same as the algebra $\mathfrak{int}_N(\mathbb{A}_n)$, but for $N = 1$ the algebra $\mathfrak{so}(N)_{\text{ST}}$ is empty, so (6.7) enforces no condition on $\mathfrak{tri}(\mathbb{A}_n)$ and thus $\mathfrak{sym}_1(\mathbb{A}_n) = \mathfrak{tri}(\mathbb{A}_n)$. Hence

$$\mathfrak{sym}_N(\mathbb{A}_n) \cong \begin{cases} \mathfrak{tri}(\mathbb{A}_n), & N = 1, \\ \mathfrak{int}_N(\mathbb{A}_n), & N > 1. \end{cases} \tag{6.8}$$

Thus the subtle difference between the algebras $\mathfrak{sym}_N(\mathbb{A}_n)$ and $\mathfrak{int}_N(\mathbb{A}_n)$ is that $\mathfrak{sym}_N(\mathbb{A}_n)$ gives the symmetries of the *dualised* (i.e. zero coupling constant, $g = 0$)

Yang-Mills theory in $D = N + 2 = 3$ dimensions: $\mathbf{tri}(\mathbb{A}_n)$. In contrast, $\mathbf{int}_N(\mathbb{A}_n)$ gives the symmetries of the *original* ‘*un-dualised*’ theory with the usual field content $\{a \in \mathbf{Re}(\mathbb{A}_n), \phi \in \mathbf{Im}(\mathbb{A}_n), \psi \in \mathbb{A}_n\}$.

| $N \backslash \mathbb{A}_n$ | \mathbb{O} $\mathcal{Q} = 16$ | \mathbb{H} $\mathcal{Q} = 8$ | \mathbb{C} $\mathcal{Q} = 4$ | \mathbb{R} $\mathcal{Q} = 2$ |
|-----------------------------|--|--|--|---|
| 8 | $\mathfrak{ex}(\mathbb{O})$ $\cong \emptyset$ | - | - | - |
| 7 | \emptyset | - | - | - |
| 6 | $\mathfrak{so}(2)$ $\cong \mathfrak{u}(1)$ | - | - | - |
| 5 | $\mathfrak{so}(3)$ $\cong \mathfrak{sp}(1)$ | - | - | - |
| 4 | $\mathfrak{so}(4)$ $\cong 2\mathfrak{sp}(1)$ | $\mathfrak{ex}(\mathbb{H})$ $\cong \mathfrak{sp}(1)$ | - | - |
| 3 | $\mathfrak{so}(5)$ $\cong \mathfrak{sp}(2)$ | $\mathfrak{sp}(1)$ | - | - |
| 2 | $\mathfrak{so}(6)$ $\cong \mathfrak{su}(4)$ | $\mathfrak{so}(2) \oplus \mathfrak{sp}(1)$ $\cong \mathfrak{u}(2)$ | $\mathfrak{ex}(\mathbb{C})$ $\cong \mathfrak{u}(1)$ | - |
| 1 | $\mathfrak{so}(8)$ $\cong \mathbf{tri}(\mathbb{O})$ | $\mathfrak{so}(4) \oplus \mathfrak{sp}(1)$ $\cong \mathbf{tri}(\mathbb{H})$ | $\mathfrak{u}(1)$ $\cong \mathbf{tri}(\mathbb{C})$ | $\mathfrak{ex}(\mathbb{R}) \cong \emptyset$ $\cong \mathbf{tri}(\mathbb{R})$ |

Table 6.1.: The internal symmetry algebras of pure super Yang-Mills on-shell: $\mathfrak{sym}_N(\mathbb{A}_n)$. Each slot corresponds to the SYM theory in $D = N + 2$ dimensions with $\mathcal{Q} = 2n$ real supercharge components. The $N = 1$ row describes the symmetries of the $D = 3$ theory when the Yang-Mills gauge field has been dualised to a scalar – compare this to Table 4.4.

To take account of this dualisation in $D = 3$, it is useful to give an alternative definition of the space of scalars $\mathbb{S}_n^N \subseteq \mathbb{A}_n$ as the space of solutions ϕ to

$$\delta(\phi, 0, 0) = T_{\text{ST}}(\phi, 0, 0) = 0, \quad \phi \in \mathbb{A}_n, \quad (6.9)$$

i.e. $\mathbb{S}_n^N := \ker[\widehat{A}_{\text{ST}}]$, where $T_{\text{ST}} = (\widehat{A}_{\text{ST}}, \widehat{B}_{\text{ST}}, \widehat{C}_{\text{ST}})$. This is just the obvious statement that scalars are invariant under spacetime rotations. Thus T_{ST} naturally decomposes

\mathbb{A}_n into two subspaces:

$$\mathbb{A}_n = \mathbb{V}^N \oplus \mathbb{S}_n^N, \quad (6.10)$$

where \mathbb{V}^N is the orthogonal complement of \mathbb{S}_n^N in \mathbb{A}_n . For $D = 3$ ($N = 1$) the spacetime little group is $\mathfrak{so}(1)_{\text{ST}} \cong \emptyset$ and hence $T_{\text{ST}} = 0$. This means that *every* $\phi \in \mathbb{A}_n$ trivially satisfies (6.9), and hence $\mathbb{S}_n^1 = \mathbb{A}_n$. For $D > 3$ ($N > 1$) the space of scalars \mathbb{S}_n^N just corresponds to the internal dimensions, $\mathbb{R}^{n-N} \cong \text{span}\{e_{\bar{i}}\}$. In summary then,

$$\mathbb{V}^N \cong \begin{cases} \emptyset, & N = 1, \\ \mathbb{R}^N, & N > 1, \end{cases} \quad \mathbb{S}_n^N \cong \begin{cases} \mathbb{A}_n, & N = 1, \\ \mathbb{R}^{n-N}, & N > 1, \end{cases} \quad (6.11)$$

and the bosonic content becomes

$$b = a + \phi \in \mathbb{A}_n, \quad a \in \mathbb{V}^N, \quad \phi \in \mathbb{S}_n^N, \quad (6.12)$$

where a is the vector and ϕ contains the scalar fields.

Just as the magic square was constructed from the triality algebras, which were the internal symmetries of super Yang-Mills theories, the magic pyramid is constructed from the algebras $\mathfrak{sym}_N(\mathbb{A}_n)$. This explicitly involves the spaces \mathbb{V}^N and \mathbb{S}_n^N , as well as the algebra \mathbb{D} , over which $\text{Cl}_0(N)$ is defined.

6.2. The Supergravity Pyramid

Tensoring an on-shell super Yang-Mills multiplet $\{b \in \mathbb{A}_L, f \in \mathbb{A}_L\}$ with another $\{\tilde{b} \in \mathbb{A}_R, \tilde{f} \in \mathbb{A}_R\}$ generates a supergravity multiplet, valued in $\mathbb{A}_L \otimes \mathbb{A}_R$. Each multiplet has global symmetry algebra

$$\mathfrak{so}(N)_{\text{ST}} \oplus \mathfrak{sym}_N(\mathbb{A}_{L,R}). \quad (6.13)$$

With respect to $\mathfrak{so}(N)_{\text{ST}}$ the tensor products are $\mathfrak{so}(N)_{\text{ST}}$ -modules, while with respect to $\mathfrak{sym}_N(\mathbb{A}_L)$ and $\mathfrak{sym}_N(\mathbb{A}_R)$ they are $\mathfrak{sym}_N(\mathbb{A}_L) \oplus \mathfrak{sym}_N(\mathbb{A}_R)$ -modules. Practically speaking, the tensoring of on-shell fields can be organised just as in $D = 3$ (see the previous chapter); the total supergravity content can be arranged into a bosonic doublet and a fermionic doublet,

$$\mathcal{B} := \begin{pmatrix} \frac{1}{E} f^* \tilde{f}^* \\ b \tilde{b} \end{pmatrix} \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}, \quad \mathcal{F} := \begin{pmatrix} f^* \tilde{b} \\ b \tilde{f}^* \end{pmatrix} \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}. \quad (6.14)$$

| $\mathbb{A}_L \setminus \mathbb{A}_R$ | $\tilde{a} \in \mathbb{V}_R^N$ | $\tilde{\phi} \in \mathbb{S}_R^N$ | $\tilde{f} \in \mathbb{A}_R$ |
|---------------------------------------|---|--|---|
| $a \in \mathbb{V}_L^N$ | $g + B + \varphi \in \mathbb{V}_L^N \otimes \mathbb{V}_R^N$ | $\mathfrak{a} \in \mathbb{V}_L^N \otimes \mathbb{S}_R^N$ | $\Psi + \chi \in \mathbb{V}_L^N \otimes \mathbb{A}_R$ |
| $\phi \in \mathbb{S}_L^N$ | $\mathfrak{a} \in \mathbb{S}_L^N \otimes \mathbb{V}_R^N$ | $\varphi \in \mathbb{S}_L^N \otimes \mathbb{S}_R^N$ | $\chi \in \mathbb{S}_L^N \otimes \mathbb{A}_R$ |
| $f \in \mathbb{A}_L$ | $\Psi + \chi \in \mathbb{A}_L \otimes \mathbb{V}_R^N$ | $\chi \in \mathbb{A}_L \otimes \mathbb{S}_R^N$ | $\varphi_{\text{RR}} + \dots \in \mathbb{A}_L \otimes \mathbb{A}_R$ |

Table 6.2.: Tensor product of left/right ($\mathbb{A}_L/\mathbb{A}_R$) SYM multiplets in $D = N + 2$ dimensions, using $\mathfrak{so}(N)_{\text{ST}}$ spacetime little group representations.

The field content – in terms of irreducible $\mathfrak{so}(N)_{\text{ST}}$ representations – contained within \mathfrak{B} and \mathfrak{F} may be found using the on-shell version of the dictionary in Section 5.1. For example, the $b\tilde{b}$ (i.e. NS-NS) sector is

$$b\tilde{b} = (a + \phi)(\tilde{a} + \tilde{\phi}) = a\tilde{a} + \phi\tilde{\phi} + a\tilde{\phi} + \phi\tilde{a}; \quad (6.15)$$

the last three terms correspond to supergravity scalars $\phi\tilde{\phi} \in \mathbb{S}_L^N \otimes \mathbb{S}_R^N$ and vectors $\mathfrak{a} := a\tilde{\phi} + \phi\tilde{a} \in (\mathbb{V}_L^N \otimes \mathbb{S}_R^N) \oplus (\mathbb{S}_L^N \otimes \mathbb{V}_R^N)$, while the first term $a\tilde{a}$ gives the graviton g , a 2-form B and a scalar as the symmetric-traceless, antisymmetric and trace parts, respectively:

$$a\tilde{a} = a_{\underline{a}}\tilde{a}_{\underline{b}} e_{\underline{a}}\tilde{e}_{\underline{b}} \equiv \left[\underbrace{(a_{\underline{a}}\tilde{a}_{\underline{b}} - \frac{1}{v}(a_{\underline{c}}\tilde{a}_{\underline{c}})\delta_{\underline{ab}})}_{g_{\underline{ab}}} + \frac{1}{v} \underbrace{(a_{\underline{c}}\tilde{a}_{\underline{c}})}_{\varphi} \delta_{\underline{ab}} + \underbrace{a_{[\underline{a}}\tilde{a}_{\underline{b}]}}_{B_{\underline{ab}}} \right] e_{\underline{a}}\tilde{e}_{\underline{b}}, \quad (6.16)$$

where $v := \dim[\mathbb{V}_L^N] = \dim[\mathbb{V}_R^N]$. The rest of the field content is listed in Table 6.2. Note that the p -form fields of the Ramond-Ramond sector $\frac{1}{E}f^*\tilde{f}^*$ in \mathfrak{B} are dimension-dependent. By virtue of working on-shell all p -forms in the following discussion will effectively always be dualised to the lowest possible rank consistent with their little group representations. Thus for example, in terms of Lorentz reps, $B_{\mu\nu} \rightarrow \phi, A_{\mu}$ in $D = 4, 5$, respectively. This ensures U-duality is manifest.

The detailed form of these tensor products for $D > 3$ are summarised in Table A.1 and Table A.2 in the Appendix, where for a given little group representation the $\mathfrak{sym}_N(\mathbb{A}_L) \oplus \mathfrak{sym}_N(\mathbb{A}_R)$ representations have been collected into the appropriate representation of \mathfrak{h} , the maximal compact subalgebra of the U-duality \mathfrak{g} . For example, consider the square of the $D = 5, \mathcal{N} = 2$ super Yang-Mills multiplet, whose

global symmetry algebra is $\mathfrak{so}(3)_{\text{ST}} \oplus \mathfrak{sym}_3(\mathbb{O}) = \mathfrak{so}(3)_{\text{ST}} \oplus \mathfrak{sp}(2)$,

| \otimes | $\tilde{A}_\mu : (\mathbf{3}; \mathbf{1})$ | $\tilde{\lambda} : (\mathbf{2}; \mathbf{4})$ | $\tilde{\phi} : (\mathbf{1}; \mathbf{5})$ | |
|--------------------------------------|--|---|---|--------|
| $A_\mu : (\mathbf{3}; \mathbf{1})$ | $(\mathbf{5}; \mathbf{1}, \mathbf{1}) + (\mathbf{3}; \mathbf{1}, \mathbf{1}) + (\mathbf{1}; \mathbf{1}, \mathbf{1})$ | $(\mathbf{4}; \mathbf{1}, \mathbf{4}) + (\mathbf{2}; \mathbf{1}, \mathbf{4})$ | $(\mathbf{3}; \mathbf{1}, \mathbf{5})$ | (6.17) |
| $\lambda : (\mathbf{2}; \mathbf{4})$ | $(\mathbf{4}; \mathbf{4}, \mathbf{1}) + (\mathbf{2}; \mathbf{4}, \mathbf{1})$ | $(\mathbf{3}; \mathbf{4}, \mathbf{4}) + (\mathbf{1}; \mathbf{4}, \mathbf{4})$ | $(\mathbf{2}; \mathbf{4}, \mathbf{5})$ | |
| $\phi : (\mathbf{1}; \mathbf{5})$ | $(\mathbf{3}; \mathbf{5}, \mathbf{1})$ | $(\mathbf{2}; \mathbf{5}, \mathbf{4})$ | $(\mathbf{1}; \mathbf{5}, \mathbf{5})$ | |

On gathering the spacetime little group representations in (6.17), the (generally reducible) $\mathfrak{sym}_3(\mathbb{O}) \oplus \mathfrak{sym}_3(\mathbb{O}) = \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$ representations they carry may be combined into irreducible $\mathfrak{h} = \mathfrak{sp}(4)$ representations, as illustrated by the following decomposition under $\mathfrak{sp}(4) \supset \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$:

$$\begin{aligned}
g_{\mu\nu} &: (\mathbf{5}; \mathbf{1}) \rightarrow (\mathbf{5}; \mathbf{1}, \mathbf{1}), \\
\Psi_\mu &: (\mathbf{4}; \mathbf{8}) \rightarrow (\mathbf{4}; \mathbf{4}, \mathbf{1}) + (\mathbf{4}; \mathbf{1}, \mathbf{4}), \\
A_\mu &: (\mathbf{3}; \mathbf{27}) \rightarrow (\mathbf{3}; \mathbf{1}, \mathbf{1}) + (\mathbf{3}; \mathbf{5}, \mathbf{1}) + (\mathbf{3}; \mathbf{1}, \mathbf{5}) + (\mathbf{3}; \mathbf{4}, \mathbf{4}), \\
\lambda &: (\mathbf{2}; \mathbf{48}) \rightarrow (\mathbf{2}; \mathbf{4}, \mathbf{1}) + (\mathbf{2}; \mathbf{1}, \mathbf{4}) + (\mathbf{2}; \mathbf{4}, \mathbf{5}) + (\mathbf{2}; \mathbf{5}, \mathbf{4}), \\
\phi &: (\mathbf{1}; \mathbf{42}) \rightarrow (\mathbf{1}; \mathbf{1}, \mathbf{1}) + (\mathbf{1}; \mathbf{4}, \mathbf{4}) + (\mathbf{1}; \mathbf{5}, \mathbf{5}).
\end{aligned} \tag{6.18}$$

The algebras $\mathfrak{g}(\mathcal{N}_L + \mathcal{N}_R, D)$ and $\mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, D)$ may always be decomposed into $\mathfrak{sym}_N(\mathbb{A}_L) \oplus \mathfrak{sym}_N(\mathbb{A}_R)$ and a direct sum of irreducible representations built up from pieces relating to the left and right Yang-Mills theories. The precise form of these representations can be obtained as follows. The spacetime little group $\mathfrak{so}(N)_{\text{ST}}$ acts on the bosonic doublet according to (6.4):

$$\delta \begin{pmatrix} \frac{1}{E} f^* \tilde{f}^* \\ b\tilde{b} \end{pmatrix} = \frac{1}{4} \theta^{ab} \begin{pmatrix} (\widehat{B}_{ab}^L + \widehat{B}_{ab}^R) & 0 \\ 0 & (\widehat{A}_{ab}^L + \widehat{A}_{ab}^R) \end{pmatrix} \begin{pmatrix} \frac{1}{E} f^* \tilde{f}^* \\ b\tilde{b} \end{pmatrix}, \tag{6.19}$$

while the fermions transform as

$$\delta \begin{pmatrix} f^* \tilde{b} \\ b \tilde{f}^* \end{pmatrix} = \frac{1}{4} \theta^{ab} \begin{pmatrix} (\widehat{B}_{ab}^L + \widehat{A}_{ab}^R) & 0 \\ 0 & (\widehat{A}_{ab}^L + \widehat{B}_{ab}^R) \end{pmatrix} \begin{pmatrix} f^* \tilde{b} \\ b \tilde{f}^* \end{pmatrix}. \tag{6.20}$$

As seen in the previous chapter, these doublets are irreducible representations of the reduced magic square algebra $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$, but in general this algebra does not commute with the $\mathfrak{so}(N)_{\text{ST}}$ transformations given above. Thus the largest internal linearly-acting symmetry compatible with the spacetime symmetry is the subalgebra of $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$ that commutes with $\mathfrak{so}(N)_{\text{ST}}$. Since any element $T_{\text{ST}} \in \mathfrak{so}(N)_{\text{ST}}$ may be written as $T_{\text{ST}} = \frac{1}{4} \theta^{ab} T_{ab} = -\frac{1}{2} \theta^{0i} T_i - \frac{1}{4} \theta^{ij} T_i T_j$, a transformation commutes with $\mathfrak{so}(N)_{\text{ST}}$ if and only if it commutes with T_i . This also extends to the non-compact

symmetries. The largest symmetry algebra is that commuting with $T_{\underline{i}}$. This will be described in detail in the following section.

6.3. The Magic Pyramid Algebra

This section is dedicated to giving a mathematical definition of the magic pyramid algebra $\mathfrak{P}_{1,2}^N(\mathbb{A}_L, \mathbb{A}_R)$. Consider $T_{\underline{i}}^L \in \mathbf{tri}(\mathbb{A}_L)$ and $T_{\underline{i}}^R \in \mathbf{tri}(\mathbb{A}_R)$, with $\underline{i} = 1, \dots, (N-1)$ as usual, satisfying

$$(T_{\underline{i}}^L T_{\underline{j}}^L + T_{\underline{j}}^L T_{\underline{i}}^L)(0, x, y) = -2\delta_{\underline{i}\underline{j}}(0, x, y), \quad \forall x, y \in \mathbb{A}_L \quad (6.21)$$

and similar for $T_{\underline{i}}^R \in \mathbf{tri}(\mathbb{A}_R)$, with $1 \leq N \leq n_L, n_R$, where $n_L = \dim[\mathbb{A}_L]$ and $n_R = \dim[\mathbb{A}_R]$. This gives a representation of the Clifford algebra $\text{Cl}(N-1) \cong \text{Cl}_0(N)$ on each of \mathbb{A}_L and \mathbb{A}_R , and up to signs and conjugations determines

$$\begin{aligned} T_{\underline{i}}^L &= (\widehat{A}_{\underline{i}}^L, \widehat{B}_{\underline{i}}^L, \widehat{C}_{\underline{i}}^L) = (\widehat{L}_{e_{\underline{i}}} + \widehat{R}_{e_{\underline{i}}}, \widehat{R}_{e_{\underline{i}}}, \widehat{L}_{e_{\underline{i}}}), \\ T_{\underline{i}}^R &= (\widehat{A}_{\underline{i}}^R, \widehat{B}_{\underline{i}}^R, \widehat{C}_{\underline{i}}^R) = (\widehat{L}_{\bar{e}_{\underline{i}}} + \widehat{R}_{\bar{e}_{\underline{i}}}, \widehat{R}_{\bar{e}_{\underline{i}}}, \widehat{L}_{\bar{e}_{\underline{i}}}). \end{aligned} \quad (6.22)$$

Then, defining

$$T_{\underline{i}} := T_{\underline{i}}^L + T_{\underline{i}}^R \in \mathbf{tri}(\mathbb{A}_L) \oplus \mathbf{tri}(\mathbb{A}_R), \quad (6.23)$$

and defining $\mathfrak{so}(N)_{\text{ST}}$ as the algebra generated by the $T_{\underline{i}}$ and their commutators $[T_{\underline{i}}, T_{\underline{j}}]$, the pyramid algebra is given by

$$\mathfrak{P}_{1,2}^N(\mathbb{A}_L, \mathbb{A}_R) := \left\{ x \in \mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R) - \mathfrak{so}(N)_{\text{ST}} \mid [x, \mathfrak{so}(N)_{\text{ST}}] = 0 \right\}, \quad (6.24)$$

where

$$\mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R) := \mathbf{tri}(\mathbb{A}_L) \oplus \mathbf{tri}(\mathbb{A}_R) + 3(\mathbb{A}_L \otimes \mathbb{A}_R) \quad (6.25)$$

is the magic square algebra, with commutators given as in Chapter 5. Just as $\mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R)$ has the reduced magic square algebra $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$ as its maximal compact subalgebra, that of the pyramid algebra is called the reduced magic pyramid algebra $\mathfrak{P}_2^N(\mathbb{A}_L, \mathbb{A}_R)$:

$$\mathfrak{P}_2^N(\mathbb{A}_L, \mathbb{A}_R) := \left\{ x \in \mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R) - \mathfrak{so}(N)_{\text{ST}} \mid [x, \mathfrak{so}(N)_{\text{ST}}] = 0 \right\}. \quad (6.26)$$

Any generator in $\mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R)$ – or its subalgebra $\mathfrak{L}_2(\mathbb{A}_L, \mathbb{A}_R)$ – commutes with $\mathfrak{so}(N)_{\text{ST}}$ if and only if it commutes with every $T_{\underline{i}}$.

To examine the structure of the pyramid algebra in more detail, the terms in the Lie-algebraic formula (6.25) can be treated individually; in each case the subspace

that commutes with T_i may be specified. By definition, the part of $\mathfrak{tri}(\mathbb{A}_{L,R})$ that commutes with $T_i^{L,R}$ is the algebra $\mathfrak{sym}_N(\mathbb{A}_{L,R})$. Therefore

$$[\mathfrak{sym}_N(\mathbb{A}_{L,R}), T_i] = 0, \quad (6.27)$$

since $[\mathfrak{tri}(\mathbb{A}_L), \mathfrak{tri}(\mathbb{A}_R)] = 0$. As well as the \mathfrak{sym} subalgebras that commute with $T_i = T_i^L + T_i^R$, the subspace $\mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R)$ also contains the orthogonal combination

$$u_i := T_i^L - T_i^R, \quad (6.28)$$

but this only commutes with T_i in the special case $N = 2$, where there is just one T_1 and one u_1 , each of which generates a $\mathfrak{u}(1)$. This corresponds to the fact that $\text{Cl}_0(2) \cong \mathbb{C}$ is the only non-trivial commutative Clifford algebra. Overall then, the condition that each generator must commute with T_i reduces $\mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R)$ to the subalgebra

$$\mathfrak{sym}_N(\mathbb{A}_L) \oplus \mathfrak{sym}_N(\mathbb{A}_R) \oplus \delta_{N,2}\mathfrak{u}(1). \quad (6.29)$$

Next, consider the terms

$$3(\mathbb{A}_L \otimes \mathbb{A}_R) = (\mathbb{A}_L \otimes \mathbb{A}_R)_v + (\mathbb{A}_L \otimes \mathbb{A}_R)_s + (\mathbb{A}_L \otimes \mathbb{A}_R)_c, \quad (6.30)$$

where the subscripts refer to elements of $3(\mathbb{A}_L \otimes \mathbb{A}_R)$ as follows:

$$(X_v, X_s^*, X_c) \in 3(\mathbb{A}_L \otimes \mathbb{A}_R). \quad (6.31)$$

Then, since the commutators of

$$\begin{aligned} T_L &= (\widehat{A}_L, \widehat{B}_L, \widehat{C}_L) \in \mathfrak{tri}(\mathbb{A}_L), \\ T_R &= (\widehat{A}_R, \widehat{B}_R, \widehat{C}_R) \in \mathfrak{tri}(\mathbb{A}_R) \end{aligned} \quad (6.32)$$

and elements of $3(\mathbb{A}_L \otimes \mathbb{A}_R)$ are given by the natural action of $\mathfrak{tri}(\mathbb{A}_L) \oplus \mathfrak{tri}(\mathbb{A}_R)$ on $3(\mathbb{A}_L \otimes \mathbb{A}_R)$,

$$\begin{aligned} [T_L, (X_v, X_s^*, X_c)] &= (\widehat{A}_L X_v, \widehat{B}_L X_s^*, \widehat{C}_L X_c), \\ [T_R, (X_v, X_s^*, X_c)] &= (\widehat{A}_R X_v, \widehat{B}_R X_s^*, \widehat{C}_R X_c), \end{aligned} \quad (6.33)$$

it follows that the condition

$$[T_i, (X_v, X_s^*, X_c)] = 0 \quad (6.34)$$

is equivalent to

$$(\widehat{A}_i^L + \widehat{A}_i^R)X_v = 0, \quad (\widehat{B}_i^L + \widehat{B}_i^R)X_s^* = 0, \quad (\widehat{C}_i^L + \widehat{C}_i^R)X_c = 0. \quad (6.35)$$

Written out in full, these conditions are very simple; respectively, they become

$$\begin{aligned} (e_i + \tilde{e}_i)X_v + X_v(e_i + \tilde{e}_i) &= 0, \\ X_s^*(e_i + \tilde{e}_i) &= 0, \\ (e_i + \tilde{e}_i)X_c &= 0. \end{aligned} \quad (6.36)$$

The solution for the X_s^* and X_c conditions is rather simple, due to the Clifford algebra relations (6.21). For concreteness, consider the X_c equation first. The Clifford algebra $\text{Cl}(N-1) \cong \text{Cl}_0(N)$ is associated with an algebra \mathbb{D} via equation (4.89), as well as an irreducible spinor representation \mathcal{S}_N with (real) dimension s_N . The representation of $\text{Cl}_0(N)$ on the division algebra $\mathbb{A}_{L,R}$ generated by $\widehat{C}_i^{L,R}$ is in general reducible with

$$\mathbb{A}_L \cong (\mathcal{S}_N)^{\mathcal{N}_L}, \quad \mathbb{A}_R \cong (\mathcal{S}_N)^{\mathcal{N}_R}, \quad (6.37)$$

where $\mathcal{N}_{L,R} := n_L/s_N$. With this in mind, by equation (4.101) the subspace of the tensor product $\mathbb{A}_L \otimes \mathbb{A}_R$ satisfying $(\widehat{C}_i^L + \widehat{C}_i^R)X_c = 0$ is isomorphic as a vector space to $\mathbb{D}_N[\mathcal{N}_L, \mathcal{N}_R]$ (the linear operators \widehat{C}_i^L and $-\widehat{C}_i^R$ correspond to the generators E_m and \tilde{E}_m in (4.101)). The same is clearly true for $(\widehat{B}_i^L + \widehat{B}_i^R)X_s^* = 0$, since \widehat{B}_i^L and \widehat{B}_i^R also give representations of the Clifford algebra on \mathbb{A}_L and \mathbb{A}_R , respectively.

For the condition $(\widehat{A}_i^L + \widehat{A}_i^R)X_v = 0$, decompose

$$\mathbb{A}_L = \mathbb{V}_L^N \oplus \mathbb{S}_L^N, \quad \mathbb{A}_R = \mathbb{V}_R^N \oplus \mathbb{S}_R^N, \quad (6.38)$$

where $\mathbb{S}_L^N := \ker[\widehat{A}_i^L] = \ker[\widehat{A}_{\text{ST}}^L]$, $\mathbb{S}_R^N := \ker[\widehat{A}_i^R] = \ker[\widehat{A}_{\text{ST}}^R]$ and $\mathbb{V}_L^N, \mathbb{V}_R^N$ are their respective orthogonal complements in $\mathbb{A}_L, \mathbb{A}_R$. Note that $\mathbb{V}_L^N \cong \mathbb{V}_R^N$. Then the tensor product algebra decomposes into four pieces:

$$\begin{aligned} (\mathbb{A}_L \otimes \mathbb{A}_R)_v &= (\mathbb{V}_L^N \oplus \mathbb{S}_L^N) \otimes (\mathbb{V}_R^N \oplus \mathbb{S}_R^N) \\ &\cong (\mathbb{V}_L^N \otimes \mathbb{V}_R^N) \oplus (\mathbb{S}_L^N \otimes \mathbb{S}_R^N) \oplus (\mathbb{V}_L^N \otimes \mathbb{S}_R^N) \oplus (\mathbb{S}_L^N \otimes \mathbb{V}_R^N). \end{aligned} \quad (6.39)$$

Examining these four subspaces, first it is clear that $X_v \in \mathbb{V}_L^N \otimes \mathbb{S}_R^N$ can never satisfy (6.36), since in this case by definition $(\widehat{A}_i^L + \widehat{A}_i^R)X_v = \widehat{A}_i^L X_v \neq 0$, and similar for $X_v \in \mathbb{S}_L^N \otimes \mathbb{V}_R^N$. The entire subspace $\mathbb{S}_L^N \otimes \mathbb{S}_R^N$ trivially satisfies the condition (6.36). This leaves only $\mathbb{V}_L^N \otimes \mathbb{V}_R^N$, whose elements are of the form $X_{\underline{ab}} e_a \tilde{e}_b$ and may

be decomposed into a traceless symmetric part, a trace part and an antisymmetric part:

$$X_{\underline{ab}} e_a \tilde{e}_b \equiv \left[(X_{(ab)} - \frac{1}{v} \text{Tr}(X) \delta_{\underline{ab}}) + \frac{1}{v} \text{Tr}(X) \delta_{\underline{ab}} + X_{[ab]} \right] e_a \tilde{e}_b, \quad (6.40)$$

where $v := \dim[\mathbb{V}_L^N] = \dim[\mathbb{V}_R^N]$. It is easy to check that the trace part satisfies (6.36), since

$$(\widehat{A}_i^L + \widehat{A}_i^R) e_a \tilde{e}_a = (e_i + \tilde{e}_i) e_a \tilde{e}_a + e_a \tilde{e}_a (e_i + \tilde{e}_i) = 0. \quad (6.41)$$

The other terms do not in general satisfy (6.36), except for the antisymmetric part in the special case of $N = 2$, since there

$$X_{[ab]} \equiv X_{[01]} \varepsilon_{ab}, \quad (6.42)$$

where $\varepsilon_{\underline{ab}}$ is the 2×2 antisymmetric symbol with $\varepsilon_{01} = 1$, which gives an invariant as follows:

$$(\widehat{A}_i^L + \widehat{A}_i^R) \varepsilon_{\underline{ab}} e_a \tilde{e}_b = (e_i + \tilde{e}_i) \varepsilon_{\underline{ab}} e_a \tilde{e}_b + \varepsilon_{\underline{ab}} e_a \tilde{e}_b (e_i + \tilde{e}_i) = 0. \quad (6.43)$$

Thus the general solution to $(\widehat{A}_i^L + \widehat{A}_i^R) X_v = 0$ is of the form

$$X_v = \Phi_{\bar{v}'} e_i \tilde{e}_{\bar{v}'} + \phi e_a \tilde{e}_a + \delta_{N,2} \varphi \varepsilon_{\underline{ab}} e_a \tilde{e}_b, \quad (6.44)$$

belonging to the following subspace of $\mathbb{A}_L \otimes \mathbb{A}_R$:

$$\mathbb{S}_L^N \otimes \mathbb{S}_R^N \oplus \text{Tr}(\mathbb{V}_L^N \otimes \mathbb{V}_R^N) \oplus \delta_{N,2} \text{Pf}(\mathbb{V}_L^N \wedge \mathbb{V}_R^N) \cong \mathbb{S}_L^N \otimes \mathbb{S}_R^N \oplus \mathbb{R} \oplus \delta_{N,2} \mathbb{R}, \quad (6.45)$$

where Pf denotes the Pfaffian, which is linear only for $N = 2$ (note that $\mathbb{V}_{L,R}^2 \cong \mathbb{R}^2$). Of course, this subspace corresponds precisely to the set of NS-NS scalars from the tensor product $b\tilde{b}$ – see equations (6.15) and (6.40) – since these are just the components of $b\tilde{b}$ invariant under spacetime rotations, i.e. $(A_{\underline{ab}}^L + A_{\underline{ab}}^R) b\tilde{b} = 0$, which is equivalent to $(A_i^L + A_i^R) b\tilde{b} = 0$. Note that the $\delta_{N,2} \mathbb{R}$ term corresponds to the extra scalar coming from dualisation of the NS-NS 2-form B in $D = 4$ ($N = 2$).

Putting all of this together, first the reduced magic pyramid algebra is

$$\mathfrak{P}_2^N(\mathbb{A}_L, \mathbb{A}_R) = \mathfrak{sym}_N(\mathbb{A}_L) \oplus \mathfrak{sym}_N(\mathbb{A}_R) + \delta_{N,2} \mathfrak{u}(1) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R], \quad (6.46)$$

with $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R] \subseteq (\mathbb{A}_L \otimes \mathbb{A}_R)_c$, which is the maximal compact subalgebra of

$$\mathfrak{P}_{1,2}^N(\mathbb{A}_L, \mathbb{A}_R) = \mathfrak{P}_2^N(\mathbb{A}_L, \mathbb{A}_R) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R] + \mathbb{S}_L^N \otimes \mathbb{S}_R^N + \mathbb{R} + \delta_{N,2} \mathbb{R}, \quad (6.47)$$

with $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R] \subseteq (\mathbb{A}_L \otimes \mathbb{A}_R)_s$ and $\mathbb{S}_L^N \otimes \mathbb{S}_R^N + \mathbb{R} + \delta_{N,2}\mathbb{R} \subseteq (\mathbb{A}_L \otimes \mathbb{A}_R)_v$. The commutators for these algebras are simply inherited from those of the magic square algebras.

Of course, by construction, for each SYM-squared supergravity in $D = N + 2$ dimensions the U-duality group G and its maximal compact subalgebra H have Lie algebras

$$\mathfrak{g}(\mathcal{N}_L + \mathcal{N}_R, D) = \mathfrak{P}_{1,2}^N(\mathbb{A}_L, \mathbb{A}_R), \quad \mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, D) = \mathfrak{P}_2^N(\mathbb{A}_L, \mathbb{A}_R). \quad (6.48)$$

Of particular interest in the supergravity theories are the scalar fields, which parameterise the coset G/H . The tangent space at a point in G/H can be identified with the vector space $\mathfrak{g} - \mathfrak{h}$. From (6.47) this is given by

$$\mathfrak{g} - \mathfrak{h} = \mathfrak{P}_{1,2}^N - \mathfrak{P}_2^N = \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R] + \mathbb{S}_L^N \otimes \mathbb{S}_R^N + \mathbb{R} + \delta_{N,2}\mathbb{R}. \quad (6.49)$$

This makes sense as the space of scalar fields, since this is the space of solutions φ to $\delta\mathcal{B} = 0$ where δ here represents the infinitesimal spacetime little group transformation given in (6.19). As described in the previous section, $\delta\mathcal{B} = 0$ if and only if

$$\begin{pmatrix} \widehat{B}_i^L + \widehat{B}_i^R & 0 \\ 0 & \widehat{A}_i^L + \widehat{A}_i^R \end{pmatrix} \begin{pmatrix} \frac{1}{E} f^* \tilde{f}^* \\ b\tilde{b} \end{pmatrix} = 0, \quad (6.50)$$

which gives precisely the first two equations in (6.36). Then, by the arguments above, the space of solutions to the $f^* \tilde{f}^*$ equation is isomorphic to $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]$ and represents the R-R scalars, while that of the $b\tilde{b}$ equation is $\mathbb{S}_L^N \otimes \mathbb{S}_R^N + \mathbb{R} + \delta_{N,2}\mathbb{R}$ and represents the NS-NS scalars – see Table 6.2.

6.4. Compact Pyramid Symmetries

The reduced pyramid algebra has more structure than is immediately obvious; $\mathfrak{P}_2^N(\mathbb{A}_L, \mathbb{A}_R)$ must always consist of two mutually commuting subalgebras: one is isomorphic to $\mathfrak{sa}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D})$ and the other is an extra piece $\mathfrak{p}_N(\mathbb{A}_L, \mathbb{A}_R)$, to be defined below. This can be seen by rewriting equation (4.111) in terms of $\mathfrak{sym}_N(\mathbb{A}_L)$ and $\mathfrak{sym}_N(\mathbb{A}_R)$:

$$\begin{aligned} \mathfrak{sym}_N(\mathbb{A}_L) &= \mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{ex}_N(\mathbb{A}_L), \\ \mathfrak{sym}_N(\mathbb{A}_R) &= \mathfrak{sa}(\mathcal{N}_R, \mathbb{D}) \oplus \mathfrak{ex}_N(\mathbb{A}_R), \end{aligned} \quad (6.51)$$

where $\mathfrak{e}\mathfrak{x}_N(\mathbb{A}_L)$ is a possible $\mathfrak{u}(1)$ or $\mathfrak{sp}(1)$ that commutes with $\mathfrak{sa}(\mathcal{N}_L, \mathbb{D})$:

$$\mathfrak{e}\mathfrak{x}_N(\mathbb{A}_L) := \mathfrak{sym}_N(\mathbb{A}_L) \ominus \mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) = \begin{cases} \mathfrak{e}\mathfrak{x}(\mathbb{A}_L), & N = 1 \\ \mathfrak{u}(1), & (N, n_L) = (2, 2), (2, 4), (6, 8) \\ \emptyset, & \text{otherwise,} \end{cases} \quad (6.52)$$

and similar for $\mathfrak{e}\mathfrak{x}(\mathbb{A}_R)$. Then

$$\begin{aligned} \mathfrak{P}_2^N &= \mathfrak{sym}_N(\mathbb{A}_L) \oplus \mathfrak{sym}_N(\mathbb{A}_R) \oplus \delta_{N,2}\mathfrak{u}(1) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R] \\ &= \mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{sa}(\mathcal{N}_R, \mathbb{D}) \oplus \delta_{N,2}\mathfrak{u}(1) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R] \oplus \mathfrak{e}\mathfrak{x}_N(\mathbb{A}_L) \oplus \mathfrak{e}\mathfrak{x}_N(\mathbb{A}_R) \\ &= \mathfrak{sa}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D}) \oplus \mathfrak{p}_N(\mathbb{A}_L, \mathbb{A}_R), \end{aligned} \quad (6.53)$$

where

$$\mathfrak{p}_N(\mathbb{A}_L, \mathbb{A}_R) := \mathfrak{P}_2^N \ominus \mathfrak{sa}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D}) = \begin{cases} \mathfrak{e}\mathfrak{x}_N(\mathbb{A}_L) \oplus \mathfrak{e}\mathfrak{x}_N(\mathbb{A}_R), & N \neq 6 \\ \mathfrak{u}(1), & N = 6. \end{cases} \quad (6.54)$$

The logic for the final equality in (6.53) is as follows. Any element $X = -X^\dagger$ of $\mathfrak{a}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D})$ may be decomposed as

$$X = \begin{pmatrix} X_L & M \\ -M^\dagger & X_R \end{pmatrix}, \quad (6.55)$$

where $X_L \in \mathfrak{a}(\mathcal{N}_L, \mathbb{D})$, $X_R \in \mathfrak{a}(\mathcal{N}_R, \mathbb{D})$, and $M \in \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]$. Hence

$$\mathfrak{a}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D}) \cong \mathfrak{a}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{a}(\mathcal{N}_R, \mathbb{D}) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R], \quad (6.56)$$

which restricts to

$$\mathfrak{sa}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D}) \cong \mathfrak{sa}(\mathcal{N}_L, \mathbb{D}) \oplus \mathfrak{sa}(\mathcal{N}_R, \mathbb{D}) \oplus \delta_{\mathbb{D}, \mathbb{C}}\mathfrak{u}(1) + \mathbb{D}[\mathcal{N}_L, \mathcal{N}_R], \quad (6.57)$$

from which equation (6.53) follows². It is then easy to see that the groups H of the pyramid diagram in Figure 6.2 correspond to those predicted by the reduced pyramid algebra construction.

Equation (6.53) is in agreement with the R-symmetry for $(\mathcal{N}_L + \mathcal{N}_R)$ -extended su-

²Some care is needed when dealing with the $D = 4$ commuting $\mathfrak{e}\mathfrak{x}_2(\mathbb{A}_{L,R})$ pieces. In general the $\mathfrak{u}(1)$ absorbed into $\mathfrak{sa}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{C}) = \mathfrak{su}(\mathcal{N}_L + \mathcal{N}_R)$ is a linear combination of that contributed by the $\delta_{N,2}\mathfrak{u}(1)$ term and an element of $\mathfrak{e}\mathfrak{x}_2(\mathbb{A}_L) \oplus \mathfrak{e}\mathfrak{x}_2(\mathbb{A}_R)$. Hence the final commuting $\mathfrak{u}(1)$ algebras that sit in $\mathfrak{p}_2(\mathbb{A}_L, \mathbb{A}_R)$ are also given by orthogonal linear combinations.

persymmetry³ (here meaning specifically the automorphisms of the supersymmetry algebra), which in general has Lie algebra

$$\mathfrak{r}(\mathcal{N}_L + \mathcal{N}_R, D) = \mathfrak{a}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D}). \quad (6.58)$$

Each term in the reduced pyramid formula (6.46), and hence the algebra \mathfrak{h} of compact U-duality symmetries of SYM-squared supergravities may be traced back to Yang-Mills:

- Of course $\mathfrak{sym}_N(\mathbb{A}_L)$ and $\mathfrak{sym}_N(\mathbb{A}_R)$ are the internal symmetries of super Yang-Mills – triality algebras in $D = 3$ and R-symmetries in $D > 3$.
- Just as in $D = 3$, the term $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]$ corresponds to the tensor product of the left and right supersymmetry generators: $Q \otimes \tilde{Q}$. Performing a supersymmetry transformation (6.3) on both the left and right Yang-Mills fields gives

$$\delta_L \delta_R \mathcal{B} = E \epsilon_a \tilde{\epsilon}_{a'} \begin{pmatrix} 0 & -e_a^* \tilde{e}_{a'} \\ e_a \tilde{e}_{a'} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{E} f^* \tilde{f}^* \\ b \tilde{b} \end{pmatrix}, \quad (6.59)$$

and

$$\delta_L \delta_R \mathcal{F} = E \epsilon_a \tilde{\epsilon}_{a'} \begin{pmatrix} 0 & -e_a^* \tilde{e}_{a'} \\ e_a \tilde{e}_{a'}^* & 0 \end{pmatrix} \begin{pmatrix} f^* \tilde{b} \\ b \tilde{f}^* \end{pmatrix}, \quad (6.60)$$

but only the components that commute with the $\mathfrak{so}(N)_{\text{ST}}$ transformations (6.19) and (6.20) can contribute to \mathfrak{h} . This condition is equivalent to

$$(\widehat{C}_i^L + \widehat{C}_i^R) \epsilon \tilde{\epsilon} = (e_i + \tilde{e}_i) \epsilon \tilde{\epsilon} = 0, \quad (6.61)$$

which is equivalent to the condition in (4.101), giving $\mathbb{D}[\mathcal{N}_L, \mathcal{N}_R]$.

- Finally, the $\delta_{N,2} \mathfrak{u}(1)$ term can be understood as a consequence of the little group $\mathfrak{so}(N)_{\text{ST}}$ being both commutative and non-trivial only for $N = 2$. In this case, any element of $\mathfrak{so}(2)_{\text{ST}}$ acting on the triple of Yang-Mills quantities (b, f^*, ϵ) may be written as

$$T_{\text{ST}} = \frac{1}{4} \theta^{ab} T_{ab} = \frac{1}{2} \theta^{01} T_{01} = \frac{1}{2} \theta^{01} T_1, \quad (6.62)$$

with $T_1 = (\widehat{L}_{e_1} + \widehat{R}_{e_1}, \widehat{R}_{e_1}, \widehat{L}_{e_1})$. Hence, as described in the last section, the little group generator $T_1^L + T_1^R$ in the supergravity theory commutes with the orthogonal combination $T_1^L - T_1^R$, corresponding to transforming the left

³Note that the commuting $\mathfrak{u}(1)$ of $\mathfrak{u}(8) \cong \mathfrak{su}(8) \oplus \mathfrak{u}(1)$ for $D = 4$, $\mathcal{N} = 8$ supergravity acts trivially and is thus not produced in $\mathfrak{P}_2^N(\mathbb{A}_L, \mathbb{A}_R)$. Otherwise, $\mathfrak{a}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D}) \subseteq \mathfrak{h}(\mathcal{N}_L + \mathcal{N}_R, D)$.

and right Yang-Mills theories under separate spacetime $\mathfrak{so}(2)_{\text{ST}}^L$ and $\mathfrak{so}(2)_{\text{ST}}^R$ rotations with *opposite* respective transformation parameters, leading to a new *internal* symmetry transformation.

As a concrete example consider $D = 4$, $\mathcal{N} = 8$ supergravity with $\mathfrak{h} = \mathfrak{su}(8)$, obtained as the tensor product of $\mathcal{N}_L = 4$ SYM and $\mathcal{N}_R = 4$ SYM, each of which has internal symmetry $\mathfrak{sym}_2(\mathbb{O}) = \mathfrak{su}(4)$. Decomposing $\mathfrak{h} = \mathfrak{P}_2^2(\mathbb{O}, \mathbb{O}) = \mathfrak{su}(8)$ into $\mathfrak{su}(4) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1)$

$$\underbrace{\mathbf{63}}_{\mathfrak{su}(8)} \rightarrow \underbrace{(\mathbf{15}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{15})_0}_{\mathfrak{sym}_2(\mathbb{O}) \oplus \mathfrak{sym}_2(\mathbb{O})} + \underbrace{(\mathbf{1}, \mathbf{1})_0}_{\delta_{N,2}\mathfrak{u}(1)} + \underbrace{(\mathbf{4}, \bar{\mathbf{4}})_1 + (\bar{\mathbf{4}}, \mathbf{4})_{-1}}_{\mathbb{C}[4, 4]}, \quad (6.63)$$

the terms of (6.63) correspond exactly to the terms in the reduced pyramid formula (6.46) (note that the $\mathfrak{u}(1)$ here corresponds to the $\delta_{N,2}\mathfrak{u}(1)$ term, which contributes since $D = 4 \Leftrightarrow N = 2$).

6.5. Non-Compact Pyramid Symmetries

The maximal supergravities populating the ‘spine’ of the magic pyramids in Figure 6.1 and Figure 6.2 are of course those familiar from dimensionally reducing $D = 11$ supergravity, whose U-duality groups $E_{d(d)}$ (where $d = 11 - D$) are listed in Table 2.2. For example, consider once again the maximal $D = 4$, $\mathcal{N} = 8$ theory with $\mathfrak{g} = \mathfrak{P}_{1,2}^2(\mathbb{O}, \mathbb{O}) = \mathfrak{e}_{7(7)}$. In terms of its maximal compact subalgebra $\mathfrak{h} = \mathfrak{su}(8)$, the adjoint of $\mathfrak{e}_{7(7)}$ decomposes as

$$\mathbf{133} \rightarrow \mathbf{63} + \mathbf{70}. \quad (6.64)$$

The non-compact generators transforming as the $\mathbf{70}$ of $\mathfrak{su}(8)$, which correspond to the 70 scalar fields, can then be decomposed in terms of the SYM symmetries $\mathfrak{sym}_2(\mathbb{O}) \oplus \mathfrak{sym}_2(\mathbb{O}) \oplus \mathfrak{u}(1) = \mathfrak{su}(4) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(8)$:

$$\mathbf{70} \rightarrow \underbrace{(\mathbf{4}, \bar{\mathbf{4}})_{-1} + (\bar{\mathbf{4}}, \mathbf{4})_1}_{\mathbb{C}[4, 4]} + \underbrace{(\mathbf{6}, \mathbf{6})_0}_{\mathbb{S}_8^2 \otimes \mathbb{S}_8^2} + \underbrace{(\mathbf{1}, \mathbf{1})_2 + (\mathbf{1}, \mathbf{1})_{-2}}_{\mathbb{R} + \delta_{N,2}\mathbb{R}}. \quad (6.65)$$

Once again, these correspond precisely to the terms given in the magic pyramid formula (6.47), with $\mathbb{C}[4, 4]$ as the R-R sector and $\mathbb{S}_8^2 \otimes \mathbb{S}_8^2 + \mathbb{R} + \delta_{N,2}\mathbb{R}$ as the NS-NS sector. Each U-duality algebra of the maximal supergravities may be built up in this way.

The non-maximal supergravities of $D = 4, 5, 6$ also have U-duality algebras fol-

lowing a more regular pattern than Figure 6.1 might at first suggest. This will be given here layer-by-layer, by setting one of the division algebras, say \mathbb{A}_R , *not* equal to \mathbb{O} .

| $\mathbb{A}_L \backslash \mathbb{A}_R$ | \mathbb{C} | \mathbb{H} | \mathbb{O} |
|--|---|---|---|
| \mathbb{C} | $\mathcal{N} = 2, \ell = 16$ $G = \mathrm{U}(1, 2)$ $H = \mathrm{U}(2) \times \mathrm{U}(1)$ | $\mathcal{N} = 3, \ell = 32$ $G = \mathrm{U}(1, 3)$ $\cong \mathrm{SO}^*(6) \times \mathrm{U}(1)$ $H = \mathrm{U}(3) \times \mathrm{U}(1)$ | $\mathcal{N} = 5, \ell = 64$ $G = \mathrm{SU}(1, 5)$ $H = \mathrm{U}(5)$ |
| \mathbb{H} | $\mathcal{N} = 3, \ell = 32$ $G = \mathrm{U}(1, 3)$ $\cong \mathrm{SO}^*(6) \times \mathrm{U}(1)$ $H = \mathrm{U}(3) \times \mathrm{U}(1)$ | $\mathcal{N} = 4, \ell = 64$ $G = \mathrm{SO}(6, 2) \times \mathrm{SL}(2, \mathbb{R})$ $\cong \mathrm{SO}^*(8) \times \mathrm{SL}(2, \mathbb{R})$ $H = \mathrm{U}(4) \times \mathrm{U}(1)$ | $\mathcal{N} = 6, \ell = 128$ $G = \mathrm{SO}^*(12)$ $H = \mathrm{U}(6)$ |
| \mathbb{O} | $\mathcal{N} = 5, \ell = 64$ $G = \mathrm{SU}(1, 5)$ $H = \mathrm{U}(5)$ | $\mathcal{N} = 6, \ell = 128$ $G = \mathrm{SO}^*(12)$ $H = \mathrm{U}(6)$ | $\mathcal{N} = 8, \ell = 256$ $G = E_{7(7)}$ $H = \mathrm{SU}(8)$ |

Table 6.3.: First floor of pyramid ($D = 4$ supergravity). The first row of each entry indicates the amount of supersymmetry \mathcal{N} and the total number of degrees of freedom ℓ . The second (third) row indicates the U-duality group G (the maximal compact subgroup $H \subset G$) and its dimension. The scalar fields in each case parametrise the coset G/H .

$D = 4$ layer: Here the non-maximal supergravity U-dualities can be obtained by setting $\mathbb{A}_R = \mathbb{C}, \mathbb{H}$.

- For $\mathbb{A}_R = \mathbb{C}$, i.e. $\mathcal{N}_R = 1$, the compact algebra \mathfrak{h} is

$$\mathfrak{h} = \mathfrak{su}(\mathcal{N}) \oplus \mathfrak{e}_{\mathbb{R}_2}(\mathbb{A}_L) \oplus \mathfrak{u}(1), \quad (6.66)$$

with $\mathcal{N} = \mathcal{N}_L + 1$. Since there are no scalars in the right Yang-Mills, $\mathbb{S}_R^2 = \emptyset$, the non-compact part $\mathfrak{g} - \mathfrak{h}$ becomes

$$\mathfrak{g} - \mathfrak{h} = \mathbb{C}[\mathcal{N}_L, 1] + \mathbb{R} + \mathbb{R} \cong \mathbb{C}^{\mathcal{N}}, \quad (6.67)$$

which gives overall⁴

$$\mathfrak{g} = \mathfrak{h} + \mathbb{C}^{\mathcal{N}} \cong \mathfrak{su}(\mathcal{N}, 1) \oplus \mathfrak{e}_{\mathbb{R}_2}(\mathbb{A}_L). \quad (6.68)$$

Of course, one must check that the commutators between \mathfrak{h} and $\mathbb{C}^{\mathcal{N}}$ match up with those of $\mathfrak{su}(\mathcal{N}, 1)$, but this is fairly straightforward. This results in the

⁴Note that the algebras $\mathfrak{u}(1)$ and $\mathfrak{e}_{\mathbb{R}_2}(\mathbb{A}_L)$ in (6.66) are again actually linear combinations of the original $\mathfrak{e}_{\mathbb{R}_2}(\mathbb{A}_R) = \mathfrak{u}(1)$ and $\mathfrak{e}_{\mathbb{R}_2}(\mathbb{A}_L)$ with the $\mathfrak{u}(1)$ from the $\delta_{N,2}$ term.

| $\mathbb{A}_L \backslash \mathbb{A}_R$ | \mathbb{H} | \mathbb{O} |
|--|--|---|
| \mathbb{H} | $\mathcal{N} = 2, \ell = 64$ $G = \mathrm{O}(5, 1) \times \mathrm{O}(1, 1)$ $\cong \mathrm{SL}(2, \mathbb{H}) \times \mathbb{R}$ $H = \mathrm{Sp}(2)$ | $\mathcal{N} = 3, \ell = 128$ $G = \mathrm{SU}^*(6)$ $\cong \mathrm{SL}(3, \mathbb{H})$ $H = \mathrm{Sp}(3)$ |
| \mathbb{O} | $\mathcal{N} = 3, \ell = 128$ $G = \mathrm{SU}^*(6)$ $\cong \mathrm{SL}(3, \mathbb{H})$ $H = \mathrm{Sp}(3)$ | $\mathcal{N} = 4, \ell = 256$ $G = E_{6(6)}$ $\cong \mathrm{SL}(3, \mathbb{O}_s)$ $H = \mathrm{Sp}(4)$ |

Table 6.4.: Second floor of pyramid ($D = 5$ supergravity). The first row of each entry indicates the amount of supersymmetry \mathcal{N} and the total number of degrees of freedom ℓ . The second (third) row indicates the U-duality group G (the maximal compact subgroup $H \subset G$) and its dimension. The scalar fields in each case parametrise the coset G/H . Note that \mathbb{O}_s here denotes the *split octonions*, which are similar to the octonions, but four of the imaginary elements square to $+1$ instead of -1 .

groups given in the pyramid of Figure 6.1 or Table 6.3.

- Then for $\mathbb{A}_R = \mathbb{H}$, i.e. $\mathcal{N}_R = 2$, the compact algebra \mathfrak{h} is once again given by (6.66), but this time with $\mathcal{N} = \mathcal{N}_L + 2$. In this case \mathfrak{g} always contains

$$\mathfrak{so}^*(2\mathcal{N}) \cong \mathfrak{su}(\mathcal{N}) \oplus \mathfrak{u}(1) + \Lambda^2(\mathbb{C}^{\mathcal{N}}). \quad (6.69)$$

For example, for $\mathcal{N}_L = 1$, the non-compact part becomes $\mathfrak{g} - \mathfrak{h} = \mathbb{C}^3 \cong \Lambda^2(\mathbb{C}^3)$, while the compact part is $\mathfrak{h} = \mathfrak{su}(3) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$, and so the U-duality group is

$$\mathfrak{g} = \mathfrak{so}^*(6) \oplus \mathfrak{u}(1). \quad (6.70)$$

Similar logic for $\mathcal{N}_L = 2, 4$ gives the remaining U-dualities of $D = 4$ displayed in Table 6.3 and Figure 6.1 (note the isomorphisms $\mathfrak{so}^*(6) \cong \mathfrak{su}(3, 1)$ and $\mathfrak{so}^*(8) \cong \mathfrak{so}(6, 2)$).

These theories were previously obtained in [72] by consistently truncating to the untwisted sector of the low-energy effective field theory of Type II superstrings on factorised orbifolds, revealing their double-copy structure. The magic $D = 4$, $\mathcal{N} = 2$ supergravities were also discussed in this context. In particular, the quaternionic theory originates from a non-factorisable \mathbb{Z}_2 -orbifold compactification [72].

$D = 5$ layer: For $\mathbb{A}_R = \mathbb{H}$, the generators of $\mathfrak{g} - \mathfrak{h}$ contain a subspace isomorphic to $\mathfrak{h}'_{\mathcal{N}}(\mathbb{H})$, the space of $\mathcal{N} \times \mathcal{N}$ Hermitian traceless quaternionic matrices. Combining this with the compact R-symmetry subalgebra $\mathfrak{h} = \mathfrak{sp}(\mathcal{N})$, which consists of $\mathcal{N} \times \mathcal{N}$

| $\mathbb{A}_L \setminus \mathbb{A}_R$ | \mathbb{H} | \mathbb{O} |
|---------------------------------------|---|--|
| \mathbb{H} | $\mathcal{N} = (1, 1), \ell = 64$ $G = \text{Spin}(4) \times \text{O}(1, 1)$ $\cong \text{SL}(1, \mathbb{H}) \times \text{SL}(1, \mathbb{H}) \times \mathbb{R}$ $H = \text{Sp}(1) \times \text{Sp}(1)$ | $\mathcal{N} = (1, 2), \ell = 128$ $G = \text{SU}^*(4) \times \text{Sp}(1)$ $\cong \text{SL}(1, \mathbb{H}) \times \text{SL}(2, \mathbb{H})$ $H = \text{Sp}(2) \times \text{Sp}(1)$ |
| \mathbb{O} | $\mathcal{N} = (2, 1), \ell = 128$ $G = \text{SU}^*(4) \times \text{Sp}(1)$ $\cong \text{SL}(2, \mathbb{H}) \times \text{SL}(1, \mathbb{H})$ $H = \text{SU}(2) \times \text{SO}(2)$ | $\mathcal{N} = (2, 2), \ell = 256$ $G = \text{SO}(5, 5)$ $\cong \text{SL}(2, \mathbb{O}_s)$ $H = \text{Sp}(2) \times \text{Sp}(2)$ |

Table 6.5.: Third floor of pyramid ($D = 6$ supergravity). The first row of each entry indicates the amount of supersymmetry \mathcal{N} and the total number of degrees of freedom ℓ . The second (third) row indicates the U-duality group G (the maximal compact subgroup $H \subset G$) and its dimension. The scalar fields in each case parametrise the coset G/H .

anti-Hermitian quaternionic matrices, gives

$$\mathfrak{sl}(\mathcal{N}, \mathbb{H}) \cong \mathfrak{sp}(\mathcal{N}) + \mathfrak{h}'_{\mathcal{N}}(\mathbb{H}). \quad (6.71)$$

Specifically, since $\mathbb{S}_4^3 = \mathbb{R}$, the space of non-compact generators is

$$\mathfrak{g} - \mathfrak{h} = \mathbb{H}[\mathcal{N}_L, 1] + \mathbb{S}_L^3 \otimes \mathbb{R} + \mathbb{R} = \mathbb{H}^{\mathcal{N}_L} + \mathbb{S}_L^3 \otimes \mathbb{R} + \mathbb{R}, \quad (6.72)$$

so that for $\mathbb{A}_L = \mathbb{H}$, i.e. $\mathcal{N}_L = 1$, this becomes

$$\mathfrak{g} - \mathfrak{h} = \mathbb{H} + \mathbb{R} + \mathbb{R} \cong \mathfrak{h}'_2(\mathbb{H}) + \mathbb{R}, \quad (6.73)$$

giving

$$\mathfrak{g} = \mathfrak{sl}(2, \mathbb{H}) \oplus \mathbb{R}, \quad (6.74)$$

while for $\mathbb{A}_L = \mathbb{O}$, i.e. $\mathcal{N}_L = 2$, the left Yang-Mills scalars belong to $\mathbb{S}_8^3 = \mathbb{R}^5 \cong \mathfrak{h}'_2(\mathbb{H})$ and hence

$$\mathfrak{g} - \mathfrak{h} = \mathbb{H}^2 + \mathfrak{h}'_2(\mathbb{H}) + \mathbb{R} \cong \mathfrak{h}'_3(\mathbb{H}), \quad (6.75)$$

giving the U-duality

$$\mathfrak{g} = \mathfrak{sl}(3, \mathbb{H}). \quad (6.76)$$

Once again this matches up with the groups shown in Table 6.4 and Figure 6.1 (note the isomorphisms $\mathfrak{sl}(\mathcal{N}, \mathbb{H}) \cong \mathfrak{su}^*(2\mathcal{N})$ and $\mathfrak{su}^*(4) \cong \mathfrak{so}(5, 1)$). The magic pyramid is plotted again in Figure 6.3 using various isomorphisms to emphasise its overall symmetries.

$D = 6$ layer: This follows the same pattern as the $D = 5$ layer, only with \mathcal{N}^+ and \mathcal{N}^- treated separately due to chirality, giving

$$\mathfrak{g} \supseteq \mathfrak{sl}(\mathcal{N}^+, \mathbb{H}) \oplus \mathfrak{sl}(\mathcal{N}^-, \mathbb{H}), \quad (6.77)$$

with equality for $\mathcal{N} = (\mathcal{N}^+, \mathcal{N}^-) = (2, 1), (1, 2)$, while if neither Yang-Mills is the maximal $(1, 1)$ theory then

$$\mathfrak{g} \cong \mathfrak{sl}(\mathcal{N}^+, \mathbb{H}) \oplus \mathfrak{sl}(\mathcal{N}^-, \mathbb{H}) \oplus \mathbb{R}. \quad (6.78)$$

Note that in $D = 6$ when tensoring two minimally-supersymmetric Yang-Mills multiplets, one may take the two multiplets to have either the same chirality or opposite chirality:

$$\begin{aligned} [(1, 0)_{\text{SYM}}^L] \times [(1, 0)_{\text{SYM}}^R] &= [(2, 0)_{\text{SUGRA}} + (2, 0)_{\text{TENSOR}}], \\ [(1, 0)_{\text{SYM}}^L] \times [(0, 1)_{\text{SYM}}^R] &= [(1, 1)_{\text{SUGRA}}], \end{aligned} \quad (6.79)$$

leading to respective U-dualities given by (6.78):

$$\begin{aligned} \mathfrak{sl}(2, \mathbb{H}) \oplus \mathbb{R} &\cong \mathfrak{so}(5, 1) \oplus \mathbb{R}, \\ \mathfrak{sl}(1, \mathbb{H}) \oplus \mathfrak{sl}(1, \mathbb{H}) \oplus \mathbb{R} &\cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}. \end{aligned} \quad (6.80)$$

The details of the above tensorings are given in section A.2. See also [94]. The chiral theory $[(2, 0)_{\text{SUGRA}} + (2, 0)_{\text{TENSOR}}]$ is anomalous since the unique anomaly-free supergravity [95] with $\mathcal{N} = (2, 0)$ consists of one $(2, 0)_{\text{SUGRA}}$ multiplet coupled to 21 $(2, 0)_{\text{TENSOR}}$ multiplets as obtained by compactifying $D = 10$ Type IIB supergravity on a $K3$. The non-chiral $[(1, 1)_{\text{SUGRA}}]$ theory was chosen for Table 6.5 and Figure 6.1. Note that the chiral $(1, 2)_{\text{SUGRA}}$ is also anomalous and adding the required compensating matter extends the theory to $(2, 2)_{\text{SUGRA}}$ [96].

Although they are not considered directly here, it should be noted that the magic $D = 6$, $\mathcal{N} = (1, 0)$ supergravities (which come coupled to 2, 3, 5, 9 tensor multiplets and 2, 4, 8, 16 vector multiplets, respectively, as well as hyper multiplets) are closely related to the magic square and constitute the parent theories of the magic $D = 5, 4, 3$ supergravities. See [33] and the references therein.

| | |
|--|--|
| | |
| $\mathbb{A}_L \backslash \mathbb{A}_R$ | \mathbb{O} |
| \mathbb{O} | $\mathcal{N} = (1, 1)$ (IIA), $\ell = 256$ $G = \mathbb{O}(1, 1)$ $H = \mathbb{1}$ |
| $\mathbb{A}_L \backslash \mathbb{A}_R$ | \mathbb{O} |
| \mathbb{O} | $\mathcal{N} = (2, 0)$ (IIB), $\ell = 256$ $G = \text{SL}(2, \mathbb{R})$ $H = \text{SO}(2)$ |

Table 6.6.: The peak of the magic pyramid: $D = 10$. The left-hand (right-hand) table is obtained by tensoring SYM of opposing (matching) chiralities, which is equivalent to applying a triality to the magic pyramid formula. Of course, there is no room for matter couplings in these theories.

$D = 10$ **layer:** In this case⁵, the compact generators form the R-symmetry

$$\mathfrak{h} = \mathfrak{so}(\mathcal{N}^+) \oplus \mathfrak{so}(\mathcal{N}^-), \quad (6.81)$$

while the non-compact generators belong to

$$\mathfrak{g} - \mathfrak{h} = \mathbb{R}[\mathcal{N}_L^+, \mathcal{N}_R^+] + \mathbb{R}[\mathcal{N}_L^-, \mathcal{N}_R^-] + \mathbb{R}. \quad (6.82)$$

Of course, once again there is a choice of chiralities:

$$\begin{aligned} [(1, 0)_{\text{SYM}}^L] \times [(1, 0)_{\text{SYM}}^R] &= [(2, 0)_{\text{SUGRA}}] \text{ i.e. Type IIB,} \\ [(1, 0)_{\text{SYM}}^L] \times [(0, 1)_{\text{SYM}}^R] &= [(1, 1)_{\text{SUGRA}}] \text{ i.e. Type IIA.} \end{aligned} \quad (6.83)$$

For Type IIB the compact part becomes $\mathfrak{h} = \mathfrak{so}(2)$, while $\mathfrak{g} - \mathfrak{h} = \mathbb{R} + \mathbb{R}$, which overall yields

$$\mathfrak{g} = \mathfrak{so}(2) + \mathbb{R} + \mathbb{R} \cong \mathfrak{sl}(2, \mathbb{R}), \quad (6.84)$$

as required; the NS-NS scalar and R-R scalar belong to an $\text{SL}(2, \mathbb{R})/\text{SO}(2)$ coset model. For Type IIA the compact subalgebra \mathfrak{h} is empty, while the non-compact part $\mathfrak{g} - \mathfrak{h} = \mathfrak{g} = \mathbb{R}$, corresponding to the single NS-NS scalar. By convention Type IIA appears in Figure 6.1 and Figure 6.2, but one could equally well place Type IIB at the summit of the pyramid.

6.6. Summary

The goal of this chapter was to obtain the field content and global symmetries of each supergravity obtained from tensoring extended SYM multiplets in $D = N + 2$

⁵The $D = 7, 8, 9$ layers have been skipped since they introduce no new considerations. The field content in these cases may be found in Table A.1 of the Appendix.

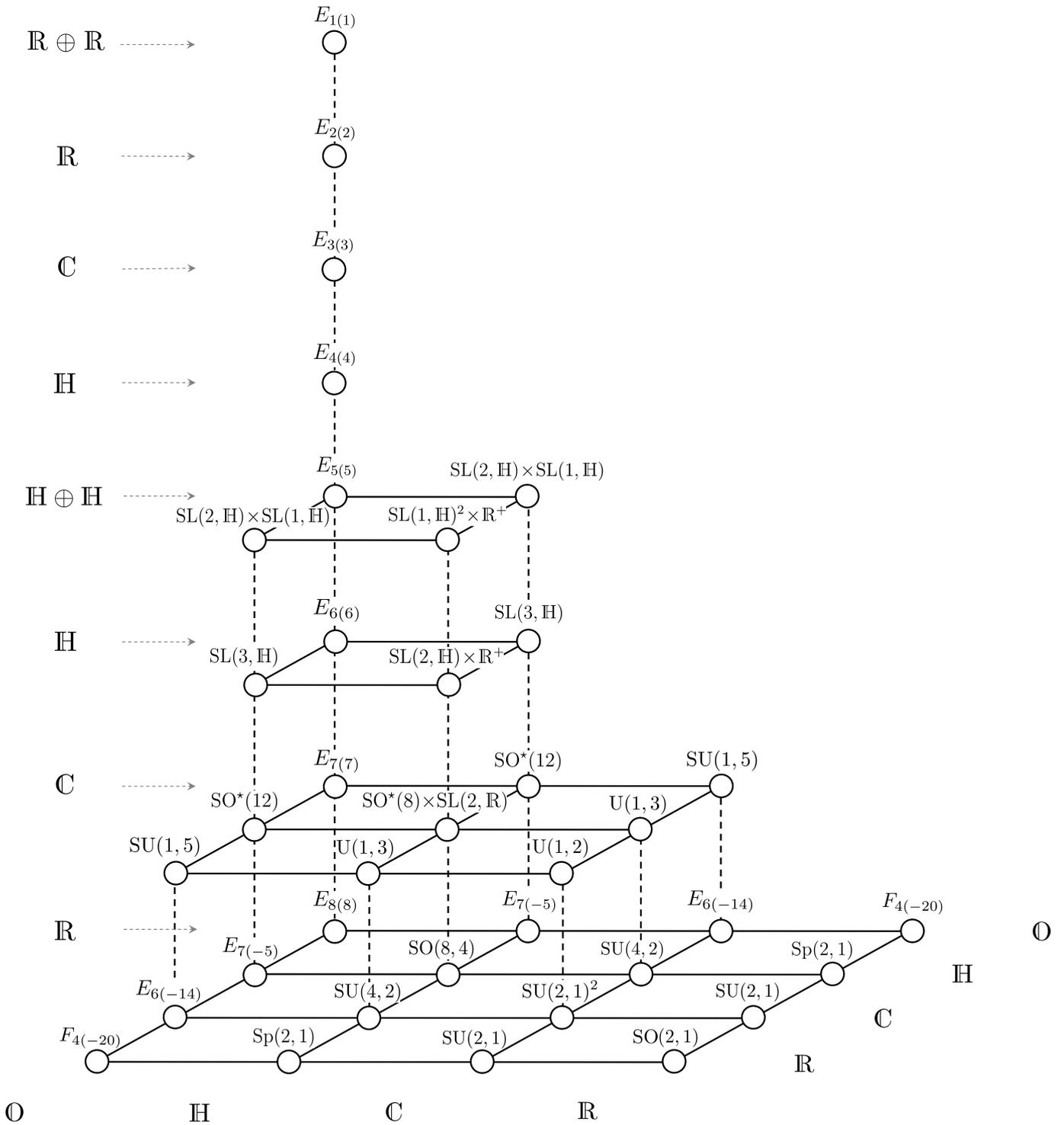


Figure 6.3.: Pyramid of U-duality groups G , with the groups rewritten using various isomorphisms in order to emphasise its overall patterns.

spacetime dimensions, with $1 \leq N \leq 8$. The division-algebraic formalism introduced throughout this thesis is particularly well-suited to finding and describing these global symmetries in a unified manner.

As demonstrated in Chapter 4, any pure Yang-Mills theory may be formulated with a single Lagrangian and single set of transformation rules, but with spacetime fields valued in the division algebra \mathbb{A}_n , where n is half the number of real supercharges \mathcal{Q} . This perspective reveals the role of the triality algebras; each SYM theory has internal symmetry (or R-symmetry) given by the algebra $\mathfrak{sym}_N(\mathbb{A}_n)$, which is defined in (6.7) directly in terms of $\mathfrak{ti}(\mathbb{A}_n)$.

Tensoring left/right SYM multiplets valued in the division algebras \mathbb{A}_L and \mathbb{A}_R then naturally leads to $\mathcal{N}_L + \mathcal{N}_R$ supergravity multiplets with spacetime fields valued in $\mathbb{A}_L \otimes \mathbb{A}_R$. For $D = 1 + 2$ this yields a set of supergravities with U-duality groups given by the magic square of Freudenthal-Rosenfeld-Tits. For $D = N + 2 > 3$, identifying a common spacetime Clifford algebra $\text{Cl}_0(N)$ truncates the magic square to 3×3 , 2×2 , and 1×1 arrays of subalgebras, corresponding precisely to the U-dualities obtained by tensoring SYM multiplets in each dimension. Together the ascending squares constitute a magic pyramid of algebras defined by the magic pyramid formula (6.24), or equivalently (6.47). The exceptional octonionic row and column of each level is constrained by supersymmetry to give the unique supergravity multiplet. On the other hand, the interior 3×3 , 2×2 and 1×1 squares can and do admit matter couplings. These additional matter multiplets are just as required to give the U-dualities predicted by the pyramid formula. Interestingly, in these cases the degrees of freedom are split evenly between the graviton multiplet and the matter multiplets, the number of which is determined by the rule⁶ $k = \min(\mathcal{N}_L, \mathcal{N}_R)$.

The magic pyramid supergravity theories are rather non-generic. Not only are they, in a sense, defined by the magic pyramid formula, they are of course generated by tensoring the division-algebraic SYM multiplets. It would therefore be interesting to explore whether they collectively possess other special properties, particularly as quantum theories, which can be traced back to their magic square origins. For example, in the maximal $[\mathcal{N}_L = 4 \text{ SYM}] \times [\mathcal{N}_R = 4 \text{ SYM}]$ case it has been shown that $\mathcal{N} = 8$ supergravity is four-loop finite [71], a result which cannot be attributed to supersymmetry alone. While $\mathcal{N} = 8$ is expected to have the best possible UV behaviour, as suggested by its connection to $\mathcal{N} = 4 \text{ SYM}$, it could still be that the remaining magic square supergravities share some structural features due to their common ‘gauge \times gauge’ origin and closely related global symmetries.

⁶Thanks to Andrew Thomson for pointing out this rule. Note the subtlety in $D = 6$ that one must treat \mathcal{N}^+ and \mathcal{N}^- separately. Hence, for example, $[(1, 0)] \times [(0, 1)]$ has $k = 0$.

One might also seek extensions of the magic pyramid construction which could account for more generic supergravities. The magic supergravities of Gunaydin, Sierre and Townsend [25, 26], for example, might be described by an extension of the present construction incorporating matter multiplets.

There are also some brief remarks to be made on the geometrical interpretation of the magic pyramid. As observed in [84] (for the exceptional cases) and described in Chapter 4, the $D = 3$ Freudenthal magic square can be regarded as the isometries of the Lorentzian projective planes $(\mathbb{A}_L \otimes \mathbb{A}_R)\tilde{\mathbb{P}}^2$. In essence the pyramid algebra describes the isometries of special submanifolds of these spaces. In $D = N + 2 > 3$, the action of the Clifford algebra $\text{Cl}_0(N) \supset \text{Spin}(N)_{\text{ST}}$ associated with the spacetime little group $\text{Spin}(N)_{\text{ST}}$ breaks the group $\text{Spin}(n_L + n_R)$ that acts (in the spinor representation) on the tangent space $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$ at each point on the projective plane down to $\text{SA}(\mathcal{N}_L + \mathcal{N}_R, \mathbb{D})$, where \mathbb{D} is the algebra over which $\text{Cl}_0(N)$ is naturally defined. This singles out a particular submanifold of $(\mathbb{A}_L \otimes \mathbb{A}_R)\tilde{\mathbb{P}}^2$ for each N , and the isometries of this submanifold yield the magic pyramid.

Finally, note that to call $(\mathbb{A}_L \otimes \mathbb{A}_R)\tilde{\mathbb{P}}^2$ a projective plane is being rather heuristic; the $\mathbb{H} \otimes \mathbb{O}$ and $\mathbb{O} \otimes \mathbb{O}$ cases do not obey the axioms of projective geometry. Unlike $\mathbb{R} \otimes \mathbb{O}$, the tensor products $\mathbb{H} \otimes \mathbb{O}$ and $\mathbb{O} \otimes \mathbb{O}$ are not division algebras, preventing a direct projective construction. Furthermore, unlike $\mathbb{C} \otimes \mathbb{O}$, Hermitian 3×3 matrices over $\mathbb{H} \otimes \mathbb{O}$ or $\mathbb{O} \otimes \mathbb{O}$ do not form a simple Jordan algebra, so the usual identification of points (lines) with trace 1 (2) projection operators cannot be made [1]. Nonetheless, they are in fact geometric spaces, generalising projective spaces, known as ‘buildings’, on which the U-dualities act as isometries. Buildings were originally introduced by Jacques Tits to provide a geometric approach to simple Lie groups, in particular the exceptional cases, but have since had far reaching implications. See, for example, [97, 98] and the references therein. Of course, it has long been known that increasing supersymmetry restricts the spaces on which the scalar fields may live, as comprehensively demonstrated for $D = 3$ in [84]. Evidently these restrictions lead to the concept of buildings in supergravity. It may be of interest to examine whether this relationship between buildings and supersymmetry has some useful implications.

7. An Octonionic Formulation of $D = 11$ Supergravity

Throughout the preceding chapters, a recurring theme has been the connection between the octonions and maximal supersymmetry. However, it is fair to say that the full significance of the octonions in string theory remains puzzling. In this chapter, in an attempt to approach this problem from a new perspective, the Lagrangian and transformation rules of $D = 11$ supergravity are presented written over the octonions.

The method relies on the fact that a $D = 11$ spinor with 32 components may be packaged as a 4-component octonionic column vector [32, 99, 100]. Dimensionally reducing to $D = 4$ and $D = 3$, where the U-duality groups are $E_{7(7)}$ and $E_{8(8)}$, respectively, the coupling of the 7 or 8 dilatons to the other scalar fields in the theory is parameterised by the sets of $E_{7(7)}$ or $E_{8(8)}$ root vectors [101, 102]. The octonionic nature of the fields in the Lagrangian suggests a new perspective in which these root vectors, or ‘dilaton vectors’, are unit-norm *octavian integers* [40] – the octonionic analogues of the integers. This involves a novel use of the *dual Fano plane*, which is obtained by interchanging points and lines on the Fano plane.

7.1. Spinors and Division Algebras in $D = 4, 5, 7, 11$

The description of the Lorentz group and its representations in terms of division algebras \mathbb{A}_n in Chapter 3 can be ‘boosted up’ by a dimension, from $D = n + 2$ to $D = n + 3$, where $n = 1, 2, 4, 8$. First consider constructing gamma matrices in $D = n + 2$ modeled on the Weyl basis (2.23) using the generalised Pauli matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \mu = 0, \dots, (n + 1). \quad (7.1)$$

These matrices clearly satisfy the Clifford algebra

$$\gamma^\mu(\gamma^\nu \lambda) + \gamma^\nu(\gamma^\mu \lambda) = 2\eta^{\mu\nu} \lambda \quad \forall \quad \lambda \in \mathbb{A}^4, \quad (7.2)$$

while each of them anti-commutes with the matrix

$$\gamma_* = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \quad (7.3)$$

Thus the set of $n + 3$ matrices $\{\gamma^M\} := \{\gamma^\mu, \gamma_*\}$ with $M = 0, 1, \dots, (n + 2)$ satisfies the Clifford algebra for $D = n + 3$:

$$\gamma^M(\gamma^N \lambda) + \gamma^N(\gamma^M \lambda) = 2\eta^{MN} \lambda \quad \forall \quad \lambda \in \mathbb{A}^4. \quad (7.4)$$

The particular case of interest for this chapter is $D = 8 + 3 = 11$. A Majorana spinor in $D = 11$ has 32 components, usually represented as a real column vector. Alternatively, viewing \mathbb{R}^{32} as a tensor product $\mathbb{R}^4 \otimes \mathbb{R}^8 \cong \mathbb{R}^4 \otimes \mathbb{O} \cong \mathbb{O}^4$, the spinor becomes a 4-component octonionic column vector

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}, \quad \lambda_\alpha \in \mathbb{O}, \quad \alpha = 1, 2, 3, 4. \quad (7.5)$$

Written out in their 4×4 form, the octonionic basis $\{\gamma^M\}$ defined above is

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, & \gamma^{a+1} &= \begin{pmatrix} 0 & 0 & 0 & e_a^* \\ 0 & 0 & e_a & 0 \\ 0 & e_a^* & 0 & 0 \\ e_a & 0 & 0 & 0 \end{pmatrix}, \\ \gamma^9 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, & \gamma^{10} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \end{aligned} \quad (7.6)$$

with $a = 0, 1, \dots, 7$. To see how these are related to a more familiar real 32×32 set one can simply take their ‘matrix elements’:

$$\begin{aligned} \langle e_a | (\gamma^\mu)_\alpha^\beta e_b \rangle &= (\gamma^\mu)_\alpha^\beta \langle e_a | e_b \rangle = (\gamma^\mu)_\alpha^\beta \delta_{ab}, & \mu &= 0, 1, 9, 10, \\ \langle e_a | (\gamma^{i+1})_\alpha^\beta e_b \rangle &= (\gamma_*)_\alpha^\beta \langle e_a | e_i e_b \rangle = (\gamma_*)_\alpha^\beta \Gamma_{ab}^i & i &= 1, \dots, 7, \end{aligned} \quad (7.7)$$

where $\gamma_{(5)}$ is defined by

$$\gamma_{(5)} = -\gamma^0 \gamma^1 \gamma^9 \gamma^{10}. \quad (7.8)$$

Thus, writing the gamma matrices over the octonions corresponds to an $\mathbf{11} = \mathbf{4} + \mathbf{7}$ split,

$$\mathrm{SO}(1, 10) \supset \mathrm{SO}(1, 3) \times \mathrm{SO}(7), \quad (7.9)$$

with the seven imaginary octonions playing the role of the $\mathrm{SO}(7)$ gamma matrices and the four real γ^μ , $\mu = 0, 1, 9, 10$, playing the role of the ('really real' Majorana) $\mathrm{SO}(1,3)$ gamma matrices. An obvious appeal of the octonionic parameterisation is that this natural split associates the seven extra dimensions of M-theory with the seven imaginary octonionic basis elements.

By equation (7.7), left-multiplying $\lambda \in \mathbb{O}^4$ by the octonionic matrix γ^M corresponds to multiplying λ 's 32 real components by an ordinary real 32×32 gamma matrix. By successive composition it is clear then that the action of the rank r Clifford algebra element on λ can be written

$$\gamma^{[M_1} \left(\gamma^{M_2} \left(\dots \left(\gamma^{M_{r-1}} \left(\gamma^{M_r} \lambda \right) \dots \right) \right) \right). \quad (7.10)$$

The positioning of the brackets fixes any ambiguities due to non-associativity. For example, an infinitesimal Lorentz transformation of a spinor λ is

$$\delta\lambda = \frac{1}{4}\omega_{MN}\gamma^M(\gamma^N\lambda), \quad (7.11)$$

where $\omega_{MN} = -\omega_{NM}$.

Define the operator $\hat{\gamma}^M$, whose action is left-multiplication by γ^M , so that the rank r Clifford algebra element becomes the operator

$$\hat{\gamma}^{M_1 M_2 \dots M_r} \equiv \hat{\gamma}^{[M_1} \hat{\gamma}^{M_2} \dots \hat{\gamma}^{M_r]}, \quad (7.12)$$

where the operators $\hat{\gamma}^M$ must be composed as

$$\hat{\gamma}^M \hat{\gamma}^N \lambda = \gamma^M(\gamma^N \lambda) \neq (\gamma^M \gamma^N) \lambda. \quad (7.13)$$

This ensures that the action of $\hat{\gamma}^{[M_1 M_2 \dots M_r]}$ on a spinor is given by (7.10), as required.

To construct the supergravity Lagrangian and transformation rules in this language will require real spinor bilinears. These are built using the charge conjugation matrix $C^{\alpha\beta}$ (which is numerically equal to γ^0 but with a different index structure):

$$C^{\alpha\beta} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}. \quad (7.14)$$

By definition this matrix satisfies

$$C(\gamma^M)^\dagger C = \gamma^M \quad \Leftrightarrow \quad (\gamma^M)^\dagger C = -C\gamma^M, \quad (7.15)$$

where the dagger denotes transposition and octonionic conjugation. Let us define

$$\bar{\lambda} := \lambda^\dagger C. \quad (7.16)$$

If λ_1 and λ_2 are octonionic spinors whose real components are anti-commuting Grassmann numbers, then the quantity

$$\text{Re}(i\bar{\lambda}_1\lambda_2) = \frac{1}{2} (i\bar{\lambda}_1\lambda_2 + (i\bar{\lambda}_1\lambda_2)^\dagger) = \frac{i}{2} (\bar{\lambda}_1\lambda_2 + \bar{\lambda}_2\lambda_1) \quad (7.17)$$

is Lorentz-invariant. Note that the dagger operation is defined here such that it also complex-conjugates the factor of i . This accounts for the anti-commuting spinor components. A general spinor bilinear may then be formed as follows:

$$\text{Re}(i\bar{\lambda}_1\hat{\gamma}^{M_1M_2\dots M_r}\lambda_2), \quad (7.18)$$

which as usual will transform as an r -index antisymmetric tensor under Lorentz transformations.

7.2. An Octonionic Formulation of the M-Theory Algebra

The fact that a $D = 11$ spinor with 32 components may be packaged as a 4-component octonionic column vector [32, 99] prompts the question of how to write the algebra of $D = 11$ supergravity (or ‘M-algebra’) using octonionic supercharges Q . This was explored in [99] where the problem was highlighted that the apparently natural choice of octonionic matrices could not provide enough degrees of freedom to account for all of M-theory’s brane charges¹.

Another fundamental question that arises when writing the $\{Q, Q\}$ algebra in this way is whether or not the usual anti-commutator is really the appropriate object to study, given that the supercharges are both fermionic *and* written over a non-commutative and non-associative algebra \mathbb{O} .

¹Brane charges are the generalisation of the central charges introduced in Chapter 2 – extra terms appearing on the right-hand side of the supersymmetry algebra. These terms contain information about the brane solutions arising in the theory.

In this chapter the above problems are tackled by introducing a novel outer product, which takes a pair of elements belonging to a division algebra \mathbb{A} and returns a real linear operator on \mathbb{A} , expressed using multiplication in \mathbb{A} . This product enables one to easily rewrite any expression involving $n \times n$ matrices and n -dimensional vectors in terms of multiplication in the n -dimensional division algebra \mathbb{A} . The problem of the octonionic M-algebra is solved using this product, which allows one to obtain the correct $\{Q, Q\}$ bracket.

7.2.1. A New Outer Product

It is interesting to see what other linear operations on \mathbb{R}^n look like when written in terms of the division-algebraic multiplication rule. This was explored in [103], but a different approach will be taken here. Consider the following general problem. Given some linear operator on \mathbb{R}^n expressed as an $n \times n$ matrix M_{ab} , one should be able to find a division-algebraic multiplication operator \hat{M} on the division algebra \mathbb{A} such that \hat{M} has the effect of multiplying the components of $x = x_a e_a \in \mathbb{A}$ by M_{ab} :

$$\hat{M}x \equiv e_a M_{ab} x_b. \quad (7.19)$$

An explicit form for this operator can be found using the inner product (3.26). First one simply rewrites

$$\begin{aligned} M_{ab} &= M_{cd} \langle e_a | e_c \rangle \langle e_b | e_d \rangle \\ &= \frac{1}{2} M_{cd} \langle e_a | e_c (e_d^* e_b) + e_c (e_b^* e_d) \rangle. \end{aligned} \quad (7.20)$$

Now it is clear that the operator

$$\hat{M} \equiv \frac{1}{2} M_{cd} \left(e_c (e_d^* \cdot) + e_c ((\cdot)^* e_d) \right) \quad (7.21)$$

(where a dot represents a slot for an octonion) has matrix elements

$$\langle e_a | \hat{M} e_b \rangle = M_{ab}. \quad (7.22)$$

Thus the outer product for division algebra elements may be expressed in terms of their multiplication rule, defining:

$$\begin{aligned} \times : \mathbb{A} \otimes \mathbb{A} &\rightarrow \text{End}(\mathbb{A}) \\ e_a \otimes e_b &\mapsto e_a \times e_b \equiv \frac{1}{2} \left(e_a (e_b^* \cdot) + e_a ((\cdot)^* e_b) \right). \end{aligned} \quad (7.23)$$

With the new product comes the power to rewrite any expression involving $n \times n$ matrices and n -dimensional vectors in terms of multiplication in the n -dimensional division algebra \mathbb{A} .

It is useful to note various equivalent ways of writing the outer product above:

$$\begin{aligned}
e_a \times e_b &= \frac{1}{2} \left(e_a (e_b^* \cdot) + e_a ((\cdot)^* e_b) \right) \\
&= \frac{1}{2} \left((\cdot e_b^*) e_a + (e_b (\cdot)^*) e_a \right) \\
&= \frac{1}{2} \left(e_a (e_b (\cdot)^*) + e_a (\cdot e_b^*) \right) \\
&= \frac{1}{2} \left(((\cdot)^* e_b) e_a + (e_b^* \cdot) e_a \right).
\end{aligned} \tag{7.24}$$

Due to the alternativity of the division algebras the brackets on each of the terms may be simultaneously shifted,

$$e_a (e_b^* \cdot) + e_a ((\cdot)^* e_b) = (e_a e_b^*) (\cdot) + (e_a (\cdot)^*) e_b, \tag{7.25}$$

and similarly for the other four possibilities above.

7.2.2. The Octonionic M-Algebra

The anti-commutator of two supercharges in the $D = 11$ supergravity theory is conventionally written as the ‘M-algebra’ [104]

$$\begin{aligned}
\{Q_{\bar{\alpha}}, Q_{\bar{\beta}}\} &= (\gamma^M C)_{\bar{\alpha}\bar{\beta}} P_M + (\gamma^{MN} C)_{\bar{\alpha}\bar{\beta}} Z_{MN} \\
&\quad + (\gamma^{MNPQR} C)_{\bar{\alpha}\bar{\beta}} Z_{MNPQR},
\end{aligned} \tag{7.26}$$

where $\bar{\alpha}, \bar{\beta} = 1, \dots, 32$, P_M is the generator of translations and Z_{MN} and Z_{MNPQR} are the brane charges. The charge conjugation matrix $C_{\bar{\alpha}\bar{\beta}}$ serves to lower an index on each of the gamma matrices.

The left-hand side is a symmetric 32×32 matrix with 528 components, while the terms on the right-hand side consist of the rank 1, 2 and 5 Clifford algebra elements, which form a basis for such symmetric matrices. In terms of $\text{SO}(1, 10)$ representations:

$$(\mathbf{32} \times \mathbf{32})_{\text{Sym}} = \mathbf{11} + \mathbf{55} + \mathbf{462}. \tag{7.27}$$

The goal is to write this algebra in terms of 4×4 octonionic matrices. However, the space of octonionic 4×4 matrices is of dimension $16 \times 8 = 128$, and hence naively does not carry nearly enough degrees of freedom to write (7.26).

The solution to this problem is to use the octonionic Clifford algebra operators

$\hat{\gamma}^{[M_1 M_2 \dots M_r]}$ defined in the previous section. These operators (including all ranks r) span a space of dimension $32 \times 32 = 1024$. In other words, their octonionic matrix elements are

$$\langle e_a | \hat{\gamma}^M_{\alpha}{}^{\beta} e_b \rangle = \gamma^M_{\alpha a}{}^{\beta b}, \quad \alpha, \beta = 1, 2, 3, 4, \quad (7.28)$$

and treating αa as a composite spinor index $\bar{\alpha} = 1, \dots, 32$, the set of $\{\gamma^M_{\bar{\alpha}}{}^{\bar{\beta}}\}$ generates the usual real Clifford algebra as in (7.26).

The octonionic matrix elements of the charge conjugation matrix are trivially

$$C_{\alpha a \beta b} = \langle e_a | C_{\alpha \beta} e_b \rangle = C_{\alpha \beta} \delta_{ab}, \quad (7.29)$$

which can be identified with the 32×32 matrix:

$$C_{\bar{\alpha} \bar{\beta}} = C_{\alpha a \beta b} = C_{\alpha \beta} \delta_{ab}. \quad (7.30)$$

Armed with these tools, the right-hand side can then be written over \mathbb{O} simply by replacing $\bar{\alpha} \rightarrow \alpha$ and putting hats on the gammas:

$$\begin{aligned} & (\hat{\gamma}^M C)_{\alpha \beta} P_M + (\hat{\gamma}^{MN} C)_{\alpha \beta} Z_{MN} \\ & + (\hat{\gamma}^{MNPQR} C)_{\alpha \beta} Z_{MNPQR}. \end{aligned} \quad (7.31)$$

With the identification $\bar{\alpha} = \alpha a$ the left-hand side of (7.26) can also be written in terms of the composite indices:

$$\{\widehat{Q_{\bar{\alpha}}}, \widehat{Q_{\bar{\beta}}}\} = \{Q_{\alpha a}, Q_{\beta b}\}. \quad (7.32)$$

Now, the expression (7.31) is an octonionic operator with matrix elements as on the right-hand side of (7.26), so on the left there must be an octonionic operator

$$\{\widehat{Q_{\alpha}}, \widehat{Q_{\beta}}\} \quad (7.33)$$

with matrix elements given by (7.32). The required operator is obtained simply by contracting (7.32) with the outer product $e_a \times e_b$ defined in (7.23):

$$\{\widehat{Q_{\alpha}}, \widehat{Q_{\beta}}\} \equiv \{Q_{\alpha a}, Q_{\beta b}\} e_a \times e_b. \quad (7.34)$$

The octonionic formulation of the M-algebra is then

$$\begin{aligned} \{\widehat{Q_{\alpha}}, \widehat{Q_{\beta}}\} &= (\hat{\gamma}^M C)_{\alpha \beta} P_M + (\hat{\gamma}^{MN} C)_{\alpha \beta} Z_{MN} \\ &+ (\hat{\gamma}^{MNPQR} C)_{\alpha \beta} Z_{MNPQR}. \end{aligned} \quad (7.35)$$

Using the first two versions of the outer product given in (7.24), the left-hand side can be written as

$$\begin{aligned} \{\widehat{Q_\alpha, Q_\beta}\} = & \frac{1}{2} \left((Q_\alpha Q_\beta^*)(\cdot) + (\cdot)(Q_\beta^* Q_\alpha) \right. \\ & \left. + (Q_\alpha(\cdot)^*) Q_\beta + Q_\beta((\cdot)^* Q_\alpha) \right). \end{aligned} \quad (7.36)$$

The first two terms look similar to the more intuitive anti-commutator $\{Q_\alpha, Q_\beta^*\}$, explored in [99], but to reproduce the full M-algebra requires all four terms above.

7.3. The Octonionic $D = 11$ Supergravity Lagrangian

With the tools described above it is not difficult to rewrite the Lagrangian and transformation rules of $D = 11$ supergravity over the octonions. Starting from the conventional Lagrangian, all one must do is exchange any 32-component real spinors with their 4-component octonionic counterparts, and exchange any bilinears with those described above. This gives the following Lagrangian:

$$\begin{aligned} \mathcal{L} = \sqrt{-g} \left[R - \text{Re} \left(i \bar{\Psi}_M \hat{\gamma}^{MNP} D_N \left(\frac{1}{2} (\omega + \tilde{\omega}) \right) \Psi_P \right) - \frac{1}{24} F_{MNPQ} F^{MNPQ} \right. \\ - \frac{\sqrt{2}}{192} \text{Re} \left(i \bar{\Psi}_R \left(\hat{\gamma}^{MNPQRS} + 12 \hat{\gamma}^{MN} g^{PR} g^{QS} \right) \Psi_S \right) \left(F_{MNPQ} + \tilde{F}_{MNPQ} \right) \\ \left. - \frac{2\sqrt{2}}{144^2} \varepsilon^{M_0 M_1 \dots M_{10}} F_{M_0 M_1 M_2 M_3} F_{M_4 M_5 M_6 M_7} A_{M_8 M_9 M_{10}} \right], \end{aligned} \quad (7.37)$$

where

$$\begin{aligned} \omega_M^{AB} &= \omega_M^{AB}(e) + K_M^{AB}, \\ K_M^{AB} &= -\frac{1}{4} \text{Re} \left(i (\bar{\Psi}_M \hat{\gamma}^B \Psi^A - \bar{\Psi}^A \hat{\gamma}_M \Psi^B + \bar{\Psi}^B \hat{\gamma}^A \Psi_M) \right), \end{aligned} \quad (7.38)$$

while

$$\begin{aligned} \tilde{\omega}_M^{AB} &= \omega_M^{AB}(e) - \frac{1}{4} \text{Re} \left(i (\bar{\Psi}_M \hat{\gamma}^B \Psi^A - \bar{\Psi}^A \hat{\gamma}_M \Psi^B + \bar{\Psi}^B \hat{\gamma}^A \Psi_M) \right), \\ \tilde{F}_{MNPQ} &= 4 \partial_{[M} A_{NPQ]} + \frac{3\sqrt{2}}{2} \text{Re} \left(i \bar{\Psi}_{[M} \hat{\gamma}_{NP} \Psi_{Q]} \right), \end{aligned} \quad (7.39)$$

and the covariant derivative $D_M(\omega)$ is defined by

$$D_M(\omega)\epsilon = \partial_M \epsilon + \frac{1}{4} \omega_M^{AB} \hat{\gamma}_{AB} \epsilon. \quad (7.40)$$

The Lagrangian (7.37) is invariant under the following supersymmetry transformations:

$$\begin{aligned}\delta e_M^A &= \frac{1}{2} \text{Re} (i\bar{\epsilon}\hat{\gamma}^A\Psi_M), \\ \delta C_{MNP} &= -\frac{3\sqrt{2}}{4} \text{Re} (i\bar{\epsilon}\hat{\gamma}_{[MN}\Psi_{P]}), \\ \delta\Psi_M &= D_M(\tilde{\omega})\epsilon + \frac{\sqrt{2}}{288}(\hat{\gamma}^{ABCD}{}_M - 8\delta_M^A\hat{\gamma}^{BCD})\tilde{F}_{ABCD}\epsilon,\end{aligned}\tag{7.41}$$

although this will not be proven here.

7.4. Interlude: The Kirmse and Octavian Integers

The results in the remainder of this chapter will require a brief discussion of octonionic number theory. By analogy with the usual set of integers $\mathbb{Z} \subset \mathbb{R}$, an octonionic integer system \mathbb{I} should be an 8-dimensional lattice embedded in \mathbb{O} , which is (preferably) closed under multiplication using the rule inherited from \mathbb{O} . The most obvious example is of course to take octonions whose components are all integers:

$$\mathbb{I} = \{x = x_a e_a \in \mathbb{O} \mid x_a \in \mathbb{Z}\}.\tag{7.42}$$

However, as shown in [105], for a richer number theory one may ask that an analogue of the unique prime factorisation theorem to hold in \mathbb{I} . For the ordinary set of integers \mathbb{Z} this theorem says that each integer is a product of positive or negative primes in a way that is unique up to order and sign change. For an analogue of this theorem to hold in \mathbb{I} , it must be ‘well-packed’ [105]; that is, the following two conditions must hold:

1. no element of \mathbb{O} has distance ≥ 1 from the nearest lattice point of \mathbb{I} ,
2. the distance between any lattice point and any other lattice point is ≥ 1 ,

where the distances are evaluated using the norm in \mathbb{O} – see equation (3.24).

One set of octonions that satisfies these two conditions is the so-called *Kirmse integers* \mathbb{K} . These can be described as follows. An octonion x can always be written as

$$x = (x_0 + x_i e_i + x_j e_j + x_k e_k) + (x_{i'} e_{i'} + x_{j'} e_{j'} + x_{k'} e_{k'} + x_{l'} e_{l'})\tag{7.43}$$

(no summation),

where $ijk \in \mathbf{L}$ is a line of the Fano plane and hence $i'j'k'l' \in \mathbf{Q}$ is the complementary quadrangle. An octonion x , written in this way, is a Kirmse integer if

{all of x_0, x_i, x_j, x_k are integers *or* all of x_0, x_i, x_j, x_k are half-integers} *and* {all of $x_{i'}, x_{j'}, x_{k'}, x_{l'}$ are integers *or* all of $x_{i'}, x_{j'}, x_{k'}, x_{l'}$ are half-integers}. As well as being well-packed, the Kirmse integers form the densest possible lattice in 8 dimensions. If a ball with radius $\frac{1}{2}$ is inserted around each lattice point in \mathbb{R}^8 , then every ball touches 240 others. In fact, this is none other than the root lattice of the largest exceptional group E_8 .

Of particular importance to this paper are the 240 unit Kirmse integers – elements of \mathbb{K} with unit norm – given by:

$$\begin{aligned} & \pm 1, \pm e_i, \\ & \frac{1}{2}(\pm 1 \pm e_i \pm e_j \pm e_k) \text{ with } ijk \in \mathbf{L}, \\ & \frac{1}{2}(\pm e_i \pm e_j \pm e_k \pm e_l) \text{ with } ijkl \in \mathbf{Q}. \end{aligned} \tag{7.44}$$

The set of Kirmse integers orthogonal to any particular basis element e_a forms a copy of the root lattice of E_7 . In particular, the set orthogonal to $e_0 = 1$ is just the pure-imaginary Kirmse integers, whose 126 unit elements are

$$\pm e_i \text{ and } \frac{1}{2}(\pm e_i \pm e_j \pm e_k \pm e_l) \text{ with } ijkl \in \mathbf{Q}. \tag{7.45}$$

This discussion of the Kirmse integers has so far overlooked the vital question of whether or not they are closed under multiplication. Kirmse himself once stated that they were. However, it is easy to find a counter-example:

$$\frac{1}{2}(1 + e_1 + e_2 + e_4)\frac{1}{2}(1 + e_2 + e_3 + e_5) = \frac{1}{2}(e_2 + e_4 + e_5 + e_7) \notin \mathbb{K}. \tag{7.46}$$

Hence \mathbb{K} is not closed under octonionic multiplication, a result sometimes referred to as *Kirmse's mistake* [105]. The mistake can be rectified by the following unusual trick. For every Kirmse integer $x = x_a e_a \in \mathbb{K}$ exchange the coefficient x_0 with any one of the seven x_i . The resulting lattice \mathbb{K}' is just a reflection of the Kirmse lattice, and so is well-packed. However, in this case it *is* closed under multiplication. In the literature, \mathbb{K}' has been referred to as the set of *octavian integers* or the *integral Cayley numbers*. They will be used in maximal supergravity theories in the following section.

7.5. Integral Octonions in $D = 3, 4$ Maximal Supergravity

7.5.1. $D = 4, \mathcal{N} = 8$ Supergravity

Next, consider dimensional reduction to $D = 4$, yielding $\mathcal{N} = 8$ supergravity. Dropping the dependence of the fields on the seven coordinates associated with the seven imaginary basis octonions (as in equation (7.6)) produces the following bosonic content:

$$\begin{aligned} g_{MN} &\rightarrow g_{\mu\nu}, \vec{\phi}, \mathcal{A}_\mu^i, \mathcal{A}_j^i \text{ (with } i < j), \\ A_{MNP} &\rightarrow A_{\mu\nu\rho}, A_{\mu\nu i}, A_{\mu ij}, A_{ijk}, \end{aligned} \quad (7.47)$$

where i, j, k run over the seven internal dimensions, while μ, ν, ρ run over the extended four, and $\vec{\phi}$ denotes the seven dilatons written as a seven-component vector. Note that the scalar fields descended from g_{MN} have not been written here so as to be covariant with respect to the $\text{SO}(7)$ (or $\text{GL}(7, \mathbb{R})$) symmetry associated with the internal dimensions; instead the dilatonic and axionic scalars have been separated as $\vec{\phi}$ and \mathcal{A}_j^i ('axionic' in this context means those that are descended from the off-diagonal components of the metric).

Denoting $(p+1)$ -form field strengths of p -form potentials with superscripts $(p+1)$, the Lagrangian for the bosonic sector is then:

$$\begin{aligned} \mathcal{L}_B = \sqrt{-g} &\left[R - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{2} \sum_i e^{2\vec{b}_i \cdot \vec{\phi}} (\mathcal{F}_i^{(2)})^2 - \frac{1}{2} \sum_{i < j} e^{2\vec{b}_{ij} \cdot \vec{\phi}} (\mathcal{F}_{ij}^{(1)})^2 - \frac{1}{2} e^{2\vec{a} \cdot \vec{\phi}} (F^{(4)})^2 \right. \\ &\left. - \frac{1}{2} \sum_i e^{2\vec{a}_i \cdot \vec{\phi}} (F_i^{(3)})^2 - \frac{1}{2} \sum_{i < j} e^{2\vec{a}_{ij} \cdot \vec{\phi}} (F_{ij}^{(2)})^2 - \frac{1}{2} \sum_{i < j < k} e^{2\vec{a}_{ijk} \cdot \vec{\phi}} (F_{ijk}^{(1)})^2 + \mathcal{L}_{FFA} \right], \end{aligned} \quad (7.48)$$

where the precise definitions of the field strengths and their 'transgression terms' are given in [106, 102] and \mathcal{L}_{FFA} denotes the terms descended from the topological term in the eleven dimensional Lagrangian. Note that in the bosonic sector all the fields are real, since the octonions have so far only been used in the description of the fermions. The constant 'dilaton vectors' $\vec{a}, \vec{a}_i, \vec{a}_{ij}, \vec{a}_{ijk}, \vec{b}_i$ and \vec{b}_{ij} parameterise the non-canonical coupling of the seven dilatons $\vec{\phi}$ to the other bosonic fields. For

the various potentials, they are given by:

$$\begin{aligned}
\text{3-potential:} & \quad \vec{a} = -\vec{g}, \\
\text{2-potentials:} & \quad \vec{a}_i = \vec{f}_i - \vec{g}, \\
\text{1-potentials:} & \quad \vec{a}_{ij} = \vec{f}_i + \vec{f}_j - \vec{g}, & \quad \vec{b}_i = -\vec{f}_i, \\
\text{0-potentials:} & \quad \vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g}, & \quad \vec{b}_{ij} = -\vec{f}_i + \vec{f}_j,
\end{aligned} \tag{7.49}$$

where the vectors \vec{g} and \vec{f}_i are given below. Here they are listed for D extended dimensions (temporarily letting $i, j = 1, 2, \dots, (11 - D)$ for the next few equations) so that the expressions may be used later on for $D = 3$:

$$\begin{aligned}
\vec{f}_i &= \frac{1}{2}(0, \underbrace{\dots, 0}_{i-1}, (10 - i)s^i, s^{i+1}, \dots, s^{11-D}) \\
\vec{g} &= \frac{3}{2}(s^1, s^2, \dots, s^{11-D}) = \frac{1}{3} \sum_i \vec{f}_i \\
s^i &= \sqrt{2/((10 - i)(9 - i))},
\end{aligned} \tag{7.50}$$

with scalar products

$$\vec{g} \cdot \vec{g} = \frac{11 - D}{2(D - 2)}, \quad \vec{g} \cdot \vec{f}_i = \frac{3}{2(D - 2)}, \quad \vec{f}_i \cdot \vec{f}_j = \frac{\delta_{ij}}{2} + \frac{1}{2(D - 2)}. \tag{7.51}$$

Returning to $D = 4$, \vec{g} and the seven \vec{f}_i are then used to build up all of the dilaton vectors. In particular, \vec{a}_{ijk} , \vec{b}_{ij} and $-\vec{a}_i$, the vectors parameterising the coupling of the dilatons to the 63 axions, are the positive roots of the U-duality group $E_{7(7)}$ (where we dualise the seven 2-forms $A_{\mu\nu i}$ to scalars, whose dilaton vectors are $-\vec{a}_i$). The dilaton vectors \vec{a}_{ij} and \vec{b}_i make up the positive weights of the **56** of $E_{7(7)}$, under which the 2-form field strengths and their duals transform.

Since there is a dilaton for each internal dimension, and the seven internal dimensions are associated with the seven imaginary octonions via (7.6), it makes sense to consider the seven dilatons $\vec{\phi}$ themselves to be components of an imaginary octonion $\phi_i e_i$. In this case, the dilaton vectors should also be viewed as a particular set of imaginary octonions in order to make sense of the scalar products that appear in the exponential couplings in (7.37). This perspective has some interesting consequences.

Consider the replacement $\vec{f}_i \rightarrow f_i \in \text{Im}(\mathbf{O})$, where

$$f_1 = \frac{1}{2}(e_1 + e_2 + e_4), \quad f_2 = \frac{1}{2}(e_2 + e_3 + e_5), \quad \dots, \quad f_7 = \frac{1}{2}(e_7 + e_1 + e_3), \tag{7.52}$$

i.e. $f_i = \frac{1}{2}(e_i + e_j + e_k)$, with $ijk \in \mathbf{L}$. This amounts only to a change of basis in

the space of dilaton vectors. The reason for this particular choice will become clear shortly. Summing all the f_i gives

$$g = \frac{1}{3} \sum_i f_i = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7), \quad (7.53)$$

and using the inner product defined in (3.26), one can check that

$$\langle g|g \rangle = \frac{7}{4}, \quad \langle g|f_i \rangle = \frac{3}{4}, \quad \langle f_i|f_j \rangle = \frac{\delta_{ij}}{2} + \frac{1}{4}, \quad (7.54)$$

as required when $D = 4$.

Now, using (7.49) the various dilaton vectors can be computed in this octonionic parameterisation. However, before doing so, it will be useful to introduce the concept of the dual Fano plane. In general, a projective plane \mathbb{P} exhibits a duality between its points and lines, whose roles may be interchanged to obtain a new space $\tilde{\mathbb{P}} \cong \mathbb{P}$. For every statement relating points and lines on \mathbb{P} there is a dual statement relating *lines and points* on $\tilde{\mathbb{P}}$. For example, just as two points on a projective plane lie on a unique line, two lines on the plane meet at a unique point. Since the (unoriented) Fano plane is the projective plane over the field \mathbb{Z}_2 , one may interchange the roles of its points and lines to obtain a dual plane – see Fig. 7.1.

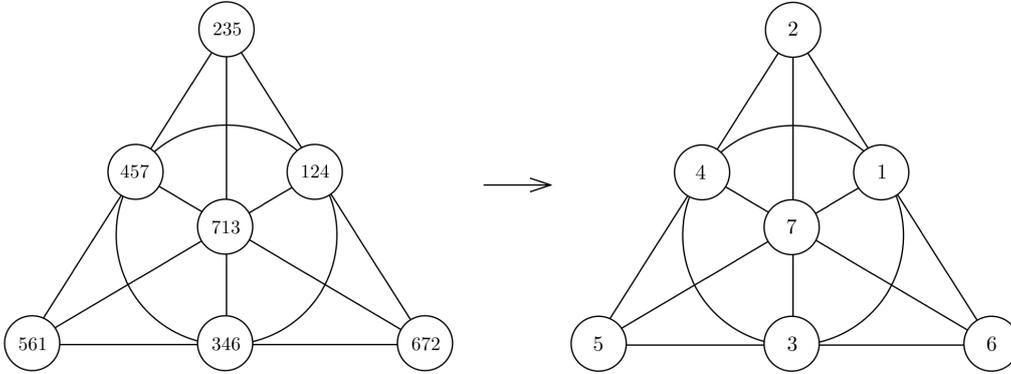


Figure 7.1.: The dual Fano plane $\tilde{\mathbb{F}}$ obtained by interchanging the roles of points and lines on the original Fano plane. Relabelling the triples 124, 235, 346, 457, 561, 672, 713 \rightarrow 1, 2, 3, 4, 5, 6, 7 gives the plane on the right. Unlike in Fig. 3.1 there are no orientations given for the lines of the dual Fano plane since it is not used for multiplication.

In practice it makes sense to relabel the lines 124, 235, 346, 457, 561, 672, 713 simply as 1, 2, 3, 4, 5, 6, 7, respectively, which leads to the plane on the right in Fig. 7.1, whose lines are given by the set $\tilde{\mathbf{L}} = \{157, 261, 372, 413, 524, 635, 746\}$. This relabelling is deliberately chosen so as to match up with (7.52).

Now it is a simple exercise to compute the $E_{7(7)}$ root dilaton vectors, starting with those whose expressions are simplest. Since a line in $\tilde{\mathbf{L}}$ corresponds to a point in the

original Fano plane \mathbb{F} , one should expect a_{ijk} with $ijk \in \tilde{\mathbf{L}}$ to correspond in some way to a point in \mathbb{F} . This is indeed the case, since

$$\begin{aligned} a_{157} &= e_1, & a_{261} &= e_2, & a_{372} &= e_3, & a_{413} &= e_4, \\ a_{524} &= e_5, & a_{635} &= e_6, & a_{746} &= e_7. \end{aligned} \quad (7.55)$$

Next consider the octonions $-a_i$, whose labels correspond to points on the dual Fano plane $\tilde{\mathbb{F}}$ and hence to lines on \mathbb{F} . Indeed, one finds that

$$\begin{aligned} -a_1 &= \frac{1}{2}(e_3 + e_5 + e_6 + e_7), & -a_2 &= \frac{1}{2}(e_4 + e_6 + e_7 + e_1), & -a_3 &= \frac{1}{2}(e_5 + e_7 + e_1 + e_2), \\ -a_4 &= \frac{1}{2}(e_6 + e_1 + e_2 + e_3), & -a_5 &= \frac{1}{2}(e_7 + e_2 + e_3 + e_4), & -a_6 &= \frac{1}{2}(e_1 + e_3 + e_4 + e_5), \\ -a_7 &= \frac{1}{2}(e_2 + e_4 + e_5 + e_6), \end{aligned} \quad (7.56)$$

which matches up with the seven quadrangles complimentary to the seven corresponding lines of \mathbb{F} . Computing the rest of the vectors, a_{ijk} ($ijk \notin \tilde{\mathbf{L}}$) and b_{ij} (see Appendix), the whole set populates the unit imaginary Kirmse integers:

$$\pm e_i \text{ and } \frac{1}{2}(\pm e_i \pm e_j \pm e_k \pm e_l) \text{ with } ijkl \in \mathbf{Q}. \quad (7.57)$$

Note that the vectors a_{ij} and b_i corresponding to the 1-form gauge potentials all have the form

$$\frac{1}{2}(\pm e_i \pm e_j \pm e_k) \text{ with } ijk \in \mathbf{L}, \quad (7.58)$$

which are the weights of the **56** representation, under which the corresponding 2-form field strengths are rotated into their electromagnetic duals.

Putting all of this together means that (after dualisation) the bosonic $\mathcal{N} = 8$ Lagrangian may be written as

$$\begin{aligned} \mathcal{L} &= \sqrt{-g} \left(R - \frac{1}{2} \langle \partial\phi | \partial\phi \rangle - \frac{1}{2} \sum_{\text{points}} e^{\langle \text{points} | \phi \rangle} (F_{\text{points}}^{(1)})^2 \right. \\ &\quad \left. - \frac{1}{2} \sum_{\text{quads}} e^{\langle \text{quads} | \phi \rangle} (F_{\text{quads}}^{(1)})^2 - \frac{1}{4} \sum_{\text{lines}} e^{\langle \text{lines} | \phi \rangle} (F_{\text{lines}}^{(2)})^2 \right), \end{aligned} \quad (7.59)$$

where the sums run over all the vectors listed in Table A.4 in the Appendix, which correspond to the points, lines and quadrangles of the Fano plane with every possible \pm sign combination. As alluded to above, this new parameterisation makes manifest the relationship between the bosonic fields and the structure of the Fano plane.

Before moving on to $D = 3$, there is a useful Fano-plane-based trick for restricting the roots of $E_{7(7)}$ to those of its maximal compact subgroup $SU(8)$. The adjoint of

$E_{7(7)}$ decomposes into $SU(8)$ as:

$$\mathbf{133} \rightarrow \mathbf{63} + \mathbf{70}, \quad (7.60)$$

so one expects the 126 roots of $E_{7(7)}$ to split into two sets: a set consisting of the 56 roots of $SU(8)$ and the remaining 70 vectors corresponding to the weights of the **70** representation. The trick is first to choose a line of the Fano plane – say 124. As shown in [107], one may then discard the unit Kirmse integers $\pm e_1$, $\pm e_2$ and $\pm e_4$, as well as those associated with the corresponding quadrangle – in this case $\frac{1}{2}(\pm e_3 \pm e_5 \pm e_6 \pm e_7)$. Take the remaining quadrangles and wherever e_1 , e_2 and e_4 appear fix their relative signs according to the following rule: choosing another point – say e_7 – the signs are the same if e_7 appears in the quadrangle and different if it does not. This is shown explicitly in Table 7.1.

| E_7 roots | $SU(8)$ roots |
|---|--|
| $\pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_5, \pm e_6, \pm e_7$ | $\pm e_3, \pm e_5, \pm e_6, \pm e_7$ |
| $\frac{1}{2}(\pm e_3 \pm e_5 \pm e_6 \pm e_7)$ | |
| $\frac{1}{2}(\pm e_4 \pm e_6 \pm e_7 \pm e_1)$ | $\pm \frac{1}{2}(+e_4 \pm e_6 \pm e_7 + e_1)$ |
| $\frac{1}{2}(\pm e_5 \pm e_7 \pm e_1 \pm e_2)$ | $\pm \frac{1}{2}(\pm e_5 \pm e_7 + e_1 + e_2)$ |
| $\frac{1}{2}(\pm e_6 \pm e_1 \pm e_2 \pm e_3)$ | $\pm \frac{1}{2}(\pm e_6 + e_1 - e_2 \pm e_3)$ |
| $\frac{1}{2}(\pm e_7 \pm e_2 \pm e_3 \pm e_4)$ | $\pm \frac{1}{2}(\pm e_7 + e_2 \pm e_3 + e_4)$ |
| $\frac{1}{2}(\pm e_1 \pm e_3 \pm e_4 \pm e_5)$ | $\pm \frac{1}{2}(+e_1 \pm e_3 - e_4 \pm e_5)$ |
| $\frac{1}{2}(\pm e_2 \pm e_4 \pm e_5 \pm e_6)$ | $\pm \frac{1}{2}(+e_2 - e_4 \pm e_5 \pm e_6)$ |

Table 7.1.: $E_7 \supset SU(8)$ roots in terms of the octavian integers.

7.5.2. $D = 3$, $\mathcal{N} = 16$ Supergravity

Dimensionally reducing to $D = 3$ means there are eight internal dimensions, which are labelled $a = 0, 1, \dots, 7$. As a result, there are eight dilatons $\vec{\phi}$, which are now written as an octonion $\phi_a e_a$. Accordingly, the eight vectors \vec{f}_a , each of which has eight entries, become eight octonions: $\vec{f}_a \rightarrow f_a \in \mathbb{O}$, with the following chosen parameterisation

$$f_0 = 1, \quad f_i = \frac{1}{2}(1 + e_i + e_j + e_k), \quad ijk \in \mathbf{L}, \quad (7.61)$$

where $i = 1, \dots, 7$ as usual. The total sum is then

$$g = \frac{1}{2}(3 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7), \quad (7.62)$$

so that the inner products are

$$\langle g|g \rangle = 4, \quad \langle g|f_a \rangle = \frac{3}{2}, \quad \langle f_a|f_b \rangle = \frac{\delta_{ab}}{2} + \frac{1}{2}, \quad (7.63)$$

consistent with (7.51) for $D = 3$.

In $D = 3$ the 2-form potentials carry no degrees of freedom and the 1-form potentials may be dualised to scalars. Since the metric contains no dynamical degrees of freedom, all the bosonic degrees of freedom of the theory are carried by the resulting 128 scalars, whose dilaton vectors are $-\vec{a}_{ab}$, \vec{a}_{abc} , $-\vec{b}_a$ and \vec{b}_{ab} . These make up the positive roots of the U-duality group of the theory, $E_{8(8)}$.

Again, due to the labelling system chosen in (7.61), one should expect the a_{abc} with $abc \in \tilde{\mathbf{L}}$ to correspond in some way to the points of the Fano plane \mathbb{F} . This is indeed the case:

$$\begin{aligned} a_{157} &= e_1, & a_{261} &= e_2, & a_{372} &= e_3, & a_{413} &= e_4, \\ a_{524} &= e_5, & a_{635} &= e_6, & a_{746} &= e_7. \end{aligned} \quad (7.64)$$

Similarly, the b_a correspond to points on $\tilde{\mathbb{F}}$ and hence to lines on \mathbb{F} : $b_a = -f_a$. The a_{0i} also reflect this simple correspondence:

$$\begin{aligned} -a_{01} &= \frac{1}{2}(e_3 + e_5 + e_6 + e_7), & -a_{02} &= \frac{1}{2}(e_4 + e_6 + e_7 + e_1), & -a_{03} &= \frac{1}{2}(e_5 + e_7 + e_1 + e_2), \\ -a_{04} &= \frac{1}{2}(e_6 + e_1 + e_2 + e_3), & -a_{05} &= \frac{1}{2}(e_7 + e_2 + e_3 + e_4), & -a_{06} &= \frac{1}{2}(e_1 + e_3 + e_4 + e_5), \\ -a_{07} &= \frac{1}{2}(e_2 + e_4 + e_5 + e_6), \end{aligned} \quad (7.65)$$

as well as the b_{0i} :

$$\begin{aligned} b_{01} &= \frac{1}{2}(-1 + e_1 + e_2 + e_4), & b_{02} &= \frac{1}{2}(-1 + e_2 + e_3 + e_5), & b_{03} &= \frac{1}{2}(-1 + e_3 + e_4 + e_6), \\ b_{04} &= \frac{1}{2}(-1 + e_4 + e_5 + e_7), & b_{05} &= \frac{1}{2}(-1 + e_5 + e_6 + e_1), & b_{06} &= \frac{1}{2}(-1 + e_6 + e_7 + e_2), \\ b_{07} &= \frac{1}{2}(-1 + e_7 + e_1 + e_3). \end{aligned} \quad (7.66)$$

Computing all positive and negative roots (see Appendix) recovers the whole set of 240 unit Kirmse integers, the roots of $E_{8(8)}$:

$$\begin{aligned} &\pm 1, \quad \pm e_i, \\ &\frac{1}{2}(\pm 1 \pm e_i \pm e_j \pm e_k) \quad \text{with } ijk \in \mathbf{L}, \\ &\frac{1}{2}(\pm e_i \pm e_j \pm e_k \pm e_l) \quad \text{with } ijkl \in \mathbf{Q}. \end{aligned} \quad (7.67)$$

Just as in $D = 4$ above, in $D = 3$ one can also write the dualised $\mathcal{N} = 16$ bosonic

Lagrangian as

$$\begin{aligned} \mathcal{L} = \sqrt{-g} & \left(R - \frac{1}{2} \langle \partial\phi | \partial\phi \rangle - \frac{1}{2} \sum_{\text{points}} e^{\langle \text{points} | \phi \rangle} (F_{\text{points}}^{(1)})^2 \right. \\ & \left. - \frac{1}{2} \sum_{\text{quads}} e^{\langle \text{quads} | \phi \rangle} (F_{\text{quads}}^{(1)})^2 - \frac{1}{2} \sum_{\text{lines}} e^{\langle \text{lines} | \phi \rangle} (F_{\text{lines}}^{(1)})^2 \right), \end{aligned} \quad (7.68)$$

where in this case the sums run over the vectors listed in Table A.5.

The parameterisation of the vectors \vec{f}_a in equation (7.61) was chosen because this leads to dilaton vectors that are easily recognisable as Kirmse integers (since the Kirmse integers take their structure from the lines and quadrangles of the Fano plane). However, it would be just as easy to parameterise so as to arrive at the octavian integers, which are closed under multiplication. In other words, in the manner above, the dilaton vectors of $D = 3$, $\mathcal{N} = 16$ supergravity curiously may be equipped with a multiplication rule, under which they form a closed algebra.

7.6. Summary

The contents of this chapter demonstrate that eleven-dimensional supergravity may be formulated over the octonions. The octonions are simply used in an alternative formulation of the usual Clifford algebra for the fermionic sector. However, the octonionic parameterisation leads to a new perspective in the bosonic sector upon dimensional reduction to the maximal supergravity theories in $D = 4$ and $D = 3$.

In the $D = 4$ case one may write the seven coordinates of the internal dimensions as an imaginary octonion, leading to an interpretation of the seven dilatons as an imaginary octonion. Upon reduction to $D = 3$, the 240 dilaton vectors may be considered as the 240 unit octavian integers, and thus they form an algebra that is closed under multiplication. This is an interesting result in its own right, although what it can be used for – or indeed whether it is useful at all – is so far a mystery. To speculate, the algebra might have some utility in working with $D = 3$ black hole solutions, in which dilaton vectors sometimes appear explicitly. However, for now this is something of an ‘answer without a question’.

8. Concluding Remarks

This thesis has explored and clarified in several contexts the close connection between division algebras and supersymmetry, as well as exploiting this relationship to classify and understand the symmetries of various supersymmetric theories. Chapter 4 began with the formulation of $D = 3, 4, 6, 10$ minimally supersymmetric Yang-Mills theories over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, which was generalised via dimensional reduction to a unified division-algebraic formalism describing all Yang-Mills theories with extended supersymmetry. Here the role of triality algebras in such theories was made manifest.

In particular, in $D = 3$ this formalism gave $\mathcal{N} = 1, 2, 4, 8$ super Yang-Mills over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, and in Chapter 5 this was combined with the notion of ‘gravity as the square of gauge theory’ to discover a magic square of supergravities. Specifically, tensoring a Yang-Mills multiplet valued over the division algebra \mathbb{A}_L with another valued over \mathbb{A}_R results in a supergravity whose U-duality is given by the magic square algebra of Freudenthal-Rosenfeld-Tits: $\mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R)$. This result presents a novel insight into the symmetries of $D = 3$ supergravity, and hence also into higher-dimensional theories related to those of $D = 3$ by means of dimensional reduction.

In Chapter 6 the work of Chapters 4 and 5 was combined; tensoring division-algebraic Yang-Mills multiplets with extended supersymmetry in all spacetime dimensions in the range $3 \leq D \leq 10$ gives a pyramid of supergravity theories with the magic square at the base in $D = 3$ and Type II at the apex in $D = 10$. The U-dualities of these supergravities are described by the magic pyramid algebra $\mathfrak{P}_{1,2}^N(\mathbb{A}_L, \mathbb{A}_R)$, a natural subalgebra of $\mathfrak{L}_{1,2}(\mathbb{A}_L, \mathbb{A}_R)$ defined in terms of the Clifford algebra $\text{Cl}_0(N)$ that acts on spinors of the spacetime little group $\text{Spin}(N)$, where $N = D - 2$. This provides a succinct summary of the U-dualities of all supergravities whose field content may be factorised into the tensor product of two super Yang-Mills multiplets.

Finally, Chapter 7 laid the foundations for an octonionic formulation of eleven-dimensional supergravity, intended to shed some light on the ‘hidden’ exceptional symmetries $E_{6(6)}$, $E_{7(7)}$ and $E_{8(8)}$ that appear after dimensional reduction down to $D = 5, 4, 3$. This leads to a surprising role for the octavian integers, as well as to

lower-dimensional Lagrangians written explicitly in terms of the Fano plane, further exposing the octonionic anatomy of maximal supergravity.

Throughout the above chapters the octonions resurface time after time in association with maximal supersymmetry, and thus also in the context of string theory and M-theory. Although the full significance of the octonions in these theories remains something of an enigma, it seems that their near-overarching function in such contexts is to make Spin(8) triality manifest. Mathematically speaking, the existence of this triality implies the existence of the octonions, and vice versa [1]; one cannot be found without the other. Since Spin(8) triality is an essential ingredient in the very construction of the superstring [39, 16], the octonions are unavoidably built-into string theory from the outset¹. Hence we find an exceptional division algebra underpinning an exceptional theory of physics, whose symmetries are frequently described by exceptional groups. The essence common to these three exceptional structures is their special feature of allowing the rotation of vector/tensor-like degrees of freedom into spinor degrees of freedom – they all leave room for ‘Boson-Fermion confusion’ [108, 109].

How far this interconnection between octonions and strings can be pushed in order to more deeply understand the latter is still not clear, but I hope that at the very least that this thesis provides some useful tools to this end.

¹Incidentally, this would make a good retort to anybody suggesting that the octonions are to be regarded as something of an obscurity or as a red herring – or are ‘not to be let out of the attic’. At worst, the octonions simply offer an alternative way of treating Spin(8) such that triality is always manifest.

A. Appendix

A.1. Division-Algebraic Spinor Decomposition

A.1.1. From $D = 10$ to $D = 7, 6, 5$: $\mathbb{O} \cong \mathbb{H}^2 \cong \mathbb{H} \oplus \mathbb{H}$

In dimensions $D = N + 2 = 7, 6, 5$, the Clifford algebra $\text{Cl}(N - 1) \cong \text{Cl}_0(N)$ is isomorphic to a matrix algebra over the quaternions \mathbb{H} , so it will be useful to write an octonion as a pair of quaternions. To see how this works, consider a general octonion $x = x_a e_a \in \mathbb{O}$, $a = 0, 1, \dots, 7$, and choose a line of the Fano plane, say 124. Then $\mathbb{H} \cong \text{span}\{e_0, e_1, e_2, e_4\}$, so write

$$\begin{aligned} x &= x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 \\ &= (x_0 + x_1 e_1 + x_2 e_2 + x_4 e_4) + e_3(x_3 - x_7 e_1 - x_5 e_2 + x_6 e_4), \end{aligned} \quad (\text{A.1})$$

where e_3 has been selected to factorise the terms corresponding to the quadrangle 3567 complementary to the line 124. The octonion x now has the form of a pair of quaternions, with e_3 singled out as the imaginary unit separating the two of them: $\mathbb{O} \cong \mathbb{H} \oplus e_3 \mathbb{H}$ (of course any of the quadrangle's basis elements could have been chosen in place of e_3 , and any line of the Fano plane to begin with).

To write (A.1) compactly, define the indices

$$\begin{aligned} \hat{a} &= 0, 1, 2, 4 \quad \text{and} \quad \hat{i} = 1, 2, 4, \\ \check{a} &= 3, 5, 6, 7 \quad \text{and} \quad \check{i} = 5, 6, 7, \end{aligned} \quad (\text{A.2})$$

so that ‘Fano line indices’ with hats correspond to spacetime directions and ‘Fano quadrangle indices’ with inverted hats correspond to internal directions. Then (A.1) becomes

$$x = x_{\hat{a}} e_{\hat{a}} + x_{\check{a}} e_{\check{a}} = (x_{\hat{a}} + e_3 \Gamma_{\hat{a}\hat{a}}^3 x_{\hat{a}}) e_{\hat{a}}, \quad (\text{A.3})$$

where $\Gamma_{\hat{a}\hat{a}}^3$ refers to the $3\hat{a}\hat{a}$ -components of the structure constants Γ_{bc}^a .

In terms of the \mathbb{H} subalgebra spanned by $e_{\hat{a}}$ and its complement $\mathbb{H}^{\mathbb{C}} \cong e_3 \mathbb{H}$ spanned by $e_{\check{a}}$, the multiplication rule can be summarised as follows.

- The \mathbb{H} subalgebra is closed: $e_{\hat{a}} e_{\hat{b}} = \Gamma_{\hat{b}\hat{c}}^{\hat{a}} e_{\hat{c}}$;

- Multiplying two elements of $\mathbb{H}^{\mathbb{C}}$ returns an element of \mathbb{H} : $e_{\tilde{a}}e_{\tilde{b}} = \Gamma_{\tilde{b}\tilde{c}}^{\tilde{a}}e_{\tilde{c}}$;
- Multiplying an element of \mathbb{H} and an element of $\mathbb{H}^{\mathbb{C}}$ returns an element of $\mathbb{H}^{\mathbb{C}}$:
 $e_{\tilde{a}}e_{\tilde{b}} = \Gamma_{\tilde{b}\tilde{c}}^{\tilde{a}}e_{\tilde{c}}$ and $e_{\tilde{a}}e_{\tilde{b}} = \Gamma_{\tilde{b}\tilde{c}}^{\tilde{a}}e_{\tilde{c}}$.

An element of $\mathbb{H}^{\mathbb{C}}$ may be mapped to a unique element of \mathbb{H} and vice versa by factorising out e_3 :

$$\begin{aligned} e_3e_{\tilde{a}} = \Gamma_{\tilde{a}\tilde{b}}^3e_{\tilde{b}} &\Leftrightarrow e_{\tilde{a}} = -e_3\Gamma_{\tilde{a}\tilde{b}}^3e_{\tilde{b}}, \\ e_3e_{\tilde{a}} = \Gamma_{\tilde{a}\tilde{b}}^3e_{\tilde{b}} &\Leftrightarrow e_{\tilde{a}} = -e_3\Gamma_{\tilde{a}\tilde{b}}^3e_{\tilde{b}}, \end{aligned} \quad (\text{A.4})$$

where the second two relations come from multiplying the first two by e_3 on the left and invoking alternativity.

Using this, the octonionic $D = 10$ spinor can be rewritten as a pair of quaternions,

$$\psi' = \psi + e_3\chi = \psi_{\tilde{a}}e_{\tilde{a}} + e_3(\chi_{\tilde{a}}e_{\tilde{a}}), \quad (\text{A.5})$$

where $\psi_{\tilde{a}} = \psi'_{\tilde{a}}$ and $\chi_{\tilde{a}} = \Gamma_{\tilde{a}\tilde{a}}^3\psi'_{\tilde{a}}$, while the vector becomes

$$a' = a + \phi = a_{\underline{a}}e_{\underline{a}} + \phi_{\bar{i}}e_{\bar{i}}, \quad (\text{A.6})$$

where the indices \underline{a} and \bar{i} are defined in (6.2). The dimensional reduction is carried out by dropping the dependence of the fields on the coordinates associated with the \bar{i} indices.

Looking at the action of the Clifford algebra on ψ' in each dimension:

- In $D = 7$ the Clifford algebra $\text{Cl}(4) \cong \text{Cl}_0(5)$ may be generated by left-multiplication by the set of imaginary basis elements $\{e_i\} = \{e_1, e_2, e_3, e_4\} = \{e_{\tilde{i}}, e_3\}$. Multiplying by e_i gives

$$e_i(\psi + e_3\chi) = (e_i\psi) + e_3(-e_i\chi) \Leftrightarrow \begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \quad (\text{A.7})$$

while multiplying by e_3 gives

$$e_3(\psi + e_3\chi) = -\chi + e_3\psi \Leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \quad (\text{A.8})$$

which exactly matches the generators of $\text{Cl}(4) \cong \mathbb{H}[2]$ given in Table 4.1, with $(\psi + e_3\chi) \in \mathbb{O}$ reinterpreted as $(\psi, \chi) \in \mathbb{H}^2 \cong \mathcal{S}_5$. This means there is just one resulting spinor in $D = 7$, so $\mathcal{N} = 1$.

- In $D = 6$ the set $\{e_i\} = \{e_i\}$ generates $\text{Cl}(3) \cong \mathbb{H} \oplus \mathbb{H}$, which does not mix ψ and χ , as (A.7) clearly demonstrates. Hence $(\psi, \chi) \in \mathbb{H} \oplus \mathbb{H} \cong \mathcal{S}_4^+ \oplus \mathcal{S}_4^-$ and this theory has $(\mathcal{N}_+, \mathcal{N}_-) = (1, 1)$.
- Finally for $D = 5$ this set becomes $\{e_i\} = \{e_1, e_2\}$, which acts as a quaternionic structure generating $\text{Cl}(2) = \mathbb{H}$. This time $(\psi + e_3\chi) \in \mathbb{O}$ is interpreted as two spinors of the same type $(\psi, \chi) \in \mathbb{H}^2 \cong (\mathcal{S}_3)^2$, so the theory has $\mathcal{N} = 2$.

In each case the higher-dimensional $\mathfrak{so}(8)_{\text{ST}}$ symmetry is broken into the lower-dimensional $\mathfrak{so}(N)_{\text{ST}} \oplus \mathfrak{so}(8 - N)$. Decomposing the $\mathfrak{so}(8)_{\text{ST}}$ parameters θ^{ab} into $\theta^{ab}, \theta^{a\bar{i}}, \theta^{\bar{i}j}$ and then setting $\theta^{a\bar{i}} = 0$ gives

$$\begin{aligned} \delta\psi' &= \frac{1}{4}\theta^{ab}e_a^*(e_b\psi') \\ &= \frac{1}{4}\theta^{ab}e_a^*(e_b\psi) + \frac{1}{4}\theta^{ab}e_a^*(e_b(e_3\chi)) + \frac{1}{4}\theta^{\bar{i}j}e_i^*(e_j\psi) + \frac{1}{4}\theta^{\bar{i}j}e_i^*(e_j(e_3\chi)), \end{aligned} \quad (\text{A.9})$$

The θ^{ab} part is just the infinitesimal action of the spacetime little group $\text{Spin}(N) \subset \text{Cl}_0(N)$, just as in equation (4.104), while the $\theta^{\bar{i}j}$ part is that of the R-symmetry group $\text{Spin}(8 - N) \cong \text{SA}(\mathcal{N}, \mathbb{D})$, with Lie algebra $\mathfrak{sa}(\mathcal{N}, \mathbb{D}) = \text{int}_N(\mathbb{O})$. For example, in $D = 6$ the R-symmetry $\text{Spin}(4)$ acts as $\frac{1}{4}\theta^{\bar{i}j}e_i^*(e_j(\psi + e_3\chi))$ with $\{e_{\bar{i}}\} = \{e_3, e_5, e_6, e_7\} = \{e_{\bar{a}}\}$; this leads to

$$\frac{1}{4}\theta^{\bar{i}j}e_i^*(e_j(\psi + e_3\chi)) = \frac{1}{4}\theta^{\bar{a}b}e_a^*(e_b(\psi + e_3\chi)) = (\psi\theta_1^-) + e_3(\chi\theta_1^+), \quad (\text{A.10})$$

where $\theta_1^-, \theta_1^+ \in \text{Im}(\mathbb{H})$ are defined by

$$\begin{aligned} \theta_1^- &:= \frac{1}{4}\theta^{\bar{a}b}(e_a^*e_b) = \frac{1}{2}(-\theta^{3i}C_{3i\hat{k}} - \frac{1}{2}\theta^{\bar{i}j}C_{\bar{i}j\hat{k}})e_{\hat{k}}, \\ \theta_1^+ &:= \frac{1}{4}\theta^{\bar{a}b}(e_a^*e_b) + \theta^{3i}(e_3e_i) = \frac{1}{2}(+\theta^{3i}C_{3i\hat{k}} - \frac{1}{2}\theta^{\bar{i}j}C_{\bar{i}j\hat{k}})e_{\hat{k}}. \end{aligned} \quad (\text{A.11})$$

This corresponds to the decomposition $\mathbf{8}_s \rightarrow (\mathbf{2}, \mathbf{1}; \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{2})$ under the subalgebra $\mathfrak{so}(4)_{\text{ST}} \oplus \mathfrak{so}(4) \subset \mathfrak{so}(8)_{\text{ST}}$. By a similar calculation for $D = 7$ the decomposition is $\mathbf{8}_s \rightarrow (\mathbf{4}; \mathbf{2})$ under $\mathfrak{so}(5)_{\text{ST}} \oplus \mathfrak{so}(3) \subset \mathfrak{so}(8)_{\text{ST}}$, while in $D = 5$ the decomposition is group-theoretically equivalent, but with the roles of spacetime and internal symmetry reversed: $\mathbf{8}_s \rightarrow (\mathbf{2}; \mathbf{4})$ under $\mathfrak{so}(3)_{\text{ST}} \oplus \mathfrak{so}(5) \subset \mathfrak{so}(8)_{\text{ST}}$.

In dimensions $D = 5, 6$, the minimal (on-shell) spinor is a single quaternion. Thus to truncate to the theories with fewer supersymmetries one must simply truncate $\mathbb{O} \rightarrow \mathbb{H}$, which practically speaking means discarding all \hat{a} components.

A.1.2. From $D = 10$ to $D = 8, 4$: $\mathbb{O} \cong \mathbb{C}^4$

Next, consider dropping dependence on six of the ten dimensions, yielding the maximal $\mathcal{N} = 4$ theory in $D = 4$ with the on-shell degrees of freedom formulated again over the octonions. Now the spacetime little group is $\text{SO}(2)_{\text{ST}} \cong \text{U}(1)_{\text{ST}}$ and the internal symmetry is $\text{Spin}(6) \cong \text{SU}(4)$ with Lie algebra $\mathfrak{int}_2(\mathbb{O})$. The on-shell field representations decompose as

$$\begin{aligned} \text{Spin}(8)_{\text{ST}} &\supset \text{U}(1)_{\text{ST}} \times \text{SU}(4) \\ \mathbf{8}_v &\rightarrow \mathbf{6}_0 + \mathbf{1}_1 + \mathbf{1}_{-1} \\ \mathbf{8}_s &\rightarrow \mathbf{4}_{1/2} + \bar{\mathbf{4}}_{-1/2}, \end{aligned} \tag{A.12}$$

so the $D = 10$ vector becomes a $D = 4$ vector $\mathbf{1}_1 + \mathbf{1}_{-1}$ and six scalars $\mathbf{6}_0$, while the spinor becomes four fermions $\mathbf{4}_{1/2} + \bar{\mathbf{4}}_{-1/2}$ transforming as the (anti-)fundamental of $\text{SU}(4)$.

In division-algebraic language the vector $\mathbf{1}_1 + \mathbf{1}_{-1}$ corresponds to a complex subspace $\mathbb{C} \subset \mathbb{O}$, which is simply $\mathbb{C} \cong \text{span}\{e_0, e_1\}$, while the scalars $\mathbf{6}_0$ correspond to the six-dimensional complement $\mathbb{S}_8^2 \cong \mathbb{R}^6 = \text{span}\{e_2, e_3, e_4, e_5, e_6, e_7\}$ of \mathbb{C} in \mathbb{O} . This time the barred and underlined indices of (6.2) are

$$\begin{aligned} \underline{a} &= 0, 1, \\ \bar{i} &= 2, 3, 4, 5, 6, 7. \end{aligned} \tag{A.13}$$

The octonionic fields a' and ψ' can then be rewritten as

$$\begin{aligned} a' &= a + \phi = a_{\underline{a}}e_{\underline{a}} + \phi_{\bar{i}}e_{\bar{i}}, \\ \psi' &= (\psi'_{\hat{a}} + e_1\psi'_{\hat{a}}\Gamma_{\hat{a}\hat{a}}^1)e_{\hat{a}} := \psi_{\hat{a}}e_{\hat{a}}, \end{aligned} \tag{A.14}$$

so there are four complex spinors $\psi_{\hat{a}}$, written in \mathbb{O} by contracting with $e_{\hat{a}}$, where here the line and quadrangle indices run over

$$\begin{aligned} \hat{a} &= 0, 2, 3, 5 \quad \text{and} \quad \hat{i} = 2, 3, 5, \\ \check{a} &= 1, 4, 6, 7 \quad \text{and} \quad \check{i} = 4, 6, 7. \end{aligned} \tag{A.15}$$

Multiplying the spinor $\psi' = \psi_{\hat{a}}e_{\hat{a}}$ by $e_{\bar{i}}$ has the following effect:

$$e_{\bar{i}}(\psi_{\hat{a}}e_{\hat{a}}) = \psi_{\hat{a}}^*(e_{\bar{i}}e_{\hat{a}}) = \psi_{\hat{a}}^*(\Gamma_{\hat{a}\hat{b}}^{\bar{i}}e_{\hat{b}} + \Gamma_{\hat{a}\hat{b}}^{\bar{i}}e_{\hat{b}}) = \psi_{\hat{a}}^*(\Gamma_{\hat{a}\hat{b}}^{\bar{i}} - e_1\Gamma_{\hat{a}\hat{b}}^{\bar{i}}\Gamma_{\hat{b}\hat{b}}^1)e_{\hat{b}}, \tag{A.16}$$

so its complex components $\psi_{\hat{a}}$ get complex-conjugated and multiplied by the matrix

$$\Upsilon_{\hat{a}\hat{b}}^{\bar{i}} = -\Upsilon_{\hat{b}\hat{a}}^{\bar{i}} := \Gamma_{\hat{a}\hat{b}}^{\bar{i}} - e_1 \Gamma_{\hat{a}\hat{b}}^{\bar{i}} \Gamma_{\hat{b}\hat{b}}^1. \quad (\text{A.17})$$

The matrices $\Upsilon^{\bar{i}}$ and $\bar{\Upsilon}^{\bar{i}} \equiv \Upsilon^{\bar{i}*}$ satisfy the relations

$$\begin{aligned} \Upsilon^{\bar{i}} \bar{\Upsilon}^{\bar{j}} + \Upsilon^{\bar{i}} \bar{\Upsilon}^{\bar{j}} &= -2\delta^{\bar{i}\bar{j}} \mathbf{1}, \\ \bar{\Upsilon}^{\bar{i}} \Upsilon^{\bar{j}} + \bar{\Upsilon}^{\bar{i}} \Upsilon^{\bar{j}} &= -2\delta^{\bar{i}\bar{j}} \mathbf{1}, \end{aligned} \quad (\text{A.18})$$

and so can be used to form the Weyl spinor generators of Spin(6), which are also a basis for anti-Hermitian, traceless 4×4 matrices, giving the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representations of SU(4):

$$\begin{aligned} T_{\hat{a}\hat{b}}^{[\bar{i}\bar{j}]} &= \frac{1}{2} \Upsilon_{\hat{a}\hat{c}}^{\bar{i}} \bar{\Upsilon}_{\hat{c}\hat{b}}^{\bar{j}} = \Sigma_{\hat{a}\hat{b}}^{[\bar{i}\bar{j}]} + e_1 \Gamma_{\hat{a}\hat{c}}^1 \Sigma_{\hat{c}\hat{b}}^{[\bar{i}\bar{j}]}, \\ \bar{T}_{\hat{a}\hat{b}}^{[\bar{i}\bar{j}]} &= \frac{1}{2} \bar{\Upsilon}_{\hat{a}\hat{c}}^{\bar{i}} \Upsilon_{\hat{c}\hat{b}}^{\bar{j}} = \Sigma_{\hat{a}\hat{b}}^{[\bar{i}\bar{j}]} - e_1 \Gamma_{\hat{a}\hat{c}}^1 \Sigma_{\hat{c}\hat{b}}^{[\bar{i}\bar{j}]}. \end{aligned} \quad (\text{A.19})$$

The transformation of the spinor is then

$$\begin{aligned} \delta\psi' &= \frac{1}{4} \theta^{ab} e_a^* (e_b \psi') \\ &= \frac{1}{2} \theta^{01} e_1 (\psi_{\hat{a}} e_{\hat{a}}) + \frac{1}{4} \theta^{\bar{i}\bar{j}} e_{\bar{i}}^* (e_{\bar{j}} (\psi_{\hat{a}} e_{\hat{a}})) \\ &= \frac{1}{2} \theta^{01} (e_1 \psi_{\hat{a}}) e_{\hat{a}} - \frac{1}{2} \theta^{\bar{i}\bar{j}} (T_{\hat{a}\hat{b}}^{[\bar{i}\bar{j}]} \psi_{\hat{b}}) e_{\hat{a}}, \end{aligned} \quad (\text{A.20})$$

using equation (A.16) twice. The four complex spinors $\psi_{\hat{a}}$ do indeed transform as the $\mathbf{4}_{1/2} + \bar{\mathbf{4}}_{-1/2}$:

$$\delta\psi_{\hat{a}} = \frac{1}{2} \theta^{01} e_1 \psi_{\hat{a}} - \frac{1}{2} \theta^{\bar{i}\bar{j}} T_{\hat{a}\hat{b}}^{[\bar{i}\bar{j}]} \psi_{\hat{b}}. \quad (\text{A.21})$$

Note that Γ_{ab}^1 , satisfying $(\Gamma^1)^2 = -\mathbf{1}$, plays the role of a complex structure, which defines an isomorphism $\mathbb{O} \cong \mathbb{C}^4$. For the spinor one may view $\text{SU}(4) \times \text{U}(1)$ as the subgroup of Spin(8) generated by Lie algebra elements $\theta^{ab} \Sigma_{cd}^{[ab]}$ that commute with the complex structure Γ^1 , that is, those transformations that treat the 8 real components of a Spin(8) spinor like 4 complex components. Any real 8×8 matrix that commutes with Γ^1 , such as $\Sigma_{ab}^{[\bar{i}\bar{j}]}$, can then be written as a complex 4×4 matrix, like $T_{\hat{a}\hat{b}}^{[\bar{i}\bar{j}]}$ above. Equivalently, the matrix Γ^1 gives a representation of the Clifford algebra $\text{Cl}(1) \cong \text{Cl}_0(2) \cong \mathbb{C}$, where each $\psi_{\hat{a}}$ lives in the spinor representation $\mathcal{S}_2 = \mathbb{C}$, and hence $\mathbb{O} \cong (\mathcal{S}_2)^4 = \mathbb{C}^4$.

To obtain the $\mathcal{N} = 2$ theory one simply truncates $\mathbb{O} \rightarrow \mathbb{H}$. This translates into discarding two fermions and four scalars. The internal symmetry of the resulting $\mathcal{N} = 2$ theory is then the subgroup of SU(4) that preserves the quaternionic sub-

algebra and commutes with the complex structure, which is just the R-symmetry $U(2)$. The $\mathcal{N} = 1$ theory with R-symmetry $U(1)$ can then be recovered by further truncating $\mathbb{H} \rightarrow \mathbb{C}$, i.e. discarding one fermion and the remaining scalars (or of course truncating directly $\mathbb{O} \rightarrow \mathbb{C}$).

A completely analogous decomposition with $\mathbb{O} \cong \mathbb{C}^4$ takes place in $D = 8$, i.e. $N = 6$, but with the roles of spacetime and internal symmetry reversed; the on-shell field representations decompose just as in (A.12), only with respect to the subgroup $SU(4)_{\text{ST}} \times U(1) \subset \text{Spin}(8)_{\text{ST}}$. Here the vector index splitting is

$$\begin{aligned} \underline{a} &= 0, 1, 2, 3, 4, 5, \\ \bar{i} &= 6, 7, \end{aligned} \tag{A.22}$$

corresponding to the decomposition $\mathbf{8}_v \rightarrow \mathbf{6}_0 + \mathbf{1}_1 + \mathbf{1}_{-1}$. Here the complex structure is the single $U(1)$ R-symmetry generator, given by the simultaneous pair of left-multiplications: $e_6(e_7\psi)$.

A.1.3. From $D = 10$ to $D = 9, 3$: $\mathbb{O} \cong \mathbb{R}^8$

Dimensional reduction to $D = 3$ results in the $\mathcal{N} = 8$ maximal theory written over the octonions. This time the algebra decomposition is simply:

$$\begin{aligned} \text{SO}(8)_{\text{ST}} &\supset \text{SO}(7) \\ \mathbf{8}_v &\rightarrow \mathbf{1} + \mathbf{7} \\ \mathbf{8}_s &\rightarrow \mathbf{8}. \end{aligned} \tag{A.23}$$

The spacetime little group here is $\text{SO}(1)$, which is trivial, so the vector, fermions and scalars each contain only a single on-shell degree of freedom. The parameters decompose as

$$\theta^{ab} \rightarrow \theta^{0i}, \theta^{ij} \tag{A.24}$$

and $\text{SO}(7)$ is the subgroup with $\theta^{0i} = 0$. The fields a' and ψ' then break into

$$\begin{aligned} a' &= a + \phi, \\ \psi' &= \psi_a e_a, \end{aligned} \tag{A.25}$$

with $a \in \mathbb{R}$ and $\phi \in \text{Im}(\mathbb{O})$, and they transform as

$$\delta a' = \frac{1}{4}\theta^{ij}(e_i(e_j^*a) - a(e_i^*e_j)) + \frac{1}{4}\theta^{ij}(e_i(e_j^*\phi) - \phi(e_i^*e_j)) = e_i\theta^{ij}\phi_j, \tag{A.26}$$

and

$$\delta\psi' = \frac{1}{4}\theta^{ij}e_i^*(e_j\psi') = -\frac{1}{2}\theta^{ij}e_b\Sigma_{ba}^{[ij]}\psi_a. \quad (\text{A.27})$$

Thus the fermions and scalars indeed transform as the **8** and **7** of $\text{SO}(7)$,

$$\begin{aligned} \delta\psi_a &= -\frac{1}{2}\theta^{ij}\Sigma_{ab}^{[ij]}\psi_b, \\ \delta\phi_i &= \frac{1}{2}\theta^{ij}\phi_j. \end{aligned} \quad (\text{A.28})$$

The fermions packaged in ψ may be considered as eight spinors of the trivial Clifford algebra $\text{Cl}(0) \cong \mathbb{R}$, using the obvious isomorphism $\mathbb{O} \cong (\mathcal{S}_1)^8 = \mathbb{R}^8$.

To obtain the $\mathcal{N} = 4$ theory over \mathbb{H} , once again, one simply truncates $\mathbb{O} \rightarrow \mathbb{H}$, leaving a quaternion of spinors $\psi_{\hat{a}}e_{\hat{a}}$ and an imaginary quaternion of scalars $\phi_i e_i$. This theory has internal symmetry $\text{Sp}(1) \times \text{Sp}(1)$. The $\mathcal{N} = 2$ theory with an internal $\text{U}(1)$ symmetry comes from truncating $\mathbb{H} \rightarrow \mathbb{C}$, further discarding two scalars and two spinors. Finally, truncating the remaining scalar and the spinor associated with the last imaginary element results in the $\mathcal{N} = 1$ theory formulated over \mathbb{R} , with no internal symmetry.

Again, the group theory of the dimensional reduction from $D = 10$ to $D = 9$ is the same as that of $D = 10$ to $D = 3$, only with the spacetime and internal symmetries interchanged: for $D = 9$ the spacetime little group is $\text{Spin}(7)_{\text{ST}} \subset \text{Spin}(8)_{\text{ST}}$ and the fields decompose as in equation (A.23). Here the Clifford algebra $\text{Cl}(6) \cong \text{Cl}_0(7) \cong \mathbb{R}[8]$ is generated by the set $\{e_{\bar{i}}\} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, defining the isomorphism $\mathbb{O} \cong \mathcal{S}_7 = \mathbb{R}^8$.

A.2. Tensor Products of Yang-Mills Multiplets

| | |
|--|--|
| $D = 10, \mathfrak{so}(8)_{ST}$ | $\mathcal{N} = (1, 0) \quad \emptyset \quad A_\mu \mathbf{8}_v \quad \lambda \mathbf{8}_s$ |
| $\mathcal{N} = (0, 1) \quad \emptyset$ $A_\mu \quad \mathbf{8}_v$ $\lambda \quad \mathbf{8}_c$ | $\mathcal{N} = (1, 1) \quad \emptyset$ $g_{\mu\nu} \quad \mathbf{35}_v$ $\psi_\mu \quad \mathbf{56}_s + \mathbf{56}_c$ $A_\mu, B_{\mu\nu}, C_{\mu\nu\rho} \quad \mathbf{8}_v + \mathbf{28}_v + \mathbf{56}_v$ $\lambda \quad \mathbf{8}_s + \mathbf{8}_c$ $\phi \quad \mathbf{1}$ |
| $D = 9, \mathfrak{so}(7)_{ST}$ | $\mathcal{N} = 1 \quad \emptyset \quad A_\mu \mathbf{7} \quad \lambda \mathbf{8} \quad \phi \mathbf{1}$ |
| $\mathcal{N} = 1 \quad \emptyset$ $A_\mu \quad \mathbf{7}$ $\lambda \quad \mathbf{8}$ $\phi \quad \mathbf{1}$ | $\mathcal{N} = 2 \quad \mathfrak{so}(2)$ $g_{\mu\nu} \quad (\mathbf{27}; 0)$ $\psi_\mu \quad (\mathbf{48}; (1) + (-1))$ $A_\mu \quad (\mathbf{7}; (2) + (0) + (-2))$ $B_{\mu\nu}, C_{\mu\nu\rho} \quad (\mathbf{21}; (2) + (-2)) + (\mathbf{35}; 0)$ $\lambda \quad (\mathbf{8}; (3) + (1) + (-1) + (-3))$ $\phi \quad (\mathbf{1}; (4) + (0) + (-4))$ |
| $D = 8, \mathfrak{so}(6)_{ST}$ | $\mathcal{N} = 1 \quad \mathfrak{u}(1)$ $A_\mu (\mathbf{6}; 0) \quad \lambda (\mathbf{4}; -1) + (\overline{\mathbf{4}}; 1) \quad \phi (\mathbf{1}; 2) + (\mathbf{1}; -2)$ |
| $\mathcal{N} = 1 \quad \mathfrak{u}(1)$ $A_\mu \quad (\mathbf{6}; 0)$ $\lambda \quad (\mathbf{4}; -1) + (\overline{\mathbf{4}}; 1)$ $\phi \quad (\mathbf{1}; 2) + (\mathbf{1}; -2)$ | $\mathcal{N} = 2 \quad \mathfrak{u}(2)$ $g_{\mu\nu} \quad (\mathbf{20}; \mathbf{1}(0))$ $\psi_\mu \quad (\mathbf{20}'; \mathbf{2}(-1)) + (\mathbf{20}'; \mathbf{2}(1))$ $A_\mu, B_{\mu\nu} \quad (\mathbf{6}; \mathbf{3}(2) + \mathbf{3}(-2)) + (\mathbf{15}; \mathbf{3}(0))$ $C_{\mu\nu\rho} \quad (\mathbf{10}; \mathbf{1}(-2)) + (\mathbf{10}; \mathbf{1}(2))$ $\lambda \quad (\mathbf{4}; \mathbf{2}(-3) + \mathbf{4}(1)) + (\overline{\mathbf{4}}; \mathbf{2}(-3) + \mathbf{4}(1))$ $\phi \quad (\mathbf{1}; \mathbf{1}(4) + \mathbf{1}(-4) + \mathbf{5}(0))$ |
| $D = 7, \mathfrak{so}(5)_{ST}$ | $\mathcal{N} = 1 \quad \mathfrak{sp}(1)$ $A_\mu (\mathbf{5}; \mathbf{1}) \quad \lambda (\mathbf{4}; \mathbf{2}) \quad \phi (\mathbf{1}; \mathbf{3})$ |
| $\mathcal{N} = 1 \quad \mathfrak{sp}(1)$ $A_\mu \quad (\mathbf{5}; \mathbf{1})$ $\lambda \quad (\mathbf{4}; \mathbf{2})$ $\phi \quad (\mathbf{1}; \mathbf{3})$ | $\mathcal{N} = 4 \quad \mathfrak{sp}(2)$ $g_{\mu\nu} \quad (\mathbf{14}; \mathbf{1})$ $\psi_\mu \quad (\mathbf{16}; \mathbf{4})$ $A_\mu, B_{\mu\nu} \quad (\mathbf{5}; \mathbf{10}) + (\mathbf{10}; \mathbf{5})$ $\lambda \quad (\mathbf{4}; \mathbf{16})$ $\phi \quad (\mathbf{1}; \mathbf{14})$ |

Table A.1.: Tensor products of left and right super Yang-Mills multiplets in $D = 10, 9, 8, 7$. Dimensions $D = 6, 5$ are given in Table A.2.

| | | |
|---|--|--|
| $D = 6, \mathfrak{su}(2) \oplus \mathfrak{su}(2)_{\text{ST}}$ | $\mathcal{N} = (1, 1) \quad \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ $A_\mu \quad \lambda \quad \phi$ $(\mathbf{2}, \mathbf{2}; \mathbf{1}, \mathbf{1}) \quad (\mathbf{2}, \mathbf{1}; \mathbf{1}, \mathbf{2}) \quad (\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{2})$ $\quad \quad \quad +(\mathbf{1}, \mathbf{2}; \mathbf{2}, \mathbf{1})$ | $\mathcal{N} = (1, 0) \quad \mathfrak{sp}(1) \oplus \emptyset$ $A_\mu \quad \lambda$ $(\mathbf{2}, \mathbf{2}; \mathbf{1}) \quad (\mathbf{1}, \mathbf{2}; \mathbf{2})$ |
| $\mathcal{N} = (1, 1) \quad \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ $A_\mu \quad (\mathbf{2}, \mathbf{2}; \mathbf{1})$ $\lambda \quad (\mathbf{2}, \mathbf{1}; \mathbf{1}, \mathbf{2})$ $\quad \quad +(\mathbf{1}, \mathbf{2}; \mathbf{2}, \mathbf{1})$ $\phi \quad (\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{2})$ | $\mathcal{N} = (2, 2) \quad \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$ $g_{\mu\nu} \quad (\mathbf{3}, \mathbf{3}; \mathbf{1}, \mathbf{1})$ $\psi_\mu \quad (\mathbf{2}, \mathbf{3}; \mathbf{4}, \mathbf{1})$ $\quad \quad +(\mathbf{3}, \mathbf{2}; \mathbf{1}, \mathbf{4})$ $A_\mu \quad (\mathbf{2}, \mathbf{2}; \mathbf{4}, \mathbf{4})$ $B_{\mu\nu} \quad (\mathbf{3}, \mathbf{1}; \mathbf{1}, \mathbf{5})$ $\quad \quad +(\mathbf{1}, \mathbf{3}; \mathbf{5}, \mathbf{1})$ $\lambda \quad (\mathbf{2}, \mathbf{1}; \mathbf{4}, \mathbf{5})$ $\quad \quad +(\mathbf{1}, \mathbf{2}; \mathbf{5}, \mathbf{4})$ $\phi \quad (\mathbf{1}, \mathbf{1}; \mathbf{5}, \mathbf{5})$ | $\mathcal{N} = (2, 1) \quad \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ $g_{\mu\nu} \quad (\mathbf{3}, \mathbf{3}; \mathbf{1}, \mathbf{1})$ $\psi_\mu \quad (\mathbf{2}, \mathbf{3}; \mathbf{4}, \mathbf{1})$ $\quad \quad +(\mathbf{3}, \mathbf{2}; \mathbf{1}, \mathbf{2})$ $A_\mu \quad (\mathbf{2}, \mathbf{2}; \mathbf{4}, \mathbf{2})$ $B_{\mu\nu} \quad (\mathbf{3}, \mathbf{1}; \mathbf{1}, \mathbf{1})$ $\quad \quad +(\mathbf{1}, \mathbf{3}; \mathbf{5}, \mathbf{1})$ $\lambda \quad (\mathbf{2}, \mathbf{1}; \mathbf{4}, \mathbf{1})$ $\quad \quad +(\mathbf{1}, \mathbf{2}; \mathbf{5}, \mathbf{2})$ $\phi \quad (\mathbf{1}, \mathbf{1}; \mathbf{5}, \mathbf{1})$ |
| $\mathcal{N} = (0, 1) \quad \emptyset \oplus \mathfrak{sp}(1)$ $A_\mu \quad (\mathbf{2}, \mathbf{2}; \mathbf{1})$ $\lambda \quad (\mathbf{2}, \mathbf{1}; \mathbf{2})$ | $\mathcal{N} = (1, 2) \quad \mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$ $g_{\mu\nu} \quad (\mathbf{3}, \mathbf{3}; \mathbf{1}, \mathbf{1})$ $\psi_\mu \quad (\mathbf{2}, \mathbf{3}; \mathbf{2}, \mathbf{1})$ $\quad \quad +(\mathbf{3}, \mathbf{2}; \mathbf{1}, \mathbf{4})$ $A_\mu \quad (\mathbf{2}, \mathbf{2}; \mathbf{2}, \mathbf{4})$ $B_{\mu\nu} \quad (\mathbf{3}, \mathbf{1}; \mathbf{1}, \mathbf{5})$ $\quad \quad +(\mathbf{1}, \mathbf{3}; \mathbf{1}, \mathbf{1})$ $\lambda \quad (\mathbf{2}, \mathbf{1}; \mathbf{2}, \mathbf{5})$ $\quad \quad +(\mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{4})$ $\phi \quad (\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{5})$ | $\mathcal{N} = (1, 1) \quad \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ $g_{\mu\nu} \quad (\mathbf{3}, \mathbf{3}; \mathbf{1}, \mathbf{1})$ $\psi_\mu \quad (\mathbf{2}, \mathbf{3}; \mathbf{2}, \mathbf{1})$ $\quad \quad +(\mathbf{3}, \mathbf{2}; \mathbf{1}, \mathbf{2})$ $A_\mu \quad (\mathbf{2}, \mathbf{2}; \mathbf{2}, \mathbf{2})$ $B_{\mu\nu} \quad (\mathbf{3}, \mathbf{1}; \mathbf{1}, \mathbf{1})$ $\quad \quad +(\mathbf{1}, \mathbf{3}; \mathbf{1}, \mathbf{1})$ $\lambda \quad (\mathbf{2}, \mathbf{1}; \mathbf{2}, \mathbf{1})$ $\quad \quad +(\mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{2})$ $\phi \quad (\mathbf{1}, \mathbf{1}; \mathbf{1}, \mathbf{1})$ |
| $D = 5, \mathfrak{so}(3)_{\text{ST}}$ | $\mathcal{N} = 2 \quad \mathfrak{sp}(2)$ $A_\mu \quad \lambda \quad \phi$ $(\mathbf{3}; \mathbf{1}) \quad (\mathbf{2}; \mathbf{4}) \quad (\mathbf{1}; \mathbf{5})$ | $\mathcal{N} = 1 \quad \mathfrak{sp}(1)$ $A_\mu \quad \lambda \quad \phi$ $(\mathbf{3}; \mathbf{1}) \quad (\mathbf{2}; \mathbf{2}) \quad (\mathbf{1}; \mathbf{1})$ |
| $\mathcal{N} = 2 \quad \mathfrak{sp}(2)$ $A_\mu \quad (\mathbf{3}; \mathbf{1})$ $\lambda \quad (\mathbf{2}; \mathbf{4})$ $\phi \quad (\mathbf{1}; \mathbf{5})$ | $\mathcal{N} = 4 \quad \mathfrak{sp}(4)$ $g_{\mu\nu} \quad (\mathbf{5}; \mathbf{1})$ $\psi_\mu \quad (\mathbf{4}; \mathbf{8})$ $A_\mu \quad (\mathbf{3}; \mathbf{27})$ $\lambda \quad (\mathbf{2}; \mathbf{48})$ $\phi \quad (\mathbf{1}; \mathbf{42})$ | $\mathcal{N} = 3 \quad \mathfrak{sp}(3)$ $g_{\mu\nu} \quad (\mathbf{5}; \mathbf{1})$ $\psi_\mu \quad (\mathbf{4}; \mathbf{6})$ $A_\mu \quad (\mathbf{3}; \mathbf{1} + \mathbf{14})$ $\lambda \quad (\mathbf{2}; \mathbf{6} + \mathbf{14}')$ $\phi \quad (\mathbf{1}; \mathbf{14})$ |
| $\mathcal{N} = 1 \quad \mathfrak{sp}(1)$ $A_\mu \quad (\mathbf{3}; \mathbf{1})$ $\lambda \quad (\mathbf{2}; \mathbf{2})$ $\phi \quad (\mathbf{1}; \mathbf{1})$ | $\mathcal{N} = 3 \quad \mathfrak{sp}(3)$ $g_{\mu\nu} \quad (\mathbf{5}; \mathbf{1})$ $\psi_\mu \quad (\mathbf{4}; \mathbf{6})$ $A_\mu \quad (\mathbf{3}; \mathbf{1} + \mathbf{14})$ $\lambda \quad (\mathbf{2}; \mathbf{6} + \mathbf{14}')$ $\phi \quad (\mathbf{1}; \mathbf{14})$ | $\mathcal{N} = 2 \quad \mathfrak{sp}(2)$ $g_{\mu\nu} \quad (\mathbf{5}; \mathbf{1})$ $\psi_\mu \quad (\mathbf{4}; \mathbf{4})$ $A_\mu \quad (\mathbf{3}; \mathbf{1} + \mathbf{5})$ $\lambda \quad (\mathbf{2}; \mathbf{4})$ $\phi \quad (\mathbf{1}; \mathbf{1})$ $[\mathcal{N} = 2]_V$ $A_\mu \quad (\mathbf{3}; \mathbf{1})$ $\lambda \quad (\mathbf{2}; \mathbf{4})$ $\phi \quad (\mathbf{1}; \mathbf{5})$ |

Table A.2.: Tensor products of left and right super Yang-Mills multiplets in $D = 6, 5$.

| | | | | |
|--|---|---|---|---|
| $D = 4, \mathfrak{so}(2)_\text{ST}$ | $\mathcal{N} = 4$ $A_\mu : (1; \mathbf{1}) + \text{c.c.}$ $\lambda : (\frac{1}{2}; \mathbf{4}) + \text{c.c.}$ $\phi : (0; \mathbf{6})$ | $\mathcal{N} = 4$ $g_{\mu\nu} : \mathfrak{su}(4)$ $\psi_\mu : (2; \mathbf{1}) + \text{c.c.}$ $A_\mu : (\frac{3}{2}; \mathbf{8}) + \text{c.c.}$ $\lambda : (1; \mathbf{28}) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{56}) + \text{c.c.}$ $(0; \mathbf{70})$ | $\mathcal{N} = 2$ $A_\mu : (1; \mathbf{1}(0)) + \text{c.c.}$ $\lambda : (\frac{1}{2}; \mathbf{2}(1)) + \text{c.c.}$ $\phi : (0; \mathbf{1}(2)) + \text{c.c.}$ | $\mathcal{N} = 1$ $A_\mu : (1; \mathbf{0}) + \text{c.c.}$ $\lambda : (\frac{1}{2}; \mathbf{1}) + \text{c.c.}$ |
| $\mathcal{N} = 4$ $A_\mu : (1; \mathbf{1}) + \text{c.c.}$ $\lambda : (\frac{1}{2}; \mathbf{4}) + \text{c.c.}$ $\phi : (0; \mathbf{6})$ | $\mathcal{N} = 8$ $g_{\mu\nu} : \mathfrak{su}(8)$ $\psi_\mu : (2; \mathbf{1}) + \text{c.c.}$ $A_\mu : (\frac{3}{2}; \mathbf{8}) + \text{c.c.}$ $\lambda : (1; \mathbf{28}) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{56}) + \text{c.c.}$ $(0; \mathbf{70})$ | $\mathcal{N} = 6$ $g_{\mu\nu} : \mathfrak{u}(6)$ $\psi_\mu : (2; \mathbf{1}(0)) + \text{c.c.}$ $A_\mu : (\frac{3}{2}; \mathbf{6}(1)) + \text{c.c.}$ $\lambda : (1; \mathbf{1}(-6) + \mathbf{15}(2)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{6}(-5) + \mathbf{20}(3)) + \text{c.c.}$ $(0; \mathbf{15}(-4)) + \text{c.c.}$ | $\mathcal{N} = 6$ $g_{\mu\nu} : \mathfrak{u}(6)$ $\psi_\mu : (2; \mathbf{1}(0)) + \text{c.c.}$ $A_\mu : (\frac{3}{2}; \mathbf{6}(1)) + \text{c.c.}$ $\lambda : (1; \mathbf{1}(-6) + \mathbf{15}(2)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{6}(-5) + \mathbf{20}(3)) + \text{c.c.}$ $(0; \mathbf{15}(-4)) + \text{c.c.}$ | $\mathcal{N} = 5$ $g_{\mu\nu} : \mathfrak{u}(5)$ $\psi_\mu : (2; \mathbf{1}(0)) + \text{c.c.}$ $A_\mu : (\frac{3}{2}; \mathbf{5}(1)) + \text{c.c.}$ $\lambda : (1; \mathbf{10}(2)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{1}(-5) + \mathbf{10}(3)) + \text{c.c.}$ $(0; \mathbf{5}(-4)) + \text{c.c.}$ |
| $\mathcal{N} = 2$ $A_\mu : (1; \mathbf{1}(0)) + \text{c.c.}$ $\lambda : (\frac{1}{2}; \mathbf{2}(1)) + \text{c.c.}$ $\phi : (0; \mathbf{1}(2)) + \text{c.c.}$ | $\mathcal{N} = 6$ $g_{\mu\nu} : \mathfrak{u}(6)$ $\psi_\mu : (2; \mathbf{1}(0)) + \text{c.c.}$ $A_\mu : (\frac{3}{2}; \mathbf{6}(1)) + \text{c.c.}$ $\lambda : (1; \mathbf{1}(-6) + \mathbf{15}(2)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{6}(-5) + \mathbf{20}(3)) + \text{c.c.}$ $(0; \mathbf{15}(-4)) + \text{c.c.}$ | $\mathcal{N} = 4$ $g_{\mu\nu} : \mathfrak{u}(4) \oplus \mathfrak{u}(1)$ $\psi_\mu : (2; \mathbf{1}(0)(0)) + \text{c.c.}$ $A_\mu : (\frac{3}{2}; \mathbf{4}(1)(1)) + \text{c.c.}$ $\lambda : (1; \mathbf{6}(2)(2)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{4}(3)(3)) + \text{c.c.}$ $(0; \mathbf{1}(4)(4)) + \text{c.c.}$ $2[\mathcal{N} = 4]_V$ $2 \times A_\mu$ $2 \times \lambda$ $2 \times \phi$ | $\mathcal{N} = 3$ $g_{\mu\nu} : \mathfrak{u}(3) \oplus \mathfrak{u}(1)$ $\psi_\mu : (2; \mathbf{1}(0)(0)) + \text{c.c.}$ $A_\mu : (\frac{3}{2}; \mathbf{3}(1)(2)) + \text{c.c.}$ $\lambda : (1; \mathbf{3}(2)(4)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{1}(3)(6)) + \text{c.c.}$ $(0; \mathbf{1}(3)(6)) + \text{c.c.}$ $[\mathcal{N} = 3]_V$ $A_\mu : (1; \mathbf{1}(2)(-6)) + \text{c.c.}$ $\lambda : (1; \mathbf{1}(2)(-4)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{4}(-5)(1)) + \text{c.c.}$ $(\frac{1}{2}; \mathbf{4}(3)(-3)) + \text{c.c.}$ $(0; \mathbf{6}(-4)(2))$ $(0; \mathbf{6}(4)(-2))$ | $\mathcal{N} = 2$ $g_{\mu\nu} : \mathfrak{u}(2) \oplus \mathfrak{u}(1)$ $\psi_\mu : (2; \mathbf{1}(0)(0)) + \text{c.c.}$ $A_\mu : (\frac{3}{2}; \mathbf{2}(1)(2)) + \text{c.c.}$ $\lambda : (1; \mathbf{1}(2)(4)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{1}(2)(4)) + \text{c.c.}$ $[\mathcal{N} = 2]_H$ $\lambda : (\frac{1}{2}; \mathbf{2}(1)(-4)) + \text{c.c.}$ $\phi : (0; \mathbf{2}(-1)(-2)) + \text{c.c.}$ |
| $\mathcal{N} = 1$ $A_\mu : (1; \mathbf{0}) + \text{c.c.}$ $\lambda : (\frac{1}{2}; \mathbf{1}) + \text{c.c.}$ | $\mathcal{N} = 5$ $g_{\mu\nu} : \mathfrak{u}(5)$ $\psi_\mu : (2; \mathbf{1}(0)) + \text{c.c.}$ $A_\mu : (\frac{3}{2}; \mathbf{5}(1)) + \text{c.c.}$ $\lambda : (1; \mathbf{10}(2)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{1}(-5) + \mathbf{10}(3)) + \text{c.c.}$ $(0; \mathbf{5}(-4)) + \text{c.c.}$ | $\mathcal{N} = 3$ $g_{\mu\nu} : \mathfrak{u}(3) \oplus \mathfrak{u}(1)$ $\psi_\mu : (2; \mathbf{1}(0)(0)) + \text{c.c.}$ $A_\mu : (\frac{3}{2}; \mathbf{3}(1)(2)) + \text{c.c.}$ $\lambda : (1; \mathbf{3}(2)(4)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{1}(3)(6)) + \text{c.c.}$ $(0; \mathbf{1}(3)(6)) + \text{c.c.}$ $[\mathcal{N} = 3]_V$ $A_\mu : (1; \mathbf{1}(2)(-6)) + \text{c.c.}$ $\lambda : (1; \mathbf{1}(2)(-4)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{4}(-5)(1)) + \text{c.c.}$ $(\frac{1}{2}; \mathbf{4}(3)(-3)) + \text{c.c.}$ $(0; \mathbf{6}(-4)(2))$ $(0; \mathbf{6}(4)(-2))$ | $\mathcal{N} = 3$ $g_{\mu\nu} : \mathfrak{u}(3) \oplus \mathfrak{u}(1)$ $\psi_\mu : (2; \mathbf{1}(0)(0)) + \text{c.c.}$ $A_\mu : (\frac{3}{2}; \mathbf{3}(1)(2)) + \text{c.c.}$ $\lambda : (1; \mathbf{3}(2)(4)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{1}(3)(6)) + \text{c.c.}$ $(0; \mathbf{1}(3)(6)) + \text{c.c.}$ $[\mathcal{N} = 3]_V$ $A_\mu : (1; \mathbf{1}(2)(-6)) + \text{c.c.}$ $\lambda : (1; \mathbf{1}(2)(-4)) + \text{c.c.}$ $\phi : (\frac{1}{2}; \mathbf{4}(-5)(1)) + \text{c.c.}$ $(\frac{1}{2}; \mathbf{4}(3)(-3)) + \text{c.c.}$ $(0; \mathbf{6}(-4)(2))$ $(0; \mathbf{6}(4)(-2))$ | $\mathcal{N} = 1$ $A_\mu : (1; \mathbf{0}) + \text{c.c.}$ $\lambda : (\frac{1}{2}; \mathbf{1}) + \text{c.c.}$ |

Table A.3.: Tensor products of left and right super Yang-Mills multiplets in $D = 4$. Each representation is labeled $(h; \mathbf{m})$, where h is the helicity under the spacetime little group $\mathfrak{so}(2)_{\text{ST}}$ and \mathbf{m} is the representation of the internal global symmetry displayed, \mathfrak{int} for the super Yang-Mills multiplets and \mathfrak{h} for the resulting supergravity + matter multiplets. Here the subscripts V and H denote vector and hyper multiplets, respectively.

A.3. Complete List of Octonionic Dilaton Vectors

In order to list the dilaton vectors in a concise form we introduce some additional terminology and notation. A *flag* on the Fano plane \mathbb{F} (or its dual $\tilde{\mathbb{F}}$) is a pair (ijk, i) , consisting of an unoriented line ijk and a point i lying on that line. Since the line ijk is unoriented, we write $\sigma(ijk) \in \mathbf{L}$ (or $\tilde{\mathbf{L}}$), where $\sigma(ijk)$ is some permutation that puts i, j and k into the appropriate order. There are $7 \times 3 = 21$ flags on the Fano plane (or its dual) and we denote the set of these as $\text{Fl}(\mathbb{F})$ (or $\text{Fl}(\tilde{\mathbb{F}})$). Note that any pair of distinct points i, j on \mathbb{F} (or $\tilde{\mathbb{F}}$) uniquely defines a flag (ijk, k) , since choosing two points i, j selects a unique line ijk , and giving preference to i and j over k singles out k .

| Dilaton Vector | Fano Plane Interpretation | Octonionic Parameterisation | Number |
|--|--|--|--------|
| $-a$ | Full Plane | $\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7)$ | 1 |
| $-a_i$ | Point i on $\tilde{\mathbb{F}}$ \leftrightarrow Line ijk on \mathbb{F} | $\frac{1}{2}(e_{i'} + e_{j'} + e_{k'} + e_{l'}),$ $i'j'k'l' \in \mathbf{Q}$ | 7 |
| $a_{ij}, i < j$ | $(ijk, k) \in \text{Fl}(\tilde{\mathbb{F}})$ $\leftrightarrow (lmn, l) \in \text{Fl}(\mathbb{F})$ | $\frac{1}{2}(e_l - e_m - e_n), \sigma(lmn) \in \mathbf{L}$ | 21 |
| $a_{ijk}, ijk \in \tilde{\mathbf{L}}$ | Line ijk on $\tilde{\mathbb{F}}$ \leftrightarrow Point i on \mathbb{F} | e_i | 7 |
| $a_{ijk}, ijk \notin \tilde{\mathbf{L}},$ $i < j < k$ | $A \in \overline{\text{Fl}}(\tilde{\mathbb{F}})$ $\leftrightarrow (lmn, l') \in \overline{\text{Fl}}(\mathbb{F})$ | $\frac{1}{2}(-e_{l'} + e_{m'} + e_{n'} + e_{p'}),$ $\sigma(l'm'n'p') \in \mathbf{Q}$ | 28 |
| b_i | Point i on $\tilde{\mathbb{F}}$ \leftrightarrow Line ijk on \mathbb{F} | $-\frac{1}{2}(e_i + e_j + e_k), ijk \in \mathbf{L}$ | 7 |
| $b_{ij}, i < j$ | $(ijk, k) \in \text{Fl}(\tilde{\mathbb{F}})$ $\leftrightarrow (lmn, l) \in \text{Fl}(\mathbb{F})$ | $\frac{1}{2}(e_{l'} + e_{m'} - e_{n'} - e_{p'}),$ $\sigma(l'm'n'p') \in \mathbf{Q},$ $\sigma(ll'm'), \sigma(ln'p') \in \mathbf{L},$ $i = n' \text{ and/or } j = l'$ | 21 |

Table A.4.: Complete list of the octonionic $D = 4$ dilaton vectors. The vectors (or Kirmse integers) a_{ijk}, b_{ij} and $-a_i$ are the positive roots of $E_{7(\tau)}$, while a_{ij} and b_i make up the positive weights of the **56** representation. The notation $\sigma(lmn) \in \mathbf{L}$ means that there exists some permutation of lmn that gives a line in \mathbf{L} (strictly speaking, the lines in \mathbf{L} consist of *ordered* triples of points).

An *anti-flag* is a pair (ijk, l) , consisting of an unoriented line ijk and a point l *not* lying on that line. There are $7 \times 4 = 28$ anti-flags and we denote the set of these as $\overline{\text{Fl}}(\mathbb{F})$ (or $\overline{\text{Fl}}(\tilde{\mathbb{F}})$ for its dual). Note that any triple of points ijk that is *not* a line

on the Fano plane defines a unique anti-flag, since the complement of that triple in the plane consists of four distinct points, three of which form a line.

| Dilaton Vector | Fano Plane Interpretation | Octonionic Parameterisation | Number |
|---|--|---|--------|
| a_{0i} | Point i on $\tilde{\mathbb{F}}$ \leftrightarrow Line ijk on \mathbb{F} | $\frac{1}{2}(e_{i'} + e_{j'} + e_{k'} + e_{l'}), i'j'k'l' \in \mathbf{Q}$ | 7 |
| $a_{ij}, i < j$ | $(ijk, k) \in \text{Fl}(\tilde{\mathbb{F}})$ $\leftrightarrow (lmn, l) \in \text{Fl}(\mathbb{F})$ | $\frac{1}{2}(-1 + e_l - e_m - e_n), \sigma(lmn) \in \mathbf{L}$ | 21 |
| $a_{ij0}, i < j$ | $(ijk, k) \in \text{Fl}(\tilde{\mathbb{F}})$ $\leftrightarrow (lmn, l) \in \text{Fl}(\mathbb{F})$ | $\frac{1}{2}(+1 + e_l - e_m - e_n), \sigma(lmn) \in \mathbf{L}$ | 21 |
| $a_{ijk}, ijk \in \tilde{\mathbf{L}}$ | Line ijk on $\tilde{\mathbb{F}}$ \leftrightarrow Point i on \mathbb{F} | e_i | 7 |
| $a_{ijk}, ijk \notin \tilde{\mathbf{L}}, i < j < k$ | $A \in \overline{\text{Fl}}(\tilde{\mathbb{F}})$ $\leftrightarrow (lmn, l') \in \overline{\text{Fl}}(\mathbb{F})$ | $\frac{1}{2}(-e_{l'} + e_{m'} + e_{n'} + e_{p'}), \sigma(l'm'n'p') \in \mathbf{Q}$ | 28 |
| $-b_0$ | — | 1 | 1 |
| $-b_i$ | Point i on $\tilde{\mathbb{F}}$ \leftrightarrow Line ijk on \mathbb{F} | $\frac{1}{2}(1 + e_i + e_j + e_k), ijk \in \mathbf{L}$ | 7 |
| b_{0i} | Point i on $\tilde{\mathbb{F}}$ \leftrightarrow Line ijk on \mathbb{F} | $\frac{1}{2}(-1 + e_i + e_j + e_k), ijk \in \mathbf{L}$ | 7 |
| $b_{ij}, i < j$ | $(ijk, k) \in \text{Fl}(\tilde{\mathbb{F}})$ $\leftrightarrow (lmn, l) \in \text{Fl}(\mathbb{F})$ | $\frac{1}{2}(e_{l'} + e_{m'} - e_{n'} - e_{p'}), \sigma(l'm'n'p') \in \mathbf{Q}, \sigma(ll'm'), \sigma(ln'p') \in \mathbf{L}, i = n' \text{ and/or } j = l'$ | 21 |

Table A.5.: Complete list of the $D = 3$ dilaton vectors ($-a_{ab}, a_{abc}, -b_a$ and b_{ab}) written as Kirmse integers. Together all the dilaton vectors make up the positive roots of $E_{8(8)}$.

Using (7.49) one can compute the full set of dilaton vectors. Because of the parameterisations (7.52) and (7.61) the resulting vectors exhibit a correspondence with the Fano plane:

- a_i carries a label i corresponding to a point of $\tilde{\mathbb{F}}$, which maps by duality to a line ijk on \mathbb{F} ; the complement of this line is a quadrangle $i'j'k'l'$ on \mathbb{F} ; we find that $a_i = \frac{1}{2}(e_{i'} + e_{j'} + e_{k'} + e_{l'})$.

- a_{ij} ($i < j$) singles out a flag in $(ijk, k) \in \text{Fl}(\tilde{\mathbb{F}})$ which maps to a flag $(lmn, l) \in \text{Fl}(\mathbb{F})$; the resulting vector is $a_{ij} = \frac{1}{2}(e_l - e_m - e_n)$, with the different signs reflecting the flag (lmn, l) .
- a_{ijk} , $ijk \in \tilde{\mathbf{L}}$ clearly singles out the line ijk on $\tilde{\mathbb{F}}$, which maps to a point i on \mathbb{F} ; we find that $a_{ijk} = e_i$.
- a_{ijk} , $ijk \notin \tilde{\mathbf{L}}$ (with $i < j < k$) corresponds to an anti-flag $A \in \overline{\text{Fl}}(\tilde{\mathbb{F}})$, which maps to an anti-flag $(lmn, l') \in \overline{\text{Fl}}(\mathbb{F})$; the complement of the unoriented line lmn is an unoriented quadrangle $l'm'n'p'$ where the point l' is distinguished by the flag (lmn, l') ; the result is $a_{ijk} = \frac{1}{2}(-e_{l'} + e_{m'} + e_{n'} + e_{p'})$.
- b_i corresponds to a point i on $\tilde{\mathbb{F}}$, which gives a line ijk on \mathbb{F} , giving $b_i = -\frac{1}{2}(e_i + e_j + e_k)$.
- b_{ij} ($i < j$) again selects a flag in $(ijk, k) \in \text{Fl}(\tilde{\mathbb{F}})$ which maps to a flag $(lmn, l) \in \text{Fl}(\mathbb{F})$; the complement of the unoriented line lmn is an unoriented quadrangle $l'm'n'p'$, which is naturally split into two halves $l'm'$ and $n'p'$ by the flag (lmn, l) , since l' and m' lie on an unoriented line $ll'm'$ with l , while n' and p' lie on another line $ln'p'$ with l ; the resulting vector is $b_{ij} = \frac{1}{2}(e_{l'} + e_{m'} - e_{n'} - e_{p'})$ with the overall sign dictated as follows: looking at the labels i and j of b_{ij} we see that in general $i = n'$ and/or $j = l'$.

This is summarised in Table A.4 and, since the $D = 3$ case is very similar, its vectors are simply listed in Table A.5. For reference, the sets of lines \mathbf{L} and quadrangles \mathbf{Q} of the Fano plane are defined again here:

$$\begin{aligned} \mathbf{L} &:= \{124, 235, 346, 457, 561, 672, 713\}, \\ \mathbf{Q} &:= \{3567, 4671, 5712, 6123, 7234, 1345, 2456\}, \end{aligned} \tag{A.29}$$

as well as the lines $\tilde{\mathbf{L}}$ of the dual Fano plane,

$$\tilde{\mathbf{L}} := \{157, 261, 372, 413, 524, 635, 746\}. \tag{A.30}$$

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