Black Hole Branes, Fluids and Entropy

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Abstract

General relativity is a thermodynamic theory that is consistent with the holographic principle. However, the precise manner in which this description emerges from an underlying theory is unknown. In this study we concentrate on the relation between fluids, gravity and the membrane paradigm as a window onto further elucidating this link. In the membrane paradigm in general relativity, black holes are viewed holographically, by excising their interior, truncating fields on the surface, and imbuing the horizon with surface properties including fluid viscosity, electrical resistivity, and thermal dissipation. The close connection between fluid and thermal behavior may also be seen by examining scaling symmetry in the Navier-Stokes equations, and considering these as perturbations to a system in thermal equilibrium, which in this case is general relativity. A better understanding of these connections may be a stepping stone towards a more complete description of gravity in "non-equilibrium spacetime".

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened.

Douglas Adams,
The Hitchhiker's Guide to the Galaxy

Contents

1 Introduction

General relativity is an effective but universal thermodynamical theory. We require the equivalence principle and local Lorentz invariance, and from this, any sufficiently coarse-grained thermal data on a boundary hypersurface evolves into the full Einstein equations in the bulk. An energy flux TdS transverse to the spatial components of the hypersurface evolves according to the Raychaudhuri-Landau equation, and a tangential momentum flux PdA is governed by the Damour-Navier-Stokes equations. The bulk Einstein equations correspondingly yield thermal and fluid behavior when constrained to the boundary, and the location of this boundary is variable; it can be null, spatial or timelike. The Bekenstein-Hawking entropy $S = A/4G\hbar$ is a thermal equilibrium condition for Einstein gravity, and is the ratio of the horizon fluid pressure to the Unruh temperature. The ratio of shear viscosity to entropy $\eta/S = 1/4\pi$ is constant when this equilibrium is satisfied. The horizon entropy and fluid pressure are also linked, implying a thermodynamical entropic force.

Chapter 1 covers geometric and background topics, fixing conventions and providing a brief introduction with the aim of increasing accessibility. Chapter 2 introduces holography, and beginning from a past null boundary in Rindler space, derives the Einstein equations in the bulk using a combination of the thermodynamic Clausius principle, holography, Lorentz invariance, the equivalence principle, and the Raychaudhuri equation, which partially governs the dynamics of null curves. In chapter 3 we project the Einstein equations from the bulk onto a future null horizon and show the 4-dimensional Einstein tensor is equivalent to a combination of the Damour-Navier-Stokes equations and the Raychaudhuri equation on a hypersurface. By examining the Raychaudhuri equation, a connection with thermodynamics is seen which is complementary to the derivation of Einstein's equations in chapter 2. The fluid pressure is then used to extend the thermal approach of chapter 2, replacing the boost Killing vector with an angular Killing vector, and the energy flux with momentum flux. Here the Unruh temperature is assumed and the holographic scaling of entropy derived. Chapter 4 extends the connection between general relativity, the Navier-Stokes equations and thermodynamics to timelike surfaces which lie in the bulk spacetime. By incorporating the Israel junction condition, valid for timelike or null surfaces, the relation between the Bekenstein-Hawking entropy, stress tensor, and dynamical entropy contributions are clarified. A fluid ansatz for the scaling of derivative expansions of the stress tensor is presented, which offers an alternate, axiomatic, perspective where fluids comprise part of a 2nd order approximation to what is presumably a general dual field theory. The action principle membrane approach offers a similar view, but arising from effective field theory. A deeper study of the close relation between fluids and gravity is briefly introduced through the example of turbulence in the Navier-Stokes equations, which is closely related to the conformal structure of null surfaces, the evolution of the expansion, shear and momenta, and, if the horizon area changes, the promotion of the entropy to an entropy current, which together correspond to the geometric turbulence of gravity. This, along with the enigmatic relation to entanglement entropy, will be topics for the future.

This thesis was originally inspired by the "Fast Scramblers" paper by Y. Sekino and L. Susskind [12], which itself builds on "Black holes as mirrors: quantum information in random subsystems" by P. Hayden and J. Preskill [11]. The goal of these papers is to find the maximum rate that Hawking radiation can release information from black holes. The bound on information retrieval can be viewed through two complementary but distinct mechanisms; the idealization of a black hole as a quantum information theoretic system, obeying the laws of ordinary quantum mechanics and providing a retrieval rate boost through the entanglement of subsystems (an external observer, infalling information, and outgoing Hawking radiation), or by considering the black hole and observer as a holographic system, where surface properties of the black hole determine the rate infalling information is thermalized. The first approach is closely related to *black hole complementarity* [16], postulating that infalling observers see unitarity evolution, and the second to external observers in the *membrane paradigm* [6] [8]. That both approaches give compatible answers is indicative of a deep link between them. In this thesis *holographic* [2] [3] is taken to mean the scaling of entropy with surface area, $S = A/4G\hbar$, which differs from the statistical behavior of matter in ordinary (nonrelativistic or special-relativistic) situations, where entropy scales as the volume. Black hole thermodynamics and the membrane paradigm display holographic scaling explicitly [6] [8], as does the AdS/CFT correspondence [17], black hole solutions in matrix theory [18], and loop quantum gravity [19].

2 Background

This thesis unifies the membrane paradigm notation used by different authors by using Kruskal-Szekeres coordinates, in which Rindler space appears as the near-horizon or infinite mass limit. There are significant differences in the approaches adapted from source material, due to a combination of differing aims and the fact that the papers span about 40 years, from the 1970's to 2013. The classical membrane paradigm is due primarily to Damour and Thorne. Damour uses a customized coordinate system which is somewhat nonstandard, begins with Eddington-Finkelstein coordinates and is expressed in terms of the generators to the black hole's null horizon. Thorne's primary aim with the membrane paradigm was to develop it into a useful tool for astrophysics, and to this end he reexpressed Damour's null horizon approach using the 3+1 formalism on a *stretched horizon* located a timelike distance outside the null horizon. The 3+1 formalism may also have been chosen because it is closer to the 3+1 formalism for quantizing the gravitational Hamiltonian, favored at the time by Wheeler and others. More recent authors, including Jacobson and Strominger, have used Rindler space since it applies generally to null horizons.

The choice of Kruskal-Szekeres coordinates is motivated by their ability to display the global behavior of Schwarzschild geometry, including the region behind the horizon. Although the membrane paradigm was originally formulated to explicitly exclude this region, the picture of having data on a causal or spacelike hypersurface and asking how it interacts with bulk spacetime is a complementary view to distributing information freely on both sides, and gives the membrane paradigm a more modern interpretation.

An important point is that the Schwarzschild metric is nonrotating, and in order for the membrane paradigm to be nontrivial an angular velocity is needed. This potential conflict is avoided because the membrane paradigm is derived based on symmetries of the Einstein tensor and can then be applied to any individual solution. Kruskal-Szekeres coordinates are well adapted to null horizons and are examined locally. We are then free to postulate the existence of global Killing vectors to normalize Kruskal-Szekeres (or any other) coordinates to have an angular velocity.

2.1 Variables and Indices

$x^i = (u, v, \theta, \phi)$ Spans (0,3) in bulk

$x^\mu =$ Spans (0,2,3) or (1,2,3) on boundary, with r or $x^0 = u$ constant

$x^A =$ Spans spatial slices (2,3)

$\eta_{ij} =$ Minkowski metric of signature $(- + ++)$

$g_{ij} =$ Metric

$h_{\mu\nu} =$ Induced metric on hypersurface

$\gamma(x^i) =$ Geodesic curve

$\lambda =$ Geodesic parameter. Not always affine.

$n =$ Parameter for selection between geodesics

$n^i =$ Normal vector to hypersurfaces

$k^i =$ Secondary null vector for null hypersurfaces

$(k^a n_a = -\alpha, n_i = \alpha du, k_i = \alpha dv) =$ Definition of Null Normals

$e^i_\mu = \dfrac{\partial x^i}{\partial x^\mu}$ Tangent vector on hypersurface

$u^i(\lambda) = \nabla_\lambda x^i$ Tangent to geodesics

$\xi^i(n) =$ Deviation vector between geodesics

$\mathcal{L}_X Y = [X, Y]$ Lie derivative of Y in X direction

$K_{ij} =$ Extrinsic curvature

$R^i_{jkl} =$ Reimann curvature tensor

$R_{jl} = R^a_{jal} =$ Ricci tensor

$R =$ Curvature scalar

${}^3R =$ Curvature scalar on hypersurface

$(t, r, \Omega) =$ Schwarzschild coordinates

$(U, V, \Omega) =$ Null Kruskal-Szekeres coordinates

$(T, X, \Omega) =$ Minkowski coordinates

$(u, v, \Omega) =$ Null Rindler coordinates

$\Omega = (\theta^2 + \sin^2\theta\phi^2)^{1/2} =$ Solid angle

$b =$ Schwarzschild radius

$r^* =$ Schwarzschild tortise coordinate

$\kappa =$ Gravitational or inertial acceleration

$\rho =$ Proper distance from event horizon

$\tau =$ Rindler time

$\tau_E =$ Euclidean time

$J =$ Charge density

$L =$ Angular momentum

(2.1)

2.2 Manifolds

A *manifold* can be roughly defined as a k-dimensional subset $M \subseteq \mathbb{R}^n$ that locally looks like $\mathbb{R}^k$ Euclidean space. It has a *Hausdorff* topology and can be covered by *diffeomorphism* invariant coordinate charts.

To be *Hausdorff* implies that for two points $a, b \subset M$ with distinct neighborhoods P_a and P_b, the neighborhoods are separable, so that their intersection is the null set $P_a \cap P_b = \emptyset$.

A function on M is *smooth*, or of *class* C^n, if its partial derivatives up to and including order n exist and are continuous.

A *diffeomorphism* is a bijective function $f : M \to M$ with a smooth inverse map.

In General Relativity we are interested in *psuedo-Riemannian* or *Lorentzian* manifolds, comprised of the pair (M, g). [1] These have the additional structure of a Lorentz metric g. Space is therefore locally Lorentzian, rather than Euclidean. The metric used for bulk geometry is g_{ij}, and η_{ij} is flat Minkowski space with metric signature $(-+++)$.

A *submanifold* is a subset $N \subseteq M$. It is defined by its *codimension*, which is just the difference between the dimension of M and dimension of N. A "codimension one" submanifold is usually referred to as a *hypersurface*.

Lowercase Latin indices $(i, j, ...)$ span $(0, 3)$ in the bulk.

Greek indices $(\mu, \nu, ...)$ are used on submanifolds.

Uppercase Latin letters isolate the spatial slices $I = (2, 3)$.

Contracted indices use letters starting from the beginning of each alphabet $(a, \alpha, ...)$.

The ordinary derivative is denoted by $\partial_i A^j$ and comma notation $A^j_{,i}$.

The covariant derivative is metric compatible and defined so that

$$\nabla_i A^j = A^j_{;i} \equiv \partial_i A^j + \Gamma^j_{ia} A^a, \tag{2.2}$$

with connection Γ given by Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{ka} \left(\partial_i g_{ja} + \partial_j g_{ia} - \partial_a g_{ij} \right). \tag{2.3}$$

The directional covariant derivative on a curve $\gamma^i(\lambda)$ along a tangent vector $u^i(\lambda)$ is

$$DA^i / d\lambda = \nabla_u A^i = u^a A^i_{;a}. \tag{2.4}$$

[1] The process of singularity formation, proven by Penrose and Hawking to occur general relativity [23] [25], violates the smoothness condition. This indicates that general relativity is not completely self-consistent. However, we will in general not work in the vicinity of singularities.

The Riemann curvature tensor, which results from parallel transporting a vector in a closed curve along directions u and v, is defined as

$$R^i_{abc} n^a u^b v^c = (\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}) n^i \tag{2.5}$$

This expression holds for any u and v, so the deviation of n in component form is

$$R^i_{abc} n^a = n^i_{;cb} - n^i_{;bc}. \tag{2.6}$$

2.3 Pushforward, Pullback and Flow

Given points on two manifolds, $p \in M$ and $q \in N$, the vectors on their tangent spaces $X \in TM$ and $Y \in TN$, and a smooth bijective function $\phi : M \to N$, the vector Y^i is called the *pushforward* of X^i by ϕ. It is a composition from

$$N \to M \to TM \to TN$$

and can be denoted

$$(\phi_* X)(q) = d\phi(\phi^{-1}(q))X(\phi^{-1}(q)).$$

Using coordinate notation, with $\phi(x^i) = y^j$, the pushforward on a $(1,0)$ tensor is a matrix of partial derivatives defined by

$$(\phi_* X)^j = \frac{dy^j}{dx^i} X^i.$$

Given a diffeomorphism ϕ we can push X forward to Y, instead of specifying Y as a map from N to its cotangent space.

Similarly, given cotangent spaces T^*M and T^*N dual to TM and TN, X_i is the *pullback* of Y_i by ϕ.

$$(\phi^* Y)(p) = d\phi^{-1}(\phi(p))Y(\phi(p))$$

The action of the pullback on a $(0,1)$ tensor is

$$(\phi^* Y)_j = \frac{dx^i}{dy^j} Y_i.$$

The *flow* along an integral curve can be viewed as a series of infinitesimal diffeomorphisms which pull back a tangent vector along the curve.

2.4 Hypersurfaces

Most of the machinery of differential geometry was developed for Riemannian manifolds with positive definite metric signatures. In these cases tools like the extrinsic curvature, Gauss, Codazzi and Ricci equations can be defined uniquely once normal vectors are selected. However, since Lorentzian manifolds have degenerate submanifolds (i.e. $g_{ab}n^a n^b = 0$) this creates a larger equivalence class that makes it impossible to distinguish certain quantities in the vicinity of null surfaces. For this reason we consider timelike and spacelike surfaces separately, and then adapt the results to null surfaces. The degeneracy of null surfaces, in particular the equivalence of tangent and normal vectors, is closely related to the emergence of fluid and thermal behavior, and also to conformal invariance.

A *hypersurface* is a submanifold of codimension one. We define this either by specifying a constraint $f(x^i) = 0$, or using an *induced metric* to pull back from a larger manifold to a submanifold.

This has the general form

$$h_{\mu\nu} = g_{ij} e^i_\mu e^j_\nu. \tag{2.7}$$

Its inverse is $(h_{\mu\nu})^{-1} \equiv h^{\mu\nu}$, and the notation $h \equiv \det h_{\mu\nu}$, and $1/h \equiv h^{-1}$. In differential geometry the induced metric is also referred to as the *first fundamental form*. As a pullback of g_{ij} by ϕ this is symbolically written

$$h_{\mu\nu} = (\phi^* g)_{\mu\nu} = \frac{\partial x^i}{\partial y^\mu}\frac{\partial x^j}{\partial y^\nu} g_{ij},$$

which also gives a parametric form for the tangent vectors e^i_μ. Distances on the hypersurface are defined as

$$ds^2 = h_{\mu\nu} dy^\mu dy^\nu.$$

Covariant derivatives can also be pulled back to hypersurfaces. We distinguish the dimensionality of a covariant derivative by its Latin or Greek indices. For example:

$$\nabla_\nu A_\mu = \phi^*(\nabla_b A_a)_{\mu\nu} = \frac{\partial x^a}{\partial y^\mu}\frac{\partial x^b}{\partial y^\nu} A_{a;b}.$$

We can associate hypersurfaces with normal vectors as follows: If g_{ij} is a Lorentzian metric and n^i is a timelike vector, then $g_{ab}n^a n^b < 0$. Tangent vectors orthogonal to n^i are spacelike and define an induced metric that is positive definite, which we therefore call a *spacelike hypersurface*. If n^i is spacelike, then $h_{\mu\nu}$ is a Lorentzian metric, and defines a *timelike hypersurface*.

For spacelike and timelike hypersurfaces the induced metric is

$$h_{\mu\nu} e^{\mu}_{i} e^{\nu}_{j} = g_{ij} - n_i n_j, \tag{2.8}$$

where the normal is unique up to sign and may be locally written as

$$n_i = \pm \alpha \mathrm{d} f,$$

for timelike and spacelike hypersurfaces, respectively. A directed area element on spacelike or timelike hypersurfaces is

$$d\Sigma_i = n_i dA = n_i \sqrt{h} d^3 y.$$

For null vectors we define normals as

$$n_i = \alpha \mathrm{d} f. \tag{2.9}$$

If n^i is null, then the induced metric is degenerate and defines a *null hypersurface*. Since null vectors are orthogonal to themselves, $g_{ab} n^a n^b = 0$, they are also tangential to null hypersurfaces. These tangents are known as *null generators* of the horizon, and are geodesics, as

$$\frac{Dn^i}{d\lambda} = n^a \nabla_a n^i = \kappa n^i.$$

Here the scalar field α has been introduced specifically so that λ will not always be affine, which lets us normalize the surface gravity κ. If α is constant then the differential form n^i is closed, as $ddf = 0$. As long as we can *foliate* the null hypersurface (see 3.2.3), for instance using the constraint $f(x^i) = u$ to specify a family of null surfaces, then λ can always be chosen as affine and $\kappa = 0$.

In the null case, an induced metric of the form used for timelike and spacelike hypersurfaces is not orthogonal to normal vectors, since $h_{ij} n^i = n_j$. To compensate for this we introduce a second, *auxiliary*, null vector which we define to have a timelike inner product with respect to n_i, as

$$k_a n^a = \alpha < 0. \tag{2.10}$$

Then, to isolate tangential components we have an induced *transverse metric*

$$h_{AB} e^{A}_{i} e^{B}_{j} = g_{ij} + k_i n_j + n_i k_j. \tag{2.11}$$

Note that the trace of this induced metric is $h^A_A = 2$. To avoid introducing additional notation, when the induced metric is degenerate we define the determinant of the spatial submanifold as $h \equiv \det h_{AB}$. Null hypersurfaces are of codimension 2, as seen by the trace, and are spanned by two null and two spatial vectors. They have topology $\mathbb{R} \times S^2$.

The inner product of the null vectors is directed in the timelike direction, which lies on the light cone of the spatial 2-surface. Using k and n we can construct a *psuedo-orthonormal* basis $\{e^1 = k(\lambda), e^2 = n(\lambda), e^3 = \theta, e^4 = \phi\}$, which is the minimum amount of structure necessary. We can parallel transport this basis along n^i to cover the null surface. Then, since normal vectors are geodesics on the null surface, we can parameterize it using coordinates $y^\mu = (\lambda, \theta, \phi)$, so that an area element is

$$d\Sigma_i = -n_i \sqrt{h} d\lambda \wedge dy^2 \wedge dy^3.$$

The spatial part of the null surface is the 2-sphere, as the original metric breaks apart into null vectors and the angular component $r^2 d^2\Omega$. By the *uniformization theorem* of differential geometry, all simply connected Riemannian surfaces are conformally equivalent to either the unit disk (*hyperbolic*: constant negative curvature), the complex plane (*parabolic*: zero curvature), or the Riemann sphere (*elliptic*: constant positive curvature). The 2-sphere has constant positive curvature and is conformally equivalent to the Riemann sphere. The holographic entropy, mentioned in the introduction, is proportional to these components of the metric. The two null vectors form a trace-free submanifold, which is therefore conformally invariant.

We can also isolate particular transverse components using a projection operator onto a subspace of the null hypersurface. For instance

$$\Pi^\mu_\nu e^i_\mu e^\nu_j = g^i_j + k^i n_j. \tag{2.12}$$

This is parallel to n^j and k_i, but orthogonal to n_i and k^j. In terms of the transverse metric

$$\Pi_{\mu\nu} e^\mu_i e^\nu_j = h_{ij} - n_i k_j \tag{2.13}$$

If we apply this projector to a 1-form, for instance the contracted Ricci tensor $R_{aj} n^a$, we get two terms, one along the null normal and another proportional to the spatial 2-surface.

$$R_{ab} n^a \Pi^b_\mu = -R_{ab} n^a n^b k_\mu + R_{ab} n^a h^b_\mu \tag{2.14}$$

The first term on the right hand side contains the null Raychaudhuri equation, and the second term the Damour-Navier-Stokes equation, but to interpret them we need more tools. The analogous situation also occurs in the non-null cases.

2.5 Lie Derivatives and Flow

Lie derivatives generalize directional derivatives of vector spaces to manifolds, and represent the derivative of a vector or tensor space along another vector space. Since Lie derivatives do not depend on the metric, they are more primitive than the covariant derivatives. Killing vectors, the extrinsic curvature, and a host of other useful operations rely on Lie derivatives. Since the tangent space T_pM of each point on a manifold has a unique vector space associated with it, in order to compare vectors at different points we need a method of moving both expressions to the same tangent space. This is done by pulling back one of the tangent spaces along the flow of the integral curves of a vector field; differentiation then proceeds normally.

Consider vector fields X and Y, and a flow ϕ^* generating integral curves to Y. The *Lie derivative* of a vector field X along Y is

$$\mathcal{L}_Y X = \lim_{\lambda \to 0} \frac{\phi_\lambda^* X^i(\phi_\lambda(x)) - X^i(x)}{\lambda}. \tag{2.15}$$

The first term is the pullback of X along the flow of Y. Fortunately for computational purposes, we can express Lie derivatives (as their name indicates) of smooth vector fields using Lie brackets. The Lie derivative of X along Y is equivalently written as

$$\mathcal{L}_Y X = [Y, X]. \tag{2.16}$$

All of these definitions are directly generalized to tensor fields. Promoting X to a tensor field $X^{i\dots}_{j\dots}$

$$\mathcal{L}_Y X^{i\dots}_{j\dots} \equiv \frac{DX^{i\dots}_{j\dots}}{dt} = Y^a \partial_a X^{i\dots}_{j\dots} + X^{i\dots}_{a\dots} \partial_j Y^a - X^{a\dots}_{j\dots} \partial^i Y_a + \dots \tag{2.17}$$

Applying this to the metric is quite useful:

$$\mathcal{L}_Y g_{ij} = Y^a \partial_a g_{ij} + \nabla_i Y_j + \nabla_j Y_i. \tag{2.18}$$

Time independent metrics admit

$$\mathcal{L}_Y g_{ij} = [\nabla_i Y_j, \nabla_j Y_i]. \tag{2.19}$$

This vanishes when the metric is independent of one of its coordinates along the integral curves of Y. In this case Y is a Killing vector, and has the conserved quantity $K_a Y^a$ associated with it.

When one phrases their equations using differential forms there is also an important relation between Lie derivatives and differential forms called *Cartan's identity*:

$$\mathcal{L}_X(Y_j) = X^a dY_{aj} + d(X^a Y_a)_j.$$

2.6 Extrinsic Curvature and Tensor Deformation

The main geometric object we will be concerned with, in addition to the induced metric, is the covariant derivative of a normal vector to a hypersurface $\nabla_j n_i$. The *shape operator*, called the *Weingarten map* by Damour, is a directional derivative, projected along a tangent.

$$\chi(u)_{i\mu} = e^a_\mu \nabla_a n_i \tag{2.20}$$

When the normal is null, the shape operator gives the deviation of the geodesic parameter from being affine. In this case it is equivalent to the geodesic equation

$$\chi(n)_{i\alpha} = n^\alpha \nabla_\alpha n_i = \kappa n_i.$$

We can symmetrize the shape operator with respect to the induced metric by projecting along a second tangent vector e^j_ν. Since $e^i_\mu e^j_\nu = \phi^*$ we have just used the pullback operation again. This is called the *extrinsic curvature* or *second fundamental form*.

$$K_{\mu\nu} = g(n, \chi(e^a_\mu)) = \nabla_b n_a e^a_\mu e^b_\nu = \frac{1}{2}\mathcal{L}_n h_{\mu\nu} \tag{2.21}$$

Extrinsic curvature is the covariant derivative of a normal vector to the brane, projected to the horizon. This is equivalent to the Lie derivative of the intrinsic metric in the normal direction. The total curvature is

$$K = K_{\mu\nu} h^{\mu\nu}.$$

A general technique used in this thesis is expressing the deformation of the extrinsic curvature in terms of Lie derivatives. Splitting 2.21 into symmetric and antisymmetric components,

$$K_{\mu\nu} = K_{(\mu\nu)} + K_{[\mu\nu]}.$$

We then define the *expansion tensor* and *torsion tensor* of the 2-surface as

$$\Theta_{\mu\nu} = \frac{1}{2}\mathcal{L}_n h_{(\mu\nu)} \tag{2.22}$$

and

$$\omega_{\mu\nu} = \frac{1}{2}\mathcal{L}_n h_{[\mu\nu]}. \tag{2.23}$$

The expansion tensor is further split into trace and trace free components, the *expansion scalar* and *shear tensor*, respectively.

$$\theta = h^{\mu\nu}\Theta_{\mu\nu}, \tag{2.24}$$

and

$$\sigma_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\theta. \tag{2.25}$$

The expansion tensor above is along the n direction. We can also include a *transverse expansion* tensor and scalar along the auxiliary null vector k, which is not proportional to the extrinsic curvature. These are

$$\Xi_{\mu\nu} = \frac{1}{2}\mathcal{L}_k h_{(\mu\nu)}$$
$$\theta_{(k)} = h^{\mu\nu}\Xi_{\mu\nu}.$$

2.7 Gauss, Codazzi and Ricci Equations

The Gauss, Codazzi, and Ricci (or tidal) equations relate the Riemann tensor to the extrinsic curvature of a hypersurface. For timelike and spacelike hypersurfaces they take the form

$$\begin{aligned} R^a_{bcd} h^\mu_a h^b_\nu h^c_\rho h^d_\sigma &= R^\mu_{\nu\rho\sigma} + K^\mu_\rho K_{\nu\sigma} - K^\mu_\sigma K_{\nu\rho} \\ R^a_{bcd} h^\mu_a h^b_\nu h^c_\rho n^d &= \nabla_\nu K^\mu_\rho - \nabla^\mu K_{\nu\rho} \\ R^a_{bcd} h_{\mu a} n^b h^c_\nu n^d &= \mathcal{L}_n K_{\mu\nu} + K_{\mu a} K^a_\nu \end{aligned} \tag{2.26}$$

By contracting the Gauss equation on the first and third indices of the Riemann tensor (and then renaming indices), we can write the Ricci tensor on a hypersurface as

$$h^a_\mu h^b_\nu R_{ab} + h_{\mu a} n^b h^c_\nu n^d R^a_{bcd} = R_{\mu\nu} + K K_{\mu\nu} - K_{\mu a} K^a_\nu .$$

Note that this actually also contains the Ricci equation, governing evolution normal to the surface. Whether the Ricci tensor operates in the bulk or on the boundary is determined by its indices; $R_{bd} = g^c_a R^a_{bcd}$ and $R_{\mu\nu} = h^\rho_\mu R^\mu_{\nu\rho\sigma}$. The contracted Codazzi equation is

$$h^a_\mu n^b R_{ab} = \nabla_\mu K - \nabla^a K_{\mu a}.$$

The trace of the contracted Gauss equation is interesting for historical reasons. It gives a relation between the Ricci scalar on a hypersurface and the extrinsic curvature of that surface; this is a generalization of Gauss's *theorema egregium*.

$$R + 2R_{ab} n^a n^b = {}^3R + K^2 - K_{\mu\nu} K^{\mu\nu}.$$

Here ${}^3R = h^{\mu\nu} R_{\mu\nu}$.

In the null case, several of the terms in the Gauss and Codazzi equations coincide, and we are left with the so-called *Gauss-Codazzi equation*. The null Ricci equation also becomes partially degenerate; tidal forces are represented in terms of both the Gauss and Codazzi equations, and the Ricci scalar projected onto the 2-surface.

These relations appear in much the same way in both the null and non-null cases; here I outline the null case. Beginning with equation 2.6 and contracting indices,

$$\begin{aligned} R_{\mu c} n^\mu &= \nabla_\mu \nabla_c n^\mu - \nabla_c \nabla_\mu n^\mu \\ &= \nabla_\mu K^\mu_c - \nabla_c K. \end{aligned}$$

The remaining index can be contracted along either a null direction or a spatial direction on the 2-surface. These two options are collectively called the Gauss-Codazzi equations.

If we again apply the null projector 2.13, the contracted Gauss-Codazzi equations can be combined as

$$R_{\mu\nu} n^\mu \Pi^\nu_\rho = (\nabla_\mu K^\mu_\nu - \nabla_\nu K) n^\nu k_\rho + (\nabla_\mu K^\mu_\nu - \nabla_\nu K) h^\nu_\rho. \tag{2.27}$$

We will reinterpret the Gauss-Codazzi equations as evolution equations in Rindler space, using 2.6, starting in sections 3.2 and 4.7.

Although not considered in the classical approach to the membrane paradigm by Damour or Thorne et al, the above equations are easily adapted to include the full Einstein tensor with a nonzero cosmological constant. This was likely neglected because historically, their primary interest was astrophysical black holes, and the cosmological constant had yet not been measured as positive. In the general case, the Ricci scalar and cosmological constant contribute to the timelike Gauss and tidal equations, and the null $R_{\alpha\beta}\Pi^\alpha_\mu \Pi^\beta_\nu$ projection.

2.8 Israel Junction Condition

The *Israel junction condition*, applying to both null and non-null hypersurfaces, is a regularity condition for the existence of smooth Lorentzian manifolds, i.e. no discontinuous changes in the metric. This relates the induced metric and extrinsic curvature to changes in the stress-energy tensor across a hypersurface. For our purposes it is interesting because it contains essentially the same information content as the Gauss-Codazzi equations, and provides another perspective on their physical interpretation.

Consider the non-gravitational case of electric and magnetic fields across a surface. The discontinuous components of the E and B fields can be used to restate Maxwell's equations. Using the notation

$$[A] \equiv A^+ - A^- \tag{2.28}$$

to represent changes in A across a hypersurface, we have

$$\begin{gathered} [E_n] = 0, \;\; [B_\perp] = 0 \\ [E_\perp] = 4\pi\sigma n^i, \;\; [B_n] = 4\pi\epsilon_{ijk} J^j n^k, \end{gathered} \tag{2.29}$$

consistent with charge conservation and $\partial_a F^{ia} = 4\pi J^a$.

In order to follow the same procedure for gravity we must require that there are no discontinuous changes in the induced metric or extrinsic curvature across a hypersurface,

$$[h_{\mu\nu}] = [K_{\mu\nu}] = 0. \tag{2.30}$$

We can, however, violate the junction condition on the extrinsic curvature. When this happens we compensate by adding a stress-energy tensor tangential to the hypersurface. Recall that the Codazzi equation 2.27 relates such a tangential Ricci tensor to the extrinsic curvature. Thus a discontinuous extrinsic curvature implies a local stress-energy tensor of the form

$$8\pi S_{\mu\nu} = [K_{\mu\nu}] - [K]h_{\mu\nu}. \tag{2.31}$$

From here, we can then expand the right-hand side to obtain evolution equations along the horizon, again consistent with the prior Gauss-Codazzi approach, but explicitly in terms of a thin, discontinuous distribution of matter or energy on the hypersurface.

2.9 The Schwarzschild metric in Kruskal-Szekeres Coordinates

The classic Schwarzschild metric describes the gravitational field outside a spherically symmetric star, planet, or black hole, and in four dimensions is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{2.32}$$

where $f(r)$ and the solid angle is Ω are given by

$$f(r) = 1 - \frac{b}{r}, \text{ with } b \equiv 2GM/c^2,$$
$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$$

My reason for not setting $b \equiv 1$ or $G \equiv 1$ is because the dependence (or one can equivalently say *definition*) of the gravitational constant on $\hbar$ and c is interesting to see, and because the black hole entropy is related to its mass. This solution describes empty space; it is assumed in the derivation that mass lies within a radial ball centered at the origin, and provided this ball is small enough all of space is covered except for a single point. The event horizon lies at radius $r = b$, and a physical singularity at $r = 0$ is caused by the Riemann curvature tensor diverging. Here bundles of locally orthonormal frames cease to exist, the equivalence principle therefore becomes invalid, and timelike and null curves cannot be extended to this point using an exponential map. The horizon acts as a "perfect unidirectional membrane"[70] preventing outflow of matter and energy, since here the escape velocity is the speed of light.

To remove the coordinate singularity at the horizon we can reparametrize our coordinate system in terms of null geodeiscs. Null vectors are given by

$$g_{ij}n^i n^j = 0 = -f(r)\left(\frac{\partial t}{\partial \lambda}\right)^2 + f(r)^{-1}\left(\frac{\partial r}{\partial \lambda}\right)^2.$$

Rearranging shows that geodesics satisfy

$$t = \pm \int \frac{dr}{f(r)} + C.$$

We introduce the tortoise coordinate

$$r^* = \int \frac{dr}{f(r)} = r + b \ln(r - b), \tag{2.33}$$

and then null coordinates

$$u^* = t - r^*$$
$$v^* = t + r^*.$$

The Schwarzschild metric becomes

$$ds^2 = -f(r) du^* dv^* + r^2 d\Omega^2.$$

By observation, the u^* and v^* are null and form a conformally flat subspace ($\mathrm{Tr} = 0$) with an associated 2-sphere at every point. Kruskal-Szekeres coordinates use the simplifying choice $U = -e^{-u^*/2b}$ and $V = e^{v^*/2b}$. Making a change of coordinates, the final form of the Schwarzschild metric is

$$ds^2 = -\frac{4b^2}{r} e^{-r/b} dU dV + r^2 d\Omega^2. \tag{2.34}$$

This last change removes the coordinate singularity at the horizon, so we are free to specify energy flux or observer motion. The event horizon corresponds to the null surface ($U = 0, V > 0$), and the timelike stretched horizon to a hyperbola at fixed radius with ($U < 0, V > 0$). The singularity, which we will avoid, is at $UV = 1$. Kruskal-Szekeres coordinates cover a larger space than the original Schwarzschild metric; this is the "analytically extended" Schwarzschild solution. The physical Schwarzschild geometry, caused by a collapsing shell of matter, is composed of a combination of the rightmost wedge and Minkowski space, but in this thesis I will consider the more general space above. Setting ($U < 0, V = 0$) gives us another null surface called the "past event horizon". Comparing these regions to their limits in Rindler space is key in connecting the differing approaches to the membrane paradigm together.

2.10 Rindler Space

Rindler space is the name given to the space seen by an accelerated observer in Minkowski space, and is equivalent to the near-horizon or infinite-mass limit of a Schwarzschild black hole. To see this, consider the near-horizon expansion of Schwarzschild space. From

$$ds^2 = -f(r)dt^2 + \frac{dr}{f(r)} + r^2 d\Omega^2$$

define the proper distance from the horizon

$$\rho = \int_b^r \sqrt{g_{rr}(r')}dr', \tag{2.35}$$

which gives

$$ds^2 = -\rho^2 d\tau^2 + d\rho^2 + r^2 d\Omega^2$$

where the time has been rescaled to $\tau \equiv \frac{t}{2b}$. The τ and ρ coordinates correspond to constant time slices and radial distance from the origin, respectively. Comparing this metric with Minkowski space via

$$\begin{aligned} T &= \rho \sinh \tau \\ X &= \rho \cosh \tau \end{aligned} \tag{2.36}$$

an observer at constant ρ lies on a timelike hyperbolic surface $X^2(\tau) = T^2(\tau) + 1/\rho^2$ and has constant acceleration $|a| = \rho^{-1}$.

To better facilitate comparison with Kruskal-Szekeres coordinates, introducing near-horizon retarded and advanced null coordinates

$$\begin{aligned} u &= T - X = -\rho e^{-\tau} \\ v &= T + X = \rho e^{\tau} \end{aligned} \tag{2.37}$$

the Rindler metric is

$$ds^2 = -dudv + \frac{1}{4}(v-u)^2 d\Omega^2, \tag{2.38}$$

with $r = \frac{1}{2}(v-u) = \rho\cosh(\tau)$. The surfaces of constant u or v are null geodesics, and have a similar horizon structure to the analytically extended Schwarzschild space. The coordinate v is also called an "outgoing null coordinate" because an observer traveling slower than c cannot cross back over the past horizon $(u < 0, v = 0)$. Since the null generators of the horizons in Rindler space are Killing vectors, the horizon can also be called a *Killing horizon*. Rindler space has a *bifuricate Killing horizon*; the region where the two Killing horizons coincide on the spatial two-surface at $u = v = 0$, with horizons generated by the *boost Killing field* $-u\partial_u + v\partial_v$.

We can also make an analytic continuation to Euclidean space, where, dropping the angular coordinate,

$${ds_E}^2 = \rho^2 {d\tau_E}^2 + d\rho^2$$

In order for this to be regular we must specify the period

$$\tau_E = \tau_E + 2\pi.$$

In quantum field theory one normally defines the temperature of a Euclidean path integral as

$$\tau_E = \tau_E + \beta, \text{ with}$$
$$\beta \equiv \frac{1}{k_B T}.$$

This is equivalent to computing the trace $\mathrm{Tr}(e^{-\beta H})$ for a system with a Hamiltonian H associated with the time translation. This implies the temperature of Rindler space is inversely proportional to proper distance from the horizon

$$T_{\text{Unruh}} = \frac{f'(b)}{4\pi} = \frac{\hbar a}{2\pi}. \tag{2.39}$$

2.11 Brown-York Stress-Energy Tensor

An important construct in later sections is the Brown-York stress tensor on a hypersurface, which is used in deriving the Navier-Stokes equations from a derivative expansion around equilibrium solutions to Einstein's equations. This is constructed in analogy with the Hamilton-Jacobi equation; it gives the stress-energy tensor the same geometric dependence on the extrinsic curvature as the Gauss-Codazzi equations and Israel junction condition give the Ricci tensor. For a nonrelativistic system the action may be written in canonical form as

$$S = \int_{\lambda_0}^{\lambda_1} d\lambda [p \frac{dx}{d\lambda} - \frac{dt}{d\lambda} \mathcal{H}(x, p, t)]. \tag{2.40}$$

The Hamilton-Jacobi equations for the energy and momentum at some λ, given appropriate fixed boundary conditions, are

$$H = -\frac{\partial S}{\partial t}, \tag{2.41}$$

and

$$p = \frac{\partial S}{\partial x}. \tag{2.42}$$

We will be interested in the energy. If the variation δt is instead promoted to a variation of the codimension-1 metric δh_{ij}, then varying the Hamilton-Jacobi action with respect to a variation in the metric leads, after normalizing by the tensor density $\sqrt{-h}$, to a generalized surface stress-energy tensor.

$$T^{ij} \equiv \frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h^{ij}} = 2(h^{ij}K - K^{ij}). \tag{2.43}$$

3 Past Null Horizons

3.1 Holography

A key insight of black hole thermodynamics is that the entropy, and therefore information, of a black hole is proportional to its horizon area. Black holes are the densest possible objects in a given volume and are therefore "maximum entropy" objects. To demonstrate this in a gedanken experiment, consider increasing the entropy of a region by adding mass; eventually this will create a black hole, with an entropy proportional to its area. If yet more mass is added, the black hole grows, and its entropy continues to depend on its horizon, satisfying the bound. [2] The *holographic principle* [2] [3] postulates that the maximum entropy of a region of space is always proportional to the surface area of its boundary.

This is essentially one line:

$$S = \frac{A}{4G\hbar}. \tag{3.1}$$

The importance of this equation is difficult to overstate. It offers a unifying principle that restricts the degrees of freedom in theories of quantum gravity, and offers a new perspective on Planck scale physics. It is also, interestingly, relevant in quantum information and condensed matter systems where the entanglement entropy of quantum fields and tensor networks in quantum information satisfy similar area laws [71] [72] [73]. Other, more common, examples of the holographic principle include black hole thermodynamics, the membrane paradigm, string theory [74] and the AdS/CFT correspondence [75], among others.

3.2 Raychaudhuri Equation

The Raychaudhuri-Landau equation categorizes the evolution of systems of non-intersecting geodesics, called geodesic congruences. This allows us to see the evolution of a family of geodesic curves due to their expansion, shear, rotation, and the effect of the stress-energy tensor. It also occurs as a fundamental lemma in the Penrose-Hawking singularity theorems [23], where, through formalizing the idea of a surfaces parameterized by geodesic congruences, it governs the evolution and collapse of integral curves of geodesics into "closed trapped surfaces". The Raychaudhuri equation is intimately related to surface

[2] If the black hole emits Hawking radiation and evaporates, then as its area decreases the total entropy of the Hawking radiation plus the black hole will continue to grow. This is because Hawking radiation is a semiclassical effect and can violate the weak energy condition, which is a constraint on the stress-energy tensor in general relativity and a requirement for the area law.

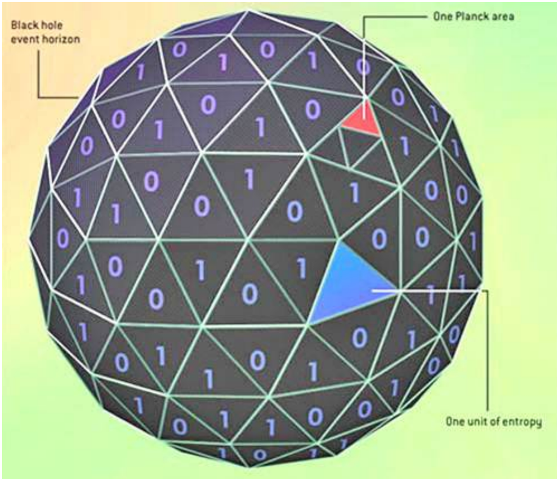

Figure 3.1: An idealized representation of a black hole with horizon described by bits of information. Credit: J.D. Bekenstein, Information in the Holographic Universe. [35]

behavior in the membrane paradigm and fluid/gravity correspondence. As will be seen, by starting on a past null horizon and considering evolution of a thermodynamic energy flux dQ which is governed by the Raychaudhuri equation, the Einstein equations may be derived in full generality. Alternatively, the Einstein equations projected from bulk space onto a future null horizon yield the Raychaudhuri equation. Thus, it is a fundamentally important process and tool. An inherent feature of the Raychaudhuri equation is also that it admits a thermal interpretation, with its expansion and shear governing the "geometric" dissipation of geodesics.

The Raychaudhuri equation takes subtly different forms for null and non-null (timelike and spacelike) geodesics. This is because null geodesics are associated with a spatial 2-surface, while timelike and spacelike geodesics have a natural 3+1 description.

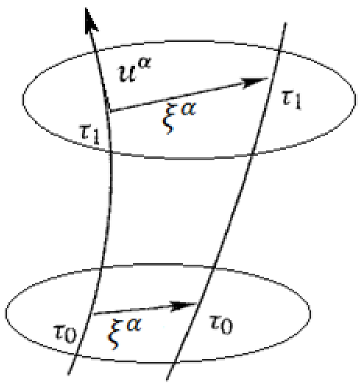

Figure 3.2: Credit: E. Poisson, A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics, p. 36. [29]

3.2.1 Jacobi Deviation Equation

Consider a collection of non-intersecting timelike geodesics $\gamma(\lambda, n)$. Varying λ moves a point initially on a geodesic along it, while varying n selects between geodesics by moving along integral curves to a particular geodesic's tangent vector $u^i = \partial x^i/\partial\lambda$. We denote the deviation vector between geodesics as $\xi^i(n)$. For simplicity the tangent vector, and therefore the geodesics, are taken as timelike. The tangent is orthogonal to the deviation vector at every point along γ, so that the flow of u along ξ is zero. In equations, these conditions are

$$\begin{aligned} u^a u_a &= -1\ ,\ u^a u^i_{;a} = 0 \\ u^a \xi_a &= 0\ ,\ \mathcal{L}_\xi u^j = [\xi^i, u^j] = 0. \end{aligned} \tag{3.2}$$

We can also express the final condition as

$$u^a \nabla_a \xi^i = \xi^a \nabla_a u^i. \tag{3.3}$$

Before deriving the Raychaudhuri equation, consider how the deviation vector evolves. Writing 3.3 as a directional derivative

$$\frac{D\xi^i}{\partial n} = \xi^a \nabla_a u^i. \tag{3.4}$$

By requiring that u^i and ξ^i be orthogonal we have implicitly restricted to a projection $h^i_a \xi^a$ of the full tangent space. Differentiating again, the relative acceleration between geodesics is

$$\begin{aligned} \frac{D^2\xi^k}{\partial n^2} &= \nabla_l \nabla_k (u^j) \xi^k u^l + \nabla_k (u^l) \nabla_l (u^k) u_a \xi^a u^l \\ &+ \nabla_k (u^j) u^k \nabla_l (u^a) \xi_a u^l + \nabla_k (u^j) \nabla_l (\xi^k) u^l. \end{aligned}$$

Exchanging the order of the covariant derivatives brings out the Riemann tensor, and enforcing the geodesic equation yields

$$\frac{D^2\xi^i}{\partial n^2} = -R^i_{jkl} \xi^k u^j u^l. \tag{3.5}$$

This says that infinitesimally separated geodesics, as measured in the tangent space orthogonal to u^i, will accelerate. The deviation equation is a generalization of the acceleration of a particle in a Newtonian potential. Two tangent vectors separated by ξ will have a relative acceleration proportional to $R^i_{jkl} u^j u^l$, while in a Newtonian potential Θ the relative acceleration of particles separated by ξ is proportional to $\nabla_\mu \nabla_\nu \Theta$. This is a tidal force, depending on the shear.

In general the deformation of a family of geodesics can be described through its expansion, torsion, and shear, as described in 2.6, of which the tidal force is an example.

3.2.2 Expansion and Area

The infinitesimal change in area of a congruence only depends on its expansion parameter, since this is the only term with a nonzero trace. By using local flatness we can express 3.3 in matrix form, following Hawking and Ellis [25], as

$$\xi_i = A_{ib} x^b,$$
$$\frac{\partial A_{ij}}{\partial n} = A_{aj} \nabla_a u_i.$$

The derivatives with respect to a parameter n can equivalently be written as Lie derivatives along the normal vector n, in which case the expansion and torsion tensors take the same form as in section 2.6. In any case, separating the above equation into expansion and torsion tensors gives the alternate form,

$$\Theta_{ij} = A^{-1}_{a(j} \frac{\partial}{\partial n} A_{i)a} \tag{3.6}$$

$$\omega_{ij} = -A^{-1}_{a[j} \frac{\partial}{\partial n} A_{i]a} \tag{3.7}$$

with an expansion scalar

$$\theta = (\det A)^{-1} \frac{\partial}{\partial n} \det A, \tag{3.8}$$

and the usual expression for the shear,

$$\sigma_{ij} = \Theta_{ij} - \frac{1}{2} A_{ij} \theta. \tag{3.9}$$

We can use h_{ij} to pull back 3.8 to a timelike or null surface, in which case we have

$$\theta = \frac{1}{\sqrt{h}} \frac{\partial \sqrt{h}}{\partial n}. \tag{3.10}$$

This prescription also immediately implies the analogous behavior for shear and torsion. For null surfaces the induced metric spans a subspace with two spatial dimensions and two associated null vectors, while timelike hypersurfaces span three spatial dimensions.

3.2.3 Frobenius's Theorem

Since the deviation vector and expansion tensor are projected orthogonal to the geodesic tangents u^i, and the direction of these tangents can change over a family of geodesics, it is helpful to formalize surfaces for these quantities to act in. One way of doing this is to construct Fermi normal coordinates, which are locally equivalent to parallel transporting gyroscopes. Another is Frobenius's theorem, which relies on integrability to generalize the idea of integral curves. This allows us to construct hypersurfaces that are orthogonal to every geodesic in a congruence.

While any one-dimensional system, for example a curve with tangent $u^i = \partial x^i(t)/\partial t$, is integrable and has an associated set of integral curves $x^i(t)$, this is not true in general in higher dimensions. The property we would like to have is the existence of *integral manifolds*. Let D be a subspace of the tangent space T_pM of a point p on a manifold M, spanned by a smooth basis in the neighborhood of p. If the tangent space T_pN of a submanifold $N \subseteq M$ is equal to D then N is an integral manifold D, and D is *integrable*. In the one-dimensional case D is spanned by u^i, and we have an integral manifold $N = x^i(t)$.

A D of two or more dimensions, spanned by smooth vector spaces $(V^1, ..., V^k)$, may fail to be integrable. As an example, consider D spanned by $X = \partial/\partial x + y\partial/\partial z$ and $Y = \partial/\partial y$. At $y = 0$ the tangent plane is $(x, y, 0)$, so if we integrate D here then N will produce this as its tangent space, which will not correspond to D for any $y \neq 0$. Geometrically, the tangent plane of X and Y has a normal in the $-\partial/\partial z$ direction at $y = 0$, but here the second derivatives depend on order of composition. This happens because the $(V^1, ..., V^k)$ do not form a Lie group under the action of $[V^i, V^j] \in D$. Although D is spanned by $(V^1, ..., V^k)$ the bracket $[V^i, V^j]$ is not, indicating they can lead to points outside the tangent space D. Thus, integrating D to get a submanifold N may not result in $TN = D$.

The necessary and sufficient condition for D to be integrable is that D be *involute*, meaning $[V^i, V^j] \in D$. Frobenius's theorem states:

A subspace D of the tangent space of M is involute if and only if it is integrable.

The proof of this essentially boils down to the fact that if a local frame for D is integrable then it can be mapped to a locally flat coordinate system, where a smooth local frame $(\partial/\partial x_1, ..., \partial/\partial x_k)$ will then commute and vectors constructed from it satisfy $[X^i, X^j] \in D$. [3] Reversing this logic is also possible. A projection operator $P : \mathbb{R}^n \to \mathbb{R}^k$ from coordinates $P(x^1, ..., x^n) = (x^1, ..., x^k)$ has an associated projection operator from the tangent space of M to D

$$dP\left(v^i \frac{\partial}{\partial x^i(p)}\right) = v^j \frac{\partial}{\partial x^j(P(p))}, \tag{3.11}$$

[3] In physics terminology, as long as the equivalence principle is valid we will have integrability. We will see later that the equivalence principle is related to the existence of local thermal equilibrium; therefore local thermal equilibrium is needed for integrability. Further, in singularity formation in general relativity a smooth basis ceases to exist; therefore neither the equivalence principle or integrability hold at these points.

where i runs over $[1, ..., n]$ and $j = [1, ..., k]$. Using this we can construct a new local frame in terms of the x^i, at a point q in a neighborhood $U \in M$ of p as

$$V_i(q) = dP^{-1}\left(\frac{\partial}{\partial x^i(P(q))}\right). \tag{3.12}$$

The involutivity of $\partial/\partial x^i$ carries over to the new basis V_i via composition

$$dP\left([V_i, V_j](q)\right) = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right](P(q)) = 0, \tag{3.13}$$

so we can integrate D in the new basis and obtain N

The relation between integrability and involutivity has an important ramification in general relativity. Since the torsion vanishes in general relativity, the involutivity criterion is satisfied and one always has integrability. Then with integrability we can construct a series of hypersurfaces that are the k-dimensional tangent spaces of some series of N_i submanifolds of M. This process is called *foliation*, and if so chosen the hypersurfaces can be orthogonal to the normals of D.

An alternate criterion for integrability is that, for the 1-forms x_i associated with a subspace of the dual tangent space T^*_pM, we can write the exterior derivative as

$$(dx)_{ij} = \alpha_i \wedge \beta_j, \tag{3.14}$$

with $\alpha_i \subset T^*_pM$.

3.2.4 Timelike Raychaudhuri equation

The Raychaudhuri equation is the rate of change of the expansion parameter in 3.10. In applying this to timelike hypersurfaces, covariant derivatives are tangent to the geodesic γ, along $u^i = e^i_\mu$. Differentiating the left hand side (alternatively, the right side of 3.10 can be resolved using the Lie group identity $\mathrm{Tr}(\ln h_{ij}) = \ln(\det h_{ij})$, but this is less efficient)

$$\begin{aligned}
\partial_\lambda \theta &= \left[h^{ab}\theta_{ab}\right]_{;c} u^c \\
&= h^{ab}\left[u_{a;cb} - R_{adbc}u^d\right] u^c \\
&= h^{ab}\left[(u_{a;c}u^c)_{;b} - (u_{a;c})(u^c_{;b}) - R_{adbc}u^d u^c\right] \\
&= h^{ab}\left[-(u_{a;c})(u^c_{;b}) - R_{adbc}u^d u^c\right] \\
&= -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{dc}u^d u^c.
\end{aligned} \tag{3.15}$$

The second line reverses the order of derivatives to get the Riemann tensor, and the third rearranges the chain rule. Negative signs indicate that the expansion scalar and the shear cause contraction, while torsion induces expansion. The torsion is zero due to Frobenius's theorem, and it is also unphysical in general relativity, so we remove it. Thus

$$\frac{\partial \theta}{\partial \lambda} = -\frac{1}{3}\theta^2 - \sigma^2 - R_{dc}u^d u^c. \tag{3.16}$$

3.2.5 Null Raychaudhuri equation

The null Raychaudhuri equation proceeds largely similar to the timelike case, except we must be careful about the change in induced metric. The congruence of geodesics evolve in the direction of the orthogonal null vector n, which is also a null tangential vector and can be written in terms of integral curves.

Proceeding as before,

$$\begin{aligned} \partial_\lambda \theta &= (h^{ab}\theta_{ab})_{;c} n^c \\ &= -\frac{1}{2}\theta^2 - \sigma^2 - R_{dc}n^d n^c. \end{aligned} \tag{3.17}$$

We can generalize this to a non-affine parameterization:

$$\partial_\lambda \theta = \kappa\theta - \frac{\theta^2}{2} - \sigma^2 - R_{dc}n^d n^c. \tag{3.18}$$

Note that we can rearrange this to write the Ricci tensor in terms of the shear and expansion. Since we can also write the Ricci tensor by using the Gauss-Codazzi equations to project the Riemann tensor onto a hypersurface, this lets us express the Gauss-Codazzi equations in terms of their shear, expansion and torsion. This is a key point in deriving thermal and fluid properties of the horizon.

As an example of the utility of the Raychaudhuri equation, consider a light ray in the region $(U < 0, V > 0)$ outside a black hole. If directed away from the black hole then its normal (and tangent) vector n^i points in the negative U direction and has an expansion

$$\begin{aligned} n^i &= \partial^i U, \\ \theta &= -\tfrac{U}{b}. \end{aligned} \tag{3.19}$$

The expansion is positive when the light ray is outside the black hole, but becomes negative when it crosses the event horizon.

We can elaborate on how the Raychaudhuri equation focuses energy by constraining the form of the matter stress-energy tensor. The *weak energy condition* is defined by the property that for a point $p \in M$ the stress-energy tensor at p obeys

$$T_{ab} n^a n^b \geq 0 \tag{3.20}$$

for any timelike or null vector $n^i \in T_p M$. Since the net contributions of the expansion and shear are always negative, the weak energy condition implies that

$$\frac{\partial \theta}{d\lambda} < 0. \tag{3.21}$$

There is also a *dominant energy condition*. In this case 3.20 holds with the additional requirement that $T_{aj} n^a$ be timelike or null. The significance of this is that any local observer will see a non-negative energy density and a timelike or null energy flow vector. Here the pressure is less than or equal to the energy density, $T^{00} \geq |T^{ij}|$. This condition holds for all known forms of matter. However, as we will see later, when Einstein's equations are constrained to a hypersurface, they behave as a fluid with a negative energy density of $-1/16\pi$. In this case their acausal behavior is due to integrating a Green's function with a fixed final boundary condition.

In the next section we will see how the Raychaudhuri equation combined with the proportionality of entropy to area $S \propto A$ leads to the thermal emergence of general relativity.

3.3 From Thermodynamics to Gravity

Here we derive general relativity from the thermodynamic relation $\delta Q = TdS$, Lorentz invariance, the equivalence principle and the scaling of entropy with area. In this way general relativity emerges as an equation of state. Inherent in this derivation is the idea that we have implicitly "coarse grained" over the underlying degrees of freedom of quantum fields in Rindler space, both through assuming holography and in several approximations, in order to get the Einstein equations.

The starting point is equilibrium thermodynamics, and our goal is to express dQ and TdS as the necessary functions of the stress-energy tensor and Einstein tensor to recover the field equations.

$$\begin{aligned} dQ &= f(T_{ij}), \\ TdS &= f(G_{ij}). \end{aligned} \tag{3.22}$$

The basic idea can be illustrated with classical thermodynamics.

$$
\begin{aligned}
dQ &= TdS \\
\delta Q &= dE + pdV \\
dS &= \left(\frac{\partial S}{\partial E}\right)dE + \left(\frac{\partial S}{\partial V}\right)dV \\
&= \left(\frac{1}{T}\right)dE + \left(\frac{p}{T}\right)dV
\end{aligned}
\tag{3.23}
$$

So the pressure takes the form $p(E,V) = T\left(\frac{\partial S}{\partial V}\right)$. Given a known scaling of entropy it is possible to obtain the equation of state to order ϵ.

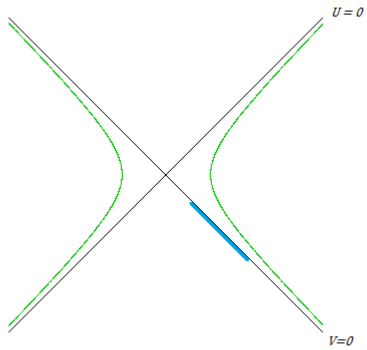

Figure 3.3: Energy flux across a Rindler horizon.

We will now adapt this to derive the Einstein equations. Let us begin with an accelerated observer on a timelike path in Rindler space, using the metric

$$ds^2 = -dudv + \frac{1}{4}(v-u)^2 d\Omega^2.$$

The past horizon at (u<0, v=0) has a tangent $n^i(\lambda)$ that can be written in terms of an affine parameter $\lambda = u$, which is zero at the origin and negative along the past horizon. An area element on the past horizon is

$$d\Sigma^i = n^i d\lambda dA. \tag{3.24}$$

The observer is associated with a boost vector $\chi^\mu = e_i^\mu$ on a timelike hypersurface with acceleration $\kappa = \rho^{-1}$, located "sufficiently" close to the past null horizon to be written in terms of λ. This involves coarse graining over a distance scale of order ρ.

$$\chi^i = -\kappa n^i \lambda \tag{3.25}$$

Energy flux across an area element on the horizon is then

$$\delta Q = -\kappa \int_H T_{ab} n^a n^b \lambda d\lambda dA. \tag{3.26}$$

To consider the TdS side of the equation we invoke the proportionality of entropy and area

$$dS = \varepsilon \delta A,$$

where the area element from 3.10 is

$$\delta A = \int_H \theta d\lambda dA.$$

This allows us to use the null Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{2} - \sigma^2 - R_{ab} n^a n^b.$$

Initial values of the shear and expansion can be taken as zero at p, implying

$$\theta = -\lambda R_{ij} n^i n^j$$

and therefore

$$\delta A = -\int_H \lambda R_{ij} n^i n^j d\lambda dA.$$

In order to make sense of $\delta Q = TdS$ in this context we identify the temperature as the Unruh temperature of an accelerating observer in Minkowski space 2.39

$$T = \frac{\hbar\kappa}{2\pi}.$$

Note that this temperature is defined for an accelerating observer a distance ρ from the horizon, while the entropy is defined *on* the horizon. It is disturbing that both the boost Killing vector and the Unruh temperature correspond to timelike observers; our "coarse graining" to shift these to the null surface requires an infinite acceleration, causing both to diverge. Clearly, we should expect additional complications at such high energies, but we surmise that so long as both variables, one relativistic, the other from quantum field theory, have the same dependence on distance from the horizon then this will cancel, and coarse graining will work. If, for example, we view quantum field theory as being an effective field theory with a short distance cutoff then we eventually require a deeper reason why these high energy contributions are irrelevant. Using the above Unruh temperature, we have

$$T_{ij} n^i n^j = \frac{\varepsilon}{2\pi} R_{ij} n^i n^j.$$

This correspondingly implies

$$T_{ij} = \frac{\varepsilon}{2\pi} R_{ij} + C g_{ij}.$$

Fixing the constant through the Bianchi constraint $\nabla^i T_{ij} = 0$ gives $C = -\frac{R}{2} + \Lambda$, which recovers the Einstein equation

$$R_{ij} + \Lambda g_{ij} = \frac{2\pi}{\hbar\varepsilon} T_{ij}. \tag{3.27}$$

The $\hbar$ is introduced because of the accelerated Minkowski observer. There is no new information on the cosmological constant Λ. The ε is needed to recover Newton's constant and the correct factors of c and π. By comparison with the standard form of Einstein's equations, we see that Newton's constant is defined as

$$G \equiv \frac{c^4}{4\hbar\varepsilon} \tag{3.28}$$

where $\varepsilon = (4l_p)^{-2}$, and l_p the Planck length.

Although Jacobson suggested that because general relativity arises as an "equation of state", it is therefore unnecessary to quantize, I think this is slightly incorrect. General relativity arises as an equation of state because it is likely related to an underlying quantized theory, and we coarse grain over these underlying degrees of freedom, thus allowing general relativity to emerge as a thermodynamical theory.

For example, the final expression for the field equations 3.27 defines Newton's gravitational constant as $G \propto 1/\hbar$. In its most naive interpretation this looks like the first term in a perturbative expansion. In quantum electrodynamics the fine structure constant is $\alpha = e^2/\hbar c \sim 1/137$. In perturbative gravity the corresponding quantity is $\varepsilon = 16\pi G/c^4 \sim 1/2.4 \times 10^{42}$, a vastly weaker interaction scale. This is also seen, from an action principle perspective, in the approach of Parikh and Wilczek [60] [61].

General relativity arises thermodynamically, but in doing so it picks up a factor of $\hbar^{-1}$, which looks like it is from a perturbative expansion. This seems consistent with expectations from string theory and canonical quantum gravity. We have here the "coarse grained" result of combining quantum field theory with thermodynamics and the equivalence principle: General Relativity. The necessary conditions for deriving general relativity are Lorentz invariance, the equivalence principle, the holographic principle, and the ability to coarse grain over degrees of freedom to obtain a thermodynamic energy. Thus it is logical that theories with these properties have general relativity as a limit and contain features such as a fluid/gravity correspondence and UV/IR connection.

3.4 Thermal Interpretation of the Equivalence Principle

The equivalence principle relates acceleration to gravitation. However, gravity can be derived from thermodynamics. Therefore the equivalence principle must also have a thermodynamical interpretation.

In its most basic form this relationship is given by the Unruh temperature $T = \kappa/2\pi$. The equivalence principle is the statement that there is always a local frame, which we can write explicitly using Riemann or Fermi normal coordinates. In this frame we are able to set the acceleration to zero, up to second order. The correspondence in terms of temperature is that every local frame is in thermodynamical equilibrium at zero temperature, which is Minkowski space with approximately no acceleration. The Hawking-Unruh temperature tells us we only need to assume one of these statements; local thermal equilibrium or the equivalence principle.

Clearly the conditions for thermal equilibrium are more general than just the gravitational equivalence principle. But the equivalence principle tell us that this finite-temperature, finite-curvature equilibrium is actually equivalent to a zero-curvature and zero-temperature equilibrium, a statement not typically found in thermodynamics.

The temperature is related to the curvature; the curvature to the equivalence principle, and the equivalence principle to special relativity and Newton's law. This has deep implications for the interpretation of inertial mass, for which I refer to [15].

By using this interpretation of the equivalence principle, general relativity is in thermal equilibrium along geodesic paths. Deviations from equilibrium in thermal systems typically imply fluid behavior, which we will see in general relativity takes the form of the Damour-Navier-Stokes equations.

4 Future Null Horizons

4.1 The Penrose Process

Now that we have crossed from the past null boundary into the bulk and derived the field equations of general relativity, let us intrinsically derive thermal properties of general relativity. In doing so we see that the laws of black hole thermodynamics, which inherently includes the holographic principle, and is constructed between bulk spacetime and the future boundary, are consistent with Jacobson's derivation of general relativity, which relies on the holographic and equivalence principles. To set this up logically it makes sense to begin with the Penrose process for Kerr black holes, which through its introduction of a minimum irreducible black hole mass lays the foundation for the thermodynamics of Schwarzschild black holes.

The Penrose process is also historically important as one of the first forays towards understanding black holes as systems that could exchange energy with the outside world (Zel'dovich's analysis of superradiant modes is the other beginning). Although not explicitly thermal, it considers infinitesimal changes in the black hole's mass, charge and spin; ideas which developed into black hole thermodynamics.

Consider a rotating Kerr-Newman black hole and a transient particle. Each have mass, spin and charge. The particle will be able to extract energy from the black hole if it splits while in the black hole's ergosphere. One component falls into the black hole, the other escapes to infinity. In this way the Penrose process concerns the global dynamics of black holes.

The Kerr-Newman metric in Boyer-Lindquist coordinates is

$$ds^2 = -\frac{\rho^2\Delta}{\Sigma}dt^2 + \frac{\Sigma}{\rho^2}sin^2\theta(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \tag{4.1}$$

with the standard definitions

$$\begin{aligned} \rho^2 =& r^2 + a^2\cos^2\theta \\ \Delta =& r^2 - 2Mr + a^2 + C^2 \\ \Sigma =& (r^2 + a^2)^2 - a^2\Delta\sin^2\theta \\ \omega \equiv& -\frac{g_{t\phi}}{g_{\phi\phi}} = a\frac{r^2 + a^2 - \Delta}{\Sigma} \\ a =& \frac{L}{M}. \end{aligned} \tag{4.2}$$

In this section $G = c = 1$ and $C = 0$. The Schwarzschild metric is recovered for $a = 0$. In order to describe energy extraction, we first need to define energy. The Kerr metric has Killing vectors $A^i = \partial_t$ and $B^j = \partial_\phi$ which correspond to a conserved global energy $E = -A_a p^a$ and angular momentum $L = B_a p^a$ for a particle of momentum $p^i = m\partial x^i/\partial\lambda$. Kerr black holes have an *ergosphere*, which is a region outside the event horizon where the black hole's rotation drags coordinates along with it, so that even at the speed of light it is impossible to remain stationary. The behavior of Killing vectors in this region will be key for extracting energy. The outer edge of the ergosphere is the surface where the norm of the time translation Killing vector vanishes.

$$A^a A_a = 0 = \frac{1}{\rho^2(\Delta - a^2\sin^2(\theta))}.$$

This implies

$$r_{\text{sls}} = M \pm \sqrt{M^2 - a^2 \cos^2(\theta)}.$$

The outer radius is known as the stationary limit surface, and for $\theta = 0$ coincides with the outer event horizon, denoted r_+. Outside r_{sls} the norm of the timelike Killing vector is negative, and so the energy of a particle on such a trajectory is positive. Inside r_{sls} the norm becomes spacelike, which means E can be either positive or negative. Inside the stationary limit surface particles can have negative energy. However, there is no way for a negative energy particle to escape the ergosphere. In order to cross the stationary limit surface it must gain enough energy to make E positive.

To exploit this and extract energy from the black hole, consider the change in mass and angular momentum of a black hole when a particle that has crossed into the ergosphere splits or decays into two particles of energy and angular momentum

$$\begin{aligned} E_0 &= E_1 + E_2 \\ L_0 &= L_1 + L_2. \end{aligned} \tag{4.3}$$

Assuming the black hole absorbs particle two, denote $E_2 = \delta M_{bh}$ and $L_2 = \delta L_H$. The most general Killing vector we have is a superposition of the previous two

$$\chi_i = A_i + \Omega_{bh} B_i,$$

and is tangent to null generators of the black hole horizon. Ω_H is an angular velocity. Since A^i and p^i are both timelike outside of the black hole, but A^i can become spacelike in the ergosphere, we can have $E_2 = -A_a p^a < 0$. Assuming this is the case, contracting χ_i with particle two gives

$$p_2^a \chi_a = p_2^a (A_i + \Omega_H B_i) < 0,$$

which results in the inequality

$$L_2 < \frac{E_2}{\Omega_H}.$$

This is equivalent to

$$\delta L_H < \frac{\delta M_{bh}}{\Omega_H}. \tag{4.4}$$

Since particle two can have negative energy, its angular momentum is negative and reduces the black hole's angular momentum and energy. When the black hole stops rotating this process ends and we obtain a Schwarzschild black hole with a minimum *irreducible mass* that, classically, can never decrease.

The irreducible mass is

$$M_{irr}^2 = \frac{M^2 + \sqrt{M^4 - L^2}}{2} \leq M^2 \tag{4.5}$$

and combined with 4.4 implies

$$\delta M_{irr} > 0.$$

This happens to be expressible in terms of the black hole's surface area. Restoring Newton's constant,

$$M_{irr} = \frac{A}{16\pi G^2}, \tag{4.6}$$

which implies the *Hawking area law* or *Second law of Black Hole Thermodynamics* for Kerr black holes

$$\delta A \geq 0 \text{ for any process.} \tag{4.7}$$

The Hawking area law emerges in general as a consequence of the Raychaudhuri equation applied to null horizons. Basically, as long as an energy condition such as the weak energy condition 3.21 holds then the null generators of a horizon will have a non-negative expansion $\theta \geq 0$ and the horizon will either grow or remain static.

The horizon area defined in terms of the irreducible mass is an extremum for a black hole of a given mass. The Penrose process was the first hint of thermal behavior of general relativity. Energy is extracted from a rotating black hole by reducing its angular momentum, and transferring it as energy to an outgoing particle. When the black hole's angular momentum vanishes its ergosphere disappears, particles of negative energy become impossible, and absorbing any further particles will increase the black hole's mass and horizon area. In this way a Schwarzschild black hole's surface area is analogous to the macroscopic entropy of ordinary thermodynamics, and creates the foundation for black hole thermodynamics.

Indeed, rearranging 4.4, in the limit of a reversible process,

$$\delta M_{bh} = \Omega_H \delta L_H. \tag{4.8}$$

This is the first law of thermodynamics without a TdS term.

4.2 Black Hole Thermodynamics

We can now develop the laws of black hole thermodynamics, as they appear on the Schwarzschild null future horizon.

The *Zeroth law*, which will not be proven here, is that in analogy with the zeroth law of thermodynamics (which states that a system in thermal equilibrium has a constant temperature), is that the surface gravity of a null horizon is constant. This has been shown by assuming the horizon is a Killing horizon but without using energy conditions or the field equations, in increasing generality, by Carter [44], Racz and Wald [45], and shown by Hawking [25] using the field equations and assuming a stationary spacetime and the dominant energy condition.

The first law of thermodynamics is

$$\delta E = TdS - PdV, \tag{4.9}$$

and we would like to find the corresponding statement for black holes. To do this we begin with the irreducible black hole mass, horizon location, and Kerr black hole area. The area is given by

$$A = \int \sqrt{-g}d\theta d\phi = 4\pi(r_+^2 + a^2).$$

Varying the horizon radius $r_+ = M + \sqrt{M^2 - a^2 - C^2}$ with respect to M, C and a, and substituting in the area gives the *First law* of black hole thermodynamics

$$\delta M = \frac{\kappa \delta A}{8\pi} + \Omega_H^A \delta L_A + \Phi \delta C - \mu \delta B, \tag{4.10}$$

where $\kappa = 2\pi(r_+ - r_-)/A$, the angular velocity on the horizon is $\Omega_H^i = a^i/A$, the Columb potential is $\Phi = C/r_+$, and μ is the induced magnetic moment.

A *physical process* version of the first law, which is local and more general, can be seen by considering an influx of mass and angular momentum across the horizon. In this case the tools involved are similar to the derivation of Einstein's equations from $\delta Q = TdS$. For mass and angular momentum flux

Work by Bekenstein [1] [4] and Jacobson [10] [39] [40] focus on the TdS term while Damour [6] [7] [9] and Thorne et al. [52] [53] [54] [55] are mostly concerned with the second term.

The second terms play the role of $-PdV$ in 4.9 and are the effect of work done on the black hole. However, notice that the dimensionality this terms scales with dV instead of dA. In the context of black hole thermodynamics, which is local to the horizon region, one would expect any spatial dependence to be proportional to the spatial two-surface since there is not a volume element. This will be elaborated on in future sections.

Now compare the first terms in 4.9 and 4.10. By observation, if we want the entropy to correspond to a dimensionless Von Neumann entropy of the form

$$S_{\mathrm{VN}} = -\operatorname{Tr} \rho \ln(\rho), \tag{4.11}$$

where ρ is here a density matrix over unknown quantum states, then we also need to make dA in 4.10 dimensionless. This can be done by dividing the area by units of the

Planck length squared, $l_p^2 = \hbar G/c^3$, so that

$$S_{\text{bh}} = \frac{k_b c^3}{4\pi\hbar G} A. \tag{4.12}$$

This is in essence the primary statement of holography, first observed in black hole physics and generalizable to the AdS/CFT correspondence, the entanglement entropy and, by postulate of the *holographic principle* [2] [3], to an arbitrary set of quantum fields. The black hole entropy is an entropy of the equivalence class of possible black hole geometries with the same external parameters but different internal states [1]. In this sense, it is rather unique in that it associates a temperature with a geometric entropy.

Along with the Hawking area law 4.7, we can now state the laws of black hole thermodynamics:

$$\begin{aligned}
&\text{0.The surface gravity } \kappa \text{ is constant on the black hole horizon.}\\
&1.\ \delta E = \frac{\kappa \delta A}{8\pi} + \Omega_H \delta L + \Phi \delta C - \mu \delta B\\
&2.\ \delta A \geq 0\\
&3.\ \kappa \nrightarrow 0 \text{ in a finite number of steps.}
\end{aligned} \tag{4.13}$$

The third law is linked to the idea of preventing naked singularities, by, for example, injecting particles with angular momentum into a Kerr black hole until $L/M^2 = 1$, at which point $\kappa = 0$.

We can express the entropy as the sum of the entropy due to bulk matter and the black hole horizon. This is sometimes called the *Generalized Second Law*

$$S_{total} = S_{bulk} + \frac{A}{4G\hbar}. \tag{4.14}$$

Bulk entropy is the ordinary thermodynamical entropy; the black hole surface area is a new contribution to the entropy.

Black hole thermodynamics provide an interesting consistency check. The holographic principle, used as a postulate valid on the past horizon in 3.3, enables us to derive the field equations of general relativity. Now, considering the field equations on their own, we have derived the holographic principle from black hole thermodynamics as an interaction between bulk matter and a null future horizon. In the next sections we will more carefully examine properties of the null future horizon.

4.3 Black Holes in Thermal Equilibrium

Now that we have a set of thermal laws, it is interesting to consider whether there are situations where a black hole can be in thermal equilibrium. To this end we will consider a black hole interacting with a radiation bath; in order to have an equilibrium configuration the box must not be infinite. This result has an interesting parallel in the AdS/CFT correspondence, which, through its negative cosmological constant, effectively provides an infinite barrier analogous to a "box".

Consider a black hole of entropy and energy

$$S_{bh} = \frac{1}{16\pi G T^2}, \;\; E_{bh} = 18\pi G T, \tag{4.15}$$

and radiation obeying the Stefan-Boltzmann law

$$E_r = aVT^4, \;\; S_r = \int \frac{dE_r}{T} = \frac{4}{3}aVT^3, \tag{4.16}$$

with $a > 0$. Note the specific heat of the black hole is negative (endothermic) since T increases as E decreases. In order for a black hole to be in thermal equilibrium with its surroundings, it needs to be constrained within a finite box. Then the Hawking radiation it emits will change the temperature outside of the black hole, and prevent a feedback loop of monotonically increasing or decreasing black hole mass.

A black hole in an infinite heat bath has two main possibilities for evolution (neglecting unstable equilibrium). It can either absorb radiation if its temperature is lower than its surroundings, in which case it will gain mass, cool down, and then continue to cool down as it absorbs more radiation. Or, if the black hole has an initial temperature higher than the heat bath, it will emit radiation, heat up, and emit higher temperature radiation. In either case due to the negative specific heat there is a feedback loop. Therefore, in order to avoid this and have local stability, we must require that when the black hole emits radiation it heats up its surroundings more than its own temperature increases. Correspondingly when the black hole absorbs radiation, the environment must cool more quickly than the black hole. By combining the above temperature and energy expressions with the entropy-mass relation $dS_{bh} = 8\pi M_{bh} dM_{bh}$, the necessary conditions for equilibria are that, if the black hole has higher temperature than the radiation we must have scaling

$$T_r \sim E_r^{1/4}, \text{ and } dT_r \sim \frac{1}{4}E_r^{-3/4} dE_r = \frac{1}{4}T_r \frac{dE_r}{E_r}, \tag{4.17}$$

and for initially lower temperature,

$$T_{bh} \sim E_{bh}^{-1}, \text{ and } dT_{bh} \sim -T_{bh}\frac{dE_{bh}}{E_{bh}}. \tag{4.18}$$

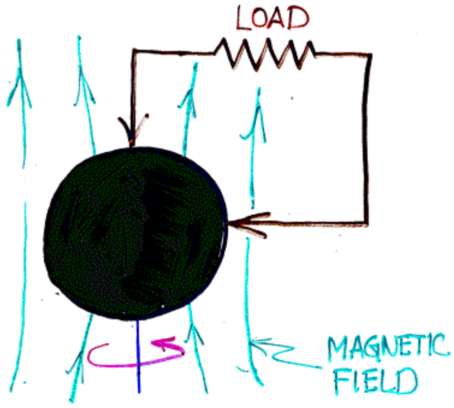

Figure 4.1: Energy can be extracted from a charged rotating black hole by using it as an electrical generator. The surface conductivity of the horizon completes the circuit. Proposed by T. Damour. [7] Image credit: J. Hartle, UCSB Plenetary Lecture, 2000. [51]

4.4 Null Electrodynamics

In this section we adapt Maxwell's equations to a null Rindler horizon. The results are characteristic of the membrane paradigm and the situation for gravity, but the implementation is simpler; by looking at a locally flat region we temporarily avoid needing to consider Einstein's equations. Observationally this implies that black holes can effectively carry surface currents and that distant observers can measure a charge density on a black hole. Black holes have a surface resistivity of $R_H = 4\pi = 377\Omega$, equal to the permittivity of free space. The event horizon acts to effectively truncate electromagnetic fields at the black hole's surface, and creates induced surface quantities on the horizon. Through its linking boundary and surface fields, this approach has some similarity to the bulk/boundary behavior of AdS/CFT.

Consider Maxwell's equations in flat space

$$\begin{aligned} \partial_a F^{ia} &= 4\pi J^i \\ \partial_a J^a &= 0, \end{aligned} \tag{4.19}$$

with antisymmetric electromagnetic tensor

$$F^{ij} = \partial^i A^j - \partial^j A^i. \tag{4.20}$$

Here we use null Rindler coordinates, on the $(u = 0, v > 0)$ horizon. The electromagnetic tensor is defined everywhere, even though quantities inside the horizon are not measurable by external observers in general relativity. In following the membrane approach, we artificially truncate F^{ij} on the horizon by introducing a step function Θ, equal to 1 on and outside the horizon and 0 inside. Making the replacement

$$F^{ij} \rightarrow F^{ij}\Theta, \tag{4.21}$$

and evaluating Maxwell's equations

$$\partial_a \left(F^{ia}\Theta\right) = 4\pi(J^i\Theta + j^i_H). \tag{4.22}$$

The step function divides Maxwell's equations into bulk and surface quantities; we define a new surface current on the horizon as

$$\begin{aligned} j^i_H &= \frac{1}{4\pi}F^{ia}\delta_H, \\ \delta_H &= \delta(-u)n_i. \end{aligned} \tag{4.23}$$

This is the most basic "membrane" result. Black holes are imbued with induced surface currents as long as electromagnetic fields are effectively truncated on the horizon.

Generators of the horizon, $x^i(v)$, have normals as in 2.9, parametrized in terms of u in Rindler space

$$n_i = \alpha \mathrm{d}u. \tag{4.24}$$

Since the horizon is null, this normal is also tangent, and we can expand in terms of the horizon basis (∂_v, ∂_A) as

$$n^a\partial_a = \frac{\partial}{\partial v} + \frac{dx^A}{dv}\frac{\partial}{\partial x^A}. \tag{4.25}$$

n^i is Lie-dragged so that for an infinitesimal displacement on the horizon $dx_i = n_i dv$. The expression for the angular component of n^i can be interpreted in a peculiar way; as the *Newtonian surface velocity* $v^A = \frac{dx^A}{dv}$ of the constituent fluid composing spacetime on the horizon. Physically $v^A = 0$ means the black hole is not rotating. To see this relation more clearly, we could change to angular coordinates and normalize ∂_A as the axisymmetric Killing vector ∂_ϕ. Then v^A corresponds to a rotational velocity of Ω_H on the horizon.

We can use the surface current to define a current density K^i on the horizon:

$$\begin{aligned} j^i_H &= \frac{1}{4\pi}F^{ia}n_a\delta_H = K^i\delta_H, \text{ with} \\ K^i &\equiv \frac{1}{4\pi}F^{ia}n_a. \end{aligned} \tag{4.26}$$

The above expressions can be combined into an equation of continuity for current density between the bulk and boundary:

$$\partial_a(\Theta_H J^a + K^a\delta_H) = 0. \tag{4.27}$$

An external current injected into the horizon will create a surface current density. This is holographic in that it depends only on Maxwell's equations outside and on the surface of the black hole, and is also local.

In analogy with electrodynamics, the current density is composed of a charge density varying in time and space.

$$K^a \partial_a = \sigma_H + K^A \partial_A \tag{4.28}$$
$$= \sigma_H n^a \partial_a + (K^A - \sigma_H v^A)\partial_A. \tag{4.29}$$

We associate σ_H with a charge density depending on v (if we were to instead use a 3+1 split of spacetime, this would be written as time dependence), and K^A with the angularly-varying component. Then the charge density is found by projecting the current density along the auxiliary null vector k^i, defined in 2.10. This gives

$$K^a n_a = \frac{1}{4\pi} F^{ab} n^a k^b$$
$$= \sigma_H n^a k_a. \tag{4.30}$$

Thus σ_H is analogous to a charge density $\sigma = \frac{1}{4\pi} E^a n_a$, and results from projecting F^{ij} onto the horizon.

Ohm's law follows by writing the current density, using $\partial_v = n^a \partial_a - v^A \partial_A$, as

$$K^a \partial_a = \sigma_H n^a \partial_a + (K^A - \sigma_H v^A)\partial_A. \tag{4.31}$$

In vector notation we have the Lorentz force for a charge q:

$$q\mathbf{E} + q\mathbf{v} \times \mathbf{B}_\perp = 4\pi q(\mathbf{K} - \sigma_H \mathbf{v}), \tag{4.32}$$

and setting $\mathbf{v} = 0$, Ohm's law

$$\mathbf{E} = 4\pi \mathbf{K} = \rho \mathbf{K}, \tag{4.33}$$

where the resistance $\rho = 377\Omega$.

Ohm's law generalizes to curved space by replacing the partial derivatives in Maxwell's equations with covariant derivatives. The equation of continuity 4.27 is then identified as a contracted Bianchi identity connecting the charge and current density with the current injected normal to the null horizon:

$$\partial_v \sigma_H + \partial_A K^A + n_a J^a = 0. \tag{4.34}$$

4.5 Scalar and Tensor Fields

The basic procedure of restricting the rank-2 electromagnetic tensor to the horizon extends to other types of fields, such as free scalar or tensor fields. The membrane approach, as long as fields near the black hole do not significantly change its mass or become so strong that they self-interact, are easily incorporated. In this case there are not correlations between the bulk and induced field on the horizon. This is essentially the *method of images* from electrodynamics, and also occurs in the general action principle approach of Parikh and Wilzcek [61] [60], which is outlined in the next section. There is also some similarity with the "brick wall" model of 't Hooft [68], which assumes all wave functions vanish at a fixed distance from the horizon, and the discussion of quantum fields outside the stretched horizon by Thorne et al in [8].

For instance, for the scalar field $\varphi(x^i)$ one may analogously define an auxiliary field $\varphi_*(x^i) = \varphi(x^i)\Theta_H$ truncated at the horizon by the same causality reasoning applied to F^{ij} previously. Using 3+1 coordinates for consistency with the typical canonical commutation relations, we can ask how φ behaves.

The canonical momenta separates into bulk and boundary terms

$$\begin{aligned}\Pi_*(x^i) &= \partial_t[\varphi(x^i)\Theta_H(r)] \\ &= \dot{\varphi}(x^i)\Theta_H(r) + \varphi(x^i)\delta_H(r).\end{aligned} \tag{4.35}$$

The derivative of the step function is treated by noting that, by virtue of being on the null horizon (which for simplicity is taken just as $ds^2 = -dt^2 + dr^2 + dx^{A^2} = 0$),

$$\partial_t\Theta_H(r) = \frac{\partial\Theta_H(r)}{\partial r}\left(1 - \left(\frac{\partial x^A}{\partial t}\right)^2\right)^{1/2},$$

which for a nonrotating black hole simplifies to

$$\partial_t\Theta_H(r) = \partial_r\Theta_H(r).$$

This also works on a timelike stretched horizon 5.1 located a finite radius α_H from the event horizon. Schwarzschild time is related to the proper time τ of a fiducial observer via the metric element

$$d\tau = \sqrt{g_{00}}dt = (1 - 2MG/r)^{1/2}\,dt.$$

This gives

$$\partial_\tau\Theta_H(r) = \partial_r\Theta_H(r)\partial r/\partial\tau.$$

Applying this to a scalar field $\varphi_*(r)$ on the horizon, the momenta separates into a bulk term and an induced boundary field on the horizon. In the bulk, with $\alpha_H > 0$, $\varphi_*(r)$ obeys standard commutation relations

$$[\varphi_*(x^i), \varphi_*(x'^i)] = [\Pi_*(x^i), \Pi_*(x'^i)] = 0, [\varphi_*(x^i), \Pi_*(x'^i)] = i\delta^3(x - x').$$

The terms in $\Pi_*(x^I)$ with delta functions decouple and do not contribute in the bulk. On the boundary, all of the commutators for $\varphi_*(x^i)$ and $\Pi_*(x^i)$ vanish. However, the canonical momenta induces a surface term $\phi(x^i)$, with standard commutation relations. Therefore $\varphi(t, x^A)$ can be written in terms of creation an annihilation operators $a(x^i)$ and $a^\dagger(x^i)$ which operate tangent to the surface. Interestingly, using the fact that a tangent vector on the null horizon is also a normal vector, this implies that the creation and annihilation operators also contribute components orthogonal to the horizon.

While tensor fields are relatively straightforward, the case of spinor fields runs into difficulty. The Dirac equation can be extended to curved space by replacing partial with covariant derivatives, but the commutation rules for spinors are defined in terms of the metric, which on null surfaces becomes degenerate. There are approaches that avoid this difficulty, such as twistor theory, and its developments in string theory, but they are out of the scope of this thesis.

4.6 Action Principle

Two authors, M. Parikh and F. Wilczek [60] have developed an action principle formulation of the membrane paradigm. Their approach is to first begin with a general gravitational action with Dirichlet boundary conditions $\delta\varphi = 0$ at the spacetime boundaries of infinity and the event horizon. The condition for the action on the stretched horizon $\mathcal{H}_S$ is then expressed in a manner reminiscent of the method of images in electrodynamics. Introducing the fictitious surface action

$$S_{total} = (S_{out} - S_{surf}) + (S_{in} - S_{surf}). \tag{4.36}$$

The variation of this becomes

$$\delta S_{out} + \delta S_{surf} \equiv 0 \Rightarrow \delta S_{in} - \delta S_{surf} = 0. \tag{4.37}$$

The results of Thorne et al are recovered. The existence of an action principle facilitates a relatively straightforward extension of the membrane approach to any series of fields on $\mathcal{H}_S$, although it does not extend to spinors on the null horizon. It does, however, recover a similar semiclassical factor of $\hbar$ to leading order in the gravitational field equations as [10], due to its appearance in the action.

4.7 Black Hole Fluid Dynamics

Historically, Damour's treatment of the electrodynamic and fluid properties of black holes synthesized and generalized results of earlier research through the 1960's and 70's. Carter [46] realized that a black hole in gravitational and electromagnetic fields had an angular velocity and electric potential that was analogous to an object with finite viscosity and conductivity, while Hawking and Hartle [43] showed that a rotating black hole would experience a "tidal" slowing due to an orbiting mass, and interpreted this as similar to a "shallow sea of incompressible viscous fluid". Bekenstein [4] further developed this as a mechanical analogy, suggesting that vibrational quasinormal modes behaved similar to a "soap bubble model".

There are three main steps in recovering fluid behavior from the Einstein equations. First, one restricts Einstein's equations to a hypersurface on or just outside the horizon using either the Gauss-Codazzi equations or the Israel Junction condition. To interpret the resulting equations, we define a surface angular momentum called the *Hajicek 1-form*. In spacetimes where an angular Killing vector and global angular momentum exist the Hajicek 1-form is equivalent to the pullback of the angular momentum to the horizon. The Gauss and Codazzi equations (the first two equations in 2.26) can be written using Lie derivatives, which are then split into trace and trace free expressions for expansion tensor and torsion tensor, which vanishes. These are then interpreted in terms of the pressure, shear and bulk viscosities of the Navier-Stokes equations.

If we start with the Einstein tensor G_{ij} we can project this onto the horizon (or any codimension-1 surface) in three main ways; along the (called in shorthand) "null-null" direction $n^a n^b$, which gives us the Raychaudhuri equation governing energy density and evolution of the expansion, along a "null-spatial" $n^a e^b_\nu$, giving evolution of the momentum density (i.e. the Damour-Navier-Stokes equations), and lastly by pulling back along a "spatial-spatial" $e^a_\mu e^b_\nu$, which gives the so-called tidal equation governing evolution of the shear. All the other possible projections are either equivalent to the above options, or vanish.

The first projection, the Raychaudhuri equation, is essentially thermodynamic in nature, the second is fluid dynamics, and the third is related to, among other things, causing shear instability leading to turbulence. Thus there is a close relationship between the membrane paradigm and black hole thermodynamics. As the membrane paradigm is the momentum counterpart to black hole thermodynamics, it could equally well have been called *black hole fluid dynamics*.

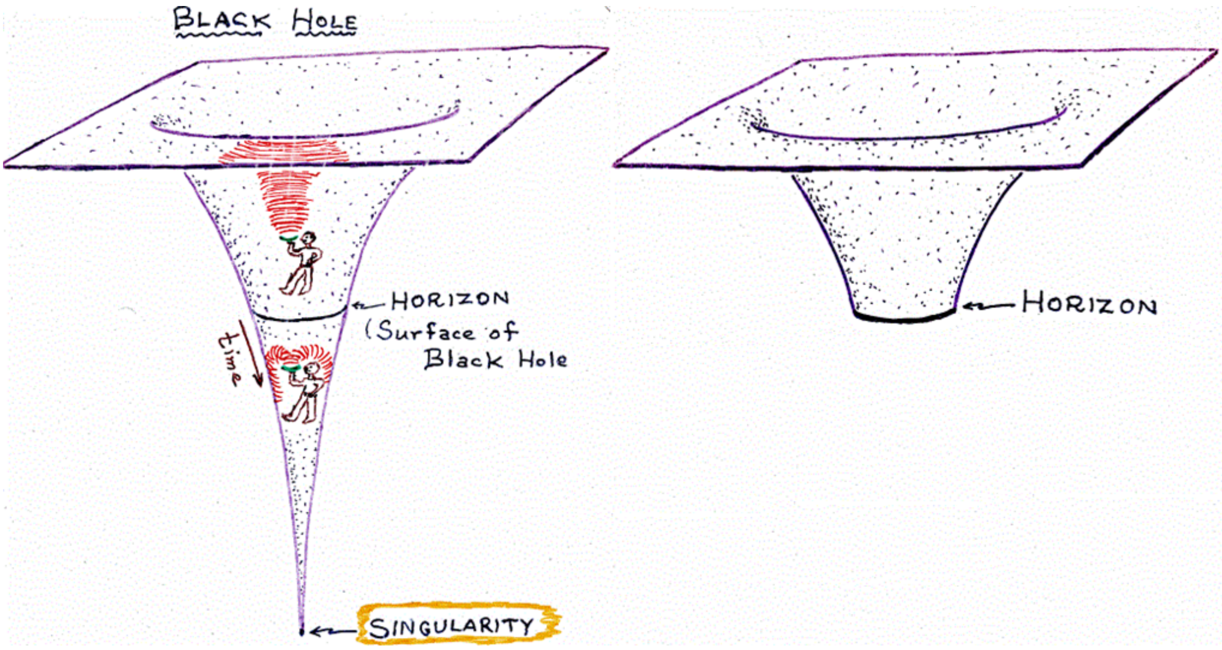

Figure 4.2: The black hole is truncated at or just outside the horizon. The horizon then has *induced* surface properties. Credit: T. Thorne, UCSB Plenetary Lecture, 1999. [50]

4.8 Damour-Navier-Stokes Equations

The existence of emergent fluid behavior on the horizon is a principle result of both the membrane paradigm and fluid/gravity correspondence of string theory. They share an underlying mechanism, which is symmetries of the Einstein equations along the null-spatial projection of a hypersurface. It is interesting that this can be interpreted in terms of fluid behavior, which typically arises from having many interacting particles in near thermal equilibrium, and this is perhaps related to what may be a dual equivalent description from general relativity. Here we will connect the Rindler horizon and null Gauss-Codazzi equations to fluid dynamics.

We first start generally: The total angular momentum of a spacetime can be defined as long as there exists a global rotational Killing vector.

$$\phi^a \partial_a = \partial_\varphi. \tag{4.38}$$

Although this will not exist in general, it is useful for motivating the definition of an angular momentum on the horizon. We can associate a conserved quantity with this via Noether's theorem

$$L = \frac{-1}{16\pi} \oint_\infty \nabla^\mu \phi^\nu d\Sigma_{\mu\nu}.$$

The angular momentum can then be divided into bulk and horizon quantities $J = J_B + J_H$, which motivates defining a horizon angular momentum as

$$L_H = \frac{-1}{8\pi} \oint_H k_\alpha \nabla_\beta n^\alpha \phi^\beta dA \equiv \oint \pi_\phi dA. \tag{4.39}$$

The quantity in the integrand is an intrinsic horizon momentum, which we can write in terms of the Hajicek 1-form Ω_μ. We define as our surface momentum density

$$\pi_\mu \equiv -\frac{1}{8\pi} k_\alpha \nabla_\mu n^\alpha = -\frac{1}{8\pi} \Omega_\mu. \tag{4.40}$$

Recall the contracted Gauss-Codazzi equations 2.27. To isolate the components on the 2-surface we can project along the induced metric, so that

$$R_{\alpha\beta} n^\alpha h^\beta_A = h^\beta_A (\nabla_\alpha \nabla_\beta n^\alpha - \nabla_\beta \nabla_\alpha n^\alpha) \tag{4.41}$$

Using the Rindler basis (∂_v, ∂_A) we expand the right hand side in terms of the extrinsic curvature, and then use 2.22 to include the expansion tensor

$$\begin{aligned} \nabla_\alpha n^\alpha &= \kappa + \theta \text{ and} \\ \nabla_\beta n^\alpha &= K^\alpha_\beta + (k_\gamma \nabla_\beta n^\gamma) n^\alpha - n_\beta k^\gamma \nabla_\gamma n^\alpha \\ &= \Theta^\alpha_\beta + \Omega_\beta n^\alpha - n_\beta k^\gamma \nabla_\gamma n^\alpha. \end{aligned} \tag{4.42}$$

Combining with 4.41

$$\begin{aligned} R_{\alpha\beta} n^\alpha h^\beta_A &= h^\beta_A \nabla_n \Omega_\beta + \Omega_A(\kappa + \theta) + h^\beta_A \nabla_\alpha \Theta^\alpha_\beta - \Theta_{A\alpha} \nabla_k n^\alpha - \nabla_A(\kappa + \theta) \\ &= h^\beta_A \mathcal{L}_n \Omega_\beta + \Omega_A \theta + \nabla_\alpha \Theta^\alpha_A - \nabla_A(\kappa + \theta). \end{aligned} \tag{4.43}$$

Here we have used the expansions

$$\begin{aligned} h^\beta_A \nabla_\alpha \Theta^\alpha_\beta &= \nabla_\alpha \Theta^\alpha_A + \Theta^\alpha_A (k^\beta \nabla_\beta n_\alpha + \Omega_\alpha) \text{ and} \\ h^\beta_A n^\alpha \nabla_\alpha \Omega_\beta &= h^\beta_A \mathcal{L}_n \Omega_\beta - \Theta^\alpha_A \Omega_\alpha - \kappa \Omega_A. \end{aligned}$$

Expanding the Lie derivative in 4.43 using $n^a \partial_a = \partial_v + v^A \partial_A$, we have the Damour-Navier-Stokes equation:

$$\partial_v \Omega_A + v^B \nabla_B \Omega_A + \Omega_B \nabla_A v^B + \theta \Omega_A = R_{\mu\nu} n^\mu h^\nu_A + \nabla_A(\kappa + \theta) - \nabla_\alpha \Theta^\alpha_A. \tag{4.44}$$

This is actually easier to interpret physically, as an evolution equation, if we observe that the Lie derivative of an area element of the horizon is

$$\mathcal{L}_n(dA) = \frac{1}{2} h^{\mu\nu} \mathcal{L}_n(h_{\mu\nu}) dy^2 \wedge dy^3 = \theta dA, \tag{4.45}$$

and the Lie derivative of the Hajicek 1-form, which plays the role of momentum density, modulo a normalization factor of -8π, is

$$\mathcal{L}_n(\Omega_A dA) = (\mathcal{L}_n\Omega_A + \Omega_A\theta)dA. \tag{4.46}$$

So that 4.44 becomes

$$\mathcal{L}_n(\Omega_A dA) = (\nabla_A(\kappa+\theta) - \nabla_\alpha\Theta^\alpha_A - R_{\mu\nu}n^\mu h^\nu_A)dA \tag{4.47}$$

Now, by substituting the stress-energy for the Ricci tensor, and replacing the Hajicek 1-form with the momentum density 4.40, the null *Damour-Navier-Stokes equation* for a momentum density on the horizon is

$$\frac{\mathcal{L}_n(\pi_A dA)}{dA} = 2\frac{\nabla_\alpha\sigma^\alpha_A}{16\pi} - \nabla_A\left(\frac{\kappa}{8\pi} + \frac{\theta}{16\pi}\right) + T_{\mu A}n^\mu. \tag{4.48}$$

The final term may be interpreted as a force density $F_A = T_{\mu A}n^\mu$.

The above equation corresponds to a fluid with a pressure $p = \frac{\kappa}{8\pi}$, shear viscosity $\eta = \frac{1}{16\pi}$, and bulk viscosity $\xi = -\frac{1}{16\pi}$. The sign on the pressure is opposite that expected in the Navier-Stokes equations. This is interpreted through the picture of the membrane phenomenologically being a "bubble" on the horizon. Conversely, the negative sign on the bulk viscosity indicates that the horizon fluid is unstable with respect to expansion. This occurs because the event horizon is specified by a future boundary condition, and teleological behavior arises in integrating the Green's function.

Observing that the Bekenstein-Hawking entropy is $1/4$ the expression for the shear viscosity, we have that the ratio of shear viscosity to entropy is

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{4.49}$$

This value is of considerable interest in applications of the fluid/gravity correspondence to condensed matter and high energy applications. It is a universal constant for Einstein gravity, but will have corrections if higher order terms are introduced into general relativity.

Also interesting is to notice that the tangential surface pressure equivalent to the normal surface gravity, as the null horizon has by definition $n^a n^b g_{ab} = 0$; this is consistent with the equipartition theorem of statistical mechanics.

4.9 Thermodynamics of the Raychaudhuri Equation

Here, we show that we can essentially perform the procedure in 3.3 backwards, projecting the Ricci tensor along a null hypersurface, and obtaining the Raychaudhuri equation. Then, we interpret the Raychaudhuri equation thermodynamically.

Beginning again with the Gauss-Codazzi equations 2.27, we contract along a null direction, and

$$
\begin{aligned}
R_{\alpha\beta}n^\alpha n^\nu &= (\nabla_\alpha K^\alpha_\beta - \nabla_\beta K)n^\beta \\
&= n^\beta \nabla_\alpha K^\alpha_\beta + n^\alpha n^\beta \nabla_\alpha \Omega_\beta + (\kappa + \theta) n^\alpha \Omega_\alpha - n^\alpha \nabla_\alpha (\kappa + \theta) \qquad (4.50) \\
&= -\Theta^{\alpha\beta}\Theta_{\alpha\beta} + \kappa\theta - n^\alpha \nabla_\alpha \theta. \qquad (4.51)
\end{aligned}
$$

Here we simplify this using the geodesic equation, the second equation in 4.42, and identities $K_{\mu\alpha}n^\alpha = 0$, and $\Omega_\alpha n^\alpha = \kappa$. Then substituting in for the extrinsic curvature $\Theta_{\alpha\beta}\Theta^{\alpha\beta} = \sigma^2 + \theta^2/2$, we obtain the null Raychaudhuri equation for non-affine parameterization

$$
R_{\alpha\beta}n^a n^b = -\nabla_n \theta + \kappa\theta - \left(\sigma^2 + \frac{\theta^2}{2}\right). \qquad (4.52)
$$

In order to link the Raychaudhuri equation with thermodynamics we can invoke the holographic principle to say $dA = 4dS$. However, before doing so, it is helpful to rewrite the expansion using Lie derivatives, via 4.45. A second Lie derivative of dA is

$$
\mathcal{L}_n(\theta dA) = (\nabla_n \theta + \theta^2) dA. \qquad (4.53)
$$

Then we can rewrite the Raychaudhuri equation, using the Unruh temperature 2.39 as

$$
(\mathcal{L}_n(dS) - \frac{1}{\kappa}\mathcal{L}_n(\theta dS)) = \frac{\dot{Q}}{T}. \qquad (4.54)
$$

We can then use this to determine the thermal dissipation for a fluid area element on the horizon,

$$
T(\mathcal{L}_n(dS) - \frac{1}{\kappa}\mathcal{L}_n(\theta dS)) = (\xi\theta^2 + 2\eta\sigma^2 + F_A)dA. \qquad (4.55)
$$

Truncating the left hand side to first order, we recover

$$
\delta Q = TdS, \qquad (4.56)
$$

exactly the relation 3.3 began with. Also, note that, approximating the Lie derivatives as ordinary derivatives, we can write a solution to the above Raychaudhuri equation (the tidal, or shear, equation evolves similarly) using a Green's function as

$$
(-\partial_v + \kappa)\, G(v, v') = \delta(v - v'), \qquad (4.57)
$$

with the form $G(v, v') = e^{\kappa(v - v')}$ for $v < v'$ and zero otherwise.

4.10 From Fluids back to Gravity

In the derivation of Einstein's equations 3.3 we take the Killing vector leading to our energy flux to be a boost Killing vector, which we parameterize as being approximately along a null tangential direction. However, the geometry of the null surface is $\mathbb{R} \times S^2$. It should also be possible to derive Einstein's equations from a momentum flux due to an angular Killing vector along the spatial surface. To do this we can consider an additional contribution which is generically of the form $\delta W = -PdA$.

Here I first generalize [10] to include work due to an angular momentum on the horizon. Instead of using the area and expansion relation 3.10 to incorporate the Raychaudhuri equation, with its "null-null" projection of the Ricci tensor, we use the Damour-Navier-Stokes equation 4.43, which gives a "null-spatial" projection of the Ricci tensor. For simplicity we take the angular Killing vector to be axisymmetric, so the momenta flux on the left hand side of $\delta E = \Omega_H \delta J$ is

$$\begin{aligned} \delta E &= \int T_{ab} \theta^A d\Sigma^b \\ &= \Omega_H \int T_{ab} n^a e^b_A d\lambda dA, \end{aligned} \tag{4.58}$$

where $y^a = (\lambda, \theta^A)$ parameterizes the null surface. On the right hand side, using $J = \int \pi_A dA$, we have

$$\begin{aligned} \Omega_H \delta J &= \Omega_H \delta \left(\int \pi_A dA \right) \\ &= \Omega_H \int \mathcal{L}_n (\pi_A dA). \end{aligned} \tag{4.59}$$

Here we write the variation as an infinitesimal directional covariant derivative,

$$\delta J = \nabla_n J = \mathcal{L}_n J, \tag{4.60}$$

which is locally equivalent to a Lie derivative. The density $\pi_A dA$ evolves according to the Damour-Navier-Stokes equation 4.47, which we substitute in and simplify

$$\begin{aligned} \delta J &= \mathcal{L}_n (\pi_A dA) \\ &= (R_{ab} n^a e^b_A + \nabla_A (\kappa + \theta) - \nabla_\alpha \theta^\alpha_A) dA \\ &= (R_{ab} n^a e^b_A) dA. \end{aligned} \tag{4.61}$$

The last line follows if we chose a slicing where the the pressure is constant, such as on the horizon, and if we look at short timescales (or Lie transport along a small dv on the horizon), then we can neglect the bulk viscosity and shear evolution.

Combining the above expressions,

$$\delta W = \Omega_H \int R_{ab} n^a e^b_A d\lambda dA.$$

Equating this with 4.58

$$8\pi T_{ab} = R_{ab} + \left(-\frac{R}{2} + \Lambda\right) g_{ab}. \tag{4.62}$$

We can now give 2.13 a thermal interpretation by contracting with a vector x^μ.

$$\begin{aligned} < (G_{ab} + \Lambda g_{ab}) n^a \Pi^b_\mu, x^\mu > &= -R_{ab} n^a n^b k_\alpha x^\alpha + R_{ab} n^a h^b_\alpha x^\alpha \\ \delta E &= T dS - P dA. \end{aligned} \tag{4.63}$$

The cosmological constant and Ricci scalar both drop out of the right hand side expression. The TdS term can be evolved backwards to recover the Ricci tensor 3.3 [10]. The same can be done for the $-PdA$ term, which is new.

This leads to an interesting general procedure. From quantum mechanics one can construct a many body quantum statistical mechanical system, after which a thermodynamical limit is taken. Then using thermodynamics, general relativity can be derived exactly. This does, however, again raise the question of how general relativity fits in *without* taking the thermal limit, a subject which is obviously much explored.

4.11 Entropy from Thermodynamics and Fluids

In the above approach, we can more generally write $\delta E = \delta Q + \delta W = TdS - PdA$. This has the same area dependence as the previously considered entropy term, and we can follow the same procedure of using the Raychaudhuri equation to evaluate it. Temporarily neglecting the entropy term, integrating the work and taking as the pressure the Damour surface pressure of the form $P = \kappa/8\pi G$,

$$\begin{aligned} \delta E &= -P dA \\ &= -\frac{\kappa}{8\pi G} \int R_{ab} n^a n^b \lambda d\lambda dA. \end{aligned} \tag{4.64}$$

Equating both sides yields the Einstein equations, but because we have a pressure, with the correct $8\pi G$ factor, rather than temperature, no factor of $\hbar$ appears.

It is interesting to note that for equilibrium situations the ratio of the fluid pressure to Unruh temperature in the above derivation is the Bekenstein-Hawking entropy.

$$dS = \frac{P}{T} dA = \frac{dA}{4G\hbar}. \tag{4.65}$$

This is consistent with the requirement of *maximum entropy production* for equilibria. This is derived on null surfaces in Rindler space, but the Gauss-Codazzi equations and Israel Junction condition contain essentially the same information, and admit both null and timelike descriptions, so it is straightforward to include timelike surfaces. Therefore, the Bekenstein-Hawking entropy/area relation is a thermal equilibrium condition on any null or timelike surface in Rindler space.

We can express the entropy due to a fluid element on the horizon, using the Damour-Navier-Stokes equations, as

$$\begin{aligned} dS &= \frac{\Omega_H}{T}\delta J \\ &= \frac{\Omega_H}{T}(R_{ab}n^a e^b_A + \nabla_A(\kappa + \theta) - \nabla_\alpha \theta^\alpha_A)dA. \end{aligned} \tag{4.66}$$

As TdS has units of force, we can also interpret the above expression as an entropic force balance of the form $TdS = -PdA$. By construction this is an entropic force that leads to classical gravity.

4.12 Slowness Parameter and Reynolds Number

The *slowness parameter* introduced by Price, Khanna and Hughes [77] has been connected to fluids by Jaramillo et al [78]. The slowness parameter is the ratio of the bulk to shear viscosity, and gives a timescale for changes in linear momentum or energy (depending on normalization) of the Rindler or black hole horizon. This can also be interpreted as the ratio of characteristic decay and oscillation timescales.

$$S = \left(\frac{\int \kappa\theta dA}{\int \sigma^2 dA}\right)^{1/2}. \tag{4.67}$$

Given deviations from Einstein gravity, the otherwise universal values for the bulk and shear viscosity can change (modulo evolution of the shear due to the tidal equation and changing surface gravity), leading to *observable* changes in this parameter through black hole or neutron star pair interactions. The physical situation is measurement of the gravitational waves emitted when the spin of massive colliding objects is aligned or anti-aligned, resulting in a *kick* due to conservation of angular momentum, and then an *anti-kick* related to the fluid response, inital spin, and trajectories.

As this is related to change in the entropy, due to the dynamical contributions mentioned above, it can be expressed in terms of changes in T and P in the previous section. This is also related to the description of inertial effects due to underlying string or entropic dynamics in [15]. Although this parameter is essentially just a ratio of terms in the Damour-Navier-Stokes, is potentially helpful for isolating situations where fluid deformation in gravity plays a predominant role.

Along with the shear viscosity to entropy ratio 4.49, we also have the Reynolds number, which characterizes inertial-dominated fluid turbulence.

Because we have a precise correspondence between the behavior of fluids and gravity, where we can both restrict to a hypersurface and obtain thermal and fluid equations, or start with fluid or thermodynamics and recover the Einstein equations, it is possible to ask a variety of interesting questions that explore these relations. For instance; given phenomena in the Navier-Stokes equations, what the corresponding behavior on the gravitational side? A prime example of such a phenomenon is turbulence in the Navier-Stokes equations, and its generalization to intrinsic turbulence in gravity.

In fluids, one has the Reynolds number characterizing regimes where turbulence typically occurs.

$$R_e = vL/\nu. \tag{4.68}$$

This is essentially the ratio of inertial to viscous forces, given a particular length scale of interest. Inertial forces in the Navier-Stokes are on the diagonal in the stress-energy tensor, and in their relativistic generalization are proportional to the expansion scalar θ. Viscous forces are symmetric and trace free, i.e. shear forces.

Therefore, the gravitational generalization of the Reynolds number is approximately

$$R_g = \frac{\theta}{\sigma^2}, \tag{4.69}$$

which is similar to the above-mentioned slowness parameter. In Rindler space, the growth in the gravitational Reynolds number mirrors the expansion scalar; close to the horizon inertial forces dominate, potentially indicating the possibility of "turbulent" underlying dynamics near the horizon. The relation of the above to conformal fluids is also a still-open topic.

5 Timelike Surfaces

Rather than repeating the Gauss-Codazzi approach discussed on null surfaces, I will use the Israel junction conditions on a timelike hypersurface to extend the fluid and thermal properties of null horizons to the stretched horizon of the membrane paradigm.

5.1 A Stretched Horizon

The *stretched horizon* is a timelike hypersurface located just outside the null horizon, at some proper distance $\rho > 0$. Like the true horizon, the stretched horizon obviates the need to consider anything beyond itself. However, because this surface is timelike, it has the advantage of allowing dynamical evolution, and it naturally truncates short distance infinities, which are generally assumed not exist in a correct theory of quantum gravity.

5.2 Stretched Horizon Dynamics

By using the Israel junction conditions, we can treat the stretched horizon as a thin shell of matter. It is relatively straightforward to examine dynamics, indeed, most of the heavy lifting is done by the formalism.

Proceeding immediately from the assumption of a distribution of matter with a discontinuous extrinsic curvature across the stretched horizon, as in 2.31, we have essentially two options.

$$8\pi S_{\mu\nu} = (K^+_{\mu\nu} - K^+ h_{\mu\nu}) - (K^-_{\mu\nu} - K^- h_{\mu\nu}). \tag{5.1}$$

We can make the physically-correct assumption that there is no surface stress-energy tensor at the horizon, set the above $S_{\mu\nu}$ to zero and equate the extrinsic curvature above and below the horizon, in which case this approach is not particularly useful. Alternatively, we can use the stress-energy tensor to account for energy which has crossed the horizon. Here we select the surface stress-energy tensor to annul the interior extrinsic curvature terms. i.e.

$$8\pi S_{\mu\nu} \equiv K^+_{\mu\nu} - K^+ h_{\mu\nu}. \tag{5.2}$$

This must also satisfy the Bianchi identity on the stretched horizon

$$\nabla_\alpha S^\alpha_\mu + [T^{\mu N}] = 0, \tag{5.3}$$

with $T^{\mu N}$ the tangent-normal component.

In the following we denote $\Sigma = S^{00}$ the energy component, and the momentum density as $\Pi = S^0_A$. Just as with the Gauss-Codazzi equations, the Bianchi constraint implies energy and momentum conservation equations on the stretched horizon. We have an energy constraint

$$\nabla_\alpha S^{\mu\alpha} = -T^{\mu N}_{+}, \tag{5.4}$$

where the right hand side is the energy flux from the stretched horizon into the bulk. Changing indices to the tangential direction

$$\nabla_\alpha S^{A\alpha} = -T^{AN}_{+}. \tag{5.5}$$

We can express these two conservation laws as evolution equations for the energy density and momentum density, in a manner reminiscent of the previously considered null cases.

The energy equation is

$$\mathcal{L}_n(\Sigma) + \theta\Sigma = -\frac{\kappa}{8\pi}\theta - \frac{\theta^2}{16\pi} + \frac{\sigma^2}{8\pi} + T^{\alpha\beta}n_\alpha n_\beta \tag{5.6}$$

and the momentum equation is

$$h^\mu_A \mathcal{L}_n(\Pi_\mu) + (\sigma^B_A + \frac{1}{2}\theta h^B_A)\Pi_B + \theta\Pi_A = -\nabla_A(\frac{\kappa}{8\pi} - \frac{\theta}{16\pi}) + \nabla_B\frac{\sigma^B_A}{8\pi} + T_{\alpha\beta}n^\alpha h^\beta_A. \tag{5.7}$$

5.3 Scaling Symmetry in the Navier-Stokes Equations

Here we begin with the Navier-Stokes equations in the form

$$\partial_t v_i - \eta\partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0. \tag{5.8}$$

The close connection between fluid and thermal behavior may also be seen by examining the scaling symmetry in the Navier-Stokes, and considering these as perturbations to a system in thermal equilibrium, which in this case is general relativity.

The Navier-Stokes equations obey a non-relativistic scaling symmetry that may be expressed in the ansatz:

$$\partial_i \to \epsilon\partial_i,\ \ \partial_t \to \epsilon^2\partial_t,\ \ v^i \to \epsilon v^i,\ \ P \to \epsilon^2 P \tag{5.9}$$

A family of solutions for the pair (v^i, P) may be parametrized by ϵ

$$\partial_i v^\epsilon_i - \eta\partial^2 v^\epsilon_i + \partial_i P^\epsilon + v^{\epsilon j}\partial_j v^\epsilon_i = 0. \tag{5.10}$$

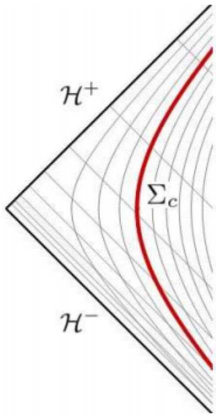

Figure 5.1: [63]

The limit $\epsilon \to 0$ is a hydrodynamic limit, and as we are doing a derivative expansion essentially any "well-behaved" theory will then yield fluid behavior. For the case of Einstein's equations, which we saw earlier arise as a thermodynamic equation of state, this can be interpreted as fluctuations about thermal equilibrium. It is interesting to consider that in general a fluid in thermal equilibrium is typically composed of a large number of interacting particles, and as we know that such a description is dual to gravity, this offers the intriguing possibility that general relativity emerges from an as yet unknown underlying theory.

5.4 Rindler Fluids

Consider the Einstein equations with the stress-energy tensor expressed in terms of a Brown-York tensor on a hypersurface

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{5.11}$$

with the Brown-York tensor given by

$$T_{\mu\nu} \equiv \frac{1}{8\pi G}(K h_{\mu\nu} - K_{\mu\nu}). \tag{5.12}$$

Perturbing the induced metric $h_{\mu\nu}$, via the fluid ansatz, will imply constraints of the form

$$\partial^\mu T_{\mu\nu} = 0 \tag{5.13}$$

on the stress-energy tensor to each order ϵ in its expansion.

The constraint to order $\mathcal{O}(\epsilon^2)$ is incompressibility of the Navier-Stokes, $\partial_i v^i = 0$. At $\mathcal{O}(\epsilon^3)$, we recover the Navier-Stokes equations.

$$\partial_t v_i - \eta \partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0. \tag{5.14}$$

We use the Rindler metric in ingoing coordinates:

$$ds^2 = -r dt^2 + 2dtdr + dx^i dx_i. \tag{5.15}$$

The induced metric is flat, and on the stretched horizon at radius constraint $r = r_H$. Following the approach in [63], we applying diffeomorphisms to obtain a general metric before applying the fluid ansatz. To this end, we require the form for a boost between two non-orthogonal axes:

$$t \to \gamma(t - \frac{\beta_i x^i}{\sqrt{r_H}}) \tag{5.16}$$

$$x^i \to x^i - \gamma \frac{\beta^i}{\sqrt{r_H}} t + \frac{(\gamma - 1)\beta^i \beta_j}{\beta^2} x^j \tag{5.17}$$

while a translation by c is:

$$r \to r - c \tag{5.18}$$

$$t \to \left(1 - \frac{c}{r_H}\right)^{-1/2} t. \tag{5.19}$$

Applying these general transformations causes the metric to deform as

$$\begin{aligned} ds^2 = &\frac{dt^2}{1 - v^2/r_H}\left(v^2 - \frac{r - c}{1 - c/r_H}\right) + \frac{2\gamma dtdr}{\sqrt{1 - c/r_H}} \\ &- \frac{2\gamma v_i dx^i dr}{r_H\sqrt{1 - c/r_H}} + \frac{2v_i dx^i dt}{1 - v^2/r_H}\left(\frac{r - c}{c - r_H}\right) \\ &+ \left(\delta_{ij} - \frac{v_i v_j}{r_H^2(1 - v^2/r_H)}\left(\frac{r - r_H}{1 - c/r_H}\right)\right) dx^i dx^j. \end{aligned} \tag{5.20}$$

The fluid pressure is identified as

$$p = \frac{1}{\sqrt{c - r_H}}, \tag{5.21}$$

and the fluid velocity is

$$u^\mu = \frac{1}{\sqrt{c - r_H}}(1, v^i). \tag{5.22}$$

Now applying the fluid ansatz to the pressure and velocity, we have that

$$v^i \to v_0 + \epsilon v^i = \epsilon v^i \tag{5.23}$$

$$P \to p_0 + \epsilon^2 P = \frac{1}{\sqrt{r_H}} + \frac{1}{r_H^{2/3}}\epsilon^2 P \tag{5.24}$$

Leading to the metric

$$\begin{aligned} ds^2 &= -r dt^2 + 2dtdr + dx_i dx^i \\ &- 2v_i\left(1 - \frac{r}{r_H}\right)dx^i dt - \frac{2v_i}{r_H}dx^i dr \\ &+ \left(1 - \frac{r}{r_H}\right)\left((v^2 + 2P)dt^2 + \frac{v_i v_j}{r_H}dx^i dx^j\right) + \left(\frac{v^2 + 2P}{r_H}\right)dtdr \\ &- O(\epsilon^3) \end{aligned} \tag{5.25}$$

Inserting this into

$$T_{\mu\nu} = \frac{1}{8\pi G}(Kh_{\mu\nu} - K_{\mu\nu}), \tag{5.26}$$

We have that

$$\begin{aligned} T_{\mu\nu}dx^\mu dx^\nu = &\frac{1}{\sqrt{r_H}}\left((-v_i dt + dx_i)^2 + \frac{v_i v_j + P\delta - 2r_H \partial_i v_j}{r_H}dx^i dx^j\right) \\ &+ O(\epsilon^3) \end{aligned} \tag{5.27}$$

Now we can check that to $O(\epsilon^2)$ the Bianchi identity gives an incompressibility condition

$$\partial^\mu T_{\mu 0} = r_H^{2/3}\partial_i v^i = 0 \tag{5.28}$$

and to $O(\epsilon^3)$ we recover the Navier-Stokes equations

$$\partial^\mu T_{\mu i} = r_H^{2/3}(\partial_t v_i - \eta\partial^2 v_i + \partial_i P + v^j \partial_j v_i) = 0. \tag{5.29}$$

The modern view of the membrane paradigm is perhaps best expressed locally in Rindler space, where consistency with the fluid/gravity correspondence of string theory, and its applicability to near horizon black holes and cosmological spacetimes such as de Sitter and anti-de Sitter are apparent.

5.5 Relativistic Fluids and Entropy Currents

The primary constraints for relativistic fluid dynamics are the conservation of the stress-energy tensor

$$\partial_\mu T^\nu_\mu = 0, \tag{5.30}$$

and the equation of continuity, expressing conservation of the number of particles,

$$\partial_\mu n^\mu = 0. \tag{5.31}$$

Here $n^\mu = nu^\mu$ is the particle flux 4-vector, with n^0 the number density of particles per volume and n^i the 3^d particle flux vector.

One can transform these constraints into a statement on the entropy of the system by using the thermodynamic relation $dQ = TdS$; the entropy current then comes by projecting the stress-energy conservation equation along a 4-vector u^μ and using the equation of continuity. The entropy current will then give us a conserved flow of entropy along a 4-vector. [4]

As an example of how this works in practice, consider an ideal relativistic fluid with

$$T^\nu_\mu = (\rho + P)u_\mu u^\nu + P\eta^\nu_\mu.$$

Conservation of the stress-energy tensor gives

$$u_\mu \partial_\nu((\rho + P)u^\nu) + (\rho + P)u^\nu \partial_\nu u_\mu + \partial_\mu P = 0.$$

Projecting this along u^μ and using the fact that $u^\mu \partial_\nu u_\mu = 0$,

$$u^\mu \partial_\nu T^\nu_\mu = -\partial_\nu((\rho + P)u^\nu) + u^\mu \partial_\mu P = 0.$$

We can incorporate the particle flux $n^\nu = nu^\nu$,

$$\partial_\nu \left(\frac{\rho + P}{n} nu^\nu \right) - \frac{1}{n} \partial_\mu(P) nu^\mu = 0,$$

and then apply the equation of continuity $\partial_\nu(nu^\nu) = 0$:

$$\partial_\nu \left(\frac{\rho + P}{n} \right) nu^\nu - \frac{1}{n} \partial_\mu(P) nu^\mu = 0$$

$$nu^\nu \left[\partial_\nu \left(\frac{\rho + P}{n} \right) - \frac{\partial_\nu P}{n} \right] = 0.$$

[4] If the stress-energy tensor is instead projected perpendicular to u^μ, then in this case one obtains the relativistic generalization of Euler's equation.

Here $\frac{1}{n} \equiv \frac{\Delta\text{vol}}{\#\text{ of particles}}$ is the molecular volume of the material.We rewrite $dQ = TdS$ as

$$d\left(\frac{\rho + P}{n}\right) - \frac{dP}{n} = Td\left(\frac{\sigma}{n}\right),$$

with σ the entropy per unit proper volume. This implies that the expression in brackets above is equivalent to dQ, giving

$$nu^\nu\left[\partial_\nu\left(\frac{\rho + P}{n}\right) - \frac{\partial_\nu P}{n}\right] = nu^\nu T\partial_\nu\left(\frac{\sigma}{n}\right) = 0.$$

Then rearranging the right-hand side and applying the equation of continuity gives a conserved entropy current.

$$\begin{aligned}
u^\nu\partial_\nu\left(\frac{\sigma}{n}\right) &= u^\nu\left[\partial_\nu(\sigma u^\mu)\frac{1}{n^\mu} - \frac{n}{n^\alpha n_\alpha}\sigma u^\mu\partial_\nu(u^\mu)\right] \\
&= u^\nu\partial_\nu(\sigma u^\mu)\frac{1}{n^\mu} \\
&= u^\nu\partial_\nu(\sigma u^\mu)\frac{1}{n^\mu}\delta_{\mu\nu} \\
&= \frac{1}{n}\partial_\nu(\sigma u^\nu) \\
&\Rightarrow \partial_\nu(\sigma u^\nu) = 0.
\end{aligned} \tag{5.32}$$

Using Einstein's equations we can rewrite conservation of the stress-energy tensor instead as conservation of the Einstein tensor $\partial_\mu G^\nu_\mu = 0$. The entropy current in this case takes on the interesting interpretation of being expressed in terms of the spatial curvature. Since we are able to link the entropy to Hawking's area theorem 4.7, it follows that the entropy current is linked to a corresponding geometric area current.

5.6 Entropy as the Noether Charge

Wald's "Black Hole Entropy is Noether charge" [79] is based on the observation that Noether's theorem links spacetime symmetries with conserved currents. We use this to link time symmetry with conservation of energy, translation symmetry with conservation of momentum, angular symmetry with conservation of angular momentum, and so on. Wald makes the observation that since the black hole entropy is proportional to its surface area, the surface area is related to its mass, the mass related to its energy, and the energy is a conserved Noether charge, it follows that the entropy can also be derived as a Noether charge with respect to a spacetime symmetry. Since black holes (and causal

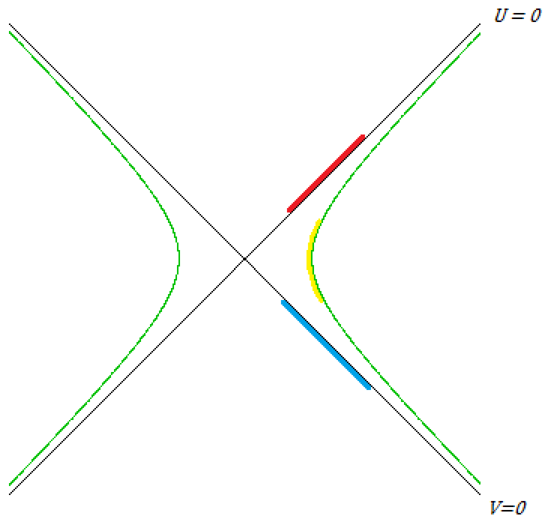

Figure 6.1: Red curve: The Damour-Navier-Stokes equations and Raychaudhuri equation result from projecting the Ricci tensor onto the horizon. The Raychaudhuri equation leads to $\delta Q = TdS$, and the Damour-Navier-Stokes evolves $\delta W = \Omega_H \delta J$ to the Einstein equations.
Yellow curve: $\delta Q = TdS$, and the Navier-Stokes equations are obtained on timelike hypersurface using either the Israel junction condition or the Gauss-Codazzi equations. The Navier-Stokes equations can also be viewed as arising via hydrodynamic scaling.
Blue curve: Starting from $dQ = TdS$ on $V = 0$ and evolving TdS with the Raychaudhuri equation yields Einstein's equations in the bulk.

horizons) are in general dynamical, the Noether charge should be too, and this gives an interesting way of approaching it. When dealing with dynamics, an entropy current is associated both with changes in the horizon area and the Noether current.

6 Conclusions

The principle result of this thesis is likely the generalization of Jacobson's thermal derivation of general relativity [10] to include a component on the spatial 2-surface. This is effectively evolving the first law of black hole thermodynamics backwards in Rindler space by using the Damour-Navier-Stokes equation to obtain the field equations. It also provides a connection to the Bekenstein-Hawking entropy (a similar approach is used in [8]

), and allows us to interpret (with inspiration from [15]) the horizon fluid and pressure as an entropic gravitational force $TdS = -PdA$.

We observe fluid dynamics in general relativity because gravity is thermal, and perturbations to systems in thermal equilibrium typically cause fluid behavior. This is the situation with gravity; perturbations on the 2-sphere, orthogonal to the null directions, obey the Damour-Navier-Stokes equation.

There is a deep relationship between turbulence on membranes, conformal field theory, and entanglement entropy. I will explore this in greater detail in the future.

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