# Relativistic Theories of Gravitation

*Proceedings of a conference held in*
*Warsaw and Jabłonna*
*July, 1962*

*Edited by*
L. INFELD

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## THE ORGANIZING COMMITTEE

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# PREFACE

The 1962 Conference on Relativistic Theories of Gravitation, the proceedings of which are before you, was preceded by a meeting of the International Gravitational Committee in Paris. Most of us there thought that the meeting should be fairly leisurely and the number of participants restricted. We decided against the usual shower of ten or fifteen minute contributions. Instead, we thought that the most suitable form would be invited lectures and extensive discussions around them. The rest was left to the Polish Organizing Committee.

Our first task was to make a list of invited participants. Here I believe we made some mistakes by omitting the names of several important workers in the field of relativity. These mistakes were hard to avoid and we can only apologize for them.

Then we had to prepare a list of lectures and invited speakers. As a rule, we planned three lectures every morning, each of them 45 minutes long. The afternoons were for discussion. However, the official discussions were not as vivid and not as time absorbing as we had hoped. On the other hand some of the guests came with prepared reports. Therefore, in the afternoon, after the discussions were finished, we introduced informal seminars. During the conference, Professor A. Schild was kind enough to devote much of his time to arranging these seminars.

We took care that the general lectures would not be the privilege of the older generation only. The younger active people also had a prominent part in them.

As it was July and Warsaw is fairly warm at that time, we decided to have our meetings outside Warsaw, at Jabłonna, in a palace which belongs to the Polish Academy of Science. However, there were not enough rooms for all the participants to live there. Therefore, about half of the guests had to stay in Warsaw and we arranged a Warsaw–Jabłonna car and bus service. The opening of the conference took place in the Academy's Pałac Staszyca in Warsaw.

Dr J. Stachel helped us greatly in preparing this report. The discussion and some of the lectures had to be transcribed from tape recordings. One day the electric current failed in Jabłonna and for that afternoon the records are not complete. Parts of the discussion were hard to transcribe for technical reasons. If certain passages of the report are not clear, the fault may be the editors' rather than speakers'.

Leopold Infeld

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## SCIENTIFIC PROGRAMME

*Wednesday, July 25th* *Pałac Staszica, Warszawa*

10.30 a. m. Opening of the Conference. Introduction by L. Infeld.

Greetings from the President of the Polish Academy of Sciences, the Minister of Higher Education, and the Chairman of the State Council for Peaceful Uses of Nuclear Energy.

FIRST SESSION

Chairman: A. LICHNEROWICZ

11.30 a. m. J. L. SYNGE

Relativistic interpretation and modification of Newtonian models

*Jabłonna*

AFTERNOON SESSION

Chairman: D. IVANENKO

4.00 p. m. S. MANDELSTAM

On the quantum theory of gravitation

SEMINARS

J. L. ANDERSON

*Q*—number coordinate transformations and the ordering problem in general relativity

C. B. RAYNER

Elasticity in general relativity

G. DAUTCOURT

Gravitationsfelder mit isotropem Killingvektor

H. J. TREDER

Über Lösungen der Einsteinschen Gravitationsgleichungen, bei denen die Determinante $g$ auf niederdimensionierten Mannigfaltigkeiten verschwindet

*Thursday, July 26th*

MORNING SESSION

Chairman: L. ROSENFELD

10.00 a. m. C. MØLLER

Conservation laws and absolute parallelism in general relativity

10.45 a. m. J. PLEBAŃSKI
Tetrads and conservation laws
12.00 a. m. V. L. GINZBURG
Experimental tests of general relativity

AFTERNOON SESSION

Chairman: J. GÉHÉNIAU
3.30 p. m. L. I. SCHIFF
Proposed gyroscope experiment to test general relativity theory
4.00 p. m. General discussion

SEMINARS

A. Z. PETROV
The type of space — the type of energy momentum tensor in general theory of relativity
E. SCHMUTZER
Spinors and bispinors in Riemannian space
N. ROSEN
Conservation laws and flat-space metric in general relativity
N. V. MITZKEVIČ
On localizability of gravitational energy,
E. T. NEWMAN
A generalization of the Schwarzschild metric

*Friday, July 27th*

MORNING SESSION

Chairman: J. A. WHEELER
10.00 a. m. R. K. SACHS
Characteristic initial value problem for gravitational theory
10.45 a. m. I. ROBINSON and A. TRAUTMAN
Exact degenerate solutions of Einstein's equations
12.00 a. m. H. BONDI
Radiation from an isolated system

AFTERNOON SESSION

Chairman: P. G. BERGMANN
3.30 p. m. General discussion

SEMINARS

M. A. TONNELAT
Energie gravitationelle et lois du mouvement dans une théorie linéaire et minkowskienne du champ de gravitation
A. L. ZELMANOV
Chronometrical invariants and some applications of them

Ya. P. Terletski
Negative mass particles
J. E. Hogarth
Cosmological considerations of the absorber theory of radiation
D. W. Sciama
Retarded potentials and the expansion of the Universe

*Saturday, July 28th*

MORNING SESSION

| | |
|---|---|
| | Chairman: H. Bondi |
| 10.00 a. m. | B. S. DeWitt<br>The quantization of geometry |
| 10.45 a. m. | P. G. Bergmann<br>Asymptotic properties of gravitating systems |
| 12.00 a. m. | P. A. M. Dirac<br>The motion of an extended particle in the gravitational field |

*Monday, July 30th*

MORNING SESSION

| | |
|---|---|
| | Chairman: C. Møller |
| 10.00 a. m. | A. Lichnerowicz<br>Commutateurs et anticommutateurs en relativité générale |
| 10.45 a. m. | C. Misner<br>Waves, Newtonian fields and coordinate functions |
| 12.00 a. m. | R. P. Feynman<br>The quantum theory of the gravitational field |

AFTERNOON SESSION

| | |
|---|---|
| | Chairman: B. DeWitt |
| 3.30 p. m. | General discussion |

SEMINARS

J. V. Narlikar
Neutrinos and the absorber theory of radiation
L. H. Thomas
Gravitation as an interaction between the small and the large
I. Robinson and A. Schild
Degeneracy and shear
H. Bondi
The steady state universe
O. Costa de Beauregard
Effet inertial de Spin en translation
D. Finkelstein
General relativity and elementary particles

R. Arnowitt
Asymptotic coordinate conditions, the wave front theorem and properties of energy and momentum
N. V. Mitzkevič
A four dimensional symmetrical canonical formalism in field theory
M. Trümper
On a characterization of non-degenerate static vacuum fields by means of test particle motion

*Tuesday, 31st July*

MORNING SESSION

Chairman: J. Weyssenhoff

10.00 a. m. J. A. Wheeler
Mach's principle as boundary condition for Einstein's equations

10.45 a. m. V. A. Fock
The uniqueness of the mass tensor and Einstein's equations

12.00 a. m. P. G. Bergmann
Concluding remarks

SEMINARS

R. Penrose
The light cone of infinity
A. L. Zelmanov
On the behaviour of the scale-factor in an anisotropic non-homogeneous universe

## L. Infeld

Dear friends,

It gives me great pleasure to open this Conference on Gravitation. Whether this is the fourth or the second such meeting depends upon whether we count them from the Berne Conference or from the founding of our organization.

I believe that when most of us use the term "Gravitation Theory" we refer to General Relativity, and this theory is about half a century old. I cannot say exactly how old it is because I do not know whether we should consider its beginning as Einstein's first paper in 1911 or the paper which laid the foundation of the theory, in 1916. In any case, the greatest interest in this discipline was evinced by scientists in the 1920's. Then, already in 1936, when I was in contact with Einstein in Princeton, I observed that this interest had almost completely lapsed. The number of physicists working in this field in Princeton could be counted on the fingers of one hand. I remember that very few of us met in the late Professor H. P. Robertson's room and then even those meetings ceased. We, who worked in this field, were looked upon rather askance by other physicists. Einstein himself often remarked to me "In Princeton they regard me as an old fool: Sie glauben ich bin ein alter Trottel". This situation remained almost unchanged up to Einstein's death. Relativity Theory was not very highly estimated in the "West" and frowned upon in the "East".

Yet the situation has changed completely in the last few years. Twenty years ago people thought that Relativity Theory was finished and that it offered no new problems. The sudden revival of General Relativity Theory and the interest shown in it by so many young people is due to several causes.

I should like to mention a few of them. First — I believe that our biannual meetings, beginning with the Berne Conference, contributed greatly to the increased interest in gravitational problems. On the other hand I am well aware that these meetings are at the same time only an indication of this growing interest.

The second reason is that we now know much more about the mathematical structure of Relativity Theory. Indeed the horizons of our knowledge are widening very much and this is mostly due to the work of young scientists. Progress has been made especially on gravitational waves and on quantizing the gravitational field. These are the chief problems of the present day and I hope that most of the discussions during our Conference will be devoted to them.

Last, but not least, the experimental evidence supporting General Relativity Theory has been much enlarged by the Mössbauer effect and we look forward to possibilities of observing new effects with artificial satellites.

Yet, much as we treasure the work of younger people, we should not forget the older men who have left us and who contributed an important share toward developing General Relativity Theory. Among the names foremost in my mind and who passed away since the Royaumont Conference are Max von Laue, H. P. Robertson, Erwin Schrödinger.

In the early twenties Professor Laue finished his two-volume work on Relativity Theory which has been studied by many physicists.

Professor Schrödinger, known mostly as the founder of wave mechanics, in his last years did much work on the unified gravitational theory.

Professor Robertson, whose loss I feel in an especially personal way, did a great deal of work on cosmology and General Relativity Theory.

I ask you now to rise and to devote a period of silence to these three men.

Thank you.

The organizing committee are especially glad that the conference is taking place in Poland near a city that was over 80% destroyed by the Nazis and is now rebuilt.

I hope you will find enough time to observe, at least in part, the progress made in reconstruction and also to experience something of the active cultural life here.

In the name of the organizing committee I welcome you to Poland and to our conference.

During J. L. Synge's lecture. In the first row, left to right: L. Infeld, V A. Fock, J. L. Anderson, E. Newman, R. Penrose, and B. Hoffmann. Far right R. Michalska-Trautman.

# ALLOCUTION INAUGURALE
# DU MINISTRE DE L'ENSEIGNEMENT SUPERIEUR,

Monsieur H. Golański

Je suis heureux de pouvoir saluer dans notre pays la conférence des spécialistes les plus éminents dans le domaine des théories relativistes.

La théorie de la relativité est sûrement l'une des plus grandes acquisitions intellectuelles du XX$^{e}$ siècle. L'oeuvre d'Albert Einstein a donné le départ à l'essor impétueux de la physique de nos temps. L'importance de ses conséquences primordiales de la conception du monde ne peut guère être surestimée.

La signification de la conférence consacrée au domaine où sont indissolublement liés les problèmes complexes du temps, de l'espace et de la gravitation, où l'essentiel est de trouver la jonction entre les grandes idées générales de la théorie de la relativité et le courant principal de la physique contemporaine des quanta—est évidente même pour un non-initié.

La conférence a lieu à une époque où la théorie de la relativité vit—si l'on peut s'exprimer ainsi—sa période de deuxième jeunesse. La preuve, c'est le nombre toujours croissant des physiciens faisant des recherches dans ce domaine, et surtout la quantité de résultats extrêmement précieux obtenus au cours des dernières années.

Il m'est fort agréable de constater, que la majorité de ces éminents hommes de science se trouvent dans cette salle.

Après les immenses succès de la mécanique des quanta, lorsque l'intérêt principal s'est porté vers la physique nucléaire et la physique des particules élémentaires, la théorie générale de la relativité s'est trouvée comme mise à l'écart. Sa profondeur et sa belle structure logique en tant que théorie fondamentale du temps et de l'espace, n'était pas liée jusqu'alors avec la réalité des quanta du microcosme.

Pour quelqu'un qui n'est pas physicien les phénomènes de gravitation ont une valeur essentielle: il y est le plus habitué.

La physique des quanta les a négligé jusqu'à maintenant comme étant super-faibles, ne s'intéressant qu'aux phénomènes électromagnétiques, forts et faibles. Il est difficile à un non spécialiste d'émettre un jugement en cette matière. Je crois cependant, ne serait-ce que d'après les thèmes de la conférence, où le problème des quanta va jouer un rôle important à coté des questions du rayonnement de gravitation, que l'espoir est aujourd'hui plus grand que jamais de voir se réaliser l'oeuvre de synthèse — l'oeuvre de jonction entre la physique quantique et la théorie générale de la relativité.

Comme en témoigne l'histoire des sciences exactes, des problèmes particulièrement intéressants se manifestent aux confins de différents domaines, ou comme résultat des manières diverses d'aborder la même question. Plus d'une découverte en est issue, qui fait progresser la science à pas de géant, aussi fertile en idées nouvelles qu'en acquisitions révolutionnaires de la technique qui transforme notre civilisation.

Il y a plus d'un demi siècle, lors de la jonction de l'électrodynamique et de la mécanique, est née la théorie particulière de la relativité, en fonction avec la nécessité de créer une théorie abordant d'une façon uniforme les deux genres de phénomènes. La mécanique fut soumise, pour ainsi dire, à l'électrodynamique. Aujourd'hui, il s'agit de combler la lacune entre la thèorie générale de la relativité et la physique des quanta. Ce problème est en liaison avec les conceptions fondamentales du temps et de l'espace; c'est donc un problème de la plus grande importance.

Je suis très heureux que la conférence qui traite des problèmes d'une telle portée, se tienne dans notre pays. Le fait que le Comité International de Gravitation a décidé d'organiser cette conférence en Pologne, pays où la vie scientifique — qui, dans la période d'après-guerre, a due être bâtie à neuf, — témoigne de l'estime portée aux réalisations scientifiques de nos savants et de nos jeunes chercheurs.

Qu'il me soit permis de vous remercier en leur nom pour cet honneur, et de souhaiter aux participants de cette conférence de fructueux débats.

# LECTURES AND DISCUSSIONS

# RELATIVISTIC INTERPRETATION AND MODIFICATION OF NEWTONIAN MODELS

J. L. Synge

*Dublin Institute for Advanced Studies*

At any conference there must be a first word and there must also be a last word. I feel much honoured by having been asked to speak the first word, and, before I proceed to scientific details, I welcome the opportunity to greet publicly my old friend and colleague Leopold Infeld and to congratulate him on his achievement in building up such an active centre of research in theoretical physics. Let me also thank him and his colleagues, in particular Professor Trautman, for their work in organizing this meeting. Anyone with experience in such matters knows how much labour is involved in organization. On behalf of us all, I wish to express our appreciation.

In broad terms, what I want to discuss on the present occasion is the relationship between Newtonian physics and the physics of general relativity. To narrow the discussion let us think only of macroscopic physics, so that the indeterminacy of quantum mechanics is entirely omitted. To narrow the matter still further, let us think of problems of celestial mechanics, in particular of the solar system.

Since the sun and the planets are nearly spherical, and are separated by distances large relative to their radii, we may decide to introduce at once an approximation based on these facts, and treat the bodies as point-particles. I prefer however to regard this as an approximation to be introduced only at a much later stage. Let us regard the bodies as finite, and write down the appropriate partial differential equations.

In Newtonian physics we have the following basic partial differential equations

$$\varrho \frac{\partial u_\alpha}{\partial t} + (\varrho u_\alpha u_\beta - s_{\alpha\beta})_{,\beta} = \varrho \mathrm{V}_{,\alpha}, \qquad \frac{\partial \varrho}{\partial t} + (\varrho u_\beta)_{,\beta} = 0. \tag{1}$$

Here we are using rectangular Cartesian coordinates $x_\alpha$, the Greek suffixes taking the values 1, 2, 3; $\varrho$ is density, $u_\alpha$ velocity, $S_{\alpha\beta}$ the stress, and $V$ the potential ($V = \int \varrho d_3 v/r$); the summation convention is understood. These equations apply for all space and time, with $\varrho = 0$, $S_{\alpha\beta} = 0$ outside the bodies. We have 4 equations for 10 dependent variables, and so we have a highly indeterminate problem. This indeterminacy is often lost sight of in Newtonian theory. Indeed it is usually avoided by adding subsidiary conditions. For example, we may assume each body to be a perfect fluid with a pressure-density relation

$$S_{\alpha\beta} = -p\delta_{\alpha\beta}, \quad f(\varrho,p) = 0. \tag{2}$$

Or we may complicate the problem by making the bodies elastic. Or we may make the bodies rigid by introducing the condition

$$u_{\alpha,\beta} + u_{\beta,\alpha} = 0, \tag{3}$$

thus making the velocity a Killing vector.

All these are in the nature of *ad hoc* assumptions. When we add the assumption that the bodies are far apart, deductions based on the several different assumptions may not be observationally distinguishable in view of the inaccuracy of astronomical observation and the comparatively short period over which observations have been made. But I want to emphasise that Newtonian theory does contain elements of indeterminacy which prevent us from writing down exact equations of motion for a system like the solar system, unless the indeterminacy is deliberately removed by special assumptions.

The field equations of general relativity read

$$G_{ij} = -\varkappa T_{ij} \quad (\varkappa = 8\pi) \tag{4}$$

the units being chosen so that the gravitational constant and the speed of light are each unity. Latin suffixes take the values 1, 2, 3, 4. In (4) we see 10 equations connecting the 20 components of two symmetric tensors — the Einstein tensor $G_{ij}$ which is to be described as chronometric since it can be calculated directly from the metric tensor $g_{ij}$ which gives the proper-time element $ds$, and energy tensor $T_{ij}$. The physical interpretation of $T_{ij}$ is delicate and important. One plan is to write down the equations

$$T_{ij}\lambda^j = -\theta g_{ij}\lambda^j \tag{5}$$

these four equations determine four eigenvalues $\theta$ and corresponding directions $\lambda^j$ and we may identify the mean velocity and density of matter with the timelike direction and its eigenvalue $\theta_{(4)}$ at the same time identifying the other (spacelike) eigendirections with the principal axes of stress and the corresponding eigenvalues with the principal stresses, reversed in sign (we regard tension as positive). However, these eigenvalues and directions may not all be real, and it is advisable to seek rather physical names for the invariants

$$T_{(ab)} = T_{ij}\lambda^i_{(a)}\lambda^j_{(b)} \tag{6}$$

where $\lambda^i_{(a)}$ is any selected orthonormal tetrad; the appropriate names are stress components, energy density (= flux of momentum), and density.[1]

[1] The inadequacy of (5) appears in the case where there is an electromagnetic field present without matter in the ordinary sense; this was pointed out to me by Professor G. Y. Rainich.

We are then to regard (4) as 10 equations connecting 20 dependent variables $g_{ij}$ and $T_{ij}$ with physical meanings. We recognize the presence of an indeterminacy somewhat greater than the indeterminacy present in the Newtonian formulae (1). As in the Newtonian case, we may try to remove the indeterminacy by taking the bodies to be perfect fluids with pressure-density relations, writing

$$T_{ij} = (\mu+p)v_i v_j + p g_{ij}, \quad f(\mu, p) = 0 \tag{7}$$

where $\mu$ is density, $p$ pressure, and $v^i$ 4-velocity, with $v_i v^i = -1$. (The signature of $g_{ij}$ is taken to be $+ + + -$.) But as in the Newtonian case, any such assumption is physically unrealistic, and it is best to work with the general equations (4) and recognize the presence of indeterminacy.

In view of inherent indeterminacy, we should, I think, not regard the problems of celestial mechanics as well-defined problems seeking solution. Rather we should try to construct model universes with properties resembling as closely as may be the properties of the actual universe, or some portion of it selected for discussion.

In constructing a model of the solar system, we seek 20 functions ($g_{ij}$ and $T_{ij}$) of the coordinates $x^i$ to satisfy the 10 equations (4) with certain requirements of a qualitative character. The first demand is that the metric tensor should have the signature $(+ + + -)$. If that were all, then we could construct at once an infinity of universes simply by choosing $g_{ij}$ arbitrarily save for this restriction, and then using (4) to calculate $T_{ij}$. Outside chosen world-tubes we might give $g_{ij}$ flat values (making $G_{ij} = 0$) and extend $g_{ij}$ smoothly into the interiors of the world tubes. However, at this point we see the relevance of the physical condition of positive density; any arbitrary approach of this type is almost certain to give negative density somewhere. In fact, the challenge in constructing models is to make density positive in the bodies (and zero outside them). I have spoken of the solar system, but it would be better to start with something simpler. So I would ask you to consider the following systems:

(i) a single body at rest;

(ii) a single body of revolution rotating about its axis of symmetry;

(iii) two bodies moving under their mutual attraction. In each case the object is to construct a relativistic model which agrees with our intuitive concept of a system of the type described.

During the past two years we have given much thought in the Dublin Institute to these problems. In the case of (i) and (ii) we have succeeded in constructing models which appear to us to be satisfactory. Our work has been published in a joint paper, Das, Florides, Synge 1961 (referred to below as DFS); further work dealing with (ii) for a fluid body by Florides and Synge has not yet been published (referred to below as FS). We have had some interesting thoughts about (iii) but we have not succeeded in constructing a model; in fact we have been led to wonder whether a model can be found at all.

It is hard to decide whether it would be more interesting to present on the present occasion the models we have succeeded in creating or our thoughts about the models we have failed to create. I shall not have time to do both. On the whole it seems best to tell you about our successful constructions and indicate briefly why the method cannot be used in the two-body problem.

Consider then a stationary system, i.e. a system in which everything is independent of time. Then Newtonian equations (1) read

$$(\varrho u_\alpha u_\beta - S_{\alpha\beta})_{,\beta} = \varrho V_{,\alpha}, \qquad (\varrho u_\beta)_{,\beta} = 0. \tag{8}$$

Observe that these equations are invariant under the following transformation, $k$ being an arbitrary constant:

$$\varrho \to k^2 \varrho, \qquad u_\alpha \to k u_\alpha, \qquad S_{\alpha\beta} \to k^4 S_{\alpha\beta}. \tag{9}$$

This tells us that if we have one Newtonian model, we have in fact a single infinity of such models, obtained from the first by making the transformation (9). This suggests that a relativistic model of a stationary system should have the same property, and that we should use a metric tensor containing an arbitrary parameter $k$. Therefore, using imaginary time $x_4 = it$, we write down

$$g_{ij} = \delta_{ij} + \underset{2}{g_{ij}} + \underset{3}{g_{ij}} + \ldots, \tag{10}$$

a numerical subscript indicating a power of $k$ contained in the term. Having committed ourselves in this way, we see that all geometrical quantities in space-time admit expansions in $k$. Thus

$$g^{ij} = \delta_{ij} + \underset{2}{g^{ij}} + \underset{3}{g^{ij}} + \ldots, \tag{11}$$

$$G^{ij} = \underset{2}{G^{ij}} + \underset{3}{G^{ij}} + \ldots \tag{12}$$

We think of $k$ as small (but not infinitesimal) and recognize that $k$ should appear first in the second power in (10) because the gravitational field arises primarily from density and in (9) density is associated with $k^2$. (I use here the modification, adopted in FS, of the notation of DFS.)

At this point some notation must be defined. The star-conjugate is (for any symmetric matrix)

$$A^*_{ij} = A_{ij} - \frac{1}{2}\,\delta_{ij} A_{kk}. \tag{13}$$

We need the linear form $L_{ij}$ defined by

$$L^*_{ij}(\gamma) = \frac{1}{2}\,(\gamma_{ij,aa} + \gamma_{aa,ij} - \gamma_{ia,aj} - \gamma_{aj,ia}). \tag{14}$$

We easily verify that

$$L_{ij,j} = 0 \tag{15}$$

identically, and that, if $\gamma_{ij}$ satisfies the 'coordinate condition'

$$\gamma^*_{ij,j} = 0 \tag{16}$$

then

$$L_{ij}(\gamma) = \frac{1}{2}\gamma^*_{ij,aa}. \tag{17}$$

Armed with this notation, we return to (12) and find that

$$\underset{N}{G}^{ij} = L_{ij}(\underset{N}{g}) + \underset{N}{M}^{ij}(\underset{2}{g}, \underset{3}{g}, \ldots, \underset{N-2}{g}), \tag{18}$$

in which the $M$-term is a rather complicated expressioh whose form need not concern us here. We have also identically

$$G^{ij}{}_{|j} = G^{ij}{}_{,j} - K^i = 0 \tag{19}$$

where the stroke indicates covariant differentiation and

$$K^i = -\Gamma^i_{aj}G^{aj} - \Gamma^j_{aj}G^{ia}, \tag{20}$$

and hence

$$\underset{N}{G}^{ij}{}_{,j} = \underset{N}{K}^i. \tag{21}$$

Hence by (18)

$$\underset{N}{M}^{ij}{}_{,j} = \underset{N}{K}^i. \tag{22}$$

Note that $\underset{N}{K}^i$ is a linear homogeneous function of

$$\underset{2}{G}^{ij}, \underset{3}{G}^{ij}, \quad \ldots, \quad \underset{N-2}{G}^{ij}. \tag{23}$$

Expanding the energy tensor in powers of $k$,

$$T^{ij} = \underset{2}{T}^{ij} + \underset{3}{T}^{ij} + \ldots, \tag{24}$$

we see that the field equations (4) are equivalent to

$$\underset{N}{G}^{ij} = -\varkappa \underset{N}{T}^{ij} \quad (N = 2, 3, \ldots). \tag{25}$$

Our problem then is to construct a suitable set of functions

$$\underset{N}{g}_{ij}, \quad \underset{N}{T}_{ij} \quad (N = 2, 3, \ldots) \tag{26}$$

to satisfy these equations and the requirements of signature and of positive density in the body and vanishing $T^{ij}$ outside it. There is also a further requirement on the boundary $B$ of the body, *viz.*

$$\underset{N}{T}^{ij} n_j = 0 \quad (N = 2, 3, \ldots) \tag{27}$$

where $n_j$ is the unit covariant normal to $B$.

The sequence of operations employed may be shown as follows:

$$\underset{2}{T} \to \underset{2}{g} \to \underset{3}{T} \to \underset{3}{g} \to \underset{4}{T} \to \underset{4}{g} \to \ldots \tag{28}$$

The arrows indicate steps which are partly deductive and partly guesswork. Guesswork is quite legitimate, because we are not engaged in *solving* some definite mathematical problem, but rather in *constructing* a model with certain desirable features. I have now to tell how the steps are carried out.

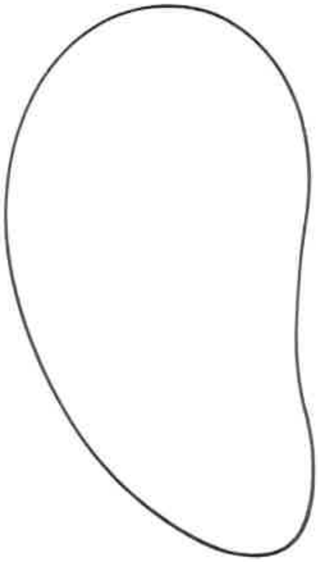

FIG. 1. Body at rest in equilibrium under its own attraction.

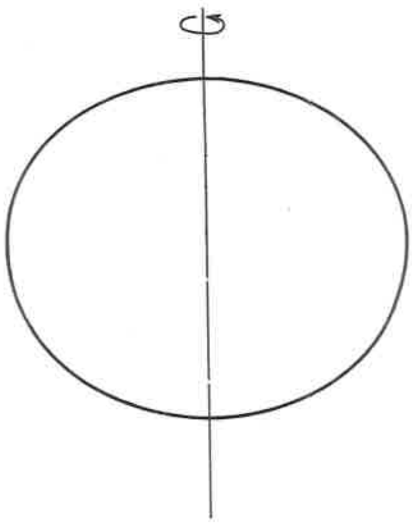

FIG. 2. Body of revolution spinning about its axis of symmetry.

Think first in a purely Newtonian way. Figure 1 shows a body of any shape in equilibrium under its own gravitational attraction. Figure 2 shows a body of revolution spinning about its axis of symmetry under its own gravitational attraction. In each case let $I$ be the interior of the body, $B$ its surface, and $E$ the domain exterior to it; $B$ is a fixed surface with equation $f(x_1, x_2, x_3)=0$, so that the direction cosines $n_\alpha$ of its normal are proportional to $f_{,\alpha}$. No particular assumptions are made about the constitution of the matter in either case.

The construction of these Newtonian models is not altogether trivial. We can deal with them both in a single argument, putting velocity equal to zero for the body at rest. Then what we need is a set of functions of position, $\varrho, u_\alpha$ and $S_{\alpha\beta}$ to satisfy (8), together with boundary conditions

$$u_\beta n_\beta = 0, \qquad S_{\alpha\beta} n_\beta = 0 \text{ on } B. \tag{29}$$

We have however complete confidence that such functions exist, and we know that, once we have set down such functions, then we can obtain by the transformation (9) a single infinity of such functions. Accordingly, we have, as basis for the construction of a relativistic model, a set of functions $\underset{2}{\varrho}$, $\underset{1}{u_\alpha}$, $\underset{4}{S_{\alpha\beta}}$, $\underset{2}{V}$ satisfying partial differential equations and boundary conditions as follows:

$$
\begin{gathered}
(\underset{2}{\varrho}\underset{1}{u_\alpha}\underset{1}{u_\beta}-\underset{4}{S_{\alpha\beta}})_{,\beta}=\underset{2}{\varrho}\underset{2}{V}_{,\alpha} \qquad (\underset{2}{\varrho}\underset{1}{u_\beta})_{,\beta}=0 \text{ in } I,\\
\underset{1}{u_\beta}n_\beta=0, \qquad \underset{4}{S_{\alpha\beta}}n_\beta=0 \text{ on } B, \\
\underset{2}{\varrho}=0, \qquad \underset{4}{S_{\alpha\beta}}=0 \text{ in } E.
\end{gathered} \tag{30}
$$

All quantities are independent of time.

In approaching the construction of relativistic models based on those Newtonian ones, we are all familiar with Einstein's linear approximation and this is indeed the first step in the construction. However, we are not to be satisfied with a first step or with a second step—we seek a procedure which can be used for any number of steps. There are two delicate questions. These involve the convergence of certain infinite integrals and the existence of certain functions in a finite domain $I$. These questions are covered by the following theorem.

THEOREM: All space is divided into a finite connected domain $I$ and an infinite domain $E$, with a surface $B$ separating them. Suppose that in $E+I$ we are given functions $\underset{2}{g_{ij}}$, $\underset{3}{g_{ij}}$, ... $\underset{N-1}{g_{ij}}$, each independent of $x_4$, of class $C^1$, small of order $r^{-1}$ at infinity in $E$, and such that, for the metric tensor

$$g_{ij}^{(N-1)}=\delta_{ij}+\underset{2}{g_{ij}}+\underset{3}{g_{ij}}+\dots+\underset{N-1}{g_{ij}}, \tag{31}$$

we have

$$\underset{2}{G^{ij}}=\underset{3}{G^{ij}}=\dots=\underset{N-1}{G^{ij}}=0 \quad \text{in} \quad E. \tag{32}$$

Then $\underset{N}{g_{ij}}$ exists, independent of $x_4$, of class $C^1$, small of order $r^{-1}$ at infinity in $E$, and such that, for the metric tensor

$$g_{ij}^{(N)}=\delta_{ij}+\underset{2}{g_{ij}}+\underset{3}{g_{ij}}+\dots+\underset{N}{g_{ij}}, \tag{33}$$

we have

$$\underset{N}{G^{ij}}=0 \text{ in } E. \tag{34}$$

Before proving this theorem, it is necessary to clarify a possible ambiguity in notation. By definition, we understand $\underset{N}{G^{ij}}$ to mean the term containing $k^N$ in the expansion of $G^{ij}$ for any metric of the form (10); accordingly this symbol has different meanings for the metrics (31) and (33).

Having committed ourselves in (34) to the metric (33), let us write $\underset{N}{\tilde{G}^{ij}}$ for the metric (31). Then we have as in (18)

$$\underset{N}{G^{ij}} = L_{ij}(\underset{N}{g}) + M^{ij}(\underset{N}{g}, \underset{2}{g}, \underset{3}{g}, \ldots, \underset{N-2}{g}), \tag{35}$$

$$\underset{N}{\tilde{G}^{ij}} = M^{ij}(\underset{N}{g}, \underset{2}{g}, \underset{3}{g}, \ldots, \underset{N-2}{g}), \tag{36}$$

the $M$-symbol having the same meaning in both cases. From (36) we extract an important fact. It is well known that for a $C^1$ metric, $G^{ij}n_j$ is continuous across any 3-space with unit normal $n_j$; then

$$\underset{N}{M^{ij}}n_j \text{ is continuous across } B, \tag{37}$$

where $n_j = (n_1, n_2, n_3, 0)$ is the unit (Euclidean) normal to $B$.

To prove the stated theorem, we now define $T$-symbols by

$$\underset{2}{G^{ij}} = -\varkappa \underset{2}{T^{ij}}, \quad \underset{3}{G^{ij}} = -\varkappa \underset{3}{T^{ij}}, \; \ldots, \underset{N-1}{G^{ij}} = -\varkappa \underset{N-1}{T^{ij}}, \tag{38}$$

and we seek to find $\underset{N}{T^{ij}}$ to satisfy

$$\begin{aligned} \underset{N}{T^{ij}}{}_{,4} &= 0 \text{ in } E+I, \\ \underset{N}{T^{ij}}{}_{,j} + \varkappa^{-1}\underset{N}{K^{i}} &= 0 \text{ in } I, \\ \underset{N}{T^{ij}}n_j &= 0 \text{ on } B, \\ \underset{N}{T^{ij}} &= 0 \text{ in } E. \end{aligned} \tag{39}$$

Here we are faced with the problem of solving 3+1 partial differential equations in $I$, with boundary conditions on $B$:

$$\underset{N}{T^{\alpha\beta}}{}_{,\beta} + \varkappa^{-1}\underset{N}{K^{\alpha}} = 0 \text{ in } I, \quad \underset{N}{T^{\alpha\beta}}n_\beta = 0 \text{ on } B; \tag{40}$$

$$\underset{N}{T^{4\beta}}{}_{,\beta} + \varkappa^{-1}\underset{N}{K^{4}} = 0 \text{ in } I, \quad T^{4\beta}n_\beta = 0 \text{ on } B. \tag{41}$$

But this is nothing but a Newtonian problem concerning a stress-field and a vector field. Necessary and sufficient conditions for the existence of solutions are

$$\int_I \underset{N}{K^{\alpha}}\xi_\alpha d_3v = 0, \quad \int_I \underset{N}{K^{4}} d_3v = 0 \tag{42}$$

where $\xi_\alpha$ is an arbitrary Euclidean Killing vector, so that

$$\xi_{\alpha,\beta} + \xi_{\beta,\alpha} = 0. \tag{43}$$

The integrals in (42) are dealt with by changing them to integrals over $B$ and then to integrals over $B_\infty$, the infinite sphere in $E$. Here the topology of $I$ is involved. The method works only if $I$ is connected. If it consisted of several disconnected parts, we would need to satisfy conditions of the form (42) for each of the parts, and we would not be able to convert each of these integrals to an integral over $B_\infty$. The existence of holes inside $I$ presents no difficulty.

The proof of the first of (42) is as follows:

$$\begin{aligned}\int_I \underset{N}{K^\alpha}\xi_\alpha d_3v &= \int_I \underset{N}{M^{\alpha\beta}_{,\beta}}\xi_\alpha d_3v \\ &= \int_B \underset{N}{M^{\alpha\beta}}\xi_\alpha n_\beta dB \\ &= \int_{B_\infty} \underset{N}{M^{\alpha\beta}}\xi_\alpha n_\beta dB_\infty. \end{aligned} \tag{44}$$

Here we have made use of (22), (23) and (32). We have still to show that the last integral in (44) vanishes, and that demands consideration of the behaviour of the $M$-term at great distance $r$. This requires examination of the structure of the $M$-term, and details will not be given here. It suffices to say that each $g$ involved in $M$ is of order $r^{-1}$, and differentiation with respect to $x_1$, $x_2$ or $x_3$ reduces it to $r^{-2}$, while differentiation with respect to $x_4$ destroys it. It results that

$$\underset{N}{M^{ij}} = O(r^{-4}) \tag{45}$$

and hence the last integral in (44) vanishes. The second integral in (42) is treated similarly, and we conclude that (42) is true.

To complete the proof of the stated theorem, we now define $\underset{N}{g^{ij}}$ by the formula

$$\underset{N}{g^*_{ij}}(x) = 4\int_{E+I} [\underset{N}{T^{ij}}(y) + \varkappa^{-1}\underset{N}{M^{ij}}(y)]\frac{d_3y}{|x-y|} \tag{46}$$

in an obvious notation. By virtue of (45) this integral converges and has moreover a value of order $r^{-1}$ for large $r=|x|$. It is easy to see that the coordinate condition

$$g^*_{ij,j} = 0 \tag{47}$$

is satisfied, and hence by (17) we get, on applying the Laplace operator to (46),

$$L_{ij}(\underset{N}{g}) = -\varkappa\underset{N}{T^{ij}} - \underset{N}{M^{ij}}, \tag{48}$$

or by (35)

$$\underset{N}{G^{ij}} = -\varkappa\underset{N}{T^{ij}}, \tag{49}$$

which vanishes in $E$. Since (46) is a potential integral, $\underset{N}{g}^{ij}$ is of class $C^1$, and thus the proof of the stated theorem is complete.

To start the process (28), we make a rather natural choice

$$\underset{2}{T}^{\alpha\beta} = 0, \quad \underset{2}{T}^{\alpha 4} = 0, \quad \underset{2}{T}^{44} = -\underset{2}{\varrho}, \tag{50}$$

using the Newtonian $\underset{2}{\varrho}$ of (30), and obtain from (46)

$$\underset{2}{g}_{\alpha\beta} = 2\underset{2}{V}\delta_{\alpha\beta}, \quad \underset{2}{g}_{\alpha_4} = 0, \quad \underset{2}{g}_{44} = -2\underset{2}{V}, \tag{51}$$

where $\underset{2}{V}$ is the Newtonian potential as in (30). For the next step we satisfy (39) by choosing

$$\underset{3}{T}^{\alpha\beta} = 0, \quad \underset{3}{T}^{\alpha 4} = i\underset{21}{\varrho u_\alpha}, \quad \underset{3}{T}^{44} = 0, \tag{52}$$

and get

$$\begin{gathered}\underset{3}{g}_{\alpha\beta} = 0, \; \underset{3}{g}_{\alpha 4} = 4i\underset{3}{U}_\alpha, \quad \underset{3}{g}_{44} = 0, \\ \underset{3}{U}_{\alpha}(x) = \int_I \underset{2}{\varrho}(y)\underset{1}{u}(y)\frac{d_3 y}{|x-y|}\end{gathered} \tag{53}$$

So far we have merely recovered the formulae of Einstein and Lense-Thirring, except for a slight generalization. We now begin to overlap with work of Fock, although the approach is different. We need a solution o (39) for $N = 4$, and for this we require to calculate the $K$-term; we find

$$\varkappa^{-1}\underset{4}{K}^\alpha = -\underset{2}{\varrho}\underset{2}{V}_{,\alpha}, \quad \varkappa^{-1}\underset{4}{K}^4 = 0. \tag{54}$$

Thus we satisfy (39) by taking from the Newtonian formulae (30)

$$\underset{4}{T}^{\alpha\beta} = \underset{2}{\varrho}\underset{1}{u}_\alpha\underset{1}{u}_\beta - \underset{4}{S}_{\alpha\beta}, \quad \underset{4}{T}^{\alpha 4} = 0, \quad \underset{4}{T}^{44} \text{ arbitrary.} \tag{55}$$

We get $\underset{4}{g}_{ij}$ from (46), which yields a very complicated expression.

As for the next step, we find that (39) with $N = 5$ are satisfied by taking

$$\underset{5}{T}^{ij} = 0 \tag{56}$$

and then $\underset{5}{g}_{ij}$ is given by (46).

There is little point in carrying the calculations any further. They get complicated and we know from the theorem stated above that nothing can go wrong in the way of convergence of integrals. But no matter how far we go (there is no question here of taking an infinite number of steps and becoming involved in a different question of convergence) we shall not achieve a perfect vacuum in $E$. However we can get as near a vacuum as we like. For suppose we stop with the metric

$$g_{ij} = \delta_{ij} + \underset{2}{g}_{ij} + \underset{3}{g}_{ij} + \dots + \underset{N}{g}_{ij}, \tag{57}$$

and then calculate the Einstein tensor $G^{ij}$; in $E$ it will contain the factor $k^{N+1}$, and so will be very small if $N$ is large and $k$ small. We may then say that in $E$ there is only a very small residual energy tensor, as small as we please if we make $N$ large enough; or preferably, since $T^{ij}$ has physical dimensions, we may say that the dimensionless product $a^2T^{ij}$ is very small, $a$ being some typical length or time. The residual energy tensor falls off like $r^{-4}$ at large distances.

That I have dealt with so far is merely a restatement in improved form of the results obtained in DFS and FS. Since a near-vacuum which can be made as close to a vacuum as we like is not to be distinguished physically from the perfect vacuum (which in fact does not exist in nature), I think we are to regard the problem of the stationary field as solved in the sense that we have a scheme for constructing models of sufficient variety. Indeed the variety is perhaps more than one might at first desire, because the solutions of (39) have a considerable degree of arbitrariness corresponding to self stress and to a vector field of vanishing divergence and vanishing normal component. Moreover (39) does not actually contain $\underset{N}{T^{44}}$ and so at each stage there enters a further indeterminacy as indicated in (55). This does not mean that the question of models of stationary fields is closed. In the above method there is a certain clumsiness about some of the formulae, so much so that the calculation of the Schwarzschild field by this method proves very tedious (cf. Florides and Synge, 1961). There may be some neater way of handling stationary fields.

However, the most interesting problems are always the unsolved ones, and so one naturally wonders whether the above method can be used for non-stationary cases, such as the spinning rod or the two-body problem. To see how we stand in the matter, we turn back to the theorem stated earlier and ask whether the stationary condition (symbolically, $\partial/\partial x_4=0$) can be removed. It cannot be removed for the following reason. If we differentiate a term of order $r^{-1}$ with respect to a space coordinate, we get a term of order $r^{-2}$, and this fact was essential in the proof of the theorem. But if we differentiate such a term with respect to $x_4$, its order remains unchanged. The divergence of integrals of the type (46) becomes a real danger. In fact, the method which has proved successful in stationary cases fails in non-stationary cases.

Does the method merely require some ingenious modifications in order to cope with non-stationary cases? Or can it be that we are looking for something that does not exist?

Let me try to put the question in proper perspective as I understand it. There is vast body of astronomical observations on the basis of which we have a pretty good idea how celestial bodies move, and within a pretty high degree of accuracy we know that these observations agree with Newtonian dynamics, so much so that most astronomers are satisfied with Newtonian models. If the system in question contains only two bodies, say for simplicity two spheres of equal masses, then an astronomer would probably accept a Newtonian model of this system in which the centres of the spheres rotate for all time on a common circle. The relativist cannot accept Newtonian theory and seeks to construct a relativistic model of such a system. Since he knows that Newtonian theory gives very good practical results, he naturally tries to construct a relativistic model which differs very little from the Newtonian one. He takes two tubes in space-time to represent the two bodies and writes down, for the exterior and interior domains,

$$G_{ij} = 0 \text{ in } E, \tag{58}$$

$$G_{ij} = -\varkappa T_{ij} \text{ in } I. \tag{59}$$

Einstein himself attached more importance to the vacuum equations (58) than to (59), and in this he has been followed by most relativists. According to that point of view, the two-body problem consists in finding solutions of (58) in all space-time except on two singular world-lines, with suitable conditions at infinity. Any such model would of course be of interest, but it could not be regarded as final, for we know from Newtonian mechanics that the finite size of bodies does play a part, in tidal effects and in other ways. Sooner or later, we must include the equations (59) in order to explore the interior field.

I want to throw out a suggestion that perhaps in the two-body problem there exists no suitable model with (58) satisfied rigorously outside the bodies. If such a system does in fact radiate energy, as is commonly supposed, perhaps that energy appears as a non-vanishing $G_{ij}$. If this sceptical view is acceptable, then we disregard (58) and deal with (59) alone, seeking a model in which $T_{ij}$ has a suitable form inside the bodies and is very small outside them.

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# QUANTIZATION OF THE GRAVITATIONAL FIELD*

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PRESENT quantum theories of the gravitational field generally work in "flat space". The original attempt at quantization was made by Gupta (1952) and carried out by him to first order. He started with the Lagrangian of the classical theory and applied the normal methods of quantization to it, treating the $g_{\mu\nu}$ as ordinary variables which have no connection with the metric. Owing to the ambiguities in the ordering of the factors, such a programme cannot be carried through exactly. However, it may be that one can write down a Lagrangian to any order in the gravitational constant and obtain consistent equations of motion up to that order.

Quantization in flat space can only be regarded as a provisional solution of the problem for several reasons. One would like to be able to formulate the equations of a theory exactly, even though approximations have to be made in their solution. This cannot be done or, at any rate, has not yet been done, in the flat space gravitation theory. Also, one has to introduce an indefinite metric and unphysical states, just as in quantizing electrodynamics with the Lorentz gauge. Both these disadvantages would be overcome if Arnowitt, Deser and Misner (1960) succeed in their programme of finding canonical variables for the quantum theory. At the moment a solution does not appear to be in sight, and in any case it would yield a theory which completely lacked manifest Lorentz covariance.

But the main objection to both these approaches lies surely in the physical sacrifices they make by going to flat space. The variables specifying the coordinates are numbers without physical significance which can be chosen in an infinite variety of ways. On the other hand distances in space-time, which are physically significant entities, are related to the coordinates and the field variables in a manner which has not been elucidated when the metric is quantized. It may be possible to add to the theory a prescription for interpreting its results physically. If it could then be shown that the physical predictions of the theory were independent of the coordinate conditions used, and that they tended to the predictions of unquantized general relativity in the classical limit, we would have a satisfactory theory. Pro-

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gress in this direction has recently been made by Thirring (1959, 1961) who, however, considered only unquantized $g$'s.

An ingeneous attempt to quantize the gravitational field has recently been made by DeWitt (1961a) who avoids going to flat space. He examines a gravitational field in interaction with a stiff elastic medium and a network of clocks, which are essential to his method in order to identify points in curved space-time. Our aim here will be to avoid the introduction of such a system, which is absent in most practical cases.

It has been realized for some time, mainly through the work of Bergmann and his collaborators, that a quantized theory of gravitation should deal only with observable quantities, in other words, with quantities independent of the coordinate system (Bergmann and Goldberg, 1955). The main problem is to find such quantities, and we shall suggest a method for doing so in what follows. It will be easiest first to examine quantum electrodynamics, with which we are much more familiar and where an analogous problem exists. The solution of the problem in that case will give us insight into the gravitational case.

The conventional methods of quantizing electrodynamics suffer either from lack of manifest covariance or from the use of unphysical states and an indefinite metric. As has been pointed out by Bergmann and Goldberg (1955, also Goldberg 1958) a satisfactory theory would have to work with gauge independent quantities only. Thus one should not introduce electromagnetic potentials at all, but should work entirely in terms of the fields. For an electromagnetic field interacting with charged particles, the particle operators $\varphi$ are also not gauge independent, as they undergo a phase transformation associated with a change of gauge. However, it has been realized for some time that the variables

$$\Phi(x,P) = \varphi(x) \exp\left\{-ie\int_P^x dz_\mu A_\mu(z)\right\} \tag{1}$$

remain unchanged under a gauge transformation. (To avoid ambiguities in ordering, we assume that the path of integration is space-like or has at most an infinitesimal time-like part). The integration in (1) will depend on the path and, in fact, it is easy to show that, if two paths are identical except for a small area $\sigma_{\mu\nu}$ between them at the point $z$,

$$\delta_z\Phi(x,P) = -ieF_{\mu\nu}(z)\Phi(x,P)\sigma_{\mu\nu}, \tag{2}$$

$\delta$ being the difference between the $\Phi$'s defined according to the two paths. It follows from (2) that, although the number of operators $\Phi$ is equal to the number of paths, the number of independent $\Phi$'s is (speaking roughly) equal to the number of points.

The proposal is now to work entirely in terms of the electromagnetic field variables and the variables $\Phi(x, P)$. Equation (2) is postulated as the fundamental equation giving the dependence of $\Phi$ on the path. For consistency one has to postulate the integrability conditions

$$\varepsilon_{\varkappa\lambda\mu\nu}\partial_\lambda F_{\mu\nu} = 0, \tag{3}$$

which are just the homogeneous Maxwell equations. One can then obtain a theory of quantum electrodynamics which is manifestly covariant and which does not introduce unphysical states and an indefinite metric. The potentials can be introduced as a mathematical aid for calculation, and current computations in the Coulomb or Lorentz gauges can be justified on our formalism.

The use of path-dependent variables defined by (1) for quantizing electrodynamics was suggested independently to the present author by DeWitt (1961) and by Pandres (1961) and both authors also mention the possibility of using similar variables for quantizing gravitation.

It may be objected that the use of path-dependent variables is a complication at least as serious as those of current quantization schemes, if not more so. We feel, however, that this path dependence is a fundamental physical property, unlike, for instance, the indefinite metric, which is completely unphysical. It is ultimately connected with the arbitrariness in the choice of phase factors associated with the operators of charged fields. One can choose the phase factors arbitrarily at one point. Once this has been done, however, the phase factors at a neighbouring point will be fixed. If one chooses the wrong phase factor here, one has to add extra terms to the equations of motion. Such terms are unphysical and can be removed by a gauge transformation, which re-establishes the correct choice of phase at the second point. In order to choose the phase factors at a field point a finite distance away from the reference point one cannot proceed directly, but one must construct a path joining the two points and go from point to point along the path. If there is an electromagnetic field present the result will depend on the path chosen. One thereby obtains the path-dependent operators of our theory.

The physical nature of the path dependence under discussion is exhibited in the Aharonov-Bohm experiment. According to the authors, the experiment indicates that electromagnetic potentials in quantum theory do have some physical significance, in that the line integral of the potential round a closed curve is measurable. Nevertheless, no local measurement would reveal the existence of the potentials; a type of non-locality appears to be indicated. In our way of looking at the problem, the effect exhibited by the experiment is fundamental, and the remaining unphysical effects of the potentials have been eliminated. The theory has no *arbitrary* non-local character.

The electromagnetic field viewed in this light bears a striking analogy to the Riemann tensor of gravitational theory. The coordinate system instead of the phase is now the arbitrary quantity. Once a local coordinate system has been specified at one point, we must obtain it at a neighbouring point by parallel displacement, otherwise there will be extra terms in the equations of motion. To obtain the local coordinate system at a distant point one cannot proceed directly, but one must take the local coordinate system from point to point along a path by parallel displacement. If the Riemann tensor is non-zero, the final result will depend on the path chosen. The electromagnetic field is thus related to the arbitrariness of the phase factor in exactly the same way as the gravitational field is related to the arbitrariness in the coordinate system.

It is now evident how one can construct variables for the gravitational field which do not depend on the coordinate system. Instead of defining a point in space-time by four numbers, which we are able to do only with respect to a coordinate system, we must give a prescription for constructing a path from the initial reference point to the field point. The prescription is always referred to a local coordinate system which is being moved along the path by parallel displacement. For instance, a specimen prescription may run as follows: Start at a reference point $P_1$, at which a local coordinate system is fixed arbitrarily. Now move a distance $d_1$ in the $x$-direction to a point $P_2$, and take the coordinate system along by parallel displacement. Next move a distance $d_2$ in the $y$-direction, defined with respect to the coordinate system which had been moved to $P_2$. Denote the point reached by $P_3$. Again take the coordinate system along by parallel displacement. Finally, move a further distances $d_3$ in the $x$-direction, defined with respect to the coordinate system which had been moved to $P_3$. Denote the point reached by $P_4$. Again we take the coordinate system along by parallel displacement, and perform the required measurement.

Thus in gravitation theory, as in electromagnetism, all variables are functions of paths rather than of points. Again we shall require an equation giving the dependence of the variable on the path, so that the number of independent variables is only equal to the number of points. The equation essentially tells us the curvature of the space. Before deriving it, let us rewrite the corresponding electromagnetic equation (2) in a more general form. As it is written at the moment, the right hand side would have the opposite sign if $\Phi$ were replaced by $\Phi^*$ and would be zero if $\Phi$ were replaced by $F_{\varkappa\lambda}$. We can however comprise all cases under the formula

$$\delta_z X(x, P) = ieF_{\mu\nu}(z)[J_0, X(x, P)]\,\sigma_{\mu\nu} \tag{4}$$

where $X$ is any operator $\Phi^*$, $\Phi$ or $F_{\varkappa\lambda}$, and $J$ is the total charge of the system.

To obtain the gravitational analogue of (4) we again examine two paths which are constructed by indentical prescriptions except that, at the point $z$, they differ by an infinitesimal area $\sigma_{\varkappa\lambda}$. According to a fundamental formula of Riemannian geometry, two vectors taken along the paths by parallel displacement whitch were identical before passing $z$ will differ after it by an amount

$$da_{\mu} = \frac{1}{2} R_{\mu\nu\kappa\lambda} a_{\nu} \sigma_{\kappa\lambda}. \tag{5}$$

As we always use local coordinate systems, we do not distinguish between upper and lower indices. Equation (5) indicates that all vectors are rotated by an amount $\frac{1}{2} R_{\mu\nu\kappa\lambda}\sigma_{\kappa\lambda}$ in the $\mu\nu$-plane. Similar formulae hold for tensors and spinors, so that we may write in general

$$\delta_z X(x, P_1) = - \frac{i}{4} R_{\mu\nu\kappa\lambda}(z, P_3)[J_{\mu\nu}(z, P_3), X(x, P_1)]\sigma_{\kappa\lambda}. \tag{6}$$

In this formula, $X$ is any operator ($R_{\alpha\beta\gamma\delta}(x, P_1)$, $\Phi(x, P_1)$ etc.), and $P_3$ is that portion of $P_1$ leading to $z$. The symbol $-i\,[J_{\mu\nu}(z, P_3), X(x,P_1)]$ is *defined* as the effect of an infinitesimal rotation of $X$ about $z$ in the $\mu\nu$-plane. For the moment we shall ignore the question of the ordering of factors.

Equation (6) is the gravitational analogue of (4). In this form of the theory, no mention is made of variables such as $g_{\mu\nu}$ or $\Gamma^{\lambda}_{\mu\nu}$ but the Riemann tensor is introduced directly as a quantum variable. The relation of this tensor to space curvature is given immediately by Eq. (6) so that besides being a dynamical variable, it has its usual geometric significance.

The type of path dependence encountered in gravitation theory is inherently more complicated than that encountered in electrodynamics. If two paths are defined by identical prescriptions except for a small area between them at one point, they will begin to diverge from one another after passing that point. The reason is that the local coordinate systems which are being moved along the paths become rotated with respect to one another when passing the area, and directions along the path are always defined with respect to these coordinate systems. As a result of this feature, it is not a simple problem to determine whether two paths lead to the same point. In classical theory, the question can in principle be answered if we have sufficient knowledge of the Riemann tensor for intermediate paths. For, in certain cases it follows from (6) that any measurement performed at the end of the paths will yield the same result. The physical criterion for two paths leading to the same point is thereby fulfilled. In the quantum theory, the elements of the Riemann tensor which one requires do not commute, so that the question does not have a meaning. It was always expected that the uncertainty principle would affect the precise location of points in a quantized gravitational theory, and this is the form the uncertainty takes.

As in electrodynamics, we require integrability conditions in order that (6) be consistent. The conditions turn out to be

$$\begin{aligned} R_{\mu\nu\varkappa\lambda}(x, P) &= R_{\varkappa\lambda\mu\nu}(x, P), \\ \varepsilon_{\mu\nu\varkappa\lambda} R_{\mu\nu\varkappa\lambda}(x, P) &= 0, \\ \varepsilon_{\varrho\sigma\varkappa\lambda}\partial_\sigma(z) R_{\mu\nu\varkappa\lambda}(x, P) &= 0. \end{aligned} \tag{7}$$

These equations are the familliar symmetry conditions and Bianchi identities which the Riemann tensor must satisfy. (It is also taken for granted that $R_{\mu\nu\varkappa\lambda}$ is anti-symmetric in $\mu$ and $\nu$ and in $\varkappa$ and $\lambda$ of course).

We now require equations of motion and commutation relations for our path-dependent variables. We have not succeeded in writing a Lagrangian for our theory in terms of path-dependent variables. The difficulty lies not in writing down a Lagrangian density, but in integrating it over volume. Our method was therefore to take the results over from the classical theory using the correspondence principle. The consistency of all equations of motion and commutation relations must then be verified directly. There is no difficulty in writing down equations of motion in the classical theory, as (for interacting gravitational and scalar fields) both the Klein-Gordon equation and the Einstein equations can immediately be written down in terms of path-dependent variables. The calculation of the space-like Poisson brackets is not quite so straightforward, but can also be done without too much difficulty, as our path-dependent variables can always be expressed in terms of the $g_{\mu\nu}$'s of the conventional theory. The Poisson brackets can then be found by a procedure given by DeWitt (1960). The calculations are greatly simplified by using Fermi coordinates along the paths.

One cannot simply quantize the classical gravitational theory of path-dependent variables as it stands, owing to ambiguities in ordering. Even in the fundamental path-dependence equation (6) we would not know how to order the two factors. Such an ambiguity does not exist in electrodynamics, where the two factors commute for space-like paths. We can overcome the difficulty by requiring for the specification of an operator, not only the spatial position of the path, but the ordering of all the elements of the path with respect to one another and to the operator itself. When dealing with a product of two operators we have to specify the orderings of the elements of both paths with respect to one another and to the operators. Only if all the elements of one path and its operator are ordered before (or after) all the elements of the other path and its operator will such a generalized product be a true product. The classical Poisson-bracket relations can then be adapted in a more or less obvious way to give space-like "commutation relations" for the difference between two operators or generalized products in which the ordering of the operators themselves, of an operator and a path element or of two path elements has been interchanged. Thus the number of independent variables is not increased by the present modification, and we have information equivalent to the space-like commutation relations of other field theories. Further, there is now no ambiguity in passing from the classical to the quantum theory. In Eq. (6), for instance, the operator $R_{\mu\nu\kappa\lambda}$ is to be ordered in the same position as the elements of the path surrounding the area $\sigma_{\kappa\lambda}$.

Another complication which occurs when writing down commutation relations is that one can only state them explicitly for space-like separated operators or path elements, i.e., for elements or operators defined either at the same point or at two points between which signals cannot propagate without exceeding the velocity of light. In a curved space we do not in general know whether two points are space-like separated without a knowledge of the Riemann tensor over a certain region, and in the quantized theory the question does not always have a meaning. It appears, however, that statement of the local commutation relations in a Lorentz-invariant theory with given equations of motion is sufficient to define the theory. One cannot, of course, prove such a theorem since, at present, nothing is known with certainty about the existence and uniqueness of solutions of quantum field theory equations. The assertion can easily be proved for free fields, and it should not be difficult to extend it to any order of perturbation theory for interacting fields. We shall, therefore, assume that it is sufficient to give the local commutation relations between operators and path elements in our theory, so that the difficulty regarding the lack of knowledge of space-like separation is not relevant.

One can thus obtain local commutation relations and equations of motion for our variables which, together with the path-dependence equation, provides us with a theory of the quantized gravitational field. The theory can be expanded in a perturbation series and, in first order, the results are equivalent to those of the Gupta theory, though the quantities in that theory are now regarded as auxiliary quantities for calculating coordinate independent functions. It is not known whether our theory would give results in higher order equivalent to the prescription of Feynman for calculating the $S$-matrix, nor have we investigated the problem of renormalization.

Though the theory of gravitation as formulated here may appear complicated, we should like to stress that it does not contain more arbitrary complications than classical gravitation theory. Once one postulates the absence of an overall inertial frame one is forced to quantize the theory in terms of coordinate independent quantities, of which our path-dependent variables appear to be an obvious choice. To avoid ambiguities in ordering and, in fact, for consistency, one must also specify the ordering of the path elements and the operator. We, therefore, feel that the ideas outlined here provide us with a natural method of quantizing a theory of curved space in which the curvature itself is a dynamical variable.

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## DISCUSSION

L. H. THOMAS:

I want to ask whether there is not a difficulty in a formulation like this from the Poisson bracket form: if you take the quantum-mechanical or the classical Poisson bracket of two functions giving operators then this gives you a function giving their commutator; but here you have a linear combination of operators given by Poisson brackets with variable coefficients, and in general this will not be in Hamiltonian form; it will not be given by the Poisson bracket with anything, unless you introduce extra variables.

S. MANDELSTAM:

I'm not quite sure if I understand the question. Anyhow, the position is, in the classical theory, that one can construct Poisson bracket between two variables which are independent of the coordinate system. One can actually take something like coordinate conditions in order to work out the Poisson brackets, but it's not difficult to prove that the result one gets would be independent of the coordinate system. So I think that the calculation of the Poisson bracket between two variables, at any rate in the classical theory, is something which is quite unambiguous, provided the variables are independent of the coordinate system. I don't know what Prof. DeWitt would say to that.

B. S. DEWITT:

Again, I'm not quite sure what is meant by adding extra variables. You certainly don't need to add specified canonical variables in order to calculate Poisson brackets.

L. H. THOMAS:

No, since the question was not clear, perhaps I should repeat it. If you have infinitesimal operators given by functions which give you infinitesimal changes as Poisson brackets of the functions and the operands, then the commutators of the operators will be got from the Poisson brackets of the functions; but if you have a linear combination of such operators, namely rotation operators or in fact any from the components of the Riemann tensor, the whole thing is not necessarily of the same form and in general cannot be of the same form. There will be difficulties in expressing the whole thing without using operators not in a canonical form or introducing new variables.

A linear combination of Poisson brackets with functions of the variables as coefficients is not immediately a Poisson bracket and usually cannot be put in that form without introducing further variables.

S. MANDELSTAM:

I'm coming to see more and more that I've been using a rather bad notation. I've used the expression $[J_{\mu\nu}{}^{(z)}, X]$ as a definition for the process of rotating the path by the amount $J_{\mu\nu}$ in the $\mu\nu$-plane; and I chose the notation by analogy with ordinary quantum mechanics. I'm beginning to think I should not have used it in this quantized gravitational theory, because I'm not thinking of it as a Poisson bracket between an angular momentum that we know and another operator that we also know; that's true in other theories, but it doesn't necessarily seem to be true in the gravitational field. I consider this whole expression as a unit which is defined by the effect of taking an infinitesimal rotation in the $\mu\nu$-plane at the point $z$; not as a Poisson bracket.

P. G. BERGMANN:

I first would like to take this opportunity to offer an apology of my own to somebody who probably doesn't expect it, and that is Prof. Rosenfeld. To the best of my knowledge, the first comprehensive attempt at quantization goes back to two papers by Rosenfeld, 1930 and 1932. The apology is due because I was totally unaware of these papers until several years after I had begun to work on this problem myself, and this meeting gives me the chance to state that as far as I know Prof. Rosenfeld was the first one to recognize the physical interest of this problem, and to make a remarkably comprehensive attempt at quantization. These papers, incidentally, work with tetrads, although probably not exactly the same tetrads that are being used in attempts that we shall hear about later in this conference. There was a paper in the *Annalen der Physik*, 1930 and one in the *Institut Henri Poincaré* about two years later, 1932.

As far as the proposal by Dr. Mandelstam is concerned, I think it is fairly unanimously the opinion, shared by you, that quantization should be attempted, as far as possible, in terms of intrinsic variables; though there may be same limitations to this program. But if you accept this objective, then I think the question may be phrased in terms of usefulness or practicability: whether a program of formulating intrinsic quantities by means of paths is likely to lead to results more convincing, or at least as convincing, as other attempts. I think that probably much more work has to be done to decide that question. The remarks that I want to make are not to say that this is either very wonderful or no good at all; but they are simply preliminary comments.

One problem that I see is this: if you try to formulate a path you must do it in term of some prescriptions, if you do not wish to introduce more than a local coordinate system, or the term of a coordinate system, you must phrase your prescription, perhaps, in terms of geodesics; that is, you say, we proceed for such-and-such a distance along a geodesic, then we turn a corner through such an angle, and so forth. If you do that, I think the following problem immediately arises. You wish to consider two paths, both leading from your reference point to a fixed end point $x$. On the other hand, it is obvious that in a non-flat space a rectangle of paths is not closed. Therefore, in the absence of knowledge of the c-number metric, you cannot *a priori* state that two certain paths will lead to the same end point. You already have to presume knowledge of the Riemann-Christoffel tensor before you can get started with your system of paths. And there I see a serious difficulty.

S. Mandelstam:

I would think that is just a fact of life. I see the point you make, but I don't think it prevents us from applying the prescription to construct the paths, does it?

P. G. Bergmann:

It seems to me that it prevents you *a priori* from considering the set of paths that has one end point in common.

S. Mandelstam:

I agree fully with that but isn't it just a real physical consequence of the uncertainty principle, that we don't know whether the two paths lead to the same point or not; that in fact in the quantum theory the question doesn't necessarily have any definite meaning?

P. G. BERGMANN:

Well, you see, the reason that I'm raising this question is that you wish to compare with each other, for instance, the Riemann-Christoffel tensor, or rather the equivalent of the R-C tensor at the same point, arrived at by different paths.

S. MANDELSTAM:

I don't know what the same point means.

P. G. BERGMANN:

Neither do I, and that was my question. So the end points of different paths may be in an unknown relationship to each other.

B. S. DEWITT:

Prof. Mandelstam hasn't had time to go on and show the details of how you perform the perturbation calculations, but at least in the case of electrodynamics it is fairly straightforward; and I wanted to tell Prof. Mandelstam that I have attempted to follow out his prescription in detail to calculate, say, the photon propagator to the lowest radiation correction order. Because the $F_{\mu\nu}$ to begin with is path independent, you should get a result which is path independent. The outcome of the calculation was that in momentum space, you get a kind of a mixed theory, where you have the paths in configuration space and the calculation in momentum space; and by the time you have shifted the origin of momentum space in the usual way to evaluate the integrals you have lost the explicit path independence of the result. It has been my experience that it is simply necessary to *reimpose* the condition that the result be path independent in order to obtain the usual gauge-invariant result. Should this be regarded as a defect, or is there any way to improve this so that the theory is better than with the usual calculations; for example, one might try, instead of doing the calculation in momentum space, to do it in coordinate space; and there may be other methods.

S. MANDELSTAM:

The comments I make are based on work by Kenneth Johnson (M.I.T.). The problem consists of finding the photon propagator in second order of perturbation theory. As I understand it, what Johnson did was instead of considering the usual current $j_\mu = e\overline{\psi}(x)\gamma_\mu\psi(x)$ to take the expression $e\overline{\psi}(x')\gamma_\mu \exp\left(ie\int_x^{x'} A_\mu dx^\mu\right)\psi(x)$. He considers the $x'$ to be separated from the $x$, and then later on at the end of the calculation he will let $x'$ tend to $x$. However, once you separate $x'$ from $x$ the first expression becomes, gauge dependent, so he does what, in my formalism, would be transforming $\psi$ to $\Psi$. Then he does the calculation, and finally makes $x'$ tend to $x$. He claims that when he does so the offending term never appears, and the calculation is completely gauge-independent without the quadratic divergence all the time.

F. J. BELINFANTE:

Why do you write $X(x, p)$ and not $X(p)$, if you admit that you cannot tell whether or not paths $P_1$ and $P_2$ do or do not end up in the same point $x$?

S. MANDELSTAM:

Strictly speaking I should write it $X(p)$. The reason I don't is just for convenience. $x$ is the total displacement in the Fermi coordinates along the path; and the only reason I keep it in is for taking Poisson bracket relations. We are dealing with two points which are infinitesimally separated, in other words with two paths which are almost the same except at their end points; so that it's worth keeping the $x$'s.

But you're right; strictly speaking we shoudn't have the $x$'s. At any rate, of two operators have the same $x$, it does not imply that they are at the same point; it just means that the displacement in the Fermi coordinate system of the two paths is the same.

A. I. JANIS:

How do you specify the beginning point of your path?

S. MANDELSTAM:

That's a hard question. As a matter of fact, if you don't have a space that is asymptotically flat at infinity I don't know how to answer it. I'm considering the case of a space that is asymptotically flat at infinity, and I just take any point in the asymptotic region.

J. A. WHEELER:

The insights that come from this very beautiful talk lead me to try to take advantage of Prof. Mandelstam's being here to ask a more general question. Over a 20 year period what would you think of as being the questions that one would try to ask of quantized general relativity? Answering problems is very difficult, but stating problems is even more difficult, and that's why I ask this particular question. Of course, you think of setting up operators and commutation relations, but this is machinery. But I would welcome very much your thoughts on what sorts of *physical questions* you would apply the machinery to.

S. MANDELSTAM:

By "you" you mean people in general? Well, I think one thing would be to find a reasonable practical prescription for writing down diagrams in perturbation theory, which I haven't really looked at yet, but which Feynman is now working at very adeptly. I think such a programme should be related to some formalism operating in terms of quantized fields. Of course, questions of renormalizability will come in. If it's found that the theory is not renormalizable in perturbation theory, or even if it is renormalizable in perturbation theory, there is the very general question as to whether the sort of fuzziness in space associated with the quantized gravitational field would manage to eliminate the divergences if one tries to use the Lagrangian formalism for any field theory. I suppose you also think of these topological questions of what wormholes and other such objects would be like in a quantized gravitational field. It's really a question I can't answer.

S. MANDELSTAM:

I think the equations analogous to the divergence equation of quantum electrodynamics are the Bianchi identities and the other symmetry relations of the Riemann tensor. I should have said that they come out as consistency conditions, just as the homogenous Maxwell equations come out as consistency conditions in the electrodynamic case.

D. IVANENKO:

You have a very general formalism. I have not understood rightly just where have you so to say lost the torsion terms? Is your formalism adequate only for curved spaces or also for spaces endowed with torsion? For the case of absolute parallelism, for instance? Indeed, torsion seems to be important, e.g. it leads to non-linear supplement in Dirac equation.

S. MANDELSTAM:

I have not introduced torsion in path dependence equation where the only path dependence is caused by the rotations in going around a small area. There is no torsion. It may be possible to include another term into this path dependence equation which takes torsion into account, but I haven't been looking at this question.

C. W. MISNER:

I just wanted to make a few comments on how this method might compare with the work of ADM, especially as regards points, and the idea of elimination of coordinates. I think that the elimination of coordinates in this work is, in a sense, equivalent to the imposition of a coordinate condition that many other workers have tried to use. That is, you haven't said in complete detail how you specify a point on a path, but it is pretty clear that the easiest thing to do is to move successively distances $x^0$, $x^1$, $x^2$, $x^3$ along the axes of a parallel-propagated frame and thus to lay out a coordinate system. I think that the most essential difference is the somewhat more local procedure used to identify points here than would be used in the attempt to define quantum canonical variables Arnowitt, Deser and I have made. There the coordinate conditions require one first to investigate essentially all of space-time, then begin laying down coordinates afterwards to say unambiguously where one is looking; but that procedure does allow one to talk about different path leading to the same point. The aspect of the points becoming fuzzy and not entirely unambiguous again does show up as a result of the quantum theory; namely the relationship between one set of physically measurable, meaningful coordinates which commute with certain of the gravitational field variables, and another such set, depends strongly on the gravitational field itself which is subject to quantum fluctuations. Therefore a point within one set of coordinate condition fluctuates relative to a fixed point in another set of coordinate conditions.

S. MANDELSTAM:

I think I agree with most of what Misner has said.

S. DESER:

I just want to add a comment to what Misner has said, and that is that the one big problem in quantization is the consistency of a given procedure. Having written down commutation relations one has to show that they match with the field equations into a consistent theory. The investigation of this problem is probably going to be a very very difficult task, in any approach to quantization.

A. PERES:

I have a technical question. If you consider two different deformations of the same path and then take them in reverse order you get something like a double commutator. I wonder whether that will satisfy something like the Jacobi identity. This is necessary for the consistency of the theory.

S. MANDELSTAM:

I think it's just a simple geometrical question. This is the effect of infinitesimal rotations; so the question is, if you perform two rotations does the result depend on the ordering. I don't see what can go wrong.

F. J. BELINFANTE:

In the case of electromagnetism the most commonly used gauge-invariant quantities are defined for a given Lorentz frame; in that case you simply take the radiation gauge. But radiation gauge can be obtained from this line definition if you use for the line first a straight leading to the field point in the surface $t$=const., and then average over all possible directions. Suppose in the gravitational case you also have this definition with this line; would there be some possibility of arriving at the type of variables that ADM have been looking for by some kind of an averaging process over paths?

S. MANDELSTAM:

Possibly; it's an interesting idea. I just dont' know.

R. SACHS:

May I ask whether you can get anything like a total energy, and a total angular momentum of the field in terms of this approach?

S. MANDELSTAM:

I don't think they could be the result of an integral of some quantity over space. I think there is a meaning to the total energy of the system; it is just the displacement operator in the time direction. But I don't think that one can express it in the form of an integral over the energy density, or anything like that.

R. SACHS:

Well, never mind how it can be expressed; would you have any way of writing down this quantity?

S. MANDELSTAM:

Yes, it's just the operator shifting all paths rigidly by a small distance in the time direction.

C. W. MISNER:

Including the initial point?

S. MANDELSTAM:

Yes, including the initial point.

B. S. DEWITT:

Can you find such an operator?

S. MANDELSTAM:

I don't think I can express such an operator in terms of the other operators, if that's what you mean.

P. G. BERGMANN:

If I may come back once more to the question of identifying points, in terms of the paths that lead to them: first of all, I would like to say that I think the application of the term 'uncertainty relation' may be a bit misleading, because the things that we are discussing here are already properties of the c-number theory. That is, it is already true in the c-number theory that, in the absence of knowledge of the metric field, by specifying several paths we may not know to which point we finally get. Now this would be quite acceptable, because I think that we do not know for certain that the concept of world point has any physically intrinsic significance. Nonetheless, in relativity one can identify points in terms of local properties without reference to paths and without reference to conventional coordinates, by some prescription based on intrinsic coordinates. This need not be the one that Komar and I gave on the basis of Géhéniau and Debever's work—there are other methods of doing it. The main point is not how to do it, but that it can be done at all. You can, therefore, specify paths leading from one point to another in terms of local intrinsic properties, rather than in terms of an arbitrary coordinate system; and thus it looks as if you might be depriving yourself of a degree of determinacy that is available in the theory.

S. MANDELSTAM:

I think that in the classical theory the problem, although it is an extremely complicated one mathematically, is one that can be carried out. In other words, if one is given two paths and one also knows the value of the Riemann tensor for sufficient intermediate paths, then I think one can in principle calculate, in the classical theory, whether or not these two paths lead to the same point. Suppose that we are working in terms of the conventional formalism, with the $g_{\mu\nu}$ and so on, and we take two paths which do lead to the same point. We then go to the path-dependent formalism, and express all variables in terms of these paths; and we also find the Riemann tensor for intermediate paths. Then, I think that one can show from the path dependence equation that if the Riemann tensors for all these intermediate paths have certain relations to one another, then any variable in the classical theory defined by means of one path is the same as that variable defined by means of the other and therefore the criteria for the two paths leading to the same point are satisfied. Now, the reason why I say it is connected with the uncertainty principle is that the components of the Riemann tensor that one requires in order to carry out this calculation do not necessarily commute with one another in the quantum theory.

C. W. MISNER:

Even if the entire construction lies on a space-like surface?

S. MANDELSTAM:

In the quantum theory you don't always know when the surface is space-like.

# CONSERVATION LAWS IN THE TETRAD THEORY OF GRAVITATION

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## 1. INTRODUCTION

During recent years the old question of the localizability of the energy in gravitational fields has been treated anew in a number of papers by different authors, and in a recent paper [1] it was finally shown that a satisfactory solution of this problem can be obtained within the framework of a tetrad description of the gravitational field. The conclusion of the foregoing investigations was that a satisfactory solution of the energy problem requires the existence ofan "energy-momentum complex" $T_i^k$ with the following properties:

I. $T_i^k$ is an affine tensor density which depends algebraically on the gravitational field variables and their derivatives and which satisfies the divergence relation

$$T_{i}{}^{k}{}_{,k} \equiv \frac{\partial T_i^k}{\partial x^k} = 0. \tag{1}$$

II. For a closed system where space-time is asymptotically flat at spatial infinity, and where we can use asymptotically rectilinear coordinates $(x^i) = (x,y,z,ct)$, the quantities

$$P_i = \frac{1}{c} \iiint\limits_{x^4 = \text{const.}} T_i^4 dx^1 dx^2 dx^3 \tag{2}$$

are constant in time, and they transform as the covariant components of a free vector under linear space-time transformations.

This property is essential for the interpretation of $P_i = \{P_i, -H/c\}$ as the total momentum and energy vector.

III. $T^k \equiv T_4{}^k$ transforms like a 4-vector density under the group of purely spatial transformations

$$\bar{x}^i = f^i(x^\alpha), \qquad \bar{x}^4 = x^4. \tag{3}$$

The last mentioned property is necessary in order to make the energy content of a finite volume of space $V$, i.e.

$$H_V = -\iiint_V T_4^4 dx^1 dx^2 dx^3, \tag{4}$$

independent of the spatial coordinates used in the evaluation of the integral (4). Thus, III is the condition of localizability of the energy in a gravitational field.

Further we must have

$$T_i^{\ k} = \mathcal{T}_i^{\ k} + \mathfrak{t}_i^{\ k} \tag{5}$$

where $\mathcal{T}_i^{\ k}$ is the matter tensor density, which appears as the source of the gravitational field in Einstein's field equations

$$\mathcal{G}_{ik} = -\varkappa \mathcal{T}_{ik} \tag{6}$$

and $\mathfrak{t}_i^{\ k}$ is the complex of the gravitational field, which vanishes in the limit of special relativity.

The expression for the energy-momentum complex given by Einstein more than forty years ago is of the form

$$\Theta_i^{\ k} = \mathcal{T}_i^{\ k} + \vartheta_i^{\ k} \tag{7}$$

$$\vartheta_i^{\ k} = \frac{1}{2\varkappa}\left(\frac{\delta \mathcal{L}_E}{\delta g^{lm}{}_{,k}}\, g^{lm}{}_{,i} - \delta_i^{\ k} \mathcal{L}_E\right) \tag{8}$$

where

$$\mathcal{L}_E = \mathcal{L}_E(g^{ik}, g^{ik}{}_{,l}) \tag{9}$$

is the usual Lagrangian of the gravitational field. It deviates from the scalar curvature density

$$\mathfrak{R} = \mathfrak{R}(g^{ik}, g^{ik}{}_{,l}, g^{ik}{}_{,l,m}) \tag{10}$$

by a divergence. Therefore the field equations may be derived from a variational principle in which either $\mathfrak{R}$ or $\mathcal{L}_E$ is taken as variant, i.e. we have

$$\mathcal{G}_{ik} = \frac{\delta \mathfrak{R}}{\delta g^{ik}} = \frac{\delta \mathcal{L}_E}{\delta g^{ik}} \tag{11}$$

where $\dfrac{\delta}{\delta g^{ik}}$ in (11) means variational derivatives of $\mathfrak{R}$ and $\mathcal{L}_E$, respectively. In contrast to $\mathfrak{R}$ which is a true scalar density $\mathcal{L}_E$ is only an affine scalar density. On the other hand, $\mathcal{L}_E$ has the advantage of being a homogeneous quadratic function of the first-order derivatives only.

Starting from the Bianchi identities which by (6) may be written

$$\mathcal{T}_i^{\ k}{}_{,k} + \tfrac{1}{2}\mathcal{T}_{lm}\, g^{lm}{}_{,i} = 0 \tag{12}$$

one is led to the Einstein expression $\Theta_i^k$ in the usual way by eliminating $\mathcal{T}_{lm}$ in the last term in (12) by means of the field equations and the last expression (11) for $\mathcal{G}_{lm}$. The equation (12) is then easily seen to take the form

$$\Theta_{i\ ,k}^{k} = 0 \tag{13}$$

where $\Theta_i^k$ is the complex defined by (7), (8). It is well-known that Einstein's expression satisfies the conditions I and II above, but *not* the localizability condition III. It is important here to note that the validity of II hinges on the property that $\mathcal{L}_E$ and therefore also $\vartheta_i^k$ is a homogeneous quadratic function of the first-order derivatives of the field variables $g^{ik}$. On the other hand, the circumstance that $\mathcal{L}_E$ is *not* a true scalar density is responsible for the non-localizability of the energy in Einstein's theory.

At the time of the last relativity conference at Royaumont I advocated a different expression $\check{\Theta}_i^k$ for the energy-momentum complex [2] which has the advantage of satisfying the localizability condition III. It can be obtained from (12) by the same procedure as that mentioned above but on using the first expression (11) instead of the second expression for $\mathcal{G}_{lm}$ [3]. In this way (12) takes the form

$$\check{\Theta}_{i\ ,k}^{k} = 0 \tag{14}$$

where

$$\check{\Theta}_i^k = \mathcal{T}_i^k + \check{\vartheta}_i^k \tag{15}$$

and $\check{\vartheta}_i^k$ is an algebraic function of $g^{ik}$ and its first- *and second-order* derivatives. The complex $\check{\Theta}_i^k$ was also obtained by applying the "method of infinitesimal coordinate transformations" to the scalar density $\mathfrak{R}$ [4], and the fact that $\mathfrak{R}$ as a true scalar density is essential for the validity of III. According to (14), $\check{\Theta}_i^k$ also satisfies I but, as was recognized in a later paper [5], the dependence of $\hat{\vartheta}_i^k$ on the second order derivatives of the metric tensor will in general invalidate II.

Thus, we are in the embarrassing situation that we have two different expressions for the energy-momentum complex, *viz.* $\Theta_i^k$ and $\check{\Theta}_i^k$, neither of which satisfies all the conditions I–III. The first satisfies the conditions I and II but not III, while the second expression satisfies I and III but in general not II. It can even be shown that if we do not allow higher order derivatives of the metric tensor then the second order to appear in our complex, the Einstein expression $\Theta_i^k$, is the only one satisfying I and II [5] and $\check{\Theta}_i^k$ is the only expression satisfying I and III [6]. As one of my students at the 1960 Summer School at Brandeis University said when confronted with this situation: "It looks as if nature wanted to tell us something". The later development [1], [9] seems to indicate that what nature wanted to tell is that the gravitational field is not a simple metric field but fundamentally a tetrad field. This means that space-time is not simply a Riemannian space but a space of the type considered first by Weitzenböck [7] which may be pictured as a Riemannian space with a built-in tetrad lattice.

As a result of the investigations in recent years we can state that a satisfactory solution of the energy problem is possible only if the gravitational equations are derivable from a variational principle where the Lagrangian density $\mathfrak{L}$ has the following properties:

a) $\mathfrak{L}$ depends algebraically on the gavitational field variables and their first-order derivatives and it is a homogeneous quadratic function of the latter quantities.

b) $\mathfrak{L}$ is a true scalar density under arbitrary space-time transformations. Now, if the components of the metric tensor are taken as the gravitational field variables such a function $\mathfrak{L}$ simply does not exist. However, as was shown in [1], the situation is quite different if one assumes that the true gravitational variables are the components of a tetrad field. This assumption is corroborated by the well-known fact that the influence of a gravitational field on a Fermion matter field is described by means of a tetrad field and not directly by the metric field.

## 2. THE TETRAD THEORY OF GRAVITATION

Let $h^i_{(a)}$, $h_{(a)i}$ denote the contravariant and covariant components, respectively, of a tetrad vector numerated by an index (a) running from 1 to 4, and let us assume that $h^i_{(4)}$ is time-like, the other three tetrad vectors being consequently space-like. With the notation

$$h^{(a)i} = \eta^{(ab)} h^i_{(b)}, \qquad h^i_{(a)} = \eta_{(ab)} h^{(b)i} \tag{1}$$

where $\eta_{(ab)} = \eta^{(ab)}$ is the constant $4\times 4$ diagonal matrix with diagonal elements $\{1, 1, 1, -1\}$, the connection between the tetrad field and the metric field $g_{ik}(x)$ is given by

$$h_{(a)i} h^{(a)}_k = g_{ik} \quad \text{or} \quad h^i_{(a)} h^{(a)k} = g^{ik} \quad \text{or} \quad h^i_{(a)} h^{(a)}_k = \delta^i_k. \tag{2}$$

For a given tetrad field the metric field is given by (2). However, for a given metric field $g_{ik}(x)$ the tetrad field is determined by (2) only up to an arbitrary Lorentz-rotation of the tetrads. For if $h_{(a)i}$ satisfies (2) with a given $g_{ik}(x)$, then

$$\lambda_{(a)i} = \Omega_{(a)}{}^{(b)}(x) h_{(b)i} \tag{3}$$

will also be a solution of (2) provided the functions $\Omega_{(a)}{}^{(b)}(x)$ at any point $(x)$ satisfy the usual orthogonality relations. Any function of the metric tensor and its derivatives may by (2) be written as a function of the tetrad functions and their derivatives and such functions will be invariant under arbitrary rotations of the type (3).

Now we assume that the gravitational field equations can be derived from a variational principle

$$\delta \int (\mathcal{L}+\mathcal{L}^{(m)})dx = 0 \tag{4}$$

where $\mathcal{L}$ is the gravitational Lagrangian with the properties a) and b) above and $\mathcal{L}^{(m)}$ is the matter Lagrangian which is a function of the tetrad field variables and the matter field variables and their first-order derivatives. Let us first consider the general consequences of this assumption.

The field equations following from (4) are

$$\frac{\delta \mathcal{L}}{\delta h_k^{(a)}} + \frac{\delta \mathcal{L}^{(m)}}{\delta h_k^{(a)}} = 0. \tag{5}$$

If we put

$$\frac{\delta \mathcal{L}^{(m)}}{\delta h_k^{(a)}} = -\mathcal{T}_{(a)}{}^k, \qquad \mathcal{T}_i{}^k = h_i^{(a)}\mathcal{T}_{(a)}{}^k \tag{6}$$

the equations (5) may be written

$$\frac{\delta \mathcal{L}}{\delta h_k^{(a)}} = \mathcal{T}_{(a)}{}^k \quad \text{or} \quad h_i^{(a)}\frac{\delta \mathcal{L}}{\delta h_k^{(a)}} = \mathcal{T}_i^k \tag{7}$$

$T_{ik} = \mathcal{T}_{ik}/\sqrt{-g}$ is the matter tensor which acts as source of the gravitational field in (7). In general $T_{ik} = T_{ki}$ is a symmetrical tensor. This is certainly the case if the matter is an electromagnetic field or an ordinary elastic body, but it also holds for a Fermion field provided the corresponding Lagrangian $\mathcal{L}^{(m)}$ is chosen so as to be invariant under arbitrary tetrad rotations (3) [8].

The condition b) that $\mathcal{L}$ is a true scalar density entails the identity (see f. inst. [4])

$$\left(h_i^{(a)}\frac{\delta \mathcal{L}}{\delta h_k^{(a)}}\right)_{,k} - \frac{\delta \mathcal{L}}{\delta h_l^{(a)}} h_{l,i}^{(a)} = 0 \tag{8}$$

which, by means of (7), leads to the conservative law

$$\mathcal{T}_{i}{}^{k}{}_{,k} - \mathcal{T}^l{}_{(a)} h_{l,i}^{(a)} = 0. \tag{9}$$

If we introduce the tensor

$$\gamma_{ikl} = h_{(a)i} h^{(a)}_{k;l} = -\gamma_{kil} \tag{10}$$

where the semicolon means covariant differentiation, (9) may be written as a tensor equation:

$$T_i{}^k{}_{;k} + T^{lm}\gamma_{lmi} = 0. \tag{11}$$

For a symetrical matter tensor, (11) reduces to the usual conservation law of the Einstein theory.

To obtain an expression for the energy-momentum complex we proceed in the same way as in section 1, i.e. we eliminate $\mathcal{T}^l_{(a)}$ in (9) by means of the field equations (7). In this way the equation (9) takes the form

$$T_i{}^k{}_{,k} = 0, \tag{12}$$

with

$$T_i^k = \mathfrak{T}_i^k + \mathfrak{t}_i^k$$

$$\mathfrak{t}_i^k = \frac{\partial \mathfrak{L}}{\partial h^{(a)}{}_{l,k}} h^{(a)}{}_{l,i} - \delta_i^k \mathfrak{L} \tag{13}$$

on the analogy of (1.7), (1.8). Like $\mathfrak{L}$ $\mathfrak{t}_i^k$ is also a homogeneous quadratic function of the first-order derivatives of the tetrad functions.

A further consequence of b) is the existence of a superpotential (see [4])

$$\mathfrak{U}_i^{kl} = -\frac{\partial \mathfrak{L}}{\partial h^{(a)}{}_{k,l}} h_i^{(a)} = \frac{\partial \mathfrak{L}}{\partial h^{(a)}{}_{l,k}} h_i^{(a)} = -\mathfrak{U}_i^{lk} \tag{14}$$

by which the complex $T_i^k$ is represented as

$$T_i^k = \mathfrak{U}_i^{kl}{}_{,l}. \tag{15}$$

From the properties a) and b) of $\mathfrak{L}$ it follows that $\mathfrak{U}_i^{kl}$ is a true tensor density and it is easy to verify that the complex given by (15) and (14) satisfies all the conditions I–III in the introduction.

If we multiply (14) by $h^i_{(a)}$ we get on account of (2)

$$\frac{\partial \mathfrak{L}}{\partial h^{(a)}{}_{k,l}} = -h^i_{(a)} \mathfrak{U}_i^{kl} = -\mathfrak{U}_{(a)}{}^{kl}, \tag{16}$$

so that the field equations (7) take the form

$$\frac{\partial \mathfrak{L}}{\partial h_k^{(a)}} + \mathfrak{U}_{(a)}{}^{kl}{}_{,l} = \mathfrak{T}_{(a)}{}^k. \tag{17}$$

Each term in this equation is a vector density. From the antisymmetry of $\mathfrak{U}_{(a)}{}^{kl}$ in $k$ and $l$ it follows that the vector density

$$\mathfrak{U}_{(a)}{}^k \equiv \mathfrak{T}_{(a)}{}^k - \frac{\partial \mathfrak{L}}{\partial h_k^{(a)}} = \mathfrak{U}_{(a)}{}^{kl}{}_{,l} \tag{18}$$

has a vanishing divergence, i.e.

$$\mathfrak{U}_{(a)}{}^k{}_{,k} = 0. \tag{19}$$

The tensor density

$$\mathfrak{U}_i^k \equiv h_i^{(a)} \mathfrak{U}_{(a)}{}^k = \mathfrak{T}_i^k - h_i^{(a)} \cdot \frac{\partial \mathfrak{L}}{\partial h_k^{(a)}} = h_i^{(a)} \mathfrak{U}_{(a)}{}^{kl}{}_{,l} \tag{20}$$

is closely connected with the energy-momentum complex $T_i^k$. In fact it is equal to the tensor part of $T_i^k$. On account of (15), (20) may be written

$$\mathfrak{U}_i^k = \mathfrak{U}_i^{kl}{}_{,l} - \mathfrak{U}_{(a)}{}^{kl} h_i^{(a)}{}_{,l} = T_i^k - \mathfrak{U}_m{}^{kl} h_{(a)}{}^{(m)} h^{(a)}{}_{i,l}.$$

Hence

$$T_i^k = \mathfrak{U}_i^k + \mathfrak{U}_m{}^{kl} \Delta^m{}_{il}, \tag{21}$$

with

$$\Delta^i{}_{kl} = h^i_{(a)} h^{(a)}{}_{k,l}. \tag{22}$$

Further, we get for the gravitational complex $\mathfrak{t}_i^k$, by comparison of the first equation (13), (21) and (20)

$$\mathfrak{t}_i^k = -h_i^{(a)} \frac{\partial \mathfrak{L}}{\partial h_k^{(a)}} + \mathfrak{U}_m{}^{kl} \Delta^m{}_{il}. \tag{23}$$

The last terms in (21) and (23) are not tensorial since $\Delta^i{}_{kl}$ is not a tensor.

In order to get an explicit expression for $\mathfrak{L}$ we started in [1] from the remark that the scalar curvature density $\mathfrak{R}$, when written in terms of the tetrad field functions, is of the form

$$\mathfrak{R} = \mathfrak{L}_1 + \mathfrak{h}^l{}_{,l}, \tag{24}$$

where the last term is the divergence of a vector density $\mathfrak{h}^l$ and the term $\mathfrak{L}_1$ has the properties a) and b) of section 1. The explicit expression for $\mathfrak{L}_1$ is

$$\mathfrak{L}_1 = |h|[\gamma_{rst}\gamma^{tsr} - \Phi_r\Phi^r], \tag{25}$$

where $h$ is the determinant

$$h = \det\{h_{(a)i}\} \tag{26}$$

with the absolute value

$$|h| = \sqrt{-g}, \tag{26$^1$}$$

and the vector $\varphi_r$ is defined by

$$\Phi_k = \gamma^i{}_{ki}. \tag{27}$$

On account of (2) and the first equation (1.11) we get from (24)

$$\frac{\delta \mathfrak{L}_1}{\delta h_k^{(a)}} = \frac{\delta \mathfrak{R}}{\delta h_k^{(a)}} = \frac{\delta \mathfrak{R}}{\delta g^{rs}} \frac{\partial g^{rs}}{\partial h_k^{(a)}} = -\mathfrak{G}_{rs}[h^r_{(a)}g^{sk} + h^s_{(a)}g^{rk}] = -2\mathfrak{G}^k_{(a)}. \tag{28}$$

Thus, choosing as Lagrangian

$$\mathfrak{L} = k_1 \mathfrak{L}_1 \tag{29}$$

with

$$k_1 = \frac{1}{2\varkappa} \tag{30}$$

the field equations (7) are seen to be identical with Einstein's field equations (1.6). Further, with $\mathfrak{L}$ given by (25), (29) the superpotential (14) becomes [1]

$$\mathfrak{U}_i{}^{kl} = \frac{|h|}{\varkappa}[\gamma^{kl}{}_i - \delta_i^k\Phi^l + \delta_i^l\Phi^k] \equiv \overset{(1)}{\mathfrak{U}}_i{}^{kl}. \tag{31}$$

The Lagrangian $\mathfrak{L}$ defined by (25), (29) has the essential property of being invariant under tetrad rotations (3) with *constant* coefficients $\Omega_{(a)}{}^{(b)}$ but it is *not* invariant under the full group of rotations (3). The same holds for the quantities $\mathfrak{U}_i{}^{kl}$ and $T_i{}^k$ given by (31) and (15). On the other hand, the field equations (7), which in this case are identical with Einstein's equations are invariant under the full group of rotations (3) and they will therefore not determine the tetrad field sufficiently accurately to give a unique expression for the energy distribution by (15) and (31). Thus, these field equations, which determine the metric only, have to be supplemented by six further equations which in [1] were given in the form

$$\varphi_{ik} = 0 \tag{32}$$

where $\varphi_{ik}$ is an antisymmetric tensor constructed from the tetrad field functions and their first and second order derivatives. The condition that the equations (32) must be generally covariant does not lead to a unique expression for the left-hand side. However, the further requirement that (32) together with Einstein's field equations and suitable boundary conditions should determine the tetrad field completely (apart from constant tetrad rotations) allowed us to arrive at a certain restricted class of expressions $\varphi_{ik}$ for which the equations (32) lead to unique expressions for the tetrad field in the two most important cases of a general weak field and of a "strong" static spherically symmetric field. Actually these are the only cases of practical importance.

In the case of a static spherically symmetric system it is convenient to use a system of isotropic coordinates in which

$$g_{ik} = g_{(ii)}{}^{(r)}\delta_{ik}, \quad r = \sqrt{\sum_{i} (x^i)^2}.$$

In this system the solution of our equations was found in [1] to be

$$h_i^{(a)} = \sqrt{|g_{aa}|}\,\delta_i^a \tag{33}$$

and the expression for $T_i^k$ following from this solution turns out to be identical with the Einstein expression $\Theta_i^k$. This identity of $T_i^k$ and $\Theta_i^k$ holds in all systems of coordinates which can be obtained from the isotropic system by a linear transformation. However, in all other coordinate systems $\Theta_i^k$ will be different from $T_i^k$ and in such systems $\Theta_4^4$ cannot be interpreted as the energy density since $\Theta_i^k$, in contrast to $T_i^k$, does not satisfy the condition III. In particular this holds in the system of harmonic coordinates which is obtained from the isotropic system by a (non-linear) transformation of the type (1.3) [1].

In the case of general weak fields, where

$$g_{ik} = \eta_{ik} + y_{ik}(x) \tag{34}$$

with $y_{ik} = y_{ki}$ small of the first order, we get in a system of harmonic coordinates the solution

$$h_{(a)i} = \eta_{ai} + \frac{1}{2} y_{ai} \tag{35}$$

and to the first order we find again $T_i^k = \Theta_i^k$. In fact both of these expressions are equal to the matter-tensor density $\mathcal{T}_i^k$ in this approximation, but the identity of $T_i^k$ and $\Theta_i^k$ holds in general only in the weak field approximation [1].

The theory developed in [1] is not quite satisfactory in so far as the field equations following from the Lagrangian (29) had to be supplemented by a set of extra equations (32) which were not even uniquely determined. Although the indeterminacy involved in the expression for $\varphi_{ik}$ was of no importance in the cases considered explicitly in [1] this situation cannot be regarded as quite satisfactory. As pointed out by Plebański the indeterminacy in $\varphi_{ik}$ is even somewhat larger than assumed in [1] where we only considered covariant functions of $\gamma_{ikl}$ and its derivatives. However, it is easily seen that the quantity

$$\eta^{iklm} = \frac{1}{h} \delta^{iklm} \tag{36}$$

where $\delta^{iklm}$ is the usual Levi-Cività symbol transforms as a tensor of rank 4 under arbitrary coordinate transformations. Further, the quantity

$$\mu \equiv \frac{\sqrt{-g}}{h} = \frac{|h|}{h} \tag{37}$$

is a constant pseudoscalar. This obviously widens the possibility of constructing covariant expressions $\varphi_{ik}$.

In order to limit the arbitrariness as much as possible it is natural to require that all the equations for the tetrad field should be derivable from a variational principle. Obviously this means that we have to look for another Lagrangian which, however, must contain the term (29) as an important part. Besides the properties a) and b) of section 1 the Lagrangian we are looking for must satisfy the following conditions:

c) $\mathcal{L}$ is invariant under constant Lorentz-rotations of the tetrads.

This means that the tetrad index (a) can appear as a dummy index only.

d) The variational equations together with suitable boundary conditions determine the tetrad field completely.

e) As regards the metric tensor the formalism must give the same result as Einstein's theory in the cases where this theory has been verified by experiments. In particular, the Newtonian theory of gravitation must follow in the limit of weak quasi-stationary fields.

The conditions a)-e) may serve as a guide in the search for a suitable Lagrangian.

This program was carried through by Pellegrini and Plebański in an interesting paper which is now in course of publication in the proceedings of the Danish Academy [9]. These authors showed that a suitable Lagrangian is given by

$$\mathcal{L} = k_1 \mathcal{L}_1 + k_2 \mathcal{L}_2 \tag{38}$$

with

$$\mathcal{L}_2 = |h| \eta^{rstu} \Phi_r \gamma_{stu} . \tag{39}$$

$k_1$ and $k_2$ are constants of which the first is again connected with Einstein's gravitational constant by (30).

If we put

$$-\frac{\delta \mathcal{L}_2}{\delta h_k^{(a)}} = \mathcal{F}_{(a)}{}^k, \quad \mathcal{F}_i{}^k = h_i{}^{(a)} \mathcal{F}_{(a)}{}^k = \sqrt{-g} F_i{}^k \tag{40}$$

the field equations (7) may, by (28) and (40), be written

$$G_{ik} + \frac{k_2}{2k_1} F_{ik} = -\varkappa T_{ik}. \tag{41}$$

Further, if we denote the even and odd parts of $F_{ik}$ by

$$F_{\langle ik \rangle} = \frac{1}{2} (F_{ik} + F_{ki}), \quad F_{[ik]} = \frac{1}{2} (F_{ik} - F_{ki}) \tag{42}$$

the 16 equations (41) fall into two sets of 10 and 6 equations, respectively:

$$G_{ik} + \frac{k_2}{\varkappa} F_{\langle ik \rangle} = -\varkappa T_{ik} \tag{43}$$

$$F_{[ik]} = 0. \tag{44}$$

Here we have made use of the symmetry of the tensors $G_{ik}$ and $T_{ik}$.

The six equations (44) are analogous with the supplementary equations (32) used in [1] and they are identical with these equations in the weak field approximation. Similarly the ten equations (43) become identical with Einstein's field equations in the limit of weak fields, [9]. Thus, the equations (41) or (43), (44) contain the Newtonian theory in the limit specified in e).

In general the field equations (41) contain an extra constant $k_2/\varkappa$ which has not hitherto been revealed in any gravitational phenomena inside our solar system. At first sight one would be inclined to believe that this requires $k_2$ to be small compared with $\varkappa$. However, this is not necessarily so. As remarked above the constant $k_2$ drops out in the weak field approximation and the solution of (41) is here the same as that found in [1], i.e. in harmonic coordinates the solution is given by (35). But also in the static spherically symmetric case the solution of (41) or (43), (44) is the same as in [1], i.e. the solution (33), for with $h_i^{(a)} = \sqrt{|g_{aa}|}\, \varepsilon_i^a$ it can be shown that the tensor $\mathcal{F}_{ik}$ defined by (40) and (39) vanishes, [9]. Now, the cases just mentioned are the only ones which are of any importance for the effects which so far have been accessible to experimental verification, and it is therefore not surprising that the terms containing the new constant $k_2$ could have escaped detection.

In this respect one might think that the situation is different when one considers cosmological problems. However, for a homogeneous isotropic model of the universe it is easily seen that the equations (41) lead to the same metric as the usual Einstein theory. For a homogeneous isotropic system it is always possible to introduce a system of coordinates for which the line-element has the form

$$\left.\begin{aligned} ds^2 &= G^2(t)A^2(r)(dx^{1^2}+dx^{2^2}+dx^{3^2})-dx^{4^2} \\ &= g_{(ii)}\delta_{ik}\, dx^i dx^k \\ A(r) &= \frac{1}{1+\lambda r^2}, \qquad r^2 = \sum_\iota (x^\iota)^2. \end{aligned}\right\} \tag{45}$$

Then, using the explicit expression for $\mathcal{F}_{ik}$ given by Pellegrini and Plebański it is easily seen that the tetrad field

$$h_i^{(a)} = \sqrt{|g_{aa}|}\,\delta_i^a \tag{46}$$

gives

$$\mathcal{F}_{ik} = 0 \tag{47}$$

so that (46) is a solution of (43), (44) if $G(t)$ is the function of the time variable corresponding to the usual Friedmann solution.

The superpotential (14) corresponding to the Lagrangian (38) is now

$$\mathfrak{U}_i^{\ kl} = \overset{(1)}{\mathfrak{U}}{}_i^{\ kl} + \overset{(2)}{\mathfrak{U}}{}_i^{\ kl}, \tag{48}$$

where $\overset{(1)}{\mathfrak{U}}{}_i^{\ kl}$ is given by (31) and

$$\begin{aligned} \overset{(2)}{\mathfrak{U}}{}_i^{\ kl} &= -\frac{\partial k_2 \mathcal{L}_2}{\partial h^{(a)}{}_{k,l}} h_i^{(a)} \\ &= k_2|h|\{(\delta_i^k \eta^{lrst} - \delta_i^l \eta^{krst})\gamma_{rst} + g_{ir}\Phi_s \eta^{rskl}\}. \end{aligned} \tag{49}$$

In the static spherically symmetric case one finds by means of (33)

$$\overset{(2)}{\mathfrak{U}}{}_i^{\ kl}{}_{,l} = 0 \tag{50}$$

so that the energy-momentum complex reduces to the complex

$$T_i^{\ k} = \overset{(1)}{\mathfrak{U}}{}_i^{\ kl}{}_{,l} \tag{51}$$

which was calculated in [1]. The same is true in the case of a weak field to the first order in the field variables. In higher-order approximations the expression for the energy-momentum complex given in [9] differs slightly from the expression derived in [1] by a term depending on the constant $k_2$.

In the preceding considerations it has been assumed that the matter tensor (6) is symmetric as is usually the case. In the paper by Pellegrini and Plebański [9] it is pointed out that the field equations (41) might be valid also if $\mathcal{T}_{ik}$ is not symmetric. As an example they consider a Fermion field with a Lagrangian which is not invariant under the full group of tetrad rotations (3) but only under rotations with constant $\Omega_{(a)}{}^{(b)}$. In this formalism which seems to be consistent and much simpler than the usual covariant theory of Fermion fields, $\mathcal{T}_{ik}$ is not symmetric. The antisymmetric part $\mathcal{T}_{[ik]}$, which is closely connected with the spin of the system, now appears as a source on the right-hand side of (44) in a similar way as the symmetric part $\mathcal{T}_{\langle ik\rangle}$, the energy-momentum stress tensor of the matter, is the source in the equations (43). It remains to be seen if this generalization of the formalism which implies an extra term in the "conservation law" (11) corresponds to anything realized in nature.

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## DISCUSSION

J. L. Anderson:

There are two comments I should like to make with regard to your requirements for a satisfactory energy-momentum complex in general relativity. While your two requirements appear to be quite natural from many points of view I would like to suggest that there are others for which one can advance strong arguments. I would like to suggest one which seems to me to be essential for any definition of a physical object and to ask if your complex satisfies it. The requirement that I would impose on any physical object is that it be represented mathematically by a corresponding geometrical object. It need not be a tensor or any other geometrical object with a linear homogeneous transformation law but I feel that it must be represented by some geometrical object before we can ascribe independent existence to it. Whether or not we have been aware of it, the whole of physical theory consists in assigning geometrical objects to physical entities and then finding relations between them. It was the realization that energy-density was not by itself a geometrical object that led Einstein to consider the metric tensor as the proper geometrical object to asociate with the gravitational field. Unless physical objects are representable by geometrical objects it is impossible to see how we can recognize them in different situations as described by different observers. I have checked that the Einstein energy-momentum complex is not a geometrical object. Is your complex a geometrical object?

The second comment I should like to make is that perhaps one should not feel too unhappy about not being able to define a local energy-momentum complex in all cases in general relativity. We have learned to live with such a situation in elementary particle physics where strangeness and isospin are conserved and therefore well-defined only for the strong interactions. It's just that strangeness and isospin haven't been around as long as energy and momentum and so we aren't disturbed when they loose their meaning in the weak interactions. It should be pointed out that an entirely analogous result obtains in the Yang-Mills theory, where it is impossible to define a local, gauge-invariant, conserved isospin current. This situation seems to be a general feature of any theory which arises from the enlargement of a non-abelian Lie group to a function group a la Utiyama.

# TETRADS AND CONSERVATION LAWS

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LET $V_4$ be a hyperbolic normal Riemann space. Let $C_\infty$ be the group of arbitrary coordinate transformations. From the assumed signature $(+,-,-,-)$ there follows the existence at any point of the space, $x^a$, of four orthonormal vectors $g^{\hat{a}}_a(x)$ (greek indices are vectorial indices; "roofed" indices, which run through $\hat{a}=\hat{0}, \hat{1}, \hat{2}, \hat{3}$ do not refer to $C_\infty$, but just label the tetrad vectors). Their orthonormality properties may be expressed as:

$$g^{a\beta}(x)g^{\hat{a}}_a(x)g^{\hat{\beta}}_\beta(x)=g^{\hat{a}\hat{\beta}} \quad \text{where} \quad ||g^{\hat{a}\hat{\beta}}||=||\,\mathrm{Diag}\,(1,-1,-1,-1,)\,||=||g_{\hat{a}\hat{\beta}}||. \tag{1}$$

By a simple algebraical argument, it follows that

$$g_{a\beta}(x)=g_{\hat{a}\hat{\beta}}g^{\hat{a}}_a(x)g^{\hat{\beta}}_\beta(x). \tag{2}$$

If $g^{\hat{a}}_a(x)$ as functions of $x$ are assumed to be of class $C^3$, one may speak of a normal hyperbolical $V_4$ with a built-in regular tetrad lattice. Observe that if $g=\mathrm{Det}\,||g_{a\beta}||$ and $\hat{g}:=\mathrm{Det}\,||g^{\hat{a}}_a||$ then (2) yields: $-g=(\hat{g})^2$. Note also that the assumption that $g_{a\beta}$ may be algebraically expressed by the right hand member of (2) is equivalent to Hilbert's conditions which assure the correct signature of the metric.

Clearly left hand member of (2) remains unchanged if tetradial indices are transformed by a Lorentz rotation,

$$g^{\hat{a}'}_a(x)=L^{\hat{a}'}{}_{\hat{a}}(x)g^{\hat{a}}_a(x), \quad g_{\hat{a}'\hat{\beta}'}L^{\hat{a}'}{}_{\hat{a}}(x)L^{\hat{\beta}'}{}_{\hat{\beta}}(x)=g_{\hat{a}\hat{\beta}}.$$

The group of $x$-dependent Lorentz rotations of the tetrads will be referred to as $L_x$. Observe at once that these do not have anything in common with Lorentz transformations of coordinates. If the $L_x$ transformations are to transform a regular tetrad lattice into another regular lattice, it is reasonable to assume that $L^{\hat{a}'}{}_{\hat{a}}$ as functions of $x$ are of class $C^3$.

Now, formula (2) shows that the 16 quantities $g^{\hat{a}}_a(x)$ determine uniquely the 10 $g_{a\beta}$'s. Conversely, the 10 $g_{a\beta}$'s determine the 16 $g^{\hat{a}}_a(x)$ only up to Lorentz rotations ($x$ dependent!) of the tetradial indices.

Therefore, there are two possible approaches to tetrads in general relativity.

The first, the conventional approach, consists in the following: because the tetrad field $g^{\hat{a}}_{\alpha}(x)$ in (2) is determined only with accuracy up to $L_x$ transformations, i.e. the concept of $g_{\alpha\beta}$ is so to speak gauge *invariant* with respect to $L_x$, the physical content of the theory has to be gauge invariant with respect to tetrad rotations. The additional 6 degrees of freedom (to the conventional 10 metrical degrees of freedom) have to be understood here as physically spurious. However, just as in electrodynamics where instead of working all the time with gauge invariant $f_{\alpha\beta}$ it is convenient to work with non-gauge invariant $A_\alpha$'s in general relativity it is also justifiable to work with tetrads as *mathematical tools*. Of course, in the framework of this philosophy, in any construction which operates with tetrads, it is neccessary to demonstrate that the final result—if physical—is $L_x$ invariant.

In electrodynamics the explicit use of potentials $A_\alpha$ allows us to forget about restrictions which impose, so to speak, half of Maxwell's equations. In relativity the explicite use of tetrads as field-theoretical degrees of freedom enables us to forget about restrictions which follow from the signature of the metric. Here is precisely the chief merit of all formalisms which operate with tetrads, if we restrict ourselves to the conventional approach discussed above.

The second, the unconventional or unorthodox approach to tetrads consists in the hypothesis that not only the 10 metrical degrees of freedom hidden in the 16 $g^{\hat{a}}_{\alpha}(x)$, but *all* of them, may represent true physical degrees of freedom. It is equivalent to saying that the 16 $g^{\hat{a}}_{\alpha}(x)$ unify the 10 metrical degrees of freedom with some additional 6, which also have a dynamical interpretation. As is well known, Einstein [1] tried to associate these with the degrees of freedom of the electromagnetic field in the framework of a unified field theory. Recently Møller [2], [3] has demonstrated that by subjecting the 6 additional tetradial degrees of freedom to some dynamical conditions one can construct an energy-momentum complex which satisfies all necessary physical requirements. That the original Møller complex [4] does not satisfy these; moreover, that no such complex can be constructed by operating only with metrical degrees of freedom, was demonstrated previously by Møller himself [5]. Rayski [6], [7] also pointed out recently that the problem of localisation of energy may be conveniently approached from the point of view of the tetrad formalism.

Why, if tetradial degrees of freedom are fixed by some restrictions, one may hope for well-defined energy and momentum (as local quantities), is intuitevely obvious. According to the general argument of Komar [8], any distinguished vector field gives rise to a conservation law. But the difficult question still has to be answered: what *physical* cause, if any, may impose the necessity of choosing the tetrad directions in a preferred manner? In a paper by C. Pellegrini and the author [9] we tried to approach this problem guided by rather formal mathematical arguments, but in the framework of a definite physical interpretation. The chief idea was the following: if the tetrad field were to unify, in fact, the metrical degrees of freedom with some 6 new (but again physical and dynamical) field-theoretical degrees of freedom which correspond to some new interaction of physical matter, then localization seems to be indeed possible. If we knew only about the electric field $\boldsymbol{E}$, not knowing about the magnetic field $\boldsymbol{H}$, it would not make any sense to talk about localizable energy. Similarly, if conventional general relativity were an incomplete theory, which fixes only the metrical degrees of freedom, and the "true" field-theoretical complex should unify these with some others, the localization impossible on the conventional level seems feasible in the "unified" version of the theory. Because of the algebraical simplicity of the tetrad field there arises the conjecture that the "true" field-theoretical complex is just $g_a^{\hat{a}}(x)$, undertermined only to a *constant* Lorentz gauge (the group of constant tetrad rotations is subsequently called $L$). Now, the reasonable extension of this idea is to study the structure of the most general Lagrange function which would determine the dynamics of the object $g_a^{\hat{a}}(x)$ with accuracy up to *constant* Lorentz rotations. This was precisely the chief objective of the paper [9], where the admissible form of such a Lagrange function was explicitly found. Of course, the physical validity of the theory studied there, depends very much on the answer to the question whether new dynamical degrees of freedom introduced that way have anything to do with reality.

It is true that conventional general relativity can do perfectly well without them. However, these degrees of freedom, although they have been introduced into our theory in purely formal fashion, lead in a straightforward way to a coupling between fermions, and not between bosons; and as such, could have meaning on the level of the quantized theory. The theories of Møller [2], [3] and our study [9] have rather as a moral the conclusion that in order to talk about a sensible localizable energy and momentum one has to go beyond conventional general relativity to some version of a unified field theory. But it has to be admitted that the fact as such that in this theory it is logical to talk about localizable energy, does not form a definite physical argument in its favour. Conventional general relativity without localizable energy is logically consistent, which must be admitted even by all those who as physicists would be more happy being able to talk about localizable energy-momentum as quantities which may be transformed by gravitational waves.

In this paper we shall not go any further into details of ideas associated with the unortodox approach (referring the reader interested in these to the more complete study given in [9]). We should like, however, to concentrate on possible advantages of the use of the tetrad formalism on the *conventional* level, where additional degrees of freedom remain spurious and energy-momentum stays unlocalizable.

First we would like to point out that even in conventional general relativity one *has to* use tetrads. There are in nature fermions as well as bosons. Therefore, the necessity of general relativistic spinors [see Eq. (10)] is evident. The "Pauli-matrices", $g^{\alpha\dot{A}B}(x)$, are fundamental in general relativistic spinor calculus (notation similar to that of [11] is adapted); together with the spinorial metric tensor $||g_{AB}|| = \left|\left|\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right|\right|$ they determine the metric

$$g^{RS}g_{\alpha\beta} = \tfrac{1}{2}\, g_{\dot{A}\dot{B}}\, (g_\alpha{}^{\dot{A}R} g_\beta{}^{\dot{B}S} + g_\beta{}^{\dot{A}R} g_\alpha{}^{\dot{B}S}). \tag{3}$$

They are just related to the "flat" Pauli matrices $g^{\hat{a}\dot{A}B}$

$$||g^{\hat{a}\dot{A}B}|| = \left( \left|\left|\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right|\right|, \left|\left|\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right|\right|, \left|\left|\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right|\right|, \left|\left|\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right|\right| \right) \tag{4}$$

by the formula

$$g^{\alpha\dot{A}B}(x) = g^{\alpha}_{\hat{a}}(x)\, g^{\hat{a}\dot{A}B}. \tag{5}$$

Formula (3) is just the spinorial counterpart of the foundamental Eq. (2). Although the group $L_x$ as introduced on the level of tetrads seems to be purely formal, one has to realize that—as is well known [12], [13]—unimodular transformations of general relativistic spinors are just the representation $D(\frac{1}{2},0)$ of $L_x$. Therefore, it can be manifestly seen, at least on the level of the spinor calculus, that the original field-theoretical degrees of the conventional theory are just tetrads defined with accuracy up to $L_x$ gauge.

Secondly, we would like to demonstrate that tetrads may be useful as mathematical tools in the construction of the *global* conserved quantities (conserved in the absence of radiation) which, as is well known (see e.g. [14], [15], [16]) have a perfectly well defined meaning in the conventional theory in the case of "insular matter". The construction given below is essentially equivalent to the usual construction which uses a reasonably defined energy momentum complex (e.g. that of Møller), but has the merit of being manifestly covariant with respect to $C_\infty$. Roughly speaking, by the use of tetrads we shall be able to keep covariance with respect to $C_\infty$ throughout the calculation. The price for that is loss of $L_x$ covariance in the intermediate steps. The proof that the final result is $L_x$ invariant seems to be simpler than the proof in the case where $C_\infty$ invariance is violated.

Let us choose $g^{\alpha}_{\hat{a}}(x)$ arbitrarily of class $C^3$. Now, define the quantity

$$t^{\alpha}{}_{\hat{\beta}}{}^{\sigma} \overset{\mathrm{df}}{=} \frac{c^4}{16\pi\varkappa}\, \delta^{\alpha\varrho\sigma}_{\lambda\nu\mu}\, g^{\hat{a}\lambda}{}_{;\varrho}\, g^{\nu}{}_{\hat{a}}\, g^{\mu}{}_{\hat{\beta}}\,. \tag{6}$$

The object defined that way is clearly $C_\infty$ covariant; it is however only $L$ covariant and is *not* $L_x$ covariant. Now

$$t^{\alpha\sigma}_{\hat\beta;\sigma} = \frac{c^4}{16\pi\varkappa}\,\delta^{\alpha\varrho\sigma}_{\lambda\nu\mu}g^{\hat\alpha\lambda}{}_{;\varrho;\sigma}g^{\nu}_{\hat\alpha}g^{\mu}_{\hat\beta} + \frac{c^4}{16\pi\varkappa}\,\delta^{\alpha\varrho\sigma}_{\lambda\nu\mu}g^{\hat\alpha\lambda}{}_{;\varrho}\,(g^{\nu}_{\hat\alpha}g^{\mu}_{\hat\beta})_{;\sigma}. \tag{7}$$

But clearly

$$g^{\hat\alpha\lambda}{}_{;\varrho;\sigma} - g^{\hat\alpha\lambda}{}_{;\sigma;\varrho} = -g^{\hat\alpha\tau}R^{\lambda}{}_{\tau\varrho\sigma}. \tag{8}$$

The remark that $\delta^{\alpha\varrho\sigma}_{\lambda\nu\mu}$ is skew in $\varrho\sigma$, and the use of (8) enable us to see at once that the term on the right-hand side of (7) which contains second derivatives is just equal to $\frac{c^4}{8\pi\varkappa}\,G^{\alpha}_{\beta}g^{\beta}_{\hat\beta}$ where $G^{\alpha}_{\beta}$ is the Einstein tensor. Hence, if Einstein's equations are assumed, $G^{\alpha}_{\beta} = \frac{8\pi\varkappa}{c^4}\,T^{\alpha}_{\beta}$ and we write

$$T^{\alpha}_{\hat\beta} = T^{\alpha}_{\beta}g^{\beta}_{\hat\beta}, \qquad t^{\alpha}_{\hat\beta} = \frac{c^4}{16\pi\varkappa}\,\delta^{\alpha\varrho\sigma}_{\lambda\nu\mu}g^{\hat\alpha\lambda}{}_{;\varrho}\,(g^{\nu}_{\hat\alpha}g^{\mu}_{\hat\beta})_{;\sigma}, \tag{9}$$

we may rewrite (7) as

$$t^{[\alpha}{}_{\hat\beta}{}^{\sigma]}{}_{;\sigma} = T^{\alpha}_{\hat\beta} + t^{\alpha}_{\hat\beta}. \tag{10}$$

Note that $T^{\alpha}_{\hat\beta}$ is both $C_\infty$ and $L_x$ covariant, whereas $t^{\alpha}_{\hat\beta}$ just as $t^{\alpha\sigma}_{\hat\beta}$, is $C_\infty$ covariant but not $L_x$ covariant. Clearly, $t^{\alpha\sigma}_{\hat\beta}$ is linear in first derivatives, when $t^{\alpha}_{\hat\beta}$ is quadratic in these. Obviously (10) may also be rewritten as (remember that $\sqrt{-g} = |g^{\hat\cdot}_{\cdot}|!$):

$$(g^{\hat\cdot}_{\cdot}\,t^{[\alpha\sigma]})_{,\sigma} = g^{\hat\cdot}_{\cdot}[T^{\alpha}_{\hat\beta} + t^{\alpha}_{\hat\beta}]. \tag{11}$$

Therefore

$$(g^{\hat\cdot}_{\cdot}[T^{\alpha}_{\hat\beta} + t^{\alpha}_{\hat\beta}])_{,\alpha} = 0. \tag{12}$$

Now let $\sigma$ be a 3-dimentional space-like surface (with time-like normal) which extends to spatial infinity. Define

$$P_{\hat\beta}[\sigma] = \frac{1}{c}\int_\sigma d_3\sigma_\alpha g^{\hat\cdot}_{\cdot}\,(t^{\alpha}_{\hat\beta} + T^{\alpha}_{\hat\beta}). \tag{13}$$

The four functionals of $\sigma$ so defined are "conserved" in the general sense:

$$\frac{\delta P_{\hat\beta}[\sigma]}{\delta\sigma(x)} = 0 \leftrightarrow (12). \tag{14}$$

If $\tau$ denotes a time-like 3-dimensional surface (i.e. one with a space-like normal) which closes the 4-region between $\sigma_2$ and $\sigma_1$ at spatial infinity, we have in general

$$P_{\hat\beta}[\sigma_2] - P_{\hat\beta}[\sigma_1] = \Delta P_{\hat\beta}, \text{ where } \Delta P_{\hat\beta} = \frac{1}{c}\int_\tau d_3\tau_\alpha g^{\hat\cdot}_{\cdot}(t^{\alpha}_{\hat\beta} + T^{\alpha}_{\hat\beta}). \tag{15}$$

If $t^{\alpha}_{\hat{\beta}}$, $T^{\alpha}_{\hat{\beta}}$, tend to zero strongly enough when one approaches spatial infinity (no radiation), $\Delta P_{\hat{\beta}}$ vanishes and $P_{\hat{\beta}}[\sigma]$ in reality does not depend on $\sigma$; and, therefore is conserved in the literal sense. We *do not* assume that, leaving room for radiation.

The quantities $P_{\hat{\beta}}$ are manifestly $C_{\infty}$ *scalars* (vectorial index contracted with vectorial index). But, at first sight since constructed with $L_x$ non-covariant quantities they seem to depend strongly on the chosen orientation of the tetrads. But, substituting into (13) the left hand member of (11) one gets after the use of the Stokes' theorem:

$$P_{\hat{\beta}}[\sigma] = \frac{1}{c}\int_{\bar{\sigma}} \alpha_2 \check{\bar{\sigma}}_{[\alpha\beta]} t^{[\alpha\sigma]}_{\hat{\beta}}; \tag{16}$$

where $\check{\bar{\sigma}}$ denotes the 2-dimensional boundary of $\sigma$ at spatial infinity, and $d\check{\bar{\sigma}}_{[\alpha\beta]}$ its (dual!) surface element ($g^{\hat{}}_{\cdot}$ is "absorbed" by the duality operation). Therefore, the $P_{\hat{\beta}}$'s do *not* depend on the orientation of tetrads "inside" of $V_4$, being sensitive only to their asymptotic orientation.

But even if "inside" of $V_4$ there were no way of making some orientations of tetrads preferable to others (which is responsible for the impossibility of the localization), asymptotically it certainly is possible to introduce preferred directions, at least in the case of insular matter when the space-time is "asymptotically flat". Speaking intuitively: if the space-time is asymptotically flat one may choose asymptotically quasi-Cartesian coordinates, so that the metric deviates from Minkowskian values $\eta_{\alpha\beta}$ only by quantities tending to zero. The reasonable asymptotical choice of tetrads would consist in orienting them along the versors of quasi-Cartesian coordinates.

Although this intuitive argument seems to be fair enough, to make it mathematically precise is not as simple. Every one, who has dealt with the problem of boundary conditions in general relativity certainly has had not quite pleasant experiences with the difficulties involved — and in our problem we again have to confront them, quite independently of the specifics of the tetrads. Recent works of Komar [17], [18] with semi-Killing vectors offer a mathematical tool which could be used in our problem (one could try to orient the tetrads asymptotically along the semi-Killing vectors). For purposes at preliminary orientation it is convenient to proceed with a less covariant approach, similar to that used by Trautman in the work where he obtained his boundary conditions [19].

Namely, if the world is asymptotically flat one usually assumes that *such coordinates* $x^{\alpha} = (x^0, x^a)$ may be introduced that

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\langle\alpha\beta\rangle}, \text{ and } h_{\langle\alpha\beta\rangle} = O\left(\frac{1}{r}\right), \text{ where } r = \sqrt{x^a x^a} \tag{17}$$

$\left(\text{But } \textit{not} \text{ necessarily } g_{\alpha\beta,\varrho} = O\left(\frac{1}{r^2}\right)!\right)$. The non-covariant element in this definition consists in the use of the concept of $r$. However, the definition is self-consistent and "fits in" with all practical experience—the theory of equations of motions for instance [20].

Suppose, therefore, that such coordinates are adopted, and the metric is given in terms of these. Introduce now a group of coordinate transformations which preserves the asymptotical form of the metric (17). Clearly, such transformations are *arbitrary* "inside" of $V_4$ and asymptotically have the form

$$x'^{\alpha} = L^{\alpha'}{}_{\beta} x^{\beta} + L^{\alpha'} + O\left(\frac{1}{r}\right), \tag{18}$$

where $L^{\alpha'}_{\beta}$ are constant Lorentz matrices and the derivatives of terms denoted as $O\left(\frac{1}{r}\right)$ are again at least of order $O\left(\frac{1}{r}\right)\left(\text{but } \textit{not} \text{ necessarily } O\left(\frac{1}{r^2}\right)!\right)$. The group of transformations of this type will subsequently be called $C^L_{\infty}$.

Now, if the tetrads have to be oriented asymptotically along the versors of quasi-Cartesian coordinates defined by (17), the following equation must hold

$$g^{\hat{a}}_{a} = \delta^{\hat{a}}_{a} + \frac{1}{2} h^{\hat{a}}_{a}, \text{ where } h^{\hat{a}}_{a} = O\left(\frac{1}{r}\right) \left(g^{\hat{a}}_{a,\varrho} \text{ not necessarily } O\left(\frac{1}{r^2}\right)!\right). \tag{19}$$

If this has to hold it is clear that we have to restrict the previously undetermined matrices $L^{\hat{a}}_{\hat{\beta}}(x)$ by the condition

$$\lim_{r\to\infty} L^{\hat{a}'}_{\hat{\beta}}(x) = L^{\infty}{}^{\hat{a}'}_{\hat{\beta}} = \text{const.}, \tag{20}$$

where matrices with superscript $\infty$ form a group: $L^{\infty} = \lim_{r\to\infty} L_x$. However, if the condition (19) has to hold in *any* coordinates which are admissible from the point of view of $C^L_{\infty}$, clearly the transformations of tetrads must be linked with coordinate transformations asymptotically. Namely, if we carry out transformation $C^L_{\infty}$, (18) simultaneously we must make an $L_x$ transformation with the "boundary" $L^{\infty}{}^{\hat{a}'}_{\hat{\beta}}$ such that

$$\delta^{\hat{a}'}_{a'} = L^{\infty}{}^{\hat{a}'}_{\hat{\beta}} \delta^{\hat{\beta}}_{\beta} L^{-1\beta}_{a'}. \tag{21}$$

This simply means that the matrices $||L^{\infty}{}^{\hat{a}'}_{\hat{\beta}}||$ have to be identical with matrices $||L^{\alpha'}_{\beta}||$.

Summarizing: adopting asymptotically, quasi-Cartesian coordinates, and the boundary condition for tetrads (19) we restrict the $L_x$ transformations to those such that the matrices $\lim L^{\hat{a}'}{}_{\hat{\beta}}$ have to be identical with these which determine the asymptotical Lorentz transformations of the quasi-Cartesian coordinates.

Now, define

$$h_{\alpha\beta} = \delta^{\hat{\alpha}}_{\alpha} g_{\hat{\alpha}\hat{\beta}} h^{\hat{\beta}}_{\beta}. \tag{22}$$

Clearly the symmetric part of this quantity, $h_{\langle\alpha\beta\rangle}$ coincides with the "correction" to the Minkowskian metric in (17). Therefore, the quantities $h_{[\alpha\beta]}$, the skew part of $h_{\alpha\beta}$ describe the unconventional degrees of freedom which are physically meaningless in the framework of the conventional interpretation.

We now obtain the following result: if one substitutes (19) into (16), and computes $P_{\hat{\beta}}$ only with accuracy up to terms *linear* in $h^{\hat{\alpha}}_{\alpha}$; then after expressing these by $h_{\langle\alpha\beta\rangle}$ and $h_{[\alpha\beta]}$ one obtains two contributions to $P_{\hat{\beta}}$; 1° depending linearly on the conventional degrees of freedom, 2° depending on the physically meaningless $h_{[\alpha\beta]}$. But it happens that the latter contribution has just the form $\oint_S {}_5 d_2 s\boldsymbol{n} \operatorname{rot} \boldsymbol{H}$, where $\boldsymbol{H}$ is constructed from $h_{[\alpha\beta]}$, and as such, in the virtue of Stokes' theorem, it vanishes.

Hence, in reality, the $P_{\hat{\beta}}$ do *not* depend on the spurious degrees of freedom, and these quantities are *invariant* with respect to $L_x$ gauge transformations provided condition (20) is adopted.

One can also easily see that, if the tetrads are transformed by an $L_x$ transformation, the $P_{\hat{\beta}}$ transform like a vector with respect to $L^{\infty}{}^{\hat{\alpha}'}_{\hat{\beta}}$. On the other hand, because of (21) this means that asymptotically Lorentz *coordinate* transformations transform $P_{\hat{\beta}}$ like a vector.

One also may add that for worlds for which the following hold asymptotically

$$h_{\langle 00\rangle} \cong 1 - \frac{2m(t)}{r} + O\left(\frac{1}{r^2}\right), \quad h_{\langle ab\rangle} = -\delta_{ab}\left(1 + \frac{2m(t)}{r}\right) + O\left(\frac{1}{r^2}\right) \tag{23}$$

$$h_{\langle 0a\rangle} = O\left(\frac{1}{r^2}\right)$$

and all $O$'s differentiated with respect to $x^a$ are one order higher in $\left(\frac{1}{r}\right)$, one obtains easily

$$P_{\hat{\beta}} = M(t) c \delta^{0}_{\hat{\beta}}, \text{ where } m(t) = \frac{2\varkappa M(t)}{c^2}. \tag{24}$$

There are reasons to believe that similar results hold under much less restrictive boundary conditions.

Of course, the material given above must be understood as only a preliminary investigation of the properties of $P_{\hat{\beta}}$. One can hope for more conclusive results after approaching the problem of boundary conditions more covariantly in the language of tetrads. That tetrads may be found to be a useful tool in the problem of boundary conditions follows from this simple argument. First derivatives of $g_{\alpha\beta}$ clearly cannot be handled covariantly; but $g^{\hat{a}}_{\alpha;\beta}$ even if not $L_x$ covariant are $C_\infty$ covariant! On the other hand the asymptotical existence of the group of motions gives grounds to hope that with the help of $g^{\hat{a}}_{\alpha;\beta}$ the boundary conditions for the metric field could be approached conveniently. On the other hand clearly investigating the properties of $P_{\hat{\beta}}$ one should learn more about the asymptotic properties of $g^{\hat{a}}_{\alpha;\beta}$

Two additional remarks about possible advantages of the tetrads formalism may be added.

If follows from (8) that:

$$R^{\alpha}{}_{\beta\gamma\delta} = g^{\alpha}_{\hat{a}}(g^{\hat{a}}_{\beta;\gamma;\delta} - g^{\hat{a}}_{\beta;\delta;\gamma}) = g^{\alpha}_{\hat{a}}\,\delta^{\varrho\sigma}_{\gamma\delta}g^{\hat{a}}_{\beta;\varrho;\sigma} \tag{25}$$

This leads straightforwardly to the result that

$$g^{\hat{\cdot}}_{\cdot}R = (g^{\hat{\cdot}}_{\cdot}\,\delta^{\varrho\sigma}_{\alpha\beta}g^{\alpha}_{\hat{a}}g^{\hat{a}\beta}{}_{;\varrho})_{,\sigma} + g^{\hat{\cdot}}_{\cdot}\,\delta^{\varrho\sigma}_{\alpha\beta}g^{\alpha}_{\hat{a};\varrho}g^{\hat{a}\beta}{}_{;\sigma}. \tag{26}$$

This result originally obtained by Møller [2], presents an interesting decomposition of the Lagrange function of the conventional theory into two $C_\infty$ scalar densities (but not $L_x$ scalars), one of which is a pure divergence. It has to be contrasted with the usual decomposition

$$\sqrt{-g}R = -\left\{\frac{1}{\sqrt{-g}}((-g)g^{\alpha\beta})_{,\beta}\right\}_{,\alpha} + \sqrt{-g}\,g^{\alpha\beta}(\Gamma^{\varrho}_{\alpha\sigma}\Gamma^{\sigma}_{\beta\varrho} - \Gamma^{\varrho}_{\alpha\beta}\Gamma^{\sigma}_{\varrho\sigma}) \tag{27}$$

into a divergence plus an "effective" Lagrange function quadratic in the first derivatives. But this usual expression for the effective Lagrange function in not any simple $C_\infty$ invariant. Hence the canonical treatment of the theory meets familiar difficulties, and is rather a non-covariant procedure.

On the other hand, adopting for the effective Lagrange function

$$L_{\text{eff}} = g^{\hat{\cdot}}_{\cdot}\,\delta^{\varrho\sigma}_{\alpha\beta}g^{\alpha}_{\hat{a};\varrho}g^{\hat{a}\beta}{}_{;\sigma} \tag{28}$$

we deal with a $C_\infty$ scalar quadratic in first derivatives, which is not an $L_x$ scalar (a scalar only up to a divergence). Hence, the canonical treatment of the theory based on $L_{\text{eff}}$ from (28) will be manifestly $C_\infty$ covariant; we would only have to worry about the lack of covariance with respect to a relatively simpler group than $C_\infty$, namely $L_x$. Clearly, increasing the number of variables to 16, also leads to an increased number of constraints. But all of these shall be $C_\infty$ covariant.

The last remark consists in the observation that if one approaches the problem of quantization of gravity from the side of functional integrals, e.g. [21], in order to compute the amplitude one integrates over all possible histories of the metric field, including situations where the signature is elliptical or ultrahyperbolical, etc. In the tetradial formalism such histories are at once excluded. Namely, if we understand in the usual formula

$$A = \int_{\text{(over histories)}} e^{\frac{i}{l^2}\int \sqrt{-g}R}\, \delta[g_{\nu\mu}], \qquad l = \sqrt{\frac{16\pi\varkappa\hbar}{c^3}} \tag{29}$$

$\sqrt{-g}\,R$ as $\hat{g}_{\cdot}^{\cdot}R$ from (26) (expressed explicitly by tetrads); and compute the usual functional measure $\delta[g_{\nu\mu}]$, which is clearly $L_x$ invariant as expressed *through tetrads*, and integrate over all possible histories of tetrads, one always gets an $L_x$ *invariant* result.

However, as soon as tetrads are explicitly introduced the correct hyperbolic normal signature of the metric is automatically secured.

I would like to express my gratitude to Prof. Møller for kind interest in this work. I owe also many thanks to Dr. C. Pellegrini for stimulating disscusions whose conclusions found their place mainly in our joint publication and partly also here. Interesting disscusions and help in preparing the English manuscript offered by Dr. J. Stachel are sincerely appreciated.

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# EXPERIMENTAL VERIFICATION OF GENERAL RELATIVITY THEORY

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THE aim of this article is to present the progress made in the experimental verification of General Relativity Theory (G.R.T.). Of course, in the first place all the facts available should be reported. But perhaps since the facts in this field are very poor, experimental verification of G.R.T. has shown itself to be particularly connected with various, so to say, collateral features. Here we mean above all a discussion of non-Einsteinian gravitational theories and also the significance of various experiments from the point of view of testing with their help the various aspects of G.R.T. In order to avoid any misunderstanding it should be stressed that in discussing these problems below, the present author will not strive either for any detailed analysis or comparison of various views found in the literature but he will only present his own views.

As is well known there is (at least in a logical plan) a profound asymmetry between the denial and proof of a theory. Indeed, it is sufficient to have one reliably established fact which definitely contradicts a theory in order for the theory to be overthrown. At the same time the coincidence between individual experiments or observations and conclusions from a theory to some extent confirms the theory, indicates that it does not contradict experiment, gives us every reason to believe that the theory is true, but it does not yet prove the theory. This last statement is obviously connected with the fact that all the conclusions considered arising from a given theory could also follow from other theories. For instance, the three well-known "critical effects" pointed out by Einstein for testing G.R.T. follow from some non-Einsteinian gravitational theories. And this is precisely the reason why sometimes one has doubts as to whether Einstein's G.R.T. is proved experimentally.

We, however, decidedly do not share this point of view, in particular because of the above-mentioned character of the statement as to the experimental testing of a theory. The only thing that is specific about G.R.T. is the small number of experiments accesible; these just allow us by some effort to combine the results of observations with some non-Einsteinian gravitational theories. Therefore, in the light of this fact is is most desirable to go beyond the limits of a study of the three "critical effects".

At the same time even now Einstein's G. R. T. cannot be considered unproved by experiment. This will be dealt with later on. First we shall proceed to an exposition of fundamental facts concerning the confirmation of G.R.T. Einstein's G.R.T. will be taken to mean the theory of the gravitational field in which the field $g_{ik}$ satisfies the equations

$$R_{ik}-\frac{1}{2}g_{ik}R=\frac{8\pi\varkappa}{c^4}T_{ik}. \qquad (1)$$

According to the comments made above, we shall start with the following statement:

I. There are no clear-cut experimental results or observations which would negate G.R.T. There is not even the smallest cloudlet on the horizon, which would foreshadow the emergence of such difficulties or restrictions as to the field of application of G.R.T. (of course this refers to macroscopic physics). It applies equally well even to cosmology where the necessity of some generalizations or other would not cause any particular surprise (the simplest known such generalization is the addition of a $\Lambda$-term to get $\Lambda g_{ik}$ on the left-hand side of Eq. (1)).

Now we shall go on to a comparison of the predictions of G.R.T. with observations.(1)

Of the three "critical" effects the simplest one is the gravitational shift of frequency pointed out by Einstein in 1907 in the very beginning of his work on G.R.T. [4]. This effect, in the first approximation follows from the principle of equivalence and from Special Relativity Theory. The corresponding formula is most conveniently expressed in quantum language: a photon with an inertial mass of $\frac{\hbar\omega}{c^2}$ would possess the same gravitational mass; in a gravitational field with a potential $\varphi$ a photon performs work, that is

$$\frac{\hbar\omega}{c^2}(\varphi_1-\varphi_2)=\hbar(\omega_2-\omega_1)\equiv\hbar\Delta\omega.$$

Hence

$$\frac{\Delta\omega}{\omega}=-\frac{\Delta\lambda}{\lambda}=\frac{\varphi_1-\varphi_2}{c^2}. \qquad (2)$$

(1) For more details see [1] and [2]. The problem of the experimental verification of the Special Theory of Relativity as well as that of experiments of the Eötvös type will not be analysed here. We shall only mention that the equality of the inertial and gravitational masses has now been established with an accuracy of the order of $10^{-10}$ (see R. H. Dicke [3]). Moreover, the study of the Mössbauer effect has shown that the rate of "ideal clocks" does not depend on their acceleration, as is assumed in G.R.T., even for accelerations of the order of $10^{16}$ $g \simeq 10^{19}$ cm/sec$^2$.

The precise formula of G.R.T., $\frac{\omega_2}{\omega_1}=\sqrt{\frac{g_{00}(1)}{g_{00}(2)}}$, goes over into (2) in a weak field, that is when

$$\frac{|\varphi|}{c^2}\ll 1, \tag{3}$$

the component $g_{00}=-1-\frac{2\varphi}{c^2}$ and everywhere we neglect all terms of order higher than $\varphi/c^2$.

For the Sun and the Earth

$$\frac{|\varphi_{\odot}|}{c^2}=\frac{\varkappa M_{\odot}}{c^2 r_{\odot}}=2.12\times 10^{-6},\quad \frac{|\varphi_{♁}|}{c^2}=\frac{\varkappa M_{♁}}{c^2 r_{♁}}=7\times 10^{-10}. \tag{4}$$

On the Earth the spectral lines of the Sun are shifted by $\frac{\Delta\lambda}{\lambda}=-\frac{|\varphi_{\odot}|}{c^2}$ and the lines in the spectrum of radiation emitted from a transmitter placed on a distant sputnik (artificial satellite) of the Earth are shifted by $\frac{\Delta\lambda}{\lambda}=\frac{|\varphi_{♁}|}{c^2}$.

It is interesting that the gravitational shift of frequency, one of the simplest and earliest effects pointed out by G.R.T., was not measured reliably until after the two other critical effects. It is true that the red shift was observed long ago in the spectra of white dwarf stars as well as in that of the Sun, but the lack of reliable (and independent) date about the radii of white dwarf stars and velocities of convectional motions in the solar photosphere hindered the use of corresponding data for the quantitative verification of formula (2). These old results will not be taken into account here (see [1], [2], [8], [10]). Recently J. E. Blamont and F. Roddier [5] carried out new measurements of the red (gravitational) shift of lines in the solar spectrum. It has been shown that for the strontium line $\lambda=4607.3$ Å the red shift observed coincides with the theoretical one $\Delta\lambda=9.76\times 10^{-3}$ Å with an accuracy which seems not to exceed a few percent (in the short communication [5] the accuracy is not given). Even before these measurements, the gravitational shift of frequency had been measured for $\gamma$-radiation with the help of the Mössbauer effect. These experiments (P. V. Pound and G. A. Rebka [6]) are probably familiar to all. The measurements were carried out with an effective difference in altitudes $H$ of 45 m, i.e. the theoretical value $\frac{\Delta\omega}{\omega}=\frac{gH}{c^2}=4.92\times 10^{-15}$. According to the latest data [7] known to us such experiments give the value

$$\frac{\Delta\omega_{\text{observ.}}}{\Delta\omega_{\text{theor.}}}=0.97+0.035,$$

i.e. the theory coincides with experiment with the accuracy of 3%.

A few years ago some authors (for instance the present author in 1954 [8]) suggested a way of measuring the gravitational shift of frequency with the help of sputniks. For a sputnik which is at an altitude of, let us say, 800 km the proportion $\frac{\Delta\omega}{\omega} = 7.7 \times 10^{-11}$ and, therefore, the effect can already be measured, in principle, with the help of methods available to-day. But now, after the other measurements mentioned above, the determination of the gravitational shift of frequency with the help of sputniks has lost its urgency, purely technical aspects aside (here we have in mind the development of precise apparatus for frequency and time measurements).

Measurement of the frequency shift with an accuracy up to the terms of order $\left(\frac{\varphi}{c^2}\right)^2$ would be of great interest. However, even for the Earth, i.e. with the help of distant satellites, $\left(\frac{\varphi_{\oplus}}{c^2}\right)^2 = 5 \times 10^{-19}$ and we do not know any way of attaining such accuracy.

The second "critical" effect is the deflection of light rays in the field of the Sun: a ray passing at the distance $R$ from the center of the Sun is deflected by the angle ($r_{\odot}$ is the radius of the photosphere)

$$\alpha = \frac{4\varkappa M_{\odot}}{c^2 R} = 1''.75 \frac{r_{\odot}}{R}. \tag{5}$$

The angle $\alpha_{max} = 1''.75$ corresponds approximately to the angle at which a match-box is seen from a distance of 5 km.

Only a little more than 10 observations of the deflection of light rays have been carried out. Perhaps there are data obtained in the last 2–3 years of which we do not know, but with this reservation the following can be said. The law of variation $\alpha \sim \frac{1}{R}$ cannot yet be considered as having been verified, but if it is assumed and extrapolated to the edge of the disk, then the mean value of many observations is $\alpha_{max} = 2''.0$, with an accuracy of about 10–20% (the sign of the effect is, of course, the same as predicted by theory—the light rays are attracted towards the Sun). Thus, within the limits of the accuracy attained, the theory agrees with experiment and we know already that the deflection of rays is not equal to twice the smaller value $\alpha'_{max} = 0''.87$, which is obtained when space curvature is disregarded.(2)

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(2) It is interesting that such a deflection of light rays in the Sun's field $\alpha'_{max} = \frac{2\varkappa M_{\odot}}{c^2 r_{\odot}}$ was predicted in 1801 by J. Soldner on the basis of the corpuscular theory of light (see [9], [10]).

It should be mentioned here that in the literature one encouters the opinion that the deflection of light rays in the Sun's field follows from the principle of equivalence and from Special Relativity Theory and, therefore, is of little importance for testing G.R.T. (see [11]). In the first place, however, the connection and, so to say, combination of the principle of equivalence with Special Relativity Theory constitutes the basis or, if you will, the foundation of G.R.T., and here any contraposition seems to be very artificial. At the same time, the deflection of rays in the Sun's gravitational field appears evidently as a more subtle effect than the gravitational shift of frequency which follows directly from the principle of equivalence and Special Relativity Theory. Indeed, the true value of the angle $\alpha$ is obtained only when the curvature of space is taken into account. Secondly, and this is of importance here, the deflection of rays in the Sun's field constitutes a global effect and cannot be removed by some suitable choice of a frame of reference near the Sun. On the other hand the principle of equivalence is of a local character and its direct use cannot lead to any explanation of global effects. It is not possible to discuss this problem here, and perhaps there is even no need to do so as it was recently analysed in detail by R. Sexl [12].

The discovery and measurement of the deflection of rays passing near the Sun is a feasible test of G.R.T. and, therefore, it is perfectly justifiable to improve the accuracy of the observations. The classical astronomical method will allow us, by good measurements during eclipses, to test the law $\alpha \sim \frac{1}{R}$ and to measure the angle $\alpha_{\max}$ with an accuracy of a few percent. A new method of measurement using balloons or, still better, artificial satellites was suggested by R. Lillestrand [13] in 1960. According to Lillestrand, his method allows an accuracy of up $0''.01$ (formula (5) would be then checked with an accuracy of more than one percent). To the present author this method appears to be very elegant, but it is difficult for us to judge the experimental aspect of the method.

Let us now proceed to the third "critical" effect: the precession of the perihelion of planets. During one revolution this precession is

$$\dot{\varepsilon} = \frac{24\pi^3 a^2}{c^2 T^2 (1-e^2)} = \frac{6\pi \varkappa M_\odot}{c^2 a (1-e^2)}. \tag{6}$$

Here, $a$ denotes the semi-major axis, $e = \sqrt{1 - \frac{b^2}{a^2}}$ —the eccentricity and $T$ — the period $\left(a^3 = \frac{\varkappa M_\odot}{4\pi^2} T^2\right)$. For Mercury the angle of precession per century ought to be equal to $\psi_{\text{theor.}} = 43''.03$ and $e\psi_{\text{theor.}} = 8''.847$. In Einstein's paper [14], in which formula (6) was obtained, the value $\psi =$

$= 45''\pm 5$ is cited as a result of observations (and astronomical computations). The best value known to us today is $\psi = 42''.56\pm 0''.94$ (G. Clemence, for references see [1], [2]). Recently a result for Venus was obtained (R. Duncombe; it is given here a value according to M. F. Subbotin): $e\psi = 0''.057+0''.033$, while according to G.R.T. $e\psi_{\text{theor.}} = 0''.059$. For the Earth $\psi_{\text{theor.}} = 3''.84$, while the value obtained from observations is $\psi = 4''.6\pm 2.7$.

Thus, for Mercury the theory agrees with the observations with an accuracy of about 3%. For Venus as well as for the Earth this effect can also be considered to have been established. Undoubtedly, the data can be made even more accurate, in particular in the case of planets (Mercury, Venus, Earth). The effect of G.R.T. can be expected to be chosen for Mars and perhaps also for the small planet Icar (in this case $e\psi_{\text{theor.}} = 8''.3$ and there is hope to obtain data even more precise than for Mercury; see [10]). The orbiting of an artifical planet (cosmic rocket) opens up still another possibility — in this case the value $\psi_{\text{theor.}}$ reaches 1000″ per century. However, the possible accuracy of measurement of $\psi$ is not clear enough. The same can be said as to the artificial satellites of the Earth. In this case the maximal angle of precession is $\psi \cong 1700''$ per century (for details see [1], [2], [10]). Other effects (perturbations) which lead to a variation in the orbits of satellites are at the same time considerably greater and there is little hope of testing the effect for satellites in the near future.

Thus

II. All the three "critical" effects of G.R.T. pointed out by Einstein about 50 years ago may be considered as having been demonstrated and the corresponding values agree with the theoretical ones within attained accuracy of a few percent (in the case of the deflection of light rays in the Sun's field the accuracy is a little lower). Naturally, this confirms the theory and is a great success for it.

However, according to what has been said above, strictly speaking, it cannot yet be maintained that G.R.T. has been verified by experiment. Of course, every further effort for greater accuracy in the values for the three critical effects is justified and highly desirable.

Since, however, at present the transition to an accuracy of the order of $\left(\frac{\varphi}{c^2}\right)^2$ is, apparently, not being considered, greater accuracy in the data concerning the three critical effects will not lead to any new essential information (it is, of course, assumed that no disagreement between theory and experiment will be found).

The most urgent problem today in the field of experimental confirmation of G.R.T. is that of measuring some new independent effects. Why is this necessary? In the first place, this is necessary for overthrowing non-Einsteinian gravitational theories.

The widely-held opinion, entirely shared by the present author, is that General Relativity Theory unquestionably holds first place among all physical theories for its internal consistency and beauty. The principle of equivalence, Special Relativity Theory, the transition in the limit to Newton's theory of gravity and some natural mathematical conditions (as, for instance, covariance, and non-occurrence of higher derivatives of the field $g_{ik}$)—this in practice is all that is necessary to obtain field equations (1) which do not contain any free parameters. The non-Euclidean character of the geometry of space-time as well as the non-linearity of the equations of the gravitational field bear a profound and clear sense. Therefore, the present author (and, apparently, the majority of physicists) absolutely does not understand the tendency to build up non-Einsteinian gravitational theories; there is no evident physical foundation for doing so, not to speak of facts.(3)

Since, however, such theories do exist (there are about ten of them) and are being discussed rather widely, this paper must take into consideration the possibility of their being distinguished from G.R.T. by observational data.

The very requirement of obtaining the correct results concerning the three critical effects in itself allows us to reject a number of non-Einsteinian theories (see G. J. Whitrow and G. E. Morduch [16]). Nevertheless, for two or three versions of such theories one is able to attain the same values as in G.R.T. for all three effects. All these versions (Birkhoff, Belinfante and Swihart, Whitehead; see [16], [17]) are connected with profound difficulties such as, for example, the arbitrariness of choosing the constants, etc. However, at this point I should like to stress another aspect of the question: it is sufficient merely to take into consideration the fourth effect of G.R.T. (namely the "rotational effect") for none of the known non-Einsteinian theories give results in strict agreement with the predictions of G.R.T. (see [16] and also a paper of W. I. Pustovoit [17]).(4)

The "rotational effect" is taken to mean the influence of the rotation of a central body upon the motion of the perihelion and the nodes of planetary orbits. This effect was investigated for the first time in 1918 (J. Lense and H. Thirring [18]). For Mercury this effect due to the rotation of the Sun,

(3) We are not discussing here an approach to G.R.T. of some methodical interest, starting from Lorentz-invariant field theory (see W. E. Thirring [15]).

(4) During the discussion at the conference Prof. F. Belinfante observed that in a version a linear theory which he proposed the deflection of light rays in the Sun's field is greater than the value (5) and $\alpha_{\max} = 2''.15$. Therefore, measurement of deflection of rays with an accuracy of a few percent is already sufficient for a decision as to fate of this non-Einsteinian theory.

gives only $\psi_{\text{rot}} \simeq 0''.02$ per century, whereas the accuracy attained nowadays in determining the angle $\psi$ for Mercury amounts to about $1''$. For the Moon the relativistic shift of its perihelion due to the Sun's field gives $1''.9$; the shift due to the Earth's field equals $0''.06$; on the other hand, the influence of the Earth's rotation leads to an additional rotation of the order of $3\times 10^{-4}$ seconds per century. Such an effect obviously cannot be measured. In this connection in 1956, before the first sputniks were launched, the author found very interesting the fact [19] that for satellites close to the Earth the additional rotation of the perihelion (i.e. due to the Earth's rotation) attains a value of $\psi_{\text{rot}} \simeq 60''$ and the rotation of their orbital nodes reaches $\Delta\Omega \simeq 20''$ per century, i.e. the values obtained are of the same order as the full effect of G.R.T. for Mercury. Unfortunately, the perturbations in the orbits of near satellites are so great that it is obviously unrealistic at present to measure the "rotational effect" for them.

Nevertheless, we have mentioned the proposal of using satellites for measurement of the "rotational effect", in particular because this proposal has induced Prof. H. Thirring to recall an interesting historic event. In the year 1918 on an evening in May Thirring was telling Einstein about his work with Lense. Einstein was complaining that the "rotational effect" is so small for the Moon and, looking at the evening sky, he exlaimed; "Wie schade, daß wir nicht einen Erdmond haben, der gerade nur außerhalb der Erdatmosphere umläuft!" Nowadays we have such a "near Moon", and not only one; but in all probability these cannot yet be used to verify the "rotational effect".

Another (in some sense congenerous) effect, the precession of a gyroscope, can apparently play the role of the fourth observed effect of G.R.T. This effect has been examined in detail by L. I. Schiff (see [11] and also [7]). At this conference L. I. Schiff is expected to read a paper on the experiment with the gyroscope, therefore, we shall limit ourselves to a few remarks.

If the angular momentum (spin) of a gyroscope in the frame of reference accompanying it is $\vec{S}_0$, then, according to G.R.T. [11],

$$\frac{d\vec{S}_0}{dt} = [\vec{\Omega}\vec{S}_0]$$
$$\vec{\Omega} = \frac{1}{2}[\vec{f}\,\vec{v}] + \frac{3\varkappa M}{2c^2r^3}[\vec{r}\,\vec{v}] + \frac{\varkappa I}{c^2r^3}\left\{\frac{3\vec{r}}{r^2}(\vec{\omega}\,\vec{r}) - \vec{\omega}\right\} \tag{7}$$

Here $\vec{v} = \dfrac{d\vec{r}}{dt}$ denotes the velocity of the gyroscope $\left(\dfrac{d\vec{v}}{dt} = \vec{f} - \dfrac{\varkappa M}{r^3}\vec{r}\right)$ and $\omega, M$ and $I$ the mean angular velocity, mass and moment of inertia of the Earth, respectively. The last term in (7) takes into account the rotation of the Earth and its measurement is entirely equivalent to the measurement of the "rotational effect". For a gyroscope placed on the Earth the influence of the Earth's rotation (in the sense that the component $g_{0\alpha}$ is involved) is essential. If the axis of the gyroscope is perpendicular to the Earth's axis, th precession of the gyroscope gives $\varphi = 3.5\times 10^9 \times (1+\cos^2 \lambda_r)$ radians per dae (here $\lambda_r$ stands for the latitude of the site of observation). Thusy $\varphi_{max} \simeq 7\times 10^{-9}$ radians per day $\simeq 0''.5$ per year. The maximum angle of ro-, tation of a gyroscope on a near satellite is $\varphi_{max} \simeq 7\times 10^{-9}$ radians per revolution $\simeq 8''$ per year, whereas the influence of the Earth's rotation does not exceed 2% of the overall effect. The question of when such an experiment could be performed successfully and whether it would be better to perform it on the Earth's surface or on a satellite, is not clear to us. From the point of view of the confirmation of G.R.T. it would be important to know what result could be obtained for the precession of the gyroscope from the non-Einsteinian gravitational theories. It is to be expected that with due account for the rotation of the Earth at least the result of G.R.T. would be different from the other ones.[5] Then, in that case the gyroscope experiment would play precisely the same role as the "rotational effect" considered above (i.e. the influence of the Earth's rotation upon the orbit of satellites). In any case, it seems to us that the gyroscope experiment is the most interesting and important of all those suggested for the purpose of further verification of G.R.T.

There are in the literature some other proposals concerning the experimental verification of G.R.T. or of its foundation. For instance, it is possible to ascertain that the velocity of light is independent of the direction of the ray relative to the line Earth–Sun, etc. This would be a verification of the principle of equivalence. There is another, even more interesting, proposal, viz. the discovery of gravitational waves, in the first place waves of cosmic origin. The present author is not sufficiently acquainted with the progress made on this problem. Of course, the discovery of gravitational waves would be interesting and perhaps would be of real use to astronomy for studying various phenomena (for instance, the motion of double stars, explosions of supernovae or creation of radio-galaxies). As to the verification of G.R.T. it must be borne in mind that the waves on the Earth will always be weak and, therefore, they can be described within the limits of linear gravitational theories.

It should be stressed in general that all the experiments and observations carried out on the Earth are directly connected with the study of only a weak gravitational field. It seems to us that the probability of any new things being discovered in this field tends to zero. Nevertheless, attempts to do this are justified insofar as their "mathematical expectation", i.e. the

[5] *Note added in proof.* The calculations by W. I. Pustovoit confirm this sugestion.

product of the probability and their possible significance (in the case of their revealing non-agreement with G.R.T.) turns out to be a large quantity.

In conclusion, we should like to make the following observations. Why, strictly speaking, must we verify a physical theory? Obviously, there are two reasons: firstly, we must know the limits of applicability of a theory and, secondly, we must learn the correct way of generalizing it (or, if you prefer, we should learn what needs to be explained beyond the limit of a given theory). From this point of view, in all probability, there is nothing to be expected from further testing of G.R.T. for weak fields. G.R.T. can already be considered as very well verified at least in that there is every basis for using Einstein's gravitational field equations (1) with sufficient confidence and without any restrictions (naturally, we have in mind only macroeffects).

These equations, of course, might be changed and generalized somehow without changing the known consequences. But the same situation exists in other cases. For instance, Maxwell's equations $\partial F_{ik}/\partial x_k = \frac{4\pi}{c} j_i$ may be generalized in at least three directions (addition of the term $\mu A_i$, owing to the rejection of the condition of gauge invariance; introduction of terms with higher derivatives; transition to non-linear theories). Nevertheless, in problems not involving quantum effects we use Maxwell's equations (in Minkowski space) with full confidence and every right. Just as, let us stress it once more, there is every right for doing this as regards the Einstein equations (1). Even if these equations should have to be generalized, in our opinion this could only be as a result of the study of strong fields. There are two possibilities here. Firstly, strong fields occur in the case of neutron stars. For a star with a mass of $M \sim M_\odot \simeq 2\times 10^{33}$ g the gravitational radius is $\varrho \sim \frac{\varkappa M_\odot}{c^2} \simeq 10^5$ cm. At the same time, the radius of such a star with a nuclear density $\sim 10^{14}$ g.cm$^{-3}$ is $r \sim 10^6$ cm. Thus, the gravitational field in the case of neutron stars (perhaps they would be better called baryon stars) can indeed be strong. Such stars are perhaps created during explosions of some supernovae. It is important to study the dynamics of the creation of neutron stars and in particular to take into account their rotation (a central-symmetric neutron star, after it has already been created, practically cannot "let one know about itself"; besides stars usually rotate and it is necessary to study the problem of neutron collapse for rotating stars). The second trend in the study of strong fields — a trend that is better known and more wide spread and, probably also more important—is the study of cosmological problems. We cannot take them into consideration here. We note only that the investigations of recent years (in particular very important investigations by E. M. Lifshitz a.o. [20]) clearly show how many various possibilities are involved in the Einsteinian equations when account is taken of the non-homogeneity and anisotropy of space. Of course, this depends upon one's point of view (the same holds for remarks further on about quantization), but the author does not at present see any basis even in cosmology to go beyond the limits of the Einstein equations (1).

As to quantum effects, it can be said that in macroscopic space-time regions they are negligible and, therefore, (as far as we know) of little importance from the point of view of their observability. What may really prove exceedingly important is, so to speak, the quantization of geometry—i.e. transition to new space-time ideas "in the small" (i.e. in the micro-world). But that already is not G.R.T., even if it proves fruitful to use the ideas and methods of Einsteinian G.R.T. in building up a new quantum theory of (microscopic) space-time.

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## DISCUSSION

F. J. Belinfante:

Regarding the observations of bending of light, I should like to point out that if one could improve just one order of magnitude, one might be able to disprove my theory, at least for all practical purposes. The theory has a Lagrangian which consists of a free field plus an interaction Lagrangian, and I just put in enough constants that I can fit all the factors, of course. Now, nobody would want to have a coefficient 9/11, or something like that, in the free field Lagrangian. But if you exclude that, and if you say only there are two constants in the interaction, if you say they may have any crazy value, then you are forced to get a bending of light which has a value which is not the Einstein value, but which is something like 2″.1. The experimental figure lies in between these two. Therefore, a slight improvement in this would exclude my theory. I mean, I would certainly not want to propose to adopt any theory which has crazy coefficients in the free field Lagrangian; and therefore it would, for all practical purposes, disprove such a theory.

L. INFELD:

You put some constants in your Lagrangian, don't you?

F. J. BELINFANTE:

My Lagrangian has some four constants to start with. I could have put in even more, I did not do that. But, there are two constants in the free field Lagrangian and two constants in the interaction; and if you make the free field constants have decent values, then either all the other effects are far from the experimental limits, or you must get something like 2″ for the bending of light.

L. INFELD:

I definitely prefer the theory that does not require any constants.

F. J. BELINFANTE:

I know that my theory can be disproved; but I'd like to see an experimental and not a philosophical disproof.

V. L. GINZBURG:

I agree, and I am quite aware of the fact that new experiments with light deflection are needed.

C. MØLLER:

May I ask you about this experiment of Blamont? Does it concern the red shift of light from the Sun?

V. L. GINZBURG:

Yes, it is the strontium line from the Sun.

C. MØLLER:

I thought that there usually was a difficulty—you know, the difference between the light from the limb of the Sun and the center?

V. L. GINZBURG:

Yes, I know this in some detail and with pleasure I will tell you more about it, though I don't understand everything here. The old observations by Miss Adam showed that on the limb there is full agreement with relativity, and in the center there is no agreement. This seems to me to be quite reasonable and possible to explain if one takes into account the motions in the solar atmosphere. In the work of Blamont and Roddier (I have only read a short note in the *Phys. Rev. Letters*) they have made some precautions, namely precautions about the calibration, etc. and they found the right results also in the middle, not in the same agreement as in the limb, but in disagreement with Miss Adam's results. But it seems to me that their work is quite a modern one, and reliable.

A. J. COLEMAN:

I did enjoy Prof. Ginzburg's talk. But I was disappointed by his treatment of the first effect. It was the treatment of a person who believes in general relativity, and nothing can convince him otherwise. When I was a student I read many textbooks, some of them written by distinguished people present here, and when they came to repeating the results of the experiments, when they come to this effect, they always said: there is no disagreement; it is a complicated effect, but there is no disagreement with relativity. Of course, after you've gone through three or four hundred pages of a book on as difficult a topic as general relativity, it would be a great disappointment if there was disagreement. Now, if you go, as I have done, and read all the papers by the astronomers, going back to St. John and Evershed and Miss Adam; the fact is that there have been many lines among these observations that gave disagreement. Now in the case of St. John, he believed in relativity, at a certain moment he was converted; and he said you could invent currents in the Sun's atmosphere to make rough agreement between observation and the theory; and this is roughly what Prof. Ginzburg has said. Currents are now more complicated, because you can now talk about all this bubbling on the surface of the Sun. However, Burns, already in 1930, Evershed before him, and Miss Adam, to my pure-mathematician's point of view, found cases of lines from the same multiplet, in which the corrections to the shift for the lines in the same multiplet differ from one another by an amount roughly of the order being predicted, or 50% of the amount being predicted. Now you cannot explain that by any theories of currents, because lines in the same multiplet come from atoms that are in the same stream, going up or coming down.

*Several Questioners*:

Why?

P. G. BERGMANN:

Atoms of the same element, but why in the same stream?

A. J. COLEMAN:

Because they are being energized at roughly the same energies. Miss Adam in a study about two years ago, involving 200 lines, came to the conclusion that the observed curve for the limb effect, when you go from the center to the edge, the observed shape of this curve, could not be explained by any effect proportional to velocity. I'm not an astrophysicist, I can't interpret that result, but that was debated at the Royal Society—in fact, Prof. McCrea was present there. It seems to me that there's a possibility that here, in this particular result, there is no proof or disproof of relativity—on that I am an agnostic—but it could be that here we do have an observational effect which is related to the interaction between the electric and the gravitational fields. This indeed is what Whitehead claimed. Whitehead's theory, which has been dismissed here, doesn't have any arbitrary constant in it, it does predict all these results that have been reported; but it also predicts the limb effect in the shift of lines coming from the sun, and it was because it did predict this that it first aroused my interest. And I do think that this is a point which still needs to be pursued.

V. L. GINZBURG:

It is rather difficult for me to exclude my own feelings and to say, perhaps, everything about the situation. You must excuse me. I was very much interested in the proof of this thing about the red shift, because of all this work of St. John, Adam, etc.; and I hoped the satellite experiments will show us the truth. But, I and perhaps everybody here, was very impressed by the new Mössbauer technique. It is quite new work, and after this it is very difficult to argue from the previous state of things. It is very difficult to question anything here. I must mention that in non-Einsteinian theories of course we can see the insensitivity of the red shift; because out of ten, perhaps eight agree in this effect; but only three in another two effects. So it is a most simple effect. Another point I wish to mention about all these astrophysical experiments. We can hope that there is some disagreement because of the granulation, the line shift due to pressure, and many other things. These people (Blamout and Roddier), as I understand from their paper, are quite advanced.

A. J. COLEMAN:

I think that, at the moment there is no known way of reconciling the actual observational findings with general relativity. I'm not saying it's a disproof. We can hope that they will be reconciled but they are not yet reconciled.

V. L. GINZBURG:

No, I rather disagree, because they (Blamont and Roddier) calculated the shift due to the pressure and the granulation (it is a complicated thing, for different lines it is not the same). So I don't think that here we have such a thing. All I can do is to give you this reference.

# PROPOSED GYROSCOPE EXPERIMENT TO TEST GENERAL RELATIVITY THEORY*

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THERE is a striking difference between the experimental bases of the special and general theories of relativity. Special relativity has been amply verified in several aspects: for example, the dynamics of electrons and protons moving with speeds close to that of light, the time dilation of the decay of rapidly moving $\pi$ mesons, the classical radiation from fast electrons in magnetic fields, and the predictions of relativistic quantum electrodynamics with respect to bremsstrahlung and more subtle radiative processes. In each of these categories, so many experiments have been found to yield results in agreement with theoretical expectation (and none in disagreement) that there can be no reasonable doubt as to the correctness of special relativity as a description of natural phenomena within its domain of validity. The situation is completely different with general relativity. Here, there are thus far only the three so-called "crucial tests": the gravitational red shift, the deflection of starlight passing close to the sun, and the precession of the perihelia of the orbits of the inner planets, especially Mercury. And of these the first, which was recently established in terrestrial experiments, [1], [2] was shown by Einstein [3] to follow directly from the equivalence principle, already established experimentally by Eötvös,(1) without employing the formalism of general relativity.

It is not surprising that it is so difficult to establish the experimental superiority of Einstein's theory of gravitation over that of Newton. Experimental situations that involve special relativity require particles moving with speeds close to that of light, and several kinds of such particles are plentifully produced by modern accelerators. The corresponding situation in general relativity would call for strong gravitational fields; the significant parameter is $GM/c^2r$, where $M$ is the mass of the gravitating object, $r$ the distance from its center, $G$ the Newtonian constant of gravitation, and $c$

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(1) These experiments are summarized in a posthumous paper of Eötvös [4].

the speed of light. This parameter is roughly $10^{-6}$ at the surface of the sun and $10^{-9}$ at the surface of the earth. Thus available gravitational fields are very weak, and Newtonian theory provides an excellent approximation.

It is because of this paucity of experimental information that a new experiment was recently proposed [5]. This would consist in moving a torque-free spherical gyroscope through the gravitational field of the earth, and observing the precession of its spin axis. According to Newtonian theory, there is no precession of gravitational origin. However, the Einstein theory predicts that the angular momentum vector $\boldsymbol{S}_0$ of the gyroscope, measured by a co-moving observer, would change with time in accordance with the equations:

$$dS_0/dt = \boldsymbol{\Omega} \times \boldsymbol{S}_0, \tag{1}$$

$$\Omega = (1/2mc^2)\boldsymbol{F}\times\boldsymbol{v}+(3GM/2c^2r^3)(\boldsymbol{r}\times\boldsymbol{v})+(GI/c^2r^3)[(3\boldsymbol{r}/r^2)(\boldsymbol{\omega}\cdot\boldsymbol{r})-\boldsymbol{\omega}]. \tag{2}$$

Here, $m$ is the mass of the gyroscope, $\boldsymbol{r}$ is its position vector with respect to the center of the earth, $\boldsymbol{v} = d\boldsymbol{r}/dt$ is its velocity vector, $\boldsymbol{F}$ is any nongravitational force that may be applied to the center of mass of the gyroscope, and $M$, $I$, and $\boldsymbol{\omega}$ are the mass, moment of inertia, and rotational angular velocity vector of the earth.

Equations (1) and (2) were calculated [5] by means of the dynamical method of Fock [6] and Papapetrou [7]. The first equation shows that the magnitude of the spin angular momentum of the gyroscope, and hence the rate of rotation measured by a co-moving observer, is constant. It also shows that the direction of the spin axis rotates with the vector angular velocity $\boldsymbol{\Omega}$. The second equation states that there are three parts to $\boldsymbol{\Omega}$. The first term does not involve $M$, and hence is not a gravitational effect; it is the Thomas precession [8], first discovered in the special relativistic treatment of atomic systems. The second term is the geodetic precession caused by motion through the gravitational field of the earth, whether or not the earth is rotating [9], [10], [11]. The third term arises from rotation of the earth, and is analogous to the rotation effect predicted in a different connection by Lense and Thirring [12].

Measurement of the precession predicted by Eq. (1) and (2) would provide a new experimental test of general relativity theory. At a conference on experimental tests held at Stanford University in July 1961,[(2)] the late Professor H. P. Robertson stated without proof the expression for the geodetic precession (second term of Eq. (2)) in an arbitrary spherically symmetric metric.[(3)] Following Robertson, we write the metric for the non-rotating earth in the most general isotropic form:

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[(2)] For a summary of this conference, see *Physics Today*, November 1961, p. 42.

[(3)] This result is also given without proof in a posthumous paper to be published shortly in the *Journal of the Society for Industrial and Applied Mathematics*.

$$ds^2 = [1-2\alpha(GM/c^2r)+2\beta(GM/c^2r)^2+\ldots]dt^2$$
$$-(1/c^2)\,[1+2\gamma(GM/c^2r)+\ldots]\,(dx^2+dy^2+dz^2), \qquad (3)$$

which includes the leading terms of an expansion in powers of the small parameter $GM/c^2r$. The dimensionless numbers $\alpha$, $\beta$, $\gamma$ are expected in general to be of order unity and are all equal to $+1$ in the Einstein theory. It should be noted that there is no loss of generality in assuming the isotropic form, since if $dx^2+dy^2+dz^2$ is expressed in spherical coordinates, the radial and angular parts can be given different coefficients by means of a transformation of $r$.

Now as is fairly well known, the measurable quantities in the three "crucial tests" referred to earlier are, in lowest order, proportional to the following combinations of the $\alpha$, $\beta$, $\gamma$ that appear in Eq. (3):(4)

| | | |
|---|---|---|
| gravitational red shift: | $\alpha$; | (4a) |
| deflection of light: | $\alpha+\gamma$; | (4b) |
| perihelion precession: | $2\alpha(\alpha+\gamma)-\beta$ | (4c) |

The number $\alpha$ not only determines the red shift in accordance with Eq. (4a), but also is responsible for the leading term in the gravitational acceleration produced by the mass $M$, and hence for the orbits predicted by Newtonian theory. Thus $\alpha$, or more precisely the product $\alpha G$, must be regarded as very well determined; with the conventional definition of $G$, $\alpha$ is equal to $+1$ with great accuracy. The observational errors associated with the measurement of the deflection of light are roughly 20% [14], so that $\gamma$ is not very well determined from Eq. (4b). On the other hand, the precession of the perihelion of the orbit of the planet Mercury agrees with the prediction of general relativity theory within about 2% [14], so that the combination $2\gamma-\beta$ is known from Eq. (4c) with this accuracy.

It is therefore of some interest to see how the geodetic precession of a gyroscope depends on $\alpha$, $\beta$, and $\gamma$. To this end, the dynamical calculation of reference [5] will be generalized to the metric given in Eq. (3). We shall do this only for the geodetic term, and quote results only for the isotropic metric and the Pirani boundary condition. The pertinent formula is then Eq. (24) of reference [5], with the nongravitational acceleration $\boldsymbol{f}$ set equal to zero and the mass parameter $m$ replaced by $GM/c^2$:

$$d\boldsymbol{S}/dt = (GM/c^2r^2)[\boldsymbol{S}(\boldsymbol{r}\cdot\boldsymbol{v})+2\boldsymbol{v}(\boldsymbol{r}\cdot\boldsymbol{S})-\boldsymbol{r}(\boldsymbol{v}\cdot\boldsymbol{S})], \qquad (5)$$

A recalculation of Eq. (5) with the metric of Eq. (3) yields:

$$d\boldsymbol{S}/dt = (GM/c^2r^3)[(2\gamma-\alpha)\boldsymbol{S}(\boldsymbol{r}\cdot\boldsymbol{v})+(\gamma+\alpha)\boldsymbol{v}(\boldsymbol{r}\cdot\boldsymbol{S})-\gamma\boldsymbol{r}(\boldsymbol{v}\cdot\boldsymbol{S})]. \qquad (6)$$

(4) For a discussion of the physical basis of (4) see [13].

Comparison of Eq. (5) and (6) shows that they agree when $\alpha$ and $\gamma$ are given the Einstein value $+1$.

It is necessary to express Eq. (6) in terms of the angular momentum $\boldsymbol{S}_0$ measured by a co-moving observer. The relation between $\boldsymbol{S}$ and $\boldsymbol{S}_0$ involves a Lorentz transformation that is independent of the metric, and a coordinate transformation that depends on the form of Eq. (3). Thus the first of these is the same as the Lorentz transformation given as Eq. (31) of reference [5]:

$$\boldsymbol{S}_0 = \boldsymbol{S} - \frac{1}{2}[v^2\boldsymbol{S} - \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{S})]. \tag{7}$$

On the other hand, the coordinate transformation involves the space part of the metric, and hence $\gamma$:

$$\boldsymbol{S}_0 = [1+2\gamma(GM/c^2r)]\boldsymbol{S}, \tag{8}$$

which reduces to Eq. (25) of reference [5] when $\gamma = +1$. Combination of Eq. (7) and (8) to first order gives the relation between $\boldsymbol{S}$ and $\boldsymbol{S}_0$:

$$\boldsymbol{S}_0 = \left[1+2\gamma(GM/c^2r) - \frac{1}{2}v^2\right]\boldsymbol{S} + \frac{1}{2}\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{S}). \tag{9}$$

The time derivative of Eq. (9) is:

$$d\boldsymbol{S}_0/dt = d\boldsymbol{S}/dt - 2\gamma(GM/c^2r^3)\boldsymbol{S}(\boldsymbol{r}\cdot\boldsymbol{v}) - \boldsymbol{S}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}}) + \frac{1}{2}\dot{\boldsymbol{v}}(\boldsymbol{v}\cdot\boldsymbol{S}) + \frac{1}{2}\boldsymbol{v}(\dot{\boldsymbol{v}}\cdot\boldsymbol{S}), \tag{10}$$

where $\dot{\boldsymbol{v}} = d\boldsymbol{v}/dt$. It is sufficient for a first-order calculation to use the Newtonian approximation for $\dot{\boldsymbol{v}}$. For the geodetic term, we again drop the non-gravitational acceleration $\boldsymbol{f}$, and note further that the gravitational acceleration must be multiplied by $\alpha$ when the metric of Eq. (3) is used; thus Eq. (34) of reference [5] is replaced by:

$$\dot{\boldsymbol{v}} = -\alpha(GM/c^2r^3)\boldsymbol{r}. \tag{11}$$

Substitution of Eq. (6) and (11) into Eq. (10) then gives:

$$d\boldsymbol{S}_0/dt = (\alpha+2\gamma)(GM/2c^2r^3)[\boldsymbol{v}(\boldsymbol{r}\cdot\boldsymbol{S}) - \boldsymbol{r}(\boldsymbol{v}\cdot\boldsymbol{S})]. \tag{12}$$

As in reference [5], the difference between the differential time intervals $dt$ in the two coordinate systems may be neglected, as can the difference between $\boldsymbol{S}$ and $\boldsymbol{S}_0$ on the right side of Eq. (12). Equation (12) is thus equivalent to the geodetic term of Eq. (1) and (2), with the number 3 replaced by $\alpha+2\gamma$. This is in agreement with Robertson's conclusion that the geodetic precession is proportional to $\alpha+2\gamma$. Our derivation also shows that the magnitude of $\boldsymbol{S}_0$ remains constant even when the general metric of Eq. (3) is used. If follows that the gyroscope precession experiment provides a method for the determination of $\gamma$ that is independent of the deflection of light; it is also slightly more sensitive, since Eq. (4b) shows that the latter depends on $\alpha+\gamma$ rather than on $\alpha+2\gamma$.

The magnitude of the precession angular velocity given in Eq. (2) is roughly 0.4″ of arc per year if the gyroscope is at rest in an earth-bound laboratory, and carried through the earth's gravitational field by rotation of the earth. In this case, the three terms of Eq. (2) are of the same order of magnitude. If the gyroscope is in a satellite at moderate altitude, the geodetic precession is about 7″ per year, and the precession caused by rotation of the earth is about 0.1″ per year; in this case the gyroscope is in nearly free fall, so that the Thomas precession is practically zero. Since both of the experimental gyroscopes now under active consideration are intended for satellite use, it follows that the first result obtained will be an independent measurement of $\gamma$. Ultimately, it is hoped that the experiment will demonstrate for the first time, through the much smaller third term of Eq. (2), the effect of the rotation of a massive object on its gravitational field; this is also a prediction of the Einstein theory that has no Newtonian counterpart.

The more advanced of the two gyroscopes referred to above is the electric vacuum gyroscope.[(5)] It consists of an electrically conducting sphere that is constrained and supported by the electric fields between its surface and three mutually perpendicular pairs of close-fitting electrodes. As is well known, such support by electric fields is dynamically unstable, so feedback loops that adjust the field strengths in accordance with the sphere-electrode spacing must be provided. This is accomplished by using alternating voltages in nearly resonant circuits with external inductances, so that the change in capacity produced by motion of the sphere with respect to one pair of electrodes automatically changes the voltages in such a way as to restore it to the desired position. This approach can be extended to a three-phase electrical system, with one phase for each of the perpendicular electrode pairs; the sphere then becomes an electrically floating neutral. It seems desirable also for the gyroscope to have a slightly larger moment of inertia about one axis than the other two, so that it will spin naturally about this axis. Then the symmetry of the support is preserved if one of the three electrode pairs is maintained along the spin axis. Readout of the direction of the spin axis is being accomplished by an optical method that consists in viewing a sinusoidal curve etched around the equator of the sphere.

The second gyroscope consists of a superconducting sphere supported in a static magnetic field.[(6)] The sphere acts as a perfect diamagnetic, so that the support is dynamically stable and no feedback loops are required. Since low temperature is required in any event in order to maintain superconductivity, ambient electric and magnetic fields can be greatly reduced by using a superconducting shield. The low temperature also decreases thermal distortion since all coefficients of thermal expansion are the very small. The readout now being developed makes use of the Mössbauer effect. A small

[(5)] A. Nordsieck, private communication.
[(6)] W. M. Fairbanks, private communication.

amount of a suitable radioactive material is placed on the sphere, and the gamma rays from it pass through an absorbing plate that rotates coaxially and synchronously with the sphere. Any misalignment of the axes of sphere and plate will result in a periodic change in the relative velocity of the two, and hence a periodic change in the Mössbauer radiation measured by the detector placed beyond the plate. Laboratory tests indicate that this method of reading out the direction of the spin axis of the sphere will have sufficient accuracy for the precession experiment.

As remarked above, both experiments are planned for satellite use. The principal reason for this is that the effective acceleration of gravity in a satellite at moderate altitude (difference between the earth's gravitational acceleration $g$ and the acceleration of the satellite) is extremely small; it arises from external forces such as light pressure and atmospheric drag, and is probably of the order of $10^{-7}$ $g$. Thus the constraining forces required (electric and magnetic in the two gyroscopes described above) are very small, and extraneous torques that arise from these forces in conjunction with imperfection in construction are hopefully small enough so that they do not obscure the general relativistic precession. A secondary reason for use of a satellite is that the precession to be observed is much larger than in an earth-bound laboratory. On the other hand, it is apparent that any experiment is more difficult to accomplish and to monitor in a satellite than on earth; however, this disadvantage is believed to be outweighed by the factors just mentioned.

An additional refinement in the satellite experiment is also being given serious consideration. It was suggested by Pugh [15] and Sherwin [16] that the satellite be made to follow the gyroscope. This would require that the position of the gyroscope with respect to the satellite be sensed without exerting a force on the gyroscope, and that forces then be applied to the satellite so that it maintains a fixed position with respect to the gyroscope. There are three main consequences of such arrangement. First, the non-gravitational force that must be exerted on the gyroscope is reduced from $10^{-7}$ $g$ times its mass to zero, thus further reducing extraneous torques. Second, the satellite will follow a true gravitational orbit about the earth, and observations of it will provide precise information on the figure of the earth. And third, the forces that must be applied to the satellite in order that it follow the gyroscope may be interpreted in terms of atmospheric density. The last two would be useful by-products of the general relativity experiment that are of interest for geodesy and high-altitude meteorology.

Even without the "slaved" satellite described in the last paragraph, it seems likely that satellite gyroscope drift rates can be reduced to less than $0.1''$ of arc per year, and that the direction of the spin axis can be read out with an accuracy considerably better than $0.1''$. There would then remain the problem of relating the direction of the spin axis of the gyroscope to some externally established direction, presumably that of a star. This would require that the satellite also contain a rather good telescope. According to present plans, it seems possible that a telescope of about one meter aperture will be in orbit by the end of 1965.

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## GENERAL DISCUSSION

J. Weber:

*Measurement of the Riemann Tensor.* At the University of Maryland, we have constructed apparatus for detection of gravitational waves. This is a device to measure the Fourier transform of the Riemann tensor component $R_{0101}$.

The theory of this method[1] is that relative displacements occur in an elastic body in a curved space. A time dependent curvarture, therefore, interacts with the normal modes of an elastic body. For one-dimensional acoustic waves set up in this body in the $x'$ direction, we may choose coordinates such that the strain $\theta$ is given by the differential equation[2]

$$y\frac{\partial^2\theta}{\partial x'^2} - \varrho\frac{\partial^2\theta}{\partial t^2} - b\frac{\partial\theta}{\partial t} = c^2\varrho R_{0101},$$

here $y$ is the elastic modulus, $\varrho$ is the density.

We have employed this equation to set limits on gravitational radiation using the earth itself as a detector, at the time the earth's normal modes were identified by the California Institute of Technology Seismology group.

Our present apparatus employs the normal modes of an aluminium cylinder having a mass of about $10^6$ grams, two meters in length and about one-third of a meter in diameter. The theory of the sensitivity[3] has been given. The ultimate value is determined by the Brownian motion of the normal modes of the cylinder. To achieve this ultimate sensitivity, two important requirements must be met. The detector has to be isolated from the motions of the environment and means must be provided for coupling out the energy without introduction of too much noise.

I wish to report that these problems have been solved for the present apparatus, which operates in the vicinity of 1657 cycles per second. Isolation is accomplished by careful design of the suspension, which rests on acoustic filters. The suspension is accomplished by milling a slot all around the cir-

(1) A detailed account of the theory is given in the book General Relativity and Gravitational Waves, by J. Weber, Chapter 8, Interscience Publishers, Inc.; New York, London 1961.

(2) This is related to the equation of geodesic deviation deduced many years ago by Synge and Levi-Cività and employed by Pirani to discuss measurement of the curvature tensor by use of free particles.

(3) Op. cit.

cumference of the cylinder in the central plane normal to the longitudinal axis. A steel wire is wrapped around in this slot, with other members to assure stability. The energy is coupled out by securing piezo electric crystals around the cylinder near the slot. These are coupled to an amplifier of unusual design, with high impendance and good noise characteristics. The input circuit of this amplifier operates at liquid helium temperatures.

If $\omega$ is the angular frequency, the mean squared relative displacement of the ends of the cylinder due to Brownian motion is given by

$$x^2 = \frac{kT}{m\omega^2}.$$

Here $m$ is the mass, $k$ is Boltzmann's constant, and $T$ is the absolute temperature. In our case, the root mean square displacement which we can detect is smaller than $10^{-14}$ centimeters if we average over times longer than minutes.

Up to the present time, gravimeters were the only objects to use for measurement of a time dependent curvature tensor. These can detect a change in the acceleration due to gravity of roughly one part in $10^9$ over periods of seconds. These values imply that our apparatus in its present form has increased the sensitivity of this type of measurement by at least a factor 100,000,000.

My colleagues, Professor David M. Zipoy and Robert L. Forward have made important contributions to the development of this device. Our researches were supported by the U. S. Natural Science Foundation.

A. Peres:

I would like to point out that there may be a more efficient method to detect gravitational radiation. Here you are measuring the spectrum of the radiation, which is an instantaneous effect (the instantaneous rate of energy flow). On the other hand, you can have a measurement of momentum flow. It is possible to make a system move in such a way that gravitational radiation is emitted in a preferred direction, and then this system will recoil like a rocket. This is a cumulative effect, and I presume this would be an easier way to detect the emission of gravitational radiation. I have no time to give detailed calculations, but I think it is easier.

J. Weber:

I have one general comment to make on this. It would be a great mistake to say that this is the best method or that this will always be the best method. The only thing we can say is that this does represent an improvement of $10^8$; and I wish you would make detailed calculations on your method and publish them, and if it gives a better improvement than $10^8$ over existing methods, then your method is better.

A. Peres:

I'll show you these calculations.[4]

J. Weber:

Also, after you have published the calculation, be sure you build one, because it's just possible you might run into some major problems. One other comment I should make. That is, when these sensitivity figures were published, it was assumed that the apparatus would work right down to the thermal fluctuations. It turns out that we were rather naïve, and it is a kind of fortuitous combination of accident and luck that this apparatus does work down to the thermal fluctuations. If we had built the apparatus in some other part of the spectrum the problem of the isolation and the acoustic filters could not have been solved, and the sensitivity would have been a good bit worse than this. So it's entirely conceivable that the other Russian workers may have been working in some other part of the spectrum. But this really does work down to the thermal fluctuations.

V. L. Ginzburg:

May I ask you to write down two figures: the component of the Riemann tensor for the double star or for anything you have as a radiator of' the energy; and your sensitivity. So one can tell, perhaps, whether it is possible to detect something.

J. Weber:

I don't carry those in my head, but I do carry them around between the covers of a book which I published and of which I have a copy here. So when I bring the book we can go over this in detail.

V. L. Ginzburg:

Yes, but the results. Can you detect something?

J. Weber:

The results are that for any known double star at known distances we cannot.

R. P. Feynman:

At what frequencies?

J. Weber:

At 1600 cycles. You see, if we went to lower frequencies we'd have a better chance of detecting something but at lower frequencies the problem of isolation and filtering are much greater. The 1600 cycle apparatus requires

(4) *Phys. Rev.* **128**, 2471 (1962).

a vacuum chamber ten feet long and seven feet in diameter. To go down two orders you would have to have a 1000 foot vacuum chamber, seven hundred feet in diameter, and the National Science Foundation might not allow us to do it.

*Unidentified questioner*:

Have you measured any Dyson neutron binaries yet?

J. WEBER:

No, we have not discovered gravitational waves. It would be right to say we have discovered nothing; it would be wrong to say we have observed nothing. We have had the apparatus working. We observed it for a period of an hour; things go along all right, and we do occasionally see bursts. I don't believe that these bursts—the recorder suddenly moving off scale in the stillness of the night—are Dyson's neutron stars. On the other hand we don't know what they are, and we'll just have to run them down, and we'll have to observe for a period of four or five months in a quiet place before we can say that we have observed nothing, and therefore this means that the gravitational radiation has certain new limits.

A. SCHILD:

Professor Ginzburg, in his very interesting lecture of this morning, stressed the importance of experiments measuring new effects of general relativity theory, especially of experiments meaning the three classical effects to second order, i.e., to order $(M/r)^2$ (with $c = G = 1$). Ginzburg referred to a paper by Whitrow and Morduch[5] which claims that several gravitational theories (including my generalization of Whitehead's theory) predict exactly the same first order effects as general relativity in the three classical tests, the gravitational red shift, the bending of light by a star, and the perihelion rotation of a planetary orbit.

I wish to point out a simple heuristic argument,[6] essentially Einstein's original argument, which leads from the first order gravitational red shift to the conclusion that space-time is curved. This is important because the gravitational red shift and the equivalence principle are by far the best verified of the post-Newtonian effects of general relativity. The red shift has been measured not only directly by the laboratory experiments of the Harwell and Harvard groups, but also indirectly[7] by the accurate experiments of

[5] G. J. Whitrow and G. E. Morduch, *Nature* **188**, 790 (1960).

[6] A. Schild, *Time, The Texas Quarterly*, vol. 3, No. 3, p. 42 (Autumn, 1960); *Am. J. Phys.* **28**, 778 (1960); lectures at International School of Physics "Enrico Fermi", course on "Evidence for Gravitational Theoreis", June 1961 (Pergamon Press, in print); *The Monist*, now series, first issue (in print).

[7] L. L. Schiff, *Proc. Nat. Acad. Sci. U. S.* **46**, 871 (1960).

Eötvös and Dicke. If my argument is valid, then these experiments are sufficient to eliminate all flat space-time theories of gravitation, such as those of Poincaré, Birkhoff and Whitehead.

The first part of the argument is illustrated by Fig. 1a, and repeats the simple argument for the red shift given by Ginzburg. In a static gravitational field, a photon of frequency $\nu$ and mass or energy $h\nu$ at level II (gravitational

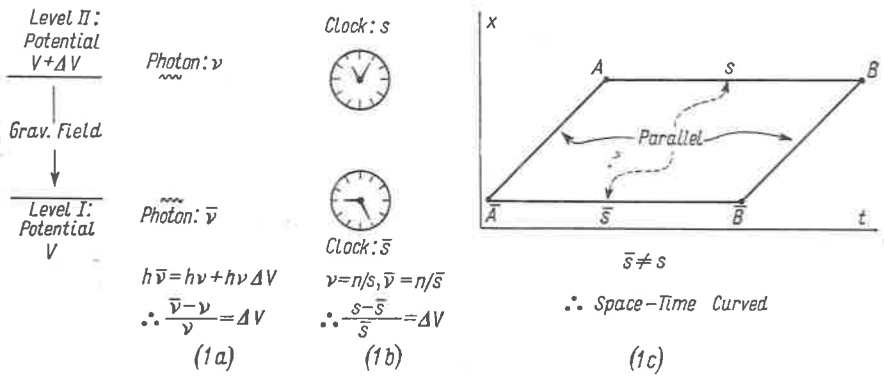


FIG. 1

potential $V+\Delta V$) falls to the lower level I (gravitational potential $V$) where its frequency is $\bar{\nu}$ and its energy $h\bar{\nu}$. The gain in energy must equal the mass $h\nu$ multiplied by the difference $\Delta V$ in the gravitational potential. Thus $h\bar{\nu} = h\nu + h\nu\Delta V$ and we obtain the gravitational red shift formula $(\bar{\nu}-\nu)/\nu = \Delta V$. It states that light moving up in a gravitational field must change in frequency from $\bar{\nu}$ to $\nu$ and thus become redder.

The second part of the argument is illustrated by Fig. 1b. We now consider the light moving up from level I to II, not from the point of view of a quantum phenomenon but from the dual point of view of a wave phenomenon. Let us think of a radio station at level I sending out a continuous monochromatic radio signal of frequency $\bar{\nu}$. A receiver, resting at level II, receives the signal at the lower frequency $\nu$. We now have an apparent contradiction. Since the whole set-up is stationary, and since wave crests cannot originate or disappear between sender and receiver, it seems that a change in frequency is impossible. The way out of this difficulty is to assume that time flows at different rates at different levels in a gravitational field. If a clock resting at level I measures a time interval $\bar{s}$ for $n$ oscillations of the radio wave and if a clock resting at level II measures a time interval $s$ for $n$ oscillations, then $\nu = n/s$ and $\bar{\nu} = n/\bar{s}$. Substituting this into the gravitational red shift formula above, we obtain the gravitational time dilatation formula $(s-\bar{s})/\bar{s} = \Delta V$.

The third part of the argument is illustrated by Fig. 1c. We begin by assuming that special relativity is valid and the space-time is Minkowskian with line-element $ds^2 = -dx^2 - dy^2 - dz^2 + dt^2$, where $ds$ is the element of proper time measured by a physical clock along its world line. We now consider a heavy, gravitating body $M$, spherically symmetric, isolated, and initially at rest the inertial frame $x$, $y$, $z$, $t$. Then, independently of any assumptions about the gravitational field, it follows from the isolation and spherical symmetry of $M$ that it will be permanently at rest, and that the world line of any particle resting near the surface of $M$ will be a straight line parallel to the $t$-axis. Let our two clocks be at rest at two different levels in the gravitational field of $M$. Figure 1c shows a space-time diagram of the two clocks. $\bar{A}\bar{B}$ is the world line of the lower clock and $AB$ that of the higher clock: the two world lines are parallel. $\bar{A}A$ is the world line of a light signal used to compare the readings of the two clocks at the beginning of an experiment, $\bar{B}B$ that of a light signal used to compare the clocks at the end of the experiment. Since the gravitational field is static, it follows that the propagation properties of light from level I to level II are independent of the time when the light was emitted; they are the same for the light signals $\bar{A}A$ and $\bar{B}B$. Therefore the lines $\bar{A}A$ and $\bar{B}B$, though not necessarily straight, are parallel. In flat Minkowski space-time it now follows that $\bar{A}\bar{B}BA$ is a parallelogram and that the two sides $\bar{A}\bar{B} = \bar{s}$ and $AB = s$ are equal. However, the gravitational time dilatation effects state precisely that $\bar{s}$ and $s$ are not equal. In geometry the absence of parallelism and of parallelograms with the usual properties is characteristic of a curved space. Therefore the results of this paragraph provide a strong heuristic argument which leads from the gravitational red shift to the conclusion that space-time is a curved Riemannian manifold whose line element $ds$ is the element of proper time measured by a physical clock along its world line.

R. P. FEYNMAN:

Frequencies of atoms can be shifted, for example, by the environment, in the sense that the electric field can affect the frequency of an atom. And so it's conceivable from another point of view, with which I don't agree, but which is at least a possibility, that in different regions, different distances from the earth, the environment is different, and the atomic frequencies are shifted thereby. So that the source is not at the same frequency as the receiver but emits at a different frequency; so that when you put the object up in the air it emits at a different frequency, and when you receive it, it doesn't check with the value you got when you were checking then on the ground. And this is a perfectly legitimate point of view at this level.

R. SACHS:

I think that it's a very bad analogy, because in the electric case you have some kinds of clocks that are not affected by the electric field; whereas here the assertion is that it's universal over all clocks.

R. P. FEYNMAN:

I understand, of course, the philosophy of Einstein. It's perfectly true that if all the clocks are shifted by the same amount that it's possible to interpret the other way; but what Dr. Schild is saying is that he doesn't understand how people can take another point of view, and have two kinds of spaces, or something. Now it's perfectly legitimate to have a theory in which the speeds of different kinds of clocks may not be exactly the same. It's not a matter of absolute principle that the two times be the same. And that it would be true that in most instances it's the same, or in the first order it's the same. So it is possible to have a theory in which the environment affects the clocks. It's possible; I don't like it, I know it's not the standard philosophy, but it's easily appreciated that it's an alternative.

F. J. BELINFANTE:

On the one hand, I agree with Feynman that there is the possibility that he mentions; in fact, we used this particular thing to get the red shift in that theory about which I spoke this morning. However, I want to point out that the argument that a change in potential should give a change in frequency is very much in contradiction to what we, in the electron case, are used to; where, for instance, if we have an electron accelerated from one potential to another, while the total energy remains the same, the kinetic and potential energies change in opposite ways; we say the frequency remains the same, only the wave length changes. A change in the frequency by a change in potential energy is something contrary to any elementary wave mechanics I have ever seen.

A. I. JANIS:

It seems to me that Feynman is not really disagreeing with Schild. Schild said that the only thing he could see as an alternative to the geometry was something *ad hoc*; and *ad hoc* seems to me to mean making a hypothesis which is not really checkable; and the hypothesis that you wish to make is that all clocks will be affected by the gravitational field in this way, and you have no other clock to bring around to see.

R. P. FEYNMAN:

This isn't a question of the meaning of *ad hoc*. The point is that you are talking about somebody else's theories. There is Birkhoff's theory and other theories which are specific theories which tell how something is supposed to move in different places; and it's a consequence of those theories that an object that's put in that place would move with a different speed, and make a different ringing speed of frequency, and that's all I said. I was only trying to explain what the other theories say, how the other theories look at it. I understand that it's the truth, that it's doomed.

A. SCHILD:

In Whitehead's theory, the way Whitehead himself did it, he doesn't get quite the red shift of general relativity. He gets a slightly different one, which depends on the mechanism of the clock. And that's what you would expect; that different types of clocks would be differently affected by the gravitational field. Now if you want to get exactly the value given by the equivalence principle argument, and let's assume the experiments come out 100% accurate, and they give just this red shift; then what it amounts to, as far as I can see, is that you're saying that there is a length $d\tau^2$, which is Minkowskian, which is there just to determine the gravitational potentials; but then everything you actually measure with clocks or rigid rods measures something else which is a $ds^2$. And that, I simply say, is artificial. If one wants to get exactly this thing which comes out of the equivalence principle out of a flat space-time theory, it seems to me it involves very artificial sorts of things.

(The following argument occurred to the speaker some days after the discussion.) I completely agree with Sachs' remarks and wish to add this: There is another important difference between the action of gravitation on a physical system and the action of all other fields. In the case of gravitation, the red shift (or the equivalence principle) teaches us that it is the potential of the field which affects physical systems, including clocks; in the case of all other fields, it is the field strength which affects a physical system or a clock.

Consider the example illustrated by Fig. 2. In a region where a field is present (region III), two distributions of the sources of the field (regions I′ and II′) are imbedded, the source distributions both being in the shape of inhomogeneous spherical shells and such that in the interior of the shells (regions I and II) the field vanishes identically; in general the constant potential (or potentials) in region I will differ from that in region II. Assume now that the field considered is an electromagnetic field in flat Minkowski space-time (special relativity), the sources in regions I′ and II′ being suitable charge and current distributions. Then two identical physical systems, one inserted in region I and the other in region II, will behave identically. If the physical systems are two clocks, they will run at the same rate (our whole set-up is macroscopic; quantum mechanical interference between the two physical systems is assumed to be negligible). The same argument applies to all field theories within the framework of special relativity. If the field considered is gravitational, then the gravitational red shift, the Mössbauer experiments at Harwell and Harvard, show that clocks I and II run at different rates.

I believe that the only possible way out of this dilemma, other than making artificial and *ad hoc* assumptions, is to give up the flat Minkowski space-time of special relativity and to assume that space-time is curved, i.e., to go a long way in the direction of Einstein's theory of general relativity. By artificial and *ad hoc* assumptions I mean, for example, gravitational theories with two space-time metrics, the first (usually flat) serving one purpose only, that of making it easy to define and calculate the gravitational

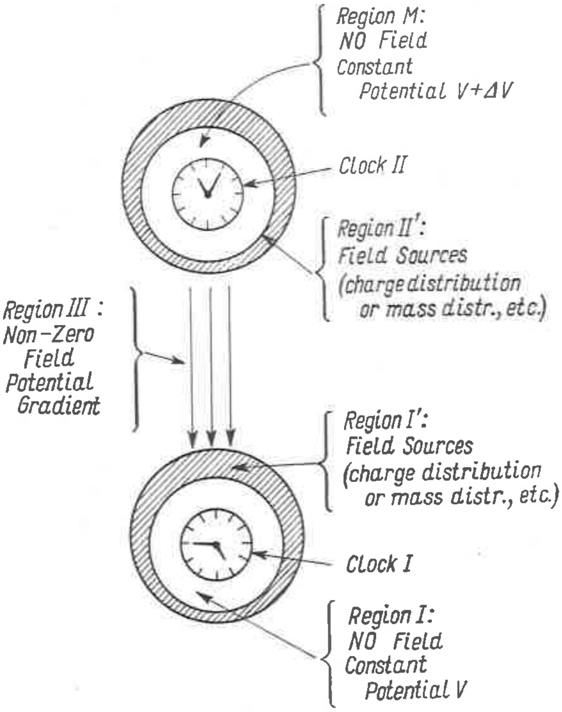


FIG. 2

field, and the second metric having physical content, i.e., being measurable by means of clocks, rigid rods, etc.

N. ST. KALITZIN:

*Expanding Galactic Systems and a New Application of Einstein's General Theory of Relativity.** As opposed to the theory of Friedmann, Einstein, Lemaitre *et al.* of an expanding or contracting universe with a constant mean density of matter in it, we propose a theory in which the vague conception of the "whole universe" is nowhere used. Our theory is based on the hierarchical structure of the clusters of galaxies. The method we use is the method of successive approximations. Thus, initially we accept that a multiple galaxy or a group of galaxies is immersed in a space which is Euclidean

* This paper has been published in the *Monthly Notices* of the Royal Astronomica Society, Vol. **122**, No 1, 41 (1961).

at infinity. Then we assume the same for a supercluster of galaxies. Finally, we accept the same for a metagalaxy. The idea of our model for a hierarchical universe is that the density within any unit is so much greater than the average density on the next larger scale that the unit may be treated as though it were in an empty space.

The model which we investigate is the following: we consider a spherical region $G$ in which we have a spherically symmetric distribution of matter. Inside the region $G$ we assume that the pressure is zero and that the density depends only on the time, whilst outside the density of matter vanishes and the field is asymptotic to the Euclidean (or rather Minkowskian) space at infinity. This model has been considered by Mc Vittie, Tolman, Datt, Einstein and Strauss, Bondi.

According to our assumption that the density $\varrho = \varepsilon/c^2$ in the region $G$ depends only on the time, it follows that in a system of coordinates, which at each point moves with the matter at this point, we have (after Datt) the following well-known particular solution of Einstein's gravitational equations in $G$:

$$ds^2 = c^2dt^2 - T^2(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2) \tag{1}$$

where $r$, $\theta$, $\varphi$ are "spherical" space coordinates, $t$ — time coordinate, $ds^2$ is the square or the four-dimensional distance between two neighbouring points, and $T^2$ is a function of $ct$ alone.

Since $T$ represents the relative measure of the metric distance between two points in $G$ as a function of the time, then

$$\frac{1}{T}\frac{dT}{d(ct)} = \frac{\dot{T}}{T} = H \tag{2}$$

gives the expansion (or the contraction) in the region $G$.

The Einstein gravitational equations for the region $G$ take, in this case, the form

$$2\dot{H} + 3H^2 = 0, \tag{3}$$

$$3H^2 = \frac{8\pi k}{c^4}\varepsilon \qquad (k\text{—Newton's constant}). \tag{4}$$

According to our assumption, the density of matter vanishes outside the region $G$. Therefore the field outside $G$ must be given by the Schwarzschild exterior solution.

The boundary conditions of our problem are that at the boundary of $G$ the field (1) shall go over continuously up to the first derivative into the Schwarzschild field in its conformally Euclidean representation. This problem for the line element (1) is solved by Einstein and Strauss (*Rev. Mod. Phys.*, 1945). This proves that (1), together with (2), (3) and (4), is a correct solution of our problem and can represent the physical properties of our model.

The relation between the expansion quantity $H$ and the average density $\varepsilon/c^2$ in the region $G$ given in equation (4) can be compared, at least with respect to the order of the quantities involved, with the experimental data.

To connect the velocity of expansion $H$ of unit distance with the quantity $\varepsilon/c^2$ we note from (2)

$$V = Hc. \tag{5}$$

From (4) and (5) we have

$$V = \sqrt{\frac{8\pi k\varrho}{3}}, \tag{6}$$

where $\varrho = \varepsilon/c^2$ is the rest density of matter in $G$.

We shall apply the expression (6) to multiple galaxies and groups of galaxies. The linear dimensions of multiple galaxies are of the order of $5 \cdot 10^4$ parsecs, whilst the distances between neighbouring multiple galaxies are of the order of $10^6 - 5 \cdot 10^6$ parsecs. Since the distances between multiple galaxies are about 50 times greater than their linear dimensions, our model can certainly by applied to multiple galaxies.

According to Vorontsov-Velyaminov (*Astronomical Journal*, 1958) the majority of multiple galaxies exhibit a clearly expressed "repulsion" among their components. This "repulsion" reveals itself in the following facts:

1. In general, the tails of galaxies are directed outside the system and are larger than the filament connections between neighbouring galaxies. The spiral structure can clearly be seen on the outer side of the system, whilst it is absent, or considerably fainter, among the galaxies of the system.

2. The destruction of the facade, i.e. the absence of spiral or other structure, or alternatively the decreased brightness of such structures, on the side of a galaxy which faces another galaxy of the system is a very widespread phenomenon. On the sides of the galaxies facing each other the brightness is generally fainter, whereas the gravitational tidal waves should give rise to a confluence of stars towards these sides and a consequent increase in brightness. The destruction of the façade is a phenomenon which has nothing to do with the tides, but which emphasises the "repulsion" between the components of the multiple galaxy. This "repulsion" is illustrated most clearly by the presence of bright tails.

3. The new data published in recent years for the radial velocities of the components of multiple galaxies has given direct confirmation that some multiple galaxies expand. The velocities of recession are of the order of 1000–3000 km/sec.

According to Ambartsumian, we can estimate the diameter of such a system to be $5\times 10^4$ parsecs. The mean density of matter in this system can be taken as $10^{-24}$ g/cm$^3$.

Substituting these values in (6) we get for the periphery velocity $V = 1150$ km/sec. Thus the velocity of recession on the periphery of the system is 1150 km/sec. The above mentioned velocities of recession of some multiple galaxies are of the same order. The effect of expansion for the multiple galaxies, which can be explained on the basis of the general theory of relativity *without additional hypotheses*, represents a new application of Einstein's theory of gravitation.

According to our theory we can have expansion as well as contraction of the matter in the region $G$. After Lifshitz (*Journal of Experimental and Theoretical Physics*, 1946) we can assume that the expanding model is stable and the contracting model unstable. This is in good agreement with the experimental data according to which, for multiple galaxies, clusters of galaxies and superclusters of galaxies, expansion predominates.

E. Schücking:

I think the problem of clusters of galaxies is a very complicated problem. Last year there was a symposium at Santa Barbara at which some of these questions were discussed at length for about a week by a hundred people or so. And I think that the data indicate that a cluster of galaxies is quite a complicated object, and these objects differ very much with respect to mass or with respect to diameter and so on. There is no indication at the moment of any expansion or the like of these clusters of galaxies. We don't know if they expand or contract or they are stable or something else. We just don't have the necessary observations. On the theoretical side much work has been done especially over the last years by Just, and many other people. And I think the model that you are considering is an extremely special model, that has already been dealt with in the literature.

D. Ivanenko:

Since we are engaged in discussing various projected experiments in general relativity, may I be permitted to draw your attention to some work being carried on at our Moscow University by Braginski and Ruckman. They have tried to detect the level of possible screening of gravitation. We have been trying once more to disprove the experiments of Majorana and others in this field. But the chief aim was to devise methods which could be applicable to further experiments. The screening was not discovered up to the level of $1.3\times 10^{-10}$. Cf. *J.E.T.P.* **43,** 50(1962). The second proposal is akin to work of Weber about which we were glad to hear today such impressive developments. Braginski and Ruckman proposed essentially to take not a single system of cylinders, excited to radiate gravitationally, but take two groups and excite them either in phase or antiphase and play on this difference; they hope that there is only a gap of two orders of magnitude beyond present day possibilities (cf. Theses of the 1st Soviet Gravit. Conference; Moscow University 1961)

Now to some effects which lie outside conventional Einstein theory. I will not enter here in any details even of most reasonable modifications suggested by tetrad formalism used with various supplementary conditions by Møller, Plebański and my collaborator V. I. Rodicev which preserve Riemannian curved geometry or of other optimistic generalizations leading to torsion (which induces among other things after V. Rodicev and R. Finkelstein, a non-linear supplement in Dirac equation, so important for unitary schemes of matter). Indeed the experimental consequences of such modifications are as yet not fully investigated. But enjoying the happy occasion of Professor Dirac participation at this Conference may I ask whether it is reasonable to consider the consequences of his well known hypothesis of slow secular diminution of gravitational constant (Here Prof. P. A. M. Dirac remarks that he indeed continues to support his old hypothesis). Well, than one may be hold enough to draw after Jordan and Dicke further consequences and we tried with M. U. Saghitov to connect the probable rate of the Earth expansion with the rate of increase of the period of diurnal rotation and even with building of gigantic rifts, treated previously by A.V. Peive (Moscow) and Dr. Heezen (Lamont Observatory, N.Y., U.S.A.). We tried to follow the course of these rifts in Siberia by means of seismic and gravitational data. (Cf. Vestnik Mosk. Universit. No 6, 1961. Theses, 1st Soviet. Gravit. Conference Moscow 1961).

B. Bertotti:

There are heurestic arguments which support the hypothesis of a variation of the gravitational constant $k$ according to the law

$$k = k_0\left(1 + \frac{aV}{c^2}\right); \qquad (1)$$

here $a$ is a dimensionless constant of the order of unity and $V$ is the gravitational potential in the region where the interacting bodies are situated. However, a way to fit the hypothesis in the framework of general relativity has not yet been suggested.

The possibility of testing Eq. (1) has been discussed by Prof. Finzi at the University of Rome in a paper submitted for publication to the *Physical Review*.

In some distant future it may be possible to test (1) through the observation of the motion of the two satellites of Mars, Phobos and Deimos.

The solar gravitational potential varies substantially along the orbit of Mars, which is strongly eccentric; one must expect therefore, according to (1), a relatively large variation in the attraction between Mars and its satellites.

A different test of (1), which seems to be within the limits of present day possibilities, is based on the following argument.

If $k$ varies, the gravitational self-energy $\Omega$ of a body also varies; accordingly, the total force acting on the body is not just $-M\dfrac{\partial V}{\partial x_i}$, but $-M\left(1+a\dfrac{\Omega}{M_c^2}\right)\dfrac{\partial V}{\partial x_i}$. However, this correction to Newton's force is of importance only in the case of a very dense star like a white dwarf.

As a consequence of this anomalous force a white dwarf should escape from a weakly bound cluster moving in the gravitational field of the Galaxy. In fact, no white dwarf has been detected in *Coma Berenices* and in the nucleus of *Ursa Major*; it must be said, however, that the total number of white dwarfs we would have expected to find in those clusters, according to a prediction of Sandage and considering the methods of observation employed, is only three. It may not be impossible to improve the statistics in the future.

Some white dwarfs have been detected in the slightly more strongly bound cluster of *Hyades*; this fact sets an upper limi for the absolute value of $a$.

# THE CHARACTERISTIC INITIAL VALUE PROBLEM FOR GRAVITATIONAL THEORY

R. K. SACHS

*Fort Monmouth, N.J.*

## 1. INTRODUCTION

I am honored to participate in this international inquiry into the nature of things.

General relativity is the best theory of gravity we have; moreover, it deals directly with the structure of space-time which, as an *a priori* given element, forms the basic background for other fundamental physical theories. Therefore in the long run general relativity, or some substitute for it, is an indispensible supplement to these other fundamental theories. Since 1916 we have had a slow, rather painful accumulation of minute technical improvements which have advanced our understanding of the mathematical content of this theory and the physics of gravity. I think that the attempt to continue obtaining such minute improvements constitutes a legitimate and fascinating part of mathematical physics. If something really exciting turns up fine; in any case routine improvements will certainly be obtained and that, for me, is exciting enough. Of course it may happen that all our rather sophisiticated attempts will be swept into obsolescence by some simple, wholly new idea or experiment; but it may also be that the only real way to understand the nature of space, time, and gravitation is to continue a careful and impartial analysis of the present theory.

Why these somewhat negative remarks? Well, I am quite enthusiastic about some of the results on which I shall here report, notably those of Penrose, of Bondi, Van der Burg, and Metzner, and of Newman and Unti [1], [2], [3]. There is the danger that you might misinterpret my enthusiasm. I don't claim that we are here dealing with a breakthrough, which will lead to a complete clarification of the main unsolved problems of gravitational theory. Such a claim would certainly be overoptimistic. In particular, I shall discuss here the *characteristic* initial value problem; that is, I shall try to count and label the different possible solutions of the Einstein field equations by analyzing properties of *lightlike* elements (lines or hypersurfaces everywhere tangent to the local lightcone). But suppose we understood lightlike elements very much better than we do at present. I think we should at least still need continued analyses of the *usual* initial value problem, particularly global analyses of the kind carried through by Arnowitt, Deser, Misner, and others [4], [5]. Otherwise definitive answers even to our present questions about the meaning of asymptotic flatness, the behavior of gravitational waves, the nature of energy conservation laws, the role of topology, and the quantum behavior of gravitational fields should not be anticipated.

There are many reasons why one might want to analyze the characteristic initial value problem; let me mention two. (i) Mathematically, the theory of hyperbolic partial differential equations is a somewhat Protean one; but it does contain one idea which serves to unify many different approaches—namely, to work with characteristic submanifolds as much as possible [6], [7]; it seems sensible to apply this idea to the Einstein field equations. (ii) Physically, we know that weak gravitational waves travel at the local velocity of light; by working with lightlike elements we can, so to speak, keep up with the waves rather than having them whizz right past us. One main result of the analysis is that one can give rather explicit, geometrically meaningful ways of avoiding constraints when one works with lightlike elements.

Given a normal-hyperbolic second order partial differential equation in one unknown $A$ and $N$ independent variables $x^{\underline{a}}$ we can define a characteristic hypersurface as an $N-1$ dimensional manifold $f(x^a)=0$ across which discontinuities of otherwise continuous derivatives of $A$ can appear; in the case of a non-linear equation it makes sense only to talk about the characteristic hypersurfaces associated with a particular solution $A(x^a)$. In the case of Einstein's equations $R_{ab}=0$ $(a,b=0\ldots,3)$ essentially the same definition can be used but of course we must ask about *physical* discontinuities, which cannot be wiped out simply by making a coordinate transformation. Then the characteristic hypersurfaces are precisely the lightlike ones, as was first discussed in detail by Darmois [8].

Let me now state essentially what we are after, leaving aside a whole host of details and qualifying statements. We want a *correspondence* between four arbitrary functions $f(\xi,\varkappa,\tau)$, $g(\xi,\varkappa,\tau)$, $h(\xi,\varkappa,\tau)$, $j(\xi,\varkappa,\tau)$ of three variables and the physically different solutions $\{S\}$ of the field equations:

$$f,g,h,j \longleftrightarrow S; \quad f',g',h',j' \longleftrightarrow S'; \quad \text{etc.} \tag{1.1}$$

whereby $S$ can be defined by giving its metric in any coordinate system $x^a$. The trivial but useful example of a vibrating string may help make this notion of a correspondence clear. Let the amplitude of the string be $A$ and the velocity of sound be unity; then

$$\frac{\partial^2 A}{\partial t^2}-\frac{\partial^2 A}{\partial x^2}=0. \tag{1.2}$$

In the usual initial value problem we give $A$ and $\dot{A}$ at time $t=0$, that is:

$$f(\xi), g(\xi) \longleftrightarrow 2A = f(x-t)+f(x+t)+\int^{x+t} g(\xi)d\xi - \int^{x-t} g(\xi)d\xi \qquad (1.3)$$

whence

$$A(x,0) = f(x), \quad \dot{A}(x,0) = g(x).$$

We see that $f$ and $g$ first appear as arbitrary functions of some auxiliary variable $\xi$; after the solution is constructed $f$ and $g$ turn out to be the initial values. For the characteristic initial value problem we give the amplitude on the two characteristic lines $x+t=0$ and $x-t=0$:

$$f(\xi), g(\varkappa) \longleftrightarrow A = f(x+t)+g(x-t). \qquad (1.4)$$

Note, incidentally, that we give only one function, rather than two, on each line; this halving of degrees of freedom sometimes puzzles people in the corresponding gravitational case. In passing note also that the correspondence (1.4) is not quite one-to-one. If we add a constant to $f(\xi)$ and subtract the same constant from $g(\xi)$ we end up with the same solution; moreover, either (1.3) or (1.4) can be altered in a trivial way by performing a two dimensional inhomogeneous Lorentz transformation. But all these transformations contain only arbitrary constants, while the data consist of arbitrary functions; thus the arbitrariness is "of measure zero" and causes no serious difficulties; we shall see that a similar situation holds in the gravitational case, where the data consist of arbitrary functions of three variables while the correspondence is unique only up to some arbitrary functions of two variables.

No matter how pedantic it may seem to insist on the distinction between $f(\xi)$, $f(x+t)$, and $f(x)$ in equation (1.3) the corresponding distinction is worth remembering in the gravitational case. Thus when I say in the following that $f(\xi,\varkappa,\tau)$ and $g(\xi,\varkappa,\tau)$ are components of the Riemann tensor in a lightlike hypersurface I shall really mean something rather more complicated. Given $f(\xi,\varkappa,\tau)$ and $g(\xi,\varkappa,\tau)$ we are to perform certain integration processes according to some set pattern. At the end we come out with a solution $S$ of the field equations. If we then look at a suitable lightlike hypersurface in $S$ using a suitable coordinate system $r$, $\theta$, $\varphi$, we find that $f(r,\theta,\varphi)$ and $g(r,\theta,\varphi)$ are actually the relevant components of the Riemann tensor in that coordinate system.

I shall give some more preliminary definitions and lemmas. By $u$ I shall always mean a scalar function for which the hypersurfaces $u$=const are lightlike; I shall often call $u$ the retarded time. The vector field normal to the $u$ hypersurfaces is given by $k_a=u_{,a}$; it obey's

$$k_a k^a = 0, \qquad (1.5)$$

$$k_{a;b}k^b = 0 \qquad (1.6)$$

and will be called the ray vector. Equation (1.5) means that $k^a$ lies within the hypersurfaces to which it is orthogonal. The lines with tangent $k^a$ are geodesics; they will be called "rays" (the words "generators" and "bicharacteristics" are also used in some references). By the "inner geometry" of a lightlike hypersurface I shall here mean the network of distances within the hypersurface, as given by a metric tensor $g_{\nu\mu}$ ($\nu, \mu=1,2,3$) as a function of any coordinates $y^\tau$ within the hypersurface. $g_{\nu\mu}$ is always degenerate; in fact, since $k^a$ lies within the hypersurface we can consider it as a vector $k^\tau$ of the hypersurface; but, since $k^a$ is also orthogonal to every direction in the hypersurface (including itself) we must have $g_{\nu\mu}k^\mu=0$. Thus only three of the six components of $g_{\nu\mu}$ are algebraically independent.

Please note that knowing the metric $g_{\nu\mu}$ in terms of some arbitrary coordinate system within the hypersurface does not by any means determine all the properties of the hypersurface which a sensible person might call "inner" properties. For example, if we know the full metric $g_{ab}$ of the imbedding four-space we can introduce along each ray in the hypersurface the preferred parameter (affine parameter) distance, defined by the property that the equation for geodesics takes its usual simple form in terms of such an affine parameter [9]. But, contrary to what one would expect off hand, knowledge of $g_{\nu\mu}$ alone does not enable one to calculate this preferred parameter as a function of an arbitrary coordinate system. For this reason, Penrose has introduced a whole hierarchy of "inner geometries" for a lightlike hypersurface; here I shall always mean the simplest of these geometries, as defined above.

## 2. THE LOCAL CHARACTERISTIC INITIAL VALUE PROBLEM

Let us next consider the results that can be obtained by assuming that we are working in a sufficiently small, finite region of four dimensional space-time. Given a normal hyperbolic Riemannian manifold and in it any point $P$ we can introduce a so-called "Riemannian Normal" (*RN*) coordinate system based on $P$ as origin [10]. The *RN* system $x^a$ is defined by the properties:

(i) at $P$ $x^a = 0$ and $g_{ab} = \eta_{ab} =$ Lorentz metric,

(ii) along every geodesic through $P$ the *RN* coordinates $x^a$ are *linear* functions of the proper distance (or of an affine parameter in the case of null geodesics).

It is well known that such coordinates always exist (in a sufficiently small neighbourhood of $P$—this qualification will always be understood during our treatment of local properties) and are unique up to a rigid homogeneous Lorentz transformation at $P$. From this high degree of uniqueness of *RN* coordinates it follows directly that the *ordinary* derivatives of the metric tensor at $P$ in an *RN* system are actually the values that certain tensors —

the so called normal tensors—take at $P$ in the $RN$ system. Now in any coordinate system the only tensors available are the Riemann tensor and its symmetrized covariant derivatives. Thus we must have

$$\begin{aligned} &(a)\quad g_{ab} = \eta_{ab} \\ &(b)\quad g_{ab,c} = 0 \\ &(c)\quad g_{ab,cd} = \mathrm{Per}\,(R_{abcd}) \text{ at } P \qquad (2.1) \\ &\quad\vdots \\ &(d)\quad g_{ab,cde\ldots fg} = \mathrm{Per}\,(R_{acbd;\,e\ldots fg}) + \text{junk} \end{aligned}$$

where Per means some suitable permutation of the indices and the junk consists of terms quadratic or higher in the derivatives of the Riemann tensor.(1) It is clear from Eq. (2.1) that if we know the Riemann tensor and all its covariant derivatives at $P$ in the $RN$ system then we know the metric throughout a region surrounding $P$, provided the metric is an analytic function of the $RN$ coordinates, as will be assumed for the time being.

Up to now we have not used the field equations in any way. These clearly now take the form of an infinite set of algebraic restrictions on the various quantities on the right (or on the left) of Eq. (2.1). For example $R_{ab}]_P = 0$ simply reads $R_{abcd}\eta^{cd} = 0$ at $P$; $R_{ab;c}]_P = 0$ places some restrictions on $R_{abcd;e}$ at $P$; and so forth. These restrictions induce corresponding restrictions on the $g_{ab,c\ldots e}$. Using the spinor calculus, which is the appropriate tool here, Penrose was able to solve these algebraic interrelations. He showed that certain algebraic combinations of the $R_{abcd;e\ldots f}$ (or of the $g_{ab,cd\ldots ef}$) can be chosen arbitrarily while the remaining combinations are then determined.

His results can be summarized as follows. We choose a particular pair of components of the Riemann tensor on the finite light cone emanating from $P$—I shall not take the time to tell you precisely which two components since that would involve introducing a Cartan normal tetrad system to go with our $RN$ coordinate system and we've had enough notation for the time being—and assemble them into a single complex function $\psi$. $\psi$ is initially given as a function of three auxiliary variables which, after the solution is constructed, turn out to be essentially the Riemannian Normal coordinates evaluated on the light cone. Then the entire metric field is determined throughout some region $R$. Conversely, a given solution $S$ determines $\psi$ (up to a homogeneous Lorentz transformation at $P$) once we have chosen $P$ according to some prescription. Thus we have a correspondence of precisely the kind we were looking for.

The algebraic manipulations described above are a little strange but the results of Penrose can only be described as unusually beautiful. The function $\psi$ can be given quite arbitrarily—no constraints appear. Moreover, $\psi$ has

(1) Equation (21) can be solved for the derivatives of the Riemann tensor in terms of the $g_{ab,cd\ldots ef}$ but this fact is not relevant to the present discussion.

a direct geometrical interpretation and the coordinates in terms of which it must be expressed likewise have a very direct geometrical interpretation. In effect the entire problem of interpreting physically the solutions of the field equations has—locally and in the analytic case—been reduced to the problem of interpreting and classifying three dimensional spaces on which there is defined a congruence of rays with their preferred parameters and a $\psi$ function. This is still a rather complicated geometric structure, but it is a far simpler one than that with which we started. Incidentally, the numerology obtained here should be compared with that for the vibrating string discussed earlier. We note the following differences: (i) the gravitational field has two degrees of freedom rather than one; (ii) we are, of course, working in three spatial dimensions rather than one in the gravitational case.

Dautcourt [24] and Penrose also discussed the data that must be set on a pair of intersecting lightlike hypersurfaces to determine a solution, and a slightly more detailed analysis was given subsequently [11]. In this case one can proceed in a much more conventional manner. We set up coordinates adapted to the pair of intersecting lightlike hypersurfaces and then use the Bianchi identities to show that really only six of the ten field equations must be integrated throughout a four dimensional region. By analyzing these six equations and also analyzing three of the remaining four on a single hypersurface (the last equation, the so-called "trivial equation", follows algebraically from the remaining ones and we need not analyze it at all) one finds out what data must be set. A uniqueness theorem can be proved generally and an existence theorem can be proved in the analytic case.

The results are the following. On each of the hypersurface one must give the inner conformal metric; that is, one must give $g_{\nu\mu}$ up to an arbitrary unknown factor. Geometrically, this amounts to specifying angles (and thus ratios of pairs of distances) at all points. As discussed above, $g_{\nu\mu}$ itself consists only of three algebraically independent functions; the inner conformal metric thus consists only of two algebraically independent functions, which is the expected number. As before, the conformal metric must be given as a function of suitably chosen coordinates. In this case, the essential thing is that one of the coordinates must be the affine parameter along each ray.

One must give some further functions of two variables on the intersection $\Gamma$ of the two lightlike hypersurfaces; $\Gamma$ is automatically a spacelike two dimensional manifold. One must give its entire inner metric and also its two mean extrinsic curvatures (the traces of the two second fundamental forms that $\Gamma$ posseses by virtue of being imbedded in a four dimensional space); finally one must give a certain extrinsic quantity of second differential order for $\Gamma$. Then the field is determined by the field equations throughout a four-dimensional region; conversely, a given field determines the data up to an arbitrariness of measure zero, and we again have the desired correspondence. I suspect that the fact that one has to give so much extra information on $\Gamma$ is basically a reflection of the fact that the gravitational field has longitudinal modes as well as its two transverse ones, but this point is not clear at present in the local treatments.

There are two more essentially similar ways to solve the characteristic initial value problem for two intersecting lightlike hypersurfaces. Instead of the inner conformal metric we can give Penrose's $\psi$ function; some additional data on $\Gamma$ must then be given. An intermediate possibility is to give instead of $\psi$ or the conformal metric the "shear" of the rays [12]. (Imagine for the moment that the rays are realized by a stream of photons which cast a shadow of some object placed in the ray congruence; the shear is a measure of the rate at which this shadow is distorted per unit parameter interval along the rays [13], [14].) The reason one has so much choice is that the $\psi$ function, the shear, and the inner conformal metric are all interrelated by simple *ordinary* differential equations along each ray separately. I do not know of any decisive reason why one of these three quantities should be preferred to either of the other two as initial data. In any case, the data consist of arbitrary functions which are geometrically meaningful, as in the case of the cone.

All these developments assume analyticity of the metric and this assumption is actually undesirable and unrealistic. As a general rule we know that a treatment which assumes analyticity may or may not be misleading when it comes to answering the very important question: under what circumstances does the solution depend in a continuous manner on the initial data. A famous example was constructed by Hadamard, who showed that the Cauchy problem for the two dimensional Laplace equation is totally unreasonable: by making arbitrarily small changes in the initial data we can cause arbitrarily large changes in the solution arbitrarily close to the initial line. For this kind of reason one also cannot argue that a non-analytic problem can be approximated by an analytic one. A few fragmentary results for the non-analytic case were obtained by Penrose and myself: in order to obtain from the $\psi$ function the entire four-dimensional metric on the initial cone or on the initial pair of lightlike hypersurfaces one need not assume analyticity. But the only detailed treatment of the outer problem — propagating the initial data away from the initial hypersurface — known to me in the non-analytic case is for simple linear analogoues of the gravitational equations.

However, three or four years ago Professor Bruhat started to consider the non-analytic case and her results appear in the L. Witten volume [15]

## 3. THE GLOBAL CHARACTERISTIC DIRICHLET PROBLEM

Let us now discuss global results, obtained by assuming that the space is asymptotically flat. There are two important preliminary questions:

(i) What, precisely, is the definition of an "asymptotically flat" manifold?

(ii) What is the structure of the "asymptotic symmetry group"; that is, which coordinate transformations preserve the boundary conditions appropriate to an asymptotically flat space?

Since the pioneering work of Fock [16] so much work has been done on these questions that I cannot here attempt to give even a summary of the main ideas. I have two opinions: first, I do not think that these two questions have yet been definitively answered; second, I think they are among the most interesting and difficult unsolved questions in gravitational theory at present.

Here I shall report on the particular treatment of these and other questions given in a series of five papers:

(a) The paper of Bondi, Van der Burg, and Metzner [1];

(b) A generalization of (a) by Sachs [17];

(c) The reprint of Newman and Penrose on the asymptotic behavior of Riemann tensors [18];

(d) A generalization of (a), (b), and (c) by Newman and Unti [3];

(e) A group theoretical analysis by Sachs [19].

The key paper (a) has just been summarized by Professor Bondi. I shall therefore concentrate attention on those points in which the subsequent four papers differ from paper (a).

First, as regards basic assumptions, you will recall that in (a) one assumes axial and reflection symmetry; that is, one assumes the existence of a hypersurface orthogonal Killing vector with everywhere spacelike closed trajectories [20]. This assumption was dropped in the four other papers. It turns out that no new ideas appear when one drops the assumption of axial symmetry. Several much less trivial modifications of the basic assumptions were given by Newman and Unti. You will also recall that in (a) one assumes the existence of four scalar fields $(u, r, \theta, \varphi)$ such that

$$\lim_{r\to\infty} ds^2 = -du^2-2dudr+r^2(d\theta^2+\sin^2\theta d\varphi^2) \tag{3.1}$$

when these scalars are introduced as coordinates; here $u$ is a retarded time, $r$ is a corresponding luminosity distance along the rays of $u$ (or preferred parameter distance; the distinction between these two different kinds of distance is trivial in the global discussion to be presented here) and $\theta$ and $\varphi$ are quantities constant along each of the rays of $u$. Now Newman and Unti start with the assumption that there exists a single scalar function $u$ such that Penrose's $\psi$ function falls off as $r^{-5}$ along the rays of $u$. They then construct the remaining three scalar fields for which Eq. (3.1) is satisfied. Thus the fundamental assumptions are more economical and this conciseness clarifies their geometric meaning. The assumptions of Newman and Unti were motivated by some results in the theory of the Petrov–Pirani classification [20], [21] and by the results of paper (c).

Even the assumptions of Newman and Unti are not yet constructive: given the metric in terms of some arbitrary coordinate system there is no algorithm for deciding whether a function $u$ meeting their requirements exists or for calculating $u$ if it exists. A treatment of similar questions which is purely constructive has been given [12] but the insistence on manifest covariance in this sense of working with constructive elements has so far proved less fruitful than the approach adopted in the five papers here under discussion.

Another important modification of the basic assumptions has been the introduction of the "uniform smoothness" assumption. The notion of uniform smoothness is implicit in the work of Fock and Trautman [16], [22], made explicit in the Newman–Pentose preprint (c), and systematically exploited in (d). Let $u$ be a retarded time whose rays have the property that one can go to the limit $r \to \infty$ along each ray; let $\theta$ and $\varphi$ be any scalars constant along each ray. Let $A$ be a scalar function of $u$, $r$, $\theta$ and $\varphi$ which is $O(r^{-M})$ as $r$ goes to infinity with $u$, $\theta$ and $\varphi$ fixed. $A(u,r,\theta,\varphi)$ will be called uniformly and radially smooth if

$$\frac{\partial A}{\partial r} = O(r^{-M-1}), \qquad \frac{\partial A}{\partial u}, \frac{\partial A}{\partial \theta}, \frac{\partial A}{\partial \varphi} = O(r^{-M}). \tag{3.2}$$

The motivation for the definition is the following: suppose $A = O(r^{-1})$ is a solution of D'Alembert's equation in Minkowski space; let $u$, $r$, $\theta$, $\varphi$ be the usual retarded time, radius and angles in some Lorentz frame; then $A$ obeys the Sommerfeld outgoing radiation condition if and only if it is uniformly and radially smooth. Now in papers (a) and (b) one assumes that the entire metric is analytic in $(1/r)$ in the specialized coordinate systems; in (d) one merely assumes that $\psi$ and its first few derivatives are uniformly and radially smooth. The second assumption is known to follow from the first but not vice versa. Using this much more realistic second assumption Newman and Unti were able to reproduce all the essential results of references (a) and (b) except one. Reference (d) also contains some further generalizations of a topological nature, which will not be discussed here.

While the basic assumptions of reference (a) have thus been modified, simplified, and generalized, the basic results have merely been generalized without essential changes.

The first result is that, to specify a particular solution of the field equations, one must specify: the inner conformal metric of the hypersurface $u = 0$, two functions of three variables; two "news functions" of three variables on the hypersurface $r = \infty$; and three functions of two variables on the two-dimensional spacelike surface $r = \infty$, $u = 0$; the latter are essentially the mass aspect and dipole-moment-cum-angular-momentum-aspect in Bondi's sense. These results, obtained in (b), show that the numerology for the global problem is very similar to that for the local problem, as one would expect. Note that the news functions cannot properly be called characteristic initial data since they are given on a timelike tube at infinity; thus the most accurate name for the problem is "the global characteristic Dirichlet problem". Newman and Unti give Penrose's $\psi$ function at $u = 0$ instead of the inner conformal metric; this formulation is an essentially equivalent one, as already mentioned in connection with the local problem. However, at this one point, the otherwise very powerful assumption of uniform smoothness is inadequate to prove a uniqueness theorem for the solutions and to obtain the theorem one must fall back one assuming analyticity in $(1/r)$; I believe that this gap in the Newman–Unti treatment vis à vis the treatments in (a) and (b) is a purely technical problem which will be eliminated in future treatments.

The next question of interest is the structure of the asymptotic symmetry group. As shown in references (a), (b), (d), and (e) the allowed coordinate transformations form a group isomorphic to the generalized Bondi–Metzner group. The generalized Bondi–Metzner group (GBM group) is defined by the equations

$$\theta' = H(\mu_1, \ldots, \mu_6; \theta, \varphi), \quad \varphi' = J(\mu_1, \ldots, \mu_6; \theta, \varphi), \quad u' = K(\mu_1, \ldots, \mu_6; \theta, \varphi)\,(u+\alpha) \tag{3.3}$$

where $H$ and $J$ represent a conformal transformation of the unit sphere into itself, $K$ is the determinant of the conformal transformation, and $\alpha$ is an *arbitrary* twice differentiable function of $\theta$ and $\varphi$. The transformations (3.3) may be visualized as taking place at the hypersurface $r = \infty$. Note that the group structure is metric-independent.

I quote without proof some theorems obtained in (b), (d), and (e):

(i) The transformations with $\alpha = 0$ form a subgroup isomorphic to the homogeneous orthochronous Lorentz group.

(ii) The transformations with $\alpha = \mu_6 + \mu_7 \cos\theta + \mu_8 \sin\theta\cos\varphi + \mu_{10}\sin\theta\sin\varphi$ form a subgroup isomorphic to the full orthochronous Lorentz group.

(iii) The transformations for which $\theta' = \theta$ and $\varphi' = \varphi$ form an (infinite dimensional) abelian normal subgroup whose factor group is the homogeneous Lorentz group (i).

(iv) The transformations for which both (ii) and (iii) hold form a four-dimensional abelian normal subgroup which I shall call the translation subgroup; the GBM group contains only this one normal four-dimensional subgroup, so the four translations are uniquely defined up to homogeneous Lorentz transformations.

(v) The rest mass operator, built from the four infinitesimal translations in the usual way, commutes with all the infinitesimal GBM transformations. The corresponding statement does not hold for the spin operator.

Why one is dealing with a group larger than the Lorentz group is still not well understood. However, the structure of the GBM group is actually quite similar to that of the Lorentz group; in particular the fact that the four translations are uniquely defined seems to open the possibility of introducing a well defined energy-momentum for the field. In any case, the most important point is that the group structure is not dependent on the metric; one therefore finds it possible to work throughout with concepts and equations that are covariant under the GBM transformations.

Further results of interest concern the asymptotic behavior of the Riemann tensor. Robinson and Trautman [14] and Newman and Tamburino [23] have shown that the asymptotic behavior of an algebraically special vacuum Riemann tensor can be given by

$$R = {}_0N/r + {}_0\mathrm{III}/r^2 + {}_0D/r^3 \tag{3.4}$$

in the generic case, where ${}_0N$, ${}_0\mathrm{III}$, ${}_0D$ are parallely displaced along each ray, are null, type three, and type one degenerate respectively, and where $r$ is a suitably defined distance along the rays, which are in this case defined by the Riemann tensor itself. In the linearized theory very similar expressions are obtained, but the series does not break off after the first three terms except in the case of the Schwarzschild metric [12]. The field of a radiating quadrupole contains five terms in the linearized theory. In papers (b), (c), and (d) wholly analogous results are obtained for the asymptotic behavior of the Riemann tensor; the expansions agree in algebraic form with the linearized equations up through the first five terms.

A further result is that time dependent periodic solutions of the field equations do not exist within the class of solutions considered in any of the above five references. One would expect this behavior, since the fields obey the outgoing radiation condition suggested by Fock [16] in the weakened form introduced by Trautman [22] and also a Riemann tensor outgoing radiation condition [12].

Finally, one has the following series of results concerning the news functions $\partial c/\partial u$ (which I here assemble into a single complex quantity): (i) at a fixed point in $u$, $\theta$, $\varphi$ space the news functions are invariant under GBM transformations up to a rigid homogeneous Lorentz transformation; (ii) they determine the loss of mass in accordance with the equation

$$\left\langle \frac{\partial M}{\partial u} \right\rangle = - \left\langle \left| \frac{\partial c}{\partial u} \right|^2 \right\rangle \tag{3.5}$$

where $M$ is the mass aspect and carets denote the average over the two-sphere at infinity; (iii) their first retarded time derivatives are the "amplitudes" (in a sense that can easily be made precise) of the outgoing asymptotically plane gravitational waves; and (iv) they can be chosen arbitrarily.

I think that these properties suffice to identify the news functions as the transverses degrees of freedom of the gravitational field. With this interpretation one can introduce a superposition principle for gravitational waves. To superimpose two fields one simply adds the news functions linearly; there by one generates a non-linear superposition of the two fields throughout a four-dimensional region. Moreover it is easy to write down for the news functions commutation relations that are GBM covariant, namely

$$[c,c'] = 0, \quad [c^{+},c'] = i\hbar\,\delta(\Omega,\Omega')S(u-u'). \tag{3.7}$$

Here $\delta(\Omega,\Omega')$ is the invariant delta function for the two-dimensional unit sphere and $S$ is the step function. One then obtains a rather primitive and naïve quantization of the radiation modes which has the advantage that it is intuitively clear. One can also construct from the news functions certain integral invariants for the gravitational field, for example

$$P[\alpha(\theta,\varphi)] = \int_{-\infty}^{\infty} du \int_{0}^{2\pi} d\varphi \int_{0}^{\pi} \sin\theta d\theta \left|\frac{\partial c}{\partial u}\right|^{2} \alpha(\theta,\varphi). \tag{3.8}$$

These integral invariants generate the GBM transformations with the commutators (37); they are in the classical theory just numbers invariant under GBM transformations.

## 4. CONCLUSION

I have discussed the local and global characteristic initial value problems. To my mind the most important result is that (leaving aside a few qualifications) one is able to describe gravitational fields in terms of a minimal, complete geometrically meaningful set of true observables. In the case of the global results, the news functions, which form half of the data, have direct intuitive interpretations similar to those one has in Lorentz covariant theories. I think these preliminary results show that the further analysis of the characteristic initial value problem is a rich and interesting problem for future investigations, and we may hope that some of the many difficult unsolved problems will yield to present mathematical techniques and physical ideas.

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# EXACT DEGENERATE SOLUTIONS OF EINSTEIN'S EQUATIONS*

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## 1. INTRODUCTION

In this lecture we wish to review recent works on gravitational fields that admit a congruence of null geodesics without shear.

Most of our present ideas on waves in general relativity are based on analogies between electromagnetism and gravitation. From the point of view of physics, among the most important features of electromagnetic waves is their ability to transport energy and to carry information. Accordingly, physicists are inclined to consider as gravitational waves those solutions of Einstein's equations *in vacuo* which correspond to a mass changing in time, or contain an arbitrary, information carrying, function. Unfortunately these properties are of such a kind that they do not seem to suggest a method for constructing such solutions. Among electromagnetic waves, there are particularly simple ones, corresponding to a null electromagnetic tensor. With null electromagnetic waves there is associated a remarkably simple and beautiful geometrical structure. Its properties can be stated independently of the electromagnetic field. Thus, one can single out the class of gravitational fields that admit a similar structure and look for waves among them. Gravitational fields belonging to this class are nowadays referred to as 'algebraically special' or 'degenerate'. Before we proceed to review these fields, we wish to describe two typical electromagnetic null fields which have close analogues among our gravitational waves.

Let us consider the following simple situation in special relativity, involving the scattering of an electromagnetic wave on the surface of a perfectly conducting paraboloid of revolution [1]. Let $x = x^1$, $y = x^2$, $z = x^3$ and

* One of us (I. R.) wishes to acknowledge the support of the U. S. Air Force.

$t = x^4$ be the Cartesian coordinates in flat space-time, $\varrho = \sqrt{x^2+y^2+z^2}$, and $l$ a positive number. Define

$$\tau = t-z-l, \quad \chi = (x+iy)/l,$$
$$\sigma = t-\varrho, \quad \zeta = (x+iy)/(\varrho-z).$$

If $A(\tau, \chi)$ is any complex function analytic in $\chi$, then the real part of[(1)]

$$A(\tau, \chi)\tau_{,[a}\chi_{,b]} \tag{1}$$

represents a solution of Maxwell's equations in empty space. The same can be said of the field

$$A(\sigma, \zeta)\sigma_{,[a}\zeta_{,b]}. \tag{2}$$

Their difference,

$$F_{ab} = A(\tau, \chi)\tau_{,[a}\chi_{b]} - A(\sigma, \zeta)\sigma_{,[a}\zeta_{,b]},$$

satisfies on the surface of the paraboloid, $\sigma = \tau$, the boundary condition

$$F_{[ab}n_{c]} = 0,$$

where $n_c$ denotes a vector orthogonal to the paraboloid. Therefore, the field given by (2) can be interpreted as resulting from reflection, on the conducting paraboloid, of the wave described by (1). If the incident field is simply a *plane* wave, i. e., if $A(\tau,\chi)$ is independent of $\chi$, the resulting field is regular throughout the region $\sigma \leqslant \tau$. The empty space field can be continued into the region $\sigma > \tau$ but this necessarily leads to singularities along the axis of the paraboloid. Both the incident and the reflected field are null. With each of them there is associated a congruence of null geodesics without shear. This is known to be the characteristic property of null electromagnetic fields [2].

The property of a congruence of null geodesics of being shear-free can be described as follows [3]. Think of the null geodesics as of rays of light. Consider a small, plane, opaque object and a plane screen, some distance apart from the object. Suppose that the object and the screen are oriented so that they are orthogonal to the rays of light in their respective rest frames and situated so that the shadow cast by the object can be observed on the screen. The congruence is non-shearing if the shadow, as observed on the screen, is similar in shape to the object.

If $f_{ab}$ is a null solution of Maxwell's equations and $^*f_{ab}$ is its dual,

$$f^{ab}{}_{;b} = 0, \quad {}^*f^{ab}{}_{;b} = 0,$$

then the null vector field $k_a$, defined up to a scalar multiplier by

$$f^{ab}k_b = 0, \quad {}^*f^{ab}k_b = 0$$

(1) Throughout this lecture the following conventions are used: Latin indices range and sum from 1 to 4. A comma followed by indices denotes ordinary differentiation, a semicolon covariant differentiation; square index-brackets denote antisymmetrization over the indices enclosed.

may be so normalized that

$$k_{a;b}k^b = 0 \tag{3}$$

and

$$(k_{a;b}+k_{b;a})k^{a;b} = (k^a{}_{;a})^2. \tag{4}$$

In this lecture we shall use the term 'ray' to denote a null geodesic belonging to a non-shearing congruence. The trajectories of a vector field $k_a$ subject to (3) and (4) are rays.

## 2. THE LINE-ELEMENT OF DEGENERATE SPACES

From now on we shall confine our attention to empty space-times, i.e., to four-dimensional Riemann spaces of signature $-2$ with vanishing Ricci tensor.

In the theory of gravitation, the analogue of null electromagnetic fields is provided by the class of metrics with degenerate Riemann tensors [3]. The curvature tensor of a non-flat, empty space-time is called algebraically special or degenerate if the equations

$$k_{[a}R_{b]cde}k^c k^d = 0$$

can be satisfied by a vector field $k_a$ which is null and different from zero. We shall also refer to a space or metric as degenerate if its Riemann tensor has this property. Sachs has shown that if such a $k_a$ exists, it must be tangent to a congruence of rays. Conversely, if an empty space-time admits a null congruence of this kind, its metric is degenerate [4], [5].

One is thus led to consider a four-dimensional, normal hyperbolic Riemann space $V_4$ that admits a null vector field $k_a$ satisfying equations (3) and (4). The curves $x^a = x^a(\varrho)$ defined by

$$\frac{dx^a}{d\varrho} = k^a$$

are null geodesics. Let us introduce coordinates in $V_4$ such that $x^3$ coincides with the affine parameter $\varrho$ and the null geodesics are coordinate lines of $x^3$ (i.e. $x^1=\xi$, $x^2=\eta$ and $x^4=\sigma$ are constant along the rays). With this choice of coordinates, $k^a=\delta^a_3$ and $g_{33}=0$. It follows from the geodetic condition, Eq. (3), that the covariant components, $k_a=g_{a3}$, are independent of $\varrho$. Let us introduce the following vectors,

$$l_a = \varrho_{,a} + \frac{1}{2} ck_a,$$

$$x^1_a = P(\xi_{,a} - ak_a),$$

$$x^2_a = P(\eta_{,a} - bk_a),$$

and choose $a$, $b$ and $c$ so that

$$l_a l^a = 0, \quad x^\lambda_a l^a = 0, \qquad \lambda = 1,2.$$

The metric tensor can then be written in the form

$$g_{ab} = k_a l_b + k_b l_a + \gamma_{\varkappa\lambda} x^{\varkappa}_a x^{\lambda}_b,$$

where $\gamma_{\varkappa\lambda}$ is a symmetric, two by two matrix. If one chooses $P$ so that det $\gamma_{\varkappa\lambda}=1$, the non-shearing condition, Eq. (4), reduces to

$$\partial\gamma_{\varkappa\lambda}/\partial\varrho = 0.$$

By a coordinate transformation one can impose the further restriction

$$\gamma_{\varkappa\lambda} = -\delta_{\varkappa\lambda}.$$

If we denote $k_a dx^a$ by $d\Sigma$ (not a perfect differential, in general), the line-element can be written as

$$ds^2 = -P^2[(d\xi - a d\Sigma)^2 + (d\eta - b d\Sigma)^2] + 2 d\varrho d\Sigma + c d\Sigma^2, \tag{5}$$

where $a$, $b$, $c$ and $P$ are functions of all the coordinates and the components of $k_a$ are independent of $\varrho$.

## 3. SPACES WITH CURLING RAY

The quantity $\omega$ defined by

$$\omega^2 = \frac{1}{2} k_{[a;b]} k^{a;b}$$

measures the amount of rotation of rays. It vanishes for a hypersurface-orthogonal congruence of rays.

The field equation

$$R_{ab} k^a k^b = 0$$

reduces for the metric (5) to

$$P^{-1} \partial^2 P / \partial\varrho^2 = \omega^2. \tag{6}$$

The case of $\omega \neq 0$ has been recently investigated by Newman, Tamburino and Unti [6]. Their paper, based on a very elegant technique developed by Newman and Penrose [7], containes an interesting class of new exact solutions in closed form. All their metrics are of degenerate type I, have curling rays and constitute a generalization of the Schwarzschild solution. Newman, Tamburino and Unti claim that all algebraically special metrics with $\omega \neq 0$ are of degenerate type I. This would be a significant result, for in the linearized gravitational theory type II null solutions can be easily constructed [8]. By an appropriate choice of coordinates, the components of the Newman-Tamburino-Unti metrics can be reduced to

$$P = p^{-1}\sqrt{\varrho^2 + \alpha^2}; \quad a = b = 0; \quad c = K - \frac{2m\varrho + 2\alpha^2 K}{\varrho^2 + \alpha^2}$$

$$k_a = (\alpha\eta p^{-1},\ -\alpha\xi p^{-1},\ 0,\ 1);$$

where

$$p = 1+\frac{1}{4}K(\xi^2+\eta^2); \quad K = -1,\ 0 \text{ or } 1;$$

$m$ and $\alpha \neq 0$ are constants.

As all these metrics are stationary, they are not interesting from the point of view of gravitational radiation theory.

## 4. SPACES WITH NON-EXPANDING RAYS

The class of known gravitational fields with non-rotating rays is much larger than that corresponding to solutions with $\omega \neq 0$. Many of these fields possess properties characteristic for waves. It should be noted that both the null electromagnetic fields described at the beginning of this lecture are connected with non-rotating rays.

If we assume that $k_a$ is hypersurface-orthogonal, coordinates in $V_4$ can be chosen so that $k_a = \sigma_{,a}$ and $d\Sigma$ becomes simply $d\sigma$. The two-dimensional surfaces $\sigma$=const., $\varrho$=const. can be interpreted as wavefronts. For $\omega=0$ equation (6) leads to

$$P = Q\varrho + R,$$

where $Q$ and $R$ are independent of $\varrho$. As

$$k^a{}_{;a} = 2P^{-1}\partial P/\partial\varrho$$

one has to consider two cases depending on whether or not $Q$ vanishes.

Let us first take the case of non-expanding rays,

$$k^a{}_{;a} = 0, \qquad \partial P/\partial\varrho = 0.$$

Sachs [3] has shown that the Riemann tensor for an empty space-time with a non-rotating, non-expanding family of rays has the form

$$R_{abcd} = \mathrm{II}_{abcd} + \varrho\,\mathrm{III}_{abcd} + \varrho^2 N_{abcd} \tag{7}$$

where $\mathrm{II}_{abcd}$ $\mathrm{III}_{abcd}$ and $N_{abcd}$ denote, respectively, tensors of Petrov's type II, III and II null, or more special tensors. They are covariantly constant along the rays.

The Einstein field equations have not yet been solved for the general case of a space endowed with non-expanding rays. However, many particular solutions with rays of this type have been known since a long time.

Brinkmann as early as in 1923 [9] had found a class of metrics which were later rediscovered(2) and described as plane-fronted gravitational

(2) In fact, they were independently discovered by I. Robinson in 1956, by J. Hely (*C. R. Acad. Sci., Paris* **249**, 1867 (1959)) and by A. Peres (*Phys. Rev. Letters* **3**, 571 (1959)). The physical meaning of these metrics and their connection with plane gravitational waves were recognized for the first time by I. Robinson (unpublished, presented at several seminars in England and in Warsaw in 1958)—*the Editors*.

waves with parallel rays [10]. They can be characterized by the statement that the corresponding propagation vector $k_a$ is covariantly constant. The Brinkmann waves are of Petrov's type II null.

Plane gravitational waves, discovered by Einstein and Rosen [11], disregarded by them on the ground that they possessed singularities, restored to good standing by Bondi, Pirani and Robinson [12], constitute a subclass of the Brinkmann metrics.

A more general class of exact solutions has been recently found by several authors [13]. The metrics are characterized by the existence of a 'recurrent' propagation vector:

$$k_{a;b} = \frac{1}{2} G k_a k_b .$$

By using the field equations, $R_{ab}=0$, their line-element can be brought to the form

$$ds^2 = -|d\zeta - f d\sigma|^2 + 2 d\varrho d\sigma + (G\varrho - 2H) d\sigma^2,$$

where $f(\zeta, \sigma) = u + iv$ is an analytic function of $\zeta = \xi + i\eta$, $G = \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta}$ and $H$ is a function of $\xi,\eta$ and $\sigma$ which may be determined from the equation

$$\frac{\partial^2 H}{\partial \xi^2} + \frac{\partial^2 H}{\partial \eta^2} + G^2 + \frac{\partial G}{\partial \sigma} + u\frac{\partial G}{\partial \xi} + v\frac{\partial G}{\partial \eta} = 0.$$

The Riemann tensor of these spaces is given by

$$R_{abcd} = \mathrm{III}_{abcd} + \varrho N_{abcd},$$

where the symbols have the same meaning as in Eq. (7).

Kundt [10] exhibited all the space-time metrics with non-expanding rays which are of type III or II null. They all have plane wave-fronts and can be characterized by this property.

## 5. SPACES WITH EXPANDING HYPERSURFACE-ORTHOGONAL RAYS

In the case of diverging rays, $k^a{}_{;a} \neq 0$, one can get

$$k^a{}_{;a} = 2/\varrho$$

by a coordinate transformation of the form $\varrho \to \varrho + \varphi(\xi, \eta, \sigma)$. $P$ can then be written as $p^{-1}\varrho$, where $p$ is a function of $\xi$, $\eta$ and $\sigma$ only. The field equations and the remaining freedom of coordinate transformations can be used to reduce the line-element to [14]

$$ds^2 = -\varrho^2 p^{-2}(d\xi^2 + d\eta^2) + 2 d\varrho d\sigma + (-2m\varrho^{-1} + K - 2H\varrho) d\sigma^2, \tag{8}$$

where

$$m = m(\sigma), \qquad K = \Delta \ln p, \qquad H = \frac{\partial}{\partial \sigma} \ln p,$$

$$\Delta \equiv p^2 \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right),$$

and $p = p(\xi, \eta, \sigma)$ is subject to the condition

$$4 \left( \frac{\partial}{\partial \sigma} - 3H \right) m - \Delta K = 0. \tag{9}$$

The curvature tensor is given now by

$$R_{abcd} = \varrho^{-1} N_{abcd} + \varrho^{-2} \mathrm{III}_{abcd} + \varrho^{-3} D_{abcd},$$

where $D_{abcd}$ denotes a tensor of Petrov's type I degenerate.

For $m = 0$, Eq. (9) reduces to $\Delta K = 0$ and the $\sigma$-dependence of $p$ is arbitrary. Solutions with $m = 0$ are of type III, II null or flat. Moreover, if $K$ is independent of $\xi$ and $\eta$ it can be reduced to $-1$, 0, or 1 by a change of $\sigma$ into a function of $\sigma$. In all three cases the Riemann tensor is of type II null, $R_{abcd} = \varrho^{-1} N_{abcd}$. For $K = 1$ the wave-fronts $\sigma = \text{const.}$, $\varrho = \text{const.}$ are spheres of radius $\varrho$ and the corresponding waves resemble the spherical null electromagnetic waves described at the beginning of this lecture. As usually waves are named after the geometry of their wave-fronts, the designation 'spherical gravitational waves' seems appropriate for the non-flat metrics (8) with $m = 0$ and $K = 1$. Spherical gravitational waves suffer from singularities, similar in nature to the line singularity that appears in null spherical electromagnetic waves, Eq. (2), when one shrinks the paraboloid by letting $l$ tend to zero.

For $m \neq 0$ one can normalize $\sigma$ so as to have $m = 1$. The solution is then completely specified by giving $p$ as a function of $\xi$ and $\eta$ for a definite value of $\sigma$, say 0. Indeed, the parabolic differential equation (9) enables us then to calculate $p$ for other values of $\sigma$. The function $p$ defines a one-parameter family of two-dimensional surfaces, $V_2(\sigma)$, with the line-element $p^{-2}(d\xi^2 + d\eta^2)$. If $V_2(0)$ is of constant curvature, $\partial p / \partial \sigma = 0$, the line-element (8) is static and the Riemann tensor is of type I degenerate and falls off as $1/\varrho^3$. In the general case, the curvature tensor is of type II and contains the $1/\varrho$ term typical for waves. Choose now for $V_2(0)$ a regular, closed surface of variable curvature, e.g., the surface of an ellipsoid. At least for a finite neighbourhood of $\sigma = 0$, Eq. (9) defines a family of regular, closed surfaces $V_2(\sigma)$. The corresponding $V_4$, with $ds^2$ given by (8), has only a point singularity at the origin $\varrho = 0$. This shows that one can construct a nearly spherical gravitational wave which, at least for a finite range of $\sigma$ is regular everywhere except the origin. A point singularity is usually interpreted as representing the source of radiation.

The authors are indebted to R. Penrose for the argument of the last paragraph.

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# RADIATION FROM AN ISOLATED SYSTEM

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In this work by Bondi, van der Burg and Metzner only the axisymmetric case is discussed, but Sachs has extended these methods to the general case and finds no real differences other than the obvious one that there are two rather than one state of polarization. By an isolated system one understands a material system entirely confined within an always finite closed surface, the rest of space being empty and tending to flatness at infinity. One studies only the variations of the gravitational field far from the system, and the assumption is later made that only outgoing waves are present.

A coordinate system is chosen, having the following properties:

(i) The azimuth $\varphi$ is invariantly defined by the axial symmetry.

(ii) Coordinates $u,\theta$ are chosen such that $u = \text{const.}$, $\theta = \text{const.}$, $\varphi = \text{const.}$ is the equation of an outgoing light ray.

(iii) The coordinate $u$ is timelike.

(iv) A coordinate $r$ is chosen so that the element of area of the 2-surface $u = \text{const.}$, $r = \text{const.}$ is $r^2 \sin\theta \, d\theta \, d\varphi$.

The metric can then be reduced to

$$ds^2 = (Vr^{-1}e^{2\beta} - U^2r^2e^{2\gamma})du^2 + 2e^{2\beta}dudr + $$
$$+ 2Ur^2e^{2\gamma}dud\theta - r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\varphi^2), \tag{1}$$

where $U,V,\beta,\gamma$ are four functions of $u$, $r$, $\theta$.

By virtue of the Bianchi identities the field equations for empty space reduce to the four 'main equations'

$$R_{11} = R_{12} = R_{22} = R_{33} = 0$$

(the coordinates $u$, $r$, $\theta$, $\varphi$ being denoted by 0, 1, 2, 3) together with two 'supplementary conditions' that the portions of $R_{00}$ and $R_{02}$ varying like $r^{-1}$ also vanish. The four main equations take the form

$$\beta_1 = \frac{1}{2} r\gamma_1^2 \tag{2}$$

$$[r^4e^{2(\gamma-\beta)}U_1] = (\beta,\gamma,1,2), \tag{3}$$

$$V_1 = (U,\beta,\gamma,1,2), \tag{4}$$

$$(r\gamma)_{01} = (U,V,\beta,\gamma,1,2), \tag{5}$$

where a suffix denotes ordinary differentiation with respect to the corresponding coordinate, while the symbol $(\beta,\gamma,1,2)$ etc. denotes an expression depending only on $\beta,\gamma$ and their derivatives with respect to (and combinations with) $r$ and $\theta$. What these equations imply is that knowledge of $\gamma$ as a function of $r$ and $\theta$ for one value of $u$ determines in turn $\beta,U,V$ and finally $\gamma_0$, so that $\gamma$ may now be constructed for the next value of $u$. However, 5 functions of integration are also involved, and the following are defined here: the parts independent of $r$ in $r\gamma$ (called $c(u,\theta)$), in $r^4 e^{2(\gamma-\beta)} U_1$ (called $-6N(u,\theta)$), and in $V$ (called $-2M(u,\theta)$).

Next the outgoing wave condition is imposed in the form, plausible from consideration of the wave equation, that each of the 4 functions $\beta,\gamma,U,V$ is of the form of a polynomial in negative powers of $r$, together with a remainder decreasing (with its derivatives) more rapidly than the lowest power of $r$ occurring. It can then be shown that coordinate conditions may be imposed which put to zero the two functions of integration not specified above. Also the otherwise exceedingly complicated supplementary conditions now take the simple form

$$M_0 = -c_0^2 + \frac{1}{2}(c_{22} + 3c_2 \cot\theta - 2c)_0, \tag{6}$$

$$-3N_0 = M_2 + 3cc_{02} + 4cc_0 \cot\theta + c_0 c_2. \tag{7}$$

The coordinate system, including the conditions just stated, still has a certain amount of freedom. The permissible transformations, however, are fully described by a single constant $v$ (corresponding to uniform motion of the coordinate system along the axis of symmetry) together with a single function $\alpha$ of the coordinate $\theta$ (corresponding to parallel displacement of the basic system of light rays at infinity).

In order to interpret our various functions we consider the static case, in which our metric must be equivalent to the Weyl metric. Then all functions are independent of $u$. The function $M(u,\theta)$ equals the mass $m$ and is thus independent of both $u$ and $\theta$, while $N(u,\theta)$ is related to the dipole moment, the coefficient of $r^{-3}$ in $\gamma$ is linked to the quadrupole moment etc.

Consider now an axisymmetric system static for $u < 0$. If $c_0 = 0$ for $u \geqslant 0$ also, then it follows from (6), (7) and the expanded form of (2)–(5) that nothing will change, while if $c_0(u,\theta)$ is a given function for $u \geqslant 0$, then these equations allow the entire development to be read off. Accordingly all the news there is will be found in $c_0$ which hence is called the *news function*. Next define the mass of system $m$ as the mean value of $M$ over the sphere so that

$$m(u) = -\frac{1}{2}\int_0^{\pi} M(u,\theta) \sin\theta d\theta. \tag{8}$$

Note than in the static case (the only one for which comparisons exist) $m$ coincides with the usual definition of mass. Now integrate (6) over the sphere

$$m_0 = -\frac{1}{2}\int_0^{\pi} c_0{}^2 \sin\theta d\theta \tag{9}$$

the second term going out owing to the axial regularity conditions on $c$. Equation (9) shows that news implies mass loss and vice versa, a most satisfactory result, displaying the energy content of information. While in the general case (9) depends to some extent on definition (8), the result (9) is wholly unambiguous if the system passes from a static state via some changes to another static state, since in static condition (8) is undoubtedly correct.

The physical components of the Riemann tensor contain points of order $r^{-1}$, type $N$ and proportion to $c_{00}$ of order $r^{-2}$, type III and depending $c_0$ and the usual Schwarzschild type terms of order $r^{-2}$ etc. Again the dependence of wave duration on the news function is clear.

In a general situation the vanishing of the news function does not imply a static field. In Eq. (7) $N_0$ will not vanish if $c = 0$ unless also $M = 0$. The equations for the higher coefficients are of similar structure, and thus we have to consider the significance of time-variable solutions with $c_0 = 0$. One and only one of these solutions is clearly understood. It corresponds to a Schwarzschild mass moving uniformly along the axis of symmetry of the system of coordinates. For this solution $M_2 \neq 0$, giving a sort of Dopplershift of the mass aspect $M$.

Other solutions of this kind await interpretation. The question may be put in the form: Is there anything to distinguish *in the matter* the solutions that at infinity are singled out by $c_0 = 0$? One would like to answer yes, and then one would perhaps think of forcefree motion (as in dust) or possibly of the motion of matter with an equation of state not explicitly dependent on the time or its direction (i.e. free of dissipation). On the other hand the linear approximation does not favour this view. Pressing as the need for an answer is, an asymptotic method does not lend itself to this purpose.

Since the source emits energy in waves with non-vanishing news function, one would like to know about the reception of these waves and the absorption of their energy. Only a rough treatment is possible, which nevertheless is quite instructive. The simplest receiver would appear to be a freely falling arrangement of two equal masses separated by a small distance, with a mechanism arranged to vary their separation so as to maximise its energy gain. The effect of the field on the masses is expressed by the appropriate component of the Riemann tensor. However, the possible gain of energy from the field is limited by the re-radiation from the receiver due to the variation of its quadrupole moment consequent upon the motion of the masses. The work is completely analogous to the electromagnetic case, and the result for the maximum rate of absorption of energy is

$$\frac{15}{16}\,\frac{(c-c_{\text{initial}})^2}{r^2}.$$

However, a difficulty in interpretation arises from the fact that, owing to the mass loss of the transmitter, $c$ does not return to its initial value except perhaps in a few isolated directions. Thus a receiver that remains switched on can absorb energy from the change in the Coulomb field, that itself is due to the emission in the wave, rather than from the wave itself. Though this can be avoided by limiting the period of activity of the receiver, in the strictest interpretation the impossibility of wholly separating the wavefield from the Coulomb field could be stated as the non-existence of a proper wave zone.

A large receiver may be considered by supposing a large spherical shell to surround the empty space which has the radiating system at its centre, the outgoing wave metric of the previous considerations applying within the shell, and a Schwarzschild metric outside it. Linking the two metrics in any way will result in describing the material of the shell, and if the mass of the Schwarzschild metric is large enough the material of the shell should everywhere be physically possible. The external metric being static, the combined mass of shell and transmitter does not vary. Hence all the energy lost by the transmitter is absorbed by the shell, which is thus a perfectly matched and tuned receiver.

The two chief remaining problems seem to be the interpretation of the news-free (i.e. non-radiative) time-variable solutions and the reformulation of the initial value problem and the coordinate conditions so that the outgoing wave condition is only introduced later in the work, and 'mixed' situations can, at least in principle be described.

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## DISCUSSION

J. L. Synge:

I wonder whether Prof. Bondi would be willing to look at this very interesting work from a slightly different point of view? He writes down a line-element involving, I think, four functions. Is that right? Now, by qualitative limitations on those functions, your line-element will have the correct signature.

H. BONDI:

Yes.

J. L. SYNGE:

And no matter what those functions are, within that qualification, we may say that that line-element represents a universe in which there is a certain distribution of energy and momentum.

H. BONDI:

Yes.

J. L. SYNGE:

Now, your job is to choose these four functions in such a way that the energy tensor will vanish outside a certain domain; and inside a certain domain will satisfy the realistic condition that the energy shall be positive—but you don't bother about that. Now in the outside domain one could wish for an algorithm which leads inevitably to certain four functions at any given point, such that as we pursued that process further and further you would reduce the energy tensor at that point to anything as small as you wished. Could you claim that your method does that?

H. BONDI:

I wish I could answer this question fully and directly. But as I haven't worked it out, I must give you a woolly answer. And this would be that if I pursue this process down a certain number of steps, if I go along my theory a certain number of steps, and claim that a certain derivative of the remainder goes down faster than a certain power of $r$, then I can work out the energy tensor up to these steps, and I can find that the energy tensor goes down faster than a corresponding power of $r$. I think that I could also show (though I haven't shown it) that, if I took the next step, the energy tensor would go down faster than before by, say, two powers of $r$.

J. L. SYNGE:

But I'm not talking about that; I'm thinking of fixed $r$.

H. BONDI:

I think it is true that at a fixed $r$, provided it is large enough, each successive step, as I've discussed it, will reduce the mass density there. I'm not convinced that this is necessarily a convergent process: it might be an asymptotic one.

J. L. SYNGE:

When you say large enough, we might be left with a body of huge extent.

H. BONDI:

Well, it's not a body, not yet. I'm always discussing a perfectly finite system, and I look at this system from sufficiently far away. Now, the need to look at the system from sufficiently far away arises from the fact that if I don't, then some of my light rays, when I pursue them, will intersect, and my coordinate system will become singular. And, in fact, the surface where the system becomes singular is, presumably, outside the actual body. So if the surface in some treatment becomes very large it doesn't necessarily mean that the body becomes very large. It only means that I must look at it from very far away. It is an asymptotic treatment, strictly, in the sense of looking at it from very far away.

J. L. SYNGE:

Looking at something that exists? Perhaps it doesn't exist.

P. G. BERGMANN:

I would like to come back to a point that was only tangentially mentioned both by Prof. Bondi and in the discussion. And that is the question of whether an $n$-body system held together by purely gravitational forces will radiate or not. Now, as I understand the situation, it is perfectly safe to argue the point on theoretical grounds because Weber is a comfortable number of orders of magnitude away from deciding this question experimentally; and we will probably all be dead by the time the decision is in. It seems to me that if one uses the weak field approximation as a point of departure to an expansion (which is admittedly not known to converge, but at least leads to meaningful operations at each step), that perhaps those here who have done appropriate weak field slow and weak field fast approximations might have some contribution to make. I don't think that we can decide this question today, except by majority vote.

J. STACHEL:

I'd like to return to the question of the mass aspect that Dr. Schücking raised. There is, I think, a big difference between the static and the non-static cases in our understanding of mass. Not only can we give an invariant meaning to the concept in the static case, but I think it's pretty clear that a small bundle of observers, represented by a rather small channel in the four space far away from the source, could measure this mass. You have indicated a method whereby a bundle of observers who completely surrounded the mass might possibly be able to measure the mass aspect—or the total mass, although that is not entirely clear to me. Do you have any way in mind by which a small bundle of observers not surrounding the mass could measure the mass aspect?

H. Bondi:

In the dynamic case, none.

R. P. Feynman:

In reply to the question whether things which interact only gravitationally can radiate, I tried the problem of two masses which come past each other from infinity at some arbitrary speed. Then we can make an expansion in powers of the gravitational constant. In the lowest order, which is already the second, because one has to generate a field which acts on the other, there is no radiation yet. It is just the beginning of the collision. In the next order, which is then the third there is a possibility of emission of the wave and this wave can come, physically speaking, from three kinds of places. It can come from the matter generated or from the field which is going between the two, from the energy of the interaction. And all this is included correctly in each order. And in this order there is definitely a radiation, in the order where it ought to come out, and there is exactly the same radiation as you'd calculate by supposing the motion to be given by some other interaction, non-gravitational, in any other manner. One gets the same degree of radiations of the gravitational wave, and so on. And so I've convinced myself at least, and that is all I expected to convince in this matter, that there is no doubt that there is radiation from a gravitationally interacting system. May I take the opportunity to ask another question, too; while I have the floor? Your work is very interesting but involves the assumption of the symmetry with regard to $\varphi$. The physical results that you have don't seem to involve this phi symmetry in any physical way; and I wondered whether you had any difficulty in generalizing it so that things would depend on phi?

H. Bondi:

Sachs had done this in detail; and nothing new, I think I'm right to say, nothing new emerges, except you have the other polarization as well. Now to your other point. If I may remark so, when you said you'd only convinced yourself, you certainly shifted me by a certain extent, even if only by an inch or so. It's a point that interests me very much. You must of course assume certain equations of state for your bodies. And I think this is something on which it all depends critically. The simplest situation seems to me to be the pressure-free gas in which we have just $T^{\mu\nu} = \varrho v^{\nu} v^{\mu}$, every particle following a geodesic. I would be most surprised if this radiated, but I don't know. In the next stage I feel it depends very strictly on the equation of state.

It seems to me absolutely obvious that if the equation state involves the time explicitly, as in the case of the time bomb, it will radiate; but I still have the hope that a purely passive equation of state might exclude radiation.

R. P. FEYNMAN:

I wanted to explain that there's a question of what is the equation of state of the bodies I use. I have to admit that this thing was done on a quantum-mechanical level and that the bodies that were used were spin zero particles, obeying the Klein–Gordon equation, and interacting in the way that would be predicted from the gravitational equations. Now, it is also possible to make the analysis and understand what happens by taking the limiting case that the frequency of the radiation that you're looking for is low, by going in that direction. And then you can show that the answer of the radiation is independent, in the low-frequency limit, of the force which generates the interaction. For instance, they could have been scattered because they were electrically charged. Giving only the angular momentum in and out is all you need to get the radiation in the low-frequency region. Spin 1/2 particles gave the same radiation as spin zero particles in the low-frequency region. A problem with three or four particles was also worked out; for instance, the disintegration of a spinless particle into three particles. The radiation from that was calculated; in the limiting case of low frequencies the formula was again simple and was exactly the formula that you would get from the momentum vectors of the various particles. It, therefore, appears to me that in the reasonable situation where you have a thing in a limited region, no matter how complicated the interaction, in the first order in which there is radiation, that radiation will be obtained, from the stress tensor variations and by the usual theory of emission. So that if a body has an equation of state, if that equation of state is due to internal interactions such as electromagnetic and so on, in other words if the body is an actual body as we understand them in physics today, with the machinery in detail inside, I still think that the whole thing will radiate at low frequency levels. The high frequency radiation, which has to do with the specific collisions, would be something like the electromagnetic radiation of a gas, which can be analyzed in two ways. If we consider motion of the charge, say, you can talk about the low frequencies due to the grand motion of the average charge. But on the other hand, the internal motions are radiating electromagnetic waves. Those are the high frequency end and they're the analogue of this. The hot gas will radiate thermal gravitational oscillations on top of the ones that we've been talking about.

## GENERAL DISCUSSION

A. Komar:

The issue I wish to discuss is where does the special significance of energy come from, that everyone is so concerned with trying to find an expression for energy. It comes originally from the Lorentz-covariant theories. For example, let's take energy, and not momentum or angular momentum, although much of what I say could be applied to them too. For energy, of course, in a Lorentz-covariant theory it is typically the expression which generates the rigid displacement in the time-like direction in the usual Minkowski frame. In other words, you can go to polar coordinates, for example, in Lorentz-covariant theories, or go to a frame which splits up the time, or use curvilinear coordinates; what really singles out the preferred directions, or the preferred conservation laws, are the Killing directions determined by the Minkowski metric. For the energy, momentum and angular momentum they're precisely the generators of the coordinate transformations in the Killing directions. When you go to curvilinear space, one can write down the generators of coordinate transformations in arbitrary directions but this does not necessarily mean that they give you physically meaningful or physically preferred conserved quantities. However, when there happen to be Killing vectors, for example to take the Schwarzschild solution in the usual coordinate system, the vector pointing in the time direction of components (0, 0, 0, 1) is a Killing vector; and this is a very natural vector to use for the definition of energy. When this is properly used you find that you get out as the energy the Schwarzschild mass. The question arises, what does one do when one does not have Killing vectors, because in general you do not.

Now, you can take several different approaches. One can say, well, physically these energy integrals are convertible into surface integrals over a two-surface which can be taken at infinity, and so that if the space is asymptotically Minkowskian the appropriate thing to take is a vector field which is in some sense asymptotically Killing. And it is a little bit tricky to define what you mean by asymptotically Killing, but it has been done in a paper which is coming out shortly; and you find that it is closely related to the choice that Trautman made in treating the radiation problem of choosing the surfaces to be asymptotically harmonic. The harmonic condition played an important role in Trautman's treatment of the radiative case, and it's essentially to select out the Killing direction. There are some qualifications about that, you need it also asymptotically minimal, and you can impose these two conditions simultaneously.

The other approach is that you see that the preferred conserved quantities are obtained by taking generators of coordinate transformations in some preferred directions, preferred by some differential equations, the Killing equation when you can find solutions. The question is when you cannot find solutions of the full Killing equation: is there some set of equations which you can impose on the descriptor vector fields which will be satisfied by Killing vectors when there are Killing vectors, but can be satisfied more generally. Then this would be a preferred set of vector fields to look at in the general case. And there are such, and they turn out to be orthogonal trajectories of families of minimal surfaces. If you choose them, you can show the following result: that the energy you get by using this preferred set of vector fields is necessarily positive definite, and that furthermore when the energy is zero the space is flat, and this is a global theorem, the proof of which goes rather similarly to the proof of the theorem that when the space is static, it's globally flat. It's slightly more complicated, but you get the same result. A third point of view possibly is that when you don't have Killing vector fields it's not really too legitimate to look for energy, and that one should not expect to find gravitational energy meaningfully localized. This can be related to the Gupta-Feynman approach to general relativity. For example, you start out with a spin two mass zero theory, and you write down the Lagrangian and construct the canonical stress tensor. Now make this the source of the field; now you have to write down a new Lagrangian, construct the canonical stress tensor from that, make that the source of the field, and reiterate this procedure *ad infinitum*. What you end up with is the Einstein theory. However, having done all this, if you now can localize energy again, one can legitimately ask: why not start the iteration procedure all over again on top of this? Somehow the Einstein theory in one fell swoop performed this infinite iteration, and it probably is not fair to require that the energy localize uniquely at each point in the gravitational field.

F. J. BELINFANTE:

I want to make two remarks. The first remark is about energy densities. Komar has just related these to generators of displacements. Bergmann has stated that there are displacement operators that naturally have the value zero. In this connection I also remind you of the use I made at Royaumont of an energy density tensor in a Schrödinger equation. As we there considered displacements from one spacelike surface to another which were different from point to point, we reached there a localization of this energy density over the surface. On the other hand, Møller and Feynman want to compute how much of some conserved quantity called "energy" is available for engineers to use at a given point. Here I want to repeat what somebody said at Royaumont. There is here no contradiction, but in the two cases we talk about two different things. This is my point number one.

Point two is related to tetrads. I would not think of introducting them in a discussion merely of the interaction of gravity with bosons. These Vierbeine, however, are used anyhow when we discuss the interaction of fermions with gravity. They occur in the covariant derivatives of the spinors we use. Just as for coordinates it is sometimes handy to pick a *particular* choice, for instance, polar coordinates or harmonic coordinates, it is then also sometimes handy to pick a particular choice of Vierbeine. In his 1952 article DeWitt made some particular choice which he called $\sqrt{g^{\nu}_{\mu}}$, but this choice was not invariant under coordinate transformations. Møller's choice is at least that, and I could imagine that Møller's choice of tetrads might at least be good for that.

P. G. BERGMANN:

You do not need tetrads in the fermion theory, and Møller's choice is not invariant under tetrad rotations.

F. J. BELINFANTE:

If you write your fermion field by means of spinors and you do not write the tetrads explicitly. They are still hiding in the quantities you do write down or in the interpretation of these quantities. The invariance of Møller's conditions under $C_\infty$ is all we want; we cannot have invariance under $L_x$ if these equations are to fix a tetrad field.

J. S. VLADIMIROV:

*Gravitational Annihilation of Electron-Positrons.* By means of the linearized gravitational field theory[1] a computation of the cross-section of electron-positron pair transmutation into 2-gravitons is performed.

The interaction Hamiltonian for the transversal gravitational field and spinor particles has the form

$$H = \frac{i}{4} \varkappa h_{\mu\nu} \left( \bar{\psi} \gamma_\mu \frac{\partial \psi}{\partial x^\nu} - \frac{\partial \bar{\psi}}{\partial x^\nu} \gamma_\mu \psi \right).$$

With the aid of the perturbation theory one can in the 2nd order derive for the differential cross-section:

$$d\sigma = \frac{\varkappa^2}{4(4\pi)^2} \cdot \frac{p^2}{16k_0} \cdot \sin^4\theta \left\{ 1 + \frac{2p^2}{k_0^2 - p^2\cos^2\theta} - \frac{2p^4 \sin^4\theta}{(k_0^2 - p^2\cos^2\theta)^2} \right\} d\Omega .$$

(1) S. N. Gupta, *Proc. Roy. Soc.* **A65**, 161, 608 (1952).

This expression has a specific factor $\sin^4\theta$ leading in contrast to electrodynamics to a maximum of the flux of gravitons at $\theta = \pi/2$.

In the ultrarelativistic case one gets $\sigma \sim \left(\frac{E}{mc^2}\right)^2$, i.e. in contrast to electrodynamics the gravidynamical cross section is increasing with energy and at $k_0 \sim 10^{12}\ mc^2$ (certainly beyond weak field approximation validity) becomes comparable with the cross section of the annihilation into photons.

In non-relativistic approximation our result coincides up to a constant factor with a similar result for the scalar field case investigated first by D. Ivanenko and A. Sokolov.[(2)]

In spite of the practical negligibility of the gravitational transmutations such processes may be of importance at cosmological scale as stressed also by J. A. Wheeler.[(3)] Using a consequent theory of Fock-Ivanenko of fermions in gravitational field, based on tetrads, one gets supplementary diagrams, which all lead to a cross section, given here for simplicity in ultra-relativistic case:

$$d\sigma' = \frac{\varkappa^2 k_0^2}{128\ (4\pi)^2}\ (3\sin^2 2\theta + 2\sin^4\theta)\, d\Omega.$$

S. Deser:

*Conditions for Flatness of an Einstein Space.** The present report is based on work carried out in collaboration with R. Arnowitt, a fuller account of which has been submitted to *Annals of Physics*.

The problem to which we address ourselves is that of finding conditions which characterize the "vacuum" state in an Einstein space,[(4)] $R_{\mu\nu} = 0$ that is, what further (physical) conditions on the metric of such a space imply that it is flat, $R_{\mu\nu\alpha\beta} = 0$. The Newtonian analogue of this problem is, of course, the classical question of what conditions on the solution of the potential equation $\nabla^2\varphi = 0$ with appropriate boundary conditions, imply $\varphi = 0$. The fact that the Einstein field equations involve time derivatives and are hyperbolic in nature, means that there will, in general, exist regular solutions of the source-free equations with asymptotically flat boundary conditions.[(5)] Thus the state of no gravitational excitation must be charac-

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(2) D. Ivanenko, A. Sokolov, *Vestnik Moscov State Univ.*, No. 8, 103 (1947). D. Ivanenko, Theories Relativ. Gravit. (Royaumont 1959).

(3) J. A. Wheeler, Geometrodynamics (N.Y. 1962) (Ed. Paris 1962).

(4) Greek indices range over 0, 1, 2, 3 and Latin over 1, 2, 3. We work with the empty-space field equations except in Theorem V.

(5) As might be expected from this argument, the analogue of the Newtonian theorem for the full theory is the absence of regular "stationary" solutions (i.e. those with $\partial_0 g_{\mu\nu} = 0$).

terized by the vanishing of appropriate field variables. In this respect, the situation is similar to Lorentz-covariant field theories, and one expects, in analogy, that the characterization of the vacuum state can be made entirely in terms of the initial Cauchy data. The field equations should then preserve the initial flatness of space-time. [Thus, in source-free electrodynamics, the vanishing of $\boldsymbol{E}$ and $\boldsymbol{B}$ initially (or alternately, the vanishing of the transverse modes $\boldsymbol{E}^{\mathrm{T}}$ and $\boldsymbol{A}^{\mathrm{T}}$) together with requirement that they vanish asymptotically, implies that they vanish for all time].

Although we shall be interested primarily in statements which can be made on a given space-like surface, a number of results have also been found in terms of certain quantities vanishing for all time. The Maxwell analogue here is the well-known theorem that if *either* $\boldsymbol{E}$ or $\boldsymbol{B}$ vanishes throughout space-time, then so does the other. The distinction between statements involving only Cauchy data at a given time (that is, only components of the metric and its first time derivatives) and those involving all times (i.e. depending effectively on second time derivatives) is exemplified by the theorem of O'Raiffertaigh and Synge[6] which states that if the four-dimensional curvature tensor vanishes initially, space is everywhere flat. Since second time derivatives, of course, appear in the Riemann tensor, the specification goes beyond that of initial data. We might mention here that a slight generalization of the O'Raiffertaigh–Synge theorem is easily established: *if a metric is conformally flat initially, it is flat everywhere*, since for empty Einstein spaces the conformal and Riemann tensors are equivalent.

We shall state below, and discuss briefly, the body of theorems that have been obtained. The proofs can be found in the publication referred to at the outset. It is convenient to have available the following 3+1 dimensional notation to discuss the results:

$$\pi_{ij} = (-{}^4g)^{1/2}(\Gamma^0_{ij} - g_{ij}\Gamma^0_{pq}\,{}^3g^{pq}).$$

In the above, ${}^3g^{pq}$ is the three-dimensional inverse of the covariant spatial metric $g_{ij}$. (Note that $\pi_{ij}$ is linear in the $\partial_0 g_{ij}$). Also, ${}^3R_{ijkl}$ is the three-dimensional curvature constructed from $g_{ij}$ and ${}^3g^{ij}$, and all three-dimensional indices are raised and lowered using $g_{ij}$ and ${}^3g^{ij}$.

T h e o r e m I. *Any regular, asymptotically flat solution of the Einstein equations for which* $\pi^{ij} = 0$ *for all space-time is flat everywhere.*

T h e o r e m II. *Any regular, asymptotically flat solution of the Einstein equations for which the three-dimensional curvature tensor,* ${}^3R_{ijkl}$ *vanishes for all space-time is flat everywhere.*[7]

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(6) L. O'Raiffertaigh and J. L. Synge, *Proc. Roy. Soc.* A **246**, 299 (1958).

(7) While the Schwarzschild metric can be cast into a form in which ${}^3R_{ij} = 0$ everywhere, it is of course not regular at the origin in such coordinates.

This pair of theorems (the proof of the first being, incidentally, more straightforward than that of the second) corresponds to the Maxwell conditions mentioned earlier that the vanishing of either $\boldsymbol{E}$ (or $\boldsymbol{B}$) for all time implies that $\boldsymbol{B}$ (or $\boldsymbol{E}$) vanishes as well. Of course, the vanishing of $\pi^{ij}$ is partly a specification of dynamical data (like the vanishing of $\partial_0\varphi$ for the scalar field) and partly a statement of coordinate conditions, since the specification of the time coordinate involves a component of $\pi^{ij}$. Similarly, the vanishing of the three-curvature tensor also contains coordinate information.

We now turn to the theorems involving only the initial Cauchy data, $g_{ij}$ and $\pi^{ij}$ at a given time. Here, we shall sometimes invoke particular coordinate conditions, such as the use of minimal surfaces. The difficult question of the global existence of such coordinates is not investigated here, though it is known that they at least exist in a perturbation expansion and that, rigorously, they exist asymptotically for a very wide class of metrics.

Theorem III. *Any regular solution containing a minimal surface* $\pi \equiv g_{ij}\pi^{ij} = 0$ *which is intrinsically flat* (${}^3R_{ijkl} = 0$) *is flat everywhere.*

The proof of this geometrically plausible theorem is not difficult, since one of the constraint equations reads

$$-({}^3g)^3R - \frac{1}{2}\pi^2 + \pi^{ij}\pi_{ij} = 0, \qquad {}^3R \equiv {}^3g^{ij3}R_{ij}$$

which by the hypotheses, implies that the surface $\pi = 0$ defines an instant of time symmetry $\pi^{ij} = 0$ (as the vanishing of $\pi^{ij}\pi_{ij}$ implies that of $\pi^{ij}$). Since the full curvature tensor may be expressed entirely in terms of ${}^3R_{ijkl}$ and $\pi^{ij}$ at any instant by means of the field equations, it vanishes at that instant. Then by the O'Raiffertaigh–Synge theorem, it vanishes for all time. The above theorem represents a particular example of a more general conjecture that an asymptotically flat system with vanishing mass is necessarily flat. More precisely, the conjecture is that if the asymptotic $1/r$ part of the component of the *spatial* metric, $h^{\mathrm{T}}$, vanishes where

$$h^{\mathrm{T}} \equiv h_{ii} - (1/\nabla^2)h_{ij,ij}, \qquad h_{ij} \equiv g_{ij} - \delta_{ij}$$

then space is flat.[8] In a particular frame (defined in theorem IV below) the result has been established for the case in which the spatial metric is conformally flat at any time.[9]

The next theorem we discuss is specifically related to the canonical formalism of general relativity[10] in which the independent dynamical excitations (or independent Cauchy data) of the gravitational field consist of two

[8] For a brief discussion as to why this component correctly measures the mass see for example, the report by R. Arnowitt in these Proceedings.

[9] R. Arnowitt, S. Deser and C. W. Misner, *Ann. Phys.* **11**, 116 (1960).

[10] R. Arnowitt, S. Deser and C. W. Misner, *Phys. Rev.* **117** 1595 (1960).

pairs of canonical variables formed from the twelve $g_{ij}$, $\pi^{ij}$, the definition being explicitly given in particular coordinate frames. Physically, then, it is important, for the canonical formalism, to show that the initial vanishing of these canonical excitations (corresponding to $\boldsymbol{E}^T$ and $\boldsymbol{A}^T$ in electrodynamics) rigorously defines the vacuum state, i.e. that if the two fundamental modes are not excited at a given instant, space is always flat. While this result has been shown earlier from perturbation arguments,[7] it has now been established in a rigorous way:

Theorem IV. *The only regular asymptotically flat solution of the Einstein equations whose two pairs of canonical variables vanish initially in the frame specified below is flat space.* This frame has spatial components closely related to isotropic coordinates, and minimal surface time specification:

$$\pi = 0 = g_{ij,jmm} - \frac{1}{4}(g_{jj,imm} + g_{mj,mji}).$$

The proof is quite short in terms of the orthogonal decomposition used to define the various components of the field, and uses the four constraint equations involving the Cauchy data $g_{ij}$ and $\pi^{ij}$ (but not their time derivatives). It may also be mentioned that general theorems concerning the vanishing of the three-momentum of a system arise simply in the canonical analysis, e.g., if either the two dynamical variables or their conjugate momenta vanish initially the field momentum vanishes always. (An analogous theorem holds for all Lorentz-covariant field theories, also).

Finally, we prove a generalization of Birkhoff's theorem which involves *only* a single time. Here, we lift the source-free requirement on the field equations.

Theorem V. *A regular asymptotically flat solution of the field equations whose initial state on a three-surface is spherically symmetric in the exterior region, defines a four-space which possesses an exterior Schwarzschild domain at all times.*

The proof consists in first showing that initially, $g_{ij}$ and $\pi^{ij}$ have the form appropriate to the spatial part of the exterior Schwarzschild solution. By the uniqueness theorem on Cauchy data in general relativity,[11] the exterior solution then coincides *always* with the exterior Schwarzschild one. More precisely, the exterior Schwarzschild region at all other times is determined as that region bounded by the light cones, leading off the initial surface's exterior region. Only in this region, of course, can the solution be Schwarzschildian. For the sources may, in time, by their own dynamical motion, penetrate into other regions (previously empty) and there alter the form of the solution. It can be shown, further, that if the mass vanishes ini-

(11) See, for example, A. Lichnerowicz, Théories Relativistes de la Gravitation, Paris 1955.

tially, space is flat everywhere, and if there are no sources, the mass must vanish by regularity conditions, so that in both cases space is flat just as is shown in the Birkhoff analysis. Note that our derivation only uses initial information on the metric, in contrast to the standard Birkhoff one. Usually, it is just *assumed* that the exterior is spherically symmetric at all times.

A. L. ZELMANOV:

The problem on the gravitational energy and momentum allows an approach which does not lead to any restrictions of the metrics or of the choice of the coordinate systems, does not require any modifications or additions to Einstein's gravitational theory and is based on a consistent use of the principles of this theory. Under this approach some requirements are imposed on the densities of gravitational energy, momentum and flow of momentum, namely not only the requirement for purely spatial covariance, but also that for chronometrical invariance (i.e. invariance under transformations of the temporal coordinate), the complex describing these quantities being a pseudo-tensor (but not a true tensor).

The physical reason for non-covariance of this complex in respect to general transformations of coordinates lies in the possibility to make it vanish at any ponit by means of a suitable choice of the time lines. But the covariance under the transformations leaving the time lines unaltered must be valid. This covariance can be splitted into purely spatial covariance and chronometrical invariance.

A number of known forms of the gravitational energy-momentum pseudo-tensor was considered, and no one of them was found to satisfy the requirement for chronometrical invariance. But this last requirement appears to be no less essential than that for purely spatial covariance. If, for instance, the energy density would not be chronometrically invariant it would be possible to create or annihilate energy solely by transformations of the time cordinate. Hence, one should have to prefer to retain the requirement for chronometrical invariance (together with that for spatial covariance) even if it would be at variance with some other requirements, usually imposed on the gravitational energy–momentum pseudo-tensor.

(See also the rreport "Chronometrical invariants and some applications of them", p. 328).

# THE QUANTIZATION OF GEOMETRY

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RELATIVITY and quantum theory have both demanded a fundamental revision in our ways of thinking about Nature. Relativity has forced us to revise our concepts of space and time. Quantum theory has forced us to change our attitude toward determinism and measurability. Both have led us to rethink, at a basic level, the problem of *what constitutes an observation.*

I should like to indicate briefly today some of the new problems to which one is led in the course of attempting to bring about a fundamental union of the two theories. The union which is envisaged goes beyond the familiar superposition of special relativity on a quantum framework and, although informally referred to as *quantum gravidynamics,* has implications by no means limited to gravitation theory.

The problem of what constitutes an observation continues to be a basic one in the quantum theory of geometry. In order that my subsequent remark be understood in proper perspective, let us, therefore, first review the elements of measurement theory in quantum mechanics and its relation to the Uncertainty Principle.

The very first and most fundamental assumption of quantum theory is that every isolated dynamical system is describable by a characteristic action functional $S$. Nearly everything else follows from this. In order to perform a measurement on the system it is necessary to couple it to an apparatus. The weaker the coupling the smaller the disturbance in the system and the more accurate, according to classical ideas, should be the information gained concerning the state of the originally undisturbed system. Quantum theory, however, imposes a limit to this accuracy.

The coupling process corresponds, at the most rudimentary level, to a transformation of the action of the form

$$S \to S + \varepsilon A, \tag{1}$$

where $A$ is an observable of the system and $\varepsilon$ is a small constant. By choosing $\varepsilon$ to be constant we are neglecting the important effect of $S$ on the apparatus, which actually constitutes the measurement. We do this here only because we are for the moment focussing our attention on the disturbance which the apparatus itself produces. Let us denote by $\delta_{\bar{A}}B$ the *retarded* change which the transformation (1) produces in a second observable $B$. In the following equation

$$D_A B \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \delta_{\bar{A}} B, \tag{2}$$

we have the definition of an important quantity $D_A B$ which was originally introduced by Peierls [1].

If the dynamical equations are invariant under an infinite dimensional group, for example, the coordinate transformation group in general relativity then the changes which the transformation (1) induces in the dynamical, variables are determined only *modulo* a group transformation. The quantity $D_A B$, however, will be well-defined and unique *provided A and B are group invariants.*

$D_A B$ together with the reciprocal quantity $D_B A$ was used by Peierls [1] to define the Poisson bracket in the form

$$(A, B) \equiv D_A B - D_B A. \tag{3}$$

From the definition (2) we see that $D_A B$ and $D_B A$ characterize effects of *infinitesimally small disturbances* in the system. Because infinitesimal disturbances are propagated by means of linear equations the superposition principle may immediately be invoked to prove that Peierls' Poisson bracket satisfies all the usual identities. Because these linear equations are *necessarily self-adjoint* when the dynamical equations themselves are derivable from a variational principle, fundamental reciprocity theorems hold, and the identities in question include the Poisson-Jacobi identity which is all important in establishing the group structure of canonical transformations.

Two characteristics of this approach to the quantum theory are especially noteworthy. Firstly, canonical Hamiltonian methods are seen to be totally unnecessary. Secondly, through the fundamental connection between the commutator and the Poisson bracket, namely

$$[A, B] \leftrightarrow i\hbar (A, B) \tag{4}$$

the quantum theory emerges as basically a theory of infinitesimal disturbances.

These disturbances play a fundamental role in the theory of measurement. Upon bringing the apparatus variables into the picture one can show, by essentially paraphrasing the arguments of the famous Bohr–Rosenfeld paper on electromagnetic measurements [2], [3] that the disturbances conspire in such a way as to insure that the uncertainty relation

$$\Delta A \, \Delta B \geqslant \hbar |(A, B)| \tag{5}$$

holds for the system observables, with the Poisson bracket given by (3), if it first holds, in exactly the same form, for the apparatus variables. It then follows, firstly, that the Uncertainty Principle, when once inserted into physics at any point, immediately extends itself by induction to all of physics; secondly, that the quantum theory is self-consistent as regards its operator formalism and interpretation; and thirdly, that the definition (4) for the commutator is essentially unique. The last point is perhaps the most important, for it permits the Uncertainty Principle to be turned around and used to define the commutator even when a canonical formalism is not readily available.

The detailed use of the definition (3) for the commutator requires the introduction of the Green's functions which describe the propagation of the infinitesimal disturbances. All pertinent Green's functions can be obtained from the well known Feynman propagator by splitting it into its real, imaginary, advanced, and retarded parts, and recombining these parts in appropriate ways. I shall denote the Feynman propagator by $G^{ij}$, the indices being the same as those which appear on the field variables which I shall denote by $\varphi^i$. For brevity I shall allow the indices $i$, $j$, etc. to do double duty as discrete labels for field components and as continuous labels, or *coordinates*, over the points of space-time. The Feynman propagator is then to be understood as a continuous matrix and the summation convention for dummy indices is extended to include integrations.

The Feynman propagator has the remarkable property that it alone, of all the Green's functions of a given operator $F_{ij}$, follows the rules of ordinary finite matrix theory. That is to say, it satisfies the identities

$$F_{ik}G^{kj} = -\delta_i{}^j, \qquad G^{ik}F_{kj} = -\delta^i{}_j, \tag{6}$$

and, under an arbitrary variation $\delta F_{ij}$ in the operator $F_{ij}$ the variational condition

$$\delta G^{ij} = G^{ik}\delta F_{kl}G^{lj}. \tag{7}$$

Furthermore, if $F_{ij}$ is symmetric in its indices, as it is when boson fields are involved, then $G^{ij}$ is likewise symmetric,

$$G^{ij} = G^{ji}. \tag{8}$$

Other Green's functions may satisfy one or the other of equations (7) and (8), but not both. The unique features of the Feynman propagator stem from the fact that it can be obtained by analytic continuation from the unique Green's function which $F_{ij}$ possesses when space-time has positive definite metric. But the fundamental role which these features play in the formal manipulations of the theory also suggest that we may be able to turn the theory around and to *define* the Feynman propagator by means of Eqs (6), (7), and (8) even when the usual definition, in terms of positive frequencies propagating into the future and negative frequencies into the past, becomes inapplicable because of space-time having non-static asymptotic curvatore or space-like cross-sections of non-Euclidean and even dynamically changing topology.

Green's functions in a curved space-time have a special characteristic that they do not possess in a flat space-time—namely, multivaluedness. The onset of multivaluedness occurs in regions where the geodesic lines emanating from a given point begin to cross one another. General expressions for the Green's functions in a given metric are exceedingly difficult to find. However, their structure both near the light cone and in asymptotic regions is well known. The simple Klein–Gordon equation

$$g^{\mu\nu}\varphi_{.\mu\nu}-m^2\varphi = 0 \tag{9}$$

leads to Green's functions which are representative of the whole class. Near the light cone the Feynman propagator for this equation has the structure

$$G(x,x') = \frac{i}{8\pi^2}\left[\frac{\Delta^{\frac{1}{2}}}{\sigma+i0}+v\log(\sigma+i0)+w\right] \tag{10}$$

which shows the typical $1/\sigma$ and logarithmic singularities. $\sigma$ is the *world-function* which has been so thoroughly investigated by Synge. $\Delta$ is essentially the Van Vleck determinant, built out of the second derivatives of $\sigma$ with respect to $x$ and $x'$. $v$ and $w$ are nonsingular biscalars which can be expressed as power series in $\sigma$, converging in the single-valued region. The multivalued region is bounded by the caustic surfaces, i.e., by the envelopes of the geodesics emanating from $x$ or $x'$. *On* the caustic surfaces the Van Vleck determinant becomes infinite. Erickson [4] has shown that the $1/\sigma$ and $\log\sigma$ singularities, as well as arbitrarily high derivatives of these light-cone singularities, can be removed in field theoretic calculations by the Feynman–Pauli-Villars regularization method, just as in the case of flat space-time. The caustic singularities, however, which occur both on and off the light cone, remain, and constitute in principle an additional problem the study of which may uncover novel properties of quantum gravidynamics, although I should hasten to add, these singularities can have no effect on the renormalization program, for the divergences which arise in the latter are those which appear when $x=x'$ and involve only the single-valued region.

I wish to emphasize the covariance of expressions such as (10) for the Feynman propagator. The directness with which this covariance can be stated in the coordinate representation contrasts sharply with the complexity of its rigorous statement in momentum space and suggests that at least parts of quantum gravidynamic calculations should be carried out in coordinate space. This is indeed feasible and, for example, Feynman's proof of the covariance of the counter terms needed for renormalization can be obtained much more directly this way. Computations in coordinate space are still rather unfamiliar, but little by little, with the aid of identities like (7), which

Mrs. DeWitt has, for example, used in a study of radiation damping, we are learning not only how to do such calculations but also more about the Green's functions themselves.

Equation (11), for example, shows the asymptotic behavior *off* the light cone when $m$ is different from zero:

$$G(x, x') \sim \frac{1}{2}\sqrt{\frac{m}{(2\pi)^3}}\frac{\Delta^{\frac{1}{2}}e^{-i(m\tau+\pi/4)}}{\tau^{3/2}}\left(1+\sum_{n=0}^{\infty}\frac{a_n}{\sigma^n}\right), \tag{11}$$

where

$$\sigma \equiv -\frac{1}{2}\tau^2. \tag{12}$$

In the case of tensor and spinor fields the expressions for the Feynman propagators are simple generalizations of Eqs. (10) and (11). $v$, $w$, and the $a_n$ become bitensors or bispinors, while the Van Vleck determinant gets multiplied by an appropriate bitensor or bispinor of geodetic parallel displacement. This simple occurrence of the parallel displacement function reflects the fact that the polarization tensor, or spinor, of any field propagates in a parallel fashion along wave fronts. The polarization may describe an orientation which the wave front possesses not only with respect to space-time but also with respect to internal spaces having their own infinite dimensional groups, such as the Yang-Mills group or the gauge group of electrodynamics. The dot in Eq. (9) and its generalizations is to be understood as designating the generalized covariant derivative, based on the affine connections associated with each invariance group and not merely the coordinate transformation group.

In the case of the pure gravitational field the pertinent Green's functions are those which describe the propagation of infinitesimal disturbances in a symmetric second rank tensor, the metric tensor. The linearized forms of these Green's functions have been used in the weak field approximation to show in detail that a Bohr–Rosenfeld analysis can be carried out for the gravitational field [5]. It is found that space-time averages of the Riemann tensor can be measured with a degree of accuracy well within the domain of quantum phenomena provided test bodies of sufficient refinement but violating no fundamental principles are used. Examination of the mutual interference of such measurements verifies in detail the statistical predictions of the quantum formalism. The gravitational field, like all other fields, therefore must be quantized.

Actually these last assertions have to be qualified in two respects. Firstly, the masses of the test bodies must be at least $10^{-5}$ g, a requirement peculiar to gravitation theory alone. Therefore, if averages are desired over domains smaller than those visible to the naked eye, the experimentally known atomic constitution of matter must be violated in the construction of the test bodies. Secondly, quite apart from such empirical limitations, a fundamental limitation exists on the size of allowable measurement domains. Below $10^{-32}$ cm it is impossible to interpret the results of measurements in terms of properties or states characterizing the individual systems under observation. The uncertainty in the energy of the devices needed to make an observation in such a small region will, in virtue of the uncontrollable gravitational disturbance which it produces, completely destroy the statistical predictive significance of the results of the observation. This is true for the measurement of any field, not only the gravitational field. Therefore $10^{-32}$ cm constitutes an absolute limit on the domain of applicability of the classical concept of field strength, even as modified by the Principle of Complementarity.

The latter restriction may be viewed with some satisfaction, as it suggests that the fundamental length of gravidynamics may yet play a role as a high energy cut-off, a point to which I shall return presently. The restriction imposed by the actual atomic structure of matter, however, is harder to evaluate. If it is to be understood as really a fundamental restriction in Nature, which should be incorporated into physics at a basic level, then a rather thoroughgoing revision of some of our theoretical ideas is going to be necessary.

In all of my discussion so far, although several suggestive issues have been raised, there is nothing startling. In the minutes which remain I should like to present evidence that things begin to get more lively in quantum gravidynamics at the level where the truly quantum phenomena associated with radiative corrections enter the picture.

Let me begin by noting that the general theory of measurement and the Uncertainty Principle, which I outlined at the beginning, is incomplete in two important respects. Firstly, there is the perennial problem of how the measurement of the *apparatus variables* is to be described, whether in terms of a collapsing wave function, a universal branching wave function, or a theory of hidden variables. This is a problem with strong philosophical overtones but which may nonetheless be of significance to physics. Secondly, the description of the measurement process à la Bohr and Rosenfeld deals only with the Correspondence Principle level. To go beyond this level it is necessary to rewrite the uncertainty relation (5) in the rigorous form

$$\Delta A \Delta B \geqslant |\langle[\boldsymbol{A},\boldsymbol{B}]\rangle|. \tag{13}$$

Here the bold face symbols are quantum operators and the quantum mechanical mean value is to be taken. Now the mean value of a product is not generally equal to the product of the mean values, and to neglect the difference is, in field theory, to neglect precisely the radiative corrections.

These corrections are conveniently described by a hierarchy of correlation functions. Without going into the technical arguments as to why, let me simply state that these correlation functions may be based on the amplitude that the vacuum in the past remains the vacuum in the future when an external source $J_i$ is coupled linearly to the field $\boldsymbol{\varphi}^i$. I shall confine my attention here to boson fields the extension to ferimon fields is straightforward. One writes the vacuum-to-vacuum amplitude in the form

$$\langle 0,\infty|0,-\infty\rangle \equiv e^{iG} \tag{14}$$

and then introduces the variational derivatives of the exponent $G$ with respect to the source, as follows

$$\varphi^i \equiv \frac{\delta G}{\delta J_i} = \langle \boldsymbol{\varphi}^i \rangle \tag{15}$$

$$G^{ij\ldots} \equiv \frac{\delta}{\delta J_i}\frac{\delta}{\delta J_j}\ldots G. \tag{16}$$

The functions $G^{ij}\ldots$ are the correlation functions, the lowest order of which, namely $G^{ij}$, is the one-particle propagator. They satisfy the hierarchy of equations

$$\langle \boldsymbol{\varphi}^i \boldsymbol{\varphi}^j \rangle = \varphi^i \varphi^j - iG^{ij}, \tag{17}$$
$$\langle \boldsymbol{\varphi}^i \boldsymbol{\varphi}^j \boldsymbol{\varphi}^k \rangle = \varphi^i\varphi^j\varphi^k - i\varphi^i G^{jk} - i\varphi^j G^{ki} - i\varphi^k G^{ij} + (-i)^2 G^{ijk},$$

which relate mean values of products to products of mean values. The mean value involved here is the Schwinger average defined by

$$\langle A[\boldsymbol{\varphi}]\rangle \equiv \frac{\langle 0,\infty|T(A[\varphi])|0,-\infty\rangle}{\langle 0,\infty|0,-\infty\rangle} \tag{18}$$

A[$\boldsymbol{\varphi}$] is an arbitrary functional of the $\boldsymbol{\varphi}$'s and $T$ denotes the chronological ordering operation. Incidentally, all of these things can be derived starting from Peierls' definition of the commutator, without ever introducing a Hamiltonian.

At this point I am sure someone is about to object that I have violated group invariance by introducing the external source. It is perfectly true that the source presents a stumbling block if one wishes to maintain the formal elegance of staying in the coordinate representation and working with covariant correlation functions which transform linearly under the group. There *are* ways of surmounting this obstacle while maintaining manifest covariance. Let me restrict myself here, however, to the more direct method of *invariant variables*. Now what do I mean by that? I mean simply that I impose a supplementary condition on the field variables at the outset and then perform only variations which maintain it. This means that I will usually go over at once to momentum space. If I choose the Lorentz condition in electrodynamics then the electromagnetic field will have three components for each momentum value, except on the mass shell where they degenerate to two components. In gravidynamics, if I adopt harmonic conditions, I have six components which degenerate to two on the mass shell. Except on the mass shell the supplementary condition uniquely determines these components. They are, therefore, true invariants and can be coupled directly to an external source. Furthermore, since radiative corrections are all due to virtual processes of the mass shell, the group flexibility which remains *on* the mass shell is irrelevant.

The continuum over which the indices $i$, $j$, etc. range is now that of momentum space rather than coordinate space. This raises the question of how the chronological ordering operation in Eq. (18) is to be defined in momentum space. The answer is simple. It is to be defined *by* the functional differentiation process with respect to the source, via Eqs. (15), (16) and (17).

The correlation functions, as well as the functional $G$, may be regarded as functionals either of $\varphi^i$ or of $J_i$. The transformation coefficients from one set of variables to the other are the components of the propagator $G^{ij}$. The propagator describes the linear response of system to source, and the higher order correlation functions, which are often called many-body propagators, describe all scattering processes, the elements of the $S$-matrix being most easily obtained via Eq. (17) and the Lehmann, Symanzik, Zimmermann reduction formulae [6]. All this is well known. What is *not* so well known, however, is that the propagator $G^{ij}$ is the Feynman propagator of an actual *c-number action functional* which I shall call $\Gamma$. The relation of $\Gamma$ to the propagator is indicated by the following equation

$$\Gamma_{,ik}G^{kj} = -\delta_i^{\ j}, \qquad G^{ik}\Gamma_{,kj} = -\delta^i_{\ j}, \tag{19}$$

where the comma followed by indices denotes functional differentiation with respect to the $\varphi^i$.

Loosely speaking, $\Gamma$ is to the quantum theory what $S$ is to the classical theory, and its introduction constitutes a complete transition from conventional quantum machanics to an independent quantum theory of fields. Its functional derivatives of order three and higher are the full irreducible vertex functions of the theory, whence the choice of the symbol $\Gamma$. The $\varphi^i$ as defined in Eq. (15) are the solutions of a set of equations derivable from a variational principle based on $\Gamma$, namely

$$\Gamma_{,i} = -J_i, \tag{20}$$

which can be shown to be just the Schwinger average of the operator equations

$$S_{,i} = -J_i. \tag{21}$$

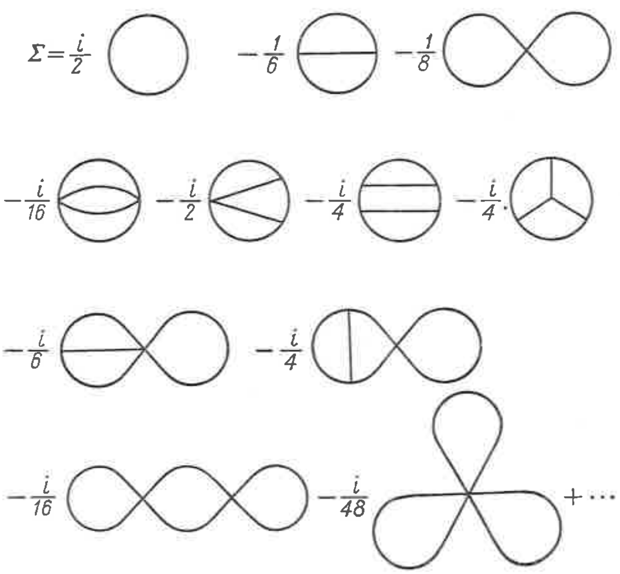
The chief difference between the classical and quantum variational principles consists in the fact that the quantum $\varphi^i$, because they satisfy Feynman boundary conditions, are *complex* even when the classical $\varphi^i$ are real, and in the fact that in the coordinate representation $\Gamma$ leads to nonlocal field equations even when $S$ leads to local ones.

By manipulating the coupled nonlinear equations satisfied by the correlation functions it is possible to show that $\Gamma$ may be expressed in the form

$$\Gamma = S - \Sigma, \tag{22}$$

where the functional $\Sigma$, which we may call the *self-energy functional*, has the graphical representation shown in the figure. The lines in the figure represent Feynman propagators and the vertices represent full vertex functions. In this form the quantum theory is seen to be a nonlocal c-number theory having a form or initial *direction* which is determined, via the Correspondence Principle, by a limiting theory based on $S$. The classical theory may be regarded as a kind of *tangent* theory, the relevance of which as a starting point for the rigorous quantum theory depends on the renormalizability of the quantum theory and the applicability of perturbation theory, which assumes that $\Sigma$ is small.

By taking the second functional derivative of Eq. (22) and inverting it to obtain the Feynman propagator, the functional $\Gamma$ may be computed by iteration starting from the "bare" propagator and vertex functions of $S$.




Diagrammatic representation of the self-energy functional.

Renormalization consists of simply throwing away at any stage the minimum local divergent parts of the self-energy functional necessary to make it finite, which is equivalent to assuming that corresponding "counter terms" are already present in $S$. The net result of this procedure may be summarized by the equation

$$\Gamma = S_R - \Sigma_F \tag{23}$$

where $S_R$ denotes the renormalized or "observed" classical action and $\Sigma_F$ denotes the finite part of the self-energy functional.

The c-number character of this formulation of quantum field theory makes it immediately possible to consider generalizations of conventional ideas which would not otherwise suggest themselves. For example, it is customary to choose a zero point for the field $\varphi^i$ which corresponds to flat empty space-time and, by regarding this as the vacuum state, to insure that $\varphi^i$ vanishes when the source vanishes. This is not necessary, however. An arbitrary macroscopic solution of the following equation

$$\Gamma_{,i} = 0 \tag{24}$$

may be taken as the "background field" and regarded as the "vacuum". This makes it easy to incorporate closed universes or universes with peculiar topology into the framework of quantum gravidynamics.

More important perhaps than this, however, are the possibilities which the existence of $\Gamma$ opens up in regard to topology in the small. Let us suppose that after having computed $\Gamma$ in terms of invariant variables we transform back to coordinate space by reintroducing the coordinate transformation group and the original dynamical variables of the theory. This we can easily do since the $\Gamma$-language is a c-number language. Since $\Gamma$ is itself an invariant the field equations (24) for the original variables are covariant equations. In the case of pure gravitation theory these equations take the form shown in Eq. (25), where they are seen to have a local part identical with Einstein's equations and a nonlocal part, coming from the self-energy functional, which describes quantum corrections important at high energies and small distances.

Now it is well known that Einstein's equations admit of solutions for which space-like cross sections have non-Euclidean topology, giving under certain conditions a particle-like or "wormhole" structure to space itself [7]. There are, however, strong reasons for believing that classical wormholes are immutable—that the topology of 3-space cannot change as long as the metric tensor satisfies Einstein's equations and retains its normal signature. On the other hand, with the equations

$$\frac{\delta\Gamma}{\delta g_{\mu\nu}} \equiv g^{\frac{1}{2}}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) - \frac{\delta\Sigma}{\delta g_{\mu\nu}} = 0. \tag{25}$$

the situation is quite different.

In the first place the metric tensor in the $\Gamma$-language is complex, and hence the question of signature loses its rigid classical significance. Secondly, since Eqs (25) are themselves complex above the threshold for real graviton production, the imaginary part of the metric can play an important role in the short wavelength limit.

If no wormhole mouths are initially present, of course, none are to be expected to develop in the course of time, *provided* no field quanta, represented by complex ripples on the metric tensor, are present either. The same should also be true for single mouths. But in the case of two or more colliding wormhole mouths it is likely that the real metric which describes them in the remote past acquires complex components in the course of time, corresponding to final gravitons, while the mouths themselves either bounce inelastically or annihilate one another.

In view of these possibilities it would seem very interesting to attempt at least a lowest order calculation of the radiative corrections to the classical field equations, and then to examine the effect which these corrections have, for example on the Schwarzschild solution for masses smaller than $10^{-5}$g, in particular, the likelihood of their eliminating both the pinch-off of the wormhole throat and the Schwarzschild singularity itself through the nonlocality of the corrected field equations below $10^{-32}$ cm. Unfortunately, serious questions would have to be raised in regard to the significance of such a calculation, even assuming all other nonlocality inducing mechanisms in Nature may be neglected.

In the first place the perturbation series gives no evidence of being even semiconvergent. The trouble is that there exists no small dimensionless coupling constant in the theory of the pure gravitational field, nor in the theory of the combined Einstein–Maxwell field. When natural units are used in which Planck's constant, the velocity of light, and the gravitation constant are all set equal to one, no physical constants are left, and all diagrams begin to diverge equally at $10^{-32}$ cm and have roughly equal magnitudes after renormalization.

Secondly, there is a rather uncomfortable situation in regard to the divergences. Owing to the special nature of gravitational coupling, whereby bare vertex functions to all orders have $p^2$ momentum space behavior, while the bare propagator has $1/p^2$ behavior, it is easy to show that to any order of perturbation theory all the renormalized full vertex functions have $p^4 \log|p^2|$ asymptotic behavior while all the self-energy diagrams diverge exactly quartically. If the exact vertex functions have the same asymptotic behavior—although, in view of the doubts about convergence, we are not obliged to believe this—then we are faced with the unpleasant fact that one counter term beyond the classical Einsteinian term is needed to effect the renormalization of every diagram. As Feynman has shown, three counter terms are needed: a quartically divergent cosmological term to compensate the zero point vacuum energy; a quadratically divergent Einsteinian term, which renormalizes the gravitation constant; and a term quadratic in the Riemann tensor involving two logarithmically divergent constants which have no counterparts in the classical theory. The latter term corresponds to an initial classical theory having field equations of the *fourth* differential order, and this leads to complications in regard to unitarity.

Of course, one might argue that this counter term precisely cancels a pre-existing one in $S$, leaving no term of the unwanted type in the renormalized action, and indeed such a proposal has some merit. The $p^4 \log|p^2|$ behavior would then mean that the light-cone singularities of the propagators have been softened without obvious violation of unitarity, and that although divergences are not eliminated, quantum gravidynamics does partially live up to the original hope that it would provide a natural cutoff in momentum space. It is worth pointing out that, unlike the situation in other field theories, this would not imply a violation of Lehmann's theorem to the effect that the singularities of the full propagator must be as strong as those of the bare one [8], for one of the assumptions of that theorem, namely, the existence of an energy-momentum 4-vector which generates infinitesimal displacements of local quantities with respect to the non-intrinsic coordinate labels, fails to hold in gravidynamics.

There is still one more difficulty accompanying field theories as nonlinear as Einstein's, which should finally be mentioned, and that is the question of which field variables should be regarded as basic when there is no clearly preferred set. It is obvious, that for example, such a simple function of the local covariant operator metric as the contravariant metric density

$$g^{\mu\nu} \rightarrow \mathfrak{g}^{\mu\nu} \equiv g^{\frac{1}{2}} g^{\mu\nu} \tag{26}$$

will have a divergent Schwinger average when expressed in terms of the complex covariant c-number metric and its associated correlation functions, and yet it should be just as good as the covariant $g_{\mu\nu}$ in describing the radiatively corrected classical field. It may happen that these divergences are exactly compensated already in the renormalization program, but if so it is by no means easy to prove. The vexing thing about the problem is that completely local field quantities are never measured in actual experiments, so that the relation between a measured covariant metric and a measured contravariant density is always a smeared-out nonsingular one, and yet in the mathematical formulation of the theory we are forced into those idealizations which lead to trouble. A similar problem arises in the Bohr–Rosenfeld analysis of both the electromagnetic and gravitational fields, in regard to the zero point oscillations of the elastic test bodies and coordinate frame-works employed. If the contribution of the field is included in the calculation of these oscillations then the result diverges. It is only by arguing that the field is not defined over regions smaller than the spacing of the constituent particles which go to make up the test bodies, that one can introduce a cut-off which allows the field contribution to be neglected. Whether the actual atomic constitution of matter can be used in a similar fashion to resolve these difficulties of basic formulation I leave as an open question.

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## DISCUSSION

S. Mandelstam:

I don't believe that use of Feynman or Schwinger formalism raises any more or any fewer difficulties with regard to factor ordering than use of a direct operator formalism. I think that when one examines what happens one sees that one gets coincident singularities, which just correspond to the same ambiguity. Now, I'm not sure the ambiguity exists at all; it may be that when one comes to renormalization the terms that give the ambiguity in factor ordering are of zero importance with regard to others. But I think if that's the case it will be so in the operator formalism as well. The second point was that if one does what you did, applying, say, Schwinger formalism with supplementary conditions, and if one takes an arbitrary operator and, tries to work out the closure properties, in other words, its $\sum_{\text{all states}} |n><n|$, then I don't believe one would find that it is equal to unity, unless one includes the unphysical states with the indefinite metric. I think one would still have to restrict oneself to coordinate independent operators in order to get the closure property. I'm pretty sure that is the case in electrodynamics. Then one question about what you said about renormalizability. Did you need one separate counter-term for each diagram so that there would be an infinite number of counter-terms?

B. S. DEWITT:

No.

S. MANDELSTAM:

So the theory would be renormalizable in perturbation theory.

B. S. DEWITT:

Yes, if you recognize in advance the asymptotic behavior of the renormalized propagators.

V. L. GINZBURG:

May I ask you a question about these two figures mentioned: $10^{-5}$ and $10^{-32}$. If you must have a body like $10^{-5}$ cm, that is macroscopical, how can we say anything about a dimension which is much smaller than this body?

B. S. DEWITT:

Already in the Bohr–Rosenfeld paper they recognized that there were experimental limitations due to the atomic structure of matter. They ignored this completely and said that if you do not consider the atomic structure of matter then there are no fundamental difficulties. If you imagine you can have a test body arbitrarily small built out of some unknown kind of matter, then everything is O.K. I quite agree with you, but I do not know how (and I don't know of anyone who does) to put the experimentally observed atomic constitution of matter fundamentally into physics.

D. IWANENKO:

Since there should be quantum corrections to the principle of equality of gravitational and inertial mass (as I see in your previous paper about gravitation, you also agree that there are corrections), then if we derive the fundamental Einstein–Infeld–Hoffman equations of motion should there also be some corrections?

B. S. DEWITT:

Well, I don't know whether it really makes very good physical sense to derive these complex equations and then to treat them in a way that we have treated the classical Einstein equations in the past, although they are c-number equations.

L. ROSENFELD:

With regard to this question of the possible limitations due to the atomic constitution of matter, we are faced here with two different factors. All this study of measurability of quantities has the logical function of testing whether the relationship between the symbols of the theory and the classical concepts is consistent. Now to define the classical concepts we use measuring apparatus which is macroscopical and in which, therefore (that is the definition of the word macroscopic), the atomic constitution is neglected. That is, the possible effects of the atomic constitution of the bodies are neglected. Then one can carry out the analysis on this assumption. This can be done in electrodynamics by steps, i.e. successive approximations in powers of Planck's constant and the effects arising therefrom; and in the first step, pure field theory, the situation is very clean, in the sense that at this stage there is no absolute scale of space and time included in the theory, because there are only two constants, $h$ and $c$. Then the question of the constitution of the test body does not arise at all, logically speaking. In the next stage, however, since we introduce masses and charges, we have an absolute scale fixed, let us say, by the Compton wave length of the electron But this does not prevent us, logically, from imagining test bodies of arbitrarily small dimensions, even much smaller than the scale given by this theory, in order to test the consistency of the interpretation of the symbols.

But, on the other hand, another argument is suggested by the consideration of the statistical fluctuations around the mean value, the measurement of which we are investigating. We are concerned, let us say, with the determination of the mean value of the charge contained in a given space-time region, which is surrounded by a shell of electrically charged test bodies; so that we measure the flux of electrical displacement issuing from this region. The first logical argument shows us that there is no limitation in the use of the concept of charge to describe the situation. But, at the same time, the theory tells us that the statistical fluctuations around the mean value which measurement by this method defines may become, when the thickness of the shell tends to zero, very large — in fact many times larger than the mean value.

Then we get the argument which you mentioned, which is of a quite different nature, that there is no help whatsoever in getting a mean value when we know that the fluctuations around this mean value are many times larger than the value in question; then the practical use of such a mean value for purposes of drawing further consequences is very reduced indeed. It is not completely useless, in theory; for instance, it remains true that if we start another measurement immediately after the first, the result of the second measurement will be exactly the same (I mean, will differ very little form the first one) irrespective of the largeness of the fluctuations. But this is a very academic connection. For all practical purposes we are here faced with a limitation, in effect, in the use of classical concepts the logical origin of which, however, is quite different from the question of consistent use of he classical concepts in interpreting the theory.

R. P. FEYNMAN:

I think that the question of the character of the divergences of the counter-terms of the theory of gravitation is still open. The thing which has to be distinguished is the orders and numbers of closed rings over which one has virtual momenta to integrate. If you have no closed rings there is no divergence, of course, and the first time you have divergences is with one closed ring. These can be of many orders because there can be any number of couplings of external lines coming in. But all of them can be summed. That increase in order doesn't make any trouble. So for a single ring what you said is right, that there is a finite number of counter-terms to the Lagrangian, or divergences, of which the last one is proportional to the square of the curvature and is logarithmically divergent. But, what happens when there are two closed loops is not clear; I mean I can't prove what I say is true, but there is no indication that it's not. It would be a miracle, it looks like, if it wasn't true, I mean if this doesn't happen that things get worse, in this sense, that, for instance, a term with three curvature tensors multiplied together which was convergent in two rings, may now be divergent, say logarithmically, but have a higher coefficient of the gravitational constant in front of it. So in the sense in which you think of the gravitational constant times a small parameter that I'm using as the cut-off length, which you have to make in a divergent analysis; then what you're saying is only that the coefficient of the third order of divergence, say the $R^3$ term, is a being infinitesimal compared to the last one you just found, which is the $R^2$ term, as long as you keep yourself within a reasonable distance away from the center. So I think as far as I can tell at the moment that the gravitational theory is not renormalizable in the usual sense of the term.

B. S. DEWITT:

I would agree, in the usual sense of the term. But mathematically, if you just forget about the physics and work it mathematically, even your $R^3$ term has a divergent coefficient, that's what you're saying.

R. P. FEYNMAN:

Yes, sir, but I have defined it to have a smaller divergence.

B. S. DEWITT:

A smaller degree of divergence or a smaller coefficient $G^3$?

R. P. FEYNMAN:

A smaller coefficient in the sense that the gravitational constant and the other constants involved occur in a higher power; so that the practical effect of such terms (if you were actually making a calculation with finite mass objects, not $10^{-5}$ g particles of $10^{-32}$ cm, but 100 BeV particles of size $10^{-15}$ cm) then the situation would be that the size of the counter-term that you'd have to take from this $R^3$ in its effect would be $10^{-39}$ times smaller than the other counter-terms; that's the sense. It's still divergent, but if we keep the numbers from blowing up then it's much smaller than the other ones. This is of no practical importance for people who want to look at regions of $10^{-33}$ cm, but this is different than what you seemed to suggest.

B. S. DeWitt:

I think the difference consists in this: it's perfectly true that if you start calculating these more complicated diagrams with more than one closed loop, and use in them the *bare propagators*, then the divergences get worse and worse. Now all I'm saying is that if you merely count momentum powers (of course all kinds of things can happen when you really calculate) and say that the propagator that you should *really* use goes something like $1/p^4$; and that the vertices themselves go like $p^4$, then any diagram you can write down diverges quartically and no worse.

R. P. Feynman:

Logarithmically is not worse than quartically.

B. S. DeWitt:

But from dimensional argument it would have to be an $R^2$ term. If you had an $R^3$ term which was logarithmic, then you'd have an $R^2$ term which was quadratic, an $R$ term which was quartic, and a cosmological term which was to the sixth.

R. P. Feynman:

No, you just proved you don't have a cosmological term to the sixth.

B. S. DeWitt:

I did; or you did?

R. P. Feynman:

No, you just did before.

B. S. DeWitt:

O. K!

# ASYMPTOTIC PROPERTIES OF GRAVITATING SYSTEMS*

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## INTRODUCTION

In this talk I shall review work that has been published over the past year [1], [2], and relate it to work that is being reported independently at this conference, work by Bondi and his collaborators [3], R. Sachs [4], and by Newman and collaborators [5], [6]. I believe that their motivation has been different from mine, and certainly their techniques have been a good deal more sophisticated. Having started not knowing about each other's work, we should, I believe, be gratified by the considerable agreement of our results.

Bondi, Sachs, and Newman, and their various coworkers, as I understand it, were interested in the examination of gravitational spherical waves far from the source, and Trautman [7], and Komar [8] achieved some new results on the rate of energy radiation. The chief idea, or at least one of the chief ideas, was the recognition that the Petrov–Pirani method of characterizing "pure radiation" tends to be useful only for fields possessing rather specialized symmetries and other properties, but that almost any mixture of diverse gravitational fields, and in particular combinations of static and radiative fields, will be of type I (non-degenerate). Early there emerged the concept of "asymptotically type null", but this concept obviously required considerable elaboration in order to become a manageable tool. I believe that this development has now achieved its goals.

All these investigations assume solutions of the field equations that satisfy certain boundary conditions at infinity. In some sense the solution of the field equations is to represent an isolated physical system, which does not distort the geometry of the universe at large distances. The belief that such solutions do exist, and that they represent physically interesting situations, has led investigators to claim that general relativity is compatible with Lorentz covariance, and to reconstruct, or to reformulate Einstein's theory of gravitation so as to make the field's Lorentz covariance explicit. My own investigation, partly alone and partly with collaborators, had as its initial purpose

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to analyze the relationship between the invariance group of general relativity and the Lorentz group. Part of the results has been published by Robinson, Schücking, and myself. I also wish to acknowledge contributions, mostly in the form of very extended discussions, with A. Komar, with J. Ehlers, and with R. Sachs, and I have had the benefit of receiving preprints and preliminary drafts of papers by the group about H. Bondi as well as those mentioned above. Thus I hope that no one will be under the misapprehension that I report on *my* work alone. I find it difficult to apportion proper credit to each individual, but the purpose of this talk is that of a connected review of work by a number of individuals, not all of which has as yet reached the stage of journal publication.

## INVARIANCE GROUPS

The assumption of asymptotic boundary conditions reduces the normal invariance group of general relativity, the group of curvilinear transformations, to a smaller group. If there exists a coordinate system in which the metric tensor at large distances approaches the Minkowski metric, then the natural invariance group is that group of coordinate transformations that preserves this asymptotic condition. In all investigations I know of it is sufficient that the coordinate transformation be defined, and a sufficient number of times continuously differentiable, in a neighborhood of infinity.

Originally, I assumed that the asymptotic condition should apply to any approach toward infinity along a space-like curve, and that the deviation of the metric from the Minkowski metric should be $O(1/r)$. The coordinate transformations which preserve this condition are those in which the derivatives of one set of coordinates with respect to the other differ from a Lorentz transformation matrix by amounts $O(1/r)$.

The London group, and those who followed them, recognized from the beginning that to examine radiation at infinity one has to follow a characteristic, or null, hypersurface, rather than a space-like direction. No physical system whose energy is to remain non-negative can radiate waves with non-diminishing amplitude forever, and thus we must assume that in effect gravitational radiation occurs in pulses; it is also very probable that the mass of the radiation itself in a wave train of infinite length is sufficiently divergent to be inconsistent with any asymptotically flat metric, whatever definition for "asymptotically flat" is adopted. Hence, if we go far enough in a space-like direction, we are bound to penetrate beyond both incoming and outgoing radiation, whereas on following a characteristic direction we can stay with the radiation progressing in that direction, though we shall also lose radiation going in other directions.

If one follows a pencil of null-directions, one can subject the coordinates to a number of conditions that reduce the remaining permissible coordinate transformations to a group which Sachs has called the GBM group(1). I shall call "my" group the AL group, the initials standing for "asymptotically Lorentz". In their structures, the GBM group and the AL group are quite different, for good reasons. They preserve inequivalent boundary conditions, and the GBM group is further restricted by very reasonable coordinate conditions. But these groups have two things in common: first, both groups define topological spaces that are function spaces, with an infinite number of dimensions; second, in describing a member of the transformation group we need not refer to properties of an individual metric field.

In this connection, I should like to make some remarks that may facilitate discussion of such proposals as Fock's restriction to harmonic coordinates [9]. In constructing a group, and more particularly an invariance group, we ordinarily think of transformation groups, though we may consider two transformation groups that are isomorphic as "the same group". At any rate, if we work with a transformation group (rather than with an abstract group only), we have the heuristic advantage that we can confirm more easily the various group axioms. In order to construct a transformation group we must first assemble a set of objects to be mapped—reversibly uniquely—on itself. Ordinarily we adopt as this set the world points as represented by their coordinate values, that is, properly speaking, quadruplets of real numbers. A coordinate transformation is then a rule that assigns to every quadruplet of real numbers another such quadruplet; we have formed a group of coordinate transformations if each of these assignments or correspondences is reversibly unique and if inverse mappings and all products are included. A group of coordinate transformations thus defined is one whose "descriptors" are all c-numbers, i.e. in which the transformation equations carry no reference to field variables.

For some purposes we may wish to consider coordinate transformations in which the new coordinates of a world point are determined by both its original coordinate values and by the field. In this case, it is better if we think of the space of objects to be mapped on itself as the function space of fields. Given a field (in relativity: a metric field) in a particular coordinate system, a coordinate transformation will convert this set of functions of the coordinates into a different set of functions, that is to say, physically the same field but in another mode of description. Provided again that this set of transformations is reversibly unique and that inverses and products are included, we have now a group of coordinate transformations whose descriptors may be q-numbers. The first type of coordinate transformation groups (the c-number transformations, for short) are isomorphic with some transformation groups constructed on the second pattern (the q-number transformations),

(1) Generalized Bondi-Metzner group.

but the second method of construction is much richer, and most of its transformation groups have no counterpart in the c-number groups.

With harmonic coordinates the situation is yet more complicated. The transformation functions $x'(x)$ must satisfy certain differential equations with boundary conditions, and the coefficients of the differential equations depend on the metric field. A given function $x'(x)$ which leads from one particular harmonic coordinate system to another, on being applied to yet another harmonic coordinate system, may yield a coordinate system that is not harmonic. Hence, we must proceed with great caution; we should have to construct mappings whose rules are such that a given mapping converts every conceivable harmonic coordinate system (more properly speaking: every conceivable Riemann–Einstein manifold represented in any conceivable harmonic coordinate system) reversibly uniquely into another one. I am not aware of this construction having been achieved by anyone, but I also know of no proof that such mappings do not exist.

At any rate, both the GBM and AL groups have been constructed so that they are c-number transformation groups, i.e. the space that is being mapped on itself is finite-dimensional. The groups themselves, however, both of them, are infinite-dimensional.

Incidentally, the representation of the invariance group of a theory by the group of transformations of a field is not irreducible, in that under any coordinate transformation fields can be transformed only into fields that are physically equivalent. That is to say, in the function space of all fields each set of equivalent fields forms a subspace that is invariant under the invariance group of the theory. The mappings of each such invariant subspace on itself under the invariance group form a representation of that invariance group; some of these representations may be faithful, others not.

## TRANSFORMATION LAWS

Once we have adopted a particular invariance group for our theory, be it the invariance group of the original general theory of relativity or the group of curvilinear transformations preserving certain boundary and coordinate conditions (such as the GBM and AL groups), we are free to replace the original field variables, the components of the metric field, by new variables that may, perhaps, conform more closely to our notions of physically intuitive variables. Our choice is, of course, subject to the requirement that every otherwise interesting physical situation can be represented in terms of the new variables (i.e. that the choice of new variables does not exclude some class of situations), and further that the values of the variables in one coordinate system (or other mode of description) uniquely determine their values in any other coordinate system.

If we are concerned with a set of variables whose transformation law represents faithfully a lower-dimensional transformation group than our original invariance group, then there are two possible relationships between the two groups in question.

(1) There may exist a homomorphism between the two groups. If so, the newly introduced field variables are sensitive to some aspects of the coordinate transformations belonging to the "permissible" group but not to others. For instance, if our coordinate systems are asymptotically Lorentzian, and hence the transformations between them also asymptotically Lorentz transformations, then the variables may transform in accordance with that asymptotic character of the transformations, while they are invariant with respect to transformations of arbitrary characteristics in the interior which asymptotically approach identity transformations. We shall look into this possibility more deeply.

(2) Among all the coordinate systems admitted to begin with we might, by means of additional coordinate conditions, select a smaller number, and then confine ourselves to those transformations that map this smaller set on itself. In general this approach will require that the original invariance group be represented by the mappings of fields on fields, because usually coordinate conditions are formulated in terms of properties of the fields themselves. If we can construct mappings (within the original invariance group) each of which maps the set of permissible fields on itself, then these surely form a group, a subgroup of the original invariance group. This subgroup in turn contains a further subgroup, the set of those mappings that leave the permissible fields individually unchanged; this second-order subgroup is invariant with respect to the first-order subgroup. It may,of course, consist of the identity mapping as its only element; if it is larger, we form the factor group, which will then consist of the mappings of the set of permissible fields on itself, without regard to the mappings of other fields. This factor group will be a new invariance group.

Suppose we construct a set of variables which have a well-defined transformation law with respect to the new invariance group. In general these variables need not be defined at all in terms of all coordinate systems admissible under the old invariance group. Consider, for instance, the relationship between arbitrary curvilinear coordinate systems and those of the AL type. With respect to the AL group we can construct quantities that are Lorentz-covariant, for instance the (total) linear momentum four vector. Its components are not even definable with respect to coordinate systems that do not behave "reasonably" at infinity, even though the metric field be one that admits AL coordinate systems.

By contrast, if we are concerned with variables that are defined with respect to all coordinate systems admissible under our original invariance group, and if they obey a well-defined transformation law among themselves, then this transformation law must represent a factor group of the original invariance group, unless it represents the invariance group faithfully.

It is well known that a transformation law should not represent a subgroup of the invariance group, whether or not that subgroup is invariant. If it does, we should be confronted with a situation similar to the one described as case (2) above: The quantities in question would be defined only for a subset of all the coordinate systems envisaged originally. Unless these coordinate systems are characterizable in their own right, that is, unless case (2) applies, this is an unacceptable state of affairs.

The existence of a factor group will permit us to construct quantities, functionals of the field variables, that are invariant with respect to transformations belonging to the invariant subgroup but transform under the factor group. With the help of the invariant subgroup we divide the set of all permissible fields (e.g. solutions of the field equations satisfying specified boundary conditions) up into equivalence classes [10]. These equivalence classes will be smaller than those obtained from the use of the full invariance group; that is to say, even the permissible equivalent forms of one particular field are divided into smaller mutually exclusive but exhaustive classes. We now introduce any set of variables that are constant in each of the new, small, equivalence classes but are not all constant as we pass from one such class to another. These variables may serve as a set of coordinates identifying small equivalence classes within one original class; they are not observables, as that term has been defined previously, but their transformation law will provide a faithful representation of the factor group. Needless to say, the actual construction of such a set of variables may be a very difficult, and even impractical, undertaking, unless some heuristic viewpoint outside of formal group theory is available.

## STRUCTURE OF THE GBM AND AL GROUPS

The two groups mentioned previously have been analyzed with considerable care, one by R. Sachs, the other by myself. Neither of these groups is simple. Both possess invariant subgroups of a kind Sachs has called "supertranslations". The members of these respective subgroups consist of coordinate transformations that do not change the directions of the coordinate axes at infinity (spatial or null infinity, as the case may be) but displace them in a manner that in general is not independent of the direction in which we go to infinity. The factor group, in either case, is the homogeneous Lorentz group.

Both Sachs and I have searched for some invariant subgroup whose factor group would be the inhomogeneous Lorentz group, but have been able to convince ourselves that no such subgroup exists. Though there is no difficulty in singling out the translations from among the supertranslations, the reverse is not true: There is no way of constructing an invariant subgroup of supertranslations that excludes the translations. As I have mentioned, Sachs's results and mine are not equivalent, because we started out from different invariance groups. I consider his more far-reaching than mine, first because he used Bondi's approach of considering the outgoing null directions, rather than space-like directions,and thus he can include the consideration of gravitational radiation in his analysis; second, he, again following Bondi and collaborators as well as others, narrows down the choice of coordinate system much more than I was able to do, and this is, of course, to be welcomed. At any rate, we both found that Lorentz-covariant, "gauge-invariant" variables are necessarily constants of the motion.

From the group-theoretical point of view, then, the invariance group of General Relativity, even of general relativity with boundary conditions, appears to be significantly different from that of Lorentz-covariant theories, and the latter is not simply a projection, or contraction, of the former. This statement should be accepted with considerable qualifications: It is a verbalization of a fairly involved group-theoretical argument, and it applies specifically to the GBM and AL invariance groups. It is conceivable that there are invariance groups, not yet discovered, which are physically germane to a theory of gravitation and which permit the construction of a truly Lorentz-covariant theory.

## ASYMPTOTICALLY FLAT MANIFOLDS

In the preceding discussion we have assumed that physically interesting solutions of the field equations admit a set of coordinate systems in which the metric tensor field approaches the Minkowski–Lorentz values at large distances as $O(1/r)$. This assumption is ordinarily described briefly as the assumption of "asymptotic flatness", and the manifold is said to be asymptotically flat. In order to see why it proves so difficult, or even altogether impossible, to construct a set of "best" coordinate systems which go over into each other by Lorentz transformations, we have attempted to analyze this concept of asymptotic flatness in some detail.

Let us consider a collection of masses which at large distances gives rise to terms in the metric field that represent a non-vanishing total rest mass. In an appropriately chosen coordinate system the leading non-Minkowski parts of the metric field will go $O(1/r)$, and the curvature tensor components will go $O(1/r^3)$. How can we exhibit the deviation of this metric field from flatness? If we attempt to construct any geometric figure (such as a triangle, a cube, and the like) at large distances from the source by choosing points and connecting them by geodesics, the angles of these figures, as is well known, will deviate from the angles of corresponding figures constructed in a perfectly flat space. The degree of deviation is generally proportional to the square of the linear dimensions of that figure, multiplied by some expression proportional to the components of the curvature tensor. Hence, in the absence of radiation, if we move such a figure out to infinity, scaling it up proportionally to the distance from the sources of the gravitational field, the deviations will decrease $O(1/r)$.

A different method of constructing figures is what I should call the method of dead-reckoning. Suppose we start our figure at one point, protract from it geodesic line segments of the length and in relative orientations that would be appropriate for the construction of the desired figure in flat space, and continue to arrange geodesic line segments one after the other until we have marked out all the corners. These corners will generally be reached by more than one polygonal trajectory; the trajectories will fail to meet at one point, because of the non-vanishing curvature of the space. How large will be those gaps? The answer is that the gaps will be proportional to the cube of the linear dimensions of the figure to be constructed, again multiplied by a factor proportional to the components of the Riemann–Christoffel (or rather Weyl) tensor. Hence, if the figure is scaled up again so as to subtend a constant solid angle as viewed from the region in which the source is located, the gaps will not tend to zero but to constant values which are of the order of magnitude of the gravitational radius belonging to the rest mass, i.e. of the order of cm for a mass of the order of our earth, or of km for a solar mass.

Dynamically, this result could also be stated as follows. Consider two test particles initially at rest relative to each other, at an initial distance of $r_1$ from each other and $r$ from the source of the field. Let both fall freely for a length of time $t = r_2/c$. At the end of that time their relative distance will have changed by an amount of the order $r_1 r_2 R/r^3$, where $R$ is the gravitational radius of the rest mass of the source.

These results were obtained on the assumption of negligible gravitational radiation. If there is radiation, and if we go to infinity along an outgoing light cone, the situation is different, in that deviations from flatness are more pronounced. If the pulse of radiation is of finite length, there is no sense in scaling up the dimensions of a figure indefinitely in the direction going through the pulse. If we merely scale up in those directions which lie within the wave front, the gap increases $O(r)$, or if we construct a figure without gaps from geodesics, angles will deviate from those of the corresponding figure in flat space by asymptotically constant amounts. Test particles that were at rest relative to each other before the passage of the wave pulse will be in a state of relative motion following its passage. Their relative velocity will be proportional to their initial distance from each other, and inversely proportional to their distance from the source of radiation.

## RADIATION AND AFFINE CONNECTION

Usually we define an affine connection in a Riemannian manifold in terms of the Christoffel symbols. We call two vectors at different points parallel to each other along a specified curve if parallel displacement of one along the curve transforms it into the other. If the curve, or rather curve segment, is geodesic, the angle between the vector and the curve is constant all along the curve, etc. Independence of the result of parallel displacement of a vector from the curve used to connect the two end points with each other is called integrability of the affine connection, or simply affine flatness. In Riemannian geometry affine flatness and metric flatness are coincident.

In the presence of radiation we may form a three-dimensional manifold that consists of the congruence of geodesic null curves on all the outgoing light cones in the course of time. This three-dimensional manifold, which Sachs has used to represent the GBM group, may be considered as the (future) celestial sphere in the course of time. Each null ray of the curve congruence may be thought of as being represented by its "end point at infinity". We may project any curve in the four-dimensional manifold of space-time on the three-dimensional manifold by moving each of its points outward along the appropriate null ray. As a result of this projection we obtain an affine connection on the (non-metric) three-dimensional manifold, which represents the limit of the Christoffel symbols for large values of $r$ as a fixed function of the Sachs coordinates $u$, $Q$, and $\phi$. This affine connection is non-integrable, and the three-dimensional manifold is not affinely flat, whereas in the absence of gravitational radiation it would be.

However, we can define on the three-dimensional manifold another affine connection, which is integrable. We do so by means of the invariant subgroup of the translations within the GBM group. An infinitesimal translation defines a vector field, and the four translations a quadruped field. Any vector at one point of the three-dimensional manifold thus defines uniquely, and invariantly, a translation, and with it a whole field of "parallel" vectors. Moreover, this identification commutes with the operations of vector addition and multiplication by a constant. Hence this identification gives rise to an integrable affine connection.

It is well known that in the presence of two affine connections on the same manifold the difference is a tensor. Thus under the GBM group there exists the possibility of defining a tensor field of rank three which is of a lower order of differentiation than the Riemann–Christoffel tensor. Moreover, as Sachs has pointed out, the transformations of the GBM group may be continued uniquely into the interior of the four-dimensional manifold, so that, for instance, the translations are well defined for finite values of $r$, though expressions for them may not be available in closed form. Thus, the tensor field in question may be defined not only on the future celestial sphere but for all values of $r > R > 0$, where $R$ is some bounding tube outside of which the metric field is well behaved. This is a remarkable result of the imposition of appropriate boundary conditions. Clearly, the tensor field which we have constructed is closely related to Bondi's "news functions", but the details have not yet been worked out.

## CONCLUSION

It appears that the systematic investigation of the asymptotic properties of "asymptotically flat" Riemann–Einstein manifolds leads to results that are not entirely trivial. Those discussed here roughly fall into two types; on the one hand the structure of invariance groups appropriate to solutions of the field equations of physical interest circumscribes their relationship to other such groups and in particular to the Lorentz group; on the other, these classes of solutions deviate from the Minkowski universe asymptotically in a manner that can be described in a mathematically acceptable manner and which may give rise to covariant fields not defined in more general Riemannian manifolds.

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## DISCUSSION

E. T. Newman:

I would like to point out that in certain situations it might be of great efficacy to introduce a coordinate system in which the metric does not appear asymptotically flat, even though the space is asymptotically flat. The prime example of this is the Robinson–Trautman metrics, where the metric grows a linear $r$ term. Now, there are other indications, in some of the Bondi solutions and Bondi–Sachs solutions, that it might be of more physical interest to introduce a similar type of coordinate system.

P. G. BERGMANN:

I believe that if you handle your affairs right (and of course whether you do or not can be decided only be the success of the undertaking) that the results should be independent of your choice of coordinate system; that is to say, you are supposedly making use of every covariant structure that is actually available, either in the form of choice of coordinate system or in some other fashion and then you can do certain things. The question in this case is, can you, for instance, introduce an inhomogenous Lorentz-covariant set of quantities. I think whether you can or not should not depend on your choice of coordinate systems. And this is once more the point that I tried to make several times. One should be cautious in claiming too much ground that one has covered. Supposing that I picked up a relatively large invariance group that could have been easily narrowed; I might have overlooked a possibility. I think that in the class of solutions that are covered by the Bondi–Metzner group this is all that's available. I mean that is my impression, I cannot swear. I think that this is all that is available, regardless of whether you choose to describe it by a different kind of coordinate system or not.

H. BONDI:

If I may intervene here, it seems to me that whether or not you get such an $r$ term as you mentioned depends essentially on whether your framework is accelerated or not, and this seems to me the crucial distinction. In the group you have studied, one assumes that there is no such acceleration.

A. LICHNEROWICZ:

I will first make an affirmation, and second ask a question. For a mathematician it is very strange to use the word "group" for "group of transformations" which are not global. Physicists use the term group for sets of transformations—local transformations—which do not constitute group.

P. G. BERGMANN:

These are pseudogroups.

A. LICHNEROWICZ:

Yes, pseudogroups. My question is the following. To define "asymptotically flat" it is necessary to define precisely the domain at infinity. You have certainly assumptions on the topology and the complete character of your manifold. But I don't know exactly what are your assumptions in these domains.

P. G. Bergmann:

Concerning the first point I have nothing to add, I agree. Concerning the second point, I tried to say that I believe that different people understand by asymptotic flatness inequivalent things, and that in order to be able to talk at all one has to be more specific. So I tried to answer your question when I said $1/r$ along all space-like directions, but not uniformly. Non-uniformly, because otherwise I would then include the light cone also, by closure.

A. Lichnerowicz:

I am in agreement on this point, but what is the domain of $r$?

P. G. Bergmann:

I think that topologically one means a Minkowski universe minus a world tube.

A. Lichnerowicz:

It is not complete in the sense of geodesics.

P. G. Bergmann:

Obviously, if you take out a world tube you have nothing complete.

R. K. Sachs:

The general topological assumption is surely the following: you consider the product of a two-sphere with an infinite half-line, including its boundary point. This product is the one that is generally assumed for such discussions.

C. W. Misner:

I think one could also replace the assumption by assuming the tube is filled with anything one pleases, which is the way it is done in Lichnerowicz's book. Then there is a Euclidean topology and one discusses questions which are independent of the metric inside a central tube.

A. Lichnerowicz:

There exist many possibilities, but it is necessary to have some precise assumption in this domain.

C. W. Misner:

Since we are only interested in what happens at infinity, you can say many different things about the interior. Our questions only imply, what can you say at infinity that is independent of the interior. So one man sets it up with no interior; another man says, let the interior be arbitrary.

J. L. Synge:

I think Professor Bergmann mentioned a two-dimensional integral which represents the whole mass? What is it?

P. G. BERGMANN:

The integral that I have in mind can be defined in several different ways; and I believe that in this particular case that all definitions agree, and lead to the same result. Namely, an integral over a superpotential, multiplied into a displacement vector field. Now, the arbitrariness that one has is in the definition of the chosen superpotential. If you try to be completely conventional, you can use the superpotentials first described by von Freud, 1939, in that famous *Annals of Mathematics* paper; or you can use the Landau–Lifshitz superpotential which is even simpler, although closely related to the von Freud expression. The sticky point is, of course, the vector field into which you dot them prior to integration; and I believe that in the case of the radiative solutions one should use one of those vectors fields which are related to each other by the second affine connection that I mentioned at the end of my paper. One can define it this way if one wishes: that connection whose components are zero, not only order $1/r$, but more strongly zero, at infinity in any appropriately chosen coordinate system of the Bondi–Metzner set.

J. WEBER:

Why asymptotically flat universe Minkowskian boundary conditons are used at infinity instead of Friedmann type of expanding universe boundary conditions?

P. G. BERGMANN:

The only answer I can give is that the investigations date back less than two years, I believe, and that people have simply started with the mathematically simplest situation, or what they hoped was the simplest situation. I think it's purely psychological-historical, and if this field should not die out in the next few years, would you suggest to Landau that it be done? However, since the chairman preceded me by several months in opening this field, I wish he would answer.

H. BONDI:

Yes, I think I can answer this; this shall be the last remark in the discussion. You cannot have a Friedmann type universe of a realistic nature without having some matter in it. You don't want to live in an empty one. Then the question of the response of the matter to the outgoing radiation arises; in other words we've got to prescribe an equation of state for the matter out there. Of course, in principle this can be done; but you realize that this is infinitely more complex than the very simple equation of state for empty space; which is $R_{\mu\nu} = 0$. I regret it as much as you do, that we haven't yet got to the point of doing the Friedmann universe.

# THE MOTION OF AN EXTENDED PARTICLE IN THE GRAVITATIONAL FIELD

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## THE MODEL

With the Einstein theory of gravitation there is a minimum size for a particle of given mass $m$, namely the radius $\varrho = 2m$ in the Schwarzschild system of coordinates. The gravitational field can be extended to smaller radii consistent with Einstein's field equations for empty space, but the region $r < 2m$ is then physically inaccessible (it would need an infinite time to send in a signal and get it out again), so it cannot be allowed in a physical theory.

Thus, to get a precise theory of the motion of a particle in the gravitational field, one cannot take the particle to be a point singularity. One must take it to have a finite size $\varrho$, such that Einstein's equations for empty space hold only for $r > \varrho$ and $\varrho$ must be $\geqslant 2m$. It is awkward to work with the case $\varrho = 2m$, because of the singular character of space-time at this radius. We shall here consider the case of $\varrho > 2m$.

One cannot very well take the particle to be a rigid sphere, because of the ambiguity in the definition of a sphere in curved space-time. We therefore assume the surface of the particle to be flxeible, so that the shape and size can vary. The simplest assumptions will be made that lead to definite equations of motion for such a particle, with stable equilibrium states.

In choosing these assumptions one can be guided by analogy with the electromagnetic field. One can get a reasonable theory of a charged particle of finite size in the electromagnetic field by assuming that the surface of the particle is a perfect conductor carrying a distribution of electric charge, and that there is a surface tension which counterbalances the electrostatic repulsion [1]. There is then no electromagnetic field inside the particle, and the electromagnetic potentials are continuous at the surface while their first derivatives are not.

We shall make analogous assumptions for our gravitational particle. We assume it carries a surface distribution of mass, which adjusts itself so that there is no gravitational field inside, i.e. space-time is flat inside. We assume also a surface pressure to counterbalance the mutual attraction of the surface distribution of mass. If these are the only forces, one can have a particle at rest in equilibrium, but it is unstable, in contradistinction to the electromagnetic case. To bring in stability we need some further force, and the simplest assumption is to take an additional energy term proportional to the total volume inside the particle.

An extended particle in the combined gravitational and electromagnetic fields has been considered by Lees [2]. His model differs from the present one through having constraints on the size and shape of the particle.

## THE ACTION PRINCIPLE

A comprehensive action principle will be set up, giving both the field equations and the equations of motion of the particle. It will determine the motion of each element of the surface, so it will give the motion of the particle as a whole as well as the changes in its size and shape. The total action is of the form

$$I = I_O + I_S + I_I$$

where $I_O$ is the action for the space outside the particle, $I_S$ is the surface action and $I_I$ is the action for the space inside.

We take $I_O$ to be the usual action for the Einstein field, namely the integral of the total curvature density

$$I_O = \int \mathcal{G} g^{\mu\nu} \{(\Gamma^{\alpha}_{\mu\sigma}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\sigma}_{\alpha\sigma}) + \Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha}\}\, d^4x$$

taken over the region outside the particle, where $-\mathcal{G}^2$ is the determinant of the $g_{\mu\nu}$. We may write it as

$$I_O = \int (\mathcal{L} + \boldsymbol{m}^{\alpha}{}_{,\alpha})\, d^4x$$

where $\mathcal{L}$ does not involve any second derivatives of the $g_{\mu\nu}$. We have then

$$\boldsymbol{m}^{\alpha} = \mathcal{G}(g^{\mu\alpha}\Gamma^{\sigma}_{\mu\sigma} - g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu}) \tag{1}$$

$$\begin{aligned} \mathcal{L} &= \mathcal{G} g^{\mu\nu}(\Gamma^{\alpha}_{\mu\sigma}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\sigma}_{\alpha\sigma}) - (\mathcal{G} g^{\mu\nu})_{/\nu}\Gamma^{\sigma}_{\mu\sigma} + (\mathcal{G} g^{\mu\nu})_{/\alpha}\Gamma^{\alpha}_{\mu\nu} \\ &= \tfrac{1}{4}\mathcal{G} g_{\mu\nu,\varrho} g_{\alpha\beta,\sigma}\{g^{\varrho\sigma}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta}) + 2g^{\nu\sigma}(g^{\mu\varrho}g^{\alpha\beta} - g^{\mu\alpha}g^{\varrho\beta})\}. \end{aligned} \tag{2}$$

We can now transform $I_O$ to

$$I_O = \int \mathcal{L} d^4x - \int \boldsymbol{m}^{\alpha} dS_{\alpha}, \tag{3}$$

where $dS_{\alpha}$ is an element of the surface of the particle. In this form it does not involve any second derivatives of the $g_{\mu\nu}$. We shall assume that $I_S$ and $I_S$ likewise involve only the $g_{\mu\nu}$ and their first derivatives.

The condition that space-time is flat inside the particle is assumed as a constraint of the action principle. We require $\delta I$ to be zero only for variations of the $g_{\mu\nu}$ that preserve this flatness.

Let the equation of the surface of the particle be

$$f(x) = 0.$$

This equation must not be varied in the variational procedure, because if it were varied, $\delta I$ would not depend linearly on the parameters that specify the variation of $f$, on account of the gravitational field being different just outside the surface and just inside. Thus $f(x)$ is kept fixed all through the calculation. For convenience we take it to be

$$x^1 = 0, \quad \text{with } x^1 > 0 \text{ outside.} \tag{4}$$

We suppose a continuous system of coordinates inside and outside the particle, and we use the suffixes $a, b, c, \ldots$ to take on the values $0, 2, 3$ only. Then the $g_{ab}$ are continuous, and also their tangential derivatives $g_{ab,c}$, but the derivatives $g_{ab,1}$ need not be continuous. Also the $g_{1\mu}$ need not be continuous, and can be varied independently on both sides of the surface. Let

$$c^{\mu\nu} = g^{\mu\nu} - \frac{g^{1\mu}g^{1\nu}}{g^{11}} \tag{5}$$

so that $c^{\mu\nu} = 0$ if $\mu$ or $\nu$ is one and $c^{ab}$ is the reciprocal matrix to $g_{ab}$.

$I_S$ is an integral over the surface of the particle,

$$I_S = \int \boldsymbol{n}^a dS_a = \int \boldsymbol{n}^1 dx^0 dx^2 dx^3$$

with the equation (4) for the surface. We must choose $\boldsymbol{n}^1$ to be a three-dimensional scalar density with respect to the coordinates $x^0, x^2, x^3$ of the surface, and to be invariant under any transformation of coordinates which does not alter the surface $x^1 = 0$ and the coordinates $x^0, x^2, x^3$ in it. The basic quantities that have this invariance property, and can, therefore, enter into $\boldsymbol{n}^1$, are the $g_{ab}$ and their tangential derivatives $g_{ab,c}$, and also the quantities $\mathcal{G}\Gamma^1_{ab}$. The latter have different values just outside the surface and just inside, on account of the discontinuity in $g_{1\mu}$ and $g_{ab,1}$. Either value for $\mathcal{G}\Gamma^1_{ab}$ has the necessary invariance property and can enter into $\boldsymbol{n}^1$. To distinguish the two values, we shall denote the inside one by $\mathcal{G}^*\Gamma^{*1}_{ab}$.

We shall now assume

$$\boldsymbol{n}^1 = -2\mathcal{G}c^{ab}\Gamma^1_{ab} + 2\omega\mathcal{M}, \tag{6}$$

where $\omega$ is a constant and $\mathcal{M}^2$ is the determinant of the $g_{ab}$. The first term in (6) is connected with the outside gravitational field, and is needed for a purpose that will become clear later. The second term gives the surface pressure.

Finally, we assume the stabilizing term in the action

$$I_I = 2\lambda \int \mathcal{G} d^4x$$

taken over the space inside the particle, $\lambda$ being a constant.

### THE VARIATION OF $I_o+I_s$

We have from (3)

$$I_o+I_s = \int \mathcal{L} d^4x + \int (n^1-m^1) dx^0 dx^2 dx^3,$$

the four-dimensional integral being taken over the region $x^1 > 0$ and the three-dimensional integral over the surface $x^1 = 0$. Hence,

$$\delta(I_o+I_s) = \int \left\{ \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}} - \left( \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,\varrho}} \right)_{|\varrho} \right\} \delta g_{\alpha\beta} d^4x + $$
$$+ \int \left\{ \left[ -\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,1}} + \frac{\partial (n^1-m^1)}{\partial g_{\alpha\beta}} - \left( \frac{\partial (n^1-m^1)}{\partial g_{\alpha\beta,c}} \right)_{|c} \right] \delta g_{\alpha\beta} + \right.$$
$$\left. + \frac{\partial (n^1-m^1)}{\partial g_{\alpha\beta,1}} \delta g_{\alpha\beta,1} \right\} dx^0 dx^2 dx^3. \tag{7}$$

In the region $x^1 > 0$, $\delta g_{\alpha\beta}$ is arbitrary, so the coefficient of $\delta g_{\alpha\beta}$ in the first term of (7) must vanish. This gives Einstein's equations for empty space, holding in the region outside the particle.

The $\delta g_{\alpha\beta,1}$ in the second term of (7) means the value of this field quantity ust outside the surface, and this value is arbitrary. Hence, its coefficient in (7) must vanish. In order not to have too many equations of motion coming from the action principle, we must arrange that this coefficient shall vanish identically. The first term for $n^1$ in (6) produces the desired effect, since from (1)

$$\begin{aligned} m^1 + 2\mathcal{G} c^{ab} \Gamma^1_{ab} &= \mathcal{G} g^{1\varrho} \{ \Gamma^\sigma_{\varrho\sigma} + (2c^{\mu\nu} - g^{\mu\nu}) \Gamma_{\mu\nu\sigma} \} \\ &= \mathcal{G} g^{1\varrho} \left\{ \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\varrho} + \left( g^{\mu\nu} - \frac{2g^{1\mu}g^{1\nu}}{g^{11}} \right) \left( g_{\mu\varrho,\nu} - \frac{1}{2} g_{\mu\nu,\varrho} \right) \right\} \\ &= \mathcal{G} g^{1\varrho} \left( g^{\mu\nu} - \frac{g^{1\mu}g^{1\nu}}{g^{11}} \right) g_{\mu\varrho,\nu} \\ &= \mathcal{G} g^{1\varrho} c^{ab} g_{a\varrho,b}, \end{aligned} \tag{8}$$

which does not involve any derivatives $g_{\alpha\beta,1}$.

We are left with

$$\delta(I_o+I_s) = \int \mathcal{Y}^{\alpha\beta} \delta g_{\alpha\beta} dx^0 dx^2 dx^3 \tag{9}$$

where

$$\mathcal{Y}^{\alpha\beta} = -\frac{\partial \mathcal{L}}{\partial g_{\alpha\beta,1}} + \frac{\partial (n^1-m^1)}{\partial g_{\alpha\beta}} - \left( \frac{\partial (n^1-m^1)}{\partial g_{\alpha\beta,c}} \right)_{|c}, \tag{10}$$

From (2), the first term of $\mathcal{Y}^{\alpha\beta}$ has the value

$$-\frac{1}{2} \mathcal{G} g_{\mu\nu,\varrho} \{ (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta}) g^{1\varrho} + g^{\mu\varrho} g^{\alpha\beta} g^{1\nu} +$$
$$+ g^{1\alpha} g^{\mu\nu} g^{\beta\varrho} - 2g^{\mu\alpha} g^{\beta\varrho} g^{1\nu} \} \tag{11}$$

This expression is written in a form not symmetrical between $\alpha$ and $\beta$, for brevity, but it should be understood as symmetrized. The same applies to the following expression. We have from (8)

$$\frac{\partial(m^1+2\mathcal{G}c^{ab}\Gamma^1_{ab})}{\partial g_{\alpha\beta}}-\left(\frac{\partial(m^1+2\mathcal{G}c^{ab}\Gamma^1_{ab})}{\partial g_{\alpha\beta,c}}\right)_{|c}=$$

$$\mathcal{G}g_{\mu\varrho,\nu}\left(\frac{1}{2}\,g^{\alpha\beta}g^{1\varrho}c^{\mu\nu}-g^{1\alpha}g^{\varrho\beta}c^{\mu\nu}-g^{1\varrho}c^{\mu\alpha}c^{\nu\beta}\right)-(\mathcal{G}g^{1\alpha}c^{\beta c})_{|c}. \tag{12}$$

Subtracting (12), from (11), we get after some reduction

$$(c^{\mu\alpha}c^{\nu\beta}-c^{\mu\nu}c^{\alpha\beta})\mathcal{G}\Gamma^1_{\mu\nu}.$$

The surface pressure term $2\omega\,\mathcal{M}$ in $\boldsymbol{n}^1$ gives as its contribution to $\mathcal{Y}^{\alpha\beta}$

$$2\omega\,\frac{\partial\mathcal{M}}{\partial g_{\alpha\beta}}=\omega\mathcal{M}c^{\alpha\beta}.$$

So altogether we get for $\mathcal{Y}^{\alpha\beta}$

$$\mathcal{Y}^{\alpha\beta}=(c^{\mu\alpha}c^{\nu\beta}-c^{\mu\nu}c^{\alpha\beta})\mathcal{G}\Gamma^1_{\mu\nu}+\omega\mathcal{M}c^{\alpha\beta}. \tag{13}$$

We see that $\mathcal{Y}^{\alpha\beta}$ vanishes when $\alpha$ or $\beta$ is one, which expresses that the surface part of $I_o+I_s$ is independent of the $g_{1\mu}$. Thus we may write (9) as

$$\delta(I_o+I_s)=\int\mathcal{Y}^{ab}\delta g_{ab}dx^0dx^2dx^3. \tag{14}$$

## THE EQUATIONS OF MOTION

The space inside the particle has to be flat, so the $g_{ab}$ at the surface are not arbitrary. They must specify a three-dimensional surface that can be embedded in a four-dimensional flat space. Any variation of the $g_{\mu\nu}$ inside the particle and of the $g_{ab}$ on the surface must be of the kind that comes merely from a change of the coordinate system. Thus for $x^1\leqslant 0$,

$$\delta g_{\mu\nu}=g_{\mu\nu,\varrho}\xi^\varrho+g_{\mu\varrho}\xi^\varrho{}_{|\nu}+g_{\nu\varrho}\xi^\varrho{}_{|\mu}, \tag{15}$$

where $\xi^\varrho$ is infinitesimal and gives the change in the coordinates.

We now get from (14)

$$\delta(I_o+I_s)=\int\mathcal{Y}^{ab}(g^*_{ab,\varrho}\xi^\varrho+2g^*_{a\varrho}\xi^\varrho{}_{|b})dx^0dx^2dx^3$$

$$=-2\int(\mathcal{Y}^{ab}\Gamma^*_{ab\varrho}+\mathcal{Y}^{ab}_{|b}g^*_{a\varrho})\xi^\varrho dx^0dx^2dx^3.$$

The * is attached to field quantities at points just inside the surface when there would otherwise be ambiguity through the corresponding field quantities just outside being different.

The variation of $I_I$ gives

$$\delta I_I=\lambda\int\mathcal{G}g^{\mu\nu}\delta g_{\mu\nu}d^4x$$

$$= 2\lambda \int \mathcal{G} g^{\mu\nu}\left(\frac{1}{2} g_{\mu\nu,\varrho}\xi_\varrho + g_{\mu\varrho}\xi^\varrho{}_{|\nu}\right) d^4x$$

$$= 2\lambda \int (\mathcal{G}_{|\varrho}\xi^\varrho + \mathcal{G}\xi^\varrho{}_{|\varrho}) d^4x = 2\lambda \int \mathcal{G}^* \xi^1 dx^0 dx^2 dx^3.$$

The integral here is, of course, merely the change of the four-dimensional volume inside the particle produced by the change $\xi^\varrho$ in the coordinate system, with the equation of the surface maintained as $x^1 = 0$.

The $\xi^\varrho$ can be arbitrary at each point of the surface, so the action principle $\delta(I_0 + I_S + I_I) = 0$ gives

$$\mathcal{Y}^{ab}\Gamma^*_{ab\varrho} + \mathcal{Y}^{ab}|_b g^*_{a\varrho} - \lambda\mathcal{G}^* g^1_\varrho = 0. \tag{16}$$

There are four equations here. For three of them, those with $\varrho = 0,2$ or 3, we may drop the *'s, so that they appear as

$$\mathcal{Y}^{ab}\Gamma_{abc} + \mathcal{Y}^{ab}|_b g_{ac} = 0. \tag{17}$$

These three must hold identically, as they merely express that the action is invariant under a change in the coordinates $x^0, x^2, x^3$ in the surface. One can easily check the identities with the help of Einstein's field equations just outside the surface.

We are left with just one equation, which we have in its most convenient form if we multiply (16) by $g^{*1\varrho}$; thus,

$$\mathcal{Y}^{ab}\Gamma^{*1}_{ab} - \lambda\mathcal{G}^* g^{*11} = 0. \tag{18}$$

We may also write it

$$\mathcal{Y}^{ab}\mathcal{G}^*\Gamma^{*1}_{ab} - \lambda\mathcal{M}^2 = 0 \tag{19}$$

when it is expressed in terms of invariants with respect to coordinate transformations which do not alter the surface $x^1 = 0$ and the coordinates $x^0, x^2, x^3$ in it.

Equation (18) or (19), with $\mathcal{Y}^{ab}$ given by (13), is the equation of motion for the surface. It is produced, together with the field equations for the outside space, by the action principle.

## THE SPHERICALLY SYMMETRIC SOLUTION

We shall apply the theory to a spherically symmetric particle with its centre at rest and its radius varying with the time. The field outside the particle is then just the Schwarzschild solution of the Einstein equations, there being no possibility of gravitational waves consistent with spherical symmetry.

Let $\varrho$ be the radius of the particle, a function of the time $t$. In terms of the Schwarzschild coordinates $r, \theta, \varphi, t$, we take $x^1 = r - \varrho$, $x^2 = \theta$, $x^3 = \varphi$, $x^0 = t$, so that the equation of the surface is $x^1 = 0$. Then for $x^1 \geqslant 0$,

$$ds^2 = \gamma dt^2 - \frac{(dx^1 + \dot{\varrho} dt)^2}{\gamma} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2, \tag{20}$$

where $\gamma = 1-2m/r$. Thus,

$$g_{00}=\gamma-\frac{\dot{\varrho}^2}{\gamma},\quad g_{10}=-\frac{\dot{\varrho}}{\gamma},\quad g_{11}=-\frac{1}{\gamma}$$

$$g_{22}=-r^2,\quad g_{33}=-r^2\sin^2\theta$$

with the other components of $g_{\mu\nu}$ vanishing.

We find

$$\mathcal{G}=r^2\sin\theta,\quad \mathcal{M}=\left(\gamma-\frac{\dot{\varrho}^2}{\gamma}\right)^{1/2}r^2\sin\theta,$$

$$c^{00}=\left(\gamma-\frac{\dot{\varrho}^2}{\gamma}\right)^{-1},\quad c^{22}=-\frac{1}{\gamma^2},\quad c^{33}=-\frac{1}{r^2\sin^2\theta},$$

with $c^{ab}$ vanishing for $a\neq b$. We find further

$$\Gamma^1_{00}=\ddot{\varrho}+m\gamma r^{-2}(1-3\dot{\varrho}^2\gamma^{-2}),$$

$$\Gamma^1_{22}=-\gamma r,\quad \Gamma^1_{33}=-\gamma r\sin^2\theta$$

with $\Gamma^1_{ab}$ vanishing for $a\neq b$. From (13) we now get, making the approximation of neglecting $\dot{\varrho}^2$ but not $\ddot{\varrho}$, and using $\gamma_0$ to denote the value of $\gamma$ when $x^1=0$ $\left(\text{namely } \gamma_0=1-\frac{2m}{\varrho}\right)$:

$$\begin{aligned}\mathcal{Y}^{00}&=-c^{00}(c^{22}\mathcal{G}\Gamma^1_{22}+c^{33}\mathcal{G}\Gamma^1_{33}-\omega\mathcal{M})\\&=-\sin\theta(2\varrho-\omega\gamma_0^{-1/2}\varrho^2),\\ \mathcal{Y}^{22}&=-c^{22}(c^{00}\mathcal{G}\Gamma^1_{00}+c^{33}\mathcal{G}\Gamma^1_{33}-\omega\mathcal{M})\\&=\sin\theta(\ddot{\varrho}\gamma_0^{-1}+\varrho^{-1}-m\varrho^{-2}\omega\gamma_0^{1/2}),\\ \mathcal{Y}^{33}&=-c^{33}(c^{00}\mathcal{G}\Gamma^1_{00}+c^{22}\mathcal{G}\Gamma^1_{22}-\omega\mathcal{M})\\&=\frac{Y^{22}}{\sin^2\theta},\end{aligned}$$

with $\mathcal{Y}^{ab}$ vanishing for $a\neq b$.

The metric inside the particle must be chosen so that it describes a flat space with the same $g_{ab}$ as (20) at $x^1=0$. The solution is easily seen to be

$$ds^2=\{\gamma_0+(1-\gamma_0^{-1})\dot{\varrho}^2\}dt^2-(dx^1+\dot{\varrho}dt)^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2 \tag{21}$$

with $r=\varrho+x^1$ as before and now $-\varrho\leqslant x^1\leqslant 0$. The metric (21) gives

$$g_{00}=\gamma_0-\frac{\varrho^2}{\gamma_0},\quad g_{10}=-\dot{\varrho},\quad g_{11}=-1,$$

$$g_{22}=-r^2,\quad g_{33}=-r^2\sin^2\theta$$

with the other components of $g_{\mu\nu}$ vanishing.

We find for $x^1 = 0$, with neglect of $\dot{\varrho}^2$ but not $\ddot{\varrho}$,

$$\mathcal{G}^* = \gamma_0^{1/2}\varrho^2 \sin\theta, \quad g^{*11} = -1,$$

$$\Gamma^{*1}_{00} = \ddot{\varrho}, \quad \Gamma^{*1}_{22} = -\varrho, \quad \Gamma^{*1}_{33} = -\varrho \sin^2\theta.$$

Substituting these results into (18), we get as the equation of motion for small $\dot{\varrho}$,

$$(2\varrho - \omega\gamma_0^{-1/2}\varrho^2)\ddot{\varrho} + 2(\ddot{\varrho}\gamma_0^{-1} + \varrho^{-1} - m\varrho^{-2} - \omega\gamma_0^{1/2})\varrho - \lambda\gamma_0^{1/2}\varrho^2 = 0. \tag{22}$$

It is of the form

$$A(\varrho)\ddot{\varrho} + B(\varrho) = 0$$

with

$$A(\varrho) = 2\gamma_0^{-3/2}(\varrho^{-1} - m\varrho^{-2}) - \frac{1}{2}\omega\gamma_0^{-1},$$

$$B(\varrho) = \gamma_0^{-1/2}(\varrho^{-2} - m\varrho^{-3}) - \omega\varrho^{-1} - \frac{1}{2}\lambda.$$

The equilibrium radius $\varrho = R$ is given by

$$B(R) = 0.$$

We may choose any value for $R$ greater than $2m$ and any $\omega$, and then choose $\lambda$ to fit this equation.

The equilibrium is stable if

$$A\frac{dB}{d\varrho} > 0$$

for $\varrho = R$. This leads to

$$\left\{\frac{4}{\left(1-\frac{2m}{R}\right)^{1/2}}\left(\frac{1}{R} - \frac{m}{R^2}\right) - \omega\right\}\left\{\omega - \frac{1}{\left(1-\frac{2m}{R}\right)^{3/2}}\left(\frac{2}{R} - \frac{6m}{R^2} + \frac{5m^2}{R^3}\right)\right\} > 0.$$

We can satisfy this condition by choosing $\omega$ to lie between the two quantities

$$\frac{4}{\left(1-\frac{2m}{R}\right)^{1/2}}\left(\frac{1}{R} - \frac{m}{R^2}\right), \quad \frac{1}{\left(1-\frac{2m}{R}\right)^{3/2}}\left(\frac{2}{R} - \frac{6m}{R^2} + \frac{5m^2}{R^3}\right),$$

except in the case when the two quantities coincide. This case occurs when

$$\frac{R}{m} = \frac{1}{2}(3+\sqrt{3}), \tag{24}$$

which gives a value for $R$ just a little greater than the Schwarzschild radius $2m$.

## CONCLUSION

We may choose any value for $R$ greater than $2m$, excluding the value (24), and then choose $\omega$ and $\lambda$ to fit the conditions. We then have a theory for the motion of a particle with the radius $R$. The particle is stable for small disturbances that preserve its spherical symmetry. Further work would be needed to check whether it is still stable if the spherical symmetry is disturbed.

## REFERENCES

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[2] A. Lees, *Phil. Mag.* **28**, 385 (1939).

## DISCUSSION

A. Schild:

Would you wish to put conditions on the particle that make the mass surface density always positive?

P. A. M. Dirac:

I would like to have a Hamiltonian which is positive definite. That would be the natural way of securing that the motion is always stable and that you don't get runaway solutions. I have big doubts as to whether it is possible to have a positive definite Hamiltonian, because the Newtonian energy is negative; but one should have the aim of getting a positive definite Hamiltonian. If that cannot be satisfied, then I would like to have, at any rate, a positive definite surface energy.

H. Bondi:

I'll abuse my position as chairman to ask a question myself. I am rather worried about the assumption of the vanishing field inside. When one considers any extended body, then clearly the motion of that body will depend on the equation of state one assumes for the material. We know this in fact from the effect of the tidal friction of the earth on the motion of the moon. Now, the type of equation of state one wants should definitely be what in electrical network theory is called passive, that is to say, that no energy from other sources of energy is fed in, but that it is a natural response, purely reactive, or, perhaps, dissipative, as in the case of tidal friction. Now in electrical theory we know that a purely reactive network, namely a perfectly conducting shell, will so distribute its charges as to give zero field inside.

But in gravitation, precisely the opposite is the case. If we assume, for example, the earth to be a shell, moving in the field of the sun, the earth falling freely; then, of course, the residual forces on the earth are the tidal forces. And if this were a shell with particles in it, that could move freely, then these particles which congregate here would make tides, which would increase the field inside, and would not abolish it. So, in the natural motion, and, probably, in any passive situation, we would get something much closer to paramagnetic behaviour, where we get an increase of the field inside, than to shielding which corresponds to electrostatics.

P. A. M. DIRAC:

Those remarks of yours would rather suggest that my particle would not be stable for disturbances which are not spherically symmetrical. I think that would be the natural interpretation of your remarks.

H. BONDI:

Actual instability would depend on the properties of the material, but an enhancement of the inhomogeneity of the field would certainly occur.

P. A. M. DIRAC:

But in the way that the theory is at present formulated there has to be no field inside, no matter what disturbances occur outside.

H. BONDI:

Well, I fear that this is in some sense unphysical.

P. A. M. DIRAC:

Yes. You could get a more physical theory by bringing in an action for the internal region corresponding to some physical conditions. That would complicate the theory, but make it more physical.

V. A. FOCK:

I should like to ask the following question. You are considering very small particles, because they are nearly point particles and their mass $m$ is very small.

P. A. M. DIRAC:

They need not be very small.

V. A. FOCK:

What kind of particles are these? Are they quantum particles or classical particles? I cannot imagine particles of such small size that are not quantum particles. And if so, how are quantum-mechanical considerations to be introduced in your theory?

P. A. M. DIRAC:

I would like to answer first that these particles do not have to be small. I have not anywhere made the assumption that they are small. We have exact equations of motion which would also apply if the particles are very far from being small. If they are small, I would agree that we ought to bring in quantum theory; and that brings in very many new problems.

V. A. FOCK:

But in the case that they are not small, they must have a very large density; so large that perhaps the notion of density is no more applicable in this case.

P. A. M. DIRAC:

I do not think the density would have to be large. The density could very well be small also. With large particles you get into the difficulty that two of them may collide, and then you would need some new equations of motion to describe that situation.

J. A. WHEELER:

The work here is in line with the old Lorentz model of the electron which, however, ran into difficulty. And it's very nice to see that if one goes into general relativity one has possibilities to construct objects that do not have that difficulty. In this connection it might be mentioned that there are also two other kinds of objects that one can construct within the framework of general relativity, namely geons and topological objects—handles or worm-holes. In those two cases one has examined the question of stability. And the questions which you have brought up in such an interesting way here have also been looked at in those two cases; namely the scattering of radiation by such objects; and the interaction of such objects with other fields. However, of course, for all three objects (the problem that you speak of here, and in the case of the geon and in the case of the wormhole) one is talking of things that have not the slightest connection with particles of the real physical world, but speaking rather of *models* which are of great interest in understanding the nature and implications of relativity. However, it is a bit puzzling to me to understand why one would introduce a *new* physical phenomenon like surface tension here, when already in the framework of well-established general relativity and electromagnetism one has the tools at hand with which to construct model objects of considerable interest in their own right. I raise this issue of this new physical term particularly because I myself do not understand what governs the law of aggregation of this substance that causes the surface tension. What decides into how many spheres it will collect? What decides — if it wants to break up into pieces — whether this is allowed or forbidden? This is why it would seem to me simpler not to bring in the surface tension to construct such models.

P. A. M. DIRAC:

You made reference to the difficulties of Lorentz. He could have avoided these difficulties if he had used an action principle. And the only way that I have succeeded in avoiding these Lorentz difficulties is by using an action principle, all the way through. Now, this introduction of the surface tension, or rather surface pressure, enables one to have particles of any size. They could be small particles, not much bigger than the Schwarzschild particle, or they could be very large. I don't think you could have the small particles without bringing in something like this surface pressure, or some other non-Einstein terms. I suppose your geons are extremely large, aren't they?

J. A. WHEELER:

Comparable in size to the sun or larger *if* they are to be analyzed without getting into the important problems of *quantizing* general relativity.

P. A. M. DIRAC:

Yes. I would agree that this model is rather remote from physical reality, but I wish to say again that we are working in a new field and we make the simplest assumptions which lead to a physically sensible theory. We can add on further terms to the action later on if we want them.

B. S. DEWITT:

I should like to make a comment about a matter of principle, and this is also in answer to Prof. Wheeler. I think the example Prof. Dirac has shown us is a very excellent example of the intimate relation which exists between the physical description of the geometry of space-time and the dynamical behaviour of bodies which occupy space-time. I should like to suggest that instead of pushing 100% in the direction of examining only empty space, we should study more, perhaps, the actual description of material objects which occupy space-time. This, for example, I found very useful in the analysis of the Bohr–Rosenfeld problem: to really describe the elastic test bodies that one uses. One learns very interesting new things this way. After all, our original ideas of distance and Riemannian geometry, are based on our experience with physical objects like rods and clocks. And, the effort to describe these objects in manifestly covariant language and to learn how they behave is worth it, I think.

P. A. M. DIRAC:

I agree completely.

C. MØLLER:

May I ask if there is in this model a definite relation between the mass, the constant in the Schwarzshild solution outside, and the radius of this object.

P. A. M. DIRAC:

No. You can have it independent by suitably choosing the surface pressure.

A. LICHNEROWICZ:

En 1946 ou 1947 j'ai fabriqué un modèle de l'électron avec de la matière à l'intérieur et une densité superficielle correspondante. Il y a certains rapports avec ce modèle; mais dans mon modèle il y a de la matière fluide à l'intérieur au lieu du vide.

P. A. M. DIRAC:

Do you have the gravitational field taken into account?

A. LICHNEROWICZ:

Yes, in the interior. But also a tensor, a repulsive tensor on the surface.

P. A. M. DIRAC:

Do you have an action principle?

A. LICHNEROWICZ:

Yes, it consists of two parts.

# PROPAGATEURS ET QUANTIFICATION EN RELATIVITÉ GÉNÉRALE

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J'AI procédé, depuis 1958, á l'étude mathématique rigoureuse d'un instrument qui doit jouer un rôle fondamental dans la théorie quantique des champs sur un espace-temps courbe. Il s'agit des propagateurs tensoriels et spinoriels qui constituent les généralisations naturelles du propagateur scalaire $D$ de Jordan–Pauli [2], [3].

Sous le nom de fonction de Green (que je n'apprécie pas) un instrument apparemment semblable a été utilisé indépendamment, pour l'étude de problèmes physiques variés, par Bryce et Cecile DeWitt [4]. Il me semble que, dans tous les travaux concernant la théorie quantique des champs en relativité générale, *doivent apparaître de manière plus ou moins consciente plus ou moins explicite, des propagateurs.*

C'est a l'étude de la notion de propagateur et de certaines de ses applications que je consacrerai cette conférence. Je me limiterai ici aux champs libres, car ce sont provisoirement les seuls pour lesquels une étude mathématique rigoureuse peut être développée. Une première partie de la conférence sera consacrée à une esquisse de la théorie mathématique des propagateurs, une seconde à la formation des commutateurs du champ électromagnétique et du champ gravitationnel varié, une troisième aux champs spinoriels, la dernière à quelques remarques sur les opérateurs de création-annihilation.

## I. THÉORIE MATHÉMATIQUE DES PROPAGATEURS

### 1. Tenseurs-distributions sur une variété riemannienne

a) Soit $V_n$ une variété riemannienne de type hyperbolique normal, orientée, de classe $C^\infty$, $D^p$ l'espace des $p$-tenseurs à support compact de classe $C^\infty$. *Un p-tenseur-distribution T est une fonctionnelle linéaire continue, à valeurs scalaires, des p-tenseurs à support compact de classe* $C^\infty$. Si $U \in D^p$, $<T, U>$ est la valeur pour $U$ du tenseur-distribution $T$. *Un tenseur ordinaire T définit un tenseur-distribution par la formule*:

$$< T, U > = \int_{V_n} T_{a_1 \dots a_p}(x) U^{a_1 \dots a_p}(x) \eta(x) = \int_{V_n} (T,U)_x \eta(x) \qquad (1\text{–}1)$$

ou $\eta$ est l'élément de volume et $(T, U)_x$ le produit contracté en $x \in V_n$.

Si $\nabla$ est l'opérateur de dérivation covariante, $\delta$ l'opérateur de *codifférentiation* sur les $(p+1)$-tenseurs

$$\delta\colon U_{\beta_1\ldots\beta_{p+1}} \to -\nabla_\varrho U^\varrho_{\alpha_1\ldots\alpha_p} \tag{1–2}$$

la dérivée covariante $\nabla T$ d'un $p$-tenseur-distribution $T$ est définie par:

$$< \nabla T, U > = < T, \delta U >, \quad U \in D^p. \tag{1–3}$$

b) Dans la théorie des opérateurs différentiels sur $V_n$, les notions de bitenseurs et bitenseurs-distributions sur $V_n \times V_n$ interviennent nécessairement. Le *biscalaire de Dirac* $\delta(x,x')$ est défini par:

$$< \delta(x,x'), f(x') > = f(x)$$

où $f$ est une fonction arbitraire. Plus généralement, on a des *bitenseurs de Dirac* $D^{(p)}(x,x')$ définis par:

$$< D^{(p)}(x,x'), U(x') > = U(x) \quad (p = 0,1,\ldots) \tag{1–4}$$

où $U$ est un $p$-tenseur arbitraire. Par antisymétrisation de $D^{(p)}$, on obtient une bi-forme distribution $\hat{D}^{(p)}$ et par symétrisation, pour $p = 2$, le bitenseur-distribution symétrique $\tilde{D}^{(2)}$. Si $d_x$ est l'opérateur de différentiation extérieure en $x$, $\delta_x$ celui de codifférentiation, on établit aisément:

$$\delta_{x'} \hat{D}^{(p+1)} = d_x \hat{D}^{(p)} \quad (p = 0, 1, \ldots, n-1). \tag{1–5}$$

Si $A$ est un vecteur, nous désignons par $\mathcal{D}A$ le tenseur symétrique $\nabla_\alpha A_\beta + \nabla_\beta A_\alpha$, On voit de même que

$$\delta_{x'} \tilde{D}^{(2)} = \mathcal{D}_x D^{(1)}. \tag{1–6}$$

**2.** L'opérateur de Klein-Gordon

La théorie des propagateurs est valable pour tous les systèmes hyperboliques au sens de Leray. Je me limiterai ici aux opérateurs de Klein-Gordon sur les tenseurs (et plus tard sur les spineurs). Dans le cas des tenseurs antisymétriques, G. de Rham a introduit le laplacien[1] $\Delta$ défini par:

$$\Delta T = (d\delta + \delta d) T$$

qui commute dans ce cas avec $d$ et $\delta$. Le laplacien peut s'écrire explicitement:

$$(\Delta T)_{\alpha_1\ldots\alpha_p} = -\nabla^\varrho \nabla_\varrho T_{\alpha_1\ldots\alpha_p} + \sum_k R_{\alpha_l p} T_{\alpha_1\ldots}{}^{\mu}{}_{\ldots\alpha_p} - \sum_{k\neq l} R_{\alpha_k\varrho,\,\alpha_l\sigma} T_{\alpha_1\ldots}{}^{\varrho\sigma}{}_{\ldots\alpha_p}. \tag{2–1}$$

J'adopte (2–1) comme *définition du laplacien d'un tenseur arbitraire.* L'opérateur $(\Delta+\mu)$ où $\mu = \text{const.}$ est l'opérateur de Klein-Gordon; $\Delta$ jouit des propriétés suivantes: il est auto-adjoint, commute avec la contraction, préserve les symétries ou antisymétries possibles de $T$. Si $T$ est à dérivée covariante nulle $\Delta(T \otimes U) = T \otimes \Delta U$.

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[1] Nous maintenons la dénomination de laplacien pour un opérateur qui généralise ce qu'on appelle usuellement un dalembertien.

Si le tenseur de Ricci de $V_n$ est à dérivée covariante nulle, on a pour tout 2-tenseur $T$ et vecteur $A$:

$$\delta\Delta T = \Delta\delta T, \quad \nabla\Delta A = \Delta\nabla A.$$

**3.** Noyaux élémentaires et propagateurs

Dans la suite, nos considérations sont relatives à un domaine ouvert $\Omega$ tel que le conoïde caractéristique $\Gamma_{x'}$, de sommet $x' \in \Omega$ soit régulier dans $\Omega$ et définisse trois régions: le futur $\mathcal{E}^+(x')$, le passé $\mathcal{E}^-(x')$ et l'ailleurs. Si $K$ est un ensemble de $\Omega$, le *futur* $\mathcal{E}^+(K)$ est l'ensemble des chemins temporels issus des points $x'$ de $K$ dans le futur de $x'$; le passé de $K$ est l'ensemble $\mathcal{E}^-(K)$ des chemins temporels aboutissant aus points $x'$ de $K$ dans le passé de $x'$.

Un sensemble $K$ est dit *compact vers le futur* (resp. le passé) si l'intersection de $\mathcal{E}^-(K)$ avec $\mathcal{E}^+(x)$ est compacte pour tout $x$; $\mathcal{E}^-(K)$ est aussi compact vers le futur. Le lemme suivant est essentiel pour les démonstrations: si *$K$ est compact vers le futur et $K'$ compact, $\mathcal{E}^-(K) \wedge \mathcal{E}^+(K')$ est compact.*

a) En ce qui concerne les solutions ou noyaux élémentaires de $(\Delta+\mu)$ nous avons le résultat suivant: *il existe deux noyaux élémentaires $E^{(p)\pm}(x, x')$, c'est-à-dire deux bi-p-tenseurs distributions satisfaisant*:

$$(\Delta_x+\mu)E^{(p)\pm}(x, x') = D^{(p)}(x, x') \tag{3–1}$$

*et qui, pour chaque $x' \in \Omega$ ont leurs supports respectivement dans $\mathcal{E}^+(x')$ ou $\mathcal{E}^-(x')$.* Nous avons aussi

$$(\Delta_{x'}+\mu)E^{(p)\pm}(x, x') = D^{(p)}(x, x').$$

L'unicité des noyaux élémentaires est un cas particulier d'un théorème général d'unicité:

*tout tenseur-distribution $T$ solution de l'équation homogène $(\Delta+\mu)T = 0$ et à support compact vers le futur (ou le passé) est nul.*

b) J'appelle *propagateur tensoriel* relatif à $(\Delta+\mu)$ le noyau différence

$$E^{(p)}(x, x') = E^{(p)-}(x, x') - E^{(p)+}(x, x'). \tag{3–3}$$

Il est solution des équations homogènes:

$$(\Delta_x+\mu)E^{(p)}(x, x') = 0, \quad (\Delta_{x'}+\mu)E^{(p)}(x, x') = 0 \tag{3–4}$$

et a son support, pour chaque $x'$, dans $\mathcal{E}^+(x') \cup \mathcal{E}^-(x')$, $(\Delta+\mu)$ étant auto-adjoint, *son propagateur est antisymétrique par rapport au couple* $(x, x')$. Tout tenseur-distribution $T$ solution de $(\Delta+\mu)T = 0$ peut être obtenu par composition de Volterra du propagateur $E^{(p)}$ et d'un tenseur-distribution *$U$ à support compact dans le passé et le futur.*

Le propagateur $E^{(p)}$ donne explicitement la solution du problème de Cauchy pour l'équation $(\Delta+\mu)0 = T$ où $T$ est ùn tenseur ordinaire. Si $\sigma$ est l'hypersurface orientée dans l'espace portant les données initiales, on peut écrire:

$$T_{\alpha_1\ldots\alpha_p}(x) = \int_\sigma \Big\{T^{\lambda'_1\ldots\lambda'_p}(y)\nabla_{\mu'}E^{(p)}_{\alpha_1\ldots\alpha_p,\,\lambda'_1\ldots\lambda'_p}(x, y) -$$

$$- E^{(p)}_{\alpha_1\ldots\alpha_p}{}^{\lambda'_1\ldots\lambda'_p}(x, y)\nabla_{\mu'}T_{\lambda'_1\ldots\lambda'_p}(y)\Big\}\, d\sigma^{\mu'}_y \tag{3–5}$$

où $d\sigma$ est l'élément d'aire de $\sigma$, $n^{\mu'}$ le vecteur unitaire normal orienté vers le futur et où $d\sigma^{\mu'} = n^{\mu'} d\sigma$, le flux qui figure au second membre se définissant aisément.

c) $(\Delta + \mu)$ opère sur les tenseurs antisymétriques. Par antisymétrisation de $E^{(p)}$, on obtient le *propagateur antisymétrique* $G^{(p)}$ relatif à $(\Delta + \mu)$. Du théorème d'unicité et des formules (1–5), on déduit:

$$\delta_{x'} G^{(p+1)} = d_x G^{(p)} \qquad (p = 0, 1, \ldots, n-1) \tag{3–6}$$

$(\Delta + \mu)$ opère aussi sur les 2-tenseurs symétriques. Par symétrisation de $E^{(2)}$, on obtient le *propagateur symétrique* $K$. On établit que par contraction en $x$ par le tenseur métrique: (1–6)

$$g^{\alpha\beta} K_{\alpha\beta,\,\lambda'\mu'} = 2 g_{\lambda'\mu'} G^{(0)}(x, x'). \tag{3–7}$$

Si le tenseur de Ricci de $V_n$ est *à dérivée covariante nulle*, on déduit du théorème d'unicité et des formules (1–6)

$$\delta_{x'} K = \mathcal{D}_x G^{(1)}. \tag{3–8}$$

Toutes ces relations sont utiles pour la quantification.

## II. CHAMPS TENSORIELS

**4.** Commutateur pour le champ électromagnétique libre

a) Dans l'espace-temps $V_4$ muni d'une métrique arbitraire *donnée*, considérons un champ électromagnétique libre $F$ correspondant à un potentiel-vecteur $a$ avec $F = da$. Dans le cas d'un photon de masse non nulle nous avons pour $a$ l'équation:

$$\delta da = \varepsilon^2 a \qquad (\varepsilon^2 = \text{const} \neq 0) \tag{4–1}$$

$\varepsilon^2$ étant $\neq 0$; (4–1) entraîne $\delta a = 0$ et (4–1) est équivalent au système:

$$(\Delta - \varepsilon^2) a = 0 \tag{4–2}$$

et

$$\delta a = 0 \tag{4–3}$$

puisque $\Delta = d\delta + \delta d$. Supposons que $a$ soit une forme linéaire à valeurs dans un espace d'opérateurs d'un espace de Hilbert. Nous cherchons à construire un *commutateur* $[a(x), a(x')]_-$ $(x, x' \in V_4)$, c'est-à-dire une bi-1-forme distribution à valeurs scalaires qui, pour chaque $x'$, a son support dans $\mathcal{E}^+(x') \cup \mathcal{E}^-(x')$ est antisymétrique par rapport au couple $(x, x')$ et vérifie le système (4–2), (4–3). La formule:

$$[a(x), a(x')]_- = \frac{\hbar}{i} \left\{ G^{(1)}(x, x') - \frac{1}{\varepsilon^2} d_x d_{x'} G^{(0)} \right\} \tag{4–4}$$

ou $G^{(1)}$, $G^{(0)}$ sont des propagateurs relatifs à $(\Delta - \varepsilon^2)$, nous fournit un commutateur rigoureusement compatible avec (4–2), (4–3) (en vertu des relations

(3–6)) et qui se réduit en relativité restreinte au commutateur classique. On en déduit:

$$[F(x), F(x')]_- = \frac{\hbar}{i} d_x d_{x'} G^{(1)}(x, x').$$

b) En l'absence de terme de masse ($\varepsilon^2 = 0$) (4–1) est invariante par transformation de jauge. Une telle transformation permet d'astreindre $\alpha$ à la condition de Lorentz $\delta\alpha = 0$. Classiquement, l'équation pour le potentiel $\alpha$ s'écrit:

$$\Delta\alpha = 0 \tag{4–5}$$

qui entraîne

$$\Delta\delta\alpha = 0 \tag{4–6}$$

et nous adoptons comme commutateur pour le potentiel:

$$[\alpha(x), \alpha(x')]_- = \frac{\hbar}{i} G^{(1)}(x, x') \tag{4–7}$$

oú $G^{(1)}$ est relatif à $\Delta$. Il en résulte d'après (3–6):

$$[\alpha(x), \delta_{x'} \alpha(x')]_- = \frac{\hbar}{i} d_x G^{(0)}(x,x'). \tag{4–8}$$

Le commutateur (4–7) est compatible avec (4–5), mais non avec la condition de Lorentz; (4–7) entraîne pour $F$ un commutateur rigoureusement compatible avec les équations de Maxwell usuelles. La situation est strictement semblable à la situation bien connue qui se présente dans le cas plat et peut être dénouée de la même façon.

**5.** Commutateur pour le champ gravitationnel varié

a) Sur un espace-temps $V_4$ *de métrique donnée g satisfaisant aux équations d'Einstein du vide* $R_{\alpha\beta} = \lambda g_{\alpha\beta}$, considérons une variation arbitraire $h_{\alpha\beta} = \delta g_{\alpha\beta}$ du tenseur métrique. La variation correspondante du tenseur de Ricci est donnée par:

$$2\delta R_{\alpha\beta} = \Delta h_{\alpha\beta} + \{\mathcal{D}k(\underline{h})\}_{\alpha\beta}$$

où l'on a posé:

$$h'_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}(g^{\lambda\mu} h_{\lambda\mu}) \qquad k_\alpha(\underline{h}) = \nabla_\varrho h'^{\varrho}_{\alpha}.$$

Par variation de l'identité d'Einstein, on a l'identité:

$$\nabla^\alpha \left(\delta R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\lambda\mu} \delta R_{\lambda\mu}\right) = \lambda k_\beta(\underline{h}).$$

Cela posé, considérons le champ décrit par le tenseur $\underline{h}$ astreint aux équations

$$\delta R_{\alpha\beta} = \mu h_{\alpha\beta} \qquad (\mu = \text{const}). \tag{5–1}$$

La théorie d'un tel champ peut être développée d'une manière strictement parallèle à celle du champ électromagnétique, les équations homologues étant munies des mêmes numéros. D'après l'identité qui la précède, (5–1) entraîne $(\mu-\lambda)k_\beta(\underline{h}) = 0$. Si $\varepsilon^2 = 2(\mu-\lambda) \neq 0$ (présence d'un terme de masse) (5–1) est équivalent au système

$$(\Delta-2\mu)\underline{h} = 0, \tag{5–2}$$

$$k(\underline{h}) = 0. \tag{5–3}$$

La formule

$$[\underline{h}(x), \underline{h}(x')]_- = \frac{\hbar}{i}\left\{K(x, x')-\underline{g}(x)\underline{g}(x')G^{(0)}(x, x')-\frac{1}{\varepsilon^2}\mathcal{D}_x\mathcal{D}_{x'}G^{(1)}(x,x')\right\} \tag{5–4}$$

où $K$, $G^{(1)}$, $G^{(0)}$ sont les propagateurs relatifs à $(\Delta-2\mu)$, nous fournit un commutateur rigoureusement compatible avec (5–2) et (5–3) (en vertu des relations (3–7), (3–8)) et se réduisant, dans le cas où $\underline{g}$ est une métrique euclidienne, au commutateur obtenu directement par la transformation de Fourier.

b) *En l'absence de terme de masse* ($\varepsilon^2 = 0$), (5–1) (où $\mu = \lambda$) est invariante par *les transformations de jauge gravitationnelles* $\underline{h} \to \underline{h}+\mathcal{D}\ A$, où $A$ est un vecteur arbitraire. Par une telle transformation on peut astreindre $\underline{h}$ à la "condition de Lorentz" (5–3). L'équation de champ pour $\underline{h}$ est:

$$(\Delta-2\lambda)\underline{h} = 0 \tag{5–5}$$

qui entraîne

$$(\Delta-2\lambda)k(\underline{h}) = 0 \tag{5–6}$$

et nous avons comme commutateur

$$[\underline{h}(x), \underline{h}(x')]_- = \frac{\hbar}{i}\{K(x, x')-\underline{g}(x)\underline{g}(x')G^{(0)}(x, x')\} \tag{5–7}$$

oú les propagateurs sont associés à $(\Delta-2\lambda)$. De (5–7) il résulte:

$$[k(\underline{h}(x)), k(\underline{h}(x'))]_- = 0.$$

Nous nous trouvons dans une situation identique à celle du cas électromagnétique en l'absence de terme de masse, situation qui peut être dénouée de la même manière. Pour assurer la compatibilité du substratum gravitationnel macroscopique décrit par $\underline{g}$ avec le champ microscopique $\underline{h}$, il suffit de supposer que la valeur moyenne de $\underline{h}$ est nulle.

## 6. Lagrangiens

Les équations de champ adoptées pour le cas électromagnétique et pour le cas gravitationnel dérivent de lagrangiens que je me borne à indiquer et qui permettent l'introduction des éléments usuels des théories de champ. Pour le cas électromagnétique avec terme de masse, (4–1) dérive de:

$$\mathcal{L}_1^{(\varepsilon)} = -\frac{1}{4}(da, da) + \frac{\varepsilon^2}{2}(a, a).$$

En l'absence de terme de masse, (4–5) dérive de:

$$\mathcal{L}_1 = -\frac{1}{4}(da, da) - \frac{1}{2}(\delta a)^2.$$

Dans le cas gravitationnel varié, posons:

$$\mathcal{L} = -\frac{1}{2}\nabla^\gamma h'^{\alpha\beta}\nabla_\gamma h'^{\alpha\beta} + \nabla^\gamma h'^{\alpha\beta}\nabla_\alpha h'_{\beta\gamma} + \frac{1}{4}\nabla^\gamma \underline{h}'\nabla_\gamma \underline{h}'$$

où $\underline{h}'$ est le tenseur associé à $\underline{h}$. L'équation (5–1) dérive de:

$$\mathcal{L}_2^{(\varepsilon)} = \mathcal{L} + \left(\lambda + \frac{\varepsilon^2}{2}\right)\left(h'^{\alpha\beta}h'_{\alpha\beta} - \frac{1}{2}h'^2\right).$$

En l'absence de terme de masse, (5–5) dérive de:

$$\mathcal{L}_2 = \mathcal{L} - k^\alpha k_\alpha + \lambda\left(h'^{\alpha\beta}h'_{\alpha\beta} - \frac{1}{2}h'^2\right).$$

## III. CHAMPS SPINORIELS

### 7. Notion de spineur en relativité générale

a) Sur un espace-temps de Minkowski rapporté à des *repères orthonormés* désignons par $\gamma_\alpha = (\gamma_\alpha{}^a{}_b)$ les matrices de Dirac[2] qui, avec l'unité $e$, engendrent une algèbre de Clifford:

$$\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = -2g_{\alpha\beta}e \qquad (\gamma^\alpha = g^{\alpha\beta}\gamma_\beta). \tag{7–1}$$

Soit $L(4)$ le groupe de Lorentz homogène complet. A ce groupe correspond le groupe Spin (4), revêtement d'ordre 2 de $L(4)$ et l'homomorphisme–projection $p$ tel que si

$$A = (A_\alpha^{\lambda'}) = pA \qquad (A \in L(4), \Lambda \in \text{Spin}\,(4))$$

on ait:

$$\Lambda\gamma_\alpha\Lambda^{-1} = A_\alpha^{\lambda'}\gamma_{\lambda'}, \tag{7–2}$$

$p$ définit un isomorphisme $p'$ entre algèbres de Lie que l'on peut expliciter de la manière suivante: si $\mu$ est un élément de l'algèbre de $L(4)$ dèfini par un tenseur antisymétrique $\mu_{\alpha\beta}$ et si $\lambda$ est l'élément correspondant de l'algèbre de Spin (4)

$$\lambda = -\frac{1}{4}\mu_{\alpha\beta}\gamma^\alpha\gamma^\beta. \tag{7–3}$$

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(2) Aux indices grecs on donnera le nom d'indices tensoriels, aux indices latins celui d'indices spinoriels.

b) L'espace-temps $V_4$ de la relativité générale est supposé *rapporté exclusivement aux repères orthonormés*, éléments d'un fibré principal $\mathcal{E}(V_4)$ de groupe structural $L(4)$. Si $A \in L(4)$, sa signature temporelle $\varrho_A$ vaut $\pm 1$ selon le signe de $A^{0'}_0$. La variété $V_4$ est supposée munie d'une *orientation temporelle p* définie relativement aux repères $y \in \mathcal{E}(V_4)$ par une composante $\varrho_y = \pm 1$ telle que si $y' = yA$, on ait $\varrho_{y'} = \varrho_y \varrho_A$.

On suppose $V_4$ telle que, de $\mathcal{E}(V_4)$, on puisse déduire par extension un fibré principal $\mathcal{S}(V_4)$ de groupe structural Spin (4) (ce n'est pas toujours le cas). Un point $z$ de $\mathcal{S}(V_4)$ est dit un *repère spinoriel*. Un 1-spineur *contravariant* $\psi$ est une application $\zeta \to \psi(\zeta)$ de $\mathcal{S}(V_4)$ dans un espace $M$ de matrices $1 \times 4$ telle que:

$$\psi(\zeta \Lambda^{-1}) = \Lambda \psi(\zeta) \qquad (\Lambda = (\Lambda^{b'}_a) \in \text{Spin}\,(4)). \tag{7–4}$$

En composantes, (7–4) peut s'écrire:

$$\psi^{b'} = \Lambda^{b'}_a \psi^a.$$

Un *1*-spineur *covariant* $\varphi$ de $V_4$ est une application $z \to \varphi(\zeta)$ de $\mathcal{S}(V_4)$ dans l'espace $M^*$ dual de $M$ telle que:

$$\varphi(\zeta \Lambda^{-1}) = \varphi(\zeta)\Lambda^{-1} \qquad (\Lambda^{-1} = (\Lambda^a_{b'}) \in \text{Spin}(4)). \tag{7–5}$$

En composantes, (7–5) s'écrit:

$$\varphi_{b'} = \Lambda^a_{b'} \varphi_a.$$

Entre le module des 1-spineurs contravariants et le module dual des 1-spineurs covariants, on a la forme de dualité:

$$(\varphi, \psi) = \varphi_a \psi^a. \tag{7–6}$$

Dans le cas où l'intersection de leurs supports sur $V_4$ est compacte, nous introduisons l'intégrale

$$< \varphi, \psi > = \int_{V_4} (\varphi, \psi) \eta. \tag{7–7}$$

La notion de *spineur-distribution* se définit dem anière analogue à celle du tenseur-distribution. J'appelle *bispineur de Dirac* le bi-1-spineur distribution $\sum^{(\ddagger)}(x,x')$ contravariant en $x$, covariant en $x'$ defini par:

$$< \textstyle\sum^{(\ddagger)}(x,x'), \psi(x') > = \psi(x), \qquad < \textstyle\sum^{(\ddagger)}(x,x'), \varphi(x') > = \varphi(x).$$

c) On démontre aisément qu'il existe

1[0] une application antilinéaire $\mathcal{A}$, *l'adjonction de Dirac*, du module des 1-spineurs contravariants sur le module des 1-spineurs covariants

$$\mathcal{A}: \psi \to \overline{\psi} = \varrho \tilde{\psi} \beta,$$

où, $\varrho$ est l'orientation temporelle de $V_4$, $\beta$ une matrice fixe convenable et où $\sim$ désigne le passage à l'adjointe ordinaire.

2⁰ une application antilinéaire $\mathcal{C}$ (avec $\mathcal{C}^2 = \text{Id.}$), la *conjugaison de charge*, du module des *l*-spineurs contravariants sur lui-même

$$\mathcal{C}: \psi \to \mathcal{C}\psi = \alpha\psi^*$$

où $\alpha$ désigne une matrice fixe convenable et* le passage aux complexes conju gués.

d) Par produit tensoriel de la représentation triviale et de la représentation de Spin (4) introduites pour définir les 1-spineurs, on obtient la définition des spineurs de type $(p, q)$, $p$ fois covariants et $q$ fois contravariants. L'adjonction de Dirac et la conjugaison de charge s'étendent de manière, naturelle aux spineurs de type quelconque. Aux représentations précédentes, on peut adjoindre la représentation définie à partir de l'homomorphsime canonique de Spin (4) sur $L(4)$. On obtient ainsi par produit tensoriel des *tenseurs-spineurs*. A l'aide de cette notion on peut interpréter la relation (7–2): elle exprime que les matrices $\gamma_\alpha = (\gamma_\alpha{}^a{}_b)$ définissent un vecteur-spineur $\gamma$, vectoriel et spinoriel de type (1,1).

Une *connexion spinorielle* est une connexion infinitésimale sur le fibré principal $\mathcal{S}(V_4)$. A la connexion riemannienne $\omega = (\omega_{\alpha\beta})$ de $V_4$ est canoniquement associée la connexion spinorielle $\sigma$ définie par la *l*-forme:

$$\sigma = -\frac{1}{4}\omega_{\alpha\beta}\gamma^\alpha\gamma^\beta \tag{7–8}$$

à valeurs dans l'algèbre de Lie de Spin (4). Le vecteur-spineur $\gamma$ est à dèrivée covariante nulle dans cette connexion.

**8.** Champ de Dirac

a) Si $\psi$ est un 1-spineur contravariant, nous posons:

$$\Delta\psi = \gamma^\alpha\gamma^\beta\nabla_\alpha\nabla_\beta\psi = -\nabla^\varrho\nabla_\varrho\psi + \frac{1}{4}R\psi. \tag{8–1}$$

où les dérivations covariantes sont évaluées dans la connexion $\sigma$ et où $R$ est la courbure riemannienne scalaire de $V_4$. De même si $\varphi$ est un 1-spineur covariant:

$$\Delta\varphi = \nabla_\alpha\nabla_\beta\varphi\gamma^\beta\gamma^\alpha = -\nabla^\varrho\nabla_\varrho\varphi + \frac{1}{4}R\varphi. \tag{8–2}$$

Si l'intersection des supports de $\varphi$ et $\psi$ est compacte:

$$< \Delta\varphi, \psi > = < \varphi, \Delta\psi >.$$

J'appelle *noyaux élémentaires* de l'opérateur de *Klein-Gordon* $(\Delta - \varepsilon^2)$ $(\varepsilon^2 = \text{const})$ les deux bi-1-spineurs distributions satisfaisant:

$$(\Delta_x - \varepsilon^2)G^{(\frac{1}{2})\pm}(x, x') = \sum\nolimits^{(\frac{1}{2})}(x, x') \tag{8–3}$$

et qui, pour chaque $x'$, ont leurs supports respectivement dans le futur ou dans le passé de $x'$. La différence $G^{(\frac{1}{2})} = G^{(\frac{1}{2})-} - G^{(\frac{1}{2})+}$ est le *propagateur spinoriel* relatif à $(\Delta - \varepsilon^2)$.

b) Introduisons sur les 1-spineurs contravariants les opérateurs de Dirac de premier ordre:

$$L\psi = \gamma^\alpha \nabla_\alpha \psi - \varepsilon\psi, \qquad L'\psi = \gamma^\beta \nabla_\beta \psi + \varepsilon\psi$$

et sur les $l$-spineurs covariants les deux opérateurs

$$\bar{L}\varphi = -(\nabla_\alpha \bar{\gamma}\gamma^\alpha + \varepsilon\varphi), \qquad \bar{L}'\varphi = -(\nabla_\beta \bar{\gamma}\gamma^\beta - \varepsilon\varphi).$$

On a immédiatement:

$$LL' = L'L = (\Delta - \varepsilon^2), \qquad \bar{L}\bar{L}' = \bar{L}'\bar{L} = (\Delta - \varepsilon^2).$$

De plus si l'intersection des supports de $\varphi$ et $\psi$ est compacte:

$$< \varphi, L\psi > = < \bar{L}\varphi, \psi >, \qquad < \varphi, L'\psi > = < \bar{L}'\varphi, \psi >.$$

Sur la variété $V_4$ de la relativité générale, le *champ de Dirac* (spin $\frac{1}{2}$) est décrit par un 1-spineur contravariant $\psi$ astreint à l'équation de champ

$$L\psi = 0. \tag{8–4}$$

Son adjoint de Dirac $\bar{\psi}$ vérifie:

$$\bar{L}\bar{\psi} = 0. \tag{8–5}$$

La formule

$$[\psi(x), \bar{\psi}(x')]_+ = \frac{\hbar}{i} S^{(\frac{1}{2})}(x, x') \tag{8–6}$$

où le propagateur spinoriel $S^{(\frac{1}{2})}$ relatif à $L$ est donné par:

$$S^{(\frac{1}{2})}(x, x') = L'_x G^{(\frac{1}{2})}(x, x') \tag{8–7}$$

nous fournit un anticommutateur rigoureusement compatible avec (8–4), (8–5), invariant par conjugaison de charge et se réduisant en relativité restreinte à l'anticommutateur usuel de la théorie du champ spinoriel libre.

c) Des résultas analogues peuvent être établis pour la théorie de tout champ physique libre en relativité générale. J'ai pu former par exemple, une généralisation de l'anticommutateur de Umezawa pour le champ de Rarita–Schwinger (spin 3/2) et j'ai développé dans le même cadre une généralisation de la théorie de Pétiau–Duffin–Kemmer [3].

## IV. REMARQUES SUR LES OPÉRATEURS DE CRÉATION-ANNIHILATION

### 9. Le noyau $G_1$

Pour simplifier les notations, je me limite ici au cas d'un champ scalaire réel astreint à l'équation de Klein–Gordon

$$(\Delta - \varepsilon^2)u = 0 \tag{9–1}$$

A l'opérateur $(\Delta - \varepsilon^2)$ supposons qu'on puisse associer un noyau réel $G_1(x, x')$, symétrique, solution de l'équation homogène (9–1) et vérifiant pour chaque hypersurface spatiale $\sigma$ la relation de composition

$$G(x, x') = \int_\sigma \{G_1(x, y')\partial_\lambda G_1(x', y) - G^1(x', y)\,\partial_\lambda G_1(x, y)\} d\sigma_4^\lambda \tag{9–2}$$

où $G$ est le propagateur relatif à $(\Delta - \varepsilon^2)$. Dans un espace-temps plat, $D$ et $D_1$ sont liés par (9–2); cette relation, jointe à une condition de norme positive, peux même caractériser $D_1$.

Pour construire $G_1$, il peut être commode de procéder de la manière suivante. Dans un espace hyperbolique de dimension impaire $n = 5$, on peut déterminer un noyau de la forme $U\Omega^{-\frac{3}{2}}$ (notations de Synge et de Guelfand et de Chilov [1]) symétrique, solution de l'équation homogène et dont le support est dans l'espace. Ce noyau $G_1$ est invariant par isométrie et, sous des conditions asymptotiques convenables, satisfait (9–2). On passe ensuite par descente de la dimension 5 à la dimension 4. La théorie mathématique de $G_1$ demande encore à être approfondie.

**10.** Les projecteurs $\oplus$ et $\ominus$

D'après la formule de résolution du problème de Cauchy, $u$ vérifie:

$$u(x) = \int_\sigma \{u(y)\partial_\lambda G(x, y) - G(x, y)\partial_\lambda u(y)\} d\sigma_y^\lambda. \tag{10–1}$$

A $u$ associons la solution $u_1$ de (9–1) définie par la relation de composition

$$u_1(x) = \int_\sigma \{u(y)\partial_\lambda G_1(x, y) - G_1(x, y)\partial_\lambda u(y)\} d\sigma_y^\lambda. \tag{10–2}$$

Nous posons:

$$2G^\oplus = G - iG_1, \quad 2G^\ominus = G + iG_1,$$

et

$$2u^\oplus = u - iu_1, \quad 2u^\ominus = u + iu_1.$$

La formule fondamentale (9–2) entraîne les résultats suivants:

1° $\oplus$ et $\ominus$ sont des projecteurs

$$(u^\oplus)^\oplus = u^\oplus, \quad (u^\ominus)^\ominus = u^\ominus \tag{10–3}$$

avec

$$(u^\oplus)^\ominus = (u^\ominus)^\oplus = 0. \tag{10–4}$$

2° La norme $\{u, u\}$ peut être définie par:

$$\{u, u\} = \int_\sigma (u(y)\partial_\lambda u_1(y) - u_1(y)\partial_\lambda u(y)) d\sigma_y^\lambda \tag{10–5}$$

avec

$$\{u^\oplus, u^\oplus\} = \{u^\ominus, u^\ominus\} = \tfrac{1}{2}\{u, u\}.$$

3° Du commutateur

$$[u(x), u(x')]_{-} = \frac{\hbar}{i} G(x, x')$$

il résulte:

$$[u^{\oplus}(x), u^{\oplus}(x')]_{-} = [u^{\ominus}(x), u^{\ominus}(x')]_{-} = 0 \qquad (10\text{–}6)$$

et

$$[u^{\oplus}(x), u^{\ominus}(x')]_{-} = \frac{\hbar}{i} G^{\oplus}(x, x'), \qquad [u^{\ominus}(x), u^{\oplus}(x')] = \frac{\hbar}{i} G^{\ominus}(x, x'). \qquad (10\text{–}7)$$

Les projecteurs $\oplus$ et $\ominus$ correspondent exactement au partage en parties de fréquences positives ou négatives du cas plat. Ainsi la théorie usuelle des opérations de création-annihilation peut être dérivée de la seule formule (9–2).

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# WAVES, NEWTONIAN FIELDS, AND COORDINATE FUNCTIONS*

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ABSTRACT—Three topics are discussed concerning fields in the neighborhood of infinity in asymptotically flat spaces: 1) the wave-front theorem which shows that the flux of energy decreases faster than $1/r^3$ on any $t =$ const. surface of an asymptotically rectangular coordinate system in an asymptotically flat space; 2) the identification of total energy-momentum $P^\mu$ and the coordinate invariant $P^\mu/r$ asymptotic behaviour of specific "Newtonian" components of the metric; 3) the definition of "gauge scalar" wave amplitudes which describe graviation radiation escaping to infinity, (or short waves in any weak field region). In addition there is brief mention of quantum theory and some indications that space-like surfaces might exist in a quantized geometry.

THE title of this lecture describes three significantly different aspects of the metric field: Newtonian (Coulomb-like) gravitational fields which are commonly measured; gravitational waves which have not yet been observed; and coordinate functions whose measurement is without interest until correlated with other observations. In the important limiting case of weak fields, there is a unique natural way to split the metric into wave, Newtonian and coordinate components. We shall be primarily concerned here with situations where weak fields in some part of space allow this natural decomposition to be used there. Within strong field regions the decomposition can be performed in many well defined but arbitrary ways, any of which will then provide a means of using physical intuition and electromagnetic analogies as a guide in studying the Einstein equations.

I will discuss three questions from this point of view. The first is the "wave front theorem" which states that at any fixed time there are no waves at spatial infinity, only Newtonian fields and (if desired) coordinate waves. The physical reason why the Einstein equations require this behaviour in asymptotically flat space is that a wave extending to infinity (amplitude $1/r$ on a $t =$ const. surface) would have infinite energy, preventing an asymptotically flat behaviour of the Newtonian components. This introduces, then,

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the second question I shall discuss, namely identifying as total energy and momentum the coefficients of the $1/r$ terms in the Newtonian fields beyond the wave-front. In particular we will see that only in exceptional circumstances is $g_{00}$ related to the energy; an asymptotically *coordinate independent* Newtonian field is obtained from the spatial components of the metric which appear in the standard (Einstein–von Freud, Landau–Lifshitz, Papapetrou–Gupta) surface integrals for total energy. We pick out four specific "components" of the metric whose leading asymptotic behaviour is $P^\mu/r$. The energy-momentum $P^\mu$ can be read off from these Newtonian fields under less restrictive coordinate conditions than are required for the convergence (i.e. surface independence in the limit of infinitely large surfaces) of the surface integrals. The third topic I consider is gravitational radiation, and the main point is to see that since waves get weak as they escape to infinity, the emitted radiation can be described by amplitudes which are *scalars* except under homogeneous Lorentz transformations.

The methods I use to treat each of the topics just mentioned are an outgrowth of the cannonical (Hamiltonian) forms of general relativity developed by Arnowitt, Deser, and myself [1]. These cannonical forms also serve to illustrate how separations of the gravitational field into waves, Newtonian fields, and coordinates can be arranged. A cannonical form means a rewrite of the Einstein equations which has the form

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}. \tag{1}$$

These Hamiltonian equations contain no constraint equations, $f(p, q) = 0$. Those components of the metric which, by solving the Einstein constraint equations, can be expressed as functionals of other components and thus eliminated in obtaining Eqs (1), we call Newtonian fields. The Hamilton equations also differ from the Einstein equations in having a *unique* solution for given initial conditions, and thus admit no effective gauge (coordinate) transformations. Some specific choice of coordinate functions achieves this. We regard the resulting coordinate functions as aspects (components) of the gravitational field, since their values at a given point will depend, via the coordinate conditions, on the geometry of space-time. The variables which remain after the Newtonian fields have been eliminated and the coordinates fixed, we call wave modes of the field, since they appear in the statement of the dynamics, Eqs (1).

The questions of wave-front, energy, and radiation will be treated using techniques which were used to obtain a canonical form for general relativity, however I shall actually make use of this canonical form only in some concluding remarks about quantization.

## THE WAVE FRONT

As is well known, the state of the gravitational field on a space-like hypersurface is determined by the spatial components of the metric $g_{ij}$ $(i, j = 1, 2, 3)$ and the second fundamental form $K_{ij}$ of this hypersurface. Once these data are given, a unique geometry satisfying the Einstein equations is determined. Thus it must be possible to see waves, or a wave front, by inspecting these quantities. The statement of the wave front theorem then is: *In an asymptotically flat space where*

$$g_{\mu\nu}-\eta_{\mu\nu} = O(1/r) \tag{2}$$

*holds on each fixed hypersurface $x^0 = t$, an appropriate choice of coordinates maintaining this asymptotic condition gives also, for each $t$,*

$$\begin{aligned} g_{ij,k} &= O\left(1/r^{\frac{3}{2}+\varepsilon}\right), \\ K_{ij} &= O\left(1/r^{\frac{3}{2}+\varepsilon}\right). \end{aligned} \tag{3}$$

The conclusion of this theorem cannot be true in every coordinate system satisfying Eq. (2), as is shown by considering a coordinate transformation. which reduces asymptotically to $\bar{x} = x+r^{-1}e^{ikr}$. But a true wave of the asymptotic form $r^{-1}e^{i(kr-\omega t)}$ is clearly excluded by this theorem, which states that any such behaviour can be eliminated by a coordinate transformation. Note also that this theorem makes it unreasonable to expect an exact solution showing a steady state motion for a two-body system. For the only plausible way to avoid radiative decay of the system is constantly to feed in energy in the form of radiation, and this steady flux of radiation (in, out, or both) is precisely what the theorem excludes.

The mathematical proof of the wave-front theorem parallels exactly the physical arguments. We look for an Einstein constraint equation which will relate the asymptotic behaviour of a Newtonian component to the energy in gravitational or other radiation. This is, of course, the $T^0_0$ equation, or more precisely the initial value equation[1] which reads

$${}^3R = K^2-K_{ij}K^{ij} = T_{**} = n^\mu T_{\mu\nu}n^\nu \tag{4}$$

where $n^\mu$ is the unit normal to the hypersurface. To simplify the presentation I first assume that coordinates can be introduced which satisfy

$$g_{ij,j} = 0, \quad K \equiv K^i{}_i = 0. \tag{5}$$

Then the pertinent Einstein equation reads

$$\begin{aligned} -\nabla^2 h_{ii} &\equiv -h_{ii,jj} \\ &= T^{00}+\frac{1}{4}(h_{ij,k})^2+\frac{1}{4}(h_{kk,i})^2+(K_{ij})^2+ \\ &\quad +\left\{\frac{1}{4}(h_{mn})^2{}_{,j}+\frac{1}{2}(h_{mi}h_{mj})_{,i}\right\}_{,j}+ \\ &\quad + \text{cubic terms} \end{aligned} \tag{6}$$

[1] See for instance reference [2] Eq. (225) or reference [3].

where $h_{ij} = g_{ij} - \delta_{ij}$. This equation we view as a Poisson equation, $-\nabla^2\Phi = 4\pi\varrho$, with $h_{ii}$ on the left viewed as a Newtonian potential, and the right-hand side as an effective energy density. The central physical idea of the argument is that any wave train of infinite energy will prevent the boundary condition $h_{ii} = O(1/r)$ from being satisfied. Mathematically, the formula

$$\Phi_0(r) \equiv \frac{1}{4\pi} \int_{r=\text{const.}} \Phi(x)\, d\Omega = \int_r^\infty \frac{1}{r^2} \left[ \int_0^r \varrho(\bar{x})\, d^3\bar{x} \right] dr \tag{7}$$

for the monopole (spherically symmetric) part $\Phi_0$ of a potential resulting from a source $\varrho$ in the Poisson equation shows that $\Phi = 0(1/r)$ requires the indefinite integral $\int_0^r \varrho d^3x$ be bounded. Thus the right hand side of Eq. (6) must have a finite integral. For the divergence term $\{\ \}_{,j}$ this is clear, since we have

$$\int \partial_j \{h_{..}\partial h_{..}\} d^3x = \int \{h_{..}\partial h_{..}\} dS_j . \tag{8}$$

This surface integral is finite as our asymptotic flatness condition (2) is to be understood as requiring not only $h_{ij}$, but all its derivatives as well, to be $0(1/r)$. By this condition, the cubic terms, which are of the form $h(\partial h)^2$ and $h(K_{..})^2$, can at most have a logarithmically divergent integral $\int r^{-3} d^3x$, but only if some component of $K_{ij}$ or $h_{ij,\kappa}$ did not vanish faster than $1/r$ at infinity. But then the leading positive definite, quadratic terms would dominate to give a linearly divergent integral of the right member of Eq. (6). Thus the cubic terms have a finite integral, so the remaining quadratic terms,

$$T^{00} + \tfrac{1}{4}(h_{ij,k})^2 + \tfrac{1}{4}(h_{kk,i})^2 + (K_{ij})^2$$

must also. Because of the positive definite character of these terms, the conclusion (3) of the wave front theorem follows, and also, by the finiteness of $\int T^{00} d^3x$, a similar $1/r^{\frac{3}{2}+\varepsilon}$ asymptotic behaviour of any other waves whose amplitudes enter $T^{00}$ in a positive definite quadratic form.

To complete the proof we need only eliminate the assumption that Eqs (5) can be satisfied. This is done by relaxing the coordinate conditions (5) to

$$g_{ij,j} = O\left(1/r^{\frac{3}{2}+\varepsilon}\right), \quad K = O\left(1/r^{\frac{3}{2}+\varepsilon}\right). \tag{9}$$

The existence of coordinates satisfying Eqs (9) is not an assumption; Arnowitt [4] in another lecture here will discuss how they can be constructed. Since Eqs (9) are quite moderate demands on the asymptotic behaviour of certain metric components, one can expect that a simple semi-linear approach to the coordinate transformation law, such as we have just now applied to a constraint, Eq. (6), would show that Eqs (9) can be satisfied. However, the $K=0$ condition cannot be satisfied in an arbitrary asymptotically flat space, as has previously been noted [5], and recourse to the Einstein equations is necessary to show that this difficulty does not occur in physical space-times.

The constraint equation (4) without coordinate conditions is

$$\begin{aligned}-\nabla^2 h^T = {} & T^{00}+\frac{1}{4}(h_{ij,k})^2+\frac{1}{4}(h_{kk,i})^2+(K_{ij})^2- \\ & -[K^2+\frac{1}{2}(h_{ik,k})^2+h_{ij,j}h_{mm,i}]+ \\ & +\Big\{-h_{ki,k}h_{ij}+\frac{1}{2}h_{kk}h_{ij,i}+ \\ & +\frac{1}{4}(h_{mm})^2_{,j}+\frac{1}{2}(h_{mi}h_{mj})_{,i}\Big\}_{,j}+ \\ & +\text{cubic terms} \end{aligned} \tag{10}$$

where $h^T$ is defined by

$$h^T = \frac{1}{\nabla^2}(h_{ii,jj}-h_{ij,ij}). \tag{11}$$

The conclusions of the wave-front theorem follow in Eq. (10) from a requirement that $h^T = O(1/r)$, discussed below, and the coordinate conditions (9). For the divergence terms $\{\ \}_{,j}$ in Eq. (10) have a finite integral as before; the negative terms in [ ] have a finite integral by Eqs (9); and the indefinite term $h_{ij,j}\, h_{mm,i}$ in [ ] would, unless $h_{mm,i} = O\left(1/r^{\frac{3}{2}+\varepsilon}\right)$ always be dominated by the positive term $(h_{kk,i})^2$. The finiteness of the integral of the right side of Eq. (10), then, does not arise by cancellations, so each term separately has a finite integral as was to be shown. The one point which remains, in establishing the wave-front theorem, is to show that $h^T = O(1/r)$ is a consequence of $h_{ij} = O(1/r)$. This is the content of Lemma 2 in the next section.

Papapetrou (2) [6] has obtained results related to the wave-front theorem given here.(3) He assumes that all matter and strong fields are restricted *for all time* to a region $r < R$ thus excluding, he notes, scattering situations. He then concludes on the basis of a perturbation argument that the metric in the region $r > R$ becomes time independent as $t \to \pm\infty$, i.e. that any radiation present is restricted to a pulse of effectively finite duration in time or space. That radiation can exist only in finite energy pulses in asymptotically

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(2) Conclusions similar to Papapetrou's were also obtained by A. Peres and N. Rosen, ref. [7]. An argument which shows that $K_{ij} = O\left(1/r^{\frac{3}{2}+\varepsilon}\right)$(as a time average of weak fields) can be found in ref. [5] Eq. (76) and in ref. [8], p. 367.

(3) The proof given here is a slight improvement over the argument given in ref. [9] and ref. [10] Eq. (2.7), which assumed coordinate conditions similar to Eq. (5).

flat spaces is not yet established. The difficult case is a source region which at $t = 0$ begins to emit radiation, and emits at a finite rate ever after; the source region must then give a more and more negative contribution to the total energy to compensate for the ever-increasing amount of radiation. The wave front theorem does not exclude this situation, and Papapetrou's work is not relevant since the unbounded radiation would eventually lead to strong fields at any fixed distance $R$ from the source. (In this connection, the open question [11] of the positive definite character of the total energy of a gravitational system is clearly relevant.) The important technical differences between the methods of proof used here and those used by Papapetrou are (A) by avoiding the de Donder coordinate condition, we obtain a Poisson equation (10) without recourse to time averaging and (B) in place of a perturbation series (where one must assume that a linearly divergent integral in one order is not cancelled by a sum of linearly divergent integrals from all higher orders) we need only the fact that the "cubic terms" in Eq. (10), ie. the difference between the full equation (4) and the terms written, has a finite integral. For this any non-singular behaviour of the metric in any bounded region is allowed, while the asymptotic behaviour can be studied in as much detail as desired if one has used, in obtaining Eq. (10), the formula

$$^3g^{ij} = \delta_{ij} - h_{ij} + h_{ik}h_{kl}\,^3g^{lj}.$$

## ENERGY

The quantity $h^T$ which played the role of a Newtonian potential in our proof of the wave-front theorem is perhaps unfamiliar. I want to show first that it really is familiar, since it appears in all the standard flux integrals for total energy, and secondly I want to explain why the strength of the $1/r$ term in $h^T$ gives precisely the total energy of the system, while the $1/r$ term in $g_{00}$ need not be related to energy. An essential part of this argument is the fact that $h^T$ is asymptotically coordinate invariant. This invariance is quite surprising since the definition of $h^T$ in Eq. (11) involves non-local and non-covariant operations. Nevertheless, we will see that the asymptotic form of $h^T$ depends only on the asymptotic form of $g_{ij}$, and not at all on the choice of asymptotically flat coordinates.

Since the right-hand side of Eq. (10) has a finite integral, it follows that $h^T$ does not merely vanish (as demanded) at infinity, but more specifically has the asymptotic form

$$h^T \sim E/4\pi r \tag{12}$$

where $E$ is independent of the space coordinates. This constant $E$ is therefore easily represented as a surface integral

$$E = \oint -h^T_{,i}\,dS_i. \tag{13}$$

Use of Gauss' theorem and the defining equation (11) gives a formula directly in terms of the metric

$$E = \int -h^T_{,ii} d^3x = \int (g_{ij,ij} - g_{ii,jj}) d^3x \tag{14}$$

and Gauss' theorem used again yields

$$E = \oint (g_{ij,j} - g_{jj,i}) dS_i . \tag{15}$$

These simple calculations show then that the asymptotic form of $h^T$ can be computed (by Eqs (12) and (15)) from that of $g_{ij}$; the apparent nonlocality of the definition (11) has essentially disappeared when we discuss only the asymptotic form of $h^T$. Our first interest in Eq. (15) however is to relate it to standard surface integrals for total energy. Following Landau–Lifshitz, the Einstein equations can be written in the form

$$H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} = T_{\text{eff}}{}^{\mu\nu} \tag{16}$$

where $H^{\mu\alpha\nu\beta}$ is a function of $g_{\mu\nu}$ with the symmetries of the Riemann tensor, thus guaranteeing that the effective stress tensor $T_{\text{eff}}{}^{\mu\nu}$ is symmetric and conserved, i.e. $T_{\text{eff}}{}^{\mu\nu}{}_{,\nu} = 0$. Integration yields a surface integral formula, since antisymmetry allows only a spatial divergence to appear

$$P^\mu \equiv \int T_{\text{eff}}{}^{o\mu} d^3x = \int H^{ok\mu\alpha}{}_{,\alpha k} d^3x$$
$$= \oint H^{ok\mu\alpha}{}_{,\alpha} dS_k . \tag{17}$$

I will discuss later why $P^\mu$ is the total energy-momentum when the integrals converge; we are now anxious to reproduce Eq. (15). The shortest formula for $H^{\mu\alpha\nu\beta}$ is that of Landau–Lifshitz:

$$H^{\mu\alpha\nu\beta} = \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta} - \mathfrak{g}^{\mu\beta}\mathfrak{g}^{\nu\alpha} \tag{18}$$

Here $\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$, and we will also write $\gamma^{\mu\nu} = \mathfrak{g}^{\mu\nu} - \eta^{\mu\nu}$). Let us assume[(4)] that all derivatives of the metric fall off faster than $1/r$ beyond the wave front, then a term $\gamma^{\cdot\cdot}\partial_\cdot\gamma^{\cdot\cdot}$ goes faster than $1/r^2$ and does not contribute to the surface integral (17). Thus this integral may be evaluated keeping only the terms in $H^{\mu\alpha\nu\beta}$ linear in $\gamma^{\mu\nu}$, i.e. using

$$H^{\mu\alpha\nu\beta} = \eta^{\mu\nu}\mathfrak{g}^{\alpha\beta} + \mathfrak{g}^{\mu\nu}\eta^{\alpha\beta} -$$
$$-\eta^{\mu\beta}\mathfrak{g}^{\nu\alpha} - \mathfrak{g}^{\mu\beta}\eta^{\nu\alpha}. \tag{19}$$

This gives the Papapetrou [12] form of the surface integrals. Evidently, since quadratic terms play no role in the surface integral, many other forms are also available. In particular, expressing $\gamma^{\mu\nu}$ in terms of $h_{\mu\nu} = g_{\mu\nu} - \eta^{\mu\nu}$ and

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(4) A slightly stronger form of the wave-front theorem than was proven here shows that it is always possible to achieve $\partial_\cdot g_{\cdot\cdot} = O(1/r^{1+\varepsilon})$ by a coordinate transformation. See ref. [4].

discarding the quadratic terms leads to the form (15) for $E = P^0$. The von Freud surface integral associated with the Einstein pseudo-tensor similarly reduces to Eq. (15) when terms quadratic in $h_{\alpha\nu}$ are discarded. The coordinate independence of these surface integrals is shown [10] as easily as their equality has been, again with the coordinate conditions ${}^{(4)}\partial.\ g.. = O\ (1/r^{1+\varepsilon})$. I will turn, however, to a more powerful method of approaching these problems, which focusses on the asymptotic form, $E/r$, of $h^T$ rather than upon surface integrals involving $h^T_{,i}$.

The definition (11) of $h^T$ shows that it is constructed from $h_{ij}$ with the aid of the operator $\nabla^{-2}\partial_i\partial_j$. In spite of the fact that this operator employs ordinary (non-covariant) derivatives and involves all space, not just asymptotic weak field regions, in requiring the solution of a Poisson equation, it is an ideal tool for the study of asymptotic fields as the following important lemmas show [9].

Lemma 1. *Let $f$ be a non-singular function which vanishes at infinity. Then the equation $\nabla^2\Phi = \partial_i\partial_j f \equiv f_{,ij}$ has a unique solution $\Phi$ which vanishes at infinity.* Thus the operator $\nabla^{-2}\partial_i\partial_j$ can be applied to functions which vanish so slowly at infinity that the operator $\nabla^{-2}$ cannot be applied.

Lemma 2. *If $f = O(1/r^\alpha)$, then $\nabla^{-2}\partial_i\partial_j f = O(1/r^\alpha)$ for $\alpha < 3$.* This second lemma shows that the behaviour of $f$ in any bounded region has no influence on the leading asymptotic behaviour of $\nabla^{-2}\partial_i\partial_j f$. For suppose $f$ and $\bar{f}$ agreed asymptotically, i.e. $f - \bar{f} = O(1/r^3)$, then $\nabla^{-2}\partial_i\partial_j f$ and $\nabla^{-2}\partial_i\partial_j \bar{f}$ also only differ by $O(1/r^3)$. The non-local aspect of $\nabla^{-2}$ is therefore harmless when we use the operator $\nabla^{-2}\partial_i\partial_j$ to discuss asymptotic fields which vanish more slowly than $1/r^3$.

Suppose we wished merely to maintain $O(1/r)$ boundary conditions on $h_{\mu\nu}$ and $g_{\mu\nu,\alpha}$. Then (excluding homogeneous Lorentz transformations) in a coordinate transformation $x^\mu = \bar{x}^\mu + \xi^\mu$, we would require $\xi^\mu{}_{,\nu} = O(1/r)$. The change in $h_{ij}$ this produces is

$$\bar{h}_{ij} = h_{ij} + \xi^i{}_{,j} + \xi^j{}_{,i} + h_{ij,\mu}\xi^\mu + O(1/r^2). \tag{20}$$

Computing $\bar{h}_{ij,ij} - \bar{h}_{ii,jj}$ we see that the linear terms $\xi^i{}_{,j}$ cancel, hence, do not contribute to $\bar{h}^T$[cf Eq. (11)]. But these are the only $O(1/r)$ terms in Eq. (20) arising from the coordinate transformation, for although $\xi^\mu = O(\ln r)$ is possible, we may assume for some frames that $h_{ij,\mu}$ vanishes faster than $1/r$. Consequently we have shown that if $h^T$ is computed in such a coordinate system, and $\bar{h}^T$ in any coordinate system preserving $\bar{h}_{\mu\nu} = O(1/r)$, then,

$$\bar{h}^T = h^T + O(1/r^{1+\varepsilon}). \tag{21}$$

In particular, *the $1/r$ term in $h^T$ is coordinate invariant.*

A formula like (20) for $g_{00}$,

$$\bar{h}_{00} = h_{00} - 2\xi^0{}_{,0} + h_{00,\mu}\xi^\mu + O(1/r^2). \tag{22}$$

shows that the $1/r$ term in $g_{00}$ is not coordinate independent. Thus our preference for thinking of $h^T$ as the Newtonian potential. A strong set of conditions under which we can clearly identify the Newtonian potential and the total mass at the system in its leading $m/r$ term are that Newtonian gravitational theory be valid—both the Poisson equation for $\Phi$ and Newton's law of motion. For Newton's law of motion this means a slowly moving particle in a weak gravitational field, conditions which reduce the geodesic equation to

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}h_{00,i} - h_{0i,0}. \tag{23}$$

For the retardationless Poisson equation to be valid, we must be able to imagine that the sources of the field can be adequately represented by a phenomenological description involving only non-relativistic velocities, $v/c \ll 1$. The metric can then be required to satisfy asymptotically

$$\partial g_{..}/\partial t \ll \partial g_{..}/\partial x^i. \tag{24}$$

Under these conditions, the metric will be Schwarzschild in its leading terms, and one can calculate that $\frac{1}{2}h_{00}$ and $\frac{1}{4}h^T$ agree asymptotically. Equation (23), now without the $h_{0i,0}$ term, thus establishes $h_{00}$ as a Newtonian $m/r$ potential under these conditions, and therefore $h^T$ also. But $h^T$, as we have seen, will continue to yield the same value of $m$ in coordinate systems which hide the Newtonian conditions. Furthermore, the obviously Lorentz invariant flux integrals of Eqs (17), (19) give a four-vector whose time component $P^0$ is, we have now seen, the total mass when velocities and time derivatives can be neglected. This establishes $P^\mu$ unambiguously as the total energy-momentum of the system. The danger of using formulas based on $g_{00}$ to obtain the energy of a system not in a center of mass coordinate system was first pointed out by Møller [13]. (see aso ref. [19]: Conditions under which this is possible are given by Arnowitt [3], who also shows that it is possible, for any asymptotically flat metric, to introduce coordinates such that slow test particles at infinity satisfy Newton's equation of motion.

In addition to this Newtonian argument identifying the surface integrals (17) with energy-momentum, there are of course other conclusive arguments based on the fact that $T_{\text{eff}}{}^{\mu\nu}$ reduces in flat space to $T^{\mu\nu}$. The classical argument notices that $\int T_{\text{eff}}{}^{0\mu} d^3x$ is a constant of motion which is energy-momentum in all those cases where we know what energy-momentum means without recourse to a theory of gravity, namely when the dynamic evolution of the system carries it through an epoch free from gravitational fields. More precisely, if at any time

$$\int \sum_{\mu\nu a} |g_{\mu\nu,a}|^2 d^3x \ll \int T^{00} d^3x \tag{25}$$

so that gravitational contributions to the energy should be negligible by any definition, then the constant $\int T_{\text{eff}}^{0\mu} d^3x$ is equal to the value of $\int T^{0\mu} d^3x$ at the gravitation-free time. By this method, considering familiar forms of energy, and processes by which they may be converted into new or unfamiliar forms, *vis viva* evolved into $T^{\mu\nu}$. It is, however, a much more difficult method than necessary in a gravitational theory. Here we should take charge, rather than energy, as our model. At present we do not know whether there is any non-trivial special relativistic quantum field theory which contains a current density $j^\mu$ such that $\int j^0 d^3x$ exists in a one proton state. This fact is recognized as irrelevant to the question of whether we can define the charge of a proton, since the surface integral $\int F^{0i} dS_i$ at infinity suffices and eliminates quantum effects as well as questions about the structure of the proton. One should adopt the same viewpoint when faced with a situation involving intense localized gravitational fields in asymptotically flat space. Rather than jumping in to a study of the structure and dynamics of the intense fields, one should pretend the intense fields were hidden in a deep fog. In the apparent absence of strong fields, then, one could apply Eq. (16) in linearized form where $T_{\text{eff}}^{\mu\nu} = T^{\mu\nu}$ and identify the surface integrals (17) with the total energy-momentum of the foggy region. Ignorance of the details of $T^{\mu\nu}$ did not affect the argument, and in fact the equation $\int T^{0\mu} d^3x = \int H^{0k\mu\alpha}{}_{,\alpha} dS_k$ is the first condition one would impose on any phenomenological $T^{\mu\nu}$ proposed as a description of the foggy region. The contrary assertion, that one must be sure no strong fields exist in the central regions before interpreting the surface integrals as indicated by linearized theory, means that we cannot define the mass of a lead ball because of the possibility that strong gravitational fields occur at $10^{-33}$ cm in its constituent nucleons.

In a relativistic theory the conserved mass of a Newtonian particle is replaced by the conserved energy-momentum of a relativistic object. Correspondingly, there is not just one $m/r$ Newtonian field, but four $P^\mu/r$ "Newtonian" fields. We have seen in some detail how the Newtonian field $h^T \sim \sim P^0/4\pi r$ enters the standard surface integrals for energy. The other Newtonian fields enter the surface integrals for momentum, which can be reduced to the form

$$P^i = -2 \oint (\Gamma^0_{ij} - \delta_{ij} \Gamma^0_{kk}) dS_j = -2 \oint \pi^{ij} dS_j \tag{25}$$

when the absence of coordinate waves ($r\partial g_{..} \to O$) allows us to neglect quadratic terms. Here

$$\pi^{ij} \equiv -({}^3g)^{\frac{1}{2}} (K^{ij} - {}^3g^{ij} K) \tag{27}$$

are the Dirac [14] momenta, components of a three-dimensional tensor density closely related to the second fundamental form $K_{ij}$. Isolation of coordinate invariant Newtonian fields from $\pi^{ij}$, whose transformation law is

$$\bar{\pi}^{ij} = \pi^{ij} + (\xi^0{}_{,ij} - \delta_{ij} \xi^0{}_{,kk}) + \pi^{ij}{}_{,\mu} \xi^\mu + O(1/r^2) \tag{28}$$

again makes use of the "asmyptotically local" operation $\nabla^{-2}\partial_i\partial_j$ to project out the linear coordinate transformation components. These techniques can be summarized by the decomposition

$$f_{ij} = f_{ij}{}^{TT} + f_{ij}{}^{T} + (f_{i,j} + f_{j,i}) \tag{29}$$

defined for any symmetric metrix field by

$$\begin{aligned} f_{i,j} &= \nabla^{-2}[f_{ik,kj} - \tfrac{1}{2}(\nabla^{-2}f_{lm,lm})_{,ij}] \\ f_{ij}{}^{T} &= \tfrac{1}{2}(\delta_{ij}f^{T} - \nabla^{-2}f^{T}{}_{,ij}) \\ f^{T} &= \nabla^{-2}(f_{ll,mm} - f_{lm,lm}), \\ f_{ij}{}^{TT} &= f_{ij} - f_{ij}{}^{T} - (f_{i,j} + f_{j,i}). \end{aligned} \tag{30}$$

Thus only the "trace of the transverse part" $\pi^T$ in $\pi^{ij}$ is affected by linear coordinate transformation terms, and only the "longitudinal part" $h_i$ of $h_{ij}$. Just as the asymptotic form $h^T \sim E/4\pi r$ was shown coordinate invariant, so for the other Newtonian fields, which are the longitudinal parts $\pi^i$ (since the transverse parts $\pi^{ijTT}$ and $\pi^{ijT}$ do not contribute to $\oint \pi^{ij} dS_j = \int \pi^{ij}{}_{,j} d^3x$), the terms $P^i/r$ can also be shown coordinate invariant. The "transverse traceless" parts $\pi^{ijTT}$ and $h_{ij}{}^{TT}$ are also obviously coordinate invariant in order $1/r$, but they represent waves as we shall see in the next section, and therefore vanish rapidly beyond the wave front.

The $P^\mu/r$ asymptotic behaviour of the Newtonian fields can be used as a method of computing $P^\mu$ even in the presence of coordinate waves, i.e. when $\partial_{.} g_{..} = O(1/r)$ at constant $t$. Under these conditions the surface integrals for $P^\mu$ do not generally converge, but undergo bounded oscillations as the surface tends to infinity. The Lorentz 4-vector character of $P^\mu$ can also be shown by studying the asymptotic Newtonian fields instead of the surface integrals.

## RADIATION

By gravitational radiation I mean physical gravitational waves escaping to infinity in the course of time. I will not try to analyse the sources of such radiation, but I will show how a gravitational disturbance behaves once it is found in a nearly flat region. The quantity $h_{ij}^{TT}$ turns out be a wave amplitude which is a scalar except under homogeneous Lorentz transformations. We have already seen that this is true formally, and it is also a simple calculation to see that the linearized Einstein equations imply that $\Box\, h_{ij}^{TT}$ vanishes in source free regions. Thus, the central question is the relevance of these formal computations. The Lemmas about the asymptotic character of $\nabla^{-2}\partial_i\partial_j$ do not suffice to define $h_{ij}^{TT}$ in an interesting region, since the wave front theorem tells us that $h_{ij}^{TT}$ vanishes asymptotically (i.e. for $r\to\infty$ with $t$=const.). Consequently we must begin by studying another way to make the decomposition (30) well defined.

The second way of defining $\nabla^{-2}\partial_i\partial_j$ as an effectively local operator is, curiously, to regard it as the operator $k^{-2}k_ik_j$ in momentum space; the locality is achieved by concentrating our attention on only that aspect of the metric for which this operation is local, namely the short wavelength part. We begin by restating locality in terms of Fourier transforms. Define $h_{\mu\nu}$ by

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} \tag{31}$$

and then assume that in a region of linear dimensions $L$ we have

$$h_{..}\ll 1. \tag{32}$$

To ignore the behaviour of a function $f(\boldsymbol{x})\equiv f(x,y,z)$ outside this nearly flat region $|\boldsymbol{x}|<L$ we multiply it by a function which goes from the value 1 for $|\boldsymbol{x}|<L$ to zero for $|\boldsymbol{x}|>L$, e.g.

$$f(\boldsymbol{x})\to\exp\left\{-\frac{1}{2}|\boldsymbol{x}|^2/L^2\right\}f(\boldsymbol{x}). \tag{33}$$

This modifies its Fourier transform $\tilde{f}(\boldsymbol{k})$ by smoothing it with a resolution $1/L$:

$$\tilde{f}(\boldsymbol{k})\to\int\tilde{f}(\boldsymbol{k}-\boldsymbol{q})\,(2/\pi)^{3/2}L^3\exp\left\{-\frac{1}{2}L^2|\boldsymbol{q}|^2\right\}d^3q. \tag{34}$$

Thus, precisely equivalent to ignoring the behaviour of $f(\boldsymbol{x})$ or $h_{\mu\nu}$ for $|\boldsymbol{x}|>L$ is ignoring fine structure in its Fourier transform on a scale $\Delta k\approx L^{-1}$. This is, of course, just the basic uncertainty principle of Fourier analysis, $\Delta k\Delta x\approx 1$. Our purpose in Fourier transforming the idea of locality was to discuss the locality of the operator $\nabla^{-2}\partial_i\partial_j$ whose Fourier transform is multiplication by $k^{-2}k_ik_j$. Locality of $\nabla^{-2}\partial_i\partial_j$ becomes a question of how $k^{-2}k_ik_j$ survives a smoothing process like (34). The effect of this smoothing on $k^{-2}k_ik_j$ can be described by a factor $[1+O\,(1/k^2L^2)]$, and is negligible if and only if

$$kL\gg 1, \tag{35}$$

i.e. for short wavelengths. Consequently, *the short wavelength part of* $h_{ij}^{TT}$ *for* $|\boldsymbol{x}|<L$ *is independent of the behaviour of* $h_{ij}$ *for* $|\boldsymbol{x}|>L$. Similar statements hold, of course, for all other terms in the decomposition (29) since they are constructed using only $\nabla^{-2}\partial_i\partial_j$.

Continuing to assume that Eq. (32), and in addition

$$(\partial_. h_{..})^2\ll\partial_.\partial_. h_{..}, \tag{36}$$

hold in a given region, we see immediately that the terms in the curvature tensor which are non-linear in $h_{\mu\nu}$ are, in this region, negligible in comparison to linear terms. (The condition (36) requires, effectively, that $h_{\mu\nu}$ not contain arbitrarily high wave number components of significant amplitude.) [9] There are now two main questions: First, what can we say about the behaviour of a field $h_{\mu\nu}$ which in a limited region (only) satisfies the linearized Einstein equations? Second, how closely will a solution of the linearized Einstein equations approximate a solution of the Einstein equations? This first question we answer by reducing these linearized equations to a more familiar form. Since the equations are linear, they hold independently for the various Fourier components of the field. Restricting ourselves to the short wavelength components satisfying (35), we may then apply to the spatial components of the equations the operator of Eq. (30) which forms the "$TT$ part". There results

$$\Box h_{ij}{}^{TT} \equiv -h_{ij,}{}^{TT}{}_{00} + h_{ij,}{}^{TT}{}_{ll} = 0, \qquad (37)$$

i.e. the shortwave part of $h_{ij}{}^{TT}$ satisfies, in the given region, the simple flat-space wave equation. Therefore we know, in terms of propagation along light cones (coordinate light cones), what the validity of Eq. (37) in a limited region allows us to conclude about the behaviour of its solution. In particular, knowledge of $h_{ij}{}^{TT}$ and $\partial h_{ij}{}^{TT}/\partial t$ at one time determines the solution uniquely in a cone based on that hypersurface, or a pulse seen entering the region on one side can be followed until it leaves the region, or interference between beams of shortwave $h_{ij}^{TT}$ entering the region can be discussed, etc. Other shortwave parts of the locally linearized Einstein equations can be similarly discussed [9]. They show $\pi^{ijTT} = \frac{1}{2}\,\partial h_{ij}{}^{TT}/\partial t$ and that the Newtonian fields $h^T$ and $\pi^i$ have no shortwave components, i.e. the Newtonian fields are slowly varying throughout the nearly flat region. The shortwave parts of $h_i$ and $\pi^T$ are arbitrary, but determine the shortwave parts of $h_{0\mu}$.

Because shortwave Newtonian fields are killed by the source-free Einstein equations, one readily computes [9] that a knowledge of shortwave $h_{ij}{}^{TT}$ is equivalent to a knowledge of the shortwave part of $R_{\mu\nu\alpha\beta}$.

For the second question above, we will estimate the difference between a solution of the linearized equations and of the Einstein equations by thinking of the quadratic terms as a source in a linear equation satisfied by the difference of the solutions. For instance, the constraint equation (10) reads $\nabla^2 h^T = (\partial h)^2 + \partial^2 hh \approx k^2h^2$ and, for the shortwave part of $h^T$ (where $\nabla^{-2}$ is a local operation) gives $h^T \approx h^2$ or by Eq. (32) then, $h^T \ll h$. That is, the shortwave part of $h^T$ produced by quadratic terms in the Einstein equations is negligible in comparison to typical components of $h_{\mu\nu}$. The same is true for the other Newtonian fields $\pi^i$. However $h_{ij}{}^{TT}$ satisfies a dynamic equation, $\Box h_{ij}^{TT} \approx \partial^2 h^2$ and errors accumulate when the source is resonant. For wave-number $k$, assuming a resonant source $\partial^2 h^2 \approx |k^2h^2|e^{ikt}$, we see from the equation $\ddot{h}^{TT} + k^2h^{TT} = |k^2h^2|e^{itk}$ that this source produces a contribution $h^{TT} \approx kth^2$. Thus, the time $t$ or distance $l \approx t$ over which the linear solution is accurate is restricted by

$$h_{..}(kl) < 1. \tag{38}$$

By Eq. (32), then, the number of wavelengths $N = kl$ for which a solution of the linear equation (37) remains accurate is very large, but because short-waves are our principle interest (35), the solution need not be usable over the whole range $L$ of weak fields. The available range of many wavelengths is, however, sufficient to discuss characteristic wave phenomena such as group velocity or interference, or to discuss detection by low density absorbers.

Let us now establish that $h_{ij}{}^{TT}$ transforms as a scalar when homogeneous Lorentz transformations are excluded. The class of coordinate transformations to be considered is characterized by

$$\xi^{\mu}{}_{,\nu} \ll 1 \tag{39}$$

when the transformation is written $x^{\mu} = \bar{x}^{\mu} + \xi^{\mu}$. In addition to excluding homogeneous Lorentz transformations, Eq. (39) is also necessary in order to preserve the coordinate conditions (32) which have been an essential part of our treatment of weak field regions. By Eqs (39) and (32) we can neglect quadratic terms $h(\partial\xi)$ and $(\partial\xi)^2$ in the transformation law which is then

$$\bar{h}_{ij} = h_{ij} + \xi^{i}{}_{,j} + \xi^{j}{}_{,i}. \tag{40}$$

The two sides of this equation are evaluated at the same point, so no transport term appears. (By contrast, Eqs (29) and (28) compare the same coordinate values in each system, and we had to show then that the transport term was small.) Since derivatives in the two coordinate systems differ negligibly, by $\bar{\partial}_{\mu} = (\delta^{\nu}_{\mu} + \xi^{\nu}{}_{,\mu})\partial_{\nu}$ we can form the shortwave $TT$ part of each side of Eq. (40) using the operator $\nabla^{-2}\partial_i\partial_j$ appropriate to each coordinate system and find

$$\overline{h_{ij}{}^{TT}} = h_{ij}{}^{TT} \tag{41}$$

as was to be shown. The shortwave parts of $\pi^{ijTT}$, $h^{T}$, and $\pi^{i}$ are similarly scalars.

The most important application of the preceding results occurs when the nearly flat region in question is a neighborhood of infinity in an asymptotically flat space. Then the linearizing conditions (32) and (36) will be valid for sufficiently large distances and from a central region which might include strong fields. The rearrangement of the linearized equations into a wave equation (37) is valid for short waves, in this case for all $k$ satisfying

$$kr \gg 1. \tag{42}$$

At sufficiently large distances, then, the propagation of arbitrarily long waves is described by the wave equation (37). Similarly, the "shortwave" part of

$h_{ij}{}^{TT}$ may include waves up to any desired wave length, and it is still, for sufficiently large $r$, a scalar against coordinate transformations with $\xi^{\mu}{}_{,\nu} = O(1/r)$. The Riemann tensor computed from $h_{ij}^{TT}$ is of Petrov type N (or [4])[15] for purely outgoing waves [9], or more generally whenever the propagation 4-vectors of all Fourier components of $h_{ij}{}^{TT}$ are parallel.

The flux of energy in the form of gravitational waves escaping to infinity can be the Poynting flux [9]

$$\mathcal{T}_{\text{grav}}{}^{io} = \pi^{lmTT}(2h_{il}{}^{TT}{}_{,m} - h_{lm}{}^{TT}{}_{,i})$$
$$= 2\pi^{lm}\Gamma_l{}^i{}_m. \tag{43}$$

The total energy flux in all forms can be measured by

$$\mathcal{T}_{\text{tot}}{}^{io} = -2\pi^{ij}{}_{,j}. \tag{44}$$

The equalities here hold only in the sense of a space-time average over the longest wave lengths and periods which it is desired to include, but this is no restriction since any absorbing medium in which a wave would damp out in a fraction of a wave length or period would be perfectly reflecting.

## QUANTIZATION

In strong fields, the Newtonian and wave components of the metric no longer possess the coordinate invariance which made them natural quantities for asymptotic or weak field situations. By use of coordinate conditions, however, various unambiguous definitions can be given for $h_{ij}{}^{TT}$, etc. This procedure can be used to put the Einstein equations in the canonical form [1] of Eqs (1). Although the method of approach is different, the resulting Hamiltonian formalism appears to be essentially equivalent to that given previously by Dirac [16].(5) It has the advantage, however, of having led to explicit sets of canonical variables so that an iteration expansion is needed only to construct the Hamiltonian, but not for all Poisson brackets. For instance, if $\pi^T = 0 = h_i$ are imposed as coordinate conditions, then $h_{ij}^{TT}$ and $\pi^{ijTT}$ become canonically conjugate variables.

To formulate reasonably a quantum theory on the basis of this canonical formalism would require that the reduction to canonical form be repeated with the factor orderings of all operators specified. This has not been done, however a "cheap quantization" is immediately possible which allows some calculations to be performed in low orders of perturbation theory without specifying yet how other problems might be formulated. In this approach one

(5) The most complete discussion of the relationship between these two methods has been given by J. L. Anderson (to be published); some remarks can be found in ref. [17]. In. ref. [20] one finds the canonical variables and coordinate conditions introduced by Arnowitt, Deser and Misner used in a theory emploing Dirac's generalized Hamiltonian formalism.

adopts the canonical commutation rules for $h_{ij}{}^{TT}$ and $\pi^{ijTT}$ on a $t =$ const. surface. This allows a representation (at that one time)

$$h_{ij}{}^{TT} = (2\pi)^{-\frac{3}{2}} \int d^3k \, (2k)^{-\frac{1}{2}} \{a_s(\boldsymbol{k}) e^s_{ij}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + h.c.\}, \tag{45}$$

and correspondingly for $\pi^{ijTT}$, where the operators $a_s(\boldsymbol{k})$ satisfy

$$[a_s(\boldsymbol{k}), a_\sigma^+(\boldsymbol{q})] = \delta_{s\sigma}\delta^3(\boldsymbol{k}-\boldsymbol{q}) \tag{46}$$

and the remaining familiar commutation rules for annihilation and creation operators. The polarization tensors might be taken as

$$e^{\pm}_{ij} = \frac{1}{2}\begin{pmatrix} 1 & \pm i & 0 \\ \pm i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{47}$$

for $\boldsymbol{k}$ in the $z$-direction. Baierlein [18] has used this method for a look at the question of the existence of space-like surface in quantized relativity. One first notes that a related question can be formulated, namely, is $t =$ const. a space-like surface in virtue of the coordinate conditions employed in this quantization! Or more specifically, with $g_{ij}$ the functional of $g_{ij}{}^{TT}$ and $\pi^{ijTT}$ given by a canonical formalism, and these operators defined by accepting the standard representation of the commutation rules (46), is $v^i g_{ij} v^j$ a positive definite operator in Hilbert space for each non-zero c-number $v^i$? Although the linear terms in $g_{ij}$ have an indefinite structure (47), Baierlein finds that the form of the quadratic terms suggests a positive definite character. In particular the classical expression for $g_{ij}$ in terms of the canonical variables used by Baierlein, through quadratic order, is a positive definite matrix for *all* values of the canonical variables.

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## DISCUSSION

J. L. Anderson:

The one thing that bothers me slightly is that these $g_{TT}$'s are highly non-local, and while you always seem to work way out at spatial infinity, they must somehow carry all the information about the "glop" that's going on inside.

C. W. Misner:

If you don't look at the details of the $g_{TT}$'s; if you are only interested in the leading asymptotic order—$1/r$, say (or beyond the wave front—$1/r^2$) then that behaviour, as long as it's not falling off faster than $1/r^3$, is independent of the interior, in spite of the non-locality. This is the crucial theorem, that the operator which forms $g^{TT}{}_{ij}$ out of $g_{ij}$ is effectively a local operator asymptotically. It's not quite a local operator but it maps the whole asymptotic region into itself. (See Lemma 2 in the published text.)

# THE QUANTUM THEORY OF GRAVITATION*

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THE quantum theory of gravitation is studied by seeing what difficulties arise if one actually tries to solve specific problems in perturbation theory to increasing orders of accuracy. We specifically keep all energies involved below some upper limit (e.g. $10^3$ GeV), and do not analyze the philosophical consequences of a quantized metric, etc. Instead, we begin by writing the classical Lagrangian for a matter field (most calculations were made for spin zero particles) in interaction with a gravitational field represented by a metric $g_{\mu\nu}$. By writing $g_{\mu\nu} = \delta_{\mu\nu}+h_{\mu\nu}$ and expanding in terms of $h_{\mu\nu}$, which we consider as the potential of the spin-two gravitational field, we find a Lagrangian involving $h$ in second order (which we call the free Lagrangian) plus third, forth etc. orders in $h$. These latter terms can be considered as interaction terms where three, four, etc. gravitations interact at a point.

Because of the invariance of the complete Lagrangian under coordinate transformations, the free wave equation is singular, and can only be solved if some gauge condition is chosen, analogous to the choice of the Lorentz gauge condition in electrodynamics. Then diagrams can be made for each process in any order and the results calculated in the usual way.

For any process in the lowest order in which it can occur, all virtual momenta are determined—such diagrams we call tree diagrams. They present no difficulties; results for the gravitational analogou of Compton effect, Bremsstrahlung, electron-electron scattering, emission of gravitons by atoms, etc. have all been worked out. The answers are satisfactorily gauge invariant and independent of the specific choice of gauge condition.

In next order (e.g. vacuum polarization, radiationless scattering correction, analogue of Lamb effect, etc.) there is one integration over an undefined momentum (one "circuit" in the diagram we shall say). Such diagrams present one problem in that they diverge, and it is a matter of some difficulty to arrange the cut-off process is gauge invariant. There is a much more serious difficulty, however. Diagrams with circuits must be related in definite ways to tree diagrams (for example, by unitarity relations). These relations are

* Abstract. The full text of Prof. Feynman's lecture will appear in *Acta Physica Polonica.*

not satisfied. This difficulty is not a particular characteristic of the gravitation theory, but exists as well, and in exact analogue, in the Yang–Mills vector meson theory with zero mass mesons. It arises from the ambiguities associated with the singular Lagrangian.

It is proposed to resolve the ambiguities in this way: A theorem is developed, valid for any field theory, by which diagrams with circuits are expressed entirely in terms of tree diagrams for more complex processes (but with all particles on the mass shell). We, then, suppose this general theorem to be equally valid for gravitation, by definition.

For diagrams of one circuit, results are now unitary, independent of the choice of gauge condition, and unique. The results are equivalent to what is computed in the usual way, provided one subtracts a certain term, equivalent to a fictitious vector particle going around the circuit involving virtual gravitons. For the Yang–Mills theory the fictitious particle is scalar, and the result is also equivalent to the result obtained from the theory with finite mass (which presents no ambiguities) in the limit that the mass approaches zero. These results are obtained by formal theoretical reasoning, checked as far as possible by specific calculations of definite problems.

Diagrams including more than one circuit (i.e. integration over two or more virtual momenta) have not yet been completely analyzed.

# GENERAL DISCUSSION

V. L. Ginzburg

As follows from the papers and discussion on the radiation of gravitational waves, there is no complete unanimity in this field. Therefore, I should like to make some observations according to which it undoubtedly seems to me that the correct answer is obtained as to the energy emitted already in linear approximation (in the sense mentioned below). We have in mind here the well-known expression obtained by Einstein back in 1918 (for details see[(1)])

$$-\frac{d\mathcal{E}}{dt}=\frac{\varkappa}{45c^5}\dddot{\mathcal{D}}^2_{\alpha\beta}$$
$$\mathcal{D}_{\alpha\beta}=\int\mu(3x^\alpha x^\beta-\delta_{\alpha\beta}x^2_\gamma)\,dV \tag{1}$$

Formula (1) is obtained by an expansion with respect to $v/c$, where $v$ is the velocity of particles (masses) performing finite motion. In that case it is apparent that $\frac{v}{c}\sim\frac{r}{\lambda}$, where $r$ denotes the dimensions of the system and $\lambda$— the wave length ($\lambda\sim cT$, $r\sim vT$, where $T$ stands for the characteristic time of the change in the quadrupole moment $\mathcal{D}_{\alpha\beta}$; in the case of harmonic motion with frequency $\omega$ obviously $\omega=2\pi/T$).

Formula (1) holds (and herein, strictly speaking, consists our assertion) with an accuracy up to terms of higher order in $v/c$ and not with an accuracy to some numerical factor as assumed by some people.

The reason why formula (1) gives rise to doubt is the following. If the particles move under the influence of gravitational forces (double star, etc.), then the derivation of formula (1) is not entirely consistent in the sense of its obtaining within the limits of linear approximation.

Nevertheless, I feel that Einstein's calculation is consistent and the answer is perfectly correct (in the sense mentioned above) and here is the reason for that.

Consider the motion of a nonrelativistic particle in an external magnetic field $\boldsymbol{H}$ where the equation of motion has the form

$$m\frac{d\boldsymbol{v}}{dt}=\frac{e}{c}[\boldsymbol{v}\boldsymbol{H}]. \tag{2}$$

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(1) L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields, Pergamon Press 1962 (§ 104).

In this case the correctness of the linear approximation does not give rise to any doubt (we assume "the particle" to have a sufficiently large radius $a$ so that its gravitational radius $\frac{\varkappa m}{c^2} \ll a$) and the same also refers to formula (1). But within the limits of classical mechanics the equation of motion $m\frac{dv}{dt} = \boldsymbol{F}$ is true without any regard as to the character of the force $\boldsymbol{F}$. Therefore, if the force $\boldsymbol{F}$ is the Newtonian force of universal gravitation $\boldsymbol{F} = -\frac{\varkappa mM}{r^3}\boldsymbol{r}$, the present case does not differ in principle from case (2). From this it is clear that formula (1) will also be valid for gravitational forces, i.e. for double stars, planetary systems, etc.

The relativistic corrections to the equations of motion are of the order of $v^2/c^2$, and the contribution of higher multiple terms is also of the same order of $v^2/c^2$. Therefore, formula (1) is true for Newtonian forces also with an accuracy up to terms of the order $1/c^7$. The argumentation given will probably not convince the followers of opposed views but it might be hoped that by the time of the next conference the problem of the accuracy of formula (1) will also have been solved with the help of other methods. Then a consensus of opinions will be reached.

I take this opportunity to observe that the problem of emission of gravitational waves in the case of a charge moving in a magnetic field has recently been solved also in the ultrarelativistic case[(2)]. It was found that the energy losses are proportional to the square of the particle energy, i.e. they depend on the energy in the same way as do the losses of radiation of electromagnetic waves. The gravitational losses are $\frac{13}{4}\frac{\varkappa m^2}{e^2}$ times smaller than the losses of radiation of electromagnetic waves ($m$ and $e$ are the mass and charge of the radiating particle). It is, thus, clear that in the case of electrons, protons and nuclei the energy losses due to the emission of gravitational waves are always to be neglected. This result is not a trivial one in the sense that it is not a universal one for arbitrary fields, but it is connected with the transversality of gravitational waves which are in this respect analogous to electromagnetic waves. If the particle moving in a magnetic field emits other particles (quanta of wave fields), in some cases the dependence on energy turns out to be more marked than for the radiation of electromagnetic and gravitational waves (i.e. photons and gravitons). What has been said follows from an analysis by G. F. Zharkov and myself of the emission of various particles due to the motion of, for instance, protons in a magnetic field (then such processes occur: $p \to n + \pi^+$, $p \to p + \pi^0$, $p \to n + e^+ + \nu$; here $\pi^{+,0}$ are pions).

(2) B. I. Pustovoit and M. E. Herzenstein, *J. Eksp. Theor. Phys.* **42**, 163 (1962).

C. W. Misner

*Radiation Gauge in General Relativity.* In situations where intense gravitational fields are all localized (i.e. ignoring cosmology), the weakness of the field at spatial infinity suggests that techniques taken from linearized theory might lead to interesting conclusions concerning solutions of the Einstein equations. Other work in this direction has made use of harmonic coordinates to make all the equations appear as wave equations. The work reported here (done in collaboration with Arnowitt and Deser) uses a different style of coordinate conditions in which the initial value (or constraint) equations appear as elliptic equations. Using this approach we discuss (1) total energy and momentum, "Newtonian" component fields, (2) the coordinate dependence of the asymptotic fields, (3) the "wave front theorem" which excludes a steady rate of radiation in asymptotically flat spaces, (4) the possibility of defining the radiation gauge fields from a knowledge of only the asymptotic (wave zone) metric, (5) coordinate invariant descriptions of the radiation fields. The novel features of these discussions are: in (1) it is explained why flux integral formulas for energy do not involve $g_{00}$ although the weak field geodesic argument suggests (misleadingly) that $g_{00}$ is a "Newtonian component" of the metric; the conclusions in (3) describe the behaviour of fields as $r \to \infty$, a variant of Papapetrou's wave front theorem describing the limit $t \to \infty$; in (4) it is shown how a non-local (Fourier analysis) method of obtaining coordinate invariant quantities can be applied usefully without requiring detailed knowledge about strong field regions.

J. Weber

*Quantization of the Coupled Maxwell Einstein Fields.* We have studied the interaction of gravitons and photons in the first approximation. Our hope was to find some consequences which might be verified experimentally.

Starting with Dirac's gravitational Hamiltonian we carried out a canonical quantization, choosing coordinates such that problems involving the constraints were minimal. We have calculated $S$ matrix elements for a number of processes. The interaction seems to allow processes in which a photon decays into another photon and a graviton. Energy and momentum can only be conserved if all three particles propagate in the forward direction. Closer study shows that the matrix elements for this process do not vanish anywhere except for the situation where all three particles propagate in the same direction. The photon cannot, therefore, decay by this process. However, at extreme energies it turns out that energy need not be exactly conserved and this leads to the possibility of decay at energies $\gg 10^{28}$ electron volts. The emitted graviton has very low energy.

Even in lowest order, self energy effects appear. We have to deal with processes in which virtual photons and gravitons are created and annihilated. It can be shown that in first order such processes contribute nothing, in consequence of the constraints.

The creation of gravitons by coulomb scattering of photons and by scattering in a magnetostatic field is shown to occur. The cross section is calculated for the case of a coulomb type scatterer in which the electric or magnetic field is uniform and all dimensions of the scatterer are large compared with a wavelength. The cross section is given by

$$S = \frac{8\pi^2 GUl}{c^4}.$$

Here $S$ the cross section, $G$ is the constant of gravitation, $U$ is the energy of the scatterer, $c$ is the speed of light and $l$ is the thickness of the scatterer in the direction of propagation of the photon.

For a galaxy the cross section is about $10^{28}$ square centimeters. For a laboratory scatterer having a volume of $10^6$ cubic centimeters containing $10^{15}$ ergs of electrical energy the cross section is about $10^{-30}$ square centimeters. The theory of the fluctuations shows that an incident photon power $P$ is required for detection, such that the power converted into gravitons is $\Delta P$ where

$$\Delta P > 2\sqrt{kTP/\tau}.$$

Here $k$ is Boltzmann's constant, $T$ is the absolute temperature and $\tau$ is the averaging time. For a cross section of $10^{-30}$ square centimeters and $\tau = 10^4$, a photon power $p > 10^{40}$ ergs per second is needed. Laboratory experiments are therefore not feasible.

This research was done in collaboration with Mr. George Hinds and was supported in part by the U. S. Natural Science Foundation and in part by the U.S. Air Force Office of Scientific Research.

D. IVANENKO

*A compensating treatment of gravitation.* As was stressed by Yang and Mills each conservation law or invariance group induces a corresponding vector field like introducing a vector potential of electromagnetic field by means of gauge transformation

$$U = \exp ie\,\Lambda(x)$$

with localized phase depending on coordinates. It may be of great interest to generalize these considerations applied already to charge iso-spin and baryonic charge conservation, to other bosonic fields, especially to gravidynamics.

With this aim in mind we consider localized homogeneous Lorentz transformations with parameters being non constant but depending on coordinates. Non-homogeneous group is considered by B. Frolov. Now, to keep invariance one must compensate the derivatives of these parameters by a coupling with a new field and this compensating field proves to be essentially gravitational field of the Riemann–Einstein general relativity.

In our work with A. M. Brodski and H. A. Sokolik[(3)] inspired by Sakurai's article[(4)] we preferred to treat transformation properties of the fields for finite transformations of a local group, which seems to be more correct as Lie's theorem is not immediately applicable to local groups. Essentially equivalent results were obtained independently by Utiyama[(5)] who used infinitesimal transformations. Analogous considerations were established also in an unpublished work of J. Schwinger and recently by J. Kibble.

This new approach to general relativity is of interest not only by its simplicity but also by the very fact of introduction of a group generalizing the Lorentz group, which means the existence of some symmetry of the Riemann-Einstein space which generally speaking is devoid of any symmetry, as was emphasized by E. Cartan. So let us require covariance of the equation for particles of arbitrary spin value

$$(h_\sigma(p)\mathcal{L}_p\partial_\sigma+im)\Psi=0$$

where $\Psi$ is transformed by means of an arbitrary represenation $S$, and the tetrapods $h_\sigma(p)$ are assumed to be functions of the coordinates. Then to compensate the term $S\partial_\sigma S^{-1}=I_{ml}\,(\partial_\sigma\varepsilon_{sp})\;N^{lm}_{ps}$ it is necessary to generalize $\partial_\sigma$ to a "compensating" (essentially covariant) derivative

$$\nabla_\sigma=\partial_\sigma-\Gamma_\sigma \quad \text{with} \quad \Gamma_\sigma=\tfrac{1}{2}I_{nm}\Delta_\sigma(m,n).$$

I being generators of the group, $\varepsilon$ — its parameters, $N=\int_0^1 \exp(t\varepsilon C)\,dt$ (indices being omitted!), $C$ — structural constants of the Lorentz group, $\Delta$ — Ricci's coefficients. For instance, for spinor Dirac field ($\mathcal{L}_p=\gamma_p$) one gets immediately the well known coefficients (Fock–Ivanenko[(6)]),

$$\Gamma_\sigma=\tfrac{1}{4}[\gamma_l\gamma_m]\Delta_\sigma(m,l).$$

Of course, a tensor purely antisymmetrical term, describing, e.g., torsion can be added to the compensating derivative. It is interesting to note that the parallel displacement of spinors which requires in a natural way application

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(3) A. M. Brodski, D. Ivanenko, H. A. Sokolik, *J. Eksp. Teor. Phys.* (1961).

(4) J. Sakurai, *Ann. Phys.* **11**, 4 (1960).

(5) R. Utiyama, *Phys. Rev.* **101**, 1597 (1956).

(6) V. Fock, D. Ivanenko, *C. R. Acad. Sci., Paris* **188**, 1470 (1929). Further development J. A. Wheeler Geometrodynamics (Academic Press, N.Y., 1962); A. Peres, *Sup. Nuovo Cimento* **24**, No 2, p. 389 (1962); E. Schmutzer, *Z. f. Naturforsch.* 1962.

of tetrads, introduces, in the case of a space endowed with torsion, a non-linear supplement in the Dirac equation, as was shown by V. Rodičev[7]. Moreover, this additional term turns out to be of the pseudo-vectorial type chosen by Heisenberg on various grounds, from all possible non-linear supplements investigated previously in our work with A. Brodski. All this seems to represent a step towards the geometrical interpretation of an unified non-linear theory which appears to be based on such most elementary entities as spinors and tetrads.

In this connection it is of great interest to analyze also a tetrad interpretation of the theory of gravitation without torsion, i.e. remaining inside Riemann-Einsteinian curved space. This was investigated by V. I. Rodičev in Moscow. This work has many common points with beautiful reports of Prof. Møller and Dr. Plebański at this conference.

We assume that passing over to non-inertial coordinate systems corresponds just to local Lorentz ratations of orthogonal tetrads and not to general coordinate transformations. With this assumption the Lagrangian (i.e. scalar density expressed by Ricci's rotation coefficients) as well the field equations must be covariant in respect to the (a) groups of general coordinate transformations and (b) groups of Lorentz rotations (both inertial as non-inertial systems of coordinates are permitted).

Riemannian–Christoffel connection represents the sum of absolute parallelism connection and Ricci's rotation coefficients, which consist of components of torsion due to absolute parallelism connection, so it is natural to propose torsion coefficients as representing gravitation field strength values. Excluding now holonomic part of tetrads and in this way presumably the inertia fields by means of condition $\nabla_\sigma h^\sigma(\alpha) = 0$, one gets for scalar density $R = \Delta(\varepsilon,\alpha\beta)\, C(\alpha\beta,\varepsilon)$ ($\Delta$ — Ricci's coefficients, C — curl of tetrad components). Then the field equations

$$\nabla_\sigma\{\Delta(\alpha)^{\mu\sigma}_{\cdot\cdot}\} = \Theta(\alpha)^\mu$$

yield at the right side the expression of the energy-momentum of gravitational field $t(\alpha)^\mu = \left\{2\Delta(\beta,\tau\varepsilon)C(\alpha\varepsilon,\beta) - \frac{1}{2}\delta(\alpha,\tau)\, R\right\} h^\mu(\tau)$. This can be also obtained by means of Noether theorem applied to parallel displacements of Lorentz tetrads.

R. P. Feynman:

I don't understand precisely what came out of this analysis of compensating fields to you. Was it the relativity theory of Einstein exactly, or was there some additional torsion or something?

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[7] V. I. Rodičev, *J. Eksp. Teor. Phys.* **40**, 1469 (1961), Dokl. A.N. (C.R. USSR) **148**, 3(1963).

D. IVANENKO:

We introduced a compensating derivative. The compensating derivative is clearly equal to the covariant derivative but an arbitrary tensor term can be added there. Of course, this pure tensor term corresponds to torsion. This is not a simple generalization, since Einsteinian theory can be generalized in many directions, but this generalization is quite a natural one. If we consider gauge transformations, we can split the electromagnetic field into transverse and longitudinal parts. Somehow analogous to this splitting is the splitting of the connection into a tensor part and a non-tensor part.

R. P. FEYNMAN:

Where do the $g$'s come from?

D. IVANENKO:

As you seem to support the general idea of quantization of Yang–Mills and Sakurai, perhaps you will not protest that an analogous method can be applied also to Lorentz transformations.

N. ST. KALITZIN

*On the existence of a non-singular stationary solution of Eddington's gravitational equations which could represent a material body.* As is well known, Einstein's equations

$$R_{ik} = 0 \tag{1}$$

do not have a non-singular solution, which could represent a material body. The same holds also for the Einstein–Maxwell equations, for Kaluza's equations and for Einstein's last unified field theory.

One obtains Einstein's equations (1) from the variational principle

$$\delta \int R\sqrt{-g}\, d\Omega = \delta \int g^{ik}R_{ik}\sqrt{-g}\, d\Omega = 0 \tag{2}$$

as is shown by Eddington in his book "Relativitätstheorie in mathematischer Behandlung"; one gets other field equations starting from the variational principle

$$\delta \int K\sqrt{-g}\, d\Omega = \int P^{ik}\delta g_{ik}\sqrt{-g}\, d\Omega \tag{3}$$

where

$$K = R^{m}{}_{isr}R_{m}{}^{isr}$$

$R^{m}{}_{isr}$—Riemannian curvature tensor, $P^{ik}$—symmetrical tensor whose divergence vanishes.

The field equations

$$P^{ik} = 0 \tag{4}$$

which we refer to as Eddington's gravitational equations, are of fourth order in $g_{ik}$ and are much more complicated than Einstein's equations (1).

In the general case they even cannot be written out explicitely but in the case of spherical symmetry they can be treated.

In this case we have for the line element the expression

$$ds^2 = -e^{\lambda}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\varphi^2 + e^{\nu}dt^2 \tag{5}$$

$\lambda$, $\nu$ — functions of $r$ only.

The variational principle (3) gives us two equations for determination of $\lambda$ and $\nu$:

$$-3r^2\lambda'^2 + 3r^2\nu'^2 + \ldots + \frac{1}{2}r^4\lambda''\nu'^2 r^4\nu'''\nu' = 0, \tag{6}$$

$$r^2\lambda'^2 + r^2\nu'^2 + \ldots + 8r^3\nu''' - r^4\nu'\lambda''' + 2r^4\,\nu'''' = 0. \tag{7}$$

As can be easily shown these equations have as a solution the Schwarzschild solution

$$e^{-\lambda} = e^{\nu} = 1 - \frac{2m}{r}$$

(all effects of general relativity hold).

But they surely have also some other solutions. We will investigate if among these solutions there exist solutions, which are non-singular in the whole space.

For this purpose we represent $\lambda$ and $\nu$ in the neighbourhood of the point $r = 0$ (center of symmetry) by the series

$$\begin{aligned} \lambda &= a_0 + a_1r + a_2r^2 + a_3r^3 + \ldots \\ \nu &= b_0 + b_1r + b_2r^2 + b_3r^3 + \ldots \end{aligned} \tag{8}$$

Equations (6) and (7) give us in this case

$$a_0 = a_1 = a_3 = a_5 = \ldots = b_1 = b_3 = \ldots = 0 \tag{9}$$

$a_2$, $a_4$, ..., $b_2$, $b_4$—can be different from zero. Relation (9) shows that at the point $r = 0$ the solution is smooth (we have not a point like $\underset{o}{\vee}$). To investigate the solution for great values of $r$ we use the series

$$\lambda = p_1u + p_2u^2 + p_3u^3 + \ldots$$

$$\nu = q_1u + q_2u^2 + q_3u^3 + \ldots$$

$$u = \frac{1}{r}$$

Here $p_0$ and $q_0$ are zero because our solution should tend to the Euclidena value at infinity.

On a sphere of radius $r = \varrho$ the solutions (8) and (9) must satisfy the conditions

$$\begin{aligned} \lambda_1 &= \lambda_2, & \nu_1 &= \nu_2, & \lambda_1' &= \lambda_2' \\ \lambda_1'' &= \lambda_2'', & \nu_1' &= \nu_2', & \nu_1'' &= \nu_2'' \end{aligned} \tag{10}$$

(1 refers to series (8), 2 to series (9)) which with equations (6) and (7) guarantee that Eddington's equations hold also for $r = \varrho$.

In the first approximation we interrupt the series on the sphere $r = \varrho$ up to $\varrho^2$ and to $\frac{1}{\varrho^2}$ and obtain from the boundary conditions (10) and from the differential equations $p^{ik} = 0$ eight algebraic equations for the 7 quantities $p_1, p_2, q_1, q_2, b_0, b_2, a_2$. We show that these algebraic equations are identically fulfilled by the solutions

$$p_1 = q_1 = -\frac{3}{2}\varrho, \quad p_2 = q_2 = \frac{9}{16}\varrho^2; \qquad a_2 = b_2 = \frac{3}{16}\frac{1}{\varrho^2}, \quad b_0 = 0 \tag{11}$$

The series (8) for $\lambda$ and $\nu$ can be majorised by the series

$$a_2r^2 + a_2^2r^4 + a_2^3r^6 + \ldots + a_2^sr^{2s} + \ldots = \frac{a_2r^2}{1-a_2r^2}. \tag{12}$$

The condition of convergence is

$$a_2r^2 = \frac{3}{16}\frac{r^2}{\varrho^2} < 1$$

and is fulfilled in the region $r \leqslant \varrho$.

The series (9) can be majorised by the series

$$\frac{3}{4}\varrho u + \left(\frac{3}{4}\varrho u\right)^2 + \ldots = \frac{\frac{3}{4}\varrho u}{1-\frac{3}{4}\varrho u} \tag{13}$$

which is convergent for $r \geqslant \varrho$.

So we have found solutions of Eddington's equations which for $r \leqslant \varrho$ and $r \geqslant \varrho$ are absolutely convergent and on the sphere approximatively fulfil the boundary conditions (10).

With the help of Einstein's equations

$$R_i^k - \frac{1}{2}\delta_i^k R = -8\pi T_i^k \tag{14}$$

we calculate the energy momentum tensor $T_i^k$ in the case of our field and with the help of the formula

$$m = 4\pi \int_0^\infty T_4^4 r^2 dr$$

we calculate the whole mass of our particle, resulting from the fields inside and outside $r = \varrho$. We obtain

$$m = 0{,}286\,\varrho \tag{15}$$

This value can be improved by calculating also the mass resulting from the fact that in this approximation the boundary conditions (10) are not fulfilled. We get finally

$$m = \frac{3}{4}\varrho$$

which mass must have in the Newtonian approximation of the gravitational potential

$$\left(q_1 = -\frac{3}{2}\varrho; \quad e^{\nu} = 1+q_1\frac{1}{r}+\dots\right)$$

We thus have obtained an approximate solution of Eddington's gravitational equations which is everywhere non-singular and which can, according to Einstein's ideas, represent a material body.

With the help of the electronic computing machine in Dubna near Moscow, together with Burneff and Nedjalkoff, we have calculated the following approximation interrupting the series up to $\varrho^4$ and $\frac{1}{\varrho^2}$ on the sphere $r = \varrho$.

The obtained results are quite near to the above mentioned results.

H. A. BUCHDAHL:

We investigated a corresponding problem for Lagrangian $R^2$ and concluded that in the spherically symmetric case and in the general case there are no asymptotically flat solutions other than those of $R_{ik} = 0$, which, of course, have a singularity at the origin. I always suspected the same to be the case here, because the Lagrangian which you have used is the same as the Lagrangian $4R_{ik}R^{ik}-R^2$, which is not very dissimilar from the Lagrangian $R^2$. I am inclined to think that all quadratic Lagrangians would lead to the same result. But the only asymptotically flat solutions are those which also satisfy the equation $R_{ik} = 0$ with a singularity somewhere. I am very surprised by your results.

N. St. KALITZIN:

Your Lagrangian is, nevertheless, quite different from Eddington's Lagrangian,

$$R^2 \neq R^m{}_{ist}R_m{}^{ist}$$

and it is no wonder that your results are different from mine.

C. MØLLER asked about the conservation law for the energy momentum tensor.

N. St. KALITZIN:

The tensor $T_i^k$ is defined by means of Einstein's gravitational equations

$$R_i^k - \frac{1}{2}\delta_i^k R = -8\pi T_i^k$$

and the divergence of such a tensor automatically vanishes.

L. ROSENFELD:

In his remarkably clear and complete exposition of the present views on the logical structure of the process of field quantization, Professor DeWitt casually stated that from this point of view the gravitation field "must" be quantized. This little word "must" prompts me to the following considerations, suggesting that the case for quantization of the gravitation field is perhaps not as obvious as it is sometimes made out to be.

(1) In view of the universality of the quantum of action, one is tempted to regard any classical theory as a limiting case of some quantal theory. However, one must not lose sight of the fact that the formulation of any theory in its application to given physical situations involves the specification of the system under consideration and of the external conditions under which it is investigated: such specifications, which represent the essential link between the theoretical description and the physical observation, are necessarily expressed in terms of classical concepts, and therefore enter into the equations in the form of c-number parameters. From this point of view, the metrical tensor, at any rate for all practical purposes, appears as such a c-number specification of conditions of observation, and there is no logical imperfection in regarding as fundamental the classical, unquantized, form of the equations expressing the connexion of the metrical or gravitation field with the other fields.

(2) This simple point has a bearing on the problem mentioned by DeWitt as one in need of further discussion, of analysing the process of measurement of field quantities entirely in terms of quantum theory, i.e., by including the experimental arrangement in the quantal description. It should be clear from the preceding remark that the requirement of such a comprehensive quantal account of the measuring process (which is doubtless possible) is actually pointless (except inasmuch as it would provide a rather trivial check of the consistency of quantum theory), since it would only obscure the essential function of the experimental arrangement in establishing the connexion between the quantal system and the classical concepts indispensable for its description. Disregard of this essential feature of quantum theory has led to futile attempts at circumventing complementarity by arguing that a solution of the wave equation for the total system including the experimental arrangement would yield a uniquely determined account of the whole measuring process and its result; in such argumentation it is not sufficiently realized that the equation in question would contain c-number parameters which in the last resort would refer to just the conditions of observation directly expressed in the classical description of the experimental arrangement. In the particular case of the measurement of gravitation quantities, it is unavoidable to have some classical metrical substratum for the localization of the test-bodies.

(3) Another point deserving close consideration concerns the limitations of validity of the concepts used in the formulation of the theory of gravitation. Thus, the classical limit of validity of the dynamics of the electron, expressed by the classical electron radius, loses any significance since quantum effects become essential for the motion of electrons within domains of much larger dimensions. In quantum electrodynamics and meson theory, an absolute limit to the possibilities of localization is strongly suggested (although its precise nature is still obscure) by the increase of the fluctuations of the charge and current contained within an unsharply defined space-time domain when the thickness of the shell limiting this domain decreases. It would seem that the critical thickness is of the order of the nuclear length unit ($10^{-13}$ cm), which is much larger than the critical length for quantal effects of gravitation to become appreciable. This again leads to a strong suspicion that quantization would be meaningless.

(4) One could still raise the question whether, for wave-lengths much larger than the critical length just mentioned, gravitational radiation, notwithstanding the smallness of any quantal effects for such wavelengths, ought not to be quantized for the sake of consistency with the quantization of the other fields. In this connexion I should like to state that an analysis of the limits of measurability of gravitational quantities, such as that carried out by DeWitt, cannot throw light on this question: all that such an analysis can tell us is whether an assumed quantization of the gravitation field is consistent with the quantization of the other fields. Thus, in the case of electromagnetism, the analysis shows that the reciprocal limitations of measurability of field components predicted by the quantized theory arise as a consequence of the impossibility of controlling the number of photons contained in the interaction of the test-bodies in the course of the measuring process; if, however, this interaction were entirely classical, it could be completely compensated, and there would be no limit to the measurability of the field even when due account is taken of the mechanical uncertainly relations to which the test bodies are subject. I insist on this because the view was expressed in conversation that, according to the analysis, the electromagnetic field quantization is necessarily entailed by the quantization of the motion of the test bodies: there is certainly no such logical necessity in either case. As regards gravitation, there is, as pointed out by DeWitt, an essential difference from the situation in electromagnetic theory: whereas the latter sets no absolute scale of space-time dimensions or mass, a quantum theory of gravitation suggests the existence of a critical mass; the consequences of this circumstance for the range of validity, or even the internal consistency of any quantization of gravitation demand closer study.

F. J. BELINFANTE:

I still feel after Rosenfeld's remarks that there is almost a "must" for quantization, certainly if one believes in Einstein's equation and in the quantization of the matter fields going into the $T_{\mu\nu}$, just like it is hard to assume that in div $E = 4\pi\varrho$ the $E$ would be a c-number while the $\psi$—field in $\varrho$ is second-quantized. One similarly cannot equate a q-number $T_{\mu\nu}$, of which the value would depend on the state vector, to a $G_{\mu\nu}$ which is a c-number and, therefore, has an invariable value.

L. ROSENFELD:

But one could use the expectation value of $T_{\mu\nu}$, instead of $T_{\mu\nu}$ itself.

F. J. BELINFANTE:

This would be most unusual, to equate a c-number to an expectation value of a q-number, and I think this violates the principles of quantum theory. Suppose we had a situation which were a superposition of states, say 40% probability for one state of the matter and 60% probability for a different state. Suppose one did a million experiments; then one would expect a correlation between the measured actual state of the matter and the surrounding gravitational field. That is, one would in 400 000 cases find one gravitational field and in 600 000 cases a different gravitational field. If, however, the gravitational field were given by the expectation value, one should in all 1 000 000 cases find the same gravitational field, obtained by a 40–60 average.

L. ROSENFELD:*

The case you consider here is rather extreme. One seldom would have fluctuations so large that their gravitational effects would be measurable.

F. J. BELINFANTE:

As you want a c-number gravitational field, I can discuss it classically and by a thought experiment I then can measure the field as accurately as I want.

L. ROSENFELD:

It is thinkable, however, that Einstein's equation has the character of a thermodynamic equation which breaks down and is not valid in case of extreme fluctuations.

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* Discussion continued on terrace, as room was to be emptied for a seminar.

F. J. BELINFANTE:

This is the reason why I feel that quantization is almost a "must", as I would feel very unsatisfied if Einstein's equation were a thermodynamic equation, while Maxwell's equations would be q-number relations, though we obtain both by varying one single Lagrangian, of which we would prefer to see all terms treated alike as q-numbers.

L. ROSENFELD:

On the other hand, since we know the validity of the gravitational equations only for macroscopic systems, it is perhaps not so impossible to look at them as a kind of thermodynamic equations.

# MACH'S PRINCIPLE AS BOUNDARY CONDITION FOR EINSTEIN'S FIELD EQUATIONS AND AS A CENTRAL PART OF THE "PLAN" OF GENERAL RELATIVITY*

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## I. THE SEARCH FOR AN ACCEPTABLE FORMULATION OF MACH'S PRINCIPLE

### *Inertia as a Consequence of an Interaction between the Accelerated Test Particle and all the Rest of the Universe*

Acceleration can have no meaning unless there is something with respect to which the acceleration takes place. The acceleration with respect to absolute space that Newton speaks about has to be understood as acceleration with respect to the stars and matter in the universe. These two sentences state in oversimplified form the argument of Mach [1]. From it he went on to make conclusions about the origin of inertia. Inertia—being tied to acceleration—must arise from interaction between the object under study and all the other mass in the universe. Thus Mach's principle might be stated in this form: (Formulation 1). The inertial properties of an object are determined by the distribution of mass-energy throughout all space.

### *Inertia as the Radiative Component of the Gravitational Force*

Mach's principle, together with Riemann's idea that the geometry of space responds to physics and participates in physics, were the two great currents of thought which Einstein, through his powerful equivalence principle, brought together into the present day geometrical description of gravitation and motion. In the course of this work Einstein identified gravitation itself as the source of the interaction by which—according to Mach—one object affects the inertial properties of another. What is important in this connection is not the familiar $1/r^2$-proportional static component of the gravitational force, but the acceleration-proportional radiative component of the interaction

* Report given at the Conference on relativistic theories of gravitation, Warszawa, Jabłonna, Poland, 25–31 July 1962, as adapted for subsequent lectures at the Summer lecture series in theoretical physics, University of Colorado, Boulder, Colorado, 8–17 August 1962.

(Table I). Einstein discussed this point a little in his book [2] in connection with the idealized experiment of Thirring. This description of the inertia of a given particle as arising from the radiative component of its interaction with all the other masses in the universe has been looked into a little further by Sciama [3] and Davidson [4]. The inertial term $ma$ is dropped from Newton's equation of motion. In its place appears the sum of the radiative interactions

$$ma \sum_k \frac{Gm_k f}{c^2 r}. \tag{1}$$

This term gives a reasonable order of magnitude account of inertia if the dimensions of the universe are of the order of $10^{10}$ light years and if the effective average density of matter is of the order of $10^{-29}$g/cm$^3$.(1)

TABLE I

Static and radiative components of electromagnetic and gravitational forces compared and contrasted. The quantity $f$ is an abbreviation for a dimensionless function of the angles between the lines of acceleration of source and receptor and the line connecting these two objects.

| | Electromagnetism | Gravitation |
|---|---|---|
| Static or near part of interaction | $\frac{e_1 e_2}{r^2}$ | $\frac{Gm_1 m_2}{r^2}$ |
| Radiative or distant component | $-\frac{e_1 e_2 a_2 f}{c^2 r}$ | $-\frac{Gm_1 m_2 a_2 f}{c^2 r}$ |

*Inertia is Tied to Geometry and Geometry is Directly Governed by the Distribution of Mass-Energy and Energy Flow*

The analysis of Thirring and Einstein brings this "sum for inertia" into closer connection with the ideas of general relativity. On the one hand the inertial properties of a test particle are expressed in terms of the metric tensor $g_{\mu\nu}$. On the other hand the agencies responsible for changes in this measure of inertia are characterized not merely by density, but by the entire stress-energy tensor $T_{\mu\nu}$. Thus, Thirring and Einstein write the change

$$h_{\mu\nu} = g_{\mu\nu} - \hat{g}_{\mu\nu}, \tag{2}$$

$$h = \hat{g}^{\mu\nu} h_{\mu\nu}, \tag{3}$$

of the metric in a local Lorentz system, due to a change $\delta T_{\mu\nu}$, in the form

$$h_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu} h = (8\pi G/c^4) \int \frac{[\delta T_{\mu\nu}]_{\text{ret}} d^3x}{r}. \tag{4}$$

This expression remains a good approximate solution of Einstein's field equation so long as the geometry of the regions where the mass-energy is

(1) For a discussion of present information on the density and size of the universe see for example [5].

located does not differ substantially from the local Lorentz geometry at the position of the test particle. Looking at Eq. (4), and recalling that in relativity theory the inertial properties of a test particle are determined by the metric one is led to formulate Mach's principle in the following form (Formulation 2): *The geometry of spacetime and therefore the inertial properties of every infinitesimal test particle are determined by the distribution of energy and energy flow throughout all space.*

*Many Objections to Mach's Principle*

That Mach's principle in anything like this form makes sense has been questioned on many sides for the following reasons:

(1) Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = (8\pi G/c^4) T_{\mu\nu} \tag{5}$$

are non-linear. It is wrong in principle to try to express the solution $g_{\mu\nu}$ as a linear superposition of effects from the $T_{\mu\nu}$ in various regions of space.

(2) The quantity $1/r$ in the integrand is not a well defined quantity in an irregularly curved space.

(3) If in the Friedmann universe one considers the contributions to the inertia at a definite point in spacetime from more and more remote points, where the retarded value of the stress-energy tensor is $[T_{\mu\nu}]_{\text{ret}}$ one is forced to go back to earlier and earlier moments of time. Ultimately one comes to a time when the system was in a singular state. What does one do then about the contribution of $[T_{\mu\nu}]_{\text{ret}}$ to the inertia.

(4) The elementary sum in Eq. (1) for the coefficient of inertia envisages a radiative interaction between particle and particle. But how can stars at distances of $10^9$ and $10^{10}$ light years respond to the acceleration of a test particle here and now in such a way as to react back upon this test particle at this very moment? Is this difficulty not argument enough that this elementary formulation should be dropped? But when one turns from this picture of two-way travel of gravitational radiation to the Thirring–Einstein calculation where only one direction of travel comes into evidence, does one not encounter an ambiguity in this sense, that one could use advanced interactions just as well as retarded interactions—or any combination of the two—in obtaining a solution of the linearized field equations? If the advanced and retarded expressions for the metric in terms of the distribution of mass-energy differ from each other—as expected—will not one be forced to conclude that one expression is wrong? And if one is wrong will it not be likely that both are wrong?

(5) Will not the $1/r$-dependence of the supposed inertial interaction make the inertial properties of a test particle depend upon the expansion and recontraction of the universe, and the proximity of nearby masses, in a physically unreasonable way?

(6) How can it make sense to speak of the distribution of mass-energy (and energy flow) as determining the geometry? One cannot specify where one mass is, let alone the entire distribution of mass, until one has been given the geometry. But then what is there to be determined?

(7) Why spoil the beautiful logical structure of relativity theory by mixing up with it anything so vague and so lacking in mathematical sharpness as Mach's principle? Why try to word it in careful 20th century language when it is an outworn 19th century idea that ought to be dropped at once and for all time?

*Solutions of Einstein's Equations not Produced but Selected by Mach's Principle*

The answer is that Einstein's equations are not enough. Differential equations in and by themselves typically do not suffice to define a solution. They must be supplemented by a boundary condition. Mach's principle is required (Formulation 3) as *a boundary condition to select allowable solutions of Einstein's equation from physically inadmissible solutions.** This kind of selection principle is so familiar in electrostatics (Table II) that it generally goes without even a name. Only when Poisson's equation is supplemented by such a boundary condition does it lead to the $(1/r)$ law of action of a charge. This $(1/r)$ law of action furnishes the usual basis for saying that the distribution of electric charge uniquely determines the distribution of electric potential.

*Cases Where the Boundary Condition Cannot be Applied Regarded as Idealizations of Cases Where It Does Apply and Where It Does Make Sense*

The boundary condition that the electrostatic potential shall fall off at large distances is noteworthy for what it does not do as well as for what it does do. It does not provide a way to *calculate* the $(1/r)$-law of action. Only the differential equation does that—giving in addition many other solutions. Moreover, one often considers in electrostatics problems where the requirement of Table II, "The potential must fall off at great distances" cannot be satisfied. By way of illustration, consider the problem: "Given $\varrho(x, y, z) = \varrho_0 \cos kz$;

* Note added after completion of this manuscript: This concept of Mach's principle as principle for the **selection** of solutions of Einstein's equations appears earlier in the discussion of J. A. Wheeler on pp. 49–51 of La structure et l'évolution de l'Univers, Bruxelles 1958, and esgecially in a recent article by H. Hönl in E. Brüchc, ed., Physikertagung Wien, Physik Verlag, Mosbach/Baden, 1962, where on p. 95 Hönl proposes two theses: (1) das Machsche Prinzip ist als kosmologisches Prinzip ein Auswahlprinzip, d.h. es gestattet, aus der großen Zahl möglicher Lösungen des kosmologischen Problems einige wenige auszusondern, die als physikalisch zinvolle Weltmodelle überhaupt in Frage kommen. (2) Das Machsche Prinzip läßt sich nur für räumlich geschloßene, endliche Weltmodelle in widerspruchloser Weise durchführen; es ist daher zu vermuten, daß die Forderung des Mach-Prinzips mit der Forderung eines endlichen Universums überhaupt identisch ist.

find $V(x, y, z)$"! Thus one can choose between accepting the problem and giving up the generality of the boundary condition; or upholding the boundary condition at all times and modifying the problem. One can say that the infinite cosine wave distribution of charge is only a mathematical idealization of a physical distribution of charge which is nearly cosine character over a great region, as illustrated, for example, by an expression of the form

$$\varrho(x, y, z) = \varrho_0 \cos kz \exp[-(x^2+y^2+z^2)/a^2], \tag{6}$$

where the Gaussian breadth $a$ is very large. On this choice of interpretation the boundary condition continues to make sense, and the potential continues to be determined uniquely by the distribution of charge.

TABLE II

Boundary conditions in electrostatic and in gravitation theory according to Formulation 3 of Mach's principle: a boundary condition to select allowable solutions of Einstein's equations from physically inadmissible solutions.

| | Electrostatics | Gravitation theory |
|---|---|---|
| Differential equations | $\nabla^2 V = -4\pi\varrho$ | The four of Einstein's equations which have to do with geometry on a space-like hypersurface. |
| Source terms | Electric charge density | Density of energy and energy flow. |
| General solution | $V = \int \frac{\varrho d^3x}{r} + \sum c_{nm} r^n Y_n^{(m)}(\theta,\varphi)$ | Geometry which (a) extends to spatial infinity or (b) is somewehre singular or (c) is closed up and free of singularity. |
| Principle of selection of physical solution | Potential must fall off at great distances | Geometry must be of class (c). (To admit singularities is to admit points where the equations are not really satisfied). |
| Consequence of this principle and also another way of formulating this principle | Potential is uniquely determined by the distribution of charge | Geometry of *spacetime must be* uniquely determined by the distribution of energy and energy flow over the original *space-like* hypersurface. |

### *Asymptotically Flat Geometry Expressed as Limit of Closed Space*

Similarly in general relativity one can find situations which are not compatible with the boundary condition of Table II—and therefore not compatible with formulation 3 of Mach's principle—and which, nevertheless, can be translated over into situations which are compatible with the boundary condition. Consider for example a single spherically symmetric concentration of mass in otherwise empty space. Associated with this mass is the familiar

Schwarzschild 4-geometry. This geometry is asymptotically flat at infinity. In this spacetime the inertial properties of an infinitesimal test particle approach indefinitely closely to the Newtonian expectations at indefinitely great distances from the mass. Consequently it is unreasonable to think of the central mass as responsible for these inertial properties. If one accepts this situation, one cannot uphold Mach's principle either as Mach originally stated it or as it is reformulated here, as a boundary condition to *select* solutions of Einstein's field equations:

(1) the inertial properties of test particles — not being attributable to the one mass that is present—are, therefore, not assignable to Mach's "distribution of mass throughout all space"; and

(2) the Schwarzschild geometry does not describe a closed universe.

Therefore, *rule out* around a center of mass a space that becomes flat at infinity. In other words, apply the geometric boundary condition of Table II to *exclude* the Schwarzschild geometry. Follow the example of electrostatics, where for example in Table II an infinite cosine distribution of charge was ruled out because it was incompatible with the boundary condition for the electrostatic potential at infinity.

The idealized situation that is pushed out of the back door as physically unacceptable comes in again at the front door in new clothes both in electrostatics and in general relativity (Table III). Consider a geometry which *is* compatible with the boundary condition—which *is* closed and free of singularity at some initial time, or more precisely on some initial space-like hypersurface. To construct such a geometry, take not a single spherically symmetrical distribution of mass, but many such mass centers. Let the number of centers and their spacing be so chosen as to curve up the space into closure.(2) The

(2) For a detailed but approximate treatment of the dynamics of such a lattice universe, see ref. [9]. For a precise analysis, consider the initial value problem at the moment of time symmetry or maximum expansion: ${}^{(3)}R = (16\pi G/c^2)\varrho$. Here $\varrho$ is the density of mass, equal for example to $\varrho_0$ inside each center of attraction, and vanishing elsewhere. Solve this equation by modifying the geometry of a 3-sphere of uniform curvature and radius $a$,

$$ds^2_{\text{ideal}} = a^2[d\chi^2 + \sin^2\chi(d\Theta^2 + \sin^2\Theta d\varphi^2)],$$

by a conformal factor $\psi$:

$$ds^2 = \psi^4 ds^2_{\text{ideal}}.$$

The initial value equation takes the form

$$\nabla^2\psi + (2\pi G/c^2)\varrho\psi^5 - (3/4a^2)\psi = 0.$$

Here the operator $\nabla^2$ is calculated from the metric of the ideal 3-sphere. This equation is to be solved throughout one lattice zone subject to the conditions (1) that $\psi$ have the appropriate symmetry within that zone and (2) that its normal derivative vanish at the zonal boundary. This is an eigenvalue problem which determines the radius a of the comparison sphere. When gravitational radiation is present the metric cannot be represented in such a simple form. However, there is still typically a factor like $\psi$ to be found—governed now not only by the distribution of mass, but also by the distribution of gravitational radiation.

dynamics of such a lattice universe before and after the moment of time symmetry agrees within a few percent or less with the dynamics of the Friedmann universe—with its filling by a uniform dust (zero pressure!) and its ideal uniform curvature. The corresponding expansion and recontraction of the lattice universe shows up not so much through any change in the geometry interior to the typical Schwarzschild zone, as through a change in the place of join between one zone and the next. The interface moves outward from the centers of attraction on each side of it, following the law of motion of a stone thrown out radially. It reaches a maximum distance. Then it falls back again towards both mass concentrations simultaneously. In this way the motion of these centers towards each other comes into evidence. The time for the expansion and contraction of the lattice universe—and of the boundaries of each Schwarzschild zone—is

$$\begin{pmatrix}\text{time for expansion}\\ \text{and recontraction}\\ \text{in length units}\end{pmatrix} = \pi \begin{pmatrix}\text{radius of lattice}\\ \text{universe at}\\ \text{maximum expansion}\end{pmatrix}$$

$$\simeq \pi \begin{pmatrix}\text{radius of one Schwarz-}\\ \text{schild zone at maximum}\\ \text{expansion}\end{pmatrix}^{3/2} \begin{pmatrix}\text{twice mass at center}\\ \text{of zone expressed in}\\ \text{length units}\end{pmatrix}^{-1/2} \qquad (7)$$

This quantity can be made arbitrarily large relative to the time required for light to cross one Schwarzschild zone by making the radius $b$ of the typical zone sufficiently large.

### *Non-Uniform Convergence to Flat Space Limit*

The order of the participants is important. Let one participant, $A$, select (1) any arbitrary but finite distance from one center of mass and (2) any arbitrary but finite lenght of time and (3) any arbitrarily small but non-zero departure from the ideal Schwarzschild geometry which he is willing to tolerate. Then the other participant, $B$, can pick an effective radius for the typical Schwarzschild zone at the moment of maximum expansion which is so great that the geometry inside that zone agrees wiht the ideal Schwarzschild geometry (1) to within the specified limits of accuracy (2) out to the stated distance and (3) for the stated time. However, if $B$ acts first, and specifies the zone radius at the moment of maximum expansion, then $A$ can always point to places so far away that the geometry there totally disagrees with the continuation of the Schwarzschild geometry of the original zone. $A$ can even point out that the space is closed and compatible with Mach's principle. Thus $A$ concludes that the geometry is asymptotically flat or closed according as he is forced to make the first move or allowed to wait until $B$ has fixed on dimensions. That $A$'s conclusions depend upon the order of his move can be said in another way: The convergence to the limit of an infinitely great lattice universe is non-uniform.

TABLE III

Schwarzschild geometry envisaged as the limit of the geometry of a closed lattice universe when the size of the typical lattice zone is allowed to go to infinity. This limiting process is compared in the table with the analogous limiting procedure in electrostatics. Notation: (1) $m^*(\text{cm}) = (G/c^2)m(\text{g})$, mass at center of each lattice cell (2) $4\pi b^3/3$, volume of lattice cell at "instant" (space-like hypersurface) of maximum expansion (3) $a$, radius of curvature of a comparison universe of uniform density and uniform curvature, also at the instant of mirror symmetry between past and future. This radius is determined as follows in terms of $m^*$ and $b$: The "Schwarzschild cells" are joined together on boundaries which are not sufficiently far out for the geometry there to be flat. The curvature of the Schwarzschild goemetry in a local Lorentz frame in a plane perpendicular to the zonal radius is $R_{2323} = 2m^*/b^3$. Identify this quantity with the curvature in a typical plane in the uniform comparison universe, $R_{2323} = 1/a^2$. Thus, find $a^2 \simeq b^3/2m^*$. Alternatively, write down the $_{00}$ component of Einstein's field equations (the principal initial value equation of Yvonne Fourès Bruhat) in the form

$$^{(3)}R + (\text{Tr}K)^2 - \text{Tr}K^2 = 2(8\pi G/c^4)\begin{pmatrix}\text{energy}\\ \text{density}\end{pmatrix}.$$

Note that the extrinsic curvature tensor $K_{ij}$ or "second fundamental form" vanishes on a time-symmetric space-like hypersurface. Note also that the scalar curvature invariant of a 3-sphere of radius $a$, expressed in terms of the physical components (carat symbol!) of the curvature is $^{(3)}\hat{R} = {}^{(3)}\hat{R}_{11} + {}^{(3)}\hat{R}_{22} + {}^{(3)}\hat{R}_{33} = (\hat{R}_{1212} + \hat{R}_{1313}) + (\hat{R}_{2121} + \hat{R}_{2323}) + (\hat{R}_{3131} + + \hat{R}_{3232}) = 6/a^2$. Identify the density of mass with $m/(4\pi b^3/3)$. Thus, have $(6/a^2) \simeq (16\pi G/c^2) \times \times (3m/4\pi b^3)$ or again the result $a^2 \simeq b^3/2m^*$. The number of lattice cells is approximately $N \simeq$ (volume of comparison universe)/(volume of cell) $\simeq 2\pi^2 a^3/(4\pi b^3/3) = (3\pi/2)^{5/2} (b/m^*)^{3/2}$ goes to infinity as size of typical cell goes to infinity).

| | Electrostatic example | Example from general relativity |
|---|---|---|
| Source (before modification) | Infinite periodic charge distribution $\varrho = \varrho_0 \cos kz$ | Single spherically symmetric concentration of mass in otherwise empty space |
| Effect of interest | Electric potential and thence the electric field | Metric of spacetime—and thence the inertial properties of every infinitesimal test particle |
| Is "effect" so uniquely associated with "source" in this idealized case that one can say effect is "produced" by source? | No—can add to $V$ any number of harmonics of form $r^n Y_n^{(m)}(\theta,\varphi)$ | No—the asymptotically flat Schwarzschild geometry and many other empty space geometries solve Einstein's equations for this source "distribution". |

| | Electrostatic example | Example from general relativity |
|---|---|---|
| Does "effect" satisfy the boundary condition listed in Table II? | No—none of these expressions for $V$ falls off as fast as $(1/r)$ at great distances | Schwarzschild geometry as normally conceived does not describe a closed universe |
| Modified situation which *is* compatible with the boundary condition | $\varrho = \varrho_0 \cos kz$ times $\exp(-r^2/a^2)$ | Many such masses spaced with reasonable uniformity through a closed universe |
| Scale factor associated with this new source | Range $a$ of charge distribution | Effective radius $b$ of typical Schwarzschild zone |
| Is source now well defined? | Yes | No. Must specify what gravitational waves if any are present—in other words, must specify otherwise undertermined features intrinsic to the three geometry in which the masses are imbedded at the moment of time symmetry.(3) |
| When specification of "source" has been completed, is it reasonable to think of "effect" as well determined by this specification plus boundary condition? | Yes—in this event can prove potential is uniquely determined by distribution of electricity. | Yes—expect other features intrinsic to this three geometry are now uniquely determined by $(_{00})$ component of Einstein's equation plus boundary condition of closure;(4) Mach's principle satisfied |
| Limiting procedure now envisaged | Range $a$ of charge distribution goes to $\infty$ | Effective radius $b$ of Schwarzschild zone goes to $\infty$ |
| For each finite value of the parameter $a$ or $b$ is the relevant boundary condition satisfied? | Yes—$V$ falls off as $1/r$ or faster at large $r$ | Yes—Schwarzschild zone is a piece of a closed universe in which Mach's principle can be censidered to apply |
| Is boundary condition satisfied for infinite value of this parameter? | No—$V$ does not fall off | No—Schwarzschild geometry is asymptotically flat |

### *Other Examples*

The ideal lattice universe is no more than one of many conceivable examples to illustrate how one can consider as closed — and compatible in general terms with Mach's principle—geometries which ostensibly are asymptotically flat. Three more examples may give a slight impression of how wide is the range of allowable geometries.

(3) See for example the "modified Taub universe" discussed in the text as an alternative to the lattice universe as a solution of Einstein's field equations which also satisfies the condition of closure.

(4) This uniqueness can be established in the case where the lattice universe contains no gravitational waves along the lines outlined in footnote.

*Lattice Universe with Gravitational Radiation*

In the lattice universe there may be present in addition to the "real" masses also the effective mass indirectly contributed by gravitational radiation. Then the inertial properties of test particles are affected by both sources of mass energy. (4)

*Modified Taub Universe*

It is not necessary to supply any "real" masses additional to the one original mass in order to secure a closed universe. Gravitational waves of sufficient strength will supply the required curvature. This one is seen from the example of the Taub universe [10]. There, gravitational radiation alone suffices to curve up the space into closure. In this 4-geometry consider the hypersurface or 3-geometry defined by the instant of time symmetry or maximum expansion. Perturb this geometry to the extent necessary to introduce a spherical ball of matter, at first arbitrarily small, eventually large or denser or both. Close to this mass the geometry is nearly Schwarzschildean. However, deviations from that limiting geometry become very great at distances comparable to the effective radius of the Taub universe.(5) In this universe it is not reasonable to speak of a geometry primarily determined by "real mass" and perturbed in only a minor way by gravitational radiation. On the contrary, the gravitational radiation is the primary determiner of the 4-geometry—and on the inertial properties of test particles. The one "real mass" produces only minor perturbations in the geometry except in its own immediate neighborhood.

*Unmodified Taub Universe*

The fourth example is the Taub universe itself, free of any "real matter" at all. This solution of Einstein's equations for a closed empty space is interpreted in the Appendix as a special case of a Tolman radiation filled universe in which (1) Tolman's electromagnetic radiation is replaced by gravitational radiation; (2) this gravitational radiation, instead of being effectively isotropic, is described by a single hyperspherical harmonic; and (3) this harmonic has the lowest possible order, or greatest possible wave length, compatible with the dimensions of the model universe.

(2) No investigation has been made of uniqueness when gravitational waves are present in this universe. However, there is a related problem where the uniqueness of the 3-geometry—for specified distribution of gravitational radiation—has been established as a consequence of the closure condition [6], [7], [8].

(5) A first order analysis of deviations from Schwarzschild geometry has been given by T. Regge and J. A. Wheeler, see ref. [11], but no attempt is made there to fit on to the Taub solution at greater distances.

*Does a Relation Between Inertia at One Place and Gravitational Radiation at Other Places Signify Circular Reasoning?*

Regardless of the details of the Taub universe, here is a closed space in which the inertial properties of every infinitesimal test particle are well determined. Yet there are no ordinary masses about, no interactions with which one can attribute the inertia of this test particle. Therefore, if Mach's principle is still to make sense, it is necessary to conclude that the distribution, not only of mass energy, but also of gravitational radiation, has to be specified in order completely to determine inertia—or, in the words of general relativity, completely to determine the geometry of spacetime. But gravitational radiation itself is described as an aspect of geometry and nothing more. Consequently one seems to be caught in a logical circle in trying to formulate Mach's principle. Apparently one has to give *the geomerty* in advance, not only in order

(1) to say in any well defined way what one means by the term "distribution of mass-energy", but also

(2) to specify what gravitational radiation is present, so that one shall thereby be enabled

(3) to determine the geometry of spacetime!

Evidently one can never feel happy about a formulation of Mach's principle that seems to contain this kind of circular reasoning. Therefore it is essential to demand a mathematically well defined statement of his principle if Mach's ideas are to be considered as having any relevance at all for present relativity physics.

*Not Circular: Specify 3-Geometry, Determine 4-Geometry*

Now for this mathematical formulation. It will be found to resolve the question of circular reasoning in this way, that what is *specified* is 3-dimensional geometry, and what is thereby *determined* is 4-dimensional geometry. At the same time it will help to clarify which features of gravitational radiation are freely disposable (field "coordinate" and its rate of change), and which features of the geometry are thereby determined (field "momentum").

## II. 3-GEOMETRY AND ITS RATE OF CHANGE AS KEYS TO THE PLAN OF GENERAL RELATIVITY

*What is the "Plan" of General Relativity?*

It is known often to help in answering one question to ask another. Therefore, it is fortunate for the search for a mathematical formulation of Mach's principle—a search now physically motivated — that another issue is currently under discussion. As Professor J. L. Synge stated it at the Warsaw conference, what is the *plan* of general relativity? What quantities can one freely specify, and what quantities are thereby determined? What is the inner structure of the dynamic theory of a geometry governed by Einstein's field equations?

*Plan 1: Initial Data on a Light-Like Hypersurface*

One plan of dynamics starts with a light-like hypersurface. In this approach as applied to the mechanics of a system of particles, one specifies the appropriate number of coordinates and momenta at the times when the respective world lines cross this null hypersurface. This formulation of mechanics has been investigated by P. A. M. Dirac and V. Fock. The corresponding formulation of geometrodynamics, particularly as relevant to the study of gravitational radiation, has been explored by R. Penrose, H. Bondi, R. Sachs and other, and has been described in a comprehensive report by Sachs at the Warsaw conference. However, this approach is not closely connected with the formulations of dynamics which are most widely used in other branches of physics. Whatever its relations with Mach's principle, they cannot be reported here because they have not been investigated.

*Plan 2: Coordinates and Momenta—or Intrinsic Geometry and Extrinsic Curvature—on a Space-Like Hypersurface*

Another plan of dynamics is more familiar. In particle dynamics give coordinates and momenta at points on the respective world lines which have a space-like relation each to the other. In electrodynamics give the field "comordinates" and "momentum"—the magnetic field $B(x^1, x^2, x^3)$ and the electric field $E(x^1, x^2, x^3)$—everywhere on a space-like hypersurface. In geometrodynamics again give on a space-like hypersurface the field coordinates and momenta—this time the 3-dimensional geometry intrinsic to this hypersurface,

$$ds^2 = {}^{(3)}g_{ik}(x^1, x^2, x^3)\, dx^i\, dx^k, \tag{8}$$

and the "extrinsic curvature" or so-called "second fundamental form"[12] telling how this hypersurface is curved—or to be curved—with respect to the enveloping—or yet to be constructed—4-dimensional geometry. When the 4-geometry is written in the form

$$\begin{aligned} d\sigma^2 = -d\tau^2 &= {}^{(4)}g_{\alpha\beta}dx^\alpha dx^\beta \\ &= {}^{(3)}g_{ik}\,(x^0, x^1, x^2, x^3)\, dx^i dx^k + \\ &\quad + 2N_i dx^i dx^0 + ({}^{(3)}g^{ik}N_i N_k - N_0^2)\,(dx^0)^2 \end{aligned} \tag{9}$$

with the condition

$$x^0 = x^{0*} \tag{10}$$

specifying the hypersurface in question, then the extrinsic curvature tensor is given by the expression[6]

$$K_{ik} = -(^1/_2 N_0)\,(\partial^{(3)}g_{ik}/\partial x^0 - N_{i|k} - N_{k|i}), \tag{11}$$

in which $x^0$ is understood as being fixed at the value $x^{0*}$. Here the vertical stroke is used to denote covariant differentiation with respect to the 3-geometry of the hypersurface, in contradistinction to the semicolon that marks covariant differentiation with respect to the 4-geometry. In terms of the extrinsic curvature tensor and its trace, the geometrodynamical momentum is[7]

$$\pi^{ik} = -(^{(3)}g)^{\frac{1}{2}}(K^{ik} - {}^{(3)}g^{ik}\mathrm{Tr}K) \tag{12}$$

*Interpretation of the Four Potentials or Metric Coefficients $N_0$ and $N_k$ as "Lapse Function" and "Shift Function"*

Some interpretation of the ADaM potentials $N_\alpha$ is appropriate. Imagine two thin ribbons of steel, distinguished from each other by the fact that one has painted on it the label $x^{0'} = 17.23$; the other, $x^{0''} = 17.27$. It is desired to construct out of these ribbons a rigid curtain. Paint cross-lines on the one ribbon at intervals which may gradually increase or gradually decrease but which never change irregularly or erratically. Label them $x' = 16$, 17, 18... Do the same on the second ribbon, taking care that the new pattern of crosslines is not widely different from the old pattern. Weld perpendicular uprights or "Lapses" to the first strip at $x' = 16$, 17, 18... As soon as these uprights have been cut to the right lengths, joined perpendicularly to the right points on the upper strip, and welded fast, the structure—with all the curves thus forced into it—will be determinate and rigid. To the waiting craftsman the architect sends two functions, $N_0(x')$ and $N'(x')$, the "lapse function" and the "shift function". The worker tabulates both at $x' = 16$, 17, 18... In two further columns he tabulates for the same values of $x'$ the product of $N_0$ and of $N'$ by the number $(x^{0''} - x^{0'}) \doteq 0.04$. The one column tells him to what heights to cut off the uprights which he has welded to the strip that is lying down. The other tells him how far one way or the other to *shift* upper ends before he welds them to the upper strip. At $x' = 18$ let the value of what might loosely be called $N'dx^0$ be 0.5. This implies that the corresponding upright is welded at its bottom to the cross line marked $x' = 18$.

[6] See R. Arnowitt, S. Deser and C. W. Misner, ref. [13] and earlier papers cited by them. This group of papers is referred to hereafter as ADaM. See also their chapter in L. Witten, editor, Gravitation: an introduction to current research, ref. [14]. This book is referred to hereafter as GIGR. See also references [15], [16] and [17].

[7] This expression comes from ADaM. Why it is most naturally expressed as a *contravariant* tensor *density* is intimately connected with the consideration of K. Kodaira see ref. [18].

The upper strip is shifted 0.5 *coordinate* units to the right. Thus the "lapse" is welded to it at a cross line marked $x' = 17.5$. How the "shift" changes from place to place—and how much the spacing between one coordinate mark and the next differs between the upper and lower steel sheets—together determine how much curvature is built into the curtain. Along this line of reasoning, generalized to three dimensions one sees at once the reason for the mathematical structure of Eq. (11).

*Interpretation in Terms of the Length of the Normal and the Difference in Space Coordinates at Its Two Ends*

To state the same interpretations of $N_0$ and $N_k$ in other words, return to expression (9) for the distance between a point $(x^0, x^1, x^2, x^3)$ that lies on one hypersurface, $x^0 = \text{const.}$ and another point $(x^0+dx^0, \ldots x^3+dx^3)$ on another hypersurface, $x^0+dx^0$. Here the $dx$'s are thought of as small but finite quantities. Let $dx^0$ be kept fixed (at the value $dx^0 = x^{0\prime\prime} - x^{0\prime} = 0.04$, for example!) but on the hypersurface so selected let one point, then another, be tried until the invariant separation between it and the fixed point on the lower surface is extremized. Vary $d\sigma^2$ with respect to $dx^k$ and set the coefficient of $\delta\, dx^k$ equal to zero:

$$2\,{}^{(3)}g_{ik}dx^i + 2N_i dx^0 = 0. \tag{13}$$

Solve for $dx^i$ and find

$$dx^i = -{}^{(3)}g^{ik}N_k dx^0 = -N^i dx^0. \tag{14}$$

The extremal value of the separation comes out—reasonably enough—to be time-like:

$$d\tau = N_0 dx^0. \tag{15}$$

Thus the "lapse function" $N_0$ represents the *proper* time separation between two hypersurfaces—measured normally—per unit of difference in their time *coordinates*. The vectorial "shift function" $N^i$ represents the *coordinates* at the base of the normal diminished by the coordinates at the summit of the normal, this difference again being referred to a unit difference between the time *coordinates* of the two hypersurfaces.

*-Lapse and Shift Functions Required in Addition to 3-Geometry to Defined 4-Geometry*

Evidently it is not enough to specify the geometries ${}^{(3)}g_{ik}$ *intrinsic* to a one parameter family of hypersurfaces in order to have a well defined 4-geometry. One must in addition tell how these hypersurfaces are *related* to each other. One must tell how far apart the surfaces are ("lapse function") and how they are displaced space-wise one with respect to another ("shift func tion").

*Arbitrary Lapse and Shift Functions Plus Arbitrary 3-Geometry Determine Field "Momentum"; but Arbitrary Field "Momentum" and Arbitrary 3-Geometry are Ordinarily Incompatible.*

From the field "coordinate" ${}^{(3)}g_{ik}$ and its rate of change with respect to the parameter $x^0$, *plus* information about the "lapse" and "shift" as function of position one can determine the "extrinsic curvature" $K_{ik}$ and the associated field "momentum" (Eq. (11)). However, the converse is not generally true. If the field "coordinate" ${}^{(3)}g_{ik}$ and the field "momentum" or the extrinsic curvature $K_{ik}$ are both specified arbitrarily, they will ordinarily be incompatible. *The independent specification of the field coordinate and the field momentum is the wrong way to define initial value conditions in general relativity.*

*The Initial Value Equations*

The incompatibility of arbitrary intrinsic geometry of field "coordinate" ${}^{(3)}g_{ik}$ with arbitrary extrinsic curvature or field "momentum" $\pi^{ij}$ follows from four of Einstein's ten equations. These *initial value equations*(8) have to do with conditions *on* the space-like hypersurface:

$$ {}^{(3)}R+(\mathrm{Tr}K)^2-\mathrm{Tr}K^2 = 2\,(8\pi G/c^4)\ \begin{pmatrix}\text{energy}\\ \text{density}\end{pmatrix}, \tag{16} $$

$$ (K_i^k-\delta_i^k\mathrm{Tr}K)_{|k} = (8\pi G/c^4)\ \begin{pmatrix}\text{density of flow of}\\ \text{energy in } i\text{-direction}\end{pmatrix}. \tag{17} $$

These initial value equations pose in sharpened form the issue, what is the *plan* of general relativity: what quantities

(1) can be freely and independently specified and yet

(2) suffice completely to specify the past and future of the 4-geometry?

*Plan 3: Specify Completely Independently the Field Coordinates on Two Hypersurfaces*

This question leads in turn directly to the two-surface formulation of dynamics, where one specifies no momenta, only coordinates (or conversely) —but coordinates on two hypersurfaces rather than one.(9) Moreover, *the field coordinates on the one surface are specified quite independently of those on the other surface.* The complete freedom that one has in this way of specifying the initial value data would seem to be what one wants when he asks for a workable statement of the *plan* of general relativity (Table IV).

(8) See references [19]–[24] and the chapter by Y. Fourès in GICR.

(9) The following is based on a paper of R. F. Baierlein, D. H. Sharp and J. A. Wheeler, ref. [25], which in turn is based on (1) the ref. [26] and (2) an analysis by R. F. Baierlein which led to the variational principle of Eq. (31).

TABLE IV

The *plans* of electromagnetism and general relativity as expressed in terms of the two surface formulation of dynamics. The field "coordinates" are specified on two space-like hypersurfaces—most simply on two hypersurfaces which have an infinitesimal separation.

| | Electromagnetism | Gravition |
|---|---|---|
| The physically significant field quantities | Components of the electromagnetic field | Components of the Riemann curvature tensor |
| The coordinate independent object which they define | A 2-form: a honeycomb-like structure of tubes of force | The intrinsic structure of the 4-geometry in the neighborhood (corrections to the Euclidean pattern of distances between one point and another in a great table of *local* "airline" (geodesic) distances) |
| The dynamic equations which tell how this object changes from place to place | Maxwell's 8 equations | Equations that refer directly to the curvature components |
| The potentials normally introduced to simplify the analysis of these equations | The 4 components of the electromagnetic potential, $A_\alpha$ | The 10 components of the metric tensor, $g_{\mu\nu}$ |
| Notation used for these potentials when spacetime is sliced into spacelike hypersurfaces | The magnetic potential $A$ with components $A_k$ and the electrostatic or scalar potential $\varphi = -A_0$ | 6 components of 3-metric ${}^{(3)}g_{ik}$ intrinsic to a slice; the normal *proper* time separation $N^0$ between two hypersurfaces per unit of difference in their *time coordinates*; and the differences $N^i$ (or more conveniently, $N_k = {}^{(3)}g_{ki}N^i$) between space coordinates at the two ends of such a normal, again per unit of difference in the time coordinates of the two hypersurfaces |
| The dynamical problem as formulated in variational language for a region of spacetime bounded by two space-like hypersurfaces $\sigma$ and $\sigma''$ | Give $A'$ on $\sigma'$ and $A''$ on $\sigma''$: in between take any trial functions $A(x^0,x^1,x^2,x^3)$ and $\varphi(x^0, x^1, x^2, x^3)$ calculate action integral; then vary the four potentials until the action is extremized | Give ${}^{(3)}g_{ik}(x^1x^2x^3)$ (this *defines* $\sigma'$) and arbitrarily call the value of $x^0$ on this surface some number $x^{0\prime}$; similarly, give ${}^{(3)}g''_{ik}$ and $x^{0\prime\prime}$. In between choose any trial values for the 10 potentials, compute action; extremize with respect to choice of the potentials |

| | Electromagnetism | Gravition |
|---|---|---|
| The simpler version of this variational problem relevant for the formulation of initial value problem and Mach's principle; the two hypersurfaces have an infinitesimal separation. | Give $A(x^1,x^2,x^3)$ and $\partial A/\partial t$; have a simpler action principle in which $\varphi(x^1,x^2,x^3)$ is the only function to be adjusted | Give ${}^{(3)}g_{ik}(x^1,x^2,x^3)$ and $\partial {}^{(3)}g_{ik}/\partial t$; have a simpler action principle in which only the "lapse function" $N^0(x^1, x^2,x^3)$ and the "shift function" $N_k(x^1,x^2,x^3)$ are to be varied |
| Variational problem well defined in an *open* space? | No | No |
| Pay-off from this extremization in a *closed* space | Value of $\varphi$ on the spacelike surface from which one can then calculate the electric field $\boldsymbol{E}$—the "momentum" conjugate to the already specified field "coordinate", $\boldsymbol{B}$ | Values of $N_0$ and $N_k$ from which one can calculate the "extrinsic curvature" $K_{ik}$ of the thin sandwich or the "momentum" conjugate to the geometrodynamical "coordinate" or intrinsic geometry ${}^{(3)}g_{ik}$ |
| What equation has automatically been solved by this extremization? | The initial value equation div $\boldsymbol{E}=4\pi\varrho$ in which there appeared superficially to be 3 unknown functions of position | The initial value equations $(K_i^k-\delta_i^k \mathrm{Tr}K)\vert_k=\dfrac{16\pi G}{c^4}\hat{T}_{\perp i}$ ${}^{(3)}R+(\mathrm{Tr}K)^2-\mathrm{Tr}K^2=\dfrac{16\pi G}{c^4}\hat{T}_{\perp\perp}$ in which there appeared ostensibly to be 6 unknown functions of position |
| Situation now in brief | Have *compatible* values for field coordinate and field momentum on initial spacelike hypersurface | Have *compatible* values for field coordinate and field momentum on initial spacelike hypersurface |
| Further pay-off? | Now have just the right amount of consistent initial value data to predict the electromagnetic field everywhere in space and at all times | Now have just the right amount of consistent initial value data to determine the geometry of *spacetime* in past, present and future—and hence the inertial properties of every infinitesimal test particle |
| Recapitulation of what information was required for this prediction | (1) Maxwell's equations (2) Law of motion of charges (3) Specification of divergence-free magnetic field and its time derivative on a closed space like hypersurface | (1) Einstein's equatiens (2) Dynamic law for the fields or objects responsible for the stress-energy tensor on the right side of Einstein's equations |

| | Electromagnetism | Gravition |
|---|---|---|
| | (4) Specification of positions and velocities of charges at points where their world lines cross this hypersurface | (3) Specification of closed space-like 3-geometry and its rate of change with respect to a parameter $x^0$—a parameter which otherwise has no direct physical meaning. (4) Specification of initial value data for fields or objects responsible for $T_{\mu\nu}$ |
| Is relation between "effect" and "source" well defined? ("Mach's principle") | "Effect" = electromagnetic field. Relation well defined only if "source" is understood to imply specification on space-like hypersurface of both (1) positions of charges *and* (2) magnetic field and its time rate of change | "Effect" = inertial properties of test particle = geometry of spacetime. Relation well defined only if "source" is understood to imply specification on space-like hypersurface of both (1) density and flow of mass-energy *and* (2) intrinsic 3-geometry and its rate of change with respect to some parameter $x^0$—this latter reasonably enough because how otherwise would one have a geometry with respect to *specify* the distribution and flow of mass? |

*Meaning of Phrase, "Independently Specifiable Coordinates?*

It is necessary to state in what sense one is to understand the phrase, "are specified quite independently of those on the other surface". What one says on this point depends upon the question whether he is thinking in the context of classical physics or quantum physics.

*"Two-Surface" Formulation of Harmonic Oscillator Problem*

By way of illustration consider the simple harmonic oscillator. Give the coordinate $x'$ at the time $t'$ and the coordinate $x''$ at the time $t''$. In this way fix the end points of a trial history,

$$x(t) = x_H(t). \tag{18}$$

The *classical* history in the intervening time interval is to be selected in such a way as to extremize the action integral

$$\begin{aligned} I_H &= \int_{x',\,t'}^{x'',\,t''} L\left(x_H(t), dx_H(t)/dt, t\right) dt \\ &= (m/2) \int (\dot{x}_H^2 - \omega^2 x_H^2)\, dt. \end{aligned} \tag{19}$$

The solution is well known—a simple harmonic motion of circular frequency $\omega$:

$$x_H(t) = x_{H\,\text{classical}}(t) = \frac{x' \sin \omega (t''-t) + x'' \sin \omega (t-t')}{\sin \omega (t''-t')}. \tag{20}$$

Associated with this "classical history" is the action—"Hamilton's principal function"—given by the expression

$$I_{H_{\text{classical}}} = [m\omega/2 \sin \omega (t''-t')] [(x'^2+x''^2) \cos \omega (t''-t') - 2x'x'']. \tag{21}$$

*The Quantum Propagator and its Relation to the Classical Action*

In quantum mechanics one gives arbitrarily, not the coordinates at two times, but the state function or probability amplitude $\psi(x',t')$ at one time, $t'$, and asks for its value $\psi(x''t'')$ at some later time, $t'$. The function at the new time can be found by solving the Schroedinger equation numerically or otherwise. The focus of attention shifts from this equation to its solution in Feynman's formulation of quantum mechanics [27]-[31]. A propagator gives the desired function in terms of the arbitrarily specified initial function:

$$\psi(x'',t'') = \int_{-\infty}^{+\infty} < x'',t''|x',t' > \psi(x',t')\, dx'. \tag{22}$$

Feynman writes this propagator as the sum of elementary propagation amplitudes,

$$< x'',t''|x',t' > = N \sum_H \exp(iI_H/\hbar). \tag{23}$$

Every conceivable history contributes with the same weight; only the phase differs from one history to another. Destructive interference automatically cuts down the *effective* contribution of the non-classical histories. The sum reduces in the case of the harmonic oscillator to an expression of the form

$$< x'',t''|x',t' > = N_1 \exp(iI_{H\text{classical}}/\hbar) \tag{24}$$

where in the exponent Hamilton's principal function has the value (21).

*Normal Compatible Versus Exceptional Incompatible Specification of "Two Surface" Data in Classical Problem*

In the classical problem a difficulty arises when the time interval $(t''-t')$ is an integral multiple of a half period. After an even number of half periods the coordinate must return to its initial value; after an odd number, it must come to the negative of its initial value. (1) If $x''$ does not agree with $x'$ in tqe one case, or with $-x'$ in the other case, the end point data have been inconsistently specified. (2) Even if they have been consistently given, the momentum with which the motion starts off at the one end point—and with which it returns to the other end point—is completely indeterminate. In both cases the variational problem is indeterminate.

*No Problem of Incompatibility in Quantum Propagator*

No such problem of compatibility of the "end point data" or "two surface data" arises in the quantum formulation. When the interval $(t''-t')$ is a half period, the propagator reduces to one type of Dirac delta function,

$$< x'', t''|x', t > = -i\delta\,(x''+x'); \tag{25}$$

and to another type when the interval is a full period:

$$< x'', t''|x', t'> = -\delta\,(x''-x'). \tag{26}$$

In other words, *the quantum propagator remains well defined for all specifications of the two surface data*, regardless of any specialities in the classical problem in one case or another.

*The Quantum Problem Always at the Background of Classical Analysis*

No one has found any way to escape the conclusion that geometrodynamics, like particle dynamics, has a quantum character. Therefore, the quantum propagator, not the classical history, is the quantity that must be well defined. Consequently it will not be considered a source of concern that one can specify the 3-geometries ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$ intrinsic to two hypersurfaces in such a way that the action functional for general relativity admits no extremum. Such cases are the geometrodynamical generalization of the special cases just encountered for the harmonic oscillator. Only on this understanding will it be justified to say that the 3-geometry on one hypersurface is specified quite independently of the 3-geometry on the other hypersurface.

*Concentration on the Case of Two Nearby Hypersurfaces*

Of greatest simplicity is the case that alone will be considered here in any detail, where the two hypersurfaces are "close together". Then the determination of the momentum from the values of the coordinate on the two surfaces is the most immediate. This step carries one halfway through the dynamic problem. Having consistent and singularity free initial value data for momentum and coordinate at the initial time, one is in a position to complete the solution—to determine without any ambiguity the history of the system for at least a finite proper time into the past and future.(10) For

(10) The proof that this can be done in the case of general relativity is given in the book of A. Lichnerowicz, ref. [22].

this purpose one uses the standard dynamical equations:

1. Hamilton's equations for a system of particles,
2. Maxwell's equations in the electromagnetic case,
3. Einstein's equations in the case of interest here.

*Alternative Ways to Apply the Two-Surface Formulation of Dynamics*

Alternative ways of applying the two-surface formulation to particle mechanics, electrodynamics and general relativity differ from one another by the apportionment of the analytic load between a variational principle and differential equations.

*Excluded Option 1: Well Separated Hypersurfaces and Exclusive Reliance Upon the Variational Method*

One can avoid any use at all of differential equations in calculating the history of the system, whether a particle, the electromagnetic field or geometry. Instead one can rely entirely on the idea of extremizing an action integral extended over the entire interval of time for which one wants to know the history. For the particle, one specifies $x'$ at $t'$ and $x''$ at $t''$. One regards as the function to be varied, either $x(t)$ alone, as in the familiar Lagrangian variational principle of Eq. (19), or *both* $x(t)$ and $p(t)$, *independently* as in the Hamiltonian formulation

$$\int_{x',t'}^{x'',t''} [p(t)\dot{x}(t) - H(p(t), x(t), t)]\, dt = \text{extremum}. \tag{27}$$

To express electrodynamics in variational language one calls on the familiar vector and scalar potentials $\boldsymbol{A}$ and $\varphi$:

$$\boldsymbol{B} = \text{curl}\, A, \tag{28}$$

$$\boldsymbol{E} = -\partial A/\partial t - \text{grad}\, \varphi, \tag{29}$$

so that half of Maxwell's equations are automatically satisfied. The other four follow from extremizing the integral

$$I = \int [(1/8\pi)(\boldsymbol{E}^2 - \boldsymbol{B}^2) + (\boldsymbol{j}\cdot\boldsymbol{A} - \varrho\varphi)](1/c)\, d^4x. \tag{30}$$

Specified in advance are

(1) the charge and current densities $\varrho$ and $\boldsymbol{j}$ (both in charge units/ (length unit)$^3$) throughout the 4-dimensional region bounded by the two hypersurfaces,

(2) $\boldsymbol{B}$ on each of the two surfaces: in such a way that div $\boldsymbol{B}$ vanishes —this specification being made by giving $\boldsymbol{A}$ on each of the two surfaces (arbitrary gauge; no effect on the physics from the change $A \to A + \text{grad}\, \lambda$).

Varied everywhere between the two surfaces to extremize $I$ are $\varphi$ (quite independently) and $\boldsymbol{A}$ (subject only to the specification of $\boldsymbol{A}'$ and $\boldsymbol{A}''$ at $t'$ and $t''$, respectively).

*Option* 1 *Continued: The Variational Principle for General Relativity*

The appropriate action principle in general relativity [9]—when supplemented with source terms—reads

$$I_4 = \int_{x^{0\prime},\, {}^{(3)}g'_{ij}}^{x^{0\prime\prime},\, {}^{3}g''_{ij}} \{\pi^{ij}\partial^{(3)}g_{ij}/\partial x^0 - N_0\,({}^{(3)}g)^{1/2}L^*(g^{\cdot\cdot}, A\,\ldots) +$$

$$+ N_0({}^{(3)}g)^{1/2}[{}^{(3)}R - {}^{(3)}g^{-1}(\mathrm{Tr}\,\pi^2 - \tfrac{1}{2}\,(\mathrm{Tr}\pi)^2)] + 2N_i\pi^{ij}{}_{|j}\}d^4x. \qquad (31)$$

This variational principle results from adding complete derivatives to the familiar Lagrange integrand of general relativity, $({}^{(4)}R+L)(-g)^{1/2}$, and translating the result into the terminology of ADaM. Here $L^*$ is $8\pi G/c^4$ times the invariant or scalar Lagrangian for whatever fields have energy and produce gravitational effects, expressed in terms of (1) the *covariant* components of that field (the field components $F_{\alpha\beta}$ in electromagnetism for example) and (2) the elements $g^{\cdot\cdot}$ of the matrix reciprocal to $g_{\alpha\beta}$:

$$g^{\mu\nu} = \left\| \begin{matrix} {}^{(3)}g^{jk} - N^jN^k/N_0^2 & (N^k/N_0^2) \\ (N^j/N_0^2) & -(1/N_0^2) \end{matrix} \right\|. \qquad (32)$$

Here ${}^{(3)}g^{jk}$ is in turn the matrix reciprocal to ${}^{(3)}g_{jk}$ and

$$N^j \equiv {}^{(3)}g^{jk}N_k. \qquad (33)$$

In (31) there are 16 functions of space and time to be varied in the region between the two surfaces in such a way as to extremize the integral. Ten of these quantities—reasonably enough—are metric coefficients: the six ${}^{(3)}g_{ik}$ free except for having to reduce to the prescribed values at the two surfaces; and the lapse and shift functions $N_0$ and $N_i$ (not $N^i$!) which are entirely freely disposable. The remaining six quantities, the momentum components $\pi^{ij}$, are also adjustable without any conditions at all. In spirit this adjustment of the momenta is like that of the particle momentum $p(t)$ in Eq. (27). At the start the function is free even to the extent that its terminal values are free. However, extremization forces in that case the condition

$$\dot{x}(t) = \partial H(p, x, t)/\partial p \qquad (34)$$

from which the momentum is completely determined in terms of the velocity. Similarly here[6] ("Palatini philosophy"). Vary (31) with respect to $\pi^{ij}$. Set the variation equal to zero for arbitrary $\delta\pi^{ij}$. Find thus six equations *determining* the six $\pi^{ij}$ in terms of the $N_\alpha$ and ${}^{(3)}g_{ik}$ and their derivatives. These equations are equivalent to Eq. (12) for the momentum in terms of the extrinsic curvature plus the definition of Eq. (11), for this extrinsic curvature. If one were concerned with translating the variational principle (31) back into differential equations, instead of using it *as* a variational principle, he would: (1) Vary the lapse and shift functions. (2) Set the coefficients of the four $\delta N_a$ equal to zero. (3) Find in this way the four initial value equations (16, 17) that have to do primarily with geometry within the successive hypersurfaces (4). Obtain the other six more "dynamic" components of Einstein's ten field equations by varying the six ${}^{(3)}g_{ik}$ and setting the coefficients of the six $\delta^{(3)}g_{ik}$ equal to zero. But in using (31) in its alternative function—to *replace* all differential equations (in the spirit of Rayleigh and Ritz)—one will (1) substitute into (31) the expressions for the six $\pi^{ij}$ in terms of the sih ${}^{(3)}g_{ij}$ and the four $N_a$ and their derivatives, and (2) use numerical methods or ten analytical trial functions (${}^{(3)}g_{ij}$, $N_a$) containing adjustable parameters to extremize the action integral $I$. Unhappily the extremum, rather than being a minimum or a maximum, is often saddle of higher order, as one can convince himself even in the simpler problem of a single particle bound in a harmonic oscillator potential. This kind of variational principle does not normally lend itself either (1) at the theoretical level to establishing existence proofs or (2) at the practical level to doing calculations.

*Most Favored Option* 2: *Two Infinitesimally Separated Hypersurfaces*; *Use of Variational Principle to Solve* 2-*Surface Initial Value Problem within the Sandwich, then Einstein Field Equations to Predict All the Rest of the* 4-*Geometry*; *Electrodynamics as an Example*

Proofs of the existence of solutions are much more widely known in manifolds with positive definite metric [32][33] than in manifolds with indefinite metric. Moreover the real problem to be treated is the initial value problem. Once it has been solved one knows from the work of Lichnerowicz [22] that the solution can be continued by way of Einstein's ten field equations. Therefore concentrate on the thin sandwich problem. The essential ideas are most easily seen in the case of electromagnetism. The magnetic potential has been specified on both surfaces ($\boldsymbol{A}'$ and $\boldsymbol{A}''$) but the separation between them has been allowed to go to zero. Therefore, the terms in $\boldsymbol{B}^2$ and in $\boldsymbol{j}\cdot\boldsymbol{A}$ *are not adjustable* in this limit. The variational principle reduces to the form

$$I = \int [(E^2/8\pi) - \varrho\varphi]\, d^3x = \text{extremum}, \tag{35}$$

to be extremized with respect to the single unknown potential $\varphi$. The theory of this variational problem is well known. Out of the extremization—conducted analytically or by the Rayleihg–Ritz method or in any other way—comes a potential $\varphi$ that satisfies the differential equation

$$\nabla^2\varphi = -4\pi\varrho - (\partial/\partial t)\,\mathrm{div}\,\boldsymbol{A}. \tag{36}$$

This potential generates an electric field

$$\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t - \mathrm{grad}\,\varphi \tag{37}$$

that automatically satisfies the initial value equation

$$\operatorname{div} \boldsymbol{E} = 4\pi\varrho. \tag{38}$$

One now has at hand $\boldsymbol{E}$ and $\boldsymbol{B}$ which can serve as the consistent starting points for the dynamic analysis. For this purpose apply the other six equations of Maxwell and predict the entire past and future of the electromagnetic field.

*Concept of Thin Sandwich in Geometrodynamics*

Similarly in relativity one seeks to adjust four potentials, the lapse function $N_0$ and the shift function $N_i$, so as to generate an extrinsic curvature tensor $K_{ik}$, according to Eq. (11), which will satisfy initial value Eqs (16) and (17). This done, the initial value problem is solved. To formulate the appropriate "thin sandwich" variational principle, proceed here as in electrodynamics to the limit in which the sandwich is idenfinitely thin. One can state this idea in two alternative ways.(9) (1) Give nearly identical ${}^{(3)}g'_{ik}$ and ${}^{(3)}g''_{ik}$. Take any arbitrary numbers $x^{0\prime}$ and $x^{0\prime\prime}$ for the labels to be applied to these two hypersurfaces. In the definition of the extrinsic curvature $K_{ik}$ (Eq. (11)) there enters the term $\partial^{(3)}g_{ik}/\partial x^0$. Adopt for this term the value $({}^{(3)}g''_{ik}-{}^{(3)}g'_{ik})/(x^{0\prime\prime}-x^{0\prime})$. Apparently the value of $K_{ik}$ will depend on $(x^{0\prime\prime}-x^{0\prime})$. Actually it will not. All that ever matters in $K_{ik}$ or anywhere else is the *product* of $(x^{0\prime\prime}-x^{0\prime})$ by the lapse function $N_0$. If a big value is used for $(x^{0\prime\prime}-x^{0\prime})$, a small value will come out of the variational principle for $N_0$, and conversely. One sees this invariance property of the product also in another way, that the normally measured interval of *proper* time between the two hypersurfaces (Eq. (15)) is $N_0(x^{0\prime\prime}-x^{0\prime})$. Therefore, in this formulation one takes as the quantities to be varied only the *products*

$$\eta_0 = N_0\,(x^{0\prime\prime}-x^{0\prime}), \tag{39}$$

$$\eta_k = N_k\,(x^{0\prime\prime}-x^{0\prime}) \tag{40}$$

and never lets the individually arbitrary quantities $N_0$, $N_k$, $x^{0\prime}$, $x^{0\prime\prime}$ show up. To this conceptually simpler formulation of what is kept fixed during the variation (${}^{(3)}g'_{ik}$ and ${}^{(3)}g''_{ik}$) there is an alternative and mathematically sharper statement. (2) Consider a continuous one-parameter ($x^0$) family of 3-geometries ${}^{(3)}g_{ik}\,(x^0; x^1, x^2, x^3)$. Then the initial value problem under consideration is defined by a knowledge of ${}^{(3)}g_{ik}$ and $\partial^{(3)}g_{ik}/\partial x^0$ for some one fixed value of $x^0$. The associated variational problem is found by dropping the factor $dx^0$ in the integrand $d^4x$ in Eq. (31).

*The "Intrahypersurface Variational Principle" for the Initial Value Problem of General Relativity*

Now that only a three-fold space integration is called for, the next to the last term in Eq. (31) can be integrated by parts:

$$2N_i\pi^{ij}{}_{|j} \to -\pi^{ij}\,(N_{i|j}+N_{j|i}). \tag{41}$$

In a non-Euclidean topology more than one coordinate system is generally required to cover a manifold without singularity. Each is defined in its own coordinate patch. It might appear that a problem of transition arises in passing from one patch to another in the integration by parts. The absence of any such difficulty is guaranteed by the covariant character of the differentiations in (41). Moreover, the surface integral disappears in the simplest example of a closed space, a manifold with the topology of the 3-sphere $S^3$. Thus, let the integration start in the neighborhood of one point $P$ in $S^3$. Let it extend out to a boundary with the topology of the 2-sphere $S^2$. As the range of integration is widened, $S^2$ at first swells more and more. Later it begins to decrease in size. Eventually, as the integration extends over the whole 3 space, the boundary collapses to nothingness at some point other than $P$. No surface integral is left. Also no derivatives of the $\pi^{ij}$ are left in Eq. (31). Therefore everywhere that these momenta appear, they are easily expressed in terms of the curvature tensor $K_{ij}$ by Eq. (12) and the $K_{ij}$ are then expressed—via Eq. (11)—in terms of the quantities that one really thinks of varying: the lapse and the shift functions. The first substitution leads by simple algebra to the formula,

$$I_3 = \int \{{}^{(3)}R - (\mathrm{Tr}\,K)^2 + \mathrm{Tr}\,K^2 - \mathrm{L}^*(g^{\cdot\cdot}, A\ldots)\} ({}^{(3)}g)^{1/2} N_0 d^3x, \tag{42}$$

for the quantity to be extremized. In this "intrahypersurface" (IHS) variational principle, as in other applications of the Lagrangian method to dynamics, the "kinetic" term $(\mathrm{Tr}K)^2 - \mathrm{Tr}K^2$, appears with a sign opposite to that of the "potential" term ${}^{(3)}R$, whereas in the initial value equation (16) for the *energy density* these terms appear—reasonably enough—with the same sign. The second substitution—writing the $K_{ij}$ and the $g^{\alpha\beta}$ in terms of the four functions to be varied, the $N_\alpha$ by using Eqs (11) and (32)—is better left understood than carried out explicitly!

*The Also Useful Option 3: Exclusive Reliance on Differential Equations to Analyze the Dynamics of General Relativity*

Option 3 for analyzing the "plan" of general relativity, like Option 2, starts with a specification of ${}^{(3)}g_{ik}$ and $\partial{}^{(3)}g_{ik}/\partial x^0$ over the entirety of a closed space-like hypersurface; in more picturesque language, it presumes a specification of two "nearby" 3-geometries ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$. Here "nearby" is to be tested after the event by calculating $N_0$ and from it (Eq.(15)) finding if the proper time separation between the two hypersurfaces is or is not small compared to the scale of the space-like variations in ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$. In addition the density of energy—and energy flow—have to be given, just as in Option 2. The difference is only that the four potentials $N_\alpha$ are to be found by solving the four Eqs (16), (17)—not by directly trying to extremize the action integral $I_3$ of Eq. (42). Once the lapse and shift have been found, however, there is no difference in what one does between Option 3 and Option 2. (1) Calculate the extrinsic curvature $K_{ik}$. (2) Calculate the field momentum $\pi^{ik}$. (3) Use all ten of Einstein's equations to predict the 4-geometry in past and future.

*Verification that the Intrasurface Variational Principle and the Initial Value Equations are Equivalent*

On the right hand side of the initial value equations stand the density of energy and energy flow, a total of four quantities. In contrast, the variational principle (42) makes reference to *all* of the covariant components of the field responsible for this energy. One could, therefore, be concerned whether the two approaches will give the same result. To check this point, vary the $N_\alpha$ in the variational principle of (42). Set the coefficients of the $\delta N_\alpha$ equal to zero. Finally, compare with the initial value equations. The variation of the field Lagrangian is the most complicated part of this program. Write

$$\delta[N_0 L^*(g^{\cdot\cdot}A\ldots)] = L^*\delta N_0 + N_0(\partial L^*/\partial^{(4)}g^{\alpha\beta})(\partial^{(4)}g^{\alpha\beta}/\partial N_\gamma)\delta N_\gamma. \tag{43}$$

Evaluate the derivatives of the components of the reciprocal metric tensor by using Eq. (32) for that tensor. Express the derivatives of the Lagrange function in terms of the stress-energy tensor of the field in question, employing for this purpose the standard formula [34]

$$T^*_{\alpha\beta} = (-g)^{1/2}(\partial/\partial g^{\alpha\beta})(-g)^{1/2}L^* = (\partial L^*/\partial g^{\alpha\beta}) - \left(\frac{1}{2}\right) g_{\alpha\beta}L^*. \tag{44}$$

Here $T^*_{\alpha\beta}$ ($m^{-2}$) is an abbreviation for $(8\pi G/c^4)$ times the usual stress-energy tensor $T_{\alpha\beta}$(kgm²/sec²m³). Find that all those terms in Eq. (43) go out which contain an undifferentiated $L$ factor. Those that remain give

$$2[T^*_{\perp\perp}\delta N_0 + T^{*k}_{\perp}\delta N_k]. \tag{45}$$

Here

$$T^*_{\perp\perp} \equiv (T^*_{00} - 2N^kT^*_{0k} + N^iN^kT^*_{ik})/N_0^2 =$$

$= T^{*\perp\perp} = (8\pi G/c^4)$ (density of energy as corrected for the ordinarily oblique coordinate system in use, a scalar with respect to coordinate changes in the hypersurface)

and (46)

$$T^{*\cdot k}_{\perp} \equiv {}^{(3)}g^{km}(T^*_{0m} - N^sT^*_{sm})/N_0;$$

$T^{*k\perp} = -T^{*k}_{\perp} = (8\pi G/c^4)$ (density of flow of energy, corrected for oblique coordinate system off surface, a contravariant vector with respect to coordinate changes in the hypersurface. (47))

The rest of the variational analysis is straightforward. One verifies the agreement with the initial value equations in all detail.

*Precisely What Features of the Energy are Specified on the Hypersurface*

As the quantities which are specified on the hypersurface in the *initial value equations* one evidently thinks most naturally of $T^{*}_{\perp\perp}$ and $T^{*k}_{\perp}$, not the much more coordinate dependent $T^{*}_{\alpha\beta}$. As regards the variational principle, it is clear that it can be changed—if only the change reproduces the initial value equations. Therefore, the Lagrange function, which may be complicated or unknown or both, can be replaced by an expression which will have the same variation (45). Thus, one comes to the modified variation principle

$$I_3^* = \int \{[{}^{(3)}R-(\operatorname{Tr} K)^2+\operatorname{Tr} K^2-2T^{*}_{\perp\perp}]\,N_0-2T^{*k}_{\perp}N_K\}\,({}^{(3)}g)^{1/2}d^3x. \quad (48)$$

*Elimination of the Lapse Function*

The lapse function $N_0$ enters only algebraically in the time component (16) of the initial value equations and in the variational principle (48). To bring this fact most clearly into evidence, introduce the abbreviation

$$\gamma_{ij} = ({}^1/_2)[N_{i|j}+N_{j|i}-\partial{}^{(3)}g_{ij}/\partial x^0] \quad (49)$$

and write

$$\gamma_2 = (\operatorname{Tr}\gamma)^2-\operatorname{Tr}\gamma^2 \quad (50)$$

("shift anomaly"). Then

$$K_{ij} = \gamma_{ij}/N_0. \quad (51)$$

$K_{ij}$ measures the true extrinsic curvature, having to do with changes in space-like distances per unit of *proper* time between two hypersurfaces. In contrast, $\gamma_{ij}$ performs a similar function when one does not yet know the lapse function, or scale of proper time, so that one has to use a purely nominal time *coordinate* $x^0$. The "kinetic" term in the variational principle becomes

$$(\operatorname{Tr} K)^2-\operatorname{Tr} K^2 = \gamma_2/N_0^2. \quad (52)$$

The modified variational principle becomes

$$I_3^* = \int \{({}^{(3)}R-2T^{*}_{\perp\perp})N_0-\gamma_2/N_0-2T^{*k}_{\perp}N_k\}\,({}^{(3)}g)^{1/2}\,d^3x. \quad (53)$$

If there exists an extremum with respect to $N_0$, it occurs for

$$N_0 = [\gamma_2/(2T^{*}_{\perp\perp}-{}^{(3)}R)]^{1/2}. \quad (54)$$

The opposite sign for the root gives nothing physically new. With this reversal in sign $N_k$ also comes out reversed in sign. All that has been changed is the convention as to the direction in which time is increasing. Reference [25] comments about the result (54): "Thus not only is the thickness of the thin sandwich from ${}^{(3)}\mathcal{G}'$ to ${}^{(3)}\mathcal{G}''$ determined by ${}^{(3)}I'$ and ${}^{(3)}I''$, but also its location in the enveloping ${}^{(4)}\mathcal{G}$ is determinate. This is the sense in which we discover a 3-geometry to be the carrier of information about time in general relativity".

*The Condensed Intrasurface Variational Principle as Mathematical Formulation of Mach's Principle*

Insert expression (54) for the lapse into Eq. (53) and obtain the "condensed intrasurface variational principle" (CIVP), (see footnote[11] p. 252)

$$\boxed{\begin{aligned} I_{\text{CIVP}} = -I_3^*/2 = \int \{[\gamma_2(2T^*_{\perp\perp} - {}^{(3)}R)]^{1/2} \\ + T^{*k}_{\perp} N_k\}({}^{(3)}g)^{1/2} d^3x = \text{extremum.} \end{aligned}} \tag{55}$$

The analogue of this intrasurface variational principle in electrodynamics is

$$\int [(\boldsymbol{E}^2/8\pi) - \varrho\varphi] \mathrm{d}^3\mathrm{x} = \text{extremum} \tag{56}$$

equivalent with

$$\boldsymbol{E} = -\partial \boldsymbol{A}/\partial \mathrm{t} - \text{grad } \varphi \tag{57}$$

to the single differential equation

$$\nabla^2\varphi = -4\pi\varrho - (\partial/\partial \mathrm{t})\mathrm{div}\boldsymbol{A} \tag{58}$$

for the single potential $\varphi$. In Eq. (55) the given quantities are still the metric ${}^{(3)}g_{ik}$ of the hypersurface, the rate of change of this metric with a parameter $x^0$, the scalar curvature invariant ${}^{(3)}R$ of the geometry, and the density of energy and energy flow. To be varied to obtain an extremum are now not four potentials but only three, the components $N_k$ of the vectorial shift function. They enter Eq. (55) (1) as coefficients of the energy flow and (2) as determiners—through their covariant derivatives—of the "shift anomaly" $\gamma_2$. The variational principle CIVP of Eq. (55) expresses in precise mathematical form the principle of Mach as formulated here (Formulation 4): *the specification of a sufficiently regular closed 3-dimensional geometry at two immediately succeeding instants, and of the density and flow of mass-energy, is to determine the geometry of spacetime, past, present and future, and thereby the inertial properties of every infinitesimal test particle.* Thus, from Eq. (55) when it possesses a solution, one obtains the shift $N_k$. Then from Eq. (54) one has immediately the lapse function. From these potentials via (49) and (51) one obtains the extrinsic curvature. Then one has in hand all the initial value data—and consistent initial value data—which one needs for the integration of Einstein's field equations and for obtaining a uniquely specified 4-geometry (the arbitrariness in the coordinate system in this spacetime having no relevance to its *geometry*!)

*Condensed Initial Value Equation*

Make small variations $\delta N_k$ in the shift components in Eq. (55). Set the coefficients of these variations equal to zero. In this way arrive at three coupled second order differential equations for the determination of the vector field $N=(N_1, N_2, N_3)$. The same equations may be obtained by solving (16) for $N_0$ (in agreement with (54)) and substituting this result into (17). The condensed initial value equations read

$$\left\{ \frac{(2T^{*\perp\perp} - {}^{(3)}R)^{1/2}[(N_{i|j} + N_{j|i} - {}^{(3)}\dot{g}_{ij}) - {}^{(3)}g_{ij}\,{}^{(3)}g^{mn}(N_{m|n} + N_{n|m} - {}^{(3)}\dot{g}_{mn})]}{[({}^{(3)}g^{ab\,(3)}g^{cd} - {}^{(3)}g^{ac\,(3)}g^{bd})(N_{a|b} + N_{b|a} - {}^{(3)}\dot{g}_{ab})(N_{c|d} + N_{d|c} - {}^{(3)}\dot{g}_{cd})]^{1/2}} \right\}_{/j}$$

$$= -T^{*}_{\perp i} = +(8\pi G/c^4) \begin{pmatrix} i\text{-th covariant component of} \\ \text{density of flow of energy} \end{pmatrix} \qquad (59)$$

*Variational Principle Equivalent to Differential Equations Plus Boundary Conditions*

These equations *plus boundary conditions* are equivalent to the condensed intrasurface variational principle (55). The *boundary conditions are essential* in geometrodynamics as in electrostatics *if* one is to obtain a unique relation between the "source" (density and flow of energy and gravitational radiation as described by ${}^{(3)}g_{ij}$ and ${}^{(3)}\dot{g}_{ij} = \partial{}^{(3)}g_{ij}/\partial x^0$) and the "effect" (the vector shift $N$ and the 4-geometry and inertial properties of test particles). The boundary conditions in a closed space are obvious: the vector field $N$ found by integration around the space one way has to join up properly with the vector field found by integration around the space another way; or more simply, the vector field (due account being taken of changes from one coordinate patch to another [8]) (1) must be everywhere *regular* and (2) must lead to a regular and *single valued* extrinsic curvature $K_{ij}$. If the space is *open*, the differential equations are still well defined; but they are not accompanied by any boundary condition. Moreover, one can no longer expect the variational integral ordinarily to have a finite and well defined value in the case of an open space. Therefore, there arises the built-in consequence of Mach's principle as formulated here, that the space should *be closed* and that the geometry ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$, or ${}^{(3)}g_{ik}$ and $\partial{}^{(3)}g_{ik}/\partial x^0$) should be everywhere regular.

## III. COMMENTS ON MACH'S PRINCIPLE AND THE INTRASURFACE VARIATIONAL PRINCIPLE

### *Issues Not Discussed Here: Uniqueness and the Question of a New Kind of Charge*

It would be an enormous labor to take up one by one all the questions that are left unanswered here and treat them systematically. Moreover, there is wanting one key element in the discussion—a proof that the solution of the variational problem in (55) (when there is a solution) is unique.[11]

### *Effect of Additional Mass on Inertia not Discussed*

On the other side of the story there are many homely questions about the physical content of Mach's principle that ought to be spelled out and that now can be spelled out. An example is the question how the inertial properties of a sun and planet are affected if centered around them at some distance is constructed a very large spherical shell of mass. Here it is necessary to recognize that in one way the inertial properties are affected and in another way they are not, according as the clocks in use are within the shell or far outside it. Again subtleties arise which are better left unmentioned than discussed inadequately.

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(11) The question of uniqueness of the solution of the initial value problem is well understood in the case of electrodynamics in a closed orientable 3-manifold. Given everywhere $\boldsymbol{E}$ and $\boldsymbol{B}$, one only then arrives at unique $\boldsymbol{E}$ when one specifies the jump $\Delta_k\varphi$ in the potential in travelling the circuit of the $k^{\text{th}}$ independent handle or "wormhole" of the topology, where $k$ runs over the values from $k = 1$ to $k = R_1 = R_2 =$ the second Betti number of the manifold. These numbers determine the charge or flux of lines of force trapped in the topology. That the numbers $\Delta_k\varphi$ have to be fixed follows most evidently from the occurrence of a surface integral $\int \delta\varphi(E \cdot dS)$ in the passage from the variational principle (35) to the differential equation (36). Does topology make an equally forceful appearance in the initial value equations of general relativity? *Is there a geometrodynamical analogue of electric charge? No argument for the existence of such a charge follows from the variational principle as discussed in the text ("coordinate representation").* The surface integral of the quantity $\pi^{ij}N_i$ shows up in the integration by parts of Eq. (41). In the discussion of the text following that equation it is remarked that surface integral vanishes when the topology is that of a 3-sphere (no handles). However, the surface integral *also* vanishes (C. W. Misner) for any closed orientable 3-manifold. The nature of the 2-surfaces encountered in these integrals is the same in geometrodynamics as in electromagnetism. Most simply, one such surface is conceived as the point of contact between two balloon-like expanding fingers that are feeling their way down into a wormhole from opposite mouths. The first factor in each integrant — $\boldsymbol{E}$ in the one case, $\pi^{ij}$ in the other case — is the same in this respect, that the quantity in question has *physical meaning* and is a *field momentum*. The difference comes in the character of the second factor — the potential jump $\delta\varphi$ in electrodynamics, the metric potential $N_i$ in geometrodynamics. Only the *gradient* of $\varphi$ has significance in electromagnetism, so that $\varphi$ itself can suffer a net change in going around the circuit of a handle. On the other hand, the quantity $N_i$ directly governs the distance between points on the two nearby hypersurfaces that have specified coordinates. Unlike the electromagnetic potential $\varphi$ this quantity must return to its orginal value after the

*Instantaneous or Retarded Effect of Source on Test Particle?*

Another question has to do with the *speed* with which the supposed inertial effects of sources are propagated to the test particles which they affect. In the equation (58) connecting source and effect even in electrodynamics, the effects of the charge distribution on the potential appear *formally* to be propagated instantaneously *within* the spacelike hypersurface. Yet the whole analysis goes back to standard Maxwell electrodynamics, in which effects are all propagated, not instantaneously, but with the speed of light. That there is no inconsistency between the instantaneous potential of (58) and the retarded potentials of usual radiation theory is well known [35]. Analogously one finds also in geometrodynamics a basically *elliptic* equation describing what appears formally to be an instantaneous propagation of effects from one place to another in a spacelike hypersurface. Yet one knows that a disturbance in a source at one point in spacetime will propagate to another point only with the speed of light [36]. In geometrodynamics as in electrodynamics the formalism itself guarantees that there can be no discrepancy between effects calculated in the two different ways from the same sources. Therefore, in principle there can be no trouble from the question mentioned earlier: How can Mach's principle make sense when it implies that the accelerated test mass acts on all the other masses in the universe and that they in turn have to act back on this particle.(12) Of course, one would like here, as in Fermi's analysis of electrodynamics, to see more of the inner workings of the machinery by which (1) the propagation in time and (2) a formally instantaneous propagation necessarily yield the same solution of Einstein's field equations!

*Do Sources Have to be Followed Back into Past when Model Universe Was in a Singular State?*

That all effects appear *formally* as propagated instantaneously within the space-like hypersurface disposes of another question about Mach's principle. Let one evaluate the inertial effects on a given test particle — that is to say, the effects on the geometry in a given neighborhood — caused in the sense of Mach by more and more remote sources of mass-energy. One appears to be forced farther and farther back in the past. On this basis one ultimately comes to regions where the geometry is singular and where it is not possible to follow back any further the dynamical evolution of the geometry by employing Einstein's field equations only at the classical level.(13) No matter! Specify the

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circuit of a handle. Therefore, a geometrodynamic analogue to electric charge—if one is to come in at all—will have to show up in the conjugate representation of the initial value problem (not analized here).

(12) For more on the equivalence between retarded and other ways of evaluating potentials in electrodynamics, see for example ref. [37].

(13) This question of singularities is raised and discussed further in an article by the author in press in the special cosmology issue of *The Monist*, ref. [38].

dynamic problem by giving the "sandwich" type of data on an initial spacelike hypersurface: give ${}^{(3)}\mathcal{G}$, $\partial^{(3)}\mathcal{G}/\partial x^0$, and the density and flow of energy. Then the integral that one has to extremize or the triplet of differential equations that one has to solve make no reference to anything going on back in the past at a time or place where the geometry—calculated classically—may be singular.

*Model Universe Clean of Constants of Motion*

Still another question is this, "what are the true physical constants of the motion" in general relativity. It is well known that total energy cannot be defined and has no meaning in a closed universe [34]. The question has recently been raised(13) whether such a system is not in principle *clean* of all constants of motion whatsoever. One can compare a model universe in some respects with a billiard ball set into motion on a triangular billiard table which has sides $e$, $\pi$ and 1. The motion is quasiergodic. Started in one way the billiard ball will come indefinitely close to repeating the motion it would have had if it were started in another way. To an observer with only a finite resolving power the only difference in the two motions might be one of rate or energy. Not even this difference can manifest itself in the case of a model universe [38]. Nevertheless, there is no more difficulty in defining the dynamics of the billiard ball (by giving $x'$, $y'$ at $t'$ and $x''$, $y''$ at $t''$) than there is in defining the dynamics of geometry (by giving ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$). In other words, if there are no constants of the motion they will hardly be missed!

*Different Masses on the Two Hypersurfaces*

Now for questions on which something more definite can be said. First, how can it possibly make sense to specify ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$ arbitrarily? Are there not all sorts of conditions of compatibility that have to be satisfied? Consider for example the case of a space that is asymptotically flat. From the rate of approach to flatness at great distances,

$$ds^2 \sim (1+2m^*/r)\,dr^2+r^2\,(d\theta^2+\sin^2\theta\, d\varphi^2) \tag{60}$$

one can evaluate the mass and energy of the system. If *this* has to be the same on both hypersurfaces, how many other constants must there not also be which have to agree between ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$? To discuss this question more fully, consider a specific example, the Schwarzschild solution of Einstein's field equations,

$$d\sigma^2 = -d\tau^2 = -(1-2m^*/r)\,dt^2+(1-2m^*/r)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2). \tag{61}$$

Let ${}^{(3)}\mathcal{G}'$ be the hypersurface $t = t' =$ const. On this the asymptotic geometry follows Eq. (60). Let the second hypersurface ${}^{(3)}\mathcal{G}''$ be described at small distances by giving $t$ as some reasonable and regular function $t''$ of $r$, $\theta$ and $\varphi$ going over at large distances into the formula

$$t'' = (8m_1{}^*r)^{1/2} \tag{62}$$

with $m_1^* =$ a constant. Taking the differential of this expression and substituting into Eq. (58), one finds that the second hypersurface has the asymptotic geometry

$$ds^2 \sim [1+2(m^*-m_1)/r]\,dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2). \tag{63}$$

The masses not only can be different but — in the example — must be different! One's first surprise at this result traces back to a semantic obscurity in the word "flat";

Meaning 1: The *intrinsic 3-geometry* is asymptotically flat.

Meaning 2: The *intrinsic* 3-geometry is asymptotically flat and also the *extrinsic curvature* is zero.

Only when "flat" is used in sense 2 do the apparent masses have to agree between two asymptotically flat geometries. However, the two-surface formulation of relativity focusses on *intrinsic* 3-geometry, so that "flat" there is no problem of compatibility between the two 3-geometries in the example. René Thom [39], [40] has even shown that one can fill in between two 3-geometries of different topology with a non-singular *topology*. Whether and when the geometry laid down on that topology can also be non-singular is a deeper question!

*Question of Effectively Elliptic Character of the Thin Sandwich Problem*

Does the CIVP (58)—or the triplet of differential equations to which it corresponds—have elliptic character. This issue brings to mind the question whether the equation

$$d^2\psi/d\theta^2+(\lambda-V_0\cos\theta)\psi = 0 \tag{64}$$

has eigenvalue character. One might think not, to look at the regions of $\theta$ where the "oscillation factor" or "effective kinetic energy factor" $(\lambda-V_0\cos\theta)$ is negative. There the solution is curved away from the $\theta$ axis. However, what counts in the end for the question of nodes and eigenvalues is the region where this factor is positive and the solution is oscillatory. The equation is *effectively* oscillatory in character (for $\lambda$ sufficiently in excess of $-V_0$). It is difficult in the case of (58), (59) to be precise at this stage; but one has the impression that it is in a comparable sense *effectively elliptic*. Space in the "thin sandwich problem" is divided up ordinarily into regions where $(2T^{*TT}-{}^{(3)}R)$ is positive—and where, therefore, also the shift anomaly $\gamma_2$ has to be positive—and regions where the second quantity has to follow the first in changing sign. At the interface between one such region and another the anomaly $\gamma_2$ has to change sign. This situation reminds one—to use another analogy—of the theory of buckling of shells, and of conditions at the boundary between one region of crumpling and another.

As the shift anomaly $\gamma_2$ now comes so centrally into the discussion, a few words about it are in order. Consider the equation for the eigenvalues of the extrinsic curvature tensor $K_{ik}$—or rather, of the closely related shift tensor $\gamma_{ik} = N_0 K_{ik}$. Consider the determinant

$$\begin{vmatrix} (\gamma_1{}^1-\lambda) & \gamma_1{}^2 & \gamma_1{}^3 \\ \gamma_2{}^1 & (\gamma_2{}^2-\lambda) & \gamma_2{}^3 \\ \gamma_3{}^1 & \gamma_3{}^2 & (\gamma_3{}^3-\lambda) \end{vmatrix} = \det\gamma_i{}^k - (\gamma_2/2)\,\lambda + (\mathrm{Tr}\,\gamma)\,\lambda^2 - \lambda^3. \quad (65)$$

A change in coordinates changes the $\gamma_i{}^k$ individually but not the eigenvalues $\lambda$ and consequently not the coefficients of the various powers of $\lambda$ on the right hand side of Eq. (65). Therefore, consider a system of coordinates such that at the particular point of interest the shift tensor $\gamma_i{}^k$ is diagonal. Let the elements down the diagonal — the eigenvalues be denoted by $A$, $B$, $C$. Then the coefficient of $-\lambda$ in the expansion of the secular determinant $(A-\lambda)(B-\lambda)(C-\lambda)$ is

$$(BC+CA+AB) = \frac{1}{2}\,[(A+B+C)^2-(A^2+B^2+C^2)] = \frac{1}{2}\,[(\mathrm{Tr}\,\gamma)^2-\mathrm{Tr}\,\gamma^2]$$
$$= (1/2)\ (\text{shift anomaly}) = \gamma_2/2. \quad (66)$$

Associated with the point in question consider a three dimensional space with coordinates $A$, $B$, $C$. Then the shift tensor is represented by a single point in this space. Moreover, this point is independent of the choice of coordinate system in the hypersurface. In the space $A$, $B$, $C$ construct through the origin a line with direction cosines $(3^{-\frac{1}{2}}, 3^{-\frac{1}{2}}, 3^{-\frac{1}{2}})$. Construct a double cone with this line as axis with an angle of opening $\theta$ such that

$$\cos\theta = 3^{-\frac{1}{2}} = \text{scalar product of } 3^{-\frac{1}{2}}(1,1,1) \text{ with} \begin{Bmatrix} (1,0,0) \text{ or} \\ (0,1,0) \text{ or} \\ (0,0,1) \end{Bmatrix}. \quad (67)$$

Then any point *on* a coordinate axis lies *on* one or other half of the cone. Every point on a coordinate axis also annuls the shift anomaly, according to (66). It takes only a few more steps to show that the shift anomaly is

(1) zero for *every* point on either cone;

(2) positive for every point *within* either cone; and

(3) negative in the neutral space *between* cones.

To each of these three cases may be said to correspond a particular character of the shift tensor $\gamma_i{}^k$. What is the detailed value of the shift tensor is only settled by extremization of the CIVP—or by integration of the initial value equations with appropriate boundary condition—and is, therefore, governed by the initial value data all over the hypersurface. However, only the *local* value of the quantity $(2T^{*\perp\perp} - {}^{(3)}R)$—read out of initial value data—is required to determine the *character* of the shift tensor. Turn now from comments on the general problem to a particular example.

*Example Where Both Hypersurfaces that Bound the Thin Sandwich Have Ideal 3-Sphere Geometry*

Let both hypersurfaces have the geometry of the ideal sphere

$$x^2+y^2+z^2+w^2 = 1;$$

thus for ${}^{(3)}\mathcal{G}'$ (give it the *name* $x^0$!)

$$ds^2 = a'^2 [d\chi^2+\sin^2 \chi\,(d\theta^2+\sin\theta d\varphi^2)] \tag{68}$$

and for ${}^{(3)}\mathcal{G}''$ (give it the *name* $x^0+\Delta x^0$!)

$$ds^2 = a''^2 [d\chi^2+\sin^2 \chi\,(d\theta^2+\sin^2\theta d\varphi^2)] \tag{69}$$

where $a'$ and $a''$ are constants. Or to use another language, consider a one parameter family of such hypersurfaces, characterized by a parameter $x^0$;

$$a = a(x^0) \tag{70}$$

and pick some fixed value of $x^0$, thus specifying

$$a \text{ and } da/(dx^0 \quad (\sim (a''-a')/\Delta x^0). \tag{71}$$

(As remarked earlier, the value of $\Delta x^0$ will drop out of the results at the end.) The remaining initial value data comprises the energy flow, which we set equal to zero, and the energy density, which we assume independent of position:

$$T^{**\perp\perp} = \text{constant (independent of } x, \theta, \varphi). \tag{72}$$

The question now is: What 4-geometry to fill in between the two hypersurfaces so as to satisfy the thin sandwich equations? The time-like perpendicular erected to ${}^{(3)}\mathcal{G}'$ at the point $\chi, \theta, \varphi$ will have to be assigned a certain length. Also it will be necessary to tell what point it touches on the hypersurface ${}^{(3)}\mathcal{G}''$, or to tell what the starred quantities are in the following formula for the coordinates of this point:

$$\chi-\chi^*, \theta-\theta^*, \varphi-\varphi^* \tag{73}$$

On account of the symmetry of the sphere it will be simplest to assume—as a trial—the same angles for both points, or to take all the starred quantities equal to zero. Thus the shift function is assumed zero:

$$N^\chi = \chi^*/\Delta x^0 = 0, \quad \text{etc. (three equations).} \tag{74}$$

Now for the shift tensor! It has to do with the fractional increase—between one hypersurface and the other—in the distance between points with corresponding coordinates, say $(\chi, \theta, \varphi)$ and $(\chi+d\chi, \theta+d\theta, \varphi+d\varphi)$. But this increase for the case we are considering is the same in all directions and at all places, and is in direct proportion to the fractional increase in the value of the radius. Thus the eigenvalues of $\gamma_i^{\,k}$ are identical:

$$A = B = C = \frac{\text{(fractional increase in radius)}}{\text{(change in the highly nominal parameter } x^0)}$$
$$= (1/a)\,(da/dx^0). \tag{75}$$

The point in the space $(A, B, C)$ lies inside one half of the double cone, right on the axis. The shift anomaly is positive:

$$\gamma_2 = (\mathrm{Tr}\,\gamma)^2 - \mathrm{Tr}\,\gamma^2 = (6/a^2)\,(da/dx^0)^2, \tag{76}$$

but independent of position. Likewise the covariant derivative of $\gamma_i^k$ is zero; and the $N_i$ vanish. These circumstances guarantee that the condensed initial value equations (59) are automatically satisfied. It only remains to find the lapse function $N_0$:

$$\gamma^2/N_0^2 = 2T^{*\perp\perp} - {}^{(3)}R \tag{77}$$

or

$$(6/a^2)\,(da/N_0\,dx^0)^2 = 2T^{*\perp\perp} - 6/a^2. \tag{78}$$

Instead of actually solving for $N_0$, it is better to recognize that $N_0 dx^0$ is the *proper* time separation—call it $dt$—between hypersurfaces, the parameters attached to which are $x^0$ and $x^0 + dx^0$, and is, therefore, directly the physical quantity of interest. Thus write

$$(da/dt)^2 = (a^2/3)T^{*\perp\perp} - 1. \tag{79}$$

The dynamics of the model universe are completely determined by (79) as soon as one puts in the law of change of energy density with expansion:

$$T^{*\perp\perp} = (8\pi G/c^4)\,(Mc^2/2\pi^2a^3) \tag{80}$$

for a universe filled with inchoate dust (Friedmann universe); and

$$T^{*\perp\perp} = \mathrm{const}/a^4 \tag{81}$$

for a system filled with isotropic radiation (Tolman universe).

*Question of Uniqueness. The Linear Approximation*

The purpose here was not to take up old problems anew, but to prepare the way in a simple example to investigate the *uniqueness* of the 4-geometry determined by ${}^{(3)}g_{ik}$, $\partial\,{}^{(3)}g_{ij}/\partial x^0$, $T^{*\perp i}$ i and $T^{*\perp\perp}$. Suppose the vector shift function $N^i = (\chi^*, \theta^*, \varphi^*)/\Delta x^0$ is *not* assumed to be zero but investigated in terms of the equations themselves. Will one find oneself with no alternative *except* the familiar solution already sketched out? Unfortunately the three coupled second order equations to be solved are only quasilinear, not linear. The problem appears difficult without some deeper mathematical considerations to draw on which do not present themselves immediately. Therefore, no decisive results can be offered here. What *has* been investigated is the case where the contribution of the shift vector $N_i$ to the shift tensor

$$\gamma_{ik} = \tfrac{1}{2}\,(N_{i|k} + N_{k|i} - \partial^{(3)}g_{ik}/\partial x^0) \tag{82}$$

is so small compared to the "main term" (Eq.(75)) that one is justified in treating the condensed initial value Eqs (59) as *linear* in $N^i$. These equations then take the form

$$(\sin\chi)^{-2}(\partial^2\chi^*/\partial\theta^2+(\sin\theta)^{-2}(\partial^2\chi^*/\partial\varphi^2)+\cot\theta\,\partial\chi^*/\partial\theta+4\chi^*)-$$
$$-(\partial/\partial\chi)\,(\partial\varphi^*/\partial\varphi+(\sin\theta)^{-1}(\partial/\partial\theta)(\theta^*\sin\theta))=0 \tag{83}$$

and

$$\sin\chi(\partial/\partial\chi)(\sin\chi\partial\theta^*/\partial\chi)+(\sin\theta)^{-2}(\partial^2\theta^*/\partial\varphi^2)+2\theta^*-$$
$$-(\sin\chi)^{-3}(\partial/\partial\chi)(\sin^3\chi\partial\chi^*/\partial\theta)-(\sin\theta)^{-2}(\partial/\partial\theta)(\sin^2\theta\partial\varphi^*/\partial\varphi)=0, \tag{84}$$

and

$$\sin^2\theta\sin\chi(\partial/\partial\chi)(\sin\chi\partial\varphi^*/\partial\chi)+\sin\theta(\partial/\partial\theta)(\sin\theta\partial\varphi^*/\partial\theta)-$$
$$-(\sin\chi)^{-3}(\partial/\partial\chi)(\sin^3\chi\partial\chi^*/\partial\varphi)-\sin\theta(\partial/\partial\theta)(\sin\theta\partial\theta^*/\partial\varphi)=0. \tag{85}$$

One can seek for a solution by writing

$$\chi^*(\chi,\,\theta,\,\varphi)=\sum f_{l,m}(\chi)Y_l^{(m)}(\theta,\,\varphi). \tag{86}$$

No thoroughgoing analysis along this line has been completed. However, Professor C. W. Misner was kind enough to point out at the Warsaw conference that the equations ought in principle to admit of *rotations*. This point has been since tested and verified. It obviously makes no difference to the geometry of the 3-sphere ${}^{(3)}\mathcal{G}''$ whether one set of hyperspherical polar coordinates $\chi$, $\theta$, $\varphi$ or a rotated set is used to describe the location of the points. However, it does make a difference to the coordinate-dependent shift vector $N_k$. To fill in between ${}^{(3)}\mathcal{G}'$ and ${}^{(3)}\mathcal{G}''$ with a thin-sandwich ${}^{(4)}\mathcal{G}$—compatible with the intrasurface variational principle or initial value equations—does not in itself fix the values of these quantities. The time-like normals that reach between the one hypersurface and the other, which start at $(\chi,\,\theta,\,\varphi)$ on one hypersurface, and also end at $(\chi,\,\theta,\,\varphi)$ on the other hypersurface, will end at *different* values of $(\chi,\,\theta,\,\varphi)$ when a rotated coordinate system is used: $(\chi-\chi^*,\,\theta-\theta^*,\,\varphi-\varphi^*)$.

*Shifts Produced by the Six Independent Rotations*

The calculation of the starred changes in the angles under a typical small rotation is most easily made by going to cartesian coordinates:

$$\begin{aligned} x &= a\sin\chi\sin\theta\cos\varphi,\\ y &= a\sin\chi\sin\theta\sin\varphi,\\ z &= a\sin\chi\cos\theta,\\ w &= a\cos\chi. \end{aligned} \tag{87}$$

There are six independent small rotations out of which the most general small rotation is constructed by linear combination. Consider as an example a turn by the small angle $\theta_{zw}$ in the $(z,w)$ plane.

$$dx = 0, \quad dy = 0,$$
$$dz = \theta_{zw}\, w, \tag{88}$$
$$dw = -\theta_{zw}\, z$$

The resulting change in the polar angle $\theta$ is

$$d\theta = \cos^2\theta\, d(\tan\theta) = \cos^2\theta\, d[(x^2+y^2)^{1/2}/z] = -\cos^2\theta\,(x^2+y^2)^{1/2}\, z^{-2}\theta_{zw} w$$
$$= -\cot\chi \sin\theta\, \theta_{zw}. \tag{89}$$

Similarly one finds the changes in all three coordinate angles under all six independent rotations (Table V).

TABLE V

Changes in polar angles on 3-sphere brought about by the six independent types of rotation

| | $\chi^*$ | $\theta^*$ | $\varphi^*$ |
|---|---|---|---|
| $\theta_{yz}$ | 0 | $\sin\varphi$ | $\cot\theta\cos\varphi$ |
| $\theta_{zx}$ | 0 | $-\cos\varphi$ | $\cot\theta\sin\varphi$ |
| $\theta_{xy}$ | 0 | 0 | $-1$ |
| $\theta_{xw}$ | $\sin\theta\cos\varphi$ | $\cot\chi\cos\theta\cos\varphi$ | $-\cot\chi\sin\varphi/\sin\theta$ |
| $\theta_{yw}$ | $\sin\theta\sin\varphi$ | $\cot\chi\cos\theta\sin\varphi$ | $\cot\chi\cos\varphi/\sin\theta$ |
| $\theta_{zw}$ | $\cos\theta$ | $-\cot\chi\sin\theta$ | 0 |

It is easy to verify that each line of Table V represents a solution of the linearized initial value Eqs (83), (84), (85). It is the conjecture that there are no other independent solutions of these equations which are free of truly geometrical singularity—as distinguished from coordinate singularity[14]—over the entire 3-sphere.

Even if and when this conjecture can be established, there will remain the question of uniqueness of the equations for this two sphere problem in their full non-linear form (59). After that will be the question of uniqueness in more general situations.

*Assessment of Mach's Principle*

Pending the investigation of these apparently difficult mathematical questions, it would not appear unreasonable to adopt as a working hypothesis the position (Formulation 4 of Mach's principle) that the specification of

[14] In principle all question of what is a coordinate singularity and what is a truly geometrical singularity can and should be eliminated by the use of two or more coordinate patches (ref. [8], p. 259) to eliminate all singularities in the coordinate systems that cover the 3-sphere.

a sufficiently regular closed 3-dimensional geometry at two immediately succeeding instants, and of the density and flow of mass-energy, is to determine the geometry of spacetime, past, present and future, and thereby the inertial properties of every infinitesimal test particle. In this sense it is proposed to view Mach's principle as the boundary condition for Einstein's field equations and an essential part of the "plan" of general relativity. The condensed intra-surface variational principe (58) is the most compact mathematical statement available of this interpretation of Mach's principle. As conceived here, it carries with it the tacit requirement that the model universe be closed.

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## APPENDIX: THE TAUB UNIVERSE INTERPRETED IN TERMS OF GRAVITATIONAL RADIATION OF MAXIMAL WAVE LENGTH

The Taub universe [10] is free of any "real matter" at all. Taub derived this solution of Einstein's equations,

$$d\sigma^2 = -d\tau^2 = \gamma_1 dx^2 + (\gamma_1 \sin^2 x + \gamma_3 \cos^2 x)\, dy^2 + 2\gamma_3 \cos x \, dy dz + \gamma_3 dz^2 - \gamma_1^2 \gamma_3 dt^2,$$

with

$$\gamma_1 = \cosh t/4 \cosh^2 (t/2)$$
$$\gamma_3 = 1/\cosh t,$$

from arguments of group theory having nothing directly to do with the kind of considerations which are the center of attention in this report. Therefore, it is of interest to see how one can be headed towards the same solution by a natural physical line of reasoning.

Replace the dust in the Friedmann universe by electromagnetic radiation distributed uniformly in space and in direction. One arrives at the Tolman universe [41]. During its expansion and recontraction the wave length of every standing wave varies as the radius a of the model universe. In consequence the density of mass-energy varies not as $1/a^3$, as in the Friedmann universe, but as $1/a^4$. Replace the electromagnetic radiation by gravitational radiation of short wave length. There is no longer any "real" density of mass-energy on the right hand side of Einstein's equation. However, the fine scale ripples in the geometry bring about the same type of larger scale curvature as would be caused by a "real" distribution of mass-energy. Let $\delta g$ denote the local root mean square amplitude of the fluctuations in the metric and let $\lambdabar = \lambda/2\pi =$ (wave length)$/2\pi$ denote their reduced wave length. Then the effective density of mass-energy associated with the gravitational radiation is of the order

$$\hat{T}_{\perp\perp\,\text{effective}} \sim (c^4/8\pi G)(\delta g/\lambdabar)^2.$$

To curve a space up into closure with a radius which at the moment of maximum expansion has the value $a_0$ requires an energy density given by the equation

$$^{(3)}R = (16\pi G/c^4)\hat{T}_{\perp\perp},$$

or

$$6/a_0^2 \sim 2(\delta g/\lambdabar)^2.$$

Thus the amplitude of the ripples need not be great

$$\delta g\,(\text{at maximum expansion}) \sim 3^{1/2}\lambdabar/a_0$$

if the wave length is short.

During the expansion and recontraction the energy density, proportional to $(\delta g/\lambdabar)^2$ necessarily varies as $1/a^4$. Consequently the amplitude of the ripples varies in accordance with the formula

$$\begin{aligned}\delta g\,(t) &\sim \text{const}_1\,\lambdabar\,(t)/a^2\,(t)\\ &\sim \text{const}_2/a\,(t)\\ &\sim 3^{\frac{1}{2}}\lambdabar_0/a\,(t)\\ &\sim (3^{\frac{1}{2}}/n)\,(a_0/a\,(t)).\end{aligned}$$

Here the last expression refers to the case where the perturbation in the otherwise ideal spherical geometry is described by a hyperspherical harmonic [42] of order $n$. It is not reasonable to consider the factor $3^{1/2}$ in this order of magnitude formula as a reliable number.

From considering a gravitational wave of very short wave length it is natural to turn to the opposite limiting case where the order $n$ has the minimum possible value and the wave length has the maximum possible value which will fit into the 3-sphere. The corresponding hyperspherical harmonic has well defined symmetry properties [42]. Possession of these symmetry properties, and of the critical amplitude required for closure, *are the features of the special* gravitational wave that gives the Taub universe.

The Taub universe is homogeneous but not isotropic: the curvature differs from one direction to another, but the principal values of the curvature do not change from place to place.

The curvature provides a more reasonable way of talking about the perturbations in the geometry than does the quantity $\delta g$ for a well known reason: neither out of the metric coefficients nor out of their first derivatives can one form coordinate-independent quantities. For the order of magnitude of typical components of the fluctuation part of the curvature in a local Lorentz frame one has the estimate

$$\begin{aligned}\hat{R}\,(t)_{\text{wave}} &\sim \delta g/\lambda\!\!\!^{-2} \\ &\sim \delta g/(a/n)^2 \\ &\sim n a_0/a^3\,(t),\end{aligned}$$

as compared to the typical component of the curvature of the background geometry,

$$\hat{R}_{\text{background}} \sim 1/a^2(t)$$

Thus, the mode of longest wave length and lowest $n$ is the one for which the perturbations in the geometry—as measured by the differences in the curvature in different directions—are not greatest (as one might have thought from the expression for $\delta g$) but least.

At early and late stages this perturbation becomes percentagewise larger and larger every spacelike 3-geometry ultimately develops infinite curvature, in accordance with what appears to be a general principle.(15)

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## DISCUSSION

P. G. BERGMANN:

I would like to talk, briefly I hope, on the question of whether in general relativity, specifying the intrinsic geometry on two surfaces that form a thin sandwich can indeed enable us to determine the geometry in between, by a scheme like the one sketched out here. That is whether it is indeed possible to specify the four-dimensional geometry between two space-like surfaces if on each surface you give the intrinsic geometry. As far as I am aware there are at present no hard proofs either that it can or cannot be done, or that it is a unique construction. So, what I want to contribute is one more plausibility argument that occured to me recently and which I want to present simply in order to give somebody else to test it out and see whether there is any merit to it. The argument is not a proof, or a counterproof, and I do not present it as such. The argument actually occured in an entirely different connection, which is immaterial here. It runs as follows: supposing we gave ourselves the $g_{mn}$ on one surface, that is the intrinsic geometry, and the $\pi_{ij}$, that is to say the extrinsic geometry of that surface. Then we know that, together with the field equations, the Riemann-Einstein manifold is determined, at least within some reasonable neighborhood of the surface; that is essentially the formulation of the Cauchy problem that, for instance, is to be found in Dirac's papers. Now, of course, these data cannot be given freely; they must be subject to four constraints at each world point (the constraints in Dirac's notation have been called $H_S$ and $H_L$). Suppose now I reduce the number of data that I give myself. First of all, I said that I cannot give all the data; there are four restrictions. I could, therefore, by a very shaky and shady argument, say that at every point of the surface I actually am free to give only the $g_{mn}$ and two out of the six $\pi_s'$, the other four being determined by the constraints. And that would still give me the full data. In fact it would give me more, because it also fixes the four-dimensional coordinate system; it also tells me how the surface lies within the four-dimensional manifold. Now, suppose that I subtract from these data two items at each space point, namely the two $\pi_{mn}$, and assume the worst loss of information that I can assume, that is not that I lose information on the coordinate system, but that I lose information on the intrinsic data. The intrinsic data, the so-called observables, which in general relativity are constants of the motion, are four per space point. If I gave myself two data less, the worst that can happen is that I lose two items; which means that, there must exist an algorithm, although I don't know it, which enables me to get half of the full data from the $g_{mn}$ alone. This same algorithm must be applicable, and it will give me the same information, regardless of which of the two surfaces I use, for the simple reason that I have no extrinsic way of locating this surface within the four-dimensional manifold; all I have to go on are the $g_{mn}$. So the contention then is that you have precisely the situation that Prof. Wheeler sketched for the harmonic oscillator, if I am mad enough to pick as my sandwich half a period. Either the data on the second surface are redundant or they are inconsistent. Now, let me repeat again that I don't claim a proof;

this is a sort of counting argument: you count the number of degrees of freedom, you count the number of data, and you subtract. And as we all know, this a very sloppy way of going about one's business. But I thought it had enough suggestive value to present it for further work by anybody who cares to.

H. Bondi:

There is a very great deal to this splendid lecture that I think has particular value in presenting underlying ideology, if I may say so, rather than the finished product, which is something most of us like to hide behind. In the fundamental ideas here, I agree very wholeheartedly with what Prof. Wheeler has so beautifully stated, that Mach's principle is a principle for selecting solutions. But beyond that point, I begin to differ from him, particularly in relation to the answer to Synge's question. Our difference, I think, is that, to Wheeler, relativity is very much a closed theory, into which I am only allowed to put simple specified rigid particles; and everything must then follow, because I've only got this to put into theory from the initial data. I like to think of it as an open theory, into which I put through the equations of state and the energy tensor, various properties of materials; and I like to think of equations of state as something very complex and difficult. I have already referred here frequently to the "time bomb" type equation of state, something which has built into it that it goes off at some stage; I have referred to that as what I regard in some ways as a typical example. And I feel that we must try to isolate the information that becomes available because of the properties of the equations of state from the initial data; this seems to me to be our central point of divergence of view. There are two rather minor ones; one is that I am much keener on the light-cone formulation, particularly, of course, in connection with the ideology; because this has enabled us to isolate the news function, which describes this; and on the second minor point my feeling is, like his, that it is a cosmological condition through which we get things in; but I like to think, that the expanding universe, through the limitation on the influence by the expansion, because everything in the distance recedes into a fog, gives us the best of both worlds between a closed and an open one.

L. Rosenfeld:

In his illuminating lecture Professor Wheeler has made the position of Mach's principle in the mathematical scheme of Einstein's theory perfectly clear. This is a very important, and hitherto too neglected, aspect of this principle, but it is not the only one. I have the impression that the more physical role of Mach's principle, to which Professor Wheeler briefly alluded in his lecture, is far from exhausted. On the contrary, I think that it forms an important part of the general cosmological problem, which is much in need of further investigation.

To make clear what I have in mind, I shall only recall the historically famous Thirring problem, which was taken up again a few years ago by Pirani. It is well known that the distribution of the far-away masses assumed by Thirring does not lead to the desired expression for the forces of the centrifugal and Coriolis type exerted by these masses on a body at the centre of the assumed distribution. Pirani has shown, however, that by modifying the distribution of the far-away masses it is possible to obtain other expressions for the forces in question. Although he did not succeed in giving an example of distribution leading to the right expression for the forces, his work is important in suggesting that since the expression for these forces critically depends on the assumed distribution, the right expression will presumably only result from distributions that approximate more or less closely the empirical one.

Wheeler's argument allows us to formulate Thirring's problem in a way which is perhaps mathematically more rigorous than is usual. We should try to find out what assumptions for the metric on Wheeler's "thin shell" lead to a general metrical field containing in particular the right expression for the Coriolis and centrifugal forces, and more generally all similar consequences of the physical Mach principle.

J. A. Wheeler:

I appreciate very much your comments, and I couldn't agree more that there is an enormous amount to be done in *spelling out* the implications of a mathematically well defined formultaion of Mach's principle, such as I have tried to give here. I tried to refrain here from discussing actual examples because there are so *many* beautiful and subtle points that can not be covered in a limited time. For example, how are the inertial properties of a particle affected by the presence of a nearby mass? This issue relates to the work of Thirring. It is such a beautiful point in the sense that in one way the inertial properties *are* affected, and in another way they are *not*. This apparent paradox comes about through the fact that one has two time scales, the time scale of the far-away asymptotically flat space and the local time scale; and so one can say it either way; both are right: the inertial properties are affected, the inertial properties are not affected. I tried to stay away from these questions because it is better to discuss them not at all than to try to say something in too short a time. I certainly don't pretend to know all the answers. I couldn't agree more that there is much still to be done in analyzing such examples.

A. H. Taub commented that in the case when $T_{00}$ and $T_{0i}$ vanish on a hypersurface there is no unique solution to the field equations.

J. A. WHEELER:

It is particularly interesting to deal with this case where the stress and energy vanish everywhere. The geometry is still determined uniquely by the specification of the two 3-geometries that make up the thin sandwich. An example is provided by Taub's own beautiful solution of the field equations. Here the space is closed. The geometry looks very much like a Friedmann universe with some ripples in it. The geometry develops in time even with no matter present. Moreover, the whole evolution of the geometry in time is completely determined.

A. H. TAUB:

But there is a variety of solutions of that sort. You're talking about only one.

J. A. WHEELER:

I am considering the situation where one gives a definite three-dimensional closed space geometry and then another geometry which is nearly the same. This information is enough to determine completely the *four*-dimensional geometry—or the entire dynamics.

# THE UNIQUENESS OF THE MASS TENSOR AND EINSTEIN'S GRAVITATIONAL EQUATIONS

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## 1. INTRODUCTION

The question I intend to consider is quite elementary, and I have to excuse myself for drawing your attention to it. This is the question of the uniqueness of the mass (or energy-momentum) tensor which occurs in Einstein's gravitational equations

$$R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R=-\varkappa T^{\mu\nu}. \tag{1}$$

These equations involve the mass tensor itself and not its divergence. In order that they have a definite meaning, it is absolutely necessary that the energy-momentum tensor should be defined in a unique way.

Another question arises. Einstein's equations are sometimes written as

$$R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R=-\varkappa T^{\mu\nu}-\lambda g^{\mu\nu}. \tag{2}$$

The question is, whether the difference between (1) and (2) is a real one or whether it is simply due to the lack of uniqueness of the mass tensor. Equations (2) are formally obtained from (1) if one makes the substitution

$$T^{\mu\nu}\rightarrow T^{\mu\nu}+\frac{\lambda}{\varkappa}g^{\mu\nu}. \tag{3}$$

Since $\lambda$ is the cosmological constant, it may be asked whether this substitution is connected with cosmological considerations; or whether the local determination of $T^{\mu\nu}$ is always incomplete and only possible apart from a term in $g^{\mu\nu}$, so that $T^{\mu\nu}$ can be replaced by

$$T'^{\mu\nu}=T^{\mu\nu}+\frac{p_0}{c^2}g^{\mu\nu}, \tag{4}$$

where $p_0$ is a constant similar in nature to the zero pressure in hydrodynamics.

Finally, the choice of the zero pressure may also be a question of cosmology.

## 2. HISTORICAL REMARK

Long before the creation of relativity theory, about 90 years ago the Russian scientist Umov introduced the general concept of energy flow and the English scientist Poynting applied this concept in electromagnetic theory. But the general concept of energy flow met with strong opposition and even recently some outstanding physicists declared the energy-momentum tensor to be a non-physical notion. The question of the uniqueness of the mass tensor has not been investigated until recently, even for the case of flat space-time.

## 3. GENERAL ASSERTION

I shall try to formulate the conditions for the mass tensor of a physical system that determine it uniquely, apart from an additive term in $g^{\mu\nu}$.

Let the state of a physical system be described by field quantities $\varphi_1, \ldots, \varphi_n$ that satisfy equations of motion (and possibly constraints) linear in the first derivatives of $\varphi_s$ with respect to the coordinates (and to the time). The quantities $\varphi_s$ may be scalars, vectors, or tensors.

Then the conditions are

a) $T^{\mu\nu}$ depends only on $\varphi_s$ (and on $g_{\alpha\beta}$ in the case of general coordinates);

b) $T^{\mu\nu}$ is a symmetric tensor;

c) $\nabla_\mu T^{\mu\nu} = 0$ is satisfied as a consequence of the field equations (identically in $\dfrac{\partial \varphi_s}{\partial x_\alpha}$).

My assertion is:

If conditions a), b), c) are fulfilled by a tensor $T^{\mu\nu}$, then the most general tensor that satisfies them is

$$aT^{\mu\nu} + bg^{\mu\nu} \tag{5}$$

where $a$ and $b$ are constants.

This assertion is supposed to be generally true, but it has been proved only for special cases, such as the Maxwell–Lorentz equations and the hydrodynamical equations in flat space. My Polish pupil, Dr. Czesław Jankiewicz, has also considered some cases of Riemannian space-time.

## 4. INFINITESIMAL OPERATORS

In the case of a *flat space-time* it is advisable to introduce infinitesimal operators $X^{\alpha\beta}$, corresponding to an infinitesimal Lorentz transformation, such that for a vector

$$\delta A^\alpha = e_\alpha \omega_{\alpha\beta} A^\beta, \quad \omega_{\alpha\beta} = -\omega_{\alpha\beta} \quad (e_0 = 1,\ e_1 = e_2 = e_3 = -1). \tag{6}$$

For an arbitrary function $f$ of the field variables we have then

$$\delta f = \frac{1}{2}\omega_{\alpha\beta} X^{\alpha\beta}(f), \quad (X^{\alpha\beta} = -X^{\alpha\beta}). \tag{7}$$

The operators $X^{\alpha\beta}$ so defined are antisymmetrical and satisfy definite commutation relations.

In the case of an electromagnetic field in vacuum we have

$$X^{10} = -X^{01} = H_2\frac{\partial}{\partial E_3} - H_3\frac{\partial}{\partial E_2} - E_2\frac{\partial}{\partial H_3} + E_3\frac{\partial}{\partial H_2}, \quad etc.$$

$$X^{ik} = -X^{ki} = E_i\frac{\partial}{\partial E_k} - E_k\frac{\partial}{\partial E_i} + H_i\frac{\partial}{\partial H_k} - H_k\frac{\partial}{\partial H_i}. \tag{8}$$

In the case of a field described by a vector $u^\alpha$ and some scalars we have

$$X^{\alpha\beta} = e_\alpha u^\beta\frac{\partial}{\partial u^\alpha} - e_\beta u^\alpha\frac{\partial}{\partial u^\beta}. \tag{9}$$

The vector $u^\alpha$ may or may not satisfy the algebraic condition

$$e_\alpha u^\alpha u^\alpha = c^2. \tag{10}$$

## 5. EQUATIONS FOR $T^{\mu\nu}$

Besides the symmetry condition

$$T^{\mu\nu} = T^{\nu\mu} \tag{11}$$

we have two systems of differential equations for the quantities $T^{\mu\nu}$ *as functions of the field quantities.* The first system is

$$X^{\alpha\beta}(T^{\mu\nu}) = e_\mu\delta_{\alpha\mu}T^{\beta\nu} - e_\mu\delta_{\beta\mu}T^{\alpha\nu} - e_\nu\delta_{\alpha\nu}T^{\mu\beta} - e_\nu\delta_{\beta\nu}T^{\mu\alpha}. \tag{12}$$

From this it follows in parcticular that

$$T^{0m} = T^{m0} = \frac{1}{2}X^{0m}T^{00},$$

$$T^{km} = T^{mk} = \left(\frac{1}{2}X^{0m}X^{0k} - \delta_{mk}\right)T^{00}, \tag{13}$$

so that all components are expressed in terms of $T^{00}$.

The second system is obtained by eliminating the derivatives $\frac{\partial \varphi_s}{\partial x_\alpha}$ between the field equations and the equations

$$\nabla_\mu T^{\mu\nu} = \sum_s \frac{\partial T^{\mu\nu}}{\partial \varphi_s}\frac{\partial \varphi_s}{\partial x^\mu} = 0. \tag{14}$$

## 6. FIRST EXAMPLE: MAXWELL EQUATIONS

Introducing the antisymmetric tensor $\varepsilon_{ikl}$ we may write

$$\frac{\partial E_k}{\partial t} = c\sum_{il}\varepsilon_{kil}\frac{\partial H_l}{\partial x_i}, \quad \sum_i \frac{\partial E_i}{\partial x_i} = 0,$$
$$\frac{\partial H_k}{\partial t} = -c\sum_{il}\varepsilon_{kil}\frac{\partial E_l}{\partial x_i}, \quad \sum_i \frac{\partial H_i}{\partial x_i} = 0. \tag{15}$$

Hence, the second set of equations is

$$\frac{\partial T^{\nu i}}{\partial E_k} - c\varepsilon_{lik}\frac{\partial T^{\nu 0}}{\partial E_k} = \lambda^{\nu}\delta_{ik}, \qquad \frac{\partial T^{\nu i}}{\partial H_k} + c\varepsilon_{lik}\frac{\partial T^{\nu 0}}{\partial E_l} = \mu^{\nu}\delta_{ik} \tag{16}$$

where $\lambda^{\nu}$ and $\mu^{\nu}$ are Lagrange's factors. From this we find

$$T^{00} = \frac{a}{2}(E^2+H^2)+b \tag{17}$$

and applying previous formulae

$$T^{0i} = a[E\times H]_i, \qquad T^{ik} = a\left\{\frac{1}{2}\,\delta_{ik}(E^2+H^2)-E_iE_k-H_iH_k\right\}-b\delta_{ik}. \tag{18}$$

The terms $b$ in $T^{00}$ and $-b\delta_{ik}$ in $T^{ik}$ indicate the degree of arbitrariness in the energy-momentum tensor.

## 7. SECOND EXAMPLE. HYDRODYNAMICS OF A PERFECT FLUID

Putting

$$a^{\alpha\beta} = e_\alpha\delta_{\alpha\beta}-\frac{1}{c^2}u^\alpha u^\beta, \qquad w^\alpha = u^\beta\frac{\partial u^\alpha}{\partial x^\beta},$$
$$P = \int_0^p \frac{dp}{\varrho} = \Pi+\frac{p}{\varrho}, \tag{19}$$

we may write the equations of hydrodynamics in the form

$$\left(1+\frac{P}{c^2}\right)w^\alpha = a^{\alpha\beta}\frac{\partial P}{\partial x^\beta}, \qquad \frac{\partial}{\partial x^\alpha}(\varrho u^\alpha) = 0. \tag{20}$$

The second set of equations for $T^{\mu\nu}$ will be of the form

$$a^{\beta\sigma}\left\{\frac{\partial T^{\mu\nu}}{\partial u^\beta}+\left(1+\frac{P}{c^2}\right)u^\nu e_\beta\frac{\partial T^{\mu\beta}}{\partial P}-\delta_{\nu\beta}\frac{\varrho}{c^2}e_\alpha u^\alpha\frac{\partial T^{\mu\alpha}}{\partial\varrho}\right\} = 0. \tag{21}$$

or

$$u^0\left\{\frac{\partial T^{\mu\nu}}{\partial u^s}+\ldots\right\}+u^s\left\{\frac{\partial T^{\mu\nu}}{\partial u^0}+\ldots\right\} = 0 \qquad (s = 1, 2, 3). \tag{22}$$

Since $T^{\mu\nu}$ is a tensor depending on a single vector $u^\alpha$ we have

$$T^{\mu\nu} = Au^\mu u^\nu+Be_\mu\delta_{\mu\nu} \tag{23}$$

and, solving the differential equations for $A$ and $B$,

$$T^{\mu\nu} = a\left[\varrho\left(1+\frac{P}{c^2}\right)u^\mu u^\nu-pe_\mu\delta_{\mu\nu}\right]+be_\mu\delta_{\mu\nu}. \tag{24}$$

We have again two constants $a$ and $b$.

## 8. CONCLUDING REMARKS

When the choice of units for the energy is made, then the only indeterminacy in the energy tensor is the constant $p_0$ in the expression (4). This constant cannot be determined from local considerations. What kind of considerations are to be used instead? How is the zero point of the pressure to be determined?

In a space that is flat at infinity we may assume that the field variables tend at infinity to such values that the mass tensor vanishes at infinity.

In a space of more general character the determination of the value of the additive constant in the energy-momentum tensor becomes a problem of cosmology, and a hypothesis concerning the form of the Einstein equations includes that on the form of the mass tensor.

## DISCUSSION

F. J. BELINFANTE:

You used certain methods in which you postulated that the tensor depends only on the field strengths and not on the derivatives. What could you do in the fermion case, for the Dirac field?

V. A. FOCK:

I have only considered the case where the field variables are scalars, vectors or tensors, but not spinors.

P. G. BERGMANN:

I should like to ask you in connection with the ambiguity of the tensor in the cosmological case, whether perhaps the requirement that has often been expressed, that the energy densities be positive-definite, does not in these circumstances, similarly as in mechanics, simply mean that the energy is bounded from below; that is, that it cannot take infinite negative values. If it has a finite negative value one can always add a constant term of the type that you have indicated, in the cosmological case.

V. A. FOCK:

I have not thought about this question.

*Peter G. Bergmann (responding to the remarks by the Chairman, Professor Weyssenhoff who said that the summary of the conference would be given as usual, by Professor P. G. Bergmann): Permit me to take strong exception to the word "usual"; I hope that this will turn out to be the last time. I have become much too much of an institution already; next time somebody else will have to do this job.*

## CONCLUDING REMARKS

P. G. BERGMANN
*Syracuse University*

BEFORE I comment on a few of our discussion topics at this conference, I should like to preface my remarks by saying, as I did at Royaumont, that it is humanly impossible, or at least impossible for me, to do justice to all the topics that have come up, or to close each of these discussions with a few words that proclaim the Truth, with a capital T. This warning is, of course, not meant primarily for the more experienced among us, who may have had to give such summaries or conclusions, themselves, elsewhere, and who are, at any rate, aware of the fallibility of all of us, but I want to make sure that there is no misunderstanding among us on this score.

The second preliminary observation of mine refers to the fortunate circumstance that we have had several concurrent seminar meetings at this conference; as a result you cannot charge my incompetence or ignorance necessarily to my negligence. I may simply plead that with but one body you cannot attend two simultaneous weddings. So much for my excuses for my shortcomings.

One more thing before I come to specific technical topics. All of us who are comfortable with English, be it our native tongue or not, owe considerable thanks to those among us who speak principally Polish, French, Russian, or what-have-you, who have graciously permitted us to conduct about ninety percent of our conference in that language in whose realm most of the linguistic illiterates are to be found. As the years go by, I expect that an increasing number of linguistic illiterates will turn up from other regions, and we shall have to find ways for distributing the conference languages more evenly. We may even have to learn each others' languages well enough that we need not impose on them with our own ignorance. In the meantime, we should certainly voice our appreciation of their forbearance.

Coming to the technical topics, I should like to comment on four discussion areas, each comprising several of the talks. These are: (1) Conservation laws and ponderomotive theory; (2) the present situation as regards experimental and observational tests of general relativity; (3) radiation theory; and (4) whether and how to quantize the gravitational field. Parenthetically, I have been told that the fifth, and by far the most important topic, has been torsion; to discuss torsion would have been personally embarrassing to me, because I am not too enthusiastic about this topic, no doubt because of my ignorance. Very fortunately, I understand, this topic has been presented, and definitively, yesterday afternoon at a seminar that I was unable to attend, and thus I am saved from having to exhibit in public my own incompetence in this area.

Let me first try to summarize my understanding of the present views of energy, and how it is conserved. At the root of this whole complex lie the theorems by Emmy Noether [1], which state that in a theory whose dynamical laws are derivable from a variational principle and which incorporates an "invariance group", to each invariant transformation there corresponds a conserved quantity. (In field theory, one might speak more advantageously of an equation of continuity.) If you consider general relativity as a local theory, that is to say without regard to boundary conditions at infinity, then the invariance group is the group of curvilinear transformations. Even in a finite four-dimensional domain these do not form a Lie group but correspond, topologically, to a function space. Therefore, the number of conserved quantities (which, incidentally, are the generators of the infinitesimal invariant transformations which belong to the group) is infinite. Our task then is to fish out, from this ocean of candidates, a few constants of the motion which may be identified with, or denoted as, the energy, the linear momentum, and the angular momentum of the field.

This can be done in several ways. But in order to succeed we need, in my opinion, to construct a number of fields of displacement vectors, one for each of the quantities to be conserved. We do not require congruences of three-dimensional hypersurfaces. (Congruences of hypersurfaces generally define covariant vector fields, but not vice versa; and a covariant vector field defines a displacement vector field only if we have available a non-singular metric tensor field.) Once we have characterized—invariantly or otherwise—a contravariant vector field, it defines for us an infinitesimal coordinate transformation. In special cases there may be particular vector fields that offer themselves and which appear particularly appropriate for this role, though in general there is an infinity of such fields. One obvious case in point, which permits the characterization of a vector field in purely local terms, is the existance of isometries; the "special" vector field is then the Killing field. For each Killing field present we can define an associated quantity that is conserved. Of course, that is not such a tremendous triumph, after all: In the presence of a Killing field every functional of the metric on a three-dimensional hypersurface is conserved. Nevertheless, the generators of an isometric displacement can be identified, and they might be interpreted as energy, linear momentum, or angular momentum, depending on the particular situation.

In this connection Professors Plebański and Møller explained what tetrads can do for you, aside from purely technological considerations: If you construct a tetrad (or "Vierbein") formalism in a theory that is equivalent to the conventional theory of relativity, then the choice of a tetrad at each world point is arbitrary (except for considerations of continuity and differentiability). Each choice results in a definite expression for energy and linear momentum; but the number of possibilities of choice is as large as if we had not introduced tetrads at all. Only, perhaps, the expressions look more persuasive than without tetrads. You will get something new only if you introduce a principle that restricts the freedom of choice of the tetrads, and with it the range of choice for the energy-momentum expressions. The work by Professor Møller and by others that is now in progress may persuade us that a particular set of restrictions is to be preferred on physical grounds; that remains to be seen. In the absence of isometries, asymptotic boundary conditions, or other "special" situations, such a fixation will constitute a significant modification of, or addition to, the conventional theory. Professor Møller, I feel certain, agrees with this analysis. In view of the fact that the search for an acceptable fixation of tetrads is in its beginnings, I shall not, as it were, uproot the new plant to see whether its roots are properly growing; let us rather wait and see.

Suppose we have chosen a particular displacement vector field, and with it a certain expression for the energy or other generator; then what can one say about the transformation laws of these quantities? I am thinking in particular about the class of solutions of the field equations that satisfy asymptotic boundary conditions at infinity along the retarded light cones. The London school, R. Sachs, and others[(1)] have shown that for this class of solutions one can define uniquely a four-parametric set of vector fields which represent the rigid translations at infinity. As required, these infinitesimal coordinate transformations all commute with each other. In view of the fact that the GBM group (R. Sachs[(1)]) also contains the homogeneous Lorentz transformations as a factor group with respect to the invariant subgroup of the "supertranslations", it is possible to state exactly how the "translations" go over into each other under a Lorentz transformation. In the absence of radiation (i.e. if the "news function" Bondi[(1)], vanishes) the fixation of the time axis determines the energy (and the linear momentum) as a constant of the mo-

(1) H. Bondi and collaborators; R. Sachs; E. Newman and collaborators; all in process of publication.

tion, so that the transformation law under the whole GBM group (which may be considered as the invariance group characteristic for this class of solutions) is exactly what one would expect.

In the presence of a pulse of radiation which is limited in (retarded) time, there are still two "half-universes", the time prior to the onset of the pulse and the time subsequent to the termination of the pulse, in each of which the energy is constant and transforms as in the previous case. While the radiation pulse takes place, the total energy changes, and the question arises whether the "rate of energy radiation" can be given a Lorentz-covariant meaning. If we could identify "the same retarded time" for all null directions uniquely, we should, presumably, be able to fix such a rate of radiation. But the meaning of the "supertranslations", which form part of the GBM group, is precisely that this is impossible, at least in our present state of knowledge. Thus, the rate of energy (or linear momentum) radiation will depend on the choice of coordinate system and will change under supertranslations in an involved manner, even though the direction of the time axis may be kept fixed. I do not believe that there is any serious difference of opinion on that score.

Whether the displacement vectors should be combined into a tetrad field is a question to which I can contribute very little. A tetrad field, at least as I understand the term, is not a combination of four unrelated vector field, but of four fields of mutually perpendicular unit vectors. Again, as Plebański has stressed, the merits of such a choice may be discussed on two levels: either as one of convenience, with no invariant method of selecting "suitable" tetrads; or, if such a method should be made available, as an enrichment of the conventional theory. In the latter case we should have to deal with a new physical theory, and no longer with an issue of technological convenience.

No discussion of ponderomotive theory, that is to say of the so-called EIH theory and its subsequent developments, had been planned by the committee of our Polish hosts, no doubt because they did not wish to impose on us with what we know is actually close to their hearts. On the whole, the problems of motion of ponderable bodies entered our discussions only by way of the back stairs, as it were, of the conservation laws. The major exception was Professor Dirac's paper. Though he concerned himself with the behavior of a particle whose internal structure was itself described by a least-action principle, the fact is that the over-all motion of his particle would obey the EIH theory; this is because outside a finite domain the vacuum field equations of the pure gravitational field are to hold. Thus, the particle may be described though not fully, as a region in which the pure gravitational equations are not satisfied. On the other hand, the EIH theory has no bearing on the internal stability of such a region, and it is with questions of internal stability that Dirac concerned himself. His examination of non-spherical disturbances is continuing; it will be most interesting to see whether a structure as simple as the one he has postulated suffices to produce a stable particle-like solution. For the time being, no similarity to real particles is claimed.

I shall now turn to a sketch of the present status of the second major area, the experimental evidence bearing on gravitational theory. From all we have heard, the three so-called classical tests appear now to be in very good shape. They have been on the agenda of every relativity conference since the one at Berne in 1955, where they had been discussed by the California astronomers. Here they were discussed by Professor Ginzburg. Some of the tests cannot yet be carried out with the accuracy we should like to see, in particular the bending of light rays. The advance of the Mercury perihelion has been confirmed with good accuracy quite some time ago. As for the gravitational red shift, it is true that the astronomical data leave much to be desired, if they are to be drawn on for a confirmation of Einstein's theory. In view of the Pound–Rebka experiment, which is purely terrestrial and which yields excellent agreement with the theory, I believe that the solar red shift data have lost all interest in that respect and have become part of the exploration of the sun's atmosphere; as such they will undoubtedly be pursued until all outstanding questions have been clarified. Apparently, the accuracy of the Pound–Rebka experiment is capable of further improvement, and an error of less than 1% may perhaps be achieved. Every critical analysis has shown, though, that the red shift is not a very discriminating test of general relativity but is essentially predicted by the principle of equivalence, regardless of the other details of the theory.

In Einstein's opinion, the three classical test were not the crucial tests of the theory, anyway. He felt that the real foundation of the theory consisted of its invariance properties, i.e. the principle of equivalence, and that the crucial fact was the equality of gravitational and inertial mass; accordingly the crucial experiment was that of Eötvös. In this connection we ought to mention the experiments now being performed by R. Dicke at Princeton, who hopes to better Eötvös' accuracy by three orders of magnitude or better; I believe that he has already gone two orders. Personally I expect that Dicke's group will simply confirm the null effect, but whether or not that will be the outcome, this effort should certainly be followed. Any discrepancy will be extremely difficult to explain away. Recently, a variation of the experiment has been proposed, one in which the ponderomotive behaviour of polarized particles is to be examined. This is an entirely different type of experiment, not one that tests the principle of equivalence, but of considerable interest in its own right. I am skeptical about the possible magnitude of such an effect of anisotropy, but the proposal should certainly be examined carefully [2].

Let me briefly mention some new test that are in various stages of development. The gyroscope experiments are fairly far along, perhaps enough so that we may hear some results at our next conference. Finally there are proposals to observe gravitational waves. Farthest along, to my knowledge, among these is the instrument by J. Weber, which is peaked in the low kilocycle range. For low frequencies, $10^{-2}$ to $10^{-3}$ cycles per second, Weber has suggested that the normal quadruple modes of the earth might serve as detectors. If one searches for gravitational waves, one ought to look in several frequency ranges, inasmuch as we do not know which systems in nature may turn out to be powerful producers of such waves.

Next I shall come to radiation theory. I think the most beautiful development is that begun by the London group about Bondi, which is also being continued by Sachs and by Newman.(1) Without engaging in unnecessarily dubious mathematical procedures, we can now get fairly directly, and quite intuitively, at the intrinsic degrees of freedom of the gravitational field. As explained by Sachs here, we can, preliminarily, split these degrees of freedom in two, the incoming and the outgoing waves. If we give the Cauchy data on a three-dimensional hypersurface that consist of one outgoing ("future") light cone and a continuing half-cylinder that extends into the future, then the incoming wave data will be found on the conical, and the outgoing on the cylindrical portion of that hypersurface. Alternatively, one can discard the incoming modes althogether, by suitable radiation conditions, and restrict oneself to the consideration of a cylindrical hypersurface at spatial infinity.

It has often been said that the total number of degrees of freedom of the gravitational field equals two per three-dimensional point, and that the Cauchy data must be four functions of three arguments. Bondi's and Sachs' "news function" represents the outgoing modes only. Accordingly, it summarizes that half of the Cauchy data in the form of one complex, or two real functions of the three arguments, $u$, $\theta$, and $\Phi$. $u$ stands for the retarded time $(t-r)$, and $\theta$ and $\Phi$ are the two angles that identify null directions leading to the points of the future celestial sphere. Certainly, the counting game of degrees of freedom comes out gratifyingly well.

Obviously, there is an intimate connection between radiation theory and Cauchy data, and some important new work is in the offing, by R. Penrose. I cannot report on that work here, because much of it will be communicated in the near future, rather than in the past. And in spite of many attempts to generalize from the orthochronous to the full Lorentz group I do not wish to comment on something that is going to happen.

I come now to the last of my four conference topics, the quantization program. In view of the great difficulties of this program, I consider it a very positive thing that so many different approaches are being brought to bear on the problem. To be sure, the approaches, we hope, will converge to one goal; but the points of departure are quite diverse. I think I should never have dared to do what Feynman is doing, but I am very happy that a, shall we say, reasonably competent physicist like him is willing to try it. As I understand it, his program consists of starting with the techniques that we know are successful in quantum electrodynamics, and of adding to them only as needed. The approach is a weak-field, or perturbation, approach, which starts from the Minkowski metric, considered to be c-numbers. At each stage of the approximation "gauge-type" covariance is required; this requirement has turned out to be a powerful instrument of checking for forgotten diagrams. Much of B. DeWitt's work also is concerned with the application of techniques already tested in coventional quantum field theory.

Most of the other approaches to quantization that have been discussed here center on the task of constructing "observables". I should like to point out one difference in approach which has not been brought out at this conference, but which I believe is worth recording. If you search for observables in the unquantized theory as your first step, to be followed subsequently by the quantization of a theory already formulated entirely in terms of invariant, or intrinsic, variables, then that means that you propose to construct a Hilbert space whose state vectors consist entirely of physically permissible states, that is to say of states that satisfy all the constraints; in other words, in such a theory each constraint expression equals zero *ab initio*. Your operator algebra is confined to such operators which map the set of permissible states on itself. All these operators are necessarily constants of the motion, and invariants. Accordingly, in such a formalism the whole original invariance group of the theory is lost. This is why Dirac has proposed a somewhat different approach. It consist of quantizing a conventional theory in its Hamiltonian version, such as the one he has worked out in his papers of 1958 and 1959 [3], [4]. The operators that appear in this theory are the $g_{mn}$ and $p^{mn}$, which Professor J. A. Wheeler also used in his talk this morning. These operators are not defined on the Hilbert space which I have just described; they will map a permissible state into one that violates the constraints, and occasionally vice versa. The Hilbert space (which consists of the permissible states only) would be a linear subspace of the linear vector space which contains the non-physical states as well, and which is needed to give the above "canonical" operators, and their standard commutation relations, any mathematical substance. Though both the "big" linear vector space and the Hilbert space are infinite-dimensional, one can say, in a certain sense, that "almost all" members of the base of the "big" vector space lie outside the Hilbert space. In the "big" space we have the means for giving expression to the invariance group of general relativity,—this is undoubtedly the principal motivation for introducing it at all,—in terms of the generators of the curvilinear coordinate transformations (Dirac's $H_s$ and $H_L$) and their commutator algebra. In the Hilbert subspace, on which alone a norm is defined and in which all expectation values must be calculated, this group is lost, because the "observables", by definition, are invariant with respect to coordinate transformations. The work presented by J. Anderson here is to be understood in the spirit of Dirac's approach: Within the Hilbert space of permissible states the constraints vanish identically, and the consistency requirements on their commutator algebra lose all substance.

Professor Mandelstam's talk was concerned with a particular technique for constructing observables. He proposes to identify world points in terms of intrinsically defined paths leading to them. Though this is a stimulating way of looking at the problem, I feel that it is too early to predict success or failure.

I must apologize to Professors V. Fock and J. Wheeler for not commenting on their two talks. As there were only a few minutes between their talks and the beginning of my remarks, I believe that it is best if I say nothing that might look very silly on further reflection. I hope that they will not interpret my silence as a polite form of negative comment, but blame it entirely on the mechanics of time, which is in fact the only reason for my omission. At any rate, I had asked the program committee to announce my comments under the heading of "concluding remarks" rather than of "summary", in order to afford me somewhat greater freedom in not attempting completeness.

As we are about to wind up our Conference, I should like to say a few words about its extra-scientific aspects. First of all, I believe that I speak in the name of all participants from outside Poland, if I express our gratitude for having been shown a country that is beautiful by nature and which has been the seat of a culture that has left its mark over the centuries. I myself am a native of Europe, though I have spent more than half of my life in the United States; every time I come to a really beautiful old European city, it is a new emotional experience, and Warsaw is no exception; it is a very beautiful city, and so are the surroundings that we were shown, on some of the excursions. Warsaw is different from many other cities in that it has become ravaged by war, not in what might legimately be called military action, but in a wanton and calculated destruction of people and of cultural values. It would be an insult to earlier epochs of humanity to call that barbarism. This is bestiality, and it is not a very reassuring experience to see what can be done; but I believe it is an experience that we must go through; and we have to thank our hosts also for having given us this experience, even if it was not entirely enjoyable. It is, perhaps, a slight compensation to see what has been rebuilt from the rubble in less than two decades, though we cannot call back to life the people who perished.

Being Jewish myself I have been deeply touched by the spirit in which the Polish people have commemorated, with the Ghetto Monument, the place where so many Jewish people lost their lives, fighting or simply being slaughtered. I am fully aware that Jews were by no means the only people who were killed off by genocide; the Polish people were just as much on the program of destruction as the Jewish people, merely for a later period, and thus this slaughter did not take place fortunately, to the full extent planned. Even so there are enough Polish people, and people of many other nationalities, who, as I said, were the victims, not of legitimate military action but of calculated murder.

I should like to talk now about some things that are a little easier to talk about. We shall try, this afternoon or in the months to come, to decide on the location of the next international meeting, and I should like to say a few words about that here, because we may receive all kinds of suggestions from various sides. As for the time, I believe that there is an informal consensus that three years from now will be a good time for another international gathering of relativists. Two years appears too brief an interval; it is desirable to have a little time between meetings to do some work, so that the performance of research may on occasion precede its being reported, instead of the other way around. But at this stage nothing is settled, and I think that comments will not be useless. As for location, we are in the happy situation that a number of very acceptable locations have been suggested by people who have combined their suggestions with commitments to do the work that goes with preparing and running such conferences. As a result, instead of having to press someone into acceding to serve as the host of the next conference, we already have a number of volunteers, and that situation also presents us with a tough problem. Fortunately, we all hope, we shall have a number of conferences in the years to come; invitations that we cannot accept for 1965 may be most welcome at some latter time.

One point that must have received a great deal of attention from our organizing committee, that is from Professor Infeld and his colleagues at Warsaw and at Cracow, is the format of a conference. The easiest method of making up a conference program is to have a number of invited papers, and then to accept as contributed papers everything that is submitted. The usual result is then that the whole conference time is taken up with ten-minute or fifteen-minute papers (there have also been conferences with five-minute, and even with two-minute papers), and there is no time available for discussion. In my opinion, the best method for presenting a finished piece of work, if all you want is to present it, is its publication in print: Your audience is larger that way, and people who may wish to comment on your work eventually have a chance to read and digest it first in the privacy of their studies. The purpose of a conference, or other meeting, is, obviously, to provide an opportunity for discussion. A great deal of effort has gone, in preparing for this conference, into limiting the number of contributions and opening up space for discussion. Incidentally, how to run a discussion so that it is neither eviscerated by the mechanical application of rules nor gets out of hand completely is a very delicate task; I know of myself that I sweated on Thursday afternoon when I served as the chairman of a "general discussion"; I am sure DeWitt sweated when his turn came. There is probably no one right solution to this problem; but it may well be worth our while to attempt to find out what has worked out well, and what not as well, so that in preparing conference programs in the future we may learn from our experiences. I have the feeling—and I certainly do not wish to presume to speak for anyone but myself in this matter—that this meeting has been on the whole a step in the right direction, and that whatever imperfections some of us may have noticed they were heavily outweighed by the positive results of the experiments initiated here.

I should also like to speak briefly on publication policies. Most of us received, several months ago, a communication from Professor André Mercier[(2)] that certain types of information of special interest to relativists will be disseminated from time to time, perhaps as a newsletter, or as a mimeographed abstract journal, and that an attempt will be made to have this venture supported by voluntary contributions, if I remember rightly, of the order of three dollars. I do not know the extent of the response. At any rate, Professor Mercier and his collaborators will continue their efforts to ascertain, and to fill, the needs of relativists on an international scale.

Very recently several publications have appeared which may not have received general attention as yet but which are of interest to our group so let me mention them here I did not hear until last night that the proceedings of the Royaumont conference of 1959 have came out well in advance of this conference, at least a week, I believe, lest you think that this is merely an idle rumor, I have seen an actual copy. The publishing organization is the Centre National de Recherche Scientifique of France, and ordering information will become available shortly.

Another very recent publication is the collection of articles that has been known informally as the "Infeld Festschrift", because it was put together to honor Professor Infeld at his sixtieth birthday. The work, whose formal

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(2) Adress: Professor André Mercier, Institut für Theoretische Physik der Universität Bern, Sidlerstrasse 5, Bern, Switzerland.

title is "Recent Developments in General Relativity", has been published jointly by the Państwowe Wydawnictwo Naukowe (PWN—Polish Scientific Publishers) and by Pergamon Press. Copies and preprints of the individual contributions have been in circulation for some time, as this volume has also been in preparation since 1959. And finally, Professor Ivanenko showed me, also last night, a handsomely put-together translation of selected non-Russian papers in general relativity into Russian, no doubt a welcome addition to the literature for those among us who read Russian more fluently than the original languages of publication.

Over the last couple of years the question has been raised repeatedly, including by representatives of commercial interests, whether one ought to organize a new journal devoted primarily to relativity. I believe that on this issue as many opinions ought to be consulted as possible. Permit me to express my personal opinion—again I wish to remind you that I do not presume to speak for anyone else. At this stage at least, I am very hesitant—I might almost say, I am opposed to the idea—about endorsing a separate journal in this area. First, the number of journals that we must watch to keep abreast of our field is enormous as it is, and I doubt that we could expect that any of these will stop accepting papers in relativity just because an additional channel of publication is being organized. Second, at least in the United States we have experienced no major resistance to the acceptance of relativity papers by existing well-established journals such as the *Physical Review* or the *Journal of Mathematical Physics*. Needless to say, papers have been rejected on occasion, but I believe that I am sufficiently well acquainted with publication policies of these two periodicals to be assured that such rejections have not been based on the grounds that they were relativity papers. I cannot judge the situation in other countries; nor can I be certain about the situation confronting authors who wish to place their papers in mathematical, rather than physical journals. My third argument is that if we organize a separate journal for relativity papers, there is considerable danger that this journal will not be read by anyone who is not specializing in our field, whereas the major physics journals now have five-digit circulations.

Finally, I am worried about the editorial policies to be adopted by such a new journal. If the editorial board tends to be overly liberal, the new journal may be inundated by papers which have been rejected elsewhere, and this is not necessarily a desirable selection though I should be the last to claim that all rejected manuscripts are poor pieces of work. If the board, on the contrary, maintains what is called "high standards", then the result may well be that of the odd-hundred relativists in this room and elsewhere everybody will hate everybody else. That is why it seems to me that one ought to organize a new specialized journal only when the community to be served has grown to such size that there is no longer enough hating power left to go around; I doubt that we have achieved that growth yet. As you see, my doubts refer, at least in part, to our present situation, as I see it, and I should be quite willing to change it in the light of future developments. There is also something to be said in favor of the unity of science, and hence against the erection of lines of demarcation as long as the broadly oriented physics and mathematics journals are still willing and able to serve our needs.

Perhaps there is a somewhat different possibility, and that is to organize a forum for very lengthy articles, semi-monographs as it were, which do not fit precisely into the pattern of a regular research journal. Such a serial publication need not tie itself to a regular schedule, in which, for instance, you must fill a certain minimum number of pages every three or four months in order to look respectable. In the course of time, as the average amount of contributed material becomes known, this serial publication may well be converted into a regular journal. Whatever we finally do, I certainly agree that we should examine and re-examine, the publication needs of our field periodically.

I shall close by saying one thing in which I am sure, even without formal consultation or vote, I am speaking for all of us, who have had the privilege of being guests. The effort that has gone into preparing this conference and in making it a pleasant experience has been enormous, and it has been well spent. I am not only thinking of the manner in which the administrative wheels have been greased, ranging from obtaining plane reservations and hotel rooms to helping with Orbis vouchers, to mention but a few popular topics of conversation, but also of giving all of us opportunities to meet informally, as well as officially, with our colleagues under congenial circumstances. Only those who have ever had anything to do with organizing a conference can fully appreciate the enormous amount of thought, and of just plain dirty labor that has gone into this. We should thank, naturally, Professor Infeld, whose has been the over-all conception; but equally our other colleagues here at Warsaw, who have given liberally of their time and effort to make this International Conference what it has turned out to be. Thank you very much.

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# SEMINARS

# Q-NUMBER COORDINATE TRANSFORMATIONS AND THE ORDERING PROBLEM IN GENERAL RELATIVITY

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**Abstract**—The coordinate transformation which effects the passage from one set of coordinate conditions to another set will, in general, depend upon the metric and thus, in the quantum version of the theory will appear as a q-number transformation. To effect such a transformation without disturbing the fundamental commutation relations it is necessary to employ the generators of the infinitesimal coordinate transformations generated by linear combinations of the constraints of the theory. These infinitesimal generator swill, however, only form a group and thus allow one to construct finite transformations from them provided that the constraints satisfy commutation relations with the same structure as the corresponding Poisson bracket relations in the classical version of the theory. Classically, the Poisson bracket of any two constraints is a linear combination of constraints. For commutators we must require that the coefficients of these combinations, if q-numbers, must all stand to the left of the constraints if the theory is to be consistent. We have been able to show that there is no ordering of factors in the quantum mechanical expressions for the constraints which satisfies this condition. Hence we conclude that quantized versions of general relativity formulated in different coordinate systems are physically inequivalent which leads in to the conclusion that the principle of general covariance is incompatible with the requirements of quantum mechanics.

## INTRODUCTION

THE problem of the quantization of theories with generalized gauge groups such as electrodynamics, the Yang–Mills field or general relativity is complicated by the fact that not all of the basic field variables are dynamically determined by the field equations. Indeed, one can assign *a priori* values to certain combinations of the field variables without regard to the field equations. Such an assignment serves, in fact, to "fit" the gauge. Thus, in electrodynamics one makes use of the radiation gauge condition $\nabla \cdot A = 0$ to fix the gauge and in general relativity one makes use of similar conditions to fix the coordinate system. Those parts of the basic field variables which are not affected by this fixation of the gauge then describe the dynamics of the system in that their evolution with time is uniquely determined from their initial values by the field equations. It would, therefore, seem most natural to consider only these dynamical parts for the field as operators in a quantized version of the theory and to take, as representatives of physical states of the system, vectors of the Hilbert space over which the operators are defined. This Hilbert space approach to the quantization of gauge invariant theories has been applied by Bergmann and co-workers [1] to the quantization of the gravitational field and recently by Schwinger [2] in connection with the Yang–Mills field.

The other approach to the quantization of gauge invariant fields is due to Fermi [3]. There, one treats all of the basic field variables as operators in a linear vector space. Not all elements of this vector space represent physical states of the system however. Only those elements which, when operated on by the constraints of the theory, give zero can serve this purpose. As a consequence it is not possible to construct a norm in the linear vector space in the usual fashion and this in turn has led to such complications as the introduction of the indefinite metric by Gupta and Bleuler into the theory. It is just this complication which makes the Hilbert space method of quantization appear attractive. There one works directly with observables operating on a well-defined Hilbert space. In addition, the constraints of the theory are dealt with prior to the quantization process and, therefore, do not appear as restrictions on permissible state vectors as they do in the linear vector space approach to quantization. While the two methods of quantization should ultimately yield the same observable results it is argued that the former approach is more physical because it does not introduce elements into the theory which must later be gotten rid off by one means or another, e.g. the part of the linear vector space referred to above which does not represent possible physical states of the system.

In spite of the above-mentioned difficulties there are certain questions having to do with the gauge invariance of the theory as a whole which seem to be discussable only in terms of a linear vector space quantization. In fact, as Dirac has emphasised in his Yeshiva lectures, one must first destroy the gauge invariance of the theory at the classical level before one can employ the Hilbert space approach. Since one works only with the dynamical parts of the field variables in this approach and since these dynamical parts are gauge invariant to start with, there is no way in which one can discuss gauge transformations here. This in itself would not be a serious objection if it were possible to show directly that, starting from two inequivalent gauge conditions in the classical theory, there existed quantum versions of the two formulations which yielded equivalent physical results. Such a proof has been given for the case of electrodynaimcs by Zumino [4] and Białynicki-Birula [5] based on a consideration of the generating functional for the Green's functions of the theory. However, this method of proof does not seem to apply to the cases of general relativity or the Yang–Mills theory.

Since this problem of equivalence is central to our whole discussion it is perhaps desirable to state it more precisely in the form in which it arises in the Hilbert space quantization before we discuss it as it appears in the linear vector approach. To this end we will assume the existence of two inequivalent gauge conditions both of which fix the gauge completely. These conditions will each lead to a different set of dynamical variables, namely the remainder of the set of basic field variables not determined by the gauge conditions, which we will denote by $y_A$ and $y'_B$. The $y_A$ and $y'_B$ will, in general, be functionals of the basic field variables and their conjugate momenta. Since, in addition, each set can be used to give a complete description of the dynamics of the system it is clear that the members of one set can be expressed as functionals of the other set alone and vice versa, i.e., we can write

$$y_A = y_A(y'_B) \quad \text{and} \quad y'_B = y'_B(y_A).$$

For similar reasons the Poisson bracket between any two members of a set will be a functional of members of that set alone. Thus,

$$(y_A, y_{A'}) = f(y)$$

and

$$(y'_B, y'_{B'}) = g(y').$$

The transition to the quantized theory is then effected by finding an operator representation for the $y_A$ which satisfies

$$(y_A, y_{A'}) = i\hbar\, f(y)$$

or a representation for the $y'_B$ which satisfies

$$(y'_B, y'_{B'}) = i\hbar\, g(y').$$

In both cases, if any representation exists at all, there will exist many such representations depending upon how we choose the sequence of factors on the right hand sides of the commutation relations. The different representations will in general result in different physical predictions even if we stick with just one of the sets of dynamical variables. However, this is just a reflection of the well-known ambiguities which arise whenever we go from a classical to a quantum mechanical version of a theory. What is not trivial is the question, given an operator representation for the $y_A$ which satisfies the basic commutation relations, does there exist an operator representation for the $y'_B$ which satisfies the commutation relations appropriate to this set and at the same time yields the same set of physical predictions as does the representation for the $y_A$.

To answer the above question in the affirmative one would have to show that, starting with the variables $y$, one could find expressions for the $y'$'s in terms of the $y$'s which yield one of the possible sets of commutation relations for the $y'$ 's. One could start with some particular expression for the $y'$'s obtained by making a choice for the sequence of factors in the classical expressions for the $y'$ 's in terms of the $y$'s and compute the commutator $(y'_B, y'_{B'})$ from the commutation relations for the $y$'s. The result would be some functional $\bar{g}$ of the $y$'s. To show that $\bar{g}(y)$ was indeed equal to a possible $g(y')$, i.e. a $g(y')$ obtained from the classical expression for $g(y')$ by choosing an ordering of factors, one would first have to show that one could solve the operator equations giving the $y'$ 's in terms of the $y$'s for the $y$'s in terms of the $y'$ 's and it is not obvious that this can be done in general. Even if this step could be accomplished it would still be necessary to check that $\bar{g}(y(y'))=g(y')$ for some sequence of factors in $g(y')$ and again it is not obvious that this would be the case in general. It is quite possible that the two expressions might differ from each other by terms proportional to powers of $\hbar$. At best it would appear that one would have to carry out an explicit proof of equivalence for each pair of gauge conditions. For this reason we have examined the equivalence problem for different gauges in the linear vector space method of quantization where it is possible to arrive at some definite conclusions concerning this problem. In particular we have shown that different gauges lead to equivalent physical theories in the case of electrodynamics and the Yang–Mills field but not for the case of general relativity.

## LINEAR VECTOR SPACE QUANTIZATION

A complete description of the linear vector space quantization of electrodynamics has been given elsewhere [6]. The general procedure is immediately applicable to any theory with a gauge group and hence only the conclusions will be stated here since our purpose is to discuss the particular case of general relativity. In particular we will state a criterion which must be met by such a theory if all gauges are to be equivalent. To this end then let us suppose that we have some theory with a gauge described by the basic canonical variables $\varphi_i$. Because of the existence of the gauge group these variables are not all independent of each other but, as is well known [7], satisfy a number of constraint equations which, in the classical theory, take the form

$$X_a(\varphi_i) = 0. \tag{1}$$

In what follows we shall refer to the expressions $X_a$ as the constraints of the theory. A knowledge of the canonical variables $\varphi_i$ together with the constraint equations and the Hamiltonian form a complete classical description of the theory under consideration. One should emphasize even here that one may not make use of the constraint equations for substitutional purposes in dynamical quantities, i.e. solving the constraint for a number of canonical variables in terms of the others and eliminating these solved for variables from dynamical quantities containing them by direct substitution, until after all desired Poisson brackets have been computed.

The quantization of our theory is effected by assuming that the canonical variables are now q-numbers satisfying the standard commutation relations for canonical variables. We must also bring in the constraint equations but we cannot simply assume that they are operator equations between the canonical variables because they would stand in contradiction to the assumed commutation relations for these variables. Thus, for example in electrodynamics one of the constraint equations is

$$p_4 = 0 \tag{2}$$

where $p_4$ is the momentum density conjugate to the fourth component of the four-potential $A^4$. It is clear that this cannot be considered as an operator equation if, at the same time, we require the commutation relation

$$[A^4(x), p_4(x')] = i\hbar\, \delta(x-x'). \tag{3}$$

What we must do is to maintain the commutation relations and restrict our attention to those elements of the linear vector space upon which all of the canonical variables can operate which are eigenvectors of $p_4$ with zero eigenvalue, that is to those elements $\Psi$ satisfying

$$p_4 \Psi = 0. \tag{4}$$

But it is just this restriction on the elements of the linear vector space which are to describe possible states of the electromagnetic field which complicates the problem of constructing a norm for them. Thus, in a coordinate representation (Schrödinger representation) where $p_4$ is represented by the operation of functional differentiation with respect to $A^4$ multiplied by $i\hbar$, the constraint equation (4) tells us that the permissible elements are independent of $A^4$. We could not then for instance construct a norm for these elements by integrating over all four components of the four-potential since the integration over $A^4$ would always lead to a divergent result. If we are not concerned with maintaining manifest Lorentz covariance we can construct a perfectly acceptable norm by eliminating all integrations with respect to $A^4$. Similarly, the other constraint equation of electrodynamics

$$(p_{r,r}+\varrho)\Psi = 0 \tag{5}$$

where $\varrho$ is the charge density, forces one to restrict the integration over only the transverse components of the vector potential. In the general case then we will assume that the quantum mechanical analogue of the constraint equations (1) are

$$X_a(\varphi_i)\Psi = 0 \tag{6}$$

for those elements $\Psi$ which are to represent physically realizable states of our system. If these equations are very complicated, as they are in general relativity, it may prove to be a difficult matter to construct a norm in practice although in principle at least there do not appear to be any complications.

We now turn our attention to the gauge invariance properties of the theory, again making use of the electrodynamical case to illustrate our remarks. As usually stated, the gauge invariance of electrodynamics is the invariance in form of the Lagrangian under the transformation of the potentials

$$A'_\mu = A_\mu + \lambda_{,\mu}. \tag{7}$$

If we impose a gauge condition on the potentials such as the radiation gauge condition or the like it is generally true that there will always exist a descriptor $\lambda$ which will effect the transformation from an arbitrary set of potentials to a set which satisfies the prescribed gauge condition. However, $\lambda$ will in general depend upon the potentials with which one started. Thus, if we impose the radiation gauge, $\lambda$ will be given in terms of the original potentials by the expression

$$\lambda = -\nabla^{-2}\,\nabla\cdot A.$$

It is just this dependence of $\lambda$ upon the potentials which causes difficulty in the canonical, and therefore the quantized, version of the theory since the transformation (7) will not usually be a canonical transformation in the classical formulation or a unitary transformation in the quantum formulation if $\lambda$ itself depends upon the potentials. What we must do is to redefine a gauge transformation so that it is in fact a canonical or unitary transformation in the linear vector space and reduces to the form (7) only in the subspace of elements which satisfy the constraint equations (4) and (5). To accomplish this we make use of the fact that the constraints themselves generate an infinitesimal transformation which has exactly this property. Thus, the generator $C$ given by

$$C = \int d^3x(\dot{\lambda}p_4 + \lambda(p_{r,r} + \varrho)) \tag{8}$$

generates the transformation

$$i\hbar\bar{\delta}A_\mu = \lambda_{,\mu} + \int d^3x\{[A_\mu, \dot{\lambda}']p'_4 + [A_\mu,\lambda'](p'_{r,r}+\varrho')\} \tag{9a}$$

$$i\hbar\bar{\delta}p_\mu = \int d^3x'\{[p_\mu, \dot{\lambda}']p'_4 + [p_\mu,\lambda'](p'_{r,r}+\varrho')\} \tag{9b}$$

which reduces to

$$(i\hbar\bar{\delta}A_\mu - \lambda_{,\mu})\Psi = 0 \tag{10a}$$

$$i\hbar\bar{\delta}p_\mu\Psi = 0 \tag{10b}$$

for $\Psi$'s satisfying Eqs (4) and (5). These results generalize immediately to the case where the constraint equations are of the form (6). The generator of the infinitesimal transformation is just

$$C = \int d^3x\, \lambda_a \cdot X_a. \tag{11}$$

In ordering the factors in this expression we must make sure that the descriptors always stand to the left of the constraints to insure the transformation reduce to the classical expression in the subspace of $\Psi$'s satisfying the constraint equations (6).

Since $C$ is the generator of only an infinitesimal transformation we must check that the commutator of two such generators is again a generator of the same kind. Otherwise we would not have the group property and could not integrate to obtain the finite generator which would lead us from one gauge frame to a finitely different frame. This condition is satisfied provided that the commutator of two constraints is a linear combination of the constraints with the additional requirement that in the linear combination the constraints must all stand to the right of the coefficients. This condition of course follows immediately from equation (6) since we must have

$$(X_\alpha X_\beta - X_\beta X_\alpha)\Psi = 0 \tag{12}$$

which will only be the case provided

$$X_\alpha X_\beta - X_\beta X_\alpha = w_{\alpha\beta}{}^{\delta} X_\delta \tag{13}$$

with the $w$'s standing to the left of all the $X$'s on the right-hand side of this equation. However, it is important to see how this condition arises as a necessary condition for the construction of finite q-number gauge transformations. Making use of these ideas one can go on to show that the theory as a whole is gauge invariant when operating on elements of the linear vector space satisfying the constraint equations (6). We will not carry through this discussion here since our main purpose was to arrive to the conditions (13) in the context of q-number gauge transformations and to examine their satisfaction for the various gauge invariant theories.

The conditions (13) are obviously satisfied for the case of electrodynamics (electromagnetic plus Dirac field) and while not obvious are also satisfied for the case of the Yang–Mills field. They are not satisfied for the case of general relativity, however, and we shall discuss this case in the next section.

## THE ORDERING PROBLEM FOR GENERAL RELATIVITY

The problem of the satisfaction of the requirements (13) for any gauge invariant theory reduces essentially to that of finding an ordering of factors in the expressions for the constraints since classically the Poisson bracket of two constraints is always a linear combination of constraints [7]. In general relativity the primary constraint equations are just

$$p_{0\mu} = 0 \tag{14}$$

so that there is no ordering problem when we go over to the quantum version of the theory. The secondary constraints divide into two groups. The first group consists of what we have called the longitudinal constraints and are of the form

$$\mathcal{H}_s = g_{ab,s}p^{ab} - 2(g_{as}p^{ab})_{,b} = 0 \tag{15}$$

where $p^{ab}$ is the momentum density conjugate to the spatial part of the metric $g_{ab}$. The Poisson bracket of two longitudinal constraints is again a linear combination of longitudinal constraint and, if we adopt the ordering indicated in Eq. (15) with the $p$'s standing to the right of the $g$'s, so is the commutator.(1) Since the coefficients of the linear combination are c-numbers the requirements (13) are automatically satisfied.

The other group of constraints which we call the Hamiltonian constraints have the classical form

$$\mathcal{H}_L = \frac{1}{K}\left(g_{ra}g_{sb} - \frac{1}{2}g_{rs}g_{ab}\right)p^{rs}p^{ab} + K^3R(g_{ab}) = 0 \tag{16}$$

where $K$ is the square root of the determinant of $g_{ab}$ and ${}^3R$ is the curvature scalar formed from $g_{ab}$ and its inverse. The ordering problem arises only in connection with the first term and for it we must find an ordering which reproduces, as commutation relations, the classical Poisson bracket relations

$$(\mathcal{H}_L, \mathcal{H}'_L) = -\mathcal{H}_r e^{rs}\delta_{,s}(x-x') + \mathcal{H}'_r e'^{rs}\delta_{,s'}(x-x') \tag{17}$$

and

$$(\mathcal{H}_L, \mathcal{H}'_s) = \{\mathcal{H}_L\delta(x-x')\}_{,s}. \tag{18}$$

There are, of course, an infinity of different possible orderings for these terms depending upon the positioning of the $p$'s. However, by making use of the commutation relations, between the $g$'s and $p$'s any ordering can be transformed to the form

$$\frac{1}{K}\left(g_{ra}g_{sb} - \frac{1}{2}g_{rs}g_{ab}\right)p^{rs}p^{ab} + K^3R - i\hbar\delta(0)\frac{a}{K}g_{rs}p^{rs} - \hbar^2\delta^2(0)\frac{b}{K}$$

where $a$ and $b$ are numerical constants which depend upon the initial ordering chosen. There are a number of possible values of $a$ and $b$, including zero, which lead to the reproduction of the relations (17). However, a rather laborious but straightforward computation shows tgat there are none which reproduce the relations (18). The consistency conditions (13) therefore cannot be satisfied for the case of general relativity.(2)

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(1) We adopt this ordering so that $\mathcal{H}_s$ will indeed be the generator of an infinitesimal transformation of coordinates in the $X^4 =$ const. surface when applied to a functional of $g_{rs}$.

(2) In a previous paper (J. L. Anderson, *Phys. Rev.* **114**, 1182 (1959)) we also discussed the ordering problem. There we made the mistake of requiring only that the commutator of two constraints again be a linear combination of constraints without regard to the position of the coefficients. We showed that there did exist orderings which satisfied this requirement. We are grateful to Prof. Dirac for pointing out the necessity of the stronger conditions (13).

The fact that the consistency conditions (13) cannot be satisfied for the case of general relativity does not mean that we cannot quantize general relativity. We could introduce a particular set of coordinate and proceed with a Hilbert space quantization as outlined in the introduction. Alternately we could impose coordinate conditions into the linear vector space quantization scheme directly. This would have the effect of converting the constraint equations into second class constraints in the terminology of Dirac. Now ordinary bracket expressions would have to be replaced by Dirac brackets [8]. Since the Dirac bracket of any two second class constraints is automatically zero we would no longer have a problem of ordering factors in the constraints; any ordering would work.

However, both procedures mentioned above introduce, *ab initio*, some particular set of coordinate conditions. If we had introduced some other set of coordinate conditions we would obtain a different theory whose results would differ from that obtained with the first set of conditions by quantities proportional to powers of $\hbar$. This in turn would mean that, at least in principle, we could distinguish between different coordinate systems by means of experimental observations. But this possibility is in flat contradiction to the principle of general covariance whereby the observable results of general relativity should be independent of the coordinate system employed. Since this principle is one of the cornerstones of the general theory we seem to be faced with an apparent contradiction when we attempt to join together the two disciplines of general relativity and quantum mechanics. If our conclusions are correct it means that we must examine in detail the whole mechanism whereby one introduces a coordinate system and the limitations imposed by the uncertainty principle on the setting up of a coordinate system.

We would like to thank Professor Dirac for several very helpful discussions of the work presented here.

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# ELASTICITY IN GENERAL RELATIVITY

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THE theory of elasticity in General Relativity which I am going to talk about was formulated initially as a modification of a theory of Professor J. L. Synge (*Math. Zeitschr.* **72**, 82–87 (1959)). The modification in question consists of using a Lie derivative where Synge used a covariant derivative. However, the two theories are logically distinct, and I confine my attention in the time available to the new theory.

For any vector field $\xi^i$ $(i = 1, 2, 3, 4)$ we may take co-moving coordinates such that $\xi^i = (0, 0, 0, 1)$, and then the Lie differential operator is defined as $\partial/\partial x^4$. In particular, for the time-like unit tangent field $\lambda^i$ to the world-lines of a moving body, we may construct the singular "projection" tensor $\tilde{g}_{ij} = g_{ij} + \lambda_i\lambda_j$ (for which $\tilde{g}_{ij}\lambda^j = 0$), and then the condition $\mathcal{L}_\lambda \tilde{g}_{ij} = 0$ is that for the field $\lambda^i$ to be rigid relative to the metric $g_{ij}$. In the classical theory we may write Hooke's law in the form $S_{\alpha\beta} = \frac{1}{2}\, e_{\alpha\beta\gamma\delta} h_{\gamma\delta}$, where $S_{\alpha\beta}$, $h_{\alpha\beta}$ are the stress and strain tensors, and $e_{\alpha\beta\gamma\delta}$ is the tensor of elastic coefficients, which remains constant in time. The matrix $e_{AB}\,(A = (\alpha\beta),\ B = (\gamma\delta))$ must be non-singular. In General Relativity we now write Hooke's law in the form $S_{ij} = \frac{1}{2}\, c_{ij}{}^{kl}(\tilde{g}_{kl} - \tilde{g}^0_{kl})$. Here, $S_{ij}$ is the stress-tensor, and $\tilde{g}_{ij} - \tilde{g}^0_{ij}$ represents the state of strain; the positive semi-definite tensor $\tilde{g}^0_{ij}$ is such that

$$\mathcal{L}_\lambda \tilde{g}^0_{ij} = 0, \qquad \tilde{g}^0_{ij}\lambda^j = 0, \qquad \tilde{g}^0_{ij} = \tilde{g}^0_{ji}. \tag{1}$$

Evidently, $\lambda^i$ is rigid with respect to the metric $\tilde{g}^0_{ij} = \tilde{g}^0_{ij} - \lambda_i\lambda_j$. The tensor $c_{ij}{}^{kl}$ is given algebraically in terms of fundamental elasticity tensor $C_{ijkl}$ and $\tilde{g}^0_{ij}$ by the conditions

$$\tilde{g}^0_{kr}\tilde{g}^0_{ls} c_{ij}{}^{rs} = C_{ijkl}. \tag{2}$$

Although $c_{ij}{}^{kl}$ is not thus uniquely defined (because $\tilde{g}^0_{ij}$ is singular), nevertheless the product $c_{ij}{}^{kl}(\tilde{g}_{kl} - \tilde{g}^0_{kl})$ is easily shown to be unique. The tensor $C_{ijkl}$ we postulate to satisfy the relations

$$C_{ijkl} = C_{jikl} = C_{klij}, \qquad C_{ijkl}\lambda^l = 0, \qquad \mathcal{L}_\lambda C_{ijkl} = 0. \tag{3}$$

These relations are interpreted respectively as meaning that $C_{ijkl}$ has the same symmetries as the classical elasticity tensor, that it is "three-dimensional", and that it is "time-independent". The interior field equations for an elastic body are thus of the form

$$G_{ij} = -\varrho\lambda_i\lambda_j + \frac{1}{2}c_{ij}{}^{kl}(\tilde{g}_{kl} - \tilde{g}^0_{kl}) \tag{4}$$

$$\varrho = -G_{rs}\lambda^r\lambda^s, \qquad \lambda^j\lambda_j = -1 \tag{5}$$

where $G_{ij}$ is the Einstein tensor and $\varrho$ the proper density; units are such that the gravitational constant is $(8\pi)^{-1}$. In particular, we say that a body is isotropic if $C_{ijkl}$ admits the representation, for some $\nu$, $\mu$,

$$C_{ijkl} = \nu\tilde{g}^0_{ij}\tilde{g}^0_{kl} + \mu(\tilde{g}^0_{ik}\tilde{g}^0_{jl} + \tilde{g}^0_{il}\tilde{g}^0_{jk}) \qquad \nu_{,j}\lambda^j = \mu_{,j}\lambda^j = 0. \tag{6}$$

To summarize, an elastic body motion in space-time is a world-tube over the interior of which the tensors $g_{ij}$, $\lambda_i$, $\tilde{g}^0_{ij}$, $C_{ijkl}$ are defined, which satisfy (1)—(5). We remark: (a) in co-moving co-ordinates (4) reduces to 9 equations in 9 unknowns. In fact, for $i = j = 4$, we get identity, by the difinition (5) of $\varrho$; the unknowns are $g_{ij}$, other than $g_{44} = -1$. (b) Contraction of (4) with $\lambda^j$ shows that $\lambda^i$ is an eigenvector of $G_{ij}$. (c) It can be shown that shock waves travel with the same speeds as arise in classical elasticity theory when the strain is small.

# GRAVITATIONSFELDER MIT ISOTROPEM KILLINGVEKTOR

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**Abstract** — The general solution of the vacuum field equations admitting an isotropic Killing field is given. The method, used first by H. Weyl [1] and later especially by Robinson and Trautman [2], consists of simplifying the metric with help of coordinate-transformations, which are only possible, if certain field equations are satisfied. There are two kinds of solutions. In the first case the Killing vector is a gradient, in which case we get the well known plane waves (Brinkmann [3]). In the other case one obtains algebraically more general solutions of Petrov-type II, in certain cases of type $I_{deg}$. The solutions contain true singularities on a submanifold.

WIR stellen uns die Aufgabe, alle Lösungen der Vakuumfeldgleichungen $R_{\mu\nu} = 0$ zu bestimmen, die (mindestens) einen isotropen Killingvektor besitzen:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0, \tag{1}$$

$$\xi_\mu \xi^\mu = 0. \tag{2}$$

Beziehen wir $\xi_{\mu;\nu}$ auf ein pseudoorthogonales Vierbein ($\xi_\mu$, $m_\mu$, $t_\mu$) so folgt aus (1), (2) die Darstellung:

$$\xi_{\mu;\nu} = i\omega t_\mu \bar{t}_\nu + \Omega(t_\mu \xi_\nu - t_\nu \xi_\mu) + \text{konj.-kompl.}; \tag{3}$$

d. h. für die betrachtete Strahlenkongruenz $\xi_\mu$ sind von den optischen Parametern nur der "Drill" $\omega$ und die "Rotation" $\Omega$ (sowie $\zeta = -\Omega$) von Null verschieden.

Zur Integration der Feldgleichungen wählen wir zunächst ein Koordinatensystem mit $\xi^\mu = \delta^\mu_0$. (1) und (2) gehen dann in $g_{\mu\nu,0} = 0$ und $g_{00} = 0$ oder

$$|g^{ik}| = 0, \quad g_{\mu\nu,0} = 0 \tag{4}$$

über. Man erkennt leicht, daß die drei zweireihigen Unterdeterminanten längst der Hauptdiagonalen der Matrix ($g^{ik}$) wegen $|g^{\mu\nu}| \neq 0$ nicht sämtlich verschwinden dürfen. Es sei also (innerhalb eines gewissen Bereiches) etwa

$$|g^{AB}| = \begin{vmatrix} g^{22} g^{23} \\ g^{23} g^{33} \end{vmatrix} \neq 0.^{(1)}$$

Setzen wir dann

$$\mu = (g^{12}g^{33} - g^{13}g^{23})/|g^{AB}|, \tag{5}$$
$$\lambda = (g^{13}g^{22} - g^{12}g^{23})/|g^{AB}|$$

(1) Die Indizes $A$, $B$ ... laufen von 2 bis 3, $i$, $k$ ... von 1 bis 3, $\mu$, $\nu$ ... von 0 bis 3; die Signatur wird als $-2$ angenommen.

so gilt

$$|g^{\mu\nu}| = -|g^{AB}|(g^{01}-\mu g^{02}-\lambda g^{03})^2, \tag{6}$$

woraus sich die schärfere Bedingung

$$|g^{AB}| > 0 \tag{7}$$

ergibt.

Die Bedingungen (4) bleiben insbesondere bei den Transformationen

$$\begin{aligned} \bar{x}^0 &= x^0, \\ \bar{x}^1 &= f(x^1, x^A), \\ \bar{x}^A &= x^A \end{aligned} \tag{8}$$

erhalten. Verlangen wir, daß durch diese Transformationen $x^1$ zu einer Nullkoordinate wird ($\bar{g}^{11} = 0$), so muß $f(x^i)$ der Gleichung

$$\begin{aligned} \sigma_A \sigma_B g^{AB} &= 0, \\ \sigma_2 &\equiv \mu f_{,1} + f_{,2}, \\ \sigma_3 &\equiv \lambda f_{,1} + f_{,3} \end{aligned} \tag{9}$$

genügen. Aus ihr folgt wegen (7) das System

$$\mu f_{,1} + f_{,2} = \lambda f_{,1} + f_{,3} = 0. \tag{10}$$

Die Integrabilitätsbedingung für diese Differentialgleichungen lautet

$$\mu_{,3} - \lambda_{,2} + \lambda \mu_{,1} - \mu \lambda_{,1} = 0 \tag{11}$$

und ist als Folge der Feldgleichung

$$R_{00} \equiv -\frac{1}{2} g_{01}^2 (\mu_{,3} - \lambda_{,2} + \mu_{,1}\lambda - \lambda_{,1}\mu)^2 = 0 \tag{12}$$

identisch erfüllt. Weiter folgt $\bar{g}^{1A} = g^{1A} f_{,1} + f_{,B} g^{AB} = 0$. Wir erhalten damit, wenn wir noch durch die Transformation $\bar{x}^0 = x^0$, $\bar{x}^1 = x^1$, $\bar{x}^A = \bar{x}^A(x^i)$ die zweidimensionale Metrik $g_{AB}$ auf Diagonalform bringen, folgendes Schema für die $g_{\mu\nu}$:

$$g_{\mu\nu} = \begin{pmatrix} 0 & g_{01} & 0 & 0 \\ g_{01} & g_{11} & g_{12} & g_{13} \\ 0 & g_{21} & -e^{2\Omega} & 0 \\ 0 & g_{31} & 0 & -e^{2\Omega} \end{pmatrix}. \tag{13}$$

Die Gestalt (13) der Metrik ist noch gegenüber den Koordinatentransformationen

$$\begin{aligned} \bar{x}^0 &= \bar{x}^0 + \varphi(x^i) \\ \bar{x}^1 &= \bar{x}^1(x^1) \\ \bar{x}^A &= \bar{x}^A(x^i) \end{aligned} \quad \text{mit} \quad \frac{\partial^2 \bar{x}^A}{(\partial x^2)^2} + \frac{\partial^2 \bar{x}^A}{(\partial x^3)^2} \equiv \Delta \bar{x}^A = 0 \tag{14}$$

invariant.

Von den Feldgleichungen $R_{\mu\nu}=0$ sind $R_{00}=0$ und $R_{0A}=0$ bereits identisch erfüllt. Die weiteren Gleichungen $R_{01}=0$, $R_{22}+R_{33}=0$, $R_{22}-R_{33}=0$, $R_{23}=0$ lauten respektive:

$$\Delta m+(m_{,2})^2+(m_{,3})^2=0 \qquad (m\equiv \ln g_{01}), \tag{15}$$

$$\Omega_{,2}m_{,2}-\Omega_{,3}m_{,3}=m_{,22}/2-m_{,33}/2+(m_{,2})^2/4-(m_{,3})^2/4\equiv A, \tag{16}$$

$$\Omega_{,3}m_{,2}+\Omega_{,2}m_{,3}=m_{,2}m_{,3}/2+m_{,23}\equiv B, \tag{17}$$

$$2\Delta\Omega+\Delta m+(m_{,2})^2/2+(m_{,3})^2/2=0, \tag{18}$$

Aus ihnen läßt sich ablesen, daß es zwei wesentlich verschiedene Arten von Lösungen gibt, je nachdem, ob $(m_{,2})^2+(m_{,3})^2=0$ oder von Null verschieden ist. $\xi_\mu$ ist nun genau dann ein Gradient, wenn $m$ von $x^A$ unabhängig ist In diesem Falle sind also, wie bekannt ist,(2) die ebenen Wellen die einzigen singularitätsfreien Lösungen der Vakuumfeldgleichungen.

Nehmen wir dagegen (innerhalb eines gewissen Gebietes) $(m_{,2})^2+(m_{,3})^2\neq 0$ an, so folgt aus (16), (17):

$$\Omega_{,2}=(Am_{,2}+Bm_{,3})/\{(m_{,2})^2+(m_{,3})^2\}, \tag{19}$$

$$\Omega_{,3}=(Bm_{,2}-Am_{,3})/\{(m_{,2})^2+(m_{,3})^2\}. \tag{20}$$

Hierbei sind sowohl die Integrabilitätsbedingung $\Omega_{,23}=\Omega_{,32}$ als auch die Feldgleichung $R_{23}=0$ infolge der Beziehung (15) identisch erfüllt.

Da für $g_{01}$ die zweidimensionale Potentialgleichung $\Delta g_{01}=0$ gilt, lassen sich "kanonische Koordinaten" [1] einführen. In ihnen gilt

$$g_{01}=x^2 \tag{21}$$

und aus (19), (20) folgt

$$e^{2\Omega}=\frac{a(x^1)}{\sqrt{x^2}}, \tag{22}$$

$a$ ist eine willkürliche Funktion von $x^1$.

Die Feldgleichungen $R_{A1}=0$ reduzieren sich nach einmaliger Integration auf die Gleichung

$$q_{3,2}-q_{2,3}=(b(x^1)-a_{,1}x^3)/(x^2)^{5/2}. \tag{23}$$

Hierbei ist $b$ eine zweite willkürliche Funktion von $x^1$ sowie

$$q_2=-g_{12}/x^2, \qquad q_3=-g_{13}/x^2. \tag{24}$$

Bei den Eichungen

$$\bar{x}^0=x^0+\varphi(x^i), \qquad \bar{x}^i=x^i \tag{25}$$

(2) Vgl. [2]. Hier stellt man dies wie folgt fest: $m=m(x^1)$ kann durch eine Transformation $\bar{x}^1=\bar{x}^1(x^1)$ zum Verschwinden gebracht werden. Dann reduzieren sich die Gleichungen $R_{AB}=0$ auf das Verschwinden des mit der Metrik $g_{AB}$ gebildeten zweidimensionalen Riccitensors, d.h. es gilt $g_{AB}=g_{AB}(x^1)$. Alle Feldgleichungen bis auf $R_{11}=0$ sind identisch erfüllt.

geht $q_A$ in

$$\bar{q}_A = q_A + \varphi_{,A} \tag{26}$$

über. Wir können somit durch geeignete Wahl von $\varphi$ erreichen, daß $q_2$ verschwindet. Aus (23) folgt dann

$$q_3 = \frac{-2}{3}(b - a_{,1}x^3)/(x^2)^{3/2} + A(x^1, x^3). \tag{27}$$

Die hierin auftretende Funktion $A(x^1,x^3)$ bringen wir ebenfalls durch Umeichung zum Verschwinden.

Die letzte noch zu befriedigende Feldgleichung $R_{11} = 0$ nimmt die Gestalt einer linearen Differentialgleichung für die Größe $\chi = g_{11}/x^2$ an:

$$\frac{\partial^2\chi}{(\partial x^2)^2} + \frac{\partial^2\chi}{(\partial x^3)^2} + \frac{1}{x^2}\frac{\partial\chi}{\partial x^2} = -\frac{(b - a_{,1}x^3)^2}{a(x^2)^{7/2}} + \frac{1}{(x^2)^{3/2}}\left(\frac{4}{3}a_{,11} - \frac{a_{,1}^2}{a}\right). \tag{28}$$

Das Linienelement für die zweite Klasse von Lösungen lautet damit endgültig:

$$ds^2 = 2x^2dx^0dx^1 + x^2\chi(dx^1)^2 - 2x^2q_3dx^1dx^3 -$$

$$- \frac{a(x^1)}{\sqrt{x^2}}[(dx^2)^2 + (dx^3)^2]. \tag{29}$$

Es enthält die beiden willkürlichen Funktionen $a(x^1), b(x^1)$. Die auf der Untermannigfaltigkeit $x^2 = 0$ auftretenden Singularitäten sind nicht durch Koordinatentransformationen eliminierbar, da sie auch in der Invarianten

$$R_{\mu\nu\varrho\sigma}R^{\mu\nu\varrho\sigma} = \frac{3a^2}{4(x^2)^3} \tag{30}$$

vorhanden sind.

Für den Riemanntensor zur Metrik (29) gilt

$$R_{\mu\nu\varrho\sigma}\xi^\nu\xi^\sigma = C\xi_\mu\xi_\varrho \tag{31}$$

mit $C = \frac{(x^2)^{-3/2}}{4a}$. Die Lösungen gehören also zu einem algebraisch speziellen Petrov-Typ, und zwar sind sie im allgemeinen vom Typ II, in Spezialfällen vom Typ $I_{ent}$. Wir bemerken noch, daß für die Lösungen (29) auch der Drill $\omega$ bezüglich der Strahlenkongruenz $\xi^\mu$ verschwindet.

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# ÜBER LÖSUNGEN DER EINSTEINSCHEN GRAVITATIONSGLEICHUNGEN, BEI DENEN DIE DETERMINANTE $g$ AUF NIEDERDIMENSIONIERTEN MANNIGFALTIGKEITEN VERSCHWINDET

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IN DEN Diskussionen über die Divergenzschwierigkeiten in der Quantenfeldtheorie ist mehrfach die Vermutung geäußert worden, daß in Gebieten mit extrem starken Feldern die Signatur der Raum-Zeit-Mannigfaltigkeit $V_4$ von der Lorentz-Minkowskischen Signatur verschieden ist. In einem solchen Weltgebiet, etwa innerhalb eines von einer Hyperfläche $S$ umschlossenen Zylinders, der die Geschichte des sehr kleinen Kerns eines Elementarteilchens darstellt, soll der $V_4$ die negativ-definite Signatur $(-1, -1, -1, -1)$ besitzen. Die Feldgleichungen werden in diesem Gebiet elliptisch und alle Wechselwirkungen geschehen momentan, so daß der Zylinder ein Modell für den sehr kleinen hard core eines Elementarteilchens sein könnte.[1] Die zur Hyperfläche $S$ benachbarten Hyperflächen $S^*$ müssen im Bereich der Minkowskischen Signatur $(-1, -1, -1, +1)$ zeitartig sein.

Nun ist für die metrische Struktur des $V_4$ die Einsteinsche Gravitationstheorie zuständig, da der metrische Tensor $g_{\mu\nu}$ durch die Einsteinschen Feldgleichungen bestimmt wird. Unabhängig von der erwähnten Hypothese über die Signatur des $V_4$ im Teilcheninneren kann man fragen, ob die Einsteinschen Gleichungen überhaupt eine Änderung der Signatur des $V_4$ gestatten und wenn ja, welche Eigenschaften für einen solchen $V_4$ aus den Einsteinschen Gleichungen folgen. Wir wollen uns hier auf die Einsteinschen Vakuumgleichungen $R_{\mu\nu}=0$ beschränken.[2] Da wir — im wesentlichen — Stetigkeit der $g_{\mu\nu}$ voraussetzen müssen, bedeutet eine Signaturänderung, daß die Determinante $g$, die im Bereich der Minkowskischen Signatur kleiner Null ist, auf der genannten Hyperfläche $S$ eine Nullstelle haben muß, da ja innerhalb des von $S$ umschlossenen Bereichs $g>0$ sein soll.

(1) Wenn sich der Zylinder auf eine Weltlinie reduziert (s.u.), hat man das Modell eines punktförmigen Teilchens.

(2) Unsere Aufgabe steht so im unmittelbaren Zusammenhang mit dem Teilchenproblem Einsteins: Ob und inwiefern die Elementarteilchen als Eigenlösungen der Gravitationsgleichungen mit Grenzbedingungen zu erhalten sind.

Es gilt also $g_s=0$ und daher sind auf $S$ einige Komponenten der kontravarianten $g^{\mu\nu}$ unendlich gross. Sind die kovarianten $g_{\mu\nu}$ überall regulär und hat $g$ an $S$ eine Nullstelle $n$-ter Ordnung, so hat mindestens eine Komponente von $g^{\mu\nu}$ einen Pol $n$-ter Ordnung an $S$. Da nun der Einstein-Ricci-Tensor $R_{\mu\nu}$ bilinear in den kontravarianten $g^{\mu\nu}$ ist und linear in ihren ersten Ableitungen, so enthält $R_{\mu\nu}$ Terme mit dem Faktor $g^{-2}$. $R_{\mu\nu}$ ist somit an der Hyperfläche $g=0$ zunächst nicht erklärt.

Unter der Voraussetzung, daß die kovarianten $g_{\mu\nu}$ überall regulär und einmal stetig differenzierbar sind, ist aber ersichtlich die Tensordichte $g^2R_{\mu\nu}$ überall erklärt und regulär. Dementsprechend haben Einstein und Rosen [1] [2] [3] vorgeschlagen, die Vakuumfeldgleichungen zu schreiben:

$$g^2R_{\mu\nu}=0. \tag{1}$$

(1) hat ersichtlich auch dann Sinn, wenn $g$ verschwindet. Setzen wir nun voraus, daß $g$ an $S$ nur mit der endlichen Ordnung $n$ verschwindet (wobei $n$ eine natürliche Zahl sein möge)(3) und verlangen wir die Gültigkeit von (1) in einem endlichen Streifen beiderseits von $S$, so verschwinden mit $g^2R_{\mu\nu}$ selbst auch sämtliche Ableitungen mit $g^2R_{\mu\nu}$ an $S$.

Geben wir etwa der Hyperfläche $S$ die Gleichung $x^1=0$, so gilt demnach:

$$(g^2R_{\mu\nu})_{x^1=0}=\left(\frac{\partial}{\partial x^1}\left[g^2R_{\mu\nu}\right]\right)_{x^1=0}=\cdots=\left(\frac{\partial^k}{\partial (x^1)^k}\left[g^2R_{\mu\nu}\right]\right)_{x^1=0}=\ldots=0. \tag{2}$$

Andererseits gilt voraussetzungsgemäß für $g$ die Entwicklung

$$g=\underset{n}{\alpha}(x^2,x^3,x^0)(x^1)^n+\underset{n+1}{\alpha}(x^2,x^3,x^0)(x^1)^{n+1}+\ldots \tag{3}$$

Wir können somit nach der l'Hôpitalschen Regel den Grenzwert

$$\lim_{x^1\to 0}\left(\frac{g^2R_{\mu\nu}}{g^2}\right)=\left(\frac{\dfrac{\partial^{2n}}{\partial (x^1)^{2n}}(g^2R_{\mu\nu})}{(2n)!\,\underset{n}{\alpha}^2}\right)_{x^1=0}=0 \tag{4}$$

bilden und damit auch an $S$ dem Tensor $R_{\mu\nu}$ den Wert Null zuschreiben. Für endliche Nullstellen von $g$ sind somit die Einsteinschen Gleichungen $R_{\mu\nu}=0$ mit (1) identisch. Hieraus folgt, daß die Nullstellen von $g$ keine echten Singularitäten der Einsteinschen Gleichungen sind.

Wir können daher bei unserem Problem einfach die Einsteinschen Vakuumgleichungen zu Grunde legen. In der Tat sieht man, daß prinzipiell zu jeder Lösung $g_{\mu\nu}$ der Einsteinschen Gleichungen mit Lorentz-Minkowskischer Signatur Lösungen angebbar sind, die nicht überall diese Signatur besitzen: Wir brauchen hierzu ja nur Koordinatentransformationen auszuführen, die in dem Sinn irregulär sind, daß an $S$ die Jacobische Determinante von der Ordnung $\frac{n}{2}$ verschwindet und im Bereich $x^1<0$ (d.h. innerhalb

(3) Eine ausführliche Diskussion enthält die in den *Annalen der Physik* erscheinende Arbeit "Gravitationsfelder mit Nullstellen der Determinante der $g_{\mu\nu}$".

von $S$) die Transformation komplex wird, während sie für $x^1>0$ (außerhalb von $S$) überall regulär und reell sein möge.

Um zu einem wohl definierten Problem zu gelangen, müssen wir also weitere Forderungen an die Metrik stellen, die nicht alle gleichzeitig durch eine irreguläre Transformation von beliebigen $g_{\mu\nu}$ der Minkowskischen Signatur befriedigt werden können. Wir fordern, daß die $g_{\mu\nu}$ im ganzen $V_4$ beschränkt sind und weiter, daß es einen dreidimensionalen Unterraum gibt, der den gewöhnliche Raum $V_3$ darstellt und eine negativ-definite Metrik $g_{ik}(i,k=1,2,3)$ besitzt, deren Determinante $|g_{ik}|$ nirgends verschwindet. Unter diesen Voraussetzungen betrachten wir nun insbesondere einen Raum mit einem Killing-Vektor $\xi^\nu$, der im Bereich der Minkowskischen Signatur zeitartig sein soll und verlangen, daß der Raum statisch ist und global $\xi^\nu=\delta^\nu_0$, gesetzt werden kann. Da die Hyperflächenschar $S^*$ im Bereich der Minkowskischen Signatur zeitartig sein soll, können wir die Hyperfläche $S$ wiederum durch die Gleichung $x^1=0$ vorgeben und in der Umgebung von $S$ ein fast-Gaussches Koordinatensystem einführen, in dem die $g_{\mu\nu}$ die Form

$$g_{\mu\nu}=\begin{pmatrix} \sum_k \underset{k}{\overline{a}}(x^2,x^3)\,(x^1)^l & 0\;0\;0 \\ 0 & \\ 0 & g_{ik}(x^i) \\ 0 & \end{pmatrix} \tag{5}$$

haben.

In (5) setzen wir voraus, daß die Matrix der $g_{ik}$ negativ definit ist. Für $g_{00}$ nehmen wir ganz allgemein die Entwicklung

$$g_{00}=\underset{n}{\overline{a}}(x^2,x^3)\,(x^1)^n+\ldots \qquad n\geqslant 1 \tag{6}$$

an, so daß für $g$ die Entwicklung

$$g=\underset{n}{\alpha}(x^2,x^3)\,(x^1)^n+\ldots \qquad n\geqslant 1 \tag{7}$$

gilt. Die kontravariante $g^{ik}$ sind, wie die $g_{ik}$ selbst überall regulär und es gilt ferner

$$g^{i0}=g_{i0}=0,$$

während $g_{00}$ für $x^1=0$ einen Pol $n$-ter Ordnung besitzt.

Geometrisch haben wir somit folgende Situation: Die Metrik besitzt ein globales Killing-Feld, das für $x^1>0$ zeitartig ist. An der Hyperfläche $S$ wird der Killing-Vektor zu einem Null-Vektor und die Signatur an $S$ ist: $(-1,-1,-1,0)$. Hat $g$ an $S$ eine Nullstelle von ungerader Ordnung, so ist der Killing-Vektor für $x^1<0$ raumartig. Für $x^1>0$ gibt es an jedem Punkt vier unabhängige Nullrichtungen, für $x^1=0$ nur eine, die Richtung des Killing-Vektors.

Geht man nun mit diesem Ansatz in die Einsteinschen Vakuumgleichungen ein und betrachtet diese in der Umgebung von $x^1=0$, indem man alle

$g_{\mu\nu}$ nach steigenden Potenzen von $x^1$ entwickelt, so kann man bereits aus diesen Voraussetzungen sukzessiv die ersten Koeffizienten der Entwicklung weitgehend bestimmen. Man findet, daß die Einsteinschen Gleichungen nur $n=2$ erlauben; ferner liefern sie Bedingungen über den Zusammenhang der Entwicklungskoeffizienten von $g_{00}$ mit denen für die $g_{ik}$.

Die Lösung ist in der Nähe von $x^1=0$ symmetrisch in Bezug auf $x^1$, so daß folgende Interpretation möglich und physikalisch notwendig ist:

Die Koordinate $x^1$ ist eine Art Radialkoordinate und die Hyperfläche $x^1=0$ ist zu einer Weltlinie, dem Ursprung des räumlichen Koordinatensystems, degeneriert. Es gibt hier kein Teilcheninneres, wir haben vielmehr ein Punktteilchen vor uns, dessen Weltlinie dadurch ausgezeichnet ist, daß auf ihr die Determinante $g$ in der 2. Ordnung verschwindet.

Ein in der Literatur bekanntes Beispiel für einen derartigen Lösungstyp ist das Schwarzschildsche Feld in der "Brückendarstellung" von Einstein und Rosen.(4) Diese "Brückenlösung" ist lokal mit der gewöhnlichen Schwarzschildschen Metrik identisch, besitzt aber eine andere Topologie, bei der der Bereich $x^1>0$ mit dem Bereich $x^1<0$ identifiziert wird.

Einstein und Rosen haben nun bemerkt, daß eine Interpretation der „Brücken" — oder allgemeiner der Weltlinien der Pseudosingularitäten $g=0$ — als Teilchen notwendig zur Folge haben muß, daß für diese Weltlinien aus den Feldgleichungen Bewegungsgleichungen herleitbar sein müssen. D.h. es dürfen nicht beliebige Anordnungen von Weltlinien mit $g=0$ mit den Feldgleichungen $R_{\mu\nu}=0$ verträglich sein. Da das Programm der Herleitung von Bewegungsgleichungen für die Pseudosingularitäten auf Schwierigkeiten führte, stellten Einstein, Infeld und Hoffmann die Teilchen später durch echte deltaartige Singularitäten dar, wobei dann an den Weltlinien der Teilchen die Vakuumfeldgleichungen nicht erfüllt sind.

Unsere weiteren Rechnungen wiesen jedoch darauf hin, daß es in der Tat möglich sein sollte, Bewegungsgleichungen für die Pseudosingularitäten $g=0$ zu erhalten. Man kann nämlich einsehen, daß es keine statischen Lösungen von (1) gibt, bei denen zwei Weltlinien mit den oben beschriebenen Nullstellen von $g$ existieren. Aus der Einstein-Rosen Metrik wissen wir nun, daß die durch $g=0$ beschriebenen Teilchen Monopole sind. Somit ist unseres Ergebnis das physikalisch notwendig zu fordernde und ermutigt nach den expliziten Bewegungsgleichungen für die Pseudosingularitäten zu suchen.

In der hier entwickelten Vorstellung werden also Massenpole, die sonst durch deltaartige Singularitäten wiedergegeben werden, durch die Pseudosingularitäten $g=0$ dargestellt in Übereinstimmung mit den Ideen Einsteins, daß in einer strengen Theorie die Singularitäten durch extrem starke Gravitationsfelder zu ersetzen sind. In der Tat ist in der Umgebung von Null-

(4) Siehe [1] [2] [3].

stellen von $g$ (und somit von $g_{00}$) das Feld extrem nichtgalileisch. Es sei darauf hingewiesen, daß an den Stellen $g=0$ die Metrik nicht einmal im Infinitesimalen durch eine reguläre Transformation auf die Minkowskische Form gebracht werden kann. Die Aussage, das Feld ist an einer Weltlinie mit $g=0$ extrem stark, besitzt hier also einen vollständig invarianten Sinn.(5)

## LITERATUR

[1] A. Einstein und N. Rosen, *Phys. Rev.* **48**, 73 (1935).
[2] A. Einstein und N. Rosen, *Phys. Rev.* **49**, 404 (1936).
[3] A. Einstein, *J. Franklin Inst.* **221**, 313 (1936).

(5) Den ersten Teil einer Untersuchung der Pseudosingularitäten enthält die in Anm.(3), Seite 305, zitierte Arbeit. Eine weitere Arbeit ist in Vorbereitung.

# THE TYPE OF SPACE—THE TYPE OF ENERGY-MOMENTUM TENSOR IN GENERAL THEORY OF RELATIVITY

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A PROBLEM which naturally arises in the modern theory of relativity is formulated in the present report. For Einstein's field equations $R_{\alpha\beta}-\frac{R}{2}g_{\alpha\beta} = T_{\alpha\beta}$ in the case when the energy-momentum tensor (EMT) describes the complex of physical phenomena and there are different fields (for instance, the electromagnetic one), a complete description of matter-energy can be reached if the following features are known: (1) eigenvalues, (2) eigenvectors, (3) algebraical structure of $T_{\alpha\beta}$. The structure is determined by the characteristics of the pair of tensors $(T_{ij}-\lambda g_{ij})$. Since the metric is not definite, non-simple an deven complex elementary divisors of the pair of tensors, as well as null and perhaps complex eigenvectors, are in principle possible. One can show that only three types of EMT can occur: [1111], [211], [31]. Only the cases corresponding to the first type have been studied in the literature so far (while in the case of the electromagnetic field the second type appears, for instance, pure radiation).

On the other hand, as has been shown by the author, three and only three types of gravitational fields exist in the general case if the algebraic structure of the space-matter tensor (which coincides in a special case with Weyl's tensor of conformal curvature) is taken as the basis of such a classification. The problem consists in the investigation of the logical interrelationship of the two classifications; in particular, it is necessary to clear up the question of the consistency between different types the two classifications. This difficult problem, which can be simply formulated and which is natural from the formal point of view is important physically, since it determinates the range of applicability of Einstein's theory.

One may think that this problem has no simple solution; however, its solution, even for simple cases, would be useful. The problem is illustrated by some special cases (the perfect fluid, pure electromagnetic radiation). Some reasons are given to clarify why this problem must play an important role for the consideration of any problem in general relativity.

# SPINORS AND BISPINORS IN RIEMANNIAN SPACE

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THE theory of spinors and bispinors in Riemannian space and curvilinear coordinates, which originated about 30 years ago [1]—[5], has assumed great importance recently in the theory of elementary particles and unified field theory [6]—[12]. The author has been occupied in the last few years with a fully covariant formulation of these theories and with the relation of one to the other [13]. We shall present here a condensed outline of the main features of this subject from a unified point of view. Greek indices (tensor indices) run from 1 to 4, while capitalized Latin indices (spinor indices) run from 1 to 2. The signature of the metric is taken to be $(+, +, +, -)$, and partial derivatives are symbolized by commas.

## 1. THEORY OF SPINORS

Spinor indices are raised and lowered by means of the metric spinor $\gamma^{AB} = -\gamma^{BA}$:

$$a_A = \gamma_{BA} a^B, \quad a^A = \gamma^{AB} a_B, \tag{1}$$

where

$$\gamma_{AB}\gamma^{CB} = \gamma_A{}^C = \delta_A{}^C. \tag{2}$$

Coordinate transformations of tensors may be described by the formulae

$$a^{\alpha'} = A^{\alpha'}_{\alpha} a^{\alpha}, \quad a_{\alpha'} = A^{\alpha}_{\alpha'} a_{\alpha}, \tag{3}$$

where $A^{\alpha'}_{\alpha} = \dfrac{\partial x^{\alpha'}}{\partial x^{\alpha}}$, and spinor transformations which are induced by continuous coordinate transformations by the formulae

$$a^{A'} = \Lambda^{A'}_{A} a^{A}, \quad a_{A'} = \Lambda^{A}_{A'} a_{A}. \tag{4}$$

In generalization of Harish Chandra's considerations [14] for Minkowski space, the metric spintensor $\sigma_{\mu\dot{A}B}$ which is Hermitean,

$$\sigma_{\mu\dot{A}B} = \sigma_{\mu B\dot{A}}, \tag{5}$$

is defined by the relation

$$\sigma_{\mu}{}^{\dot{B}}{}_{A}\sigma_{\nu\dot{B}C} = g_{\mu\nu}\gamma_{AC} \pm \frac{i}{2}\varepsilon_{\mu\nu\varrho\tau}\sigma^{\varrho\dot{B}}{}_{A}\sigma^{\tau}{}_{\dot{B}C} \tag{6}$$

where

$$\varepsilon_{\mu\nu\varrho\tau} = \sqrt{g}\, \delta^{\mu\nu\varrho\tau} \tag{7}$$

is the Levi-Cività pseudotensor and $g = -|g_{\mu\nu}|$. All further formulae for higher $\sigma$-products may be deduced from (6).

The metric spintensor connects the tensorial basis vectors $\mathfrak{n}^{\varkappa}$ with the spinorial basis vectors $\mathfrak{n}_{\dot{A}B}$ according to

$$\mathfrak{n}_{\dot{A}B} = -\sigma_{\varkappa\dot{A}B}\mathfrak{n}^{\varkappa} \quad \text{or} \quad \sigma_{\varkappa\dot{A}B} = -\mathfrak{n}_{\dot{A}B}\mathfrak{n}_{\varkappa}. \tag{8}$$

So Bergmann's representation vectors get an obvious geometrical interpretation. The covariant derivative of spinors is defined by means of the spinor affinity $\Gamma^{B}_{A\lambda}$ in the form

$$a_{A;\lambda} = a_{A,\lambda} - \Gamma^{B}_{A\lambda}a_{B}, \qquad a^{A}{}_{;\lambda} = a^{A}{}_{,\lambda} + \Gamma^{A}_{B\lambda}a^{B}. \tag{9}$$

For the metric spintensor we postulate

$$\sigma_{\varkappa\dot{A}B;\varrho} = 0; \tag{10}$$

while for the metric spinor we put

$$\gamma_{AB;\varkappa} = -i\varphi_{\varkappa}\gamma_{AB} \qquad (\varphi_{\varkappa} \text{ real}) \tag{11}$$

so that

$$\gamma_{AB,\varkappa} = \gamma_{CB}\Gamma^{C}_{A\varkappa} + \gamma_{AC}\Gamma^{C}_{B\varkappa} - i\varphi_{\varkappa}\gamma_{AB}. \tag{12}$$

Splitting up the spinor affinity according to the formula

$$\Gamma^{D}_{A\lambda} = [{}_{A}{}^{D}{}_{\lambda}] + \frac{i}{2}\gamma_{A}{}^{D}\varphi_{\lambda} \tag{13}$$

gives

$$\gamma_{AB,\varkappa} = [A\varkappa,B] - [B\varkappa,A]. \tag{14}$$

Because of $\mathfrak{n}_{\dot{A}B;\lambda} = 0$, it follows that

$$[{}_{A}{}^{D}{}_{\lambda}] = -\frac{1}{4}\mathfrak{n}_{\dot{B}A,\lambda}\mathfrak{n}^{\dot{B}D} - \frac{1}{2}\gamma_{A}{}^{D}[{}_{\dot{B}}{}^{\dot{B}}{}_{\lambda}]; \tag{15}$$

and from this

$$\Gamma_{A}{}^{A}{}_{\lambda} + \Gamma_{\dot{A}}{}^{\dot{A}}{}_{\lambda} = [{}_{A}{}^{A}{}_{\lambda}] + [{}_{\dot{A}}{}^{\dot{A}}{}_{\lambda}] = -\frac{1}{4}\mathfrak{n}_{\dot{B}A,\lambda}\mathfrak{n}^{\dot{B}A} = 2\Gamma_{,\lambda} \tag{16}$$

where $\Gamma = \frac{1}{2}\ln(\gamma_{\dot{1}\dot{2}}\gamma_{12})$. By means of the decomposition

$$[{}_{A}{}^{A}{}_{\lambda}] = \Gamma_{,\lambda} + i\Pi_{\lambda} \qquad (\Pi_{\lambda} \text{ real}^{(1)}), \tag{17}$$

and the abbreviation

$$\{{}_{A}{}^{D}{}_{\lambda}\} = -\frac{1}{4}\mathfrak{n}_{\dot{B}A,\lambda}\mathfrak{n}^{\dot{B}D} \tag{18}$$

the gravitational spinor affinity (13) can be written in the form

$$\Gamma^{D}_{A\lambda} = \{{}_{A}{}^{D}{}_{\lambda}\} + \frac{1}{2}i\gamma_{A}{}^{D}\Omega_{\lambda}, \tag{19}$$

---

(1) $\pi_{\lambda} = \frac{1}{2}\theta_{,\lambda}$, where $\theta$ is given by $\gamma_{12} = \sqrt{\gamma}\, e^{i\theta/2}$.

with $\Omega_\lambda = \varphi_\lambda + \Pi_\lambda + i\Gamma_{,\lambda}$. For $\{_A{}^D{}_\varkappa\}$ one finds

$$\{_A{}^D{}_\varkappa\} = \frac{1}{2}\{_\varrho{}^\lambda{}_\varkappa\}\sigma_{\lambda\dot{B}A}\sigma^{\varrho\dot{B}D} - \frac{1}{4}\sigma^{\varrho\dot{B}D}\sigma_{\varrho\dot{B}A,\varkappa}. \tag{20}$$

So the connection of the spinor affinity with the Christoffel symbols is given. From (20) there results the relation

$$\{_A{}^A{}_\varkappa\} = -\frac{1}{2}\{_\lambda{}^\lambda{}_\varkappa\} - \frac{1}{4}\sigma^{\varrho\dot{B}D}\sigma_{\varrho\dot{B}D,\varkappa}, \tag{21}$$

from which the reality of

$$\{_{\dot{A}}{}^{\dot{A}}{}_\lambda\} = \{_A{}^A{}_\lambda\} = 2\Gamma_{,\lambda} \tag{22}$$

can be seen once more. Formula (15) now may be written in the form

$$[_A{}^D{}_\lambda] = \{_A{}^D{}_\lambda\} - \frac{1}{2}\gamma_A{}^D(\Gamma_{,\lambda} - i\Pi_\lambda). \tag{23}$$

Then the covariant derivative takes the form

$$a^A{}_{;\lambda} = a^A{}_{,\lambda} + \frac{1}{2}i\Omega_\lambda a^A + \{_B{}^A{}_\lambda\}a^B. \tag{24}$$

In Galilean coordinates in Minkowski space, with the choice $\gamma_{AB}$, $\sigma_{\mu\dot{A}B}$, $\pi_\mu$, $\pi_{\dot{A}B} =$ const., we get from (24):

$$a^A{}_{;\lambda} = a^A{}_{,\lambda} + \frac{1}{2}i\varphi_\lambda a^A. \tag{25}$$

Comparison with the well-known gauge derivative gives the connection with the electromagnetic potential

$$A_\lambda = -\frac{\hbar c}{2e}\varphi_\lambda. \tag{26}$$

While $\varphi_\lambda$ is a tensor, $\Gamma_{,\lambda}$ and $\Pi_\lambda$ transform as follows:

$$\Gamma'_{,\lambda'} = \Gamma_{,\lambda}A^\lambda_{\lambda'} + \frac{1}{2}(\Lambda^A_{A',\lambda'}\Lambda^{A'}_A + \Lambda^{\dot{A}}_{\dot{A}',\lambda'}\Lambda^{\dot{A}'}_{\dot{A}}), \tag{27}$$

$$\Pi_{\lambda'} = \Pi_\lambda A^\lambda_{\lambda'} - \frac{1}{2}i(\Lambda^A_{A',\lambda'}\Lambda^{A'}_A - \Lambda^{\dot{A}}_{\dot{A}',\lambda'}\Lambda^{\dot{A}'}_{\dot{A}}). \tag{28}$$

On the other hand, from the definition of $\Gamma$ we get

$$\Gamma'_{,\lambda'} = \Gamma_{,\lambda}A^\lambda_{\lambda'} - \frac{1}{2}[\ln(\Lambda^*\Lambda)]_{,\lambda'} \quad (\Lambda = |\Lambda^{A'}_B|), \tag{29}$$

so that one finds, by comparison of (27) and (29),

$$d\Lambda^A_{A'}\Lambda^{A'}_A + d\Lambda^{\dot{A}}_{\dot{A}'}\Lambda^{\dot{A}'}_{\dot{A}} = -d[\ln(\Lambda^*\Lambda)]. \tag{30}$$

By use of the relation

$$a_{A;\nu;\varkappa} - a_{A;\varkappa;\nu} = a_{A,\nu,\varkappa} - a_{A,\varkappa,\nu} + a_B P^B{}_{A\nu\varkappa}, \tag{31}$$

the curvature spintensor $P^B{}_{A\nu\varkappa}$ may be put in the form

$$P^B{}_{A\nu\varkappa} = \Gamma^B_{A\varkappa,\nu} - \Gamma^B_{A\nu,\varkappa} + \Gamma^B_{C\nu}\Gamma^C_{A\varkappa} - \Gamma^B_{C\varkappa}\Gamma^C_{A\nu}. \tag{32}$$

The gravitational curvature spintensor

$$O^B{}_{A\nu\varkappa} = \{{}_A{}^B{}_\varkappa\}_{,\nu} - \{{}_A{}^B{}_\nu\}_{,\varkappa} + \{{}_C{}^B{}_\nu\}\{{}_A{}^C{}_\varkappa\} - \{{}_C{}^B{}_\varkappa\}\{{}_A{}^C{}_\nu\} \tag{33}$$

is connected with $P^B_{A\nu\varkappa}$ by the equation

$$P^B{}_{A\nu\varkappa} = O^B{}_{A\nu\varkappa} + \frac{1}{2} i\gamma_A{}^B \Omega_{\nu\varkappa}, \tag{34}$$

where

$$\Omega_{\nu\varkappa} = \Omega_{\varkappa,\nu} - \Omega_{\nu,\varkappa}.$$

From (33) and (34) it follows that

$$P^A{}_{A\nu\varkappa} = i\Omega_{\nu\varkappa}, \quad O^A{}_{A\nu\varkappa} = 0. \tag{35}$$

The Riemannian curvature tensor is connected with the gravitational curvature spintensor by

$$O^B{}_{M\nu\varkappa} = \frac{1}{4} R_{\alpha\mu\nu\varkappa}\sigma^\alpha{}_{\dot{A}M}\sigma^{\mu\dot{A}B}, \tag{36}$$

and

$$R^\beta{}_{\mu\nu\varkappa} = \frac{1}{2} O^{\dot{A}}{}_{\dot{N}\nu\varkappa}\sigma_{\mu\dot{A}M}\sigma^{\beta\dot{N}M} + \frac{1}{2} O^A{}_{M\nu\varkappa}\sigma_{\mu\dot{N}A}\sigma^{\beta\dot{N}M}. \tag{37}$$

Finally, the cyclic relation

$$[O^D_{B\nu\sigma;\tau}]_{<\nu\sigma\tau>} = 0 \tag{38}$$

is worth noting.

## 2. THEORY OF BISPINORS

The metric bispintensors (generalized Dirac matrices) are defined through

$$\gamma_\mu\gamma_\nu = g_{\mu\nu} + \frac{1}{2} i\varepsilon_{\mu\nu}{}^{\varkappa\lambda}\gamma_\varkappa\gamma_\lambda\gamma_5, \tag{39}$$

where

$$\gamma_5 = \frac{1}{4!\,i}\varepsilon^{\nu\mu\varkappa\lambda}\gamma_\nu\gamma_\mu\gamma_\varkappa\gamma_\lambda. \tag{40}$$

The definition of $\gamma_5$ is chosen in such a way that

$$(\gamma_5)^2 = 1. \tag{41}$$

In addition

$$\gamma_5\gamma_\tau = -\gamma_\tau\gamma_5. \tag{42}$$

All the other formulae of $\gamma$-algebra can be deduced from (39). The covariant derivative of the bispinor $\Psi$, which transforms according to

$$\Psi' = S\Psi, \tag{43}$$

is defined by

$$\Psi_{;\nu} = \Psi_{,\nu} + \Gamma_\nu \Psi. \tag{44}$$

So for the bispinor affinity the transformation law

$$\Gamma'_{\nu'} = A^\nu_{\nu'}(S\Gamma_\nu S^{-1} - S_{,\nu}S^{-1}) \tag{45}$$

results, while the $\gamma_\nu$ are transformed according to

$$\gamma'_{\nu'} = A^\nu_{\nu'} S\gamma_\nu S^{-1}. \tag{46}$$

Following Bergmann, we choose the covariant definition for the adjoint bispinor:

$$\bar{\Psi} = \Psi^+\beta, \tag{47}$$

where $\beta$ possesses the following properties:

$$\begin{gathered} \beta' = \beta, \ \ \beta^+ = \beta, \ \ \beta_{;\nu} = 0, \\ S^+\beta = \beta S^{-1}, \ \ \beta\Gamma_\nu + \Gamma_\nu^+\beta = 0, \ \ \beta\gamma^\mu + \gamma^{\mu+}\beta = 0. \end{gathered} \tag{48}$$

The explicit expression for $\beta$ is $\beta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then no Hermiticity postulate for the $\gamma_\mu$ is necessary and all difficulties connected with it vanish.

So the covariant derivative of the adjoint bispinor is given by

$$\bar{\Psi}_{;\nu} = \bar{\Psi}_{,\nu} - \bar{\Psi}\Gamma_\nu. \tag{49}$$

For the covariant derivatives of the $\gamma_\mu$ we postulate:

$$\gamma_{\mu;\nu} = \gamma_{\mu,\nu} - \left\{ {}_\mu{}^\alpha{}_\nu \right\}\gamma_\alpha + [\Gamma_\nu, \gamma_\mu] \equiv 0. \tag{50}$$

The entity

$$\Phi_{\mu\nu} = \Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} + [\Gamma_\nu \Gamma_\mu], \tag{51}$$

defined from the relation

$$\Psi_{;\mu;\nu} - \Psi_{;\nu;\mu} = \Phi_{\mu\nu}\Psi, \tag{52}$$

is connected with the curvature tensor in the following way

$$\Phi_{\nu\lambda} = \frac{1}{4} i\, R^{\alpha\beta}{}_{\lambda\nu}\gamma_{\alpha\beta} + f_{\nu\lambda}, \tag{53}$$

where

$$\gamma_{\alpha\beta} = -\frac{1}{2} i\, [\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha]. \tag{54}$$

The following relation with the electromagnetic field strength $F_{\mu\nu}$ exists

$$f_{\nu\lambda} = \frac{1}{4} \operatorname{trace} \Phi_{\nu\lambda} = -\frac{ie}{\hbar c} F_{\lambda\nu}. \tag{65}$$

## 3. CONNECTION BETWEEN THE THEORIES OF SPINORS AND BISPINORS

Coincidence of the spinor and the bispinor apparatus can be obtained in the following way: From (6) and (39) there results the preferred representation (we call it "standard representation")

$$\gamma^{\mu} = -i\begin{pmatrix} 0 & \sigma^{\mu A\dot{B}} \\ \sigma^{\mu}{}_{\dot{A}B} & 0 \end{pmatrix}, \tag{56}$$

and from this formula one gets

$$\gamma_5 = \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{57}$$

The bispinor is split up into spinors:

$$\Psi = \begin{pmatrix} \chi^{A} \\ \Psi_{\dot{A}} \end{pmatrix}. \tag{58}$$

For the transformation matrix $S$ this give

$$S = \begin{pmatrix} \Lambda^{A'}_{B} & 0 \\ 0 & \Lambda^{\dot{B}}_{\dot{A}'} \end{pmatrix}, \tag{59}$$

while the bispinor affinity takes the form

$$\Gamma_{\mu} = \begin{pmatrix} \Gamma^{A}_{B\mu} & 0 \\ 0 & -\Gamma^{\dot{B}}_{\dot{A}\mu} \end{pmatrix}, \tag{60}$$

and

$$\Phi_{\nu\lambda} = \begin{pmatrix} P^{A}{}_{B\lambda\nu} & 0 \\ 0 & -P^{\dot{B}}{}_{\dot{A}\lambda\nu} \end{pmatrix}. \tag{61}$$

Thus the bispinor affinities are expressed through the Christoffel symbols and the metric spintensors with the help of (19) and (20).

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# CONSERVATION LAWS AND FLAT-SPACE METRIC IN GENERAL RELATIVITY

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**Abstract**—The formalism involving two metric tensors, one corresponding to a Riemannian space, the other to a flat space, is reviewed. It is pointed out that one can define an energy-momentum density tensor for the true gravitational field, as distinguished from the inertial field.

It was proposed some time ago [1] [2] to introduce within the framework of the general theory of relativity, along with the usual metric tensor $g_{\mu\nu}$, second metric tensor $\gamma_{\mu\nu}$ corresponding to flat space. This can be done without attributing any special properties to the space-time and leads to improvements in the formalism. It is possible [3] to reinterpret the formalism so that $\gamma_{\mu\nu}$ can be considered as describing the geometry of space-time while $g_{\mu}$ plays the role of a gravitional potential, but one does not have to do this.

If one has the two metric tensors, one can define two kinds of covariant derivatives, a $g$-derivative (denoted by a semicolon) based on $g_{\mu\nu}$, and a $\gamma$-derivative (denoted by a comma) based on $\gamma_{\mu\nu}$. If we denote the Christoffel 3-index symbols formed from $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ by $\{{}^{\lambda}_{\mu\nu}\}$ and $\Gamma^{\lambda}_{\mu\nu}$ respectively, we find that

$$\{{}^{\lambda}_{\mu\nu}\} = \Delta^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\mu\nu}, \tag{1}$$

where $\Delta^{\lambda}_{\mu\nu}$ is a tensor and has the same form as $\{{}_{\mu}{}^{\lambda}{}_{\nu}\}$ except for the fact that ordinary partial derivatives have been replaced by $\gamma$-derivatives. One also finds that the Riemann–Christoffel tensor can be written in the form

$$R^{\lambda}{}_{\mu\nu\sigma} = -\Delta^{\lambda}_{\mu\nu,\sigma} + \Delta^{\lambda}_{\mu\sigma,\nu} + \Delta^{\lambda}_{\beta\nu}\Delta^{\beta}_{\mu\sigma} - \Delta^{\lambda}_{\beta\sigma}\Delta^{\beta}_{\mu\nu}, \tag{2}$$

and hence the Ricci tensor in the form

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\mu\alpha} = -\Delta^{\alpha}_{\mu\nu,\alpha} + \Delta^{\alpha}_{\alpha\mu,\nu} - \Delta^{\alpha}_{\alpha\beta}\Delta^{\beta}_{\mu\nu} + \Delta^{\alpha}_{\beta\mu}\Delta^{\beta}_{\alpha\nu}. \tag{3}$$

Comparing these expressions with the usual forms, one sees that $\{{}_{\mu}{}^{\lambda}{}_{\nu}\}$ has been replaced by $\Delta^{\lambda}_{\mu\nu}$ and ordinary differentiation by $\gamma$-differentiation.

One finds that, in general, one can rewrite the equations of general relativity theory so that $\{{}_{\mu}{}^{\lambda}{}_{\nu}\}$ is replaced by $\Delta^{\lambda}_{\mu\nu}$, ordinary derivatives by $\gamma$-derivatives, $(-g)^{\frac{1}{2}}$ by $\varkappa = (g/\gamma)^{\frac{1}{2}}$ where $g$ and $\gamma$ are determinants of the metric tensors and, in integrations, $d\tau$ by $(-\gamma)^{\frac{1}{2}}d\tau$ where $d\tau = dx^1dx^2dx^3dx^4$. One can go from this modified formalism to the usual one by taking $\gamma_{\mu\nu} = \eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is the metric tensor of special relativity theory.

For example, the equation of motion of a test-particle, i.e. the equation of the geodesic, can be written

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}+\Delta^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0. \tag{4}$$

The field equations for a given energy-momentum density tensor $T^{\mu\nu}$ can be written, as usual,

$$G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-8\pi T_{\mu\nu}, \tag{5}$$

where $R_{\mu\nu}$ is given by Eq. (3). The left-hand member of Eq. (5) can be derived from a variational principle

$$\delta\int\bar{L}(-\gamma)^{\frac{1}{2}}\,d\tau=\int\varkappa G^{\mu\nu}\delta g_{\mu\nu}(-\gamma)^{\frac{1}{2}}\,d\tau, \tag{6}$$

where

$$\bar{L}=\varkappa g^{\mu\nu}(\Delta^\alpha_{\beta\mu}\Delta^\beta_{\alpha\nu}-\Delta^\alpha_{\alpha\beta}\Delta^\beta_{\mu\nu}). \tag{7}$$

Since $\bar{L}$ is a scalar, one can derive an energy-momentum density tensor for the gravitational field, analogous to the Einstein pseudo-tensor. However, this tensor is not symmetric and therefore does not lead to a conservation law for angular momentum.

To get a symmetric energy-momentum density tensor one can follow the method of Landau and Lifshitz [4]. Let us first define the tensors

$$\bar{g}^{\mu\nu}=\varkappa g^{\mu\nu},\qquad \bar{g}_{\mu\nu}=\frac{1}{\varkappa}g_{\mu\nu}, \tag{8}$$

and

$$\tilde{T}^{\mu\nu}=\varkappa^2T^{\mu\nu}. \tag{9}$$

Now we define a tensor $\tilde{t}^{\mu\nu}$ by means of the relation

$$\tilde{T}^{\mu\nu}+\tilde{t}^{\mu\nu}=k^{\mu\nu\lambda}{}_{,\lambda}, \tag{10}$$

where

$$k^{\mu\nu\lambda}=-k^{\mu\lambda\nu}=\frac{1}{16\pi}(\bar{g}^{\mu\nu}\bar{g}^{\lambda\tau}-\bar{g}^{\mu\lambda}\bar{g}^{\nu\tau})_{,\tau}. \tag{11}$$

It then follows that

$$(\tilde{T}^{\mu\nu}+\tilde{t}^{\mu\nu})_{,\nu}=0. \tag{12}$$

The explicit form of $\tilde{t}^{\mu\nu}$ is given by

$$\begin{aligned}16\pi\tilde{t}^{\mu\nu}=\;&\bar{g}^{\mu\alpha}{}_{,\beta}\bar{g}^{\nu\underline{\alpha}}{}_{,\underline{\beta}}+\frac{1}{2}\bar{g}^{\alpha\beta}{}_{,\underline{\mu}}\bar{g}^{\underline{\alpha\beta}}{}_{,\underline{\nu}}-\bar{g}^{\alpha\beta}{}_{,\underline{\mu}}\bar{g}^{\nu\underline{\beta}}{}_{,\alpha}\\&-\bar{g}^{\alpha\beta}{}_{,\underline{\nu}}\bar{g}^{\mu\underline{\beta}}{}_{,\alpha}-\bar{g}^{\mu\alpha}{}_{,\alpha}\bar{g}^{\nu\beta}{}_{,\beta}-\varkappa_{,\underline{\mu}}\varkappa_{,\underline{\nu}}\cdot\frac{1}{\varkappa^2}\\&+\bar{g}^{\mu\nu}{}_{,\alpha}\bar{g}^{\alpha\beta}{}_{,\beta}+\bar{g}^{\mu\nu}\bar{L},\end{aligned} \tag{13}$$

where underlined indices are to be raised or lowered with $\bar{g}^{\mu\nu}$ or $\bar{g}_{\mu\nu}$ after differentiation.

While other expressions are possible for the energy-momentum density of the gravitational field, the tensor $\tilde{t}^{\mu\nu}$ appears to be the most satisfactory because it contains no derivatives of $g_{\mu\nu}$ higher than the first and because it is symmetric. It should be pointed out that Eq. (12) contains, in general, a $\gamma$-divergence, and not an ordinary divergence so that it does not lead directly to an integral form of conservation law. However, if one goes over to a Galilean coordinate system (which is always possible in principle) one obtains an ordinary divergence.

In order to get a completely determined set of variables (except for the possibility of an arbitrary coordinate transformation) it is necessary to impose four relations between the $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ in addition to the general requirement that for an insular physical system we should have $g_{\mu\nu} = \gamma_{\mu\nu}$ at infinity. It is desirable to take these relations in the form of four covariant equations. One reasonable possibility is to take

$$\bar{g}^{\mu a}{}_{,a} = 0. \tag{14}$$

This resembles the De Donder, or harmonicity, condition, but since this is a covariant set of equations it does not fix the coordinate system.

It should be pointed out that the introduction of the flat-space metric enables one to separate the true gravitational field (i.e. that arising from the presence of masses) from the inertial field. This can be seen, for example, in Eq. (1) where, on the right-hand side, $\Delta^{\lambda}_{\mu\nu}$ describes the true gravitational field since it is a tensor, while $\Gamma^{\lambda}_{\mu\nu}$ describes the inertial field since it can be made to vanish in an appropriate coordinate system. Similarly, the energy-momentum density tensor $\tilde{t}^{\mu\nu}$ refers to the true gravitational field, while the effect of the inertial field appears in the presence of the $\gamma$-divergence in Eq. (12).

Møller [5] has shown that if one looks for an energy-momentum density complex for the gravitational field depending only on the tensor $g_{\mu\nu}$, it is impossible to satisfy all the conditions which it is reasonable to impose in order that one obtain a localization of energy and momentum. He has been led, therefore, to introduce tetrads to describe the field. However, it should be pointed out that the tetrad formalism, among other things, brings in a flat-space metric in addition to the Riemannian metric. It would appear that it is enough to introduce the flat-space metric without the additional complexity of the tetrads, if one wants a theory only of the gravitational field, and not a unified field theory.

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# ON LOCALIZABILITY OF GRAVITATIONAL ENERGY

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THE localizability problem for gravitational energy-momentum has been widely discussed since the very beginnings of the general theory of relativity. This problem is closely related to the material aspect of the gravitational field—i.e., to the possibility of energy transport by gravitational waves or gravitons and the possibility of transmutation of gravitons into other known particles.

The various proposed conserved quantities suffered from several ambiguities in their transformation properties and from assymmetry between the "matter" and gravitation parts. Møller has proved that it is insufficient to use only the 10 components of the metric $g_{\mu\nu}$ for constructing a complex with satisfactory transformation properties. It is, therefore, clear that the introduction of a new set of functions (i.e., of a new field!) seems necessary for obtaining a localized formulation of gravitational energy.

There are in fact various ways to satisfy the requirements of Møller. Besides Møller's proposal to introduce 16 components of a tetrad field, i.e. 6 new functions, the present author has revived the old idea of N. Rosen and used the bi-metrical formalism. In this case 4 general transforms $x'^{\mu} = f^{\mu}(x^{\nu})$ leading to the centre of mass system which put the 2nd metric $e_{\mu\nu}$ into the form diag (1, −1, −1, −1) form the set of additional functions; there are only 4 in number, and not 6 as required by Møller.

It is shown that the deduction of the conserved quantities by means of Noether's theorem is also possible in the bi-metrical case. One can get either non-tensorial (but covariant) Noether relations or tensorial ones; the two are mutually connected in a simple way. Our aim was to obtain the tensorial continuation of the pseudotensor of Einstein already given by Rosen, but to use the Noether method.

However, it is necessary either to narrow the class of transformations in the Noether theorem while still using the tensorial continuation of the Einsteinian Lagrangian of the gravitational field

$$\Lambda_{\text{cov}} = \frac{\sqrt{-g}}{2}\, g^{\mu\nu}(\Pi^{\alpha}_{\mu\nu}\Pi^{\beta}_{\alpha\beta} - \Pi^{\beta}_{\mu\alpha}\Pi^{\alpha}_{\nu\beta})$$

or, retaining the general transformations, to pass over to a new (and unnaturally complicated) gravitational Lagrangian. We prefer here the first alternative.

The restricted transformation class which may be called "the group of generalized orthogonal linear transforms" (GOLT) is formulated here in a covariant way. In the case of GOLT

$$\delta^* e_{\mu\nu} = 0,$$

which means that

$$\delta x_{\dot{\nu}|\mu} = -\delta x_{\dot{\mu}|\nu}$$

(dotted indices are lowered with use of $e_{\mu\nu}$; a bar is used for the $e$-covariant derivative), and therefore

$$\delta x_{\dot{\alpha}|\beta|\gamma} = 0.$$

These formulae justify the name of GOLT.

The problem is now to formulate an unambiguous prescription for finding the $e_{\mu\nu}$'s. Interpreting the 2nd metric field as an inertia field and trying to use Mach's principle, one could put $e_{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$ in the centre of gravity system of the entire Universe, but there is still a lot of ambiguity in this definition. In this connection we may mention some ideas recently expressed by I. Gutman which could improve the situation: his "parametrization" group. There is a pecular synthesis of locality and non-locality in this case: the determination of $e_{\mu\nu}$ is strongly non-local.

Several authors have contributed independently to this problem, in particular Ya. I. Pugachov and I. I. Gutman. We have discussed here mainly the method of derivation of this important result.

Finally we stress that the bi-metric formalism by no means goes beyond the limits of the traditional Einstein field theory only at *first* sight an absolute inertia field seem to contradict the conventional treatment.

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# A GENERALIZATION OF THE SCHWARZSCHILD METRIC*

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IT HAS been shown [1] [2] that in empty space (i.e., $R_{\alpha\beta} = 0$) the algebraic equation for the unknown vector $l_\mu$, $l_{[\mu}R_{\alpha]\beta\gamma[\delta}l_{\nu]}l^\beta l^\gamma = 0$ possesses four solutions such that $l_\mu l^\mu = 0$. Such a vector is called a principle null vector. If two or more principle null vectors coincide the Riemann tensor is said to be algebraically special. A necessary and sufficient [3] [4] condition for this to occur is that the shear $\sigma$ of $l^\mu$ vanishes $\left(\text{the magnitude of } \sigma \text{ is defined by } \sigma\bar{\sigma} = \frac{1}{2}\left[l_{(\mu;\nu)}l^{\mu;\nu} - \frac{1}{2}(l^\mu{}_{;\mu})^2\right]\right)$ and $l^\mu$ be tangent to a null geodesic.

One can give a subclassification of algebraically special metrics by stating certain differential properties of $l^\mu$. The simplest method is to state whether $l_\mu$ is proportional to a gradient field (hypersurface orthogonal) or not (in the latter case $l_\mu$ is said to be curling) and whether the divergence, $l^\mu{}_{;\mu}$, is zero or not. The case of $l_\mu$ being hypersurface orthogonal with non-vanishing divergence has been completely analyzed by Robinson and Trautman [5] [6]. The hypersurface orthogonal case with vanishing divergence has been extensively studied by W. Kundt. The remaining case, curling $l_\mu$, with non-vanishing divergence will be discussed here. (Solutions with curling $l_\mu$ but with vanishing divergence do not exist.) A subset of this class has been found, all being Petrov type I degenerate. They depend on two arbitrary parameters and one parameter which takes on the discrete values 1, 0, $-1$.

Using null coordinates (i.e., three coordinates labeling points on a null surface and one coordinate labeling the null surfaces) the metric can be put in the following form:

$$g_{\mu\nu} = \left\| \begin{matrix} U, & 1, & -AU, & -BU \\ 1, & 0, & -A, & -B \\ -AU, & -A, & A^2U-R^2, & ABU \\ -BU, & -B, & ABU, & B^2U-R^2 \end{matrix} \right\|, \tag{1}$$

$$U = \varepsilon - \varrho\bar{\varrho}(mr + 2\,\varepsilon\varrho^{02}), \quad \varrho = (r + i\varrho^0)^{-1},$$

* The work reported here is a joint effort of Dr. L. Tamburino, Mr. T. Unti and the present speaker.

$$A = \frac{\varrho^0 y}{1+\frac{\varepsilon}{4}(x^2+y^2)}, \qquad B = \frac{-\varrho^0 x}{1+\frac{\varepsilon}{4}(x^2+y^2)},$$

$$R^{-2} = \varrho\bar{\varrho}\left[1+\frac{\varepsilon}{4}(x^2+y^2)\right]^2,$$

where $m$ and $\varrho^0$ are the continuous parameters and $\varepsilon$ takes the values 1, 0, $-1$. $x^1 = u$ labels the null surfaces, $x^2 = r$ is the affine parameter along the null geodesics, and $x^3 = x$ and $x^4 = y$ are the remaining coordinates on the null surface. $x$ and $y$ are actually stereographic coordinates on the sphere $u = \text{const.}$, $r = \infty$. (One can look upon them as two angular coordinates.)

The special case $\varrho^0 = 0$, which has been investigated by Robinson and Trautman, contains the Schwarzschild metric. This is obtained by setting $\varepsilon = 1$. We, therefore, consider the case $\varrho^0 \neq 0$, $\varepsilon = 1$ as the simplest generalization of the Schwarzschild metric.

The general solution (Eq. 1) contains a four parameter group of motions. In the special case of the generalized Schwarzschild metric the time-like Killing vector is not hypersurface orthogonal, meaning that the metric is stationary rather than static. The generators of the space-like motions have the same commutator algebra as the components of angular momentum. The paths are however not closed.

If a time coordinate, and polar coordinates are introduced the metric can be put into the following form (similar to the usual Schwarzschild coordinates),

$$ds^2 = U\left(dt+4\varrho^0 \sin^2\frac{1}{2}\,\theta d\varphi\right)^2 - U^{-1}dr^2 - (d\theta^2 + \sin^2\theta d\varphi^2)(\varrho\bar{\varrho})^{-1},$$

$$U = 1-\varrho\bar{\varrho}(mr+2\varrho^{02}), \qquad \varrho = (r+i\varrho^0)^{-1}.$$

It was initially believed that this metric represented the field of a rotating point particle. It now appears as if this is not so; the symmetries being to high.

Further details including a discussion of the bound orbits will appear in a paper being prepared for the *Journal of Math. Phys.*(1)

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(1) The notation used here differs from that used in the more detailed paper to appear in the *Journal of Math. Phys.*

# ÉNERGIE GRAVITATIONELLE ET LOIS DU MOUVEMENT DANS UNE THÉORIE LINÉAIRE ET MINKOWSKIENNE DU CHAMP DE GRAVITATION

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À PARTIR de la Relativité Générale, on peut étudier, aux divers ordres d'approximation, le comportement de la partie non minkowskienne,

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu},$$

du tenseur métrique. On assimile alors $h_{\mu\nu}$ à une quantité petite devant l'unité ($h_{\mu\nu} \sim \varepsilon$), susceptible d'une description linéaire au premier ordre dans l'espace minkowskien muni de la métrique $\eta_{\mu\nu}$. On obtient ainsi une version phénoménologique qui suppose, bien entendu, la donnée de la Relativité Générale et en constitue une retranscription locale et approchée.

D'autre part, on peut se proposer le but assez différent d'édifier, dans un espace strictement minkowskien, une théorie phénoménologique mais, en principe, rigoureuse du champ de gravitation. La description des forces de gravitation forme alors le pendant d'une électrodynamique. Parmi les recherches de ce type, citons celles de Nordström [1], de W. Pauli et de M. Fierz [2], celles de l'auteur [3], enfin celles de A. Papapetrou [4], de Birkhoff [5], de Moshinsky [6] et de Belinfante [8].

La plupart de ces théories n'essaient pas de tirer les équations du champ et la définition d'une impulsion-énergie gravitationelle en utilisant une déduction lagrangienne. Les équations du mouvement résultent alors d'une loi de force posée a priori. Tel est le grave défaut de la théorie de Birkhoff qui sacrifie, selon la remarque de H. Weyl [7], la conservation de l'énergie à la conservation de la masse. Un inconvénient mineur de cette théorie réside enfin dans l'obligation de choisir une impulsion-énergie matérielle d'un type très particulier: l'obtention d'une avance correcte du périhélie des planètes exige en effet une équation d'état ($p = \mu c^2/2$) entre la pression et la densité de masse du schéma fluide parfait. Sa justification est loin d'être décisive.

C'est pour éviter ces difficultés que nous proposons ici une théorie de type maxwellien susceptible d'une déduction lagrangienne bien déterminée.

## 1. CHAMP DE GRAVITATION ET LAGRANGIEN

La métrique $g_{\mu\nu}$ est totalement dissociée du potentiel de gravitation $\psi_{\mu\nu}$. Celui-ci est une donnée phénoménologique (analogue au potentiel électromagnétique $\varphi_\mu$) réprésentée par un tenseur symétrique du second rang. Après variations — mais après variations seulement — on peut toujours réduire $g_{\mu\nu}$ à sa valeur minkowskienne $\eta_{\mu\nu}$.

Le champ de gravitation est défini (cf. [3]) par

$$\psi_{\mu\nu,\varrho} = \nabla_\mu \psi_{\nu\varrho} - \nabla_\nu \psi_{\mu\varrho}. \tag{1}$$

Les dérivées covariantes s'introduisent du fait que la métrique est quelconque, avant variations. Contrairement au champ électromagnétique rotationnel, le champ de gravitation $\psi_{\mu\nu,\varrho}$ dépend donc des dérivées $\partial_\lambda g_{\sigma\gamma}$.

Introduisons la densité lagrangienne

$$\mathcal{L} = \sqrt{-g}\, L$$

en postulant que $L$ est un scalaire, fonction quadratique du champ $\psi_{\mu\nu,\varrho}$ de sa contraction $\psi_{\mu\varrho,}{}^{\varrho}$ et aussi du champ $\partial_\mu \psi$ défini à partir du potentiel scalaire

$$\psi = g^{\mu\nu} \psi_{\mu\nu}.$$

Nous poserons ainsi

$$L = \frac{1}{4}\psi_{\mu\nu,\varrho}\psi^{\mu\nu,\varrho} + \frac{a}{2}\psi_{\mu\lambda,}{}^{\lambda}\psi^{\mu\sigma}{}_{,\sigma} + b\psi_{\mu\lambda,}{}^{\lambda}\partial^\mu\psi + \frac{c}{2}\partial_\mu\psi\partial^\mu\psi. \tag{2}$$

## 2. ÉQUATIONS DU CHAMP

Les conditions

$$\delta \int \mathcal{L} d\tau = 0$$

imposées pour des variations $\delta\psi^{\mu\nu}$, nulles à la limite du domaine d'intégration, se traduisent par les équations d'Euler

$$\sqrt{-g}\, A_{\mu\nu} \equiv -\frac{\delta\mathcal{L}}{\delta\psi^{\mu\nu}} = \partial_\varrho \frac{\partial\mathcal{L}}{\partial(\partial_\varrho\psi^{\mu\nu})} - \frac{\partial\mathcal{L}}{\partial\psi^{\mu\nu}} = 0$$

qui expriment le comportement du champ dans le vide.

En présence de matière, l'adjonction usuelle du terme

$$\chi\mathcal{M} = \chi\sqrt{-g}\, M \quad \text{avec} \quad M = \psi^{\mu\nu} M_{\mu\nu}$$

conduit aux équations du champ

$$\begin{aligned} A_{\mu\nu} \equiv \Box\, \psi_{\mu\nu} - \tfrac{1}{2}(\partial_\mu \Gamma_\nu + \partial_\nu \Gamma_\mu) - (a+b)(\partial_\mu\partial_\nu\psi + \eta_{\mu\nu}\partial_\lambda\Gamma^\lambda) + \\ + (a+2b+c)\eta_{\mu\nu}\Box\,\psi = -\chi M_{\mu\nu} \end{aligned} \tag{3}$$

en posant

$$\Gamma_\mu = \partial^\lambda \psi_{\lambda\mu}$$

et en revenant, après variations, à la métrique adaptée $\eta_{\mu\nu}$.

## 3. ÉNERGIE GRAVITATIONELLE

Par définition, l'énergie-impulsion du champ de gravitation est représentée par le tenseur du second rang toujours symétrique

$$\frac{2\sqrt{-g}}{\lambda}\Sigma_{\mu\nu} = -\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} \equiv \partial_\varrho \frac{\partial\mathcal{L}}{\partial(\partial_\varrho g^{\mu\nu})} - \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}.$$

L'avant dernier terme qui disparait en théorie de Maxwell (le champ — et par conséquent $\mathcal{L}$ — sont indépendants de $\partial_\varrho g_{\mu\nu}$) donne ici une contribution non nulle. On obtient

$$-\frac{2}{\lambda}\Sigma_{\mu\nu} = \psi_{\mu\alpha,\beta}\psi_\nu^{\alpha,\beta} + \frac{1}{2}\psi_{\alpha\beta,\mu}\psi^{\alpha\beta}{}_{,\nu}$$

$$-\frac{1}{4}\eta_{\mu\nu}\psi_{\alpha\beta,\varrho}\psi^{\alpha\beta,\varrho} - \frac{1}{2}\partial_\sigma\{\psi_{\mu\alpha}(\psi_\nu{}^{\alpha,\sigma} + \psi^{\sigma\alpha}{}_{,\nu}) +$$

$$+ \text{sym}\ \mu,\nu + \psi^{\alpha\sigma}(\psi_{\alpha\mu,\nu} + \psi_{\alpha\nu,\mu})\} + \text{termes en } (a, b, c) \qquad (4)$$

en revenant, après variations, à la métrique $\eta_{\mu\nu}$. La divergence de ce tenseur d'impulsion-énergie se met alors (quels que soient $a$, $b$, $c$) sous la forme très simple:

$$\partial^\mu \Sigma_{\mu\nu} = -\lambda[\alpha\beta, \nu]A^{\alpha\beta} - \lambda\psi_{\nu\beta}\partial_\alpha A^{\alpha\beta}. \qquad (5)$$

$A_{\alpha\beta}$ représente le premier membre — linéaire — des équations du champ. On a posé par analogie formelle:

$$[\alpha\beta, \nu] = \frac{1}{2}(\partial_\alpha\psi_{\beta\nu} + \partial_\beta\psi_{\alpha\nu} - \partial_\nu\psi_{\alpha\beta}).$$

En dehors de la présence de matière ($M_{\mu\nu} = 0$), les équations du champ dans le vide ($A_{\alpha\beta} = 0$) entraînent

$$\partial^\mu \Sigma_{\mu\nu} = 0.$$

Modulo les équations du champ, la divergence de l'impulsion-énergie gravitationnelle est alors identiquement nulle.

## 4. MOUVEMENT D'UNE PARTICULE D'ÉPREUVE

Les équations du mouvement d'une particule d'épreuve soumises au champ de gravitation $\psi_{\mu\nu,\varrho}$ s'obtiendront, comme en électrodynamique, en imposant à l'énergie totale (énergie matérielle $m_{\mu\nu} = \mu c^2 u_\mu u_\nu +$ énergie gravitationnelle $\Sigma_{\mu\nu}$) les conditions de conservation

$$\partial^\mu(m_{\mu\nu} + \Sigma_{\mu\nu}) = 0. \qquad (6)$$

L'espace minkowskien est muni de la métrique $\eta_{\mu\nu}$. Il est commode de définir d'une façon purement formelle l'expression

$$a_{\mu\nu} = \eta_{\mu\nu} - \lambda\chi\psi_{\mu\nu}$$

qui joue le rôle de "métrique associée". On posera

$$f = a_{\mu\nu}u^\mu u^\nu = 1 - \lambda\chi\psi_{\mu\nu}u^\mu u^\nu \quad \left(u^\mu = \frac{dx^\mu}{ds}\right),$$

$$v_\varrho = a_{\varrho\sigma}u^\sigma .$$

Ces définitions permettent de simplifier les formes déduites de (6). À partir de cette condition, on obtient en effet

a) Une équation de continuité

$$\partial_\varrho(\mu \overset{*}{u}{}^l) = 0. \tag{7}$$

b) Des équations dynamiques

$$\overset{*}{u}{}^\varrho \partial_\varrho \overset{*}{v}_\mu = \frac{1}{2} u^\varrho u^\sigma \partial_\mu a_{\varrho\sigma} \tag{8}$$

en posant

$$\overset{*}{u}{}^\varrho = u^\varrho \sqrt{f}, \quad \overset{*}{v}_\varrho = v_\varrho / \sqrt{f}.$$

Le mouvement d'une particule neutre reste donc indépendant de la masse de cette particule. Enfin les principes de conservation relatifs à la masse et à l'impulsion-énergie peuvent être maintenus simultanément. Tel est le but que s'était proposé H. Weyl [7] après les critiques adressées à la théorie de Birkhoff. Notons enfin que le scalaire $f = 1 - \lambda\chi\psi_{\mu\nu}u^\mu u^\nu$ $\left(1 - \frac{Gm}{c^2 r}\right.$ pour les champs statiques faibles) intervient comme un indice ($n \neq 1$) pour produire une polarisation gravitationnelle du vide (sans être aucunement lié comme dans la théorie de M. Moshinsky à l'obtention d'équations électromagnétiques adéquates).

## 5. PROLONGEMENTS ET CONCLUSIONS

Le calcul de solutions particulières (solution statique à symétrie sphérique) permet d'appliquer la théorie à la détermination des trajectoires des planètes dans le champ solaire. Les données expérimentales concernant Mercure conduisent à la prévision d'une avance séculaire correcte (cf. rapport de S. Mavridès).

Enfin cette théorie dont le développement reste tout à fait parallèle à celui de l'électrodynamique permet d'étudier très simplement les questions concernant la radiation gravitationnelle. On peut, en particulier, calculer la self réaction de radiation par un procédé tout à fait analogue à ceux que l'on utilise en électrodynamique.

Pour terminer par des considérations plus personnelles, j'ajouterai que cette théorie ne poursuit pas le but caché de se substituer à la Relativité Générale mais d'explorer, de façon plutôt heuristique, quelques domaines tout spécialement coriaces et complexes dès que l'on adopte les principes d'une théorie non euclidienne: Etude de solutions présentant des symétries spatiales particulières; quantification du champ; définition d'une énergie gravitationnelle localisable dans une variété où la notion de vecteur libre n'a plus de sens.

Bien entendu, le retour à des principes purement minkowskiens est payé de quelque rançon: le Principe d'Equivalence n'a plus le même sens dès que les actions de gravitation sont dissociées de la métrique; l'interaction lumière-gravitation doit s'expliciter comme toute autre forme d'interaction par des termes supplémentaires introduits dans le lagrangien: ainsi les 2$^e$ et 3$^e$ tests découlent encore de la théorie mais d'une façon moins immédiate que dans le cas de la Relativité Générale.

Ainsi l'introduction d'une gravitodynamique, tout en offrant des avantages considérables et très suffisants pour la justifier, permet d'apprécier l'originalité profonde, irremplaçable, de la Relativité Générale.

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# CHRONOMETRICAL INVARIANTS AND SOME APPLICATIONS OF THEM

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CHRONOMETRICAL invariants (i.e. three-dimensional tensors and other quantities and differentiation operators, invariant under transformations of the temporal coordinate) were introduced some time ago by the author. The requirement for chronometrical invariance is essential because any transformation of the time coordinate alone, as well as purely spatial transformations express no transition between different systems of reference. The use of chronometrical invariants is especially expedient in cases when either a physically privileged system of reference exists (e.g. co-moving system in cosmology) or the quantities under consideration are not covariant in respect to general transformations of coordinates (e.g. the gravitational energy-momentum pseudotensor).

In cosmology the present and other authors use chronometrical invariants as a formalism for the theory of anisotropic non-homogeneous Universe.

The criterion of chronometrical invariance was applied to a number of proposed forms of the gravitational energy-momentum pseudo-tensor (by Einstein, Landau-Lifshitz, Møller, Mitzkévič, Goldberg). Investigations performed by D. V. Belov, A. G. Malov, I. D. Novikov and R. Th. Polishchuk have shown that all these forms do not satisfy the requirement for chronometrical invariance.

Chronometrical invariants are also used at considering wave solutions of Einstein's equations. A chronometrically invariant criterion for the existence of gravitational-inertial waves as well as a general covariant criterion for gravitational waves are proposed. Recently V. D. Zakharov has applied both criteria to a number of exact wave solutions of Einstein's equations and found most of them do satisfy such criteria.

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# NEGATIVE MASS PARTICLES

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SOME physical and cosmological consequences of the introduction of negative mass particles (minus-particles) are considered.

1. It is shown that minus-particles cannot be detected by apparatus of the conventional type (cloud chamber, photoemulsion, etc.). Some instruments are, however, possible in principle for the detection of minus-particles. The detecting element of such an instrument must have a negative temperature.

2. It is shown that minus-particles can exist in a thermodynamical equilibrium state only at negative temperatures. Thus if minus-particles are present, the universe cannot be kept in a thermodynamical equilibrium state, i.e. it represents a system possessing no thermodynamical equilibrium state at all. Therefore, negative masses are inconsistent with the conventional thermodynamics.

3. According to Shirokov's paper [1] we can prove that an arbitrary weak gravitational field has negative energy density. So we can suppose that some kinds of gravitational waves transport negative energy.

Some consequences of gravitational waves carrying negative energy are considered. It is shown that such gravitational radiation can exist in a thermodynamical equilibrium state only at negative temperatures. Thus, an ordinary substance in weak interaction with gravitational radiation of such a kind has to be steadily heated.

4. The pecularities of the motion of positive and negative mass particles in the presence of a gravitational field of a large system possessing either positive or negative mass are considered. It is shown that the antigravitation of negative masses does not contradict the equivalence principle of Einstein.

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# COSMOLOGICAL CONSIDERATIONS OF THE ABSORBER THEORY OF RADIATION*

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THIS paper examines the hypotheses that the physically significant solutions of Maxwell's equations are those which exhibit perfect time-symmetry, and that radiation is not an intrinsic property of charge. It concludes that radiation and the electrodynamical arrow-of-time may be dependent upon the large-scale cosmological properties of the universe.

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# RETARDED POTENTIALS AND THE EXPANSION OF THE UNIVERSE

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As Hogarth pointed out in the previous lecture, the retarded nature of electrodynamic interactions is probably connected with the expansion of the universe. In his discussion he uses the Wheeler–Feynman theory, and arrives at the curious result that in the Einstein–de Sitter model (amongst others) advanced interactions would prevail, that is, charges would radiate in the sense of time in which the universe is contracting (whereas in the steady state model (amongst others) retarded interactions are obtained). This result leads to theoretical difficulties as well as observational ones, for there exist distributions of sources in the Einstein–de Sitter model which would give rise to an infinite radiation-density (as a result of the *blue* shift[(1)]). For this and other reasons, we have studied Maxwellian electrodynamics in expanding world-models, and have been able to derive retarded potentials for the steady state model and, under certain circumstances, for the Einstein–de Sitter model.

The basic idea is to use the Kirchhoff integral theorem, which facilitates the formulation of the correct boundary conditions. For the wave equation $\Box^2\varphi=\varrho$ in Minkowski space (we go over to cosmological space later), Kirchhoff's theorem expresses the value of $\varphi$ at the point $P$ and the time $t$ in terms of quantities defined inside and over the closed surface $S$ of a volume $V$ surrounding $P$. This expression has the form

$$\varphi(P,t) = \frac{1}{4\pi}\int_V \frac{[\varrho]}{r}\,dV + \frac{1}{4\pi}\int_S \left\{[\varphi]\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial r}{\partial n}\left[\frac{\partial\varphi}{\partial t}\right] - \frac{1}{r}\left[\frac{\partial\varphi}{\partial n}\right]\right\} dS,$$

where

$r$ is the distance from $P$ to a point of $S$,

$\frac{\partial}{\partial n}$ represents differentiation along the inward normal to $S$, and

[ ] represents retarded or advanced values.

A similar expression holds for each component of the electromagnetic vector potential, or of the electromagnetic field itself, if Cartesian coordinates are used.

(1) This is proved in a paper which is being submitted for publication.

The first point we want to stress is that by including the surface integral we have ensured that the expression for $\varphi$ is quite general, that is, no boundary conditions have yet been introduced. Moreover, this general field $\varphi$ is expressed in two *equivalent* ways, namely, in terms of retarded quantities and in terms of advanced quantities. By linear combination we can also express $\varphi$ in terms of half-retarded half-advanced quantities. To proceed further we must study in more detail the significance of the surface integral. As is well-known, it can be regarded as the contribution to $\varphi$ coming from an effective pole and dipole layer on the surface which represents the combined effects of

(i) the sources outside $V$,

(ii) any source-free radiation that may be present (called by Dirac (1938) [1] $F_{\text{in}}$ and $F_{\text{out}}$).

In conventional discussions of the retarded (Liénard–Wiechert) potentials it is assumed that all the sources are confined within a finite volume. Accordingly we can take $V$ to enclose all the sources, so that they make no contribution to the surface integral. This integral is then made to vanish in the retarded formula by adopting Sommerfeld's radiation conditions, which assert that in the distant past there was no source-free radiation entering the volume. On the other hand, the surface integral in the advanced formula does not in general vanish. In fact, we must have

$$4\pi\varphi = \int_V \text{adv} + \int_S \text{adv} = \int_V \text{ret}.$$

So the advanced surface integral is given by

$$\int_S \text{adv} = \int_V \text{ret} - \int_V \text{adv}.$$

We deviate from this conventional discussion in two respects. First of all we permit the sources to be distributed throughout space-time, as is natural in a cosmological context. Secondly, we do not assume the Sommerfeld conditions, which from our point of view beg the question. Rather we shall try to derive them from the cosmological boundary conditions. These two deviations require us to consider the behaviour of the surface integrals as $V$ tends to infinity.

For either of the surface integrals to tend to zero two conditions must be satisfied:

(i) the contribution from sources outside $V$ should tend to zero as $V$ tends to infinity (an Olbers-type condition),

(ii) there should be no source-free radiation outside $V$ in the appropriate half of $P$'s light cone (a Mach-type condition).

In order to study these conditions we must go over to expanding world-models. Fortunately Kirchhoff's theorem still holds in all Robertson-Walker models (and therefore in the Einstein–de Sitter and steady state models in particular). The reason is that all these models are conformally flat, while Maxwell's equations are conformally invariant. The only difference from the Minkowskian case is that in non-static models there will be red shift effects in the retarded formula and blue shift effects in the advanced formula (choosing the sense of time so that the universe is expanding). These shifts will play a decisive role in what follows.

We first consider the surface integrals in the Einstein–de Sitter model. As we mentioned at the outset there exist in this model distributions of sources whose total advanced fields are infinite. In reaching this result allowance has been made for the possible *coherence* of the sources. It turns out that if phase relations between the sources are produced by either retarded or advanced interactions, the resulting interference effects do not eliminate the infinity. It follows that in the advanced formulation of Kirchhoff's theorem there exist distributions of sources for which the Olbers-type condition is not satisfied. In these cases both $\int_V \text{adv}$ and $\int_S \text{adv}$ diverge as $V$ tends to infinity. However, as we shall see, their sum can be finite.

To see this, we examine the retarded formula. As is well-known, the Olbers-type condition is now satisfied. On the other hand we can say nothing about the Mach-type condition—there may be an arbitrary amount of source-free radiation present. It is true that because of the red shift the density of this radiation would diverge at $t=0$. But since the metric itself is singular then, the existence of source-free radiation cannot be ruled out. Indeed in the $\alpha$—$\beta$—$\gamma$ cosmology, such radiation plays a decisive role. What we can assert is that this source-free radiation does not cancel out $\int_V \text{ret}$, that is, we cannot have

$$\int_S \text{ret} = \int_V \text{adv} - \int_V \text{ret},$$

if we require a finite solution (since $\int_V \text{adv}$ is divergent).

We conclude that for certain source-distributions the Einstein–de Sitter model leads uniquely to retarded potentials plus an arbitrary amount of source-free radiation. For other distributions the Maxwellian theory leads to indeterminate boundary conditions. It is just in the former case (of determinate boundary conditions) that the Wheeler–Feynman theory breaks down. For in effect it starts from the mixed Kirchhoff formula

$$4\pi\varphi = \frac{1}{2}\left(\int_V \text{ret} + \int_V \text{adv}\right) + \frac{1}{2}\left(\int_S \text{ret} + \int_S \text{adv}\right),$$

and then assumes that the sum of the surface integrals vanishes in the limit as $V \to \infty$. However, in the former case $\int_S$ ret is convergent and $\int_S$ adv is divergent, so that their sum cannot vanish. If the actual universe conforms to the Einstein–de Sitter model, the galaxies constitute such a case if their brightness decreases no faster than $t^{-\frac{5}{3}}$ as $t \to \infty$(2). As regards the source-free radiation, which the Einstein–de Sitter theory leaves arbitrary, optical and radio astronomical observations indicate that there is little or none of it in the actual universe.

Finally we consider the steady state model. In the retarded Kirchhoff formula the Olbera-type condition is satisfied as before. But now the Mach-type condition is also satisfied, since any non-zero amount of source-free radiation would grow monotonically and without limit along the past light-cone, in contradiction to the assumption of uniformity in time. We thus obtain a *pure* retarded potential. The advanced Kirchhoff formula also behaves differently from the Einstein–de Sitter case for, although there is now a uniform distribution of sources up the future light-cone, the phase relations resulting from the retarded potentials prevent $\int_V$ adv from diverging, and in fact in the limit as $V \to \infty$ $\int$ adv becomes equal to $\int$ ret (since their difference is a constant source-free field, and so must vanish). Accordingly $\int_S$ adv tends to zero, and all the conditions of the Wheeler–Feynman theory are satisfied. We conclude that in the steady state model Maxwell's theory implies the Wheeler–Feynman theory, and also retarded potentials with no source-free radiation (in agreement with observation).

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# NEUTRINOS AND THE ABSORBER THEORY OF RADIATION

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THIS is a short report of the application of the absorber theory of radiation to the neutrino field. Before considering the neutrinos it is interesting to look at the case of the electromagnetic fields. This has already been described by Hogarth [1]. His analysis will be given here in a somewhat different form.

In the conventional electrodynamics only the retarded potentials are used to describe the fields of charged particles. As Maxwell's equations are time symmetric, this preference for retarded solutions only appears to be arbitrary. In their absorber theory of radiation Wheeler and Feynman start with the assumption that the intrinsic field of a particle is the time symmetric field $\frac{1}{2}(F_{\text{ret}}+F_{\text{adv}})$ instead of the usual retarded field $F_{\text{ret}}$. The observed retarded field is then explained in the following way. The observed field $F_{\text{ret}}$ travels into the future light cone of the particle and sets the particles in the universe (known collectively as the "absorber") into motion. The combined advanced field of the particles in the absorber is then shown to be equal to the field $\frac{1}{2}(F_{\text{ret}}-F_{\text{adv}})$ near the source particle. This field supplies the radiative reaction and when added to the intrinsic field of the particle it gives the total field $F_{\text{ret}}$. Thus the solution is self consistent.

However, owing to the time symmetric nature of the equations the above argument when applied to a static universe, also leads to other consistent solutions. For instance, $F_{\text{adv}}$ is also a possible solution. Indeed, any linear combination of the form $AF_{\text{ret}}+BF_{\text{adv}}$ is also a solution, where $A$, $B$ are constants such that $A+B=1$. This was realized by Wheeler & Feynman and a way out of this difficulty was suggested by them in their original paper [2]. This involved the use of unsymmetrical initial conditions that, on statistical grounds, would favour retarded rather than advanced solutions.

Hogarth has shown that this extra postulate is not in general necessary in a non-static universe. He considers conformally flat expanding universes with line element of the form

$$ds^2 = \exp\left[2\,\zeta(t)\right]\cdot\{-dr^2-r^2(d\theta^2+\sin^2\theta\,d\varphi^2)+dt^2\}. \tag{1}$$

Suppose fields are Fourier analysed and only a single monochromatic component is considered. Taking the total field of a particle to be of the form

$AF_{\text{ret}}+BF_{\text{adv}}$ we see that $AF_{\text{ret}}$ interacts with the future absorber and $BF_{\text{adv}}$ with the past absorber. The reactions of these absorbers will be of the form $Af\cdot\frac{1}{2}(F_{\text{ret}}-F_{\text{adv}})$ and $Bp\cdot\frac{1}{2}(F_{\text{adv}}-F_{\text{ret}})$ respectively where $f, p$ are complex numbers. For consistency we require

$$AF_{\text{ret}}+BF_{\text{adv}} = \tfrac{1}{2}(F_{\text{ret}}+F_{\text{adv}})+(Af-Bp)\cdot\tfrac{1}{2}(F_{\text{ret}}-F_{\text{adv}}).$$

Equating coefficients on both sides gives the relations

$$A=\tfrac{1}{2}+\tfrac{1}{2}(Af-Bp), \qquad B=\tfrac{1}{2}-\tfrac{1}{2}(Af-Bp). \tag{2}$$

Except when $f=p=1$, the solution is

$$A=\frac{1-p}{2-f-p}, \qquad B=\frac{1-f}{2-f-p}. \tag{3}$$

When $f=p=1$, there is no unique solution; we only have the one equation $A+B=1$. This is the situation encountered by Wheeler & Feynman for the static universe. In an expanding universe $f$, $p$ are in general different and a unique solution exists for $A$, $B$.

It is illuminating to look at the above picture in terms of particles instead of fields. We then have each source emitting photons into past and future. The interaction with the absorber takes the form of scattering—which in the electromagnetic case is the classical Thomson scattering.

When looked at in this way it is possible to describe the analogue in the case of neutrinos, even though not much is known about them as in the case of photons. All we need to know are the following three properties: (i) the mode of transmission of neutrinos in curved space time; (ii) the scattering properties and (iii) the refractive index owing to the presence of the scatterers. All these properties are known in the case of neutrinos.

It is then possible to work out $f$, $p$ for neutrinos in the various cosmological models. It is seen that the form of energy dependence of the cross section makes the condition $f=1$, $p\neq 1$ (necessary for purely retarded neutrinos) more difficult to satisfy than in the case of photons. For example, the steady state model, which easily satisfied this condition for photons, only "just" manages to do so for neutrinos. The Einstein-de Sitter model satisfies it in neither case. The details of the calculation may be found elsewhere [3].

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# GRAVITATION AS AN INTERACTION BETWEEN THE SMALL AND THE LARGE

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IT APPEARS that a Lorentz-invariant theory can be constructed to tend in the non-relativistic limit to any Newtonian Theory, but that this requires correction of the terms giving particle-particle interaction [1] or particle-field interaction by introducing simultaneous many-body interaction terms.

In the case of one dimension of space and time it has been shown that, if a particle-field interaction not including higher terms is exactly Lorentz-invariant, it must have the usual local form (Appendix). Some progress has been made in extending this theorem to the actual three-dimensional case.

When the same assumption is made for general relativity, it seems that in the one-dimensional case uniform curvature is not possible and that the curvature must depend on the local distribution of matter. If this proves true and can be extended to three dimensions, gravitation would appear as a necessary consequence, when there is space-curvature in the large, of particle-field interactions having this special form.

A difficulty in carrying out this program is that in general relativity the neighbour to whom an infinitesimal transformation leads depends on the state of the universe and it is not usually possible to put all the transformations in Hamiltonian form simultaneously without introducing new variables not determined by the state of the universe [2].

## APPENDIX

If $P(u)$, $Q(u)$, are canonical variables for radiation oscillator $u$ of a continous set, and if $p$, $q$, are representive particle variables, the functions giving infinitesimal transformations in canonical form,

$$U = q + \int P \frac{\partial Q}{\partial u} du + \int \mathcal{U}(p, q, u)(Q+iP)\, du + \text{conj.}$$

$$H = h(p) + \int \eta(u) \tfrac{1}{2}(P^2+Q^2)\, du + \int \mathcal{H}(p, q, u)(Q+iP)\, du + \text{conj.}$$

$$X = x(p) + \int \xi(u) \tfrac{1}{2}(P^2+Q^2)\, du + \int \mathcal{X}(p, q, u)(Q+iP)\, du + \text{conj.}$$

satisfy the Poisson bracket relations

$$(U, H) = X, \quad (U, X) = \frac{1}{c^2} H, \quad (X, H) = 0,$$

only if

$$h'(p) = x(p), \quad x'(p) = \frac{1}{c^2} h(p); \quad \eta'(u) = \xi(u), \quad \xi'(u) = \frac{1}{c^2} \eta(u)$$

and

$$\mathcal{U} = i\lambda \int e^{i(\eta(u)A(s)+\xi(u)B(s))} \frac{1}{2}\big(B'(s)B(s) - A'(s)A(s)\big) ds + \mu'(u),$$

$$\mathcal{X} = -i\lambda \int e^{i(\eta(u)A(s)+\xi(u)B(s))} A'(s) ds + i\mu(u)\xi(u),$$

$$\mathcal{H} = i\lambda \int e^{i(\eta(u)A(s)+\xi(u)B(s))} B'(s) ds + i\mu(u)\eta(u),$$

where $s$ is determined by $q = x(p)A(s) + \dfrac{1}{c^2} h(p)B(s) + C$.

The constant $C$ and the function $\mu(u)$ can be transformed away, and one of the functions $A(s)$, $B(s)$ can be removed by redefining $s$. What is left reduces to the usual local interaction referred to one or another space-like line of reference according to the remaining function.

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# DEGENERACY AND SHEAR

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J. M. Goldberg and R. K. Sachs proved the following interesting theorem: [1] [2] a space-time which satisfies Einstein's field equation for the vacuum,

$$R_{ij} = 0 \tag{1}$$

is algebraically degenerate in the sense of the Petrov classification if and only if it contains a shear-free congruence of null geodesics [3].

Algebraic degeneracy, null geodesic congruences and their shear are purely conformal properties of space-time: they are properties of a space-time endowed only with null cones or, equivalently, they are invariant under a conformal change of the metric $g_{ij}(x) \to \lambda(x) g_{ij}(x)$ [4]. The vacuum equation (1) is not a conformal property of space-time. It is thus clear that a conform invariant generalization of the Goldberg–Sachs theorem must exist, where the field equations (1) are replaced by weaker equations. In the following we find the weakest possible such field equations.

Consider a field of null directions $k_i$ on space-time.

We denote by "*gs*" the property that the congruence of null curves which has $k_i$ as tangents is geodesic and shear-free. We denote by "$D_{(1)}$" the property that, throughout space-time, $k_i$ is a double Penrose–Debever direction of Weyl's conformal curvature tensor, and similarly, by "$D_{(2)}$" and "$D_{(3)}$" that $k_i$ is respectively a triple and quadruple Penrose–Debever direction. We denote by $d_{(1)}$ the property that $k_i$ is a degenerate Penrose–Debever direction, i.e., that it is at least double throughout space-time, and similarly by $d_{(2)}$ and $d_{(3)}$ that $k_i$ is respectively at least triple and at least quadruple (in the last case, stronger degeneracy means that the conformal curvature tensor is zero).

Let $P_{abc} = -R_{a[b;c]} + \frac{1}{6} g_{a[b} R_{,c]}$ and $P^{-}_{abc} = P_{abc} + iP^{*}_{abc}$. Let $V_{ab}$ be a field of self-dual ($V^{*}_{ab} = -iV_{ab}$) null bivectors such that $V_{ab}k^{b} = 0$. We denote by "$f_{(1)}$"

the property that, throughout space-time, the field equation $V^{ab}P^{-}_{bcd}\ V^{cd} = 0$ is satisfied, and similarly, by "$f_{(2)}$" that $P^{-}_{acd}V^{cd} = 0$, and by "$f_{(3)}$" that $P^{-}_{abc}V^{cd} = 0$. These field equations are clearly progressively stronger, but all three are weaker than the vacuum equations, i.e.,

$$R_{ij} = 0 \Rightarrow f_{(3)} \Rightarrow f_{(2)} \Rightarrow f_{(1)}.$$

The following theorems hold:

Theorem I: $d_{(2)} \Rightarrow f_{(1)}$, $\quad d_{(3)} \Rightarrow f_{(2)}$.

Theorem II: $D_{(1)} \Rightarrow (gs \leftrightarrow f_{(1)})$, $\quad D_{(2)} \Rightarrow (gs \leftrightarrow f_{(2)})$, $\quad D_{(3)} \Rightarrow (gs \leftrightarrow f_{(3)})$.

Theorem III: ($gs$ and $f_{(1)}) \Rightarrow d_{(1)}$.

Theorem IV: $d_{(1)} \Rightarrow (f_{(1)}$ is conform invariant), $d_{(2)} \Rightarrow (f_{(3)}$ is conform invariant), $d_3 \Rightarrow (f_3$ is conform invariant).

Theorem I states that strong degeneracy implies weak field equations, no other assumption being needed. Theorems II and III constitute the generalization of the Goldberg–Sachs theorem. Theorem II shows that this generalization is best possible in the sense that the field equations are as weak as possible. Theorem IV shows that the generalized Goldberg–Sachs theorem is conform invariant. Proofs of the theorems will be published elsewhere.

A more symmetrical generalization of the Goldberg–Sachs theorem may exist, stating that for each $\alpha$ in the range 1, 2, 3, any two of the three properties $gs_{(\alpha)}$, $f_{(\alpha)}$, $D_{(\alpha)}$ imply the third. Here $gs_{(\alpha)}$ would be expected to be a stronger form of $gs$ which incorporates additional properties of the shear-free congruence of null geodesics.

Some time after we had obtained our results [5] W. Kundt and A. Thompson obtained the same results [6] by using the elegant spinor techniques developed by Penrose.

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# THE STEADY STATE UNIVERSE

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Professor Bondi put the case for the steady-state theory. He agreed with most of what Professor Ginzburg had said about the scientific approach but stressed that a theory could be abandoned, not only because of contradiction with experiment, but also if a more testable theory was proposed. The chief claim of the steady-state theory was indeed that it was far more testable than any other cosmological theory. This was due to the fact that most cosmological tests involved looking at distant objects and that if the average characteristics of astronomical objects changed with the age of the universe, then a knowledge of this variation was required before any such observation could be interpreted. Only on the basis of the steady-state theory, which denied any such change of average properties, could astronomical tests be interpreted unambiguously as for or against the theory. This was its main claim, and it was borne out by the frequency with which observers argued that of all cosmological theories their observations disproved the steady-state theory alone. Fortunately, in almost every such case the observations themselves were later shown to be in doubt. Even in the case of the very serious criticism put forward by Ryle and his colleagues on the basis of their radio-astronomical investigations the issue was in doubt, since other radio-astronomers were at present disputing the interpretation of these results. The most promising evidence of research seemed to lie in further study of the properties of such far distant objects where related to the radio-astronomical number counts, optical number counts (now much out of favour), radio-astronomical diameter measurements, variations of colour with distance (the excellent agreement of red shifts of over 0·4 from the continuous spectrum and spectral lines is in splendid agreement with the steady-state theory), the richness of clusters and the like. Perhaps of even more interest is the possibility of investigating whether the age distribution of near galaxies is in accordance with that demanded by the steady-state theory. Problems concerning the density of matter in intergalactic space and the mechanism of the formation of new galaxies are also promising fields.

# EFFET INERTIAL DE SPIN EN TRANSLATION

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## I. INTRODUCTION

Plusieurs auteurs [1] [2] [3] [4] [5] sont arrivés à la conclusion que le tenseur d'impulsion-énergie d'un milieu matériel doué de spin est asymétrique; la formule

$$T^{ij} - T^{ji} + \partial_k \sigma^{ijk} = 0 \tag{1}$$

que Tetrode [6] avait déduite de la théorie de l'électron de Dirac, a pu ainsi être interprétée et justifiée axiomatiquement; nous raisonnons en métrique de Minkowski, avec $x^4 = ict$ et $i, j, k, l = 1, 2, 3, 4$;

$$\sigma^{ijk} = -\frac{c\hbar}{2}\bar{\psi}\gamma^{ijk}\psi \tag{2}$$

désigne la densité de spin de Dirac et

$$T^{ij} = -\frac{c\hbar}{2}\bar{\psi}[\partial^i]\gamma^j\psi + ieA^i\bar{\psi}\gamma^j\psi \tag{3}$$

le tenseur d'impulsion-énergie asymétrique de Tetrode;

$$\gamma^{ij\cdots}\begin{cases}\gamma^i\gamma^j\ldots \text{ si } i \neq j \neq \ \ldots,\\ 0 \text{ si deux indices sont égaux;}\end{cases}$$

$[\partial^i] = \overrightarrow{\partial^i} - \overleftarrow{\partial^i}$ opérateur du courant de Gordon; $-e$, charge de l'électron en u. e. m. c. g. s.; $A^i$, potentiel électromagnétique.

On considère généralement que les effets de l'asymétrie du $T^{ij}$, s'ils existent, sont essentiellement du domaine de la microphysique. Il est cependant légitime de rechercher si, dans des conditions appropriées, il n'y aurait pas sommation des micro-effets, et manifestation de la non-symétrie de $T^{ij}$ de l'électron de Dirac. L'idée naturelle est de s'adresser pour cela au ferro ou au ferrimagnétisme, et nous avons proposé [7] le principe d'une expérience-test. Depuis, nous nous sommes avisé [8] que l'existence de l'effet que nous avions postulé semble bien être rigoureusement déductible des principes généraux de la mécanique quantique, comme nous allons l'exposer.

Auparavant nous devons faire une importante remarque. Deux tenseurs d'impulsion-énergie $T_1^{ij}$ et $T_2^{ij}$ dont la différence est de divergence nulle,

$\partial_j(T_1^{ij}-T_2^{ij})=0$, sont dits mathématiquement équivalents en ceci qu'ils admettent la même intégrale d'impulsion-énergie

$$P^i = \oiiint_S T_1^{ij}du_j = \oiiint_S T_2^{ij}du_j \tag{4}$$

*sur un contour fermé* $S$ ($du_j$, quadrivecteur élément de volume sur $S$). Il ne s'ensuit *pas* que les tenseurs $T_1^{ij}$ et $T_2^{ij}$ seront inconditionnellement physiquement équivalents. En effet, si les circonstances sont telles qu'on ait à calculer la variation d'impulsion-énergie d'un corps matériel de dimensions finies entre un état initial $S_1$ et un état final $S_2$ ($S_1$ et $S_2$, hypersurfaces du genre espace), *l'on doit également considérer le flux d'impulsion-énergie à travers le contour extérieur du corps,* c'est à dire, dans l'espace-temps, à travers la paroi $S_3$ du tube d'univers comprise entre les cloisons $S_1$ et $S_2$ (Fig. 1).

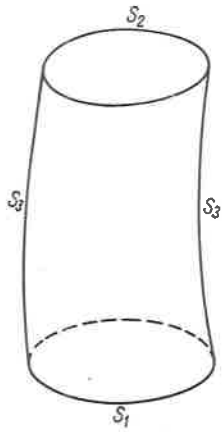


FIG. 1

La formule (4) reste bien entendue vraie pour $S=S_2-S_1+S_3$(1) mais, la variation d'impulsion-énergie physique étant calculée sur le contour *ouvert* $S_2-S_1$, *les tenseurs* $T_1^{ij}$ *et* $T_2^{ij}$ *ne seront pas en général équivalents du point de vue physique.*

## II. DÉDUCTION ABRÉGÉE DE L'EFFET EN THÉORIE DE L'ÉLECTRON DE DIRAC(2)

**1.** Des deux principes fondamentaux suivants: a) la valeur moyenne $R$ à l'instant $t_0$ d'une grandeur d'opérateur $R$ est

$$\overline{R} = \iiint_{t=t_0} \psi^+ R\,\psi\, dxdydz; \tag{5}$$

(1) Le signe "$-$" apparait si l'on oriente $S_1$ dans le même sens que $S_2$ relativement aux lignes de temps.

(2) Théorie non-superquantifiée.

b) l'opérateur de l'impulsion est ($u$, $v$, $w = 1, 2, 3$)

$$P^u = i\hbar\partial^u + eA^u = \frac{i\hbar}{2}[\partial^u] + eA^u, \tag{6}$$

il suit nécessairement que *le tenseur d'impulsion-énergie de l'électron est le tenseur asymétrique de Tetrode* (3). On le voit en portant (6) dans (5), en remplaçant $\psi^+$ par $\bar{\psi}\gamma^4$, $dx\,dy\,dz$ par $-i\,du_4$, l'hyperplan d'intégration à temps constant par une hypersurface arbitraire du genre espace, l'expression dégénérée $\gamma^4 du_4$ par l'expression complète $\gamma^j du_j$, enfin l'indice $u$ à 3 valeurs par l'indice $i$ à 4 valeurs.

L'argument de la fin de l'introduction prouve qu'*on n'a pas en général le droit d'ajouter arbitrairement à ce tenseur un autre tenseur de divergence nulle.*

**2.** Une caractéristique essentielle de l'expérience que nous proposons est que *les valeurs moyennes dans le corps d'épreuve du courant de Dirac*

$$j^i = -ie\bar{\psi}\gamma^i\psi \tag{7}$$

*et du courant de Gordon*

$$k^i = \frac{ie}{2\varkappa}\,\bar{\psi}[\partial^i]\psi - \frac{e^2}{\varkappa\hbar}\,A^i\bar{\psi}\psi \tag{8}$$

*sont différents* ($\varkappa$, fréquence propre de l'onde électronique). Il nous suffira de raisonner ici en termes prérelativistes, les caractères gras désignant des vecteurs de l'espace ordinaire.

$\boldsymbol{M}$ représentant la densité de polarisation magnétique de l'électron, on a la formule de Gordon

$$\boldsymbol{j} = \boldsymbol{k} + \mathrm{rot}\,\boldsymbol{M} \tag{9}$$

et par conséquent

$$\iiint (\boldsymbol{k}-\boldsymbol{j})\,du = -\iiint \mathrm{rot}\,\boldsymbol{M}\,du = \oint\!\!\oint \boldsymbol{M}\times\boldsymbol{ds}; \tag{10}$$

($du$, $\boldsymbol{ds}$, éléments de volume et de surface au sens ordinaire). *Dans les conditions de notre expérience, l'intégrale* (10) *est essentiellement non-nulle dans l'état final.*

Par ailleurs, puisque $\psi^+\psi$ représente la densité de probabilité de présence de l'électron, le produit scalaire $\boldsymbol{j}\cdot\boldsymbol{ds}$ est nul sur tout le contour du corps matériel d'épreuve; il s'ensuit que l'intégrale $\iiint \boldsymbol{j}du$ calculée à temps constant dans le repère propre du corps sera en général petite. Un cas parmi d'autres où l'on aura rigoureusement

$$\iiint \boldsymbol{j}du = 0 \tag{11}$$

dans le repère propre du corps d'épreuve sera celui de notre expérience décrite plus loin (n° IV). Dans ce cas, l'on aura rigoureusement dans le même repère

$$\iiint \boldsymbol{k}du = \oint\!\!\oint \boldsymbol{M}\times\boldsymbol{ds} \tag{12}$$

3. Comparons les deux intégrales[3]

$$P_2^i = \iiint_{S_2} T^{ij} du_j, \quad L_2^i = \iiint_{S_2} T^{ji} du_j, \tag{13}$$

avec l'expression (3) de $T^{ij}$ (tenseur asymétrique de Tetrode).

Plaçons nous dans un cas statique, $\psi(x) = \varphi(x) \exp(iWt)$ et adoptons la "représentation des vitesses faibles", $\gamma^4 = (+1, +1, -1, -1)$ avec deux "grandes" ($\psi_1$ et $\psi_2$) et deux "petites" ($\psi_3$ et $\psi_4$) composantes du $\psi$. En intégrant à temps constant, l'on voit immédiatement que *dans chaque élément de volume, la "vraie impulsion-énergie $P_2^i$ sera presque colinéaire au courant de Gordon $K^{i(3)}$ et la "fausse" impulsion-énergie $L_2^i$ rigoureusement colinéaire au courant de Dirac $j^i$.*

Comme on a vu, à propos des formules (11) et (12), que la valeur moyenne du courant de Dirac coïncide avec la vitesse macroscopique du corps d'épreuve, la "fausse" impulsion-énergie $L_2^i$, mérite le nom *d'impulsion-énergie longitudinale.* Il suit de ce qu'on vient de voir et du commentaire des formules (11) et (12) que *la valeur moyenne $P_2^i$ de l'impulsion-énergie physique fera, dans les circonstances que nous avons définies, un angle notable avec la quadrivitesse du corps d'épreuve.*

4. Pour confirmer ce point calculons l'*impulsion-énergie transversale*[3]

$$T_2^i = P_2^i - L_2^i, \tag{14}$$

$$T_2^i = \iiint_{S_2} (T^{ij} - T^{ji}) du_j = - \iiint_{S_2} \partial_k \sigma^{ijk} du_j = \frac{1}{2} \oiint \sigma^{ijk} ds_{jk} \tag{15}$$

ou, en langage prérelativiste[4] (et l'indice 2 étant négligé)

$$\boldsymbol{T} = - \iiint \text{rot}\, \boldsymbol{\sigma}\, \text{du} = \oiint \boldsymbol{\sigma} \times \boldsymbol{ds}, \quad T^4 = 0; \tag{16}$$

les vecteurs d'espace $\boldsymbol{M}$ et $\boldsymbol{\sigma}$ étant, comme il est bien connu, colinéaires dans le repère propre du courant de Dirac[5], la compatibilité de la première formule (16) avec (12) est manifeste en vertu du contenu des §§ 2 et 3 ci-dessus.

5. Si, entre l'état initial 1 et l'état final 2, le corps d'épreuve contenant

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(3) L'indice 2 précise que nous considérons l'état final du corps d'épreuve; voir à ce sujet le n° IV.

(4) La seconde formule (16) suit d'une intégration à temps constant *avec un $\sigma^{ijk}$ complètement antisymétrique.*

(5) Les deux identités du type Pauli [9]—Kofink [10] $(m^{ik})(j_k) = (\omega_2)(\sigma^i)$, $(\overline{m}^{ik})(j_k) = (\omega_1)(\sigma^i)$, montrent que, dans le repère propre du courant de Dirac (qui est du genre temps), les trois trivecteurs densités de spin, de polarisation magnétique et de polarisation électrique sont colinéaires;

$$(m^{ij}) = \overline{\psi}\gamma^{ij}\psi, \quad (\omega_1) = \overline{\psi}\psi, \quad (\omega_2) = \overline{\psi}\gamma^5\psi,$$

$\sigma^i$ et $\overline{m}^{ij}$ désignent les duals de $\sigma^{jkl}$ et de $m^{kl}$; les parenthèses indiquent qu'on néglige les facteurs physiques.

le nuage électronique ne reçoit aucune impulsion-énergie de l'extérieur, et si dans l'état initial $T_1^i = 0$, les équations de son mouvement s'écrivent

$$P_2^i = P_1^i, \quad L_1^i = P_1^i, \quad L_2^i = P_2^i - T_2^i \tag{17}$$

ou encore, dans le repère propre initial du corps d'épreuve,

$$\boldsymbol{P}_2 = \boldsymbol{P}_1 = 0, \quad \boldsymbol{L}_1 = 0, \quad \boldsymbol{L}_2 = -\boldsymbol{T}_2. \tag{18}$$

En conclusion, si, *entre l'état initial* 1 *et l'état final* 2, *la distribution de la densité de spin varie dans le corps d'épreuve sans que lui soit imprimée une impulsion-énergie extérieure, une vitesse de recul apparaitra en conformité avec les formules* (17), (18) et (16)[6] (Fig. 2).

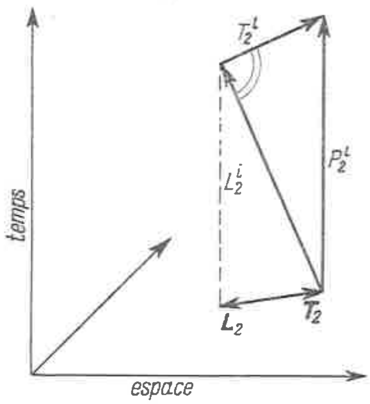


FIG. 2

Deux remarques avant de changer de sujet:

A. *Si l'on représentait la densité d'impulsion-énergie par le* $T^{\underline{ij}}$ *symétrisé, il suivrait de là non pas un effet nul comme on l'a toujours admis implicitement jusqu'ici, mais (en vertu des précédents raisonnements) la moitié de l'effet qu'on vient de calculer.*

B. D'après la précédente théorie l'impulsion-énergie physique est la somme d'une impulsion-énergie longitudinale ou *orbitale* et d'une impulsion-énergie transversale sans support cinématique, qu'on peut (en ce sens) dire *potentielle.* Ceci décalque le statut maintenant classique de la théorie du spin, où le moment angulaire physique est la somme d'un moment angulaire orbital et d'un moment angulaire propre sans support cinématique; en ce sens, le spin peut être dit une entité potentielle.

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(6) Pour établir la connexion entre cette Section II et l'Introduction, remarquons qu'en vertu de la conséquence $\partial_j \partial_k \sigma^{ijk} = 0$ de l'antisymétrie complète de la densité de spin

$$\triangle T^i = \iiint_{S_2 - S_1} (T^{ij} - T^{ji})\, du_j = -\iiint_{S_3} (T^{ij} - T^{ji})\, du_j;$$

"les sources de l'impulsion-énergie transversale sont purement superficielles".

## III. PASSAGE DU MICROSCOPIQUE AU MACROSCOPIQUE

Nous avons jusqu'ici raisonné avec la théorie de l'électron unique non superquantifié.

En réalité, trois prises de moyenne successives sont nécessaires pour décrire la situation à l'échelle macroscopique;

1° dans chaque domaine de Weiss il y a de nombreux électrons actifs;

2° dans chaque cristal il y a de nombreux domaines de Weiss;

3° dans le corps macroscopique il y a de nombreux cristaux.

La première prise de moyenne ressortit à la théorie superquantifiée; le problème est en effet celui de la distribution des électrons actifs sur des états orthogonaux conformément à la statistique de Fermi. Si l'on néglige l'interaction entre le champ de Dirac et le champ des photons libres, la description à utiliser est celle de Heisenberg, avec un vecteur d'état constant; toutes les précédentes formules subsistent en moyenne.

La 3ème prise de moyenne se fait aisément en substituant à l'idéalisation du corps continu un pavage de cristaux non jointifs, et en postulant (ce qui est très sensiblement vrai) qu'à l'intérieur des "pavés" les lignes de champ se superposent à celles du corps continu. L'on a donc a comparer deux intégrales $\iiint \mathrm{rot}\, \boldsymbol{\sigma}\, du$ et $\iiint \mathrm{rot}\, \boldsymbol{\sigma}_0\, du$ différant par le domaine d'intégration (le volume des lacunes) et la valeur en chaque point de $|\sigma|$ ($\boldsymbol{\sigma}_0$ est la moyenne de $\boldsymbol{\sigma}$): ces deux intégrales sont manifestement égales.

La seconde prise de moyenne est la plus délicate. Cependant, si l'on admet que la distribution réelle à l'intérieur d'un cristal peut être considérée comme la limite d'une distribution sans discontinuités sur $\boldsymbol{\sigma}$ et où rot $\boldsymbol{\sigma}$ peut être définie, alors la seconde prise de moyenne se fait, elle aussi, sans incidents.

## IV. L'EXPÉRIENCE PROPOSÉE

On utilisera le ferro ou le ferrimagnétisme, en tant que dus au spin de l'électron. Dans son état initial le corps sera non-aimanté. Dans son état final il sera aimanté de telle manière que les intégrales doubles (12) et (16) ne soient pas nulles. Enfin *la procédure d'aimantation sera telle qu'aucune impulsion ne sera communiquée au corps d'épreuve.*

Prenons comme corps d'épreuve (Fig 3) un petit anneau de ferrite ou de fer-cobalt, qu'on aimantera à quasi-saturation[7] par une brève impulsion de courant dans un fil dirigé suivant son axe $z'z$. D'après l'électroma-

[7] Il est pratiquement impossible de saturer un ferrite dans nos conditions expérimentales, mais on peut réaliser un état d'aimantation où $M$ varie très peu en fonction de la distance radiale.

gnétisme classique, aucune impulsion n'est ainsi transmise au corps d'épreuve(8).

Dans l'état final 2, il y aura une impulsion "transversale" $T$ dirigée suivant $z'z$ et valant, d'après la formule (16),

$$T_2 = 2\pi ab\sigma \tag{19}$$

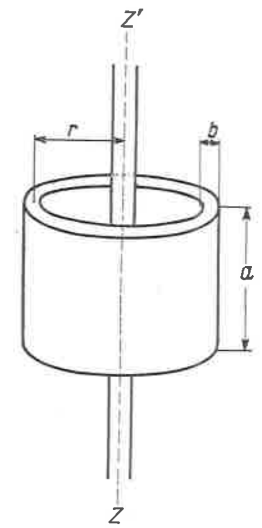


FIG. 3

($a$, hauteur, $b$, épaisseur de l'anneau). D'aprés les formules (18), le corps d'épreuve acquerra donc, dans le référentiel du laboratoire, une impulsion "longitudinale" $L_2 = -T_2$; $\varrho$ désignant sa densité et $r$ son rayon moyen, sa masse vaut

$$M = 2\pi rab\varrho; \tag{20}$$

la vitesse de recul sera donc

$$v = \frac{\sigma}{\varrho r}. \tag{21}$$

Mais, $\boldsymbol{M}$ désignant toujours la densité de polarisation magnétique,

$$\frac{\sigma}{\varrho} = \frac{\sigma}{M} \cdot \frac{M}{\varrho}, \tag{22}$$

$$\frac{\sigma}{M} = \frac{\text{moment cinétique de l'électron}}{\text{moment magnétique de l'électron}} = \frac{m}{e} \tag{23}$$

(8) Il faut examiner le cas d'un corps d'épreuve en ferrite, qui peut porter une charge électrique $Q$. Le champ électrique $-\partial \boldsymbol{A}/\partial t$ créé par le courant est, comme $\boldsymbol{A}$, parallèle à $z'z$. Mais l'impulsion totale créée par une impulsion de courant, $-Q\Delta \boldsymbol{A}$ est nulle. Donc, même dans ces conditions défavorables, aucun "artifact" n'est à craindre.

est une constante universelle valant $5{,}7 \cdot 10^{-8}$ u.e.m.c.g.s.; $M/\varrho$ est *l'intensité d'aimantation spécifique*, qui peut valoir 70 pour un bon ferrite ou 210 pour un bon alliage Fe—Co. Avec un rayon $r = 0{,}1$ cm, on trouve ainsi $v \simeq 3{,}9 \cdot 10^{-5}$ ou $1{,}17 \cdot 10^{-4}$ cm/sec.

On doublera l'effet en alternant le sens des impulsions de courant, et on l'amplifiera par résonance mécanique. Une détection interférométrique des déplacements sera suffisante pour trancher de l'existence ou de la non existence de l'effet. L'expérience est actuellement montée suivant ce schéma par Ch. Goillot.

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# GENERAL RELATIVITY AND ELEMENTARY PARTICLES*

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I PRESENT here some attempts to bring general relativity into closer grip with the elementary particle problem, drawing heavily on published or to-be-published work with Misner, and with Jauch, Schiminovich and Speiser.

Examine any table of the elementary particles with their properties tabulated in columns next to them, and it becomes clear that the first problem of any theory of matter is to explain the column *headings*, such as mass, spin, baryon number, isospin. Only later need it concern itself with the numerical table entries. But where are the theoretical constructs in the general theory of relativity to correspond to these semi-empirical constructs of microphysics? A geometrical theory, even a quantized one, seems very deficient in raw materials.

## CHARGE AND ISOSPIN

While the metric structure makes possible the formulation of mass and spin, and particle numbers might conceivably have a topological origin, where in a Lorentzian manifold are charge and isospin? We propose to look to a more flexible process of quantization for those quantities that are not already to be found in geometry. We should explain what we mean by this:

Let us call a theory *rigid* if it is determined essentially uniquely, like *the* integers, or *the* Euclidean solid geometry; *flexible* otherwise, if essential elements are unspecified, like group theory or Riemannian geometry. (Rigid =categorical, in another parlance.) The histories of the development of a flexible geometry within mathematics and of its subsequent utilization in physics are instructive because in both cases the process begins merely with a transition from one rigid theory to another. The questioning of the Fifth Postulate of Euclid by Bolyai, Gauss, and Lobatchewsky led in each case to an alternative geometry that was non-Euclidean but still rigid; only later did Riemann change the problem from a decision between various rigid theories to the formulation of a flexible theory that as it happens admits them all as degenerate cases. Likewise the flat Minkowskian world-geometry,

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an important first step on the route from Euclidean solid geometry to the Einstein field equations is itself still a rigid theory, contained within the flexible world-geometry of Einstein as a degenerate case.

Implicit in any physical theory is a calculus of propositions or assertions, delimiting the collection of physically meaningful propositions and the logical relations between them. It is well known that the propositional calculi of classical mechanics and quantum mechanics are quite different, one corresponding to the calculus of sets in a phase space, the other to the calculus of projections in a Hilbert space. The key difference between these calculi is the distributive law of the propositional calculus; indeed it is possible to formulate them so that a certain weakening of the distributive law is the sole difference and can be regarded as the essential step in quantization. However the propositional calculus of conventional quantum mechanics is as rigid as that of classical mechanics, that is, the Hilbert space is uniquely defined. Recently another kind of quantum mechanics was put forward using quaternion Hilbert space but the associated propositional calculus is still rigid.

Is it possible that we are in an arrested stage of a development from the rigid "Aristotelian" world-logic to a flexible one, analogous to the development from the rigid Euclidean world-geometry to the flexible Einsteinian one? What sort of effects would variations in the world-logic cause, as variations in the world-geometry cause gravity?

A simple example of a flexible propositional calculus or flexible quantization has been set up. It turned out to be more natural to work in a quaternion quantum mechanics than a complex, for much the same reason that a non-trivial Riemannian geometry is not possible in one dimension. A new fundamental field $C^{\mu}{}_{m}$ (analogous to the affine connection $\Gamma^{\mu}_{\varkappa\lambda}$) is required for this development ($m$=1,2,3). With the simplest choice for the action of this new field, it is found to contain a Maxwell electromagnetic field $A^{\mu} = C^{\mu}{}_{3}$ rigorously and an additional charged *massive* vector field $B^{\mu} = C^{\mu}{}_{1} + iC^{\mu}{}_{2}$.

It is most surprising to find this asymmetry between the massless $A^{\mu}$ and the massive $B^{\mu}$ arising naturally. This asymmetry happens because in addition to $C^{\mu}{}_{m}$ analogous to the affine connection, the logical structure of the theory requires a field $\eta_m$ analogous to the metric tensor. The field $\eta_m$ takes the place of the usual $i/h$ in Schrödinger's equation much as $g_{\mu\nu}$ replaces the flat metric in the geodesic equation. $A^{\mu}$ and $B^{\mu}$ are essentially components of $C^{\mu}{}_{m}$, $\| \eta_m$ and $\perp \eta_m$ respectively.

While this example of a flexible world-logic is of interest in itself for phenomenological applications, our purpose here is to explore the possibilities it opens for the space theory of matter. It makes it possible to describe charge and isospin within a framework that can still be regarded as nothing but quantized geometry—only now both the quantization and the geometry contribute dynamical elements.

## PARTICLE NUMBERS

We come now to the problem of the conserved particle numbers, e.g., baryon number and the lepton number (*s*). These are remarkable because unlike charge they do not seem to belong to a natural gauge group, yet they have very strict conservation laws.

Evidently these quantities are invariant under a large group of transformations. We are therefore trying to find counterparts for these quantities in the *homotopy invariants* of the theory: functionals of a classical history which are invariant under any deformation (homotopy) whatsoever of the history. Such quantities are bound to correspond to a strictly conserved quantity in the quantum theory of the field but, for all the conventional field theories but gravity there are no such homotopy invariants except numerical constants, because any history $\Phi$ can be deformed into $\Phi \equiv 0$ (and therefore into any other $\Phi'$) via the intermediates $s\Phi$, $0 \leqslant s \leqslant 1$. This is not so for gravity: $g_{\mu\nu} \equiv 0$ is not an acceptable history! Let us call a (maximal) class of 4-manifolds with Lorentzian metrics that *can* be deformed into one another (subject possibly to boundary conditions) a (*world*) *class* $w$. A homotopy invariant then defines, and is defined by, a function of a class. The collection $W$ of all classes may thus be an object of importance for quantum general relativity. A quantum-mechanical integral over all histories decomposes into a discrete sum over $W$ according to

$$\int Dg_{\mu\nu} = \sum_{w \in W} \int_w Dg_{\mu\nu},$$

of which the perturbation theory based on the flat space catches only one term ($w = 0$, in the notation to follow). The enumeration of all the *topologically trivial* classes $W_0$ has already been carried out, and led to a single "topological conservation law" or homotopy invariant, an integer $M$ ("number of $M$-geons"). In that case we found a natural group structure (infinite cyclic) for $W_0$, corresponding to addition of $M$, which could be interpreted as juxtaposition of particle systems. Accordingly we termed the generator $w_0$ of $W_0$ an *M-geon*. Now we consider the possibility of topologically nontrivial worlds to find further topological conservation laws (and also because as yet we have only integer spins). For any two classes of worlds $u$, $v$ we define a class

$$w = u + v$$

called their sum by choosing representatives for $u$ and $v$ that coincide with (say) the flat space of Minkowski, outside disjoint world-tubes $U$ and $V$ respectively. Then $w$ is the class of the world that contains both $U$ and $V$ and coincides with the Minkowski space outside them.

With this definition of $+$, $W$ is a semi-group. The class of the Minkowski space-time is the identity and will be written **0**. There is a maximal sub-group $W_0$ of $W$. $W_0$ contains all worlds that possess inverses (negatives), or equivalently, it appears, all topologically trivial worlds. $W_0$ is infinite cyclic. Its generator, previously called $M$-geon, will be designated by **1**.

As candidates for particle numbers we take the *additive* integer-valued functions on $W$. We can call a class $w$ a *unit* if there is such a function that assumes the value **1** on $w$. Do units exist?

The handle or "wormhole" is *not* a unit. This is the class $W_2$ with topology $S^2 \times S^1 \times R^1$ and metric $ds^2 = dt^2 - du^2$ where $dt^2$ is the usual metric on $R^1$, and $du^2$ is a positive-definite metric on $S^2 \times S^1$. To see that $w_2$ is not a unit, we use the relation

$$w_2 + w_1 = w_1 + w_1 + w_1. \tag{1}$$

Here $w_1 = P^3 \times R^1$ ($P^n$ is the real projective $n$-space) and a metric $ds^2 = dt^2 - du^2$ is used, $dt^2$ and $du^2$ being positive-definite on $R^1$ and $P^3$. For any additive functional $N(w)$, the relation (1) implies that

$$N(w_2) = 2N(w_1),$$

which must be even, hence $N(w_2) \neq 1$. Maybe **1** and $w_1$ are units!

In addition to the metric of $w_2$, the topology $S^2 \times S^1 \times R^1$ could have been given the metrics $w_2 + mw_0$, $m = 1, 2, \ldots$ The metrics on the handle that do not fall into this sequence can all be found in a second sequence of the form $w_2' + mw_0$. In these second-sequence worlds a future-directed vector is continuously carried into a past-directed vector when continuously transported "through the handle".

## SPIN $\frac{1}{2}$

For a classical mechanical system to admit a canonical quantization which permits it to have spin $\frac{1}{2}$ like the rigid rotator, it is necessary and sufficient that the orbits of a $2\pi$ rotation in the classical configuration space not be shrinkable to a point. The natural generalization of this criterion to field theory has been indicated elsewhere and it was shown that the worlds of the form

$$m\mathbf{1} + nw_2 + n'w_2', \quad m = 0, \pm 1, \pm 2, \ldots; \quad n, n' = 0, 1, \ldots$$

fail to satisfy it. This is an extension of the usual result on the absence of spin $\frac{1}{2}$, which applies to the special case $m = n = n' = 0$, the **0** world, and appears to dispose of all worlds of the form $M^3 \times R^1$ without topological torsion. As a next step we have considered worlds of the form

$$m\mathbf{1}+nw_1+n'w_1'+pw_2+p'w_2', \quad n, n', p, p', = 0, 1, \ldots,$$

where $w_1'$ is another world with topology $P^3 \times R^1$ not expressible in the form $w_1+nw_0$. These worlds possess torsion but are still orientable. We have found that the criterion for spin $\frac{1}{2}$ is *not* satisfied. Since this is not the most general world, being orientable, the question of the existence of spin $\frac{1}{2}$ in this framework is still open.

# ASYMPTOTIC COORDINATE CONDITIONS, THE WAVE FRONT THEOREM, AND PROPERTIES OF ENERGY AND MOMENTUM*

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## I. INTRODUCTION

This report is based on work done in collaboration with S. Deser and C. W. Misner. A detailed account of the material will be published in three papers currently in preparation. The subject matter is concerned with the derivation of the so-called "wave-front" theorem, the existence of a Newtonian limit for asymptotically flat systems, and the correct definition of energy and momentum of a gravitational system. With respect to the latter question, relativists generally fall into two classes: those who feel that this is a difficult and as yet unsolved problem, and those who offer expressions for energy and momentum (indeed, several new ones have been suggested at this conference). Part of the purpose of this report is to give some of the reasons why the canonical formalism's definitions are indeed the valid ones, and to give some general tests for the validity of any definition. The work to be described is an extension and generalization of previous analyses [1](1). Most of the results were already obtained in these papers, but use was made there of the canonical coordinate conditions. While these frames probably exist for a reasonably wide class of metrics, their domain of validity has not been vigorously established. The present discussion, while stimulated by the canonical formalism, does not in any way depend upon it.

We begin with a description of the class of metrics to be considered. We assume that asymptotically, i.e. at *spatial* infinity, space is flat. (Cartesian coordinates in a Lorentz frame will be employed there.) More precisely,

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(1) See also report by C. W. Misner in the Proceedings of this conference.

we require that the metric and its derivatives approach their flat values no slower than order[2] $\frac{1}{r}$:

$$h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu} \sim O(r^{-\beta_0}), \quad h_{\mu\nu,\varrho} \sim O(r^{-\beta_1}), \quad h_{\mu\nu,\varrho\sigma} \sim O(r^{-\beta_2}), \qquad (1)$$
$$\beta_n \geqslant 1.$$

We further assume that $h_{\mu\nu}(x)$ possesses an asymptotic series in descending powers of $r$:

$$h_{\mu\nu} = \sum_{n=0} \frac{A_{\mu\nu}{}^{(n)}}{r^{\alpha_n}}, \quad \alpha_n \geqslant 1, \quad A_{\mu\nu}{}^{(n)} \sim O(1) \qquad (2)$$

and that it is valid to integrate and differentiate this series. In Eq. (2), the $\alpha_n$ need not be integers and logarithms are allowed in $A^{(n)}$ (except in $A^{(0)}$ if $\alpha_0 = 1$, so that Eq. (1) is not violated). These conditions must, of course, hold in any of the frames to be considered, a requirement which puts restrictions on the class of coordinate transformation functions allowed. Writing the general transformation as $\bar{x}^\mu = x^\mu + \eta^\mu(x)$, one may separate out the Lorentz transformation part according to

$$\eta^\mu(x) = a^\mu_\nu x^\nu + a^\mu + \xi^\mu(x) \qquad (3)$$

where $a^\mu_\nu$ is a Lorentz matrix, $a^\mu$ is a constant, and the remainder, $\xi^\mu(x)$, are the "gauge transformation" functions. With the boundary conditions imposed, it is possible to show that $\xi^\mu{}_{,\nu}$ must go asymptotically as $O\left(\frac{1}{r}\right)$. Characteristic examples of $\xi^\mu$ (in the asymptotic region) are

$$\xi^\mu: \quad \frac{x^i}{r}, \quad \ln r, \quad \frac{e^{ik_\mu x^\mu}}{r}, \quad \frac{e^{ikr^{1/2}}}{r^{1/2}}. \qquad (4)$$

The first two structures are of $O(1)$, the third is a "coordinate wave" of $O\left(\frac{1}{r}\right)$, while the last is in between these two extremes.[3] The general type of term considered in Eq. (4) can be characterized by the expression

$$\xi^\mu: \quad \frac{f_{(ab)}{}^{(x)}}{r^c}, \quad f_{ab} \sim O(1) \qquad (5a)$$

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[2] Latin indices run from 1 to 3 and Greek from 0 to 3. The Lorentz metric $\eta_{\mu\nu}$ has the form $\eta_{\mu\nu} = \text{diag}\,(-1, +1, +1, +1)$. Repeated indices are summed whether or not one is covariant and the other contravariant. We use units such that $16\pi\gamma c^{-4} = 1 = c$ where $\gamma$ is the Newtonian constant. A comma denotes ordinary differentiation.

[3] Note that the Lorentz transformations plus structures like the first example in Eq. (4) form the Bondi-Metzner group (though it should be stressed that we are here examining the metric in the *spatial* asymptotic direction, *not* along the light cone). Thus the coordinate transformation group being considered is much larger than the B−M group.

where the "amplitude function" $f_{(ab)}$ obeys

$$f_{(ab),i} \sim \frac{f_{(ab)}}{r^{1-a}}, \quad f_{(ab),0} \sim \frac{f_{(ab)}}{r^{1-b}}. \tag{5b}$$

Thus, the first example in Eq. (4) is of the type $a = 0$ and the last $a = \frac{1}{2}$, The boundary conditions on $\xi^{\mu}{}_{,\nu}$ then require that $a, b \leqslant c$. We also have that $a, b \leqslant 1$. This upper limit is needed so that higher derivatives do not violate the $\frac{1}{r}$ boundary conditions. We now assume that the most general form for $\xi^{\mu}$ and for the asymptotic series for $h_{\mu\nu}$ of Eq. (2) is a sum of terms of the type given in Eq. (5):

$$\xi^{\mu} = \sum \frac{f^{(n)}}{r^{c_n}}, \quad f^{(n)} \equiv f_{(a_n, b_n)}(x). \tag{6}$$

The class of metrics being allowed is quite general and an arbitrary asymptotic behaviour of the type given in Eq. (1) can be simulated by a series for $h_{\mu\nu}$ of the type given in Eq. (6).

## II. WAVE FRONT THEOREM

The key result needed to establish the other theorems in the asymptotic domain is the so-called "wave front theorem". By this is meant the theorem that the modes of the field representing radiation (and carrying the radiated energy) must fall off *faster* than $O\left(\frac{1}{r}\right)$, (more precisely as $O(r^{-\frac{3}{2}-\varepsilon})$) beyond some closed two-surface (the "front of the wave"). On physical grounds, one can easily see why this should occur. As one goes into the asymptotic domain, the field amplitudes become weaker, and the situation should be governed, to first approximation, by linearized theory, where the energy density is quadratic in the field. A radiation situation then, with the fields going to infinity as $O\left(\frac{1}{r}\right)$ would possess infinite energy. A similar situation arises in other field theories. There one *imposes* a "wave front" condition upon the solutions of the field equations on physical grounds, as one does not wish to consider systems with infinite energy. Such a condition, though, is not a direct consequence of the field equations there, and generally is not needed in the usual discussions of radiation questions. In general relativity, however, the wave front arises in a more natural fashion. Here energy is the source of the gravitational field, and by the principle of equivalence, gravitational energy is also a source. Thus, in the component of the metric representing the Newtonian potential, $\varphi \sim \frac{\gamma m}{r}$, a situation where $m \to \infty$ violates the $\frac{1}{r}$ boundary conditions. We expect, therefore, that our boundary conditions should yield the gravitational wave front naturally (and, indeed, again by the principle of equivalence, imply it also for all other fields).

Let us now sketch briefly how the above physical argument works mathematically. According to the canonical formalism, the fourth constraint equation, $G_0{}^0 = 0$, determines the part of the metric that contains, asymptotically, the Newtonian potential. As we shall see, this equation does, in fact, enforce the wave front condition. Introducing the notation[4]

$$\pi_{ij} \equiv (-{}^4g)^{1/2}(\Gamma^0_{ij} - g_{ij}\Gamma^0_{mn}{}^3g^{mn}), \qquad (7)$$
$$\pi \equiv {}^3g^{ij}\pi_{ij},$$

this constraint equation reads

$$({}^3g)^3R + \frac{1}{2}\pi^2 - \pi_{ij}\pi^{ij} = 0. \qquad (8)$$

To analyse Eq. (8) further, we make the orthogonal decomposition (so useful in the canonical formalism description) into transverse and longitudinal parts:

$$h_{ij} = h_{ij}{}^{TT} + h_{ij}{}^T + h_{i,j} + h_{j,i},$$
$$h_{ij}{}^T \equiv \frac{1}{2}\,[h^T\delta_{ij} - \nabla^{-2}(h^T_{,ij})]. \qquad (9a)$$

Similarly, one has for $\pi^{ij}$:

$$\pi^{ij} = \pi^{ijTT} + \pi^{ijT} + \pi^i{}_{,j} + \pi^j{}_{,i}. \qquad (9b)$$

In Eq. (9), the functions $h_{ij}{}^{TT}$ and $h_{ij}{}^T$ are divergenceless ($h_{ij,}{}^{TT}{}_j = 0 = h_{ij,}{}^T{}_j$) while $h_{i,j}$ is longitudinal. The first transverse quantity, $h_{ij}{}^{TT}$, has vanishing trace ($h_{ii}{}^{TT} = 0$) while $h_{ii}{}^T = h^T$ is in general non-zero. The operator $\nabla^{-2}$ is the inverse laplacian with vanishing boundary conditions. The basic properties of the decomposition are: (1) it is unique for any $h_{ij}$ which vanishes asymptotically as $O(1/r^\varepsilon)$, $\varepsilon > 0$ (i.e., the different orthogonal parts, $h_{ij}{}^{TT}$ $h^T$, and $h_{i,j}$ are uniquely expressible as *linear* functionals of $h_{ij}$); and (2) each orthogonal part asymptotically depends *only* on the corresponding *asymptotic* part of $h_{ij}$ down to $O(1/r^3)$ (i.e. the $\frac{1}{r}$ part

[4] The metric ${}^3g^{ij}$ is the *three*-dimensional inverse of $g_{ij}$. Similarly ${}^3R$ is the three-dimensional scalar curvature formed from $g_{ij}$ and ${}^3g^{ij}$, and ${}^3g$ is the determinant of $g_{ij}$. All three dimension indices are raised and lowered by $g_{ij}$ and ${}^3g^{ij}$.

of $h^T$ is determined from the $\frac{1}{r}$ part of $h_{ij}$, etc.). In terms of these variables, one may re-express Eq. (8) as

$$-\nabla^2 h^T = \left[\frac{1}{4}(h_{ij,m})^2 + (\pi^{ij})^2\right] + \partial_i D_i$$
$$- \frac{1}{2}\left\{\pi^2 + (h_{ij,j})^2 + \frac{1}{2} h_{ij,j} h_{mm,i}\right\} + C \qquad (10)$$

In Eq. (10), the term labeled $C$ represents the cubic and higher non-linearities (which approach zero *a priori*, as $O(1/r^3)$ according to Eq. (1)). The remaining terms on the right hand side (r.h.s.) are the quadratic structures. (One of these, $\partial_i D_i$, can be arranged into a divergence.) *A priori*, Eq. (1) says that these structures approach zero as $O(1/r^2)$. The one linear structure, $\nabla^2 h^T$, has been isolated on the l. h. s. Now since $h_{ij}$ asymptotically approaches zero no slower than $O(1/r)$, the same must be true of the orthogonal component $h^T$. The requirement then is that $\nabla^{-2}$ (r.h.s.) $\sim O(1/r)$. This is in general true only if the r. h. s. $\sim O(1/r^{3+\varepsilon})$, $\varepsilon > 0$, or is in the form of a divergence of a vector which goes as $O(1/r^2)$. We see then that only the $\partial_i D_i$ term *a priori* satisfies this requirement.

(Since $C \sim O(1/r^3)$ one expects that $\nabla^{+2} C \sim \frac{\ln r}{r}$ for its leading term (violating the boundary condition on $h^T$) while the first and third terms, being quadratic, could conceivably contribute structures of $O(1)$ to $h^T$.)

Thus, conditions more restrictive than Eq. (1) must actually hold on some components of the metric so that the boundary conditions on $h^T$ are not violated. The derivation is simplified by noting *it is always possible to find a frame such that the third term (the brace) vanishes as* $O(1/r^{3+\varepsilon})$, $e > 0$. The contribution of this term to $h^T$ then satisfies the boundary conditions. Under these circumstances, the first term can go no *worse* than $O(1/r^3)$ (so that any possible $O(1/r^3)$ part in $C$ is canceled). Since the bracket is a sum of squares one has at least that

$$h_{ij,m} \sim O(1/r^{\frac{3}{2}}), \qquad \pi^{ij} \sim O(1/r^{\frac{3}{2}}). \qquad (11)$$

However, if one then reexamines term $C$ in light of the more restrictive condition (11) (rather than merely Eq. (1)), one finds[5] that actually $C$ goes

[5] The proof consists simply in noting that all cubic and higher terms each contain precisely two first derivatives of the metric (either $h_{ij,k}$ or $\pi_{ij} \approx h_{ij,0}$) times a number of $h_{ij}$'s. Thus, by (11), $C$ goes faster than $O(1/r^3)$.

at least as $O(1/r^4)$. Hence the bracket term must preserve the $\frac{1}{r}$ boundary condition on $h^T$ by itself, leading to the basic result:(6)

$$h_{ij,k} \sim O(1/r^{\frac{3}{2}+\varepsilon}), \quad \pi^{ij} \sim O(1/r^{\frac{3}{2}+\varepsilon'}), \quad \varepsilon, \varepsilon' > 0. \tag{12}$$

The non-trivial part of the derivation is to show that a frame can always be found in which the brace term of Eq. (10) goes faster than $O(1/r^3)$ (so that one is left to deal with only the positive definite bracket expression). One may in fact show that a frame always exists where

$$\pi \sim O(1/r^2), \quad h_{ij,j} \sim O(1/r^2) \tag{13}$$

which is more than enough to establish the theorem. Equation (13) is derived by starting in an arbitrary frame and finding appropriate coordinate transformations to the asymptotically better behaved frames. Since the relation $\pi = 0$ is the minimal surface condition, it is interesting to note that this condition can always be achieved asymptotically to a very high accuracy (one may even find frames where $\pi \sim O(1/r^4)$). In fact, almost any coordinate condition allowing Cartesian coordinates at spatial infinity can be shown to be achievable asymptotically for the arbitrary metrics of our class.

In previous analyses [1], it has been established quite generally that $\pi^{ijTT}$ and $h_{ij}{}^{TT}$ are the modes which carry all the information describing the radiation in the wave zone. Thus, these quantities, which propagate according to ordinary wave equations in the wave zone, give one the radiation pattern, determine the energy being propagated, etc., for an *arbitrary* radiation situation. Within the wave zone, these modes are characteristically of $O(1/r)$. On account of Eq. (12), however, one has that *past* the wave front, these radiation modes must fall off as least as $O(1/r^{\frac{3}{2}+\varepsilon})$ in the frame of Eq. (13). However, it is easy to see that this "cutoff" condition on the "$TT$" modes is left invariant when one transforms out this frame to some arbitrary (asymptotically Cartesian) one. Thus, the physical arguments given at the beginning of this section can be verified in detail.

## III. NEWTONIAN LIMIT

As was mentioned at the beginning of Sec. II, the wave front theorem is a consequence of requiring that our system possesses finite energy. In fact,

(6) In the above derivation, a separate argument must actually be given to account for the *a priori* possibility that the sum of squares in Eq. (10) may behave as $\sim e^{ikr}/r^n$, $2 \leqslant n \leqslant 3$, or as $P_l(\theta)/r^3$, $l > 0$; either of these forms yields a contribution to $h^T \sim 1/r$ and would seem to require no cancellation from the $1/r^3$ parts of $C$. However, for a sum of squares, it is easy to show that a "wavy" term is necessarily accompanied by a "static" one of the some $O(r^{-n})$; and also that if there are any $P_l/r^3$ terms at all, there is one with $P_0/r^3$. Therefore, since such "dangerous" terms must necessarily occur in the sum of squares if it is at all to go as $1/r^3$, the argument in the text goes through as before.

the first term (the bracket) on the r.h.s. of Eq. (10) is essentially the linearized theory's expression for energy density, and thus condition (12) just guarantees the finiteness of the energy in the wave zone (where linearized theory is a good approximation). According to the canonical formulation of the *full* theory [2] the total energy, $E$, can be obtained from the coefficient of the asymptotic $\frac{1}{r}$ part of $h^T$ according to

$$h^T \sim \frac{E}{4\pi\gamma}. \tag{14}$$

Thus, the full theory's energy is given by the integral of the entire r.h.s. of Eq. (10) and may be shown to be a constant of motion. That this definition of energy is a valid one can be seen from a number of arguments [1]. We discuss one here which is a consequence of the previous analysis.

Let us consider a test particle at spatial infinity moving slowly with respect to some Lorentz frame there. In Newtonian physics, the particle's force law reads

$$\frac{d^2x^i}{dt^2} = \varphi_{,i} \sim -\frac{\gamma m}{r^2} \tag{15}$$

where $\varphi$ is the Newtonian potential. In general relativity, the geodesic equation for this situation reduces to

$$\frac{d^2x^i}{dt^2} \cong -\Gamma^i_{00} \tag{16}$$

where

$$\Gamma^i_{00} \cong -\frac{1}{2}h_{00,i} + [h_{0i,0} + h_{0\mu}(h_{0\mu,0} - \frac{1}{2}h_{00,\mu}) + \ldots]. \tag{17}$$

Now for our class of asymptotically flat metrics, one may show that it is always possible to find a frame such that

$$h_{0i,0} \sim O(1/r^{2+\varepsilon}), \quad h_{00,0} \sim O(1/r^{1+\varepsilon'}), \quad (\varepsilon, \varepsilon' > 0)$$
$$h_{00,i} \sim O(1/r^2). \tag{18}$$

The derivation of Eq. (18) is a direct consequence of the wave front theorem. In such a frame, the general relativistic force law (16) takes on a form (and order) identical to the Newtonian one (through order $1/r^2$) with potential $\varphi = \frac{1}{2}h_{00}$. What has been established is the physically desirable result that: *there always exists a "correspondence principle" limit to Newtonian physics in the asymptotic domain when the metric approaches* $\eta_{\mu\nu}$ *as* $O(1/r)$ *in some* *frame*[7]. Now, according to the equivalence principle, all the inertial mass of the interior gravitating system should manifest itself as gravitational mass attracting the test body at infinity. Thus, if in the frame of Eq. (18) we write

$$\varphi_{,i} = \frac{1}{2} h_{00,i} \sim -\frac{\gamma E}{r^2}, \tag{19}$$

the parameter $E$ should correctly be the energy of the interior system, i.e. one may read off the energy from the coefficient of the $\frac{1}{r}$ part of $h_{00}$. However, in frame (18), it is possible to show that the metric is asymptotically Schwarzschildian, and hence obeys the relation $h_{00} \sim \frac{1}{2} h^T$. From Eq. (14), then, one has[2]

$$\frac{1}{2} h_{00} \sim \frac{1}{4} h^T \sim \frac{E}{16\pi r} \sim \frac{\gamma E}{r} \tag{20}$$

where $E$ is now the canonical theory's definition of energy, obtained from the asymptotic part of $h^T$. Thus, the canonical theory's definition correctly agrees with the physical definition in terms of the asymptotic validity of the Newtonian force law. It is, of course, possible to make coordinate transformations which modify the $\frac{1}{r}$ part of $h_{00}$. Under these circumstances one can no longer use this term to calculate the energy of the system. However, any such transformations will also introduce non-Newtonian terms into the geodesic equation's force law. Thus, the energy can be correctly determined from the asymptotic part of $h_{00}$ *only* in the class of frames in which the system looks Newtonian asymptotically, and one may always recognize, from the geodesic equation, when this is the case.

## IV. LORENTZ TRANSFORMATION PROPERTIES OF ENERGY AND MOMENTUM

The discussion of Sec. III allows one to formulate a criterion for energy of the gravitational system which identifies it uniquely in a fixed asymptotic Lorentz frame: we require that an expression for energy agree with the coefficient of the $\frac{1}{r}$ part of $\frac{1}{2} h_{00}$ in a "Newtonian frame" (i.e. one obeying con-

[7] It should perhaps be emphasized that in the usual discussion of correspondence to the Newtonian limit, the result (18) is implicitly *assumed* by restricting oneself to (asymptotically) Schwarzschildian systems. The point of the present discussion has been to show that, indeed, a satisfactorily wide class of physical systems have an asymptotic "correspondence" limit.

dition (17)), and be numerically invariant under any transformation to another frame related to a Newtonian one by a coordinate transformation such that $\eta^{\mu}{}_{,\nu} \to 0$ asymptotically (and, of course, maintaining our boundary conditions, etc.). That the canonical formalism's definition obeys the first half of the criterion was discussed in Sec. III. The invariance of the coefficient of the $\frac{1}{r}$ part of $h^T$ (for coordinate transformations that leave the Lorentz frame at infinity unchanged) can also be established. The simplest way of proving this is to start in a Newtonian frame and explicitly make an arbitrary transformation (of the required type) out of it. One finds, in fact, that $h^T$ is preserved numerically to $O(1/r^{2-\varepsilon})$, $\varepsilon > 0$.

Once an expression for the energy has been obtained, the basic principles of Lorentz-covariant physics lead, as for other systems, to a criterion which *uniquely* determines the momentum, $P^i$, of the gravitational field. Thus, one requires that $P^i$ be defined in such a way that the four quantities $E$, $P^i$ transform as a Lorentz four-vector under any transformation that is asymptotically Lorentz (and, of course, maintains our boundary conditions, etc.). According to the canonical formalism [1] [2] the momentum of the gravitational field can be obtained from the $\frac{1}{r}$ part of $\pi^i$ in the orthogonal decomposition (9b), according to

$$\pi^i \sim \frac{1}{4\pi r}\frac{1}{12}\left[5P^i + \frac{1}{2}\left(\frac{3x^ix^j}{r^2} - \delta_{ij}\right)P^j\right]. \tag{21}$$

As was the case with $E$, the coefficient $P^i$ can be shown to be numerically invariant under coordinate transformations that have $\eta^{\mu}{}_{,\nu} \to 0$ asymptotically. More generally, it can be shown explicitly that $E$ and $P^i$ do indeed transform as a Lorentz four-vector under any transformation which is asymptotically a Lorentz transformation. (Again, the derivation of these theorems is most easily achieved by starting in a Newtonian frame and transforming out to an arbitrary frame.) These results show that the $E$ and $P^i$ of the canonical formalism correctly and uniquely represent the energy and momentum of the gravitational field.

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# A FOUR-DIMENSIONAL SYMMETRICAL CANONICAL FORMALISM IN FIELD THEORY

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THE standard formulation of Hamiltonian theory possesses a strong asymmetry with respect to time and space coordinates in formal contradiction to the spirit of the General Relativity. It is shown in this report that a four-symmetrical extension of the canonical Hamiltonian formalismfrom ordinary mechanics into field theory is possible. The single mechanical parameter (time $t$) has to be split into 4 independent parameters (coordinates $x^{\mu}$) and canonical velocities must be split in an analogous way. On the other hand, instead of the $3N$ coordinates of classical mechanics, field theory conteins all components of the field potentials on the same footing.

We have therefore a new contravariant index for our "canonical momentum" (no connection to the conventional energy-momentum!)

$$\Pi^{B\alpha} = \frac{\partial \mathfrak{L}}{\partial \mathcal{A}_{B,\alpha}},$$

index $B$ running through all wave functions.

Then the canonical energy-momentum quasi-tensor density is written in the form

$$f^{\alpha}_{\beta} = \Pi^{B\alpha} \mathcal{A}_{B,\beta} - \mathfrak{L}\, \delta^{\alpha}_{\beta}.$$

Using Stokes' theorem one can obtain the following expression for a variation of the energy-momentum integral:

$$\delta P_{\beta} = \int \left[ \mathcal{A}_{B,\beta} \delta \Pi^{B\alpha} - \Pi^{B\alpha}{}_{,\beta} \delta \mathcal{A}_B \right] dS_{\alpha};$$

from this follow the four-symmetrical Hamiltonian equations:

$$\frac{\overset{\alpha}{\Delta} P_{\beta}}{\Delta \Pi^{B\gamma}} = \delta^{\alpha}_{\gamma} \mathcal{A}_{B,\beta}, \quad \frac{\overset{\alpha}{\Delta} P_{\beta}}{\Delta \mathcal{A}_B} = -\Pi^{B\alpha}{}_{,\beta}.$$

Here variational derivatives for a functional on a hypersurface are introduced. These derivatives differ from the conventional ones only in an additional index (here $\alpha$).

In this formalism it is easy to formulate the Poisson brackets

$$\frac{\partial F(\mathscr{A}_B;\Pi^{B\alpha})}{\partial x^\nu}=\{F,P_\nu\}=\frac{1}{4}\frac{\partial F}{\partial \mathscr{A}_B}\frac{\overset{\alpha}{\Delta}P_\nu}{\Delta \Pi^{B\alpha}}-\frac{\partial F}{\partial \Pi^{B\alpha}}\frac{\overset{\alpha}{\Delta}P_\nu}{\Delta \mathscr{A}_B}.$$

The factor $\frac{1}{4}$ has to be mentioned as reflecting the number of canonical parameters (coordinates of space-time); in classical mechanics a single parameter $t$ leads to the usual factor unity.

It is worth mentioning that not only the energy-momentum integral but also the generalized spin integral [1] is of importance for Poisson brackets leading to the transformation coefficients cf the canonical coordinates (components of the field potentials):

$$\{\mathscr{A}_B;S^{\alpha}_{\beta}\}=\mathscr{A}_C a^{C\alpha}_{B\beta}.$$

The Hamilton–Jacobi equations can also be formulated in the present formalism.

It is also possible to introduce canonical transformation which may simplify computational processes in some cases.

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# ON A CHARACTERIZATION OF NON-DEGENERATE STATIC VACUUM FIELDS BY MEANS OF TEST PARTICLE MOTION

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THERE are two definitions of static fields in Einstein's theory:

A normal hyperbolic Riemannian 4-space is called static

1. if it admits a one parametric group of isometric motions whose trajectories are time-like and hypersurface-orthogonal, or

2. if there exists a cloud of test particles moving rigidly and irrotationally, and if the acceleration vector is fermi-propagated along the world lines [1]. It should be noted that these test particles need not travel along geodesics, because there may be non-gravitational forces between them.

In principle, methods of experimental verification of static fields are given by both definitions. In the first case a suitable chosen observer can measure the metric form at various times making use of clocks and rigid rods. Thus, the first definition is a geometrical one. In the second case, the static field is verified by the observation of relative velocities in a cloud of test particles; in this sense the second definition is a kinematical one.

Here we want to treat the following problem: What conditions on the kinematical properties of test particles are necessary in order to ensure that a vacuum field (special Einstein space) is static?

The answer is given by the following theorem:

Theorem: A cloud of test particles moving shearfree and irrotationally in a non-degenerate vacuum field must be rigid; the field itself must be static.

In other words: if, for a cloud of test particles moving shearfree and irrotationally with four-velocity $u_a$, the expansion scalar $\theta \equiv u^a{}_{;a}$ does not equal zero, then the field must be type D. We remark that a vacuum field admitting a timelike vector field with vanishing shear and rotation is always type I or, in the case of degeneracy, type D.

The theorem is proved in the following way:(1)

A. We consider a non-degenerate special Einstein space admitting a time-like vector field $u_a$ ($u_a u^a = -1$) whose trajectories form a rigid irrotational congruence of curves. Then it follows from the Bianchi identities that the

(1) The detailed calculations of the proof are given in [2].

acceleration vector $\dot{u}_a \equiv u_{a;c}u^c$ is fermipropagated along $u_a$. Therefore, the field must be static.

B. Now we assume $u_a$ to be a timelike vector field the trajectories of which are shearfree and irrotational. The special Einstein space is assumed nondegenerate. In view of A, we assume the expansion scalar $\theta$ to be different from zero. Then from the Bianchi identities we conclude that $u_a$ is tangent to the trajectories of a conformal motion, i.e. there exists a vector field $\xi_a$ collinear with $u_a$ obeying the equations

$$\xi_{a;b}+\xi_{b;a}-\frac{1}{2}\xi^c{}_{;c}\,g_{ab}=0.$$

C. We consider a special Einstein space (degenerate or not) admitting a one-parametric group of conformal motions, whose trajectories are hypersurface orthogonal. Using the Einstein conditions, we can show that the space-time must be flat, if the motion is not an isometric one. Now, combining the results of A, B, and C we obtain the proof of our theorem.

The theorem shows that the existence of a cloud of test particles with a particular kind of motion imposes rather stringent conditions on its kinematical behaviour and on the vacuum field in which it is moving. A result similar to this was obtained by C. B. Rayner [3]. He proved, in particular, that a rigid body in a vacuum field cannot alter its angular velocity.

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# THE LIGHT CONE AT INFINITY

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**Abstract**—From the point of view of the conformal structure of space-time, "points at infinity" can be treated on the same basis as finite points. Minkowski space can be completed to a highly symmetrical conformal manifold by the addition of a null cone at infinity—the "absolute cone". Owing to their conformal invariance, zero rest-mass fields can be studied on the whole of this conformal manifold. The behaviour of the fields on the absolute cone can be used to define the radiation fields. General relativity can be treated from this point of view if the gravitational field is represented by an expression originally equal to the Weyl tensor but which transforms slightly differently under conformal transformation.

QUESTIONS concerning radiation, or asymptotic flatness of space-time, involve statements about events in the "neighbourhood of infinity". It would appear, therefore, that some deeper understanding of the mathematical nature of this "infinity" might be of great conceptual value to physics. I wish to describe here an idea which seems to have been partially suggested by a number of people, and to me particularly by E. Schücking. The essential construction can apparently be traced back to Möbius [1].(1)

The idea is that if space-time is considered from the point of view of its conformal structure only, points at infinity can be treated on the same basis as finite points. It should be recalled that the concepts of angle, of the light cone at any event and a null geodesic, are conformally invariant and therefore pertinent to the conformal geometry of space-time. The concepts of infinitesimal distance and of space-like or time-like geodesics are not.

I shall be concerned here primarily with Minkowski space and its completion to a compact conformal manifold. The construction is analogous to the completion of a Euclidean plane to a projective plane by the addition of a "line at infinity", or alternatively, to its completion to an inversive plane by the addition of a "point at infinity". Some considerations applicable to curved space-time will be given at the end.

(1) Another article of much greater immediate relevance has recently been brought to my attention. This is: Hans Rudberg, The compactification of a Lorentz space and some remarks on the foundation of the theory of conformal relativity; dissertation (1957), University of Uppsala; Physics Abstracts No 30 Vol. 61 (1958). In it, the geometry of the conformally completed Minkowski space is discussed in some considerable detail.

Let $x^\mu$ be the position vector of a general event in Minkowski space-time relative to a given origin $O$. Then the transformation to new Minkowskian coordinates $\hat{x}^\mu$ given by

$$\hat{x}^\mu = \frac{x^\mu}{x_a x^a}, \qquad x^\mu = \frac{\hat{x}^\mu}{\hat{x}_a \hat{x}^a}, \tag{1}$$

is *conformal* ("inversion with respect to $O$"). Observe that the whole null cone of $O$ is transformed to infinity in the $\hat{x}^\mu$ system and that infinity in the $x^\mu$ system becomes the null cone of the origin $\hat{O}$ of the $\hat{x}^\mu$ system. ("Space-like" or "time-like" infinity become $\hat{O}$ itself but "null" infinity becomes spread out over the null cone of $\hat{O}$.) Thus, from the conformal point of view "infinity" must be a *null cone*. The two systems $x^\mu$, $\hat{x}^\mu$ related by (1) may be regarded as two coordinate systems, each of which covers part of a conformally flat and *compact* manifold $\mathcal{M}$. Together they do not cover quite the whole of $\mathcal{M}$ since the points at infinity on the null cone of $O$ are excluded in both coordinate systems, but these points can easily be covered by choosing a third coordinate system which is related to $x^\mu - a^\mu (a_a a^a > 0)$ in the same way that $\hat{x}^\mu$ is related to $x^\mu$ in (1).

The geometry of $\mathcal{M}$ is then briefly as follows. $\mathcal{M}$ contains $\infty^4$ points and $\infty^5$ null (straight) lines. The $\infty^2$ null lines through each point generate the null cone of that point. These null cones are all closed, the null lines being topologically circles. Each cone has just one vertex. $\mathcal{M}$ admits a transitive $\infty^{15}$ group of motions, so that all its points are on the same footing. Thus all these null cones are also on the same footing. If any one of these cones is chosen, it may be regarded as an *absolute cone* (cone at infinity) for a Minkowski metric structure consistent with the given conformal structure. The metrical concepts can all be defined in relation to this absolute cone. Thus, "parallel" null lines are null lines which meet the same generator of the absolute cone. If they meet at the same point of the absolute cone, they are not only "parallel" but they lie in the same "null hyperplane". Thus, a "null hyperplane" is simply a null cone whose vertex lies on the absolute cone (other than at its vertex). $\mathcal{M}$ also contains $\infty^9$ *space-like circles* and $\infty^9$ *time-like circles*—a time-like circle being, in general, the world-line of a uniformly accelerating particle (together with the other branch of the "hyperbola"). A space-like or time-like "straight line" is simply a space-like or time-like circle which passes through the vertex of the absolute cone. (Note the characteristic fact that on the other hand "null straight lines" will *not* pass through the vertex of the absolute cone, if "finite".) A limiting case of a space-like or time-like circle is a pair of intersecting null lines.

To picture $\mathcal{M}$ consider first the two-dimensional case. Imagine the whole of two-dimensional Minkowski space-time to be mapped continuously onto the interior of a square, with the null lines parallel to the sides (see Fig. 1).

Then infinity is represented by the sides of the square and to complete the picture, opposite sides must be identified preserving sense. The resultant compact manifold is topologically a torus. Next, consider the three-dimen-

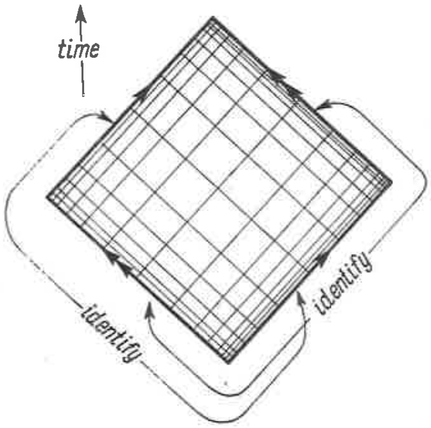


FIG. 1.

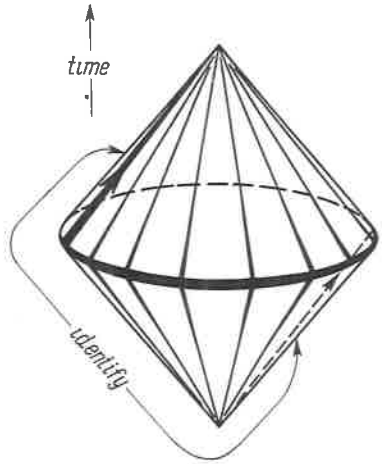


FIG. 2.

sional case (two space and one time dimension). This time we map the space-time continuously onto the interior of a region bounded by two portions of cones joined base to base (see Fig. 2). Each generator of the top cone is to be identified with the opposite generator of the bottom cone, preserving future sense. It follows that both the top vertex and the bottom vertex must be identified with the whole of the "equator"—which must therefore be considered as a point. The resultant compact manifold is non-orientable and has the topology of a three-dimensional analogue of Klein's bottle. The four-dimensional case is very similar, except that in this case the manifold turns out to be orientable again and has the topology $S^1 \times S^3$.

Note the fact that the removal of any null cone from $\mathcal{M}$ leaves a (simply-) connected set of points. This is most easily seen if the cone is chosen to be the absolute cone. Thus the three regions "past", "future", "elsewhere" into which a null cone divides normal Minkowski space are *connected* to each other in $\mathcal{M}$. There is, thus, no invariant distinction between a space-like or a time-like separation for two general points in $\mathcal{M}$.

The possibility of applying these ideas to physics rests on the fact that any zero rest-mass free field can be regarded (with a suitable interpretation) as being conformally invariant. Thus, under the conformal transformation

$$\tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}, \qquad (2)$$

$\lambda$ being a function of position, the source-free Maxwell equations, in particular, are preserved if we put

$$\tilde{F}_{\mu\nu} = F_{\mu\nu}.$$

For the case of a source-free spin-two field we have a tensor $K_{\mu\nu\varrho\sigma}$ with the symmetries and trace-free conditions of an empty-space Riemann tensor (or Weyl tensor) satisfying

$$K_{\mu\nu[\varrho\sigma;\tau]} = 0. \tag{3}$$

This is preserved under the conformal transformation (2) if we put

$$\tilde{K}_{\mu\nu\varrho\sigma} = \lambda\, K_{\mu\nu\varrho\sigma} \tag{4}$$

so that the field must be regarded as a suitable kind of density. The case of general spin is most easily treated in the two-component spinor formalism. For any spin $s > 0$ we can describe the field by a symmetric spinor $\theta_{AB\cdots E}$ with $2s$ indices satisfying

$$\theta_{AB\cdots E;\mu}\sigma^{\mu A\dot{P}} = 0. \tag{5}$$

Then, (5) is preserved under the conformal transformation (2) (accompanied by $\tilde{\sigma}_{\mu}{}^{A\dot{P}} = \lambda\sigma_{\mu}{}^{A\dot{P}}$) provided

$$\theta_{AB\cdots E} = \lambda^{-s-1}\theta_{AB\cdots E}.$$

Spin-zero, of course, requires special consideration.

Using this conformal invariance it is possible to give a meaning to the concept of a zero rest-mass field defined over the *whole* of $\mathcal{M}$. The condition that the field be *finite* on the absolute cone is a simple statement of an asymptotic condition that is reasonable to impose on the field. Also, it is known that initial data for such a field can be given on any null cone:—one complex number per point of the cone [2]. Hence, we can use the *absolute* cone on which to specify initial data. This initial data then simply measures the strength of the radiation field.

If interactions are to be present, we will not expect the incoming field to match the outgoing field. Therefore, the identification of the past infinity with the future infinity that was done to define the manifold $\mathcal{M}$ seems inadvisable in this case. If the identification is not carried out, we have two absolute cones which bound the Minkowski space, one in the past and one in the future. The comparison between the data on the past cone with that on the future cone determines a kind of $S$-matrix theory.

In general relativity, the Weyl tensor $C_{\mu\nu\varrho\sigma}$ in empty space satisfies the spin-two zero rest-mass free field equations (3), so we may set

$$K_{\mu\nu\varrho\sigma} = C_{\mu\nu\varrho\sigma}.$$

However, under transformation (2) we have

$$\tilde{C}_{\mu\nu\varrho\sigma} = \lambda^2 C_{\mu\nu\varrho\sigma}$$

which by comparison with (4) gives

$$\tilde{C}_{\mu\nu\varrho\sigma} = \lambda \tilde{K}_{\mu\nu\varrho\sigma} \tag{6}$$

so that the Weyl tensor transforms differently from a zero rest-mass spin-two field under conformal transformation. (This is perhaps not surprising, since Ricci tensor, i.e. "source" for the Weyl field, may be introduced by a conformal transformation). Thus, we must choose for our expression representing the gravitational field, not exactly the Weyl tensor, but the $K_{\mu\nu\varrho\sigma}$ which equals the Weyl tensor in the original metric space and which transforms according to (4) under conformal transformation. A reasonable asymptotic condition on the field—a condition of *asymptotic flatness*—can then be stated as the fact that $K_{\mu\nu\varrho\sigma}$ must be finite (and suitably regular) on the absolute cone(s). The gravitational field is a (self-)interacting field, so it seems unreasonable to carry out an identification of the infinite past with the infinite future in general relativity—we have two cones, one in the past and one in the future. The strength of the incoming, or outgoing, radiation field is then measured by the initial data for $K_{\mu\nu\varrho\sigma}$ on the past, or future, absolute cone.

From (6) it follows that the Weyl tensor *vanishes* on the absolute cones. This is very fortunate since it implies that the conformal structure of infinity is the same in such asymptotically flat curved spaces as in Minkowski space—with the one important difference that the (double) vertex of each absolute cone becomes singular and so is best removed. The absolute cones are then perhaps better thought of as *cylinders* ($S^2 \times E^1$). The possibility of an $S$-matrix theory for gravitation suggests itself.

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# ON THE BEHAVIOUR OF THE SCALE-FACTOR IN AN ANISOTROPIC NON-HOMOGENEOUS UNIVERSE

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THE chronometrical invariants and a semi-reciprocal method were used in the relativistic theory of anisotropic non-homogeneous universe. A possibility was shown for new (in comparison with homogeneous isotropic universe) types of temporal behaviour of the co-moving space and in particular for new behaviour types of the so-called scale-factor.

In particular the possibility of a combination of volume expansion of the co-moving space in one region with volume contraction in another one was found which means that the expansion of Metagalaxy is possibly of a local character although the non-stationarity being probably universal.

It is also proved that the scale-factor can pass through a regular minimum instead of a singularity, which means that a contraction of the Metagalaxy could lead to its expansion without any passage through a high density state. In this way the well-known time-scale difficulties of the theory could be eliminated.

In order to get a regular minimum of the scale-factor the presence factors of anisotropy, namely the absolute rotation or pressure gradient are required. A tendency of anisotropy factors of some to decrease with the expansion of the Metagalaxy could lead to observational inappreciability of such factors.

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# EXTRACTS FROM THE PLAY PRESENTED AT THE CONCLUDING DINNER

Chairman: B. S. DeWitt

B. S. DeWitt: Our first speaker is Professor Alfred Schild of the Billie Sol Estes Peoples Friendship University of Texas. Prof. Schild, who is unfortunately unable to speak the official language of the conference, feels that it would be more friendly to deliver his address in English. We are fortunate to have with us this evening an English speaking scholar, Professor Doctor Engelbert Schücking, also of the Billie Sol Estes Peoples Friendship University of Texas. Professor Schücking will translate Professor Schild's paper into the official language of the conference, American.

A. Schild: Mach's principle states: space is necessarily flat in the absence of physicists.

B. S. DeWitt: Translation, please:

E. Schücking: I translate: "Mach's principle states: the ideas of a physicist are completely determined by the literature he has read. At this conference I gave a beautiful example that this formulation is wrong. I produced an exact solution which showed that ideas can also be communicated orally. This solution will be published in the forthcoming 1960 volume of the memoirs of the Belgian Academy of Sciences. Since Mach's principle has to be true, we have to reformulate it slightly. It reads now, using the principle of equivalence and of general covariance, though not completely including Huygen's principle: all publications in the Physical Review are on an equal footing. A convincing example for the validity of this refined version has been published by Bergmann, Robinson and Schücking in the Physical Review—and by the Science News Service. Bergmann and coworkers found the famous gap in the Schwarzschild literature. They showed through several and independent lines of reasoning that curved space is not necessarily flat. This result that so far had never been proven rigorously showed how right Mach was to die before the advent of Mach's principle." (*Turning to Schild*) Was that the correct translation?

A. Schild: Exactly.

I. Robinson: Excuse me, please!

B. S. DeWitt: Excuse me!

I. Robinson (*with a fixed smile and a more audibly Russian accent than usual*): Excuse me, please. We are very happy, yes. It is a great pleasure to hear the friendly work of Professor Schild. Here in our country, yes, we are very interested in these problems. Many distinguished scientists work on them. These are my students. The others are just babies in relativity. I have the papers here. Excuse me, please. With your permission I translate my question into American, perhaps. I wish to say how happy I am to hear these problems so well formulated in terms of torsion.

R. P. FEYNMAN: It ain't that simple. You got to look at things physically, see! You got these particles, and they attract each other, see. They bounce photons off each other, they get closer, and they go round each other, see! It's a kind of screwing motion (*illustrative gestures*). The situation gets real complicated. You got to try to think of it physically. You can't describe all this by just one word: TOISION.

B. S. DEWITT: Are there any questions?

(*All rise and speak simultaneously*).

B. S. DEWITT: Since there are no further questions I call on Mme. Tonnere-Lichnerowicz to deliver the concluding report on this session.

R. P. FEYNMAN: Infeld ... relativité ... Infeld ... gravitation ... Infeld ... conférence ... Infeld ... Infeld.

(*Ed. note*: Professor Feynman's talk delivered in the classic tongue of Racine, Molière, Corneille, and J. L. Synge so affected the French-speaking members of the conference that the text recorded for posterity seems to be incomplete in several respects. See above).

B. S. DEWITT: A further comment on the conference will be provided by Professor Arnold Komar of the New York Theological Seminary.

A. KOMAR:

THE BALLAD OF THE ULTIMATE SECTARIAN

Jim Johnson was a student of physics
Gravitation was his specialty
To every Conference he would come
and this is what he'd say:

"Oh you may be a colleague of John Wheeler,
Ivanenko you may think fine,
You may have studied with Bergmann or Fock,
But you're no colleague of mine"

Jim Johnson hung up a shingle,
"Albert Einstein School" it did read
And to every student who knocked at his door
He would propound this creed:

"Oh you may be a colleague of Møller,
Lichnerowicz you may think fine,
You may agree with Infeld or Synge,
But you're no colleague of mine!"

Jim Johnson died and went to heaven,
Einstein welcomed him with outstretched hands,
But Jim Johnson refused to collaborate
And firmly announced his stand;

"Oh you may be a colleague of Christcffel,
Emmy Noether you may think fine,
You may have studied with Riemann or Weyl,
But you're no colleague of mine!"

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# AUTHORS' INDEX

Bold-face figures denote lectures, ordinary figures denote seminars and contributions to the general discussions, while figures in italics denote contributions to the discussions after lectures.