General Relativity and Gravitation (2011)
DOI 10.1007/s10714-008-0705-6

RESEARCH ARTICLE

W. Muschik · H.-H. v. Borzeszkowski

Entropy identity and equilibrium conditions in relativistic thermodynamics

Received: 28 April 2008 / Accepted: 8 October 2008
© Springer Science+Business Media, LLC 2008

Abstract Starting out with the balance equations for energy-momentum, spin, particle and entropy density, an approach is considered which represents a framework for special- and general-relativistic continuum thermodynamics. A general entropy density 4-vector, containing particle, energy-momentum, and spin density contributions, is introduced. This makes possible, firstly, to test special entropy density
4-vectors used by other authors with respect to their generality and validity and, secondly, to determine entropy supply and entropy production. Using this entropy density 4-vector, material-independent equilibrium conditions are discussed. While in literature, generally thermodynamic equilibrium is determined by introducing a variety of conditions by hand, the present approach proceeds as follows: For a comparatively wide class of space–time geometries, the necessary equilibrium conditions of vanishing entropy supply and vanishing entropy production are exploited. Because these necessary equilibrium conditions do not determine the equilibrium, supplementary conditions are added systematically motivated by the requirement that also all parts of the necessary conditions have to be fixed in equilibrium.

Keywords General relativistic thermodynamics, Relativistic entropy, Relativistic thermodynamical equilibria, Spin in thermodynamics

1 Introduction

The present paper is dealing with equilibrium in an extended version of the so-called relativistic Theory of Irreversible Processes (rTIP). In its original formulation given by Eckart [1], this theory (or better: framework) is based on relations

W. Muschik H.-H. v. Borzeszkowski Institut für Theoretische Physik Technische Universität Berlin Hardenbergstr. 36 10623 Berlin, Germany muschik@physik.tu-berlin.de

that are generated by a transfer of the non-relativistic field formulation of Irreversible Thermodynamics into a Lorentz-covariant form from which its general-covariant version can be obtained as usual. The primary ingredients of the covariant theory are the basic dynamical variables: (symmetric) energy-momentum tensor T^{ki}, particle flux N^k, and entropy vector S^k, satisfying the balance relations

$$N^k_{\cdot;k} = 0, \quad T^{ki}_{\cdot\cdot;k} = 0 \tag{1}$$

and the dissipation inequality

$$S^k_{\cdot;k} \geq 0. \tag{2}$$

In literature, there are several ansatzes for the entropy vector: these are mostly motivated by certain heuristic considerations in equilibrium thermodynamics. The relations (1) and (2) are interpreted as a near-equilibrium framework presupposing that the (macroscopic) concept of entropy is valid also for near-equilibrium states. Theories of this framework diverge not only by different ansatzes for the entropy vector, but also by different assumptions for the heat flux, for the equilibrium properties of the kinematical invariants characterizing a fluid, etc. Accordingly, there exists a variety of formulations of rTIP today (for a review, see [2; 3; 4; 5]). In these theories, the question concerning the definition of equilibrium states is not always posed and, if posed, different answers occur. The only condition which is common to all approaches is that the entropy vector is divergence-free in equilibrium. All these considerations are performed on the level of rTIP for the theories of Special and General Relativity but not for relativistic gravitational theories going beyond General Relativity or beyond rTIP. Quite complete considerations of thermodynamic equilibrium in General Relativity can be found in [6; 7], where beside (1) and (2), Einstein's equations are taken into account. Vanishing divergence of the entropy vector implies here that the temperature vector is Killing or conform Killing [7].[1] By assuming the Onsager scheme of forces and fluxes, the Killing-property of the temperature vector is used to distinguish between complete and frozen equilibria [6].

However, here we are not dealing with definite complete gravitational theories and specified matter (constitutive) equations. Rather, we are here concerned with an extension of Eckart's framework to the case of balance equations which firstly are valid in Minkowski, Riemann and post-Riemann geometries, which secondly regard spin as well as production and supply terms in the energy-momentum and spin balances and which thirdly are not necessarily restricted to near-equilibrium situations.

Thus for an 1-component material, we start out with the balance equations of the particle flux 4-vector N^k, the energy-momentum tensor T^{ik} and the spin tensor

[1] The considerations made in [7] also show that the use of Einstein's equations allows to derive equilibrium conditions which generally are assumed to be ad-hoc. In our context, it is of particular interest that the often made asumption of a vanishing heat flux is not justified, if the kinematic fluid invariant "rotation" is unequal to zero. This illustrates that the equilibrium problem cannot be solved by defining ad-hoc conditions which are only justified by certain heuristic arguments. Rather, equilibrium conditions should be introduced in accordance with the fundamentals of the relativistic theory under consideration.

$S_{ji}^{\cdot\cdot k}$

$$N^k_{\cdot;k} = 0, \quad T^{ki}_{\cdot\cdot;k} = G^i + K^i, \quad S_{ji;k}^{\cdot\cdot k} = H_{[ji]} + L_{[ji]}{}^2 \tag{3}$$

and with the balance of the 4-entropy S^k

$$S^k_{\cdot;k} = \varphi + \sigma, \quad \sigma \geq 0. \tag{4}$$

As usual, the semicolon "$_;$" denotes the covariant derivative, T^{ik} is the energy momentum-tensor of a material which is not necessarily symmetric with vanishing covariant derivative, the spin tensor $S_{ji}^{\cdot\cdot k}$ is skew-symmetric in the lower indices, and φ and σ are the entropy supply and the entropy production, respectively. The force K^i and the angular momentum $L_{[ji]}$ are the external sources of the energy momentum tensor and of the spin tensor.

In the balance equation for the particle flux, we omit production and supply terms because we do not consider processes by which the particle number is not conserved, like pair creation. Rather, we want to construct a wide framework for competing relativistic theories of gravity and only the terms arising on the right-hand side of the energy-momentum and spin balances model theories which are different from general relativity theory (see below).

The supply terms K^i and $L_{[ji]}$ are substitutes for cases in which the energy-momentum tensor and/or the spin tensor do not include all fields, so that additional fields come into account by external sources. Of course, in a field theory describing systems completely by the equations of the fundamental fields, these external sources do not occur. If one is forced to introduce supply terms, this shows that the theory is not field-theoretically complete. To complete it, one has to describe the supply terms by additional fundamental fields in such a way that they can be absorbed by the other expressions in the balances (3) [8; 9].

The G^i and $H_{[ij]}$ are internal source terms caused by the choice of a special space–time and by the spin-momentum-energy coupling (SMEC). For instance in Einstein–Cartan geometry, the G^i and $H_{[ij]}$ are caused by the torsion and depend as coupling terms on the energy-momemtum and on the spin tensor. We call a theory for which the G^i and $H_{[ij]}$ vanish identically SMEC-free.

In contrast to Special Relativity Theory (SRT) and Einstein–Cartan Theory (ECT) [10; 11; 12] based on [13], General Relativity Theory (GRT), makes no general statements on the structure of spin and spin balances, except for that a spin tensor as an explicit source of gravity does here not occur. The spin of the matter source has only an implicit influence on the gravitational field insofar, as the source term in Einstein's equations (the symmetric metrical energy-momentum tensor), differs for different kinds of spinorial matter.

In some cases, where the total set of equations consists of Einstein's equations coupled to field equations of phenomenological matter, one can derive from this set, beside the energy-momentum balance, also a spin balance. For a Weyssenhoff fluid, particularly follows beside $K^i = 0$ the SMEC-term $H_{[ik]} = T_{[ik]}$ [14].

The non-negative entropy production σ in (4) represents the strong formulation of the Second Law of thermodynamics in field theories. The inequality

$$S^k_{\cdot;k} - \varphi \geq 0 \tag{5}$$

[2] Square brackets are also used to emphasize that the tensor is antisymmetric, $L_{(ji)} = 0$, especially for H_{ji} and L_{ji}.

is called the dissipation inequality.

The relations (3) and (4) are the relativistic generalization of the balance equations of non-relativistic continuum thermodynamics. In their special-relativistic version—with vanishing spin tensor, vanishing supply terms and vanishing SMEC-terms—they were introduced by Eckart [1] and Kluitenberg [15]. The quantities appearing in (3) and (4) are tensors with respect to Lorentz transformations, and the derivatives denoted by the semicolon have to be read as partial derivatives, if one refers to inertial systems with cartesian coordinates, and as covariant derivatives (with the Christoffel symbols as components of the Levi–Civita connection), if non-inertial reference systems are considered.

In this paper, we do not assume a special theory of gravitation, but we discuss the balance equations (3) and (4) for the general case of a curved space–time with a given, but arbitrary background specifying metric and connection. Attention is concentrated on the analysis of the dissipation inequality (4) from the point of view that it has to be satisfied by any ansatz for the entropy vector. The results are generally valid in SRT and have to be specified by additional conditions, especially for G^i and $H_{[ij]}$, in the case of non-SMEC-free relativistic gravitational theories.

According to this program, we start out with the balance equations (3) and (4) which are most general for the following reasons:

1. The balances (3) and (4) are valid in Minkowski space–time as well as in curved space–times which are characterized by a connection defining a covariant derivative (i.e. they are true also in Riemann-, Riemann-Cartan-, and metric-affine space–times),
2. The balances (3) and (4) imply external inputs (the right-hand sides of (3) and (4)), the so-called supplies of energy-momentum, spin, and entropy, and they imply the internal source terms caused by the spin-momentum-energy couplimg (SMEC) depending on the chosen space–time.
3. Entropy supply and entropy production are distiguished in the entropy balance. Therefore, in contrast to other approaches, the dissipation inequality takes the correct form (5) including the supplies.
4. By proving an identity for the entropy density 4-vector, an expression for the 4-entropy can be derived which is different from and more correct than ansatzes in the literature.

A (material-independent) theory only based on (3) and (4) is necessarily incomplete in a multiple manner, namely for the supplies, the missing specification of the state space describing the material, and due to the yet missing specification of the considered gravitational theory. In particular, the equilibrium conditions depend on the gravitational equations, too. For instance, as was shown for GRT [7], most equilibrium conditions ad-hoc introduced in [6] result from Einstein's equations. The advantage of an approach considered here is, that it represents a comparatively general framework for possible relativistic continuum thermodynamics.

For solving the system of differential equations (3) and (4) in chosen geometries, constitutive (matter) equations are needed, because the balances and field equations are valid for arbitrary, for the present unspecified materials. Here N^k, T^{ik}, S^k and $S_{ik}^{\cdot\cdot l}$ are constitutive mappings defined on a large state space (no after-effects) [16]

$$\mathbf{z} = (g_{ik}, \mathscr{T}_{ik}^{\cdot\cdot l}, n, e, s_{ik} \Xi_k, u^k, \ldots), \tag{6}$$

which may contain the geometrical fields, such as the metric g_{ik}, the torsion $\mathscr{T}_{ik}^{\cdot\cdot l}$, and the wanted basic fields $(n, e, s_{ik}, \Xi_k, u^k)$ (particle number density, energy density, spin density, spin density vector and another time-like vector field u^k which is arbitrary for the present) and beyond them other fields which depend on the considered material and which are of no interest here, because we are looking for *material-independent* properties. Consequently, a special constitutive equation will not appear in this paper.

In rTIP, stable thermodynamical equilibria are characterized by the fact, that the temperature 4-vector is Killing [6; 17], and is (conform) Killing in General Relativity [6; 7]. But this is only true, if T^{ik} is symmetric and if the space–time is SMEC-free, properties which are not valid in general and which are not presupposed here. Therefore the question arises and will be answered: Are there equilibrium conditions independently of constitutive properties in the framework of a general gravitation theory?

The paper is organized as follows: starting out with the 3-1-split of the quantities appearing in (3) and (4), we derive an identity for the entropy 4-vector [18] in Sect. 3. Using Eckart's interpretation of the time-like vector u^k as 4-velocity of the material [1] in Sect. 4, we are formulating material-independent equilibrium conditions in Sect. 6.

2 3-1-Split

The normalized time-like vector field u^k included in the state space (6) (signature of the metric is -2)

$$u^k u_k = a^2 > 0 \quad \longrightarrow \quad u^k u_{k;m} = 0 \tag{7}$$

is for the present arbitrary and can therefore be chosen in different ways. Here, it is introduced for spitting the quantities into their parts parallel and perpendicular to u^k. This split allows for a special interpretation later on. By introducing the projector belonging to (7)

$$h_l^k := g_l^k - \frac{1}{a^2} u^k u_l, \tag{8}$$

we obtain as shown in [19]

$$N^k = \frac{1}{a^2} n u^k + n^k, \quad S^k = \frac{1}{a^2} s u^k + s^k, \tag{9}$$

$$T^{ik} = \frac{1}{a^4} e u^i u^k + \frac{1}{a^2} u^i p^k + \frac{1}{a^2} q^i u^k + t^{ik}, \tag{10}$$

$$S_{ik}^{\cdot\cdot l} = \left(\frac{1}{a^2} s_{ik} + \frac{1}{a^4} u_{[i} \Xi_{k]} \right) u^l + s_{ik}^{\cdot\cdot l} + \frac{1}{a^2} u_{[i} \Xi_{k]}^{\cdot l}. \tag{11}$$

Here the following abreviations are introduced

$$n := N^k u_k, \quad n^k := h_l^k N^l, \tag{12}$$

$$s := S^k u_k, \quad s^k := h_l^k S^l, \tag{13}$$

$$e := u_l u_m T^{lm}, \quad p^k := h_l^k u_m T^{ml}, \quad q^k := h_l^k u_m T^{lm}, \quad t^{ik} := h_l^i h_m^k T^{lm} \tag{14}$$

$$s_{ik} := S_{ab}^{\cdot\cdot c} h_i^a h_k^b u_c, \quad \Xi_k := 2S_{ab}^{\cdot\cdot c} u_c u^a h_k^b, \tag{15}$$

$$s_{ik}^{\cdot\cdot l} := S_{ab}^{\cdot\cdot c} h_i^a h_k^b h_c^l, \quad \Xi_k^{\cdot l} := 2S_{ab}^{\cdot\cdot c} h_c^l u^a h_k^b. \tag{16}$$

The physical interpretation of the quantities in (12)–(16) remains uncertain, as long as the the time-like vector field u^k is not interpreted. Later on, the quantities in (15) and (16) are recognized as follows [20]: s_{ik} is the spin density, $s_{ik}^{\cdot\cdot l}$ the couple stress, Ξ_k the spin density vector and Ξ_k^l is the spin stress. These quantities are not independent of each other, but they are coupled by the spin axioms [20] which we will use later. Independently of any interpretation, the 4-entropy satisfies an identity which is derived in the next section.

3 The entropy identity

In literature, one finds different approaches to a special- and general-relativistic conception of entropy. Most of them is in common that entropy is described by a 4-vector, but there are proposed different expressions for it (which generally do not incorporate spin terms). For instance, in [2] the non-relativistic expression of the internal energy U for constant temperature and composition is generalized

$$U = TS - \mu n \tag{17}$$

(T = rest temperature, S = entropy, μ = chemical potential) which results by differentiation in classical thermodynamics of discrete systems together with the Gibbs equation in the Gibbs–Duhem equation. This yields an entropy 4-vector

$$S^k = \mu N^k + \frac{u_m}{T} T^{km} + p\frac{u^k}{T} = \mu N^k + \frac{u_m}{T}\left[T^{km} + p\delta^{km}\right] \tag{18}$$

(p = pressure).

Other authors make an ansatz for the entropy vector such that its covariant divergence becomes a relativistic generalization of the Carnot–Clausius relation,

$$d_e S = \frac{\delta Q}{T}. \tag{19}$$

Here “d_e” denotes a change caused by an external supply (see e.g. [6]). Accordingly, they assume[3]

$$S^k = \mu N^k + \frac{u_m}{T} T^{km}. \tag{20}$$

The procedure in this paper is quite different: we do not make ansatzes of the entropy vector S^k, but we start out with an identity which runs as follows:

◇ Independently of the special interpretation of the time-like vector field u^k, the following identity for the 4-entropy is valid:

$$S^k \equiv \left(s^k - \lambda q^k - \mu n^k - \Lambda^m \Xi_m^{\cdot k}\right) + \left(\mu N^k + \xi_l T^{kl} + \zeta^{nm} S_{nm}^{\cdot\cdot k}\right), \tag{21}$$

[3] For the present, a vector O_m is introduced which later on is identified to be equal to u_m/T.

with the following abbreviations:

$$\lambda \text{ arbitrary scalar, } \Lambda^k \text{ arbitrary tensor field of 1st order,} \tag{22}$$

$$\mu := \frac{1}{n}(s - \lambda e - \Lambda^m \Xi_m), \quad \xi_l := \lambda u_l, \quad \zeta^{nm} := 2u^n \Lambda^p h_p^m. \qquad \diamond \tag{23}$$

□ The proof is easy: Starting out with the relations (9)

$$s^k = S^k - \frac{s}{n}\left(N^k - n^k\right), \tag{24}$$

we obtain from (8), $(9)_1$, $(14)_1$ and $(14)_3$

$$q^k = u_m T^{km} - \frac{1}{a^2} e u^k = u_m T^{km} - \frac{e}{n}\left(N^k - n^k\right). \tag{25}$$

From $(16)_2$ follows by use of (8), $(9)_1$ and $(15)_2$

$$\begin{aligned} \Xi_m^{\cdot k} &= 2u^p h_m^q S_{pq}^{\cdot\cdot k} - \frac{2}{a^2} u^p h_m^q S_{pq}^{\cdot\cdot r} u^k u_r \\ &= 2u^p h_m^q S_{pq}^{\cdot\cdot k} - \frac{1}{n} \Xi_m \left(N^k - n^k\right). \end{aligned} \tag{26}$$

Summing up the last three equations multiplied with λ and Λ^m, we obtain

$$\begin{aligned} s^k - \lambda q^k - \Lambda^m \Xi_m^{\cdot k} = {} & S^k - \lambda u_m T^{km} - 2\Lambda^m u^p h_m^q S_{pq}^{\cdot\cdot k} \\ & + \frac{1}{n}(-s + \lambda e + \Lambda^m \Xi_m)\left(N^k - n^k\right) \end{aligned} \tag{27}$$

which is identical to (21). □

Consequently, the identity (21) is valid for arbitrary λ and Λ^m and for all time-like vector fields u^k.

The ad-hoc chosen entropy vector (20) is in accordance with the identity (21) by setting

$$\Lambda^m := 0, \quad \zeta^{nm} := 0, \quad \lambda := \frac{1}{T}, \quad s^k := \lambda q^k + \mu n^k. \tag{28}$$

But it is not quite clear, if (20) represents the most general ansatz also without spin, since the identity (21) allows for adding a space-like vector, the first bracket in (21). To clarify this question and for incorporating spin, we do not start out in Sect. 5 with a specific ansatz for the entropy vector, but with the identity (21).

In contrast to the expression (20) for the entropy 4-vector, (18) is not in accordance with the identity (21). If the chemical potential μ and the energy-momentum tensor T^{km} in (18) are the same quantities as in (21), we obtain for the spin-free case by comparing (18) with (21) the false equation

$$p\frac{u^k}{T} \stackrel{f!}{=} s^k - \lambda q^k - \mu n^k. \tag{29}$$

Consequently, μ and T^{km} in (18) are different from those in (21), or (18) is wrong.

Without any restriction of generality, from $(15)_2$, $(16)_2$, (21) and (23) follows, that Λ^m can be chosen orthogonal to u^m

$$\Lambda^m \doteq \Lambda^p h_p^m. \tag{30}$$

Later on, this choice makes an interpretation of Λ^m more easy.

In the next section, we will identify the time-like u^k field, thus resulting in an interpretation of the quantities (12)–(16).

4 Eckart and Landau–Lifshitz interpretation

Two different interpretations of the u^k can be found in literature: the first one is due to Landau–Lifshitz [4], the second one due to Eckart [1].

Landau–Lifshitz choose u^k as an eigenvector of the energy-momentum tensor

$$u_m T^{km} = \frac{e}{a^2} u^k. \tag{31}$$

By (25), this choice results in

$$q^k \equiv 0. \tag{32}$$

That means, this choice fixes 3 of the 16 free components of the energy-momentum tensor. Because this tensor represents a constitutive mapping, (31) is introducing a special constitutive property, a matter equation. Because we are looking for material independent statements, we do not accept the Landau–Lifshitz choice (31) of u^k.

Eckart's choice of u^k along $(9)_1$

$$u^k := \frac{a^2}{n} N^k, \quad a \equiv c, \quad \text{or} \quad n^k \equiv 0, \tag{33}$$

is more general than (31): It does not restrict the energy-momentum tensor or the spin tensor, because N^k is not a part of T^{ik} or $S_{ik}^{\cdot\cdot l}$. A second advantage is its illustrative interpretation: because the particle flux is purely convective and has no conductive part, u^k is according to (33) the material 4-velocity, and we obtain for the particle number flux according to Eckart

$$N^k = \frac{1}{c^2} n u^k, \tag{34}$$

an expression which is widely accepted in relativistic continuum physics.

5 Entropy balance

We now introduce Eckart's version into the entropy identity (21)

$$S^k \equiv (s^k - \lambda q^k - \Lambda^m \Xi_m^{\cdot k}) + (\mu N^k + \xi_l T^{kl} + \zeta^{nm} S_{nm}^{\cdot\cdot k}), \tag{35}$$

and (22) and (23) are still valid.

In order to determine the entropy vector in accordance with this identity, one can exploit the entropy balance (4)

$$S^k_{;k} = \varphi + \sigma, \tag{36}$$

and by differentiating (35) and by use of the balance equations (3), we obtain

$$\begin{aligned} S^k_{;k} = {} & (s^k - \lambda q^k - \Lambda^m \Xi_m^{\cdot k})_{;k} + \mu_{,k} N^k + \xi_{l;k} T^{kl} + \zeta^{nm}_{;k} S_{nm}^{\cdot\cdot k} \\ & + \xi_l [G^l + K^l] + \zeta^{nm}(H_{[nm]} + L_{[nm]}). \end{aligned} \tag{37}$$

To interpret this balance by physics, one has to identify the supply and production terms φ and σ. To this end, we refer to classical thermodynamics which defines the entropy supply as the energy supply r times the reciprocal rest temperature

$$\varphi := \frac{r}{T}. \tag{38}$$

The energy supply itself is caused by the external forces K^i and by the external moments $L_{[ik]}$. Consequently, we have by definition

$$r := u_i K^i + s_{lm} \Theta^{[lm][ik]} L_{[ik]}. \tag{39}$$

The tensor $\Theta^{[lm][ik]}$ which connects the spin to the external moments does not need to be specified for our purposes. Interesting is that the rest temperature T is introduced by $T = r/\varphi$ according to (38).

The entropy supply can be read off from (37), and a comparison with (38) results in

$$\varphi = \xi_l K^l + \zeta^{nm} L_{[nm]} = \frac{1}{T} u_i K^i + \frac{1}{T} s_{lm} \Theta^{[lm][ik]} L_{[ik]}. \tag{40}$$

This enables one to determine λ and Λ^m which were arbitrary up to now. From $(23)_2$ and $(23)_3$ follows

$$\xi_i = \frac{u_i}{T} = \lambda u_i, \tag{41}$$

$$\zeta^{[ik]} = \frac{1}{T} s_{lm} \Theta^{[lm][ik]} = 2u^{[i} h^{k]}_m \Lambda^m = u^i h^k_m \Lambda^m - u^k h^i_m \Lambda^m. \tag{42}$$

Multiplication of (42) with u_i and taking (30) into consideration results in

$$\lambda = \frac{1}{T}, \quad \Lambda^k = \frac{1}{c^2} \frac{s_{lm} u_i}{T} \Theta^{[lm][ik]}. \tag{43}$$

The vector (41) which is in accordance with the former definition $(28)_3$ is called the 4-temperature. The vector $(43)_2$ which later on will play a role for formulating the equilibrium conditions of the spin is called the temperature-spin.

After having determined the supply terms according to (41) and (42), the remaining terms on the left-hand side of (37) have to be considered as the entropy production according to (36)

$$\begin{aligned} \sigma = {} & \left(s^k - \frac{1}{T} q^k - \Lambda^m \Xi_m^{\cdot k} \right)_{;k} + \frac{1}{T} u_l G^l + \frac{2}{T} u^{[i} \Lambda^{k]} H_{[ik]} + \mu_{,k} N^k \\ & + \left(\frac{1}{T} u_l \right)_{;k} T^{kl} + 2(u^{[n} \Lambda^{m]})_{;k} S_{nm}^{\cdot\cdot k} \geq 0. \end{aligned} \tag{44}$$

This expression includes three terms of different characters, a divergence term of a space-like vector, the SMEC-terms and terms according to the usual flux-force scheme [21] of the entropy production. The divergence term contains fluxes which does not contribute to the entropy production. Therefore we define the entropy flux by

$$s^k := \frac{1}{T}q^k + \Lambda^m \Xi_m^{\cdot k}. \tag{45}$$

Finally taking (45) into account, the entropy production (44) results in

$$\sigma = \frac{1}{T}u_l G^l + \frac{2}{T}u^{[i}\Lambda^{k]}H_{[ik]} + \mu_{,k}N^k + \left(\frac{1}{T}u_l\right)_{;k}T^{kl} + 2(u^{[n}\Lambda^{m]})_{;k}S_{nm}^{\cdot\cdot k} \geq 0. \tag{46}$$

The entropy follows from (35), (41) and (42)

$$S^k = \mu N^k + \frac{1}{T}u_l T^{kl} + 2u^{[n}\Lambda^{m]}S_{nm}^{\cdot\cdot k}. \tag{47}$$

In contrast to the entropy production, the entropy does not contain SMEC-terms which are generated by differentiation. For vanishing spin density, (47) coincides with the ansatz (20). But (47) is a derived relation and not only a guessed ansatz. Beyond that, it includes the spin, and also the entropy flux density (45) and the entropy production density (46) follow consistently by the same procedure including the spin.

In [6] the possibility is briefly discussed, if the ansatz (20) for the entropy can be extended by adding a time-like vector. This possibility is excluded by the identity (35) which allows to add only a space-like vector, the first bracket in (35).

6 Equilibrium conditions

To obtain equilibrium conditions by this quasi-axiomatic approach to relativistic continuum thermodynamics in a consistent way, we proceed as follows: We analyze the conditions induced by vanishing entropy production density and by vanishing entropy supply density. Both the quantities are independent of each other zero in equilibrium, because entropy production and entropy supply are defined differently: Entropy supply vanishes for isolated sub-systems, whereas entropy production does not. Taking into account that introducing isolating partitions into an equilibrium system do not disturb its (local) equilibrium state, the entropy supply has to be zero in equilibrium. Because the entropy production describes the irreversible behavior of the system according to the Second Law, it has to be zero in equilibrium, too.

We call these conditions “necessary”, because without vanishing entropy production and supply there is no equilibrium at all (“necessary” is here used as in mathematical logic: If the necessary condition is not satisfied, the statement for which the condition is necessary is also not satisfied). Of course, these necessary conditions do not guarantee equilibrium. Consequently, we need “supplementary” (additional) conditions for characterizing equilibrium which we formulate here step-by-step. These supplementary conditions are motivated by the requirement that quantities occurring in the necessary conditions have to be to specified for equilibrium without assuming special classes of materials. If the necessary condition as well as the supplementary conditions are satisfied, they do not enforce equilibrium, because there may be further supplementary equilibrium conditions by specifying space–time and material.

We will mark both kinds of equilibrium conditions differently: the necessary ones by $\stackrel{\bullet}{=}$, the supplementary ones by $\doteq$. For the present, we consider the necessary conditions in the next section.

6.1 Necessary equilibrium conditions

The necessary equilibrium conditions are given by vanishing entropy production density (46) and vanishing entropy supply density (40)

$$\sigma_{eq} \stackrel{\bullet}{=} 0, \quad \varphi_{eq} \stackrel{\bullet}{=} 0 \quad \longrightarrow \quad S^{k\ eq}_{;k} = 0. \tag{48}$$

(equilibrium quantities are marked by $_{eq}$ or by ^{eq} in the sequel) and vanishing entropy flux density

$$s^k{}_{eq} \stackrel{\bullet}{=} 0. \tag{49}$$

The implication in (48) follows from (36).

For the present, we will exploit the entropy supply density $(48)_2$ by starting out with (40). Because the force K^i is independent of the momentum $L_{[ik]}$, the part of the necessary equilibrium conditions belonging to the entropy supply splits into two parts and using (42)and (30), we obtain

$$u_i{}^{eq} K^i{}_{eq} = 0, \quad 2(u^{[i}\Lambda^{k]})_{eq} L_{[ik]}{}^{eq} = 0. \tag{50}$$

From (50) we read off, that for the present neither the external forces nor the external moments have to be zero in equilibrium. Using the balance equations $(3)_{2,3}$, we obtain

$$u_i{}^{eq}[T^{ki}_{;k} - G^i]_{eq} = 0, \quad (u^{[i}\Lambda^{k]})_{eq}[S_{ik}^{\cdot\cdot j}{}_{;j} - H_{[ik]}]_{eq} = 0. \tag{51}$$

From (47) follows by $(48)_3$

$$0 = \Big(\mu N^k\Big)^{eq}_{;k} + \left(\frac{1}{T} u_l T^{kl}\right)^{eq}_{;k} + 2\left(u^{[n}\Lambda^{m]} S_{nm}^{\cdot\cdot k}\right)^{eq}_{;k}. \tag{52}$$

The N^k, T^{kl} and $S_{nm}^{\cdot\cdot k}$ are not independent of each other, because they are coupled by constitutive equations and by the SMEC-terms. Therefore we cannot state that each term of the sum (52) vanishes. The equilibrium condition (52) is only one equation which cannot describe equilibrium completely. Therefore we need supplementary equilibrium conditions beyond (48)and (49). These conditions will be considered in the next section.

6.2 Supplementary equilibrium conditions

6.2.1 Supply conditions

According to the necessary condition $(50)_1$, the power of the external forces is zero in equilibrium. From that one cannot conclude that the external forces vanish themselves in equilibrium. There exist an easy criterion for testing whether the external forces vanish in equilibrium: Starting out again with $(50)_1$, we see that in equilibrium the 4-component of the force is zero in the rest system, marked by ^{R},

$$^{R}K^{4}{}_{eq} = 0. \tag{53}$$

If now also the 3-components of the force vanish in the rest system

$$^{R}K^{\alpha}{}_{eq} = 0, \quad \alpha = 1,2,3, \tag{54}$$

we obtain the very special supplementary equilibrium condition

$$K^{i}{}_{eq} \doteq 0 \tag{55}$$

for the external forces.

According to $(43)_2$, the necessary equilibrium condition $(50)_2$, depends on the spin density $(15)_1$ and the temperature. There may be non-zero $\Lambda^{k}{}_{eq}$-fields depending on the external moments in such a way that $(50)_2$ is satisfied, but this situation is so strange, that we do not take this seriously into consideration. Consequently, we obtain two supplementary equilibrium conditions

$$\Lambda^{k}{}_{eq} \doteq 0 \cup L_{[ik]}{}^{eq} \doteq 0, \tag{56}$$

that means, the external moments have to vanish in equilibrium in systems of non-vanishing spin. If the external moments do not vanish, the system must be spin-free in equilibrium. These statements are true except for the exotic situation that $(50)_2$ is satisfied for non-vanishing $\Lambda^{k}{}_{eq}$ and $L_{[ik]}{}^{eq}$.

6.2.2 N^k-condition

To begin with the supplementary equilibrium conditions, we consider by use of $(3)_1$, (34) and the abbreviation $^{\bullet} :=_{;k} u^k$

$$\left(\mu N^k\right)_{;k} = \mu_{,k}\frac{1}{c^2}nu^k = \frac{1}{c^2}n\mu^{\bullet}. \tag{57}$$

Because it is obvious that there are no non-vanishing material time derivatives in equilibrium except that of the acceleration $u^{k\bullet}$,[4] we demand as a first supplementary equilibrium condition

$$\boxplus^{\bullet} \doteq 0, \quad \boxplus \neq u^k \quad \longrightarrow \quad \left(\mu N^k\right)_{;k}{}^{eq} = 0. \tag{58}$$

[4] The $^{\bullet}$ is the relativistic analogue to the non-relativistic material time derivative d/dt which describes the time rates of a rest-observer. Therefore, d/dt is observer-independent and zero in equilibrium [22; 23].

Consequently, we obtain by (34)

$$0 = \Big(\mu N^k\Big)_{;k}^{eq} = \Big(\mu \frac{1}{c^2} n u^k\Big)_{;k}^{eq} = \Big(\mu \frac{1}{c^2} n\Big)^{\bullet}_{eq} + \mu \frac{1}{c^2} n u^{k\ eq}_{;k} \tag{59}$$

which by $(58)_1$ results in

$$u^{k\ eq}_{;k} = 0. \tag{60}$$

Further we obtain by $(58)_1$ and (60)

$$\Big(\frac{1}{T} u^k\Big)_{;k} = \frac{1}{T} u^k_{;k} + \Big(\frac{1}{T}\Big)^{\bullet} \quad \longrightarrow \quad \Big(\frac{u^k}{T}\Big)_{;k}^{eq} = 0. \tag{61}$$

Hence, the vanishing first term in (52) is exploited by applying the supplementary equilibrium condition (58). Now we will consider the next term.

6.2.3 T^{kl}-condition

Using $(25)_1$, we obtain for the second term in (52)

$$\Big(\frac{u_l}{T} T^{kl}\Big)_{;k} = \Big(\frac{q^k}{T} + \frac{1}{c^2 T} e u^k\Big)_{;k} = \Big(\frac{q^k}{T}\Big)_{;k} + \Big(\frac{e}{c^2 T}\Big)^{\bullet} + \frac{e}{c^2 T} u^k_{;k} \tag{62}$$

which by use of $(58)_1$ and (60) results in

$$\Big(\frac{u_l}{T} T^{kl}\Big)_{;k}^{eq} = \Big(\frac{q^k}{T}\Big)_{;k}^{eq} = \Big(\frac{1}{T}\Big)_{,k}^{eq} q^k_{eq} + \Big(\frac{1}{T}\Big)^{eq} q^{k\ eq}_{;k}. \tag{63}$$

The first term of the right-hand side represents the dissipation due to heat conduction which is zero in equilibrium, a statement which represents an other supplementary equilibrium condition

$$\Big(\frac{1}{T}\Big)_{,k}^{eq} q^k_{eq} \doteq 0 \quad \longrightarrow \quad q^k{}_{eq} = 0. \tag{64}$$

Because there are equilibria with non-vanishing temperature gradient (e.g. in gravitational fields) and because the dissipation due to heat conduction is always not negative

$$\Big(\frac{1}{T}\Big)_{,k} q^k \geq 0, \tag{65}$$

and the heat flux density depends continuously on the temperature gradient, the conclusion in (64) is the only possible one [24]. But it is also obvious that there are no heat fluxes in equilibrium. From $(64)_2$ follows

$$q^{k\ eq}_{;k} = 0, \quad \Big(\frac{q^k}{T}\Big)^{eq} = 0. \tag{66}$$

Taking $(64)_2$ into account, (63) results by use of $(51)_1$ in

$$0 = \left(\frac{u_l}{T}T^{kl}\right)_{;k}^{eq} = \left(\frac{u_l}{T}\right)_{;k}^{eq} T^{kl}_{eq} + \frac{u_l}{T}^{eq} G^l_{eq}. \tag{67}$$

If we consider the special case that the energy-momentum tensor is symmetric in equilibrium (what is not the case in general), then (67) results in

$$T^{[kl]}_{eq} \equiv 0 \quad \longrightarrow \quad \left(\frac{1}{T}u_{(l}\right)_{;k)}{}^{eq} T^{kl}{}_{eq} = -\frac{u_l}{T}^{eq} G^l_{eq}. \tag{68}$$

In general, we cannot conclude from $(68)_2$, that the temperature 4-vector u_l/T is Killing in equilibrium even for SMEC-free space–times,

$$\left(\frac{1}{T}u_l\right)_{;k}{}^{eq} + \left(\frac{1}{T}u_k\right)_{;l}{}^{eq} \stackrel{?}{=} 0, \tag{69}$$

because we do not presuppose a symmetric T^{kl}, as it was assumed in [6]. Presupposing (69), no additional equilibrium conditions would follow for the symmetric T^{kl}, because (68) is satisfied identically for SMEC-free space–times. We now treat the general case.

After a short calculation, we obtain in non-equilibrium by using (10)

$$\left(\frac{u_l}{T}\right)_{;k} T^{kl} = \frac{1}{T}u_l{}^{\bullet}\frac{1}{c^2}p^l + \frac{1}{T}u_{l;k}t^{kl} + \left(\frac{1}{T}\right)^{\bullet}\frac{1}{c^2}e + \left(\frac{1}{T}\right)_{,k} q^k. \tag{70}$$

Taking (8), $(14)_2$, $(58)_1$ and $(64)_2$ into account, (67) results in

$$0 = u_{l;k}{}^{eq} t^{kl}{}_{eq} + u_l^{eq} G^l_{eq}. \tag{71}$$

As we can see easily, the following identity is valid

$$0 \ = u_{l;k}{}^{eq} t^{kl}{}_{eq} + u_l^{eq} G^l_{eq} = u_{l;k}{}^{eq}\left[t^{kl}{}_{eq} + \frac{u_p^{eq} G^p_{eq}}{u_{p;q}{}^{eq} A^{qp}} A^{kl}\right], \tag{72}$$

for all A^{kl} with

$$u_{p;q}{}^{eq} A^{qp} \neq 0, \quad u_p{}^{\bullet}{}_{eq} u_q{}^{eq} A^{qp} = 0. \tag{73}$$

We need the second property for later use. Consequently, we can introduce non-unique modified stress tensors which include the SMEC-term

$$\tau^{kl} := t^{kl}{}_{eq} + J^{kl}, \quad J^{kl} := \frac{u_p^{eq} G^p_{eq}}{u_{p;q}{}^{eq} A^{qp}} A^{kl} \tag{74}$$

and (71) results in

$$0 = u_{l;k}{}^{eq} \tau^{kl}, \tag{75}$$

a result which can be also expressed in another way.

As usual [6], we introduce the kinematical invariants shear σ_{ab}, rotation ω_{ab}, accelerationu $u^{\bullet}{}_a$ and expansion Θ satisfying the relations

$$h_{ab}u^b = u^{\bullet}{}_a u^a = \sigma_{ab}u^b = \omega_{ab}u^b = 0, \tag{76}$$

$$u_{a;b} = \sigma_{ab} + \omega_{ab} + \frac{1}{3}\Theta h_{ab} + \frac{1}{c^2}u^{\bullet}{}_a u_b. \tag{77}$$

By use of (60) and $(73)_2$, (75) results in

$$u_{l;k}{}^{eq}\tau^{kl} = [\sigma_{lk} + \omega_{lk}]^{eq}\tau^{kl} = \sigma_{lk}^{eq}\tau^{(kl)} + \omega_{lk}^{eq}\tau^{[kl]} = 0. \tag{78}$$

Because the symmetric and the antisymmetric part of the stress tensor are independent of each other, we can split (78) into

$$\sigma_{lk}^{eq}\tau^{(kl)} = 0, \quad \omega_{lk}^{eq}\tau^{[kl]} = 0. \tag{79}$$

Because the tensor A^{kl} in $(74)_2$, and consequently also J^{kl} in $(74)_1$, can be chosen arbitrarily, the SMEC-term can be distributed freely on the shear or rotation terms: If A^{kl} is chosen to be symmetric, no part of the SMEC-term appears in the rotation part and vice-versa.

The equilibrium conditions (79) can be interpreted differently: if we are looking for equilibrium conditions which are the same for all space–times and materials, that means, they are valid for arbitrary $\tau^{(kl)}$ and $\tau^{[kl]}$, we obtain

$$\sigma_{lk}^{eq} \doteq 0, \quad \omega_{lk}^{eq} \doteq 0 \tag{80}$$

as supplementary equilibrium conditions.

The second interpretation is as follows: because (79) are derived material-independently, there may be shear and rotation fields different from zero satisfying (79) for special chosen space–times and materials. That means, there are special material- and space–time-dependent equilibria having non-vanishing shear and/or rotation. By these remarks, the second necessary equilibrium condition (67) is exploited, and we have now to consider the equilibrium conditions belonging to the spin.

6.2.4 $S_{nm}^{\cdot\cdot k}$-condition

Taking $(66)_2$ and the necessary equilibrium condition (49) into account, we obtain from (45) and $(58)_1$

$$\Lambda_{eq}^{m}\Xi_m^{\cdot keq} = 0. \tag{81}$$

Because the spin stress $\Xi_m^{\cdot k}$ is not regular

$$\Xi_m^{\cdot k}u_k = 0, \quad u^m\Xi_m^{\cdot k} = 0, \tag{82}$$

according to $(16)_2$, and

$$\Xi_m^{\cdot keq}u_k{}^{\bullet}{}_{eq} = 0, \quad u^{m\bullet}{}_{eq}\Xi_m^{\cdot keq} = 0, \tag{83}$$

according to $(58)_1$ and (82), there is the possibility that in equilibrium non-zero temperature-spins are in the kernel of the spin stress as solution of (81).

We obtain from (52) by $(58)_2$ and (67) another necessary equilibrium condition

$$\left(u^{[n}\Lambda^{m]}S_{nm}^{\cdot\cdot k}\right)_{;k}^{eq} = 0. \tag{84}$$

As derived in [19],

$$u^n\Lambda^m S_{nm}^{\cdot\cdot k} = \frac{1}{2}\Lambda^m\left(\Xi_m^{\cdot k} + \frac{1}{c^2}\Xi_m u^k\right) \tag{85}$$

is valid. Consequently, by taking $(58)_1$, (60) and (81) into account, (84) results in

$$0 = \left[\Lambda^m\Xi_m u^k\right]_{;k}^{eq} = (\Lambda^m\Xi_m)^{\bullet}{}_{eq} \tag{86}$$

and according to $(43)_2$, we obtain

$$\Lambda^k\Xi_k = \frac{1}{Tc^2}s_{lm}\Theta^{[lm][ik]}u_i\Xi_k. \tag{87}$$

The spin variables (15), that are the spin density s_{nm} and the spin vector Ξ_m, and the constitutive equations (16), that are the couple stress $s_{nm}^{\cdot\cdot k}$ and the spin stress Ξ_m^k, are related by the spin axioms [20]

$$s_{nm} = \frac{1}{2}\eta_{nmpq}u^p\Xi^q, \tag{88}$$

$$s_{nm}^{\cdot\cdot k} = \frac{1}{2}\eta_{nmpq}u^p\Xi^{qk}. \tag{89}$$

Here η is the Levi-Civita symbol. The spin axioms are caused by the fact that there are only three spin fields and only nine constitutive spin equations [20].

Inserting (88) into (87) results in

$$\Lambda^j\Xi_j = \frac{1}{2Tc^2}\eta_{lm}^{\cdot\cdot pq}\Theta^{[lm][ik]}u_p\Xi_q u_i\Xi_k, \tag{90}$$

and by taking $(58)_1$ into account, (86) becomes

$$0 = \eta_{lm}^{\cdot\cdot pq}\Theta_{eq}^{[lm][ik]}\Xi_q{}^{eq}\Xi_k{}^{eq}(u_p{}^{\bullet}{}_{eq}u_i{}^{eq} + u_p{}^{eq}u_i{}^{\bullet}{}_{eq}). \tag{91}$$

The case of a non-linear coupling tensor is also included, because

$$\Theta_{eq}^{[lm][ik]\bullet}(\Xi_p, \Xi_p^q) = 0 \tag{92}$$

is valid.

Because in (91) the antisymmetric parts of the quadratic forms in (q,k) and (p,i) do not contribute, we obtain

$$\begin{aligned}&[\eta_{lm}^{\cdot\cdot pq}\Theta_{eq}^{[lm][ik]} + \eta_{lm}^{\cdot\cdot pk}\Theta_{eq}^{[lm][iq]} + \eta_{lm}^{\cdot\cdot iq}\Theta_{eq}^{[lm][pk]} + \eta_{lm}^{\cdot\cdot ik}\Theta_{eq}^{[lm][pq]}]\\ &\quad\Xi_q{}^{eq}\Xi_k{}^{eq}(u_p{}^{\bullet}{}_{eq}u_i{}^{eq} + u_p{}^{eq}u_i{}^{\bullet}{}_{eq}) \ = 0.\end{aligned} \tag{93}$$

The tensor of 4th order in the square bracket has the following properties: it is symmetric in (q,k) and in (p,i), and it has an empty kernel according to the coupling property (39). In [19] is proven that the only solutions of (93) are

$$u_p{}^\bullet{}_{eq} \neq 0 \quad \longrightarrow \quad \Xi_q{}^{eq} = 0 \quad \longleftrightarrow \quad \Lambda^q_{eq} = 0, \tag{94}$$

or

$$\Xi_q{}^{eq} \neq 0 \quad \longrightarrow \quad u_p{}^\bullet{}_{eq} = 0. \tag{95}$$

Thus, we proved the following remarkable statement

> If the acceleration does not vanish in equilibrium, the system has to be spin-free, and if the system is not spin-free, the acceleration has to vanish in equilibrium.

The equilibrium conditions (94) and (95) have to be comparable with (81) and (83). This results in

$$u_p{}^\bullet{}_{eq} \neq 0 \quad \longrightarrow \quad u_p{}^\bullet{}_{eq} \in \ker \Xi_m^{\cdot p}{}_{eq} \cap \Xi_p{}^{eq} = 0, \tag{96}$$
$$\Xi_p{}^{eq} \neq 0 \quad \longrightarrow \quad \Xi_p{}^{eq} \in \ker \Xi_m^{\cdot p}{}_{eq} \cap u_p{}^\bullet{}_{eq} = 0. \tag{97}$$

We obtain from (93) to (97) that equilibrium is possible in the following cases

$$u_p{}^\bullet{}_{eq} = 0 \cap \Xi_q{}^{eq} = 0, \tag{98}$$
$$u_p{}^\bullet{}_{eq} \neq 0 \cap \Xi_q{}^{eq} = 0 \cap u_p{}^\bullet{}_{eq} \in \ker \Xi_m^{\cdot p}{}_{eq}, \tag{99}$$
$$\Xi_q{}^{eq} \neq 0 \cap u_p{}^\bullet{}_{eq} = 0 \cap \Xi_p{}^{eq} \in \ker \Xi_m^{\cdot p}{}_{eq}. \tag{100}$$

As (99) and (100) show, constitutive properties may prevent equilibrium. Whereas in equilibrium the acceleration is always in the kernel of the spin stress according to (83), it depends of the material, if the spin density vector is an element of the kernel of the spin stress in equilibrium. According to (98), the equilibrium is material independent only in spin-free materials with zero acceleration. There are no equilibria with $u_p{}^\bullet{}_{eq} \neq 0$ and $\Xi_q{}^{eq} \neq 0$.

7 Conclusions

After having proved the unrenouncable entropy identity (21), for the present the most general relativistic expression for the entropy density 4-vector was derived. It contains three parts belonging to particle current, energy-momentum and spin. After that, arguments are given in favor of Eckart's ansatz of the particle flux density 4-vector being parallel to the 4-velocity of the material under consideration. As a consequence,
entropy supply and entropy production can be determined as expressions of relativistic invariant terms given by the balances (3) of energy-momentum and spin.

As a further implication of the entropy identity (21) and Eckart's ansatz, it can be shown, that the entropy density 4-vector, ad-hoc introduced in [6], is the correct one (in case of General Relativity) except the missing spin part, while the entropy expression given in [2] contradicts the entropy identity (21). The latter follows from the fact that the expression correctly given in [6] must not be supplemented by a time-like vector, as it was supposed in [6] and done in [2].

After the more general considerations, the second part of the paper is devoted to material-independent equilibria in relativistic thermodynamics. For the present, equilibrium is defined by necessary equilibrium conditions: According to the second law, entropy supply, entropy production and consequently, entropy 4-flux vanish in equilibrium. From this demand, four equations [$(48)_1$, (49) and (50)] follow.

The above mentioned four necessary equilibrium conditions are not sufficient for equilibrium. Consequently, we have to complete these necessary equilibrium conditions by supplementary ones. These *supplementary equilibrium conditions* are

- The vanishing entropy supply results in
  1. the power $(50)_1$ generated by the forces has to vanish in equilibrium. Sufficient for vanishing power is the supplementary equilibrium condition that the forces themselves are zero in equilibrium (55)

$$u_i{}^{eq} K^i{}_{eq} = 0 \quad \longleftarrow \quad K^i{}_{eq} = 0. \tag{101}$$

  2. if the material is not spin-free, the external moments have to vanish (56). If they do not, the system has to be spin-free in equilibrium

$$\Lambda^k{}_{eq} \neq 0 \quad \longrightarrow \quad L_{[ik]}{}^{eq} = 0, \tag{102}$$

$$\Lambda^k{}_{eq} = 0 \quad \longleftarrow \quad L_{[ik]}{}^{eq} \neq 0. \tag{103}$$

- Stemming from the entropy production and generated by particle flux density,
  3. the material time derivatives have to vanish in equilibrium, except that of the 4-velocity

$$\boxplus_m{}^{\bullet eq} := \boxplus_{m;k}{}^{eq} u^k{}_{eq} = 0, \quad \boxplus_m{}^{eq} \neq u_m. \tag{104}$$

  4. the expansion (60) has to vanish in equilibrium

$$u^{k\ eq}_{;k} = 0. \tag{105}$$

- Stemming from the entropy production and generated by the energy-momentum tensor
  5. the heat 4-flux density $(64)_2$ and the entropy 3-flux density $(66)_2$ have to vanish in equilibrium

$$q^k{}_{eq} = 0 \quad \longrightarrow \quad \left(\frac{q^k}{T}\right)^{eq} = 0. \tag{106}$$

  6. independently of material and space time, shear and rotation (80) have to be zero in equilibrium

$$\sigma^{eq}_{lk} = 0, \quad \omega^{eq}_{lk} = 0. \tag{107}$$

- Stemming from the entropy production and generated by the spin tensor
  7. equilibrium is possible in the following cases

$$u_p{}^{\bullet}{}_{eq} = 0 \cap \Xi_q{}^{eq} = 0, \tag{108}$$

$$u_p{}^{\bullet}{}_{eq} \neq 0 \cap \Xi_q{}^{eq} = 0 \cap u_p{}^{\bullet}{}_{eq} \in \ker \Xi^{\cdot p}_{m\ eq}, \tag{109}$$

$$\Xi_q{}^{eq} \neq 0 \cap u_p{}^{\bullet}{}_{eq} = 0 \cap \Xi_p{}^{eq} \in \ker \Xi^{\cdot p}_{m\ eq}. \tag{110}$$

8. according to (103), external moments need not be zero in equilibrium

Acknowledgments W. Muschik thanks R. Wulfert for his co-working in Sects. 2 and 3 of a former unpublished version of this paper.

References

1. C. Eckart (1940) Thermodynamics of irreversible proceses III *Phys. Rev.* **58** 919
2. Israel, W., Stewart, J.M.: Progress in relativistic thermodynamics and electrodynamics of continuous media. In: Held, A. (ed.) General Relativity and Gravitation, vol. 2, p. 491. Plenum Press, New York (1997)
3. Israel, W.: Covariant fluid mechanics and thermodynamics: an introduction. In: Anile, A., Choquet-Bruhat, Y. (eds.) Relativistic Fuid Mechanics. Lecture Notes in Mathematics, vol. 1385, pp. 152–210. Springer, Heidelberg (1987)
4. Landau, L.D., Lifshitz, E.M.: Fluid Mechanics, p. 505. Addison-Wesley, Reading/Massachusetts (1959)
5. E. Calzetta (1998) Relativistic fluctuating hydrodynamics *Class. Quantum Gravit.* **15** 653
6. Neugebauer, G.: Relativistische Thermodynamik. Vieweg (1980). ISBN 3-528-06863-9
7. T. Chrobok H.-H.v. Borzeszkowski (2006) Thermodynamical equilibrium and spacetime geometry *Gen. Relativ. Gravit.* **38** 397
8. G.A. Maugin (1978) On the covariant equations of the relativistic electrodynamics of continua I *J. Math. Phys.* **19** 1198
9. Borzeszkowski, H.-H.v., Chrobok, T.: On special- and general-relativistic thermodynamics. Atti dell' Academia Peloritana del Pericolanti. Classe di Scienze Fisiche, Matematiche e Natural, Bd. LXXXV, C1S0701002, Suppl. 1, 2007
10. T.W.B. Kibble (1961) Lorentz invariance and the gravitational field *J. Math. Phys.* **2** 212
11. Sciama, D.W.: On the analogy between charge and spin in general relativity theory. In: Recent Developments in General Relativity, p. 415. Pergamon Press, Oxford and PWN-Polish Scientific Publishers, Warszawa (1962)
12. Hehl, F.W., McCrea, J.D., Mielke, E.W., Ne'eman, Y.: Metric-affine theory of gravity: Field equations, Noether identities, world spinors and breaking of dilatation invariance. Physics Reports 258, No. 1, Sect. 2 (1995)
13. J.A. Schouten (1954) Ricci-Calculus Springer Berlin
14. Yu.N. Obukhov O.B. Piskareva (1989) Spinning fluid in general relativity *Class. Quantum Gravit.* **6** L15
15. G.A. Kluitenberg S.R. de Groot P. Mazur (1953) Relativistic thermodynamics of irreversible processes I, II *Physica* **689** 1071
16. Muschik, W., Papenfuss, C., Ehrentraut, H.: Concepts of Continuum Thermodynamics. Kielce University of Technology. ISBN 83-905132-7-7 (1996)
17. Neugebauer, G.: Black hole thermodynamics. In: Hehl, F.W., Kiefer, C., Metzler, R.J.K. (eds.) Black Holes: Theory and Observation. Springer, Berlin (1998)

18. Muschik, W.: In: Relativitätstage Berlin, Jena, 20–22 June 1990, TU Berlin, Verhandl. DPG (VI) 25, p. 314 (1990)
19. Muschik, W., Borzeszkowski, H.-H.v.: Entropy identity and material-independent equilibrium conditions in relativistic thermodynamics, arXiv:0804.2659v1 [gr-qc] 16 Apr 2008
20. H. Herrmann W. Muschik G. Rückner H.-H.v. Borzeszkowski (2004) Spin axioms in different geometries of relativistic continuum physics *Found. Phys.* **34** 1005
21. de Groot, S.R., Mazur, P.: Non-Equilibrium Thermodynamics, Chap. III, Sect. 2. North-Holland, Amsterdam (1961)
22. W. Muschik L. Restuccia (2002) Changing the observer and moving materials in continuum physics: objectivity and material frame-indifference *Tech. Mech.* **22** 152
23. Muschik, W., Restuccia, L.: Systematic remarks on objectivity and frame-indifference: liquid crystal theory as an example. Arch. Appl. Mech. **78**, 837 (2008). doi:10.1007/s00419-007-0193-2
24. Muschik, W., Papenfuss, C., Ehrentraut, H.: A sketch of continuum thermodynamics. J. NonNewtonian Fluid Mech. **96**, 255 (2001) (proposition on p. 279)