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Anomaly and Mass Spectrum of Tensionless String in Light-cone Gauge

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Abstract

Recently the method of dualities as the AdS/CFT correspondence has been used in higher spin gauge theories to understand gauge theories, gravities and M theory. Because higher spin gauge theories contain the infinite tower of higher spin fields, their relations with the tensionless limit of string theories have been studied. However the relations are not clear. One of the reasons is probably the poor understanding of the tensionless string theory. Therefore its understanding is very important to investigate the relation between the tensionless string theory and higher spin gauge theories and to consider dualities on the tensionless string theory.

Classical string theories have the spacetime symmetry as the global symmetry, as well as point particle theories. The theory of a string with tension, tensionful string theory, has the Poincare symmetry and the tensionless string theory has the spacetime conformal symmetry. As is well known, the quantization of a tensionful string usually causes the anomaly in the Poincare symmetry, the Lorentz anomaly. To avoid this anomaly, the spacetime dimensions and the operator ordering constant are determined.
Such anomalies and restrictions are known also in the quantum tensionless string theory. For example, in the light-cone gauge quantization method, the existence of the spacetime conformal anomaly is known in a certain operator order.

Because there is only one transverse direction in three dimensional light-cone coordinate, the Lorentz anomaly vanishes trivially for the light-cone string theory in three dimensions. Recently this specialty of three dimensional light-cone gauge has gathered attention and then the mass spectra of some 3-dim. tensionful strings in the light-cone gauge have been investigated in detail. Because there is not any restriction except for $D = 3$ in the case of the bosonic tensionful string, the operator ordering constant is undetermined. Such an ambiguity is removed for the tensionful superstring. Similarly, the ambiguity in three dimensional theories might be removed by the requirement of a larger symmetry.

In this thesis based on [1], we consider a tensionless closed bosonic string in the light-cone gauge in various dimensions and operator orders. The main products of our study are the construction of the non-separable mass eigenfunctions in the Reference order(R-order), the avoidance of the spacetime conformal anomaly in the Hermitian R-order and the explicit calculation of the spacetime conformal anomaly in the XP-normal order.
In three dimensions, we make use of the above specialty of three dimensions to investigate the spacetime conformal symmetry for the tensionless string in the light-cone gauge. In the Hermitian Refernce order, we verify the absence of the spacetime conformal anomaly and the removing of the ambiguity coming from the operator ordering. Furthermore we investigate separable and non-separable mass eigenstates in the theory without the anomaly and then we consider the relations between such eigenstates in terms of the Poincaré invariant corresponding to spin. In the XP-normal order, we calculate a dangerous commutator in the spacetime conformal symmetry concretely to find anomalous terms. To avoid the anomaly, we interpret the tensionless string in the XP-normal order as the point-like theory.
Furthermore, in $D > 3$, we investigate the anomaly in the spacetime conformal symmetry in the same way as three dimensional case. In the Hermitian R-order, we find the possibility of avoiding the anomaly. In the XP-normal order, we verify the existence of the anomaly concretely.

Contents

Notation and Conventions

Units

$$c = \hbar = 1 \tag{1}$$

Minkowski metric $\eta_{\mu\nu} = \eta^{\mu\nu}$ in D dimensions

In Orthogonal coordinate :

$$\eta_{00} = -1, \ \eta_{ij} = \delta_{ij} \quad (i, j = 1, \cdots D-1). \tag{2}$$

In Light-cone coordinate :

$$\eta_{+-} = \eta_{-+} = 1, \ \eta_{IJ} = \delta_{IJ} \quad (I, J = 2, \cdots D-1). \tag{3}$$

The indices are raised and lowered with metric η.

Definition of Components in Light-cone Coordinates

Vector in Orthogonal coordinates :

$$\mathbb{V}^\mu = \eta^{\mu\nu}\mathbb{V}_\nu \quad (\mu, \nu = 0, 1, \cdots, D-1). \tag{4}$$

Light-cone components:

$$V^\pm = V_\mp \equiv \frac{1}{\sqrt{2}}\left(\mathbb{V}^1 \pm \mathbb{V}^0\right), \quad V^I = V_I \equiv \mathbb{V}^I \quad (I = 2, \cdots, D-1). \tag{5}$$

Inner product :

$$\mathbb{V}^\mu \mathbb{U}_\mu = \eta_{\mu\nu}\mathbb{V}^\mu \mathbb{U}^\nu = -\mathbb{V}^0\mathbb{U}^0 + \sum_{i=1}^{D-1}\mathbb{V}^i\mathbb{U}^i = V^+U^- + V^-U^+ + \sum_{I=2}^{D-1} V^I U^I = V^\mu U_\mu. \tag{6}$$

Square :

$$\mathbb{V}^2 = -(\mathbb{V}^0)^2 + \sum_{i=1}^{D-1}(\mathbb{V}^i)^2 = 2V^+V^- + \sum_{I=2}^{D-1}(V^I)^2 = V^2 \tag{7}$$

Dot product in light-cone coordinate :

$$V \cdot U = \sum_{I=2}^{D-1} V^I U^I. \tag{8}$$

More Definitions in 3 Dimensions

Totally antisymmetric tensor :

$$\epsilon^{012} = -\epsilon_{012} = 1, \tag{9}$$
$$\epsilon^{+-2} = -\epsilon_{+-2} = 1. \tag{10}$$

Two rank anti-symmetric tensor $S_{\mu\nu} = -S_{\nu\mu}$ as the vector :

$$S^{\mu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho}S_{\nu\rho} \tag{11}$$

Symbol of Summation

Unless otherwise noted, we use the following notations for summation.

$$\sum_{n} \equiv \sum_{n=-\infty}^{\infty}, \tag{12}$$
$$\sum_{n\neq 0} \equiv \sum_{n=1}^{\infty} + \sum_{n=-\infty}^{-1}, \tag{13}$$
$$\sum_{n>0} \equiv \sum_{n=1}^{\infty}, \tag{14}$$
$$\sum_{|n|\leq N} \equiv \sum_{n=-N}^{N}. \tag{15}$$

Representation of Gamma matrix in three dimensions

$$\mathbf{\Gamma}^0 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ ,\ \mathbf{\Gamma}^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ ,\ \mathbf{\Gamma}^2 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ , \tag{16}$$
$$\Gamma^{+} = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}\ ,\quad \Gamma^{-} = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}\ ,\ \Gamma = \mathbf{\Gamma}^2\ . \tag{17}$$

Chapter 1

Introduction

Recently the method of dualities as the AdS/CFT correspondence has been used in higher spin gauge theories to understand gauge theories, gravities and M theory: [2–7] and [8–16]. Because higher spin gauge theories contain the infinite tower of higher spin fields, their relations with the tensionless limit of string theories have been studied [17–22]. However the relations are not clear. One of the reasons is probably the poor understanding of the tensionless string theory, as well as higher spin gauge theories[1]. Therefore its understanding is very important to investigate the relation between the tensionless string theory and higher spin gauge theories and to consider dualities on the tensionless string theory.

Classical string theories have the spacetime symmetry as the global symmetry, as well as a point particle [33]. The tensionful string theory[2] has the Poincare symmetry and the tensionless string theory has the spacetime conformal symmetry in addition to the Poincaré symmetry. As is well known, the quantization of the tensionful string usually causes the anomaly in the Poincare symmetry, the Lorentz anomaly. To avoid this anomaly, the spacetime dimensions and the operator ordering constant are determined [27–32].

Such anomalies and restrictions are known also in the quantum tensionless string theory. There are two kinds of anomalies in the tensionless string theory, the Lorentz anomaly and the spacetime conformal anomaly. The Lorentz anomaly in the tensionless string theory is investigated with various methods. In the BRST[3] quantization formalism, the absence of the critical dimension is verified in a hermitian operator order [34] and the same critical dimension and ordering constant as the tensionful string theory are verified in the normal order [35]. On the other hand, in the light-cone gauge quantization method, the absence of the critical dimension in some operator orders and the critical dimension in the XP-normal order[4] are verified [34, 36].

The spacetime conformal anomaly in the tensionless string theory is also investigated with various methods. In the BRST formalism, the critical dimension of the conformal

[1] Higher spin gauge theories are studied in detail by M. Vasiliev et al. [23–26].

[2] "Tensionful string" means "string with tension" in this thesis.

[3] The BRST refers to Becchi, Rouet, Stora and Tyutin.

[4] In the XP-normal order, all positive modes of coordinate X and momentum P are to the right of negative ones.

string theory is two [37][5]. On the other hand, in the light-cone gauge quantization method, the case of two dimensions is not under consideration[6] because of no stringy dynamical variable. Furthermore it is verified by the calculation without mode the expansion that the case of the light-cone gauge in higher dimensions causes the anomaly in the Reference order, called R-order for short[7] [38–40].
Similarly, tensionless string theories with the supersymmetry [41–44] and higher dimensional objects [48, 49] are studied.

There is only one transverse direction in three dimensional light-cone coordinate. Therefore the Lorentz anomaly vanishes for the string theory in three dimensional light-cone gauge, in addition to the usual critical dimension, $D = 26$ or $D = 10$. Such an avoidance of the anomaly in three dimensions cannot be obtained with the BRST formalism. The difference between two quantization methods is strange and interesting. Of course, there is no necessity for the coincidence of results in two methods, and we just expect it.

Recently the specialty of three dimensional light-cone gauge has gathered attention and then the mass spectra of some 3-dim. tensionful string theories in the light-cone gauge have been investigated in detail [50]. Because there is not any restriction except for $D = 3$ in the case of a bosonic tensionful string, the operator ordering constant is undetermined. Such an ambiguity is removed for a tensionful superstring [50–53]. Similarly, the ambiguity in three dimensional theories might be removed by the requirement of a larger symmetry.

In this thesis based on [1], we consider a tensionless closed bosonic string in the light-cone gauge in various dimensions and operator orders. The main products of our study are the construction of the non-separable mass eigenfunctions in the (Hermitian) R-order and the Weyl order, the avoidance of the spacetime conformal anomaly in the Hermitian R-order[8] and the explicit calculation of the spacetime conformal anomaly in the XP-normal.

Because the appearance of the anomaly in the spacetime symmetry are different between the case of $D = 3$ and one of $D > 3$, we must consider their cases separately.

In $D = 3$, we make use of the above specialty of the three dimensions to investigate the spacetime conformal symmetry for a tensionless string in the light-cone gauge. The absence of the Lorentz anomaly is the same as the tensionful string theory.
In the Hermitian R-order, we verify the absence of the spacetime conformal anomaly and the removing of the ambiguity coming from the operator ordering. Furthermore we investigate separable and non-separable mass eigenstates in the theory without the anomaly and then we consider the relations between such eigenstates in terms of the Poincaré invariant corresponding to spin.
In the XP-normal order, we calculate a dangerous commutator in the spacetime conformal symmetry concretely to find anomalous terms. To avoid the anomaly, we interpret the

[5]The BRST quantization of the tensionless string tells only the information of the Lorentz anomaly. Therefore the conformal string, which has the extra coordinates to have the manifest spacetime conformal symmetry, has been considered instead of the tensionless string. Of course, the action of the conformal string is classically equivalent are to that of the tensionless string.

[6]Or non-anomalous trivially or topological.

[7]In the R-order, all P-modes are to the right of X-modes.

[8]The Hermitian R-order is defined as the Hermitian version of the R-order. It will be explained in section 3.

tensionless string theory in the XP-normal order as the point-like theory.

In $D > 3$, we investigate the anomaly in the spacetime conformal symmetry in the same way as three dimensional case. In the Hermitian R-order, we find the avoidance of the anomaly. In the XP-normal order, we verify the existence of the anomaly concretely.

Finally we discuss the supersymmetric string case and the open string case.

Organization of this thesis

In this thesis we consider mainly a tensionless closed bosonic string in the light-cone gauge. Because the appearance of the anomaly in the spacetime symmetry are different between the case of $D = 3$ and one of $D > 3$, we discuss their cases separately.

In chapter 2 we consider the classical theory for a tensionless string. First, from the analogy of a massless point particle, we give the action for the tensionless string and the generators of the spacetime conformal symmetry, which is the global symmetry. Next we choose the light-cone gauge to fix gauge symmetries in the tensionless string theory. After we solve constraints with the mode expansion, we represent the action and classical generators of the spacetime symmetry in the light-cone gauge.

In chapter 3 the tensionless closed bosonic string in the light-cone gauge is quantized. Then some candidates for the string ground state and operator order which are used in the following chapter are shown. One of the orders is based on the Reference order, in which all momentum P modes are to the right of coordinate X modes. Another one is the XP-normal order, in which all positive modes of X and P are to the right of negative ones.

In chapter 4 we consider the quantum 3-dim. tensionless closed bosonic string with the Poincaré symmetry in the light-cone gauge. Because a dangerous commutator for the Lorentz anomaly vanishes trivially in three dimensions and there is no restriction, the constant of the ambiguity coming from operator order is undetermined. Therefore we investigate the mass spectrum to consider the natural choice of the ordering constant. In the R-order, its hermitian version and the Weyl order, we find the continuous spectrum in addition to zero. Furthermore in these orders we find non-separable mass eigenstates, which are not factorized with each Fourier-mode, as well as separable ones. From such informations, we consider the natural choice of the ordering constant. In the XP-normal order, we investigate mass eigenstates and their norms to consider the natural choice of the ordering constant.

In chapter 5 we consider the quantum D-dim. tensionless closed bosonic string with the Poincaré symmetry in the light-cone gauge. In $D > 3$, the avoidance of the Lorentz anomaly restricts the theory. In the R-order and similar ones, the ordering constant is determined to be zero but the dimension is free. On the other hand, in the XP-normal order, the critical dimension is 26 and the ordering constant is determined. Furthermore we give the mass spectra in theories without the anomaly.

In chapter 6 we consider the structures of commutators in some examples of the operator order. Then we find the structures of dangerous commutators to use them in the following chapters.

In chapter 7 we consider the quantum 3-dim. tensionless closed bosonic string with the spacetime conformal symmetry in the light-cone gauge. We calculate explicitly a dangerous commutator in the spacetime conformal symmetry. Although we need some regularization in the Hermitian R-order, in chapter 7, we verify that there is no anomaly which is irrelevant to the choice of the regularization. The concrete calculation with the cut-off regularization is shown in appendix. On the other hand, in the XP-normal order, we find the anomaly which can not be removed by the choice of the ordering constant.

In chapter 8 we consider the quantum D-dim. tensionless closed bosonic string with the spacetime conformal symmetry in the light-cone gauge. In [38–40], the anomaly in the spacetime conformal symmetry is considered from the calculation without the mode expansion. In chapter 8 of this thesis, we give the calculations of dangerous commutators with the mode expansion. Then we find the vanishing of the anomaly in the Hermitian R-order and give the concrete form of the anomalous commutator in the XP-normal order.

In chapter 9 we discuss other types of string, super and open. We investigate the difference from the tensionless bosonic closed string theory in supersymmetric and open string theories respectively. Then we estimate whether the anomaly in commutators exists or not.

Finally, we summarize this thesis and give the outlook for future works.

In appendix, we give the following contents: the lists of the generators of the spacetime symmetry and useful commutation relations in some operator orders, the calculations and considerations which have been omitted in the main body, and the short review of a 3-dim. tesionful string in the light-cone gauge and so on.

Chapter 2

Classical Theory of Tensionless String

In this chapter we investigate the classical theory of a tensionless string. First we consider the massless point particle for reference and then show the generators of the spacetime conformal symmetry.
Next we define the action of a tensionless closed string. Then we find the gauge symmetry corresponding to the world sheet diffeomorphism and the spacetime conformal symmetry as the global symmetry. Furthermore we give the explicit representations for generators of the spacetime conformal symmetry.
Finally we fix the gauge symmetry with the light-cone gauge to solve the constraints. Then we find the time-like motion of the tensionless string as well as the null motion. Furthermore we represent generators of the spacetime conformal symmetry in the light-cone gauge.

2.1 Massless Point Particle and Global Symmetry

First we consider a massless point particle to get an useful help in constructing a tensionless string theory. We start from the next action of a massless point particle.

$$S = -m \int dt \sqrt{-\dot{x}^{\mu}\dot{x}_{\mu}} \ , \tag{2.1}$$

where x^{μ} is the position of a particle, m is mass and the overdot ($\dot{}$) means t-derivative. This is written as

$$S = \int dt \left[\dot{x}^{\mu} p_{\mu} - \frac{1}{2} v \left(p^2 + m^2 \right) \right] , \tag{2.2}$$

where p_{μ} is the conjugate momentum of x^{μ} and v is a Lagrange multiplier for the on-shell condition, $p^2 + m^2 = 0$. In the case of $m^2 \neq 0$, we solve p_{μ} and v in order and then get the original action (2.1).
We can choose $m = 0$ in the second action (2.2) to obtain

$$S = \int dt \left[\dot{x}^{\mu} p_{\mu} - \frac{1}{2} v p^2 \right] . \tag{2.3}$$

This is the action of massless point particle. (2.3) has the next gauge symmetry.

$$\begin{cases} \delta x^\mu = \alpha p_\mu \\ \delta p_\mu = 0 \\ \delta v = \dot{\alpha} \end{cases} \tag{2.4}$$

where α is the parameter which depends on time t and the upper dot $\dot{}$ means t-derivative. In fact, the action (2.3) is invariant under this gauge transformation.

$$\delta S = \int dt\ \partial_\tau \left(\frac{1}{2}\alpha p^2\right) = 0 \tag{2.5}$$

The field equations of (2.3) for x^μ, p_μ and v are

$$\begin{aligned} &\dot{p_\mu} = 0, \\ &\dot{x^\mu} - vp^\mu = 0, \\ &p^2 = 0. \end{aligned} \tag{2.6}$$

From these equations, we find that the motion of the massless point particle is restricted on the light-cone.

The action (2.3) has the global symmetry as well as the local gauge symmetry. It is the spacetime conformal symmetry which consists of the translation, the Lorentz transformation, the dilatation and the special conformal transformation. We give the explicit representations for these transformations and their generators below.

The translation is

$$\begin{cases} \delta_a x^\mu = a^\mu \\ \delta_a p_\mu = \delta_a v = 0 \end{cases} \tag{2.7}$$

where a^μ are parameters independent of t. The generators of the translation are

$$\mathcal{P}_\mu = p_\mu. \tag{2.8}$$

The Lorentz transformation is

$$\begin{cases} \delta_\omega x^\mu = \omega^\mu{}_\nu x^\nu \\ \delta_\omega p_\mu = \omega_\mu{}^\nu p_\nu \\ \delta_\omega v = 0 \end{cases} \tag{2.9}$$

where $\omega^\mu{}_\nu = -\omega_\nu{}^\mu$ are anti-symmetric parameters independent of t. The generators of the Lorentz transformation are

$$\mathcal{J}^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu. \tag{2.10}$$

The scale transformation, dilatation is

$$\begin{cases} \delta_\lambda x^\mu = \lambda x^\mu \\ \delta_\lambda p_\mu = -\lambda p_\mu \\ \delta_\lambda v = 2\lambda v \end{cases} \tag{2.11}$$

where λ is a parameter independent of t. The generator of the dilatation is

$$\mathcal{D} = x^\mu p_\mu. \tag{2.12}$$

The special conformal transformation is

$$\begin{cases} \delta_b x^\mu = x^\mu (b_\nu x^\nu) - \frac{1}{2} b^\mu x^2 \\ \delta_b p_\mu = -b_\mu (x^\nu p_\nu) - (b_\nu x^\nu) p_\mu + x_\mu (b^\nu p_\nu) \\ \delta_b v = 2(b_\nu x^\nu) v \end{cases} \tag{2.13}$$

where b^μ are parameters independent of t. The generators of special conformal transformation are

$$\mathcal{K}^\mu = x^\mu (x^\nu p_\nu) - \frac{1}{2} x^2 p^\mu. \tag{2.14}$$

The translation, $\mathcal{P}_\mu$, and the Lorentz transformation, $\mathcal{J}^{\mu\nu}$, are invariant under the gauge transformation in (2.4). The dilatation, $\mathcal{D}$, and the special conformal transformation, $\mathcal{K}^\mu$, are proportional to p^2 under the gauge transformation (2.4). Therefore they are weakly invariant.
Thanks to the gauge invariance of generators, we can use these generators in the light-cone gauge only by changing the spacetime index.

2.2 Tensionless String and Global Symmetry

In this section we consider a tensionless closed string.

2.2.1 Action of tensionless string from Nambu-Goto action

First we change the Nambu-Goto action to the equivalent action on the phase space in the same way as the point-particle case. The Nambu-Goto action for a string with tension T [1] is

$$S[\mathbf{X}] = -T \int d\tau \oint \frac{d\sigma}{2\pi} \sqrt{\left(\left(\dot{\mathbf{X}}^\mu \mathbf{X}'_\mu \right)^2 - \dot{\mathbf{X}}^2 (\mathbf{X}')^2 \right)}, \tag{2.15}$$

where τ is the time-like coordinate on world sheet and σ is the space-like coordinate with a period of 2π on world sheet. Here the overdot ($\dot{}$) means τ-derivative and the prime ($'$) means σ-derivative. This action is classically equivalent to

$$S[\mathbf{X}, \mathbf{P}; V, U] = \int d\tau \oint \frac{d\sigma}{2\pi} \left\{ \dot{\mathbf{X}}^\mu \mathbf{P}_\mu - \frac{1}{2} V \left[\mathbf{P}^2 + (T\mathbf{X}')^2 \right] - U \mathbf{X}^{\mu\prime} \mathbf{P}_\mu \right\}, \tag{2.16}$$

where P_μ is conjugate momentum of X^μ, and V and U are Lagrange multipliers. In the case of $T \neq 0$, we solve P_μ, U and V in order and then get the original action (2.15).

We can choose $T = 0$ in (2.16) in the same way as the case of a point particle.

$$S[\mathbf{X}, \mathbf{P}; V, U] = \int d\tau \oint \frac{d\sigma}{2\pi} \left\{ \dot{\mathbf{X}}^\mu \mathbf{P}_\mu - \frac{1}{2} V \mathbf{P}^2 - U \mathbf{X}^{\mu\prime} \mathbf{P}_\mu \right\}. \tag{2.17}$$

[1] In our convention, $T = \frac{1}{\alpha'}$.

This is one of the action of a tensionless closed string.
Of course, there are other equivalent actions for a tensionless string [33, 38, 40]. For example,

$$S'[\mathbf{X}, V^a] = \int d^2\xi \ V^a\partial_a\mathbf{X}^\mu V^b\partial_b\mathbf{X}_\mu, \tag{2.18}$$

where $a = \{0,1\}$or $= \{\tau, \sigma\}$. This is obtained from (2.17) by using $\mathbf{P}_\mu = \frac{1}{V}(\dot{\mathbf{X}}_\mu - U\mathbf{X}'_\mu)$, $V^\tau \equiv \frac{1}{\sqrt{2V}}$ and $V^\sigma \equiv -\frac{U}{\sqrt{2V}}$. Therefore (2.17) and (2.18) are classically equivalent.
We can easily find the world sheet diffeomorphism for the action in (2.18),

$$\begin{cases} \delta\mathbf{X}^\mu = \gamma^a\partial_a\mathbf{X}^\mu \\ \delta V^a = -V^b\partial_b\gamma^a + \gamma^b\partial_b V^a + \frac{1}{2}(\partial_b\gamma^b)V^a \end{cases}. \tag{2.19}$$

(2.18) is a better representation for the covariant treatment such as the BRST quantization.

Because the tensionlessness is clear and we can easily solve the constraints in the light-cone gauge, we consider the first action, (2.17).

The first action, (2.17), has the next gauge symmetry corresponding to the world sheet diffeomorphism.[2]

$$\begin{cases} \delta\mathbf{X}^\mu = \alpha\mathbf{P}^\mu + \beta\mathbf{X}^{\mu\prime} \\ \delta\mathbf{P}_\mu = (\beta\mathbf{P}_\mu)' \\ \delta V = \dot{\alpha} + U'\alpha - U\alpha' + V'\beta - V\beta' \\ \delta U = \dot{\beta} + U'\beta - U\beta'. \end{cases} \tag{2.20}$$

In fact, the action in (2.17) is invariant under this transformation.[3]

$$\begin{aligned} \delta S &= \int d\tau \oint \frac{d\sigma}{2\pi}\Big[\partial_\tau\Big(\frac{1}{2}\alpha\mathbf{P}^2\Big) + \partial_\sigma\Big(-\frac{1}{2}\alpha U\mathbf{P}^2 + \beta\big(\dot{\mathbf{X}}^\mu\mathbf{P}_\mu - \frac{1}{2}V\mathbf{P}^2 - U\mathbf{X}^{\mu\prime}\mathbf{P}_\mu\big)\Big)\Big] \\ &= 0 \end{aligned} \tag{2.21}$$

The field equations of X^μ, P_μ, V and U are

$$\begin{aligned} &\dot{\mathbf{P}}_\mu - (U\mathbf{P}_\mu)' = 0, \\ &\dot{\mathbf{X}}^\mu - U\mathbf{X}^{\mu\prime} - V\mathbf{P}^\mu = 0, \\ &\mathbf{P}^2 = 0, \\ &\mathbf{X}^{\mu\prime}\mathbf{P}_\mu = 0. \end{aligned} \tag{2.22}$$

2.2.2 Global symmetry and generators

As well as a massless point particle, the action in (2.17) is expected to have the spacetime conformal symmetry as the global symmetry. The explicit transformations, their generators and the gauge invariance of them are shown below.

[2]It is obtained by choosing $\gamma^\tau = \frac{\alpha}{V}$, $\gamma^\sigma = \beta - \alpha\frac{U}{V}$ in (2.19).

[3]Because the closed string has no boundary, the total derivative terms vanish. For the tensionless open string, we have to select some boundary condition.

The translation is

$$\begin{cases} \delta_a \mathbf{X}^\mu = a^\mu \\ \delta_a \mathbf{P}_\mu = \delta_a V = \delta_a U = 0 \end{cases} \tag{2.23}$$

where a^μ are parameters independent of τ and σ. From the Noether charge for this transformation, the generators of the translation are obtained as

$$\mathcal{P}_\mu = \oint \frac{d\sigma}{2\pi} \mathbf{P}_\mu. \tag{2.24}$$

The Lorentz transformation is

$$\begin{cases} \delta_\omega \mathbf{X}^\mu = \omega^\mu{}_\nu \mathbf{X}^\nu \\ \delta_\omega \mathbf{P}_\mu = \omega_\mu{}^\nu \mathbf{P}_\nu \\ \delta_\omega V = \delta_\omega U = 0 \end{cases} \tag{2.25}$$

where $\omega^\mu{}_\nu = -\omega_\nu{}^\mu$ are anti-symmetric parameters independent of τ and σ. The generators of the Lorentz transformation are

$$\mathcal{J}^{\mu\nu} = \oint \frac{d\sigma}{2\pi} \left[\mathbf{X}^\mu \mathbf{P}^\nu - \mathbf{X}^\nu \mathbf{P}^\mu \right]. \tag{2.26}$$

The dilatation transformation is

$$\begin{cases} \delta_\lambda \mathbf{X}^\mu = \lambda \mathbf{X}^\mu \\ \delta_\lambda \mathbf{P}_\mu = -\lambda \mathbf{P}_\mu \\ \delta_\lambda V = 2\lambda V \\ \delta_\lambda U = 0 \end{cases} \tag{2.27}$$

where λ is parameter independent of τ and σ. The generator of the dilatation is

$$\mathcal{D} = \oint \frac{d\sigma}{2\pi} \mathbf{X}^\mu \mathbf{P}_\mu. \tag{2.28}$$

The special conformal transformation is

$$\begin{cases} \delta_b \mathbf{X}^\mu = \mathbf{X}^\mu (b_\nu \mathbf{X}^\nu) - \frac{1}{2} b^\mu \mathbf{X}^2 \\ \delta_b \mathbf{P}_\mu = -b_\mu (\mathbf{X}^\nu \mathbf{P}_\nu) - (b_\nu \mathbf{X}^\nu) \mathbf{P}_\mu + \mathbf{X}_\mu (b^\nu \mathbf{P}_\nu) \\ \delta_b V = 2(b_\nu \mathbf{X}^\nu) V \\ \delta_b U = 0 \end{cases} \tag{2.29}$$

where b^μ are parameters independent of τ and σ. The generators of the special conformal transformation are

$$\mathcal{K}^\mu = \oint \frac{d\sigma}{2\pi} \left[\mathbf{X}^\mu (\mathbf{X}^\nu \mathbf{P}_\nu) - \frac{1}{2} \mathbf{X}^2 \mathbf{P}^\mu \right]. \tag{2.30}$$

These generators change under the gauge symmetry in (2.20) as follows:

$$\begin{aligned} \delta \mathcal{P}_\mu &= \oint \frac{d\sigma}{2\pi} (\beta \mathbf{P}_\mu)' = 0 \\ \delta \mathcal{J}^{\mu\nu} &= \oint \frac{d\sigma}{2\pi} \left(\beta \mathbf{X}^\mu \mathbf{P}^\nu - \beta \mathbf{X}^\nu \mathbf{P}^\mu \right)' = 0 \\ \delta \mathcal{D} &= \oint \frac{d\sigma}{2\pi} \left[\alpha \mathbf{P}^2 + (\beta \mathbf{X}^\mu \mathbf{P}_\mu)' \right] = \oint \frac{d\sigma}{2\pi} \alpha \mathbf{P}^2 \approx 0 \\ \delta \mathcal{K}^\mu &= \oint \frac{d\sigma}{2\pi} \left\{ \left(\beta \left[\mathbf{X}^\mu (\mathbf{X}^\nu \mathbf{P}_\nu) - \frac{1}{2} \mathbf{X}^2 \mathbf{P}^\mu \right] \right)' + \alpha \mathbf{X}^\mu \mathbf{P}^2 \right\} = \oint \frac{d\sigma}{2\pi} \alpha \mathbf{X}^\mu \mathbf{P}^2 \approx 0, \end{aligned} \tag{2.31}$$

where $\approx$ means "weakly equal" which is satisfied by using field equations. Thus the generators of the translation and the Lorentz transformation are invariant and the generators of the dilatation and the special conformal transformation are weakly invariant. Thanks to the gauge invariance of generators, we can use these generators in the light-cone gauge only by changing the spacetime index.

2.3 Tensionless String in Light-cone Gauge

In this section we fix the gauge symmetry in (2.20). First we define the light-cone coordinate as

$$\begin{aligned} X^{\pm} &\equiv \frac{1}{\sqrt{2}}(\mathbf{X}^1 \pm \mathbf{X}^0),\ X^I \equiv \mathbf{X}^I, \\ P_{\pm} &\equiv \frac{1}{\sqrt{2}}(\mathbf{P}_1 \pm \mathbf{P}_0) = P^{\mp},\ P_I \equiv \mathbf{P}_I, \end{aligned} \tag{2.32}$$

where I is the index of transverse directions, $I = 2, \ldots, D-1$.
The light-cone gauge is defined as

$$X^+ = \tau\ , \quad P_- = p_-(\tau) \neq 0. \tag{2.33}$$

This gauge choice fixes most of the gauge freedoms such that $\alpha = 0$ and $\beta = \beta_0(\tau)$. The residual gauge by $\beta_0(\tau)$ corresponds to σ-shift and is related to the unique constraint as seen later.
The Lagrangian density in the light-cone gauge is

$$\mathcal{L} = P_+ + \dot{X}^- p_- + \dot{X} \cdot P - \frac{1}{2} V\left(2P_+ p_- + P \cdot P\right) - U\left((X^-)' p_- + X' \cdot P\right), \tag{2.34}$$

where $\cdot$ means the product over the indexes of the transverse directions.
In order to clarify the constraint corresponding to β_0 and solve the rest constraints, we separate the function of σ into the center of mass part and the rest part as follows:

$$\begin{aligned} f(\tau) &\equiv \oint \frac{d\sigma}{2\pi} F(\tau, \sigma), \\ \bar{F}(\tau, \sigma) &\equiv F(\tau, \sigma) - f(\tau), \quad \oint \frac{d\sigma}{2\pi} \bar{F}(\tau, \sigma) = 0. \end{aligned} \tag{2.35}$$

where $F = X^I, P_I, X^-, P_+, U$. By using the above separation, the Lagrangian is written as

$$\begin{aligned} L = p_+ + \dot{x}^- p_- + \dot{x} \cdot p + \oint \frac{d\sigma}{2\pi} \dot{\bar{X}} \cdot \bar{P} - u \oint \frac{d\sigma}{2\pi} \bar{X}' \cdot \bar{P} \\ - p_- \oint \frac{d\sigma}{2\pi} \left[V\left(P_+ + \frac{1}{2p_-} P \cdot P\right) + \bar{U}\left((\bar{X}^-)' + \frac{1}{p_-} \bar{X}' \cdot P\right)\right]. \end{aligned} \tag{2.36}$$

In this Lagrangian, $\bar{X}^-$ is not dynamical variable and then its variation gives $\bar{U}' = 0$. Because of $\oint \frac{d\sigma}{2\pi} \bar{U}(\tau, \sigma) = 0$, we obtain $\bar{U} = 0$. On the contrary, the variation of $\bar{U}$ gives the next relation.[4]

$$p_-(\bar{X}^-)' = -\bar{X}' \cdot P + \oint \frac{d\sigma}{2\pi} \bar{X}' \cdot \bar{P}. \tag{2.37}$$

Therefore $\bar{X}^-$ is represented with other fields.
The variation of P_+ gives $V = \frac{1}{p_-}$ and the variation of V gives

$$P_+ = -\frac{1}{2p_-} P \cdot P. \tag{2.38}$$

Therefore we can write P_+ with other fields. The c.o.m. part of P_+ is the Hamiltonian,

$$H \equiv -p_+ = \frac{1}{2p_-}\left(p \cdot p + \mathcal{M}^2\right), \tag{2.39}$$

and the mass square operator is

$$\mathcal{M}^2 = -p^2 = -2p_+p_- - p \cdot p = \oint \frac{d\sigma}{2\pi} \bar{P} \cdot \bar{P}. \tag{2.40}$$

After most of constraints are solved by (2.37) and (2.38), the Lagrangian is written as

$$L = \dot{x} \cdot p + \dot{x}^- p_- + \oint \frac{d\sigma}{2\pi} \dot{\bar{X}} \cdot \bar{P} - H - u \oint \frac{d\sigma}{2\pi} \bar{X}' \cdot \bar{P}. \tag{2.41}$$

We can choose $u = 0$ with the residual gauge symmetry by β_0, and then we obtain the next Lagrangian with the unique constrain.

$$L = \dot{x} \cdot p + \dot{x}^- p_- + \oint \frac{d\sigma}{2\pi} \dot{\bar{X}} \cdot \bar{P} - H, \tag{2.42}$$

and the unique constraint is

$$\oint \frac{d\sigma}{2\pi} \bar{X}' \cdot P = 0, \tag{2.43}$$

(2.43) corresponds to the level-matching condition for the tensionful string theory. In the quantum theory, this constraint is imposed on states.

2.3.1 Equation of motion in light-cone gauge

The field equations in (2.42) are

$$\begin{aligned} &\dot{x}^- = -\frac{H}{p_-},\ \dot{p}_I = \dot{p}_- = 0,\ \dot{x}^I = \frac{p^I}{p_-} \\ &\dot{\bar{X}}^I = \frac{\bar{P}^I}{p_-}, \quad \dot{\bar{P}}_I = 0. \end{aligned} \tag{2.44}$$

From these, we obtain the τ-dependences of X^I and x^- as

$$\begin{aligned} &X^I(\tau, \sigma) = X^I(\sigma) + \frac{P^I(\sigma)}{p_-}\tau \\ &x^-(\tau) = x^- - \frac{H}{p_-}\tau. \end{aligned} \tag{2.45}$$

To find τ-dependence of $\bar{X}^-$, we need the concrete representation of $\bar{X}^-$.

[4]The second term in the r.h.s. have been added because $\oint \frac{d\sigma}{2\pi} \bar{X}^- = 0$. We find soon that this is the same as the unique constraint.

2.3.2 Fourier mode expansions of fields

In order to write $\bar{X}^-$ and P_+ concretely, we use the periodicity of the closed string. $X(\sigma)$ and $P(\sigma)$ are expanded as follows:

$$\begin{aligned} X^I(\sigma) &= \sum_n X^I_n e^{in\sigma} \ , \quad X^I_0 = x^I, \\ P_I(\sigma) &= \sum_n P_{I,n} e^{in\sigma} \ , \quad P_{I,0} = p_I, \end{aligned} \tag{2.46}$$

where $\sum_n \equiv \sum_{n=-\infty}^{\infty}$. From the reality condition of $X^I(\sigma)$ and $P_I(\sigma)$, the next relations are satisfied.

$$(X^I_n)^* = X^I_{-n} \ , \quad (P_{I,n})^* = P_{I,-n}, \tag{2.47}$$

where A^* means complex conjugate of A.
By using (2.46), P_+ is solved concretely as

$$P_+(\sigma) = -\frac{1}{2p_-} \sum_n L_n e^{in\sigma}, \tag{2.48}$$

where

$$L_n \equiv \sum_m P_m \cdot P_{n-m} \ , \quad (L_n)^* = L_{-n}. \tag{2.49}$$

In particular, its zero mode is $L_0 = 2p_- H = p \cdot p + \mathcal{M}^2$ and the mass square operator is

$$\mathcal{M}^2 = \sum_{n \neq 0} P_n \cdot P_{-n}. \tag{2.50}$$

Similarly, by using (2.46), $\bar{X}^-$ is written as

$$\bar{X}^-(\tau, \sigma) = -\frac{1}{p_-} \sum_{n \neq 0} \frac{i}{n} M_n(\tau) e^{in\sigma}, \tag{2.51}$$

where

$$\begin{aligned} M_n(\tau) &\equiv -i \sum_m m\big(X_m + \frac{1}{p_-} P_m \tau\big) \cdot P_{n-m} = M_n - i\frac{n}{2p_-} L_n \tau \ , \quad \text{for } n \neq 0 \\ M_n &\equiv M_n(\tau = 0) = -i \sum_m m X_m \cdot P_{n-m} \ , \quad (M_n)^* = M_{-n}. \end{aligned} \tag{2.52}$$

Because the c.o.m part of X^- is x^-, we find $X^- = x^- + \bar{X}^-$.
Furthermore the unique constraint in (2.43) is written as

$$M_0 \equiv -\oint \frac{d\sigma}{2\pi} \bar{X}' \cdot P = -i \sum_n n X_n \cdot P_{-n} = 0. \tag{2.53}$$

2.3.3 Motion of tensionless string in light-cone gauge

With the mode expansion, the Lagrangian (2.42) is written as

$$L = \dot{x} \cdot p + \dot{x}^- p_- + \sum_{n \neq 0} \dot{X}_n \cdot P_{-n} - H, \tag{2.54}$$

and then the field equations for zero modes and non-zero modes are

$$\begin{aligned} \dot{x}^I &= \frac{p^I}{p_-} \ , \ \dot{x}^- = -\frac{H}{p_-} \ , \ \dot{p}_I = \dot{p}_- = 0, \\ \dot{X}^I_n &= \frac{P^I_n}{p_-} \ , \ \dot{P}_{I,n} = 0. \end{aligned} \tag{2.55}$$

Because of $\dot{M}_n = -i \frac{1}{2p_-} L_n$, we obtain

$$\begin{aligned} X^I(\tau, \sigma) &= X^I(\sigma) + \frac{P^I(\sigma)}{p_-} \tau, \\ X^-(\tau, \sigma) &= X^-(\sigma) + \frac{P_+(\sigma)}{p_-}. \end{aligned} \tag{2.56}$$

From (2.56), we obtain

$$\dot{X}^\mu(\tau, \sigma) \dot{X}_\mu(\tau, \sigma) = \frac{1}{p_-^2} \left(2P_+(\sigma) p_- + P(\sigma) \cdot P(\sigma) \right) = 0, \tag{2.57}$$

where we use the constraint in (2.43). This means that each point of the tensionless string moves on the light-cone.
However, the c.o.m. of the string does not always move on the light-cone unlike massless point particle.

$$\dot{x}^\mu \dot{x}_\mu = -\frac{\mathcal{M}^2}{p_-^2} \tag{2.58}$$

2.3.4 Classical generators in light-cone gauge

Finally we give the representations for generators of the spacetime conformal symmetry in the light-cone gauge. Because all generators are invariant under the gauge symmetry in (2.19), all we have to do is to change the spacetime indexes in generators to indexes in the light-cone base. Furthermore we will verify that generators are independent of τ by using field equations, or constraints.

From (2.24), the generators of the translation in the light-cone gauge are

$$\mathcal{P}_- = p_- \ , \ \mathcal{P}_I = p_I \ , \ \mathcal{P}_+ = p_+ = -\frac{L_0}{2p_-} \tag{2.59}$$

Because $P_\mu(\sigma)$ is independent of τ, these generators are clearly independence of τ.

From (2.26), the generators of Lorentz transformation in the light-cone gauge are

$$
\begin{aligned}
&\mathcal{J}^{+I} = -x^I p_- \ , \ \mathcal{J}^{+-} = -x^- p_- \ , \ \mathcal{J}^{IJ} = \sum_n (X_n^I P_{-n}^J - X_n^J P_{-n}^I) \\
&\mathcal{J}^{-I} = x^- p^I - \frac{i}{p_-} \sum_{n\neq 0} \frac{1}{n} M_n P_{-n}^I + \frac{1}{2p_-} \sum_n X_n^I L_{-n}.
\end{aligned}
\tag{2.60}
$$

We find that these generators are independent of τ from τ-dependence of X_n^I and x^-, (2.56).
In three dimensions, $\mathcal{J}^{IJ}$ does not exist and we can write Lorentz generators as vector,

$$
\mathcal{J}^+ = \mathcal{J}^{+2} \ , \ \mathcal{J} = -\mathcal{J}^{+-} \ , \ \mathcal{J}^- = -\mathcal{J}^{-2}, \tag{2.61}
$$

where $\mathcal{J}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}\mathcal{J}_{\nu\rho}$ and $\epsilon^{+-2} = -\epsilon_{+-2} = 1$.

From (2.28), the generator of the dilatation in the light-cone gauge is

$$
\mathcal{D} = x^- p_- + \sum_n X_n \cdot P_{-n}. \tag{2.62}
$$

We find that this is independent of *tau* from (2.56) and the definition of Hamiltonian, H, in (2.39).

From (2.30), the generator of special conformal transformation in the light-cone gauge are

$$
\begin{aligned}
\mathcal{K}^+ =& -\frac{1}{2}\sum_n X_n \cdot X_{-n} p_- \\
\mathcal{K}^I =& x^I x^- p_- + i \sum_{m\neq 0} \frac{1}{m} X_m^I M_{-m} + \sum_n \sum_m \left[X_n^I X_m^J P_{J,-n-m} - \frac{1}{2} X_n^J X_m^J P_{-n-m}^I \right] \\
\mathcal{K}^- =& x^- x^- p_- + x^- \sum_n X_n P_{-n} + \frac{1}{p_-} \sum_{m\neq 0} \frac{1}{m^2} M_m M_{-m} \\
& - \frac{i}{p_-} \sum_n \sum_{m\neq 0} \frac{1}{m} X^I M_m P_{I,-n-m} + \frac{1}{4p_-} \sum_n \sum_m X_n^I X_m^I L_{-n-m}.
\end{aligned}
\tag{2.63}
$$

We find that these generators are independent of τ by using the next relation: $2P_+(\sigma)p_- + P(\sigma) \cdot P(\sigma) = 0$.
In three dimensions, $\mathcal{K}^I$ becomes simple a little.

$$
\mathcal{K} = \mathcal{K}^{I=2} = x x^- p_- + i \sum_{m\neq 0} \frac{1}{m} X_m M_{-m} + \frac{1}{2} \sum_n \sum_m X_n X_m P_{-n-m}, \tag{2.64}
$$

where we omit the index of transverse direction.

In quantum theory, the representation of the generators for the spacetime conformal symmetry become complicated, according to the choice of the operator order. Furthermore there are quantum effect's terms in $\mathcal{K}^-$, which become anomalous. The detail is in the later chapter.

Chapter 3

Quantization, Operator Order and String Ground State

In the last chapter we have considered the classical theory of a tensionless string in the light-con gauge. In this chapter we quantize the tensionless string in the light-cone gauge and then consider some candidates for the operator order and the string ground state. In the following chapters we calculate the infamous dangerous commutator, $[\mathcal{J}^{-I}, \mathcal{J}^{-J}]$, and other dangerous commutators in the operator orders given in this chapter and then investigate whether the anomaly for the spacetime symmetry exists or not.

3.1 Quantization of Tensionless String in Light-cone Gauge

The dynamical variables of the tensionless string theory in the light-cone gauge in chapter 2 are $X^I(\sigma)$ and x^-, and their conjugate momenta, $P_I(\sigma)$ and p_-. They are quantized as follows:

$$[x^-, p_-] = i,\ \ [X^I(\sigma)\ ,\ P_J(\sigma')] = 2\pi i\delta^I_J\delta(\sigma - \sigma'), \tag{3.1}$$

and otherwise zero. For Fourier modes, the second commutators is written as

$$[x^I, p_J] = i\delta^I_J\ ,\ \ [X^I_n, P_{J,m}] = i\delta^I_J\delta_{n+m,0}, \tag{3.2}$$

where the delta function is one for the periodic function,

$$\delta(\sigma) = \frac{1}{2\pi}\sum_n e^{in\sigma}. \tag{3.3}$$

3.2 Some Candidates for Operator Order and String Ground State

Next we consider some operator orders and the string ground state corresponding to them. The vacuum for the zero mode part is the eigenstate of momenta, p_I and p_-, $|p_I, p_-\rangle$. The operator order for the zero mode part is given by the Weyl order or the *hermitian*

average[1] of terms ordered so that p and p_- are to the right of x and x^-.
We generally represent the ground state for the part of non-zero modes as $|0\rangle_{stringy}$. Then we collect both parts to obtain the total ground state as

$$|p_I, p_-\rangle \otimes |0\rangle_{stringy}. \tag{3.4}$$

Now we consider the operator order of the non-zero part and the string ground state explicitly. Because the principle which determines the operator order and the string ground state for the tensionless string theory is not unclear, we refer to the case of a usual tensionful string at first.

Tensionful string : $T \neq 0$

In the case of a usual string with non-zero tension, $T \neq 0$, we compose the left- and right-moving modes from X-mode and P-mode and then consider the normal order of them. For $T \neq 0$, the left- and right-moving modes are defined as

$$\begin{aligned} \alpha^I_{(T)n} &= -i\sqrt{\frac{T}{2}}nX^I_n + \frac{1}{\sqrt{2T}}P_{I,n}, \\ \tilde{\alpha}^I_{(T)n} &= -i\sqrt{\frac{T}{2}}nX^I_{-n} + \frac{1}{\sqrt{2T}}P_{I,-n}, \\ [\alpha^I_{(T)n}, \alpha^J_{(T)m}] = [\tilde{\alpha}^I_{(T)n}&, \tilde{\alpha}^J_{(T)m}] = n\delta^{IJ}\delta_{n,-m},\ [\alpha^I_{(T)n}, \tilde{\alpha}^J_{(T)m}] = 0. \end{aligned} \tag{3.5}$$

In the normal order, where all positive modes are to the right of negative ones, the mass square operator is

$$\mathcal{M}^2 = 2T\Big[\sum_{n>0}(\alpha_{(T)-n}\cdot\alpha_{(T)n} + \tilde{\alpha}_{(T)-n}\cdot\tilde{\alpha}_{(T)n}) - a - \tilde{a}\Big], \tag{3.6}$$

where a and $\tilde{a}$ are constants of the ambiguity arising from the operator order.[2] (3.6) is the set of the harmonic oscillators. Therefore, to be the eigenstate of (3.6) corresponding to the minimum eigenvalue, the string ground state, $|0\rangle_{(T)}$, must be annihilated by positive modes.

$$\alpha^I_{(T)n}|0\rangle_{(T)} = \tilde{\alpha}^I_{(T)n}|0\rangle_{(T)} = 0. \tag{3.7}$$

Furthermore the unique constraint in the tensionful string theory is

$$M_0 = \Big(\sum_{n>0}\alpha_{(T)-n}\cdot\alpha_{(T)n} - a\Big) - \Big(\sum_{n>0}\tilde{\alpha}_{(T)-n}\cdot\tilde{\alpha}_{(T)n} - \tilde{a}\Big) \approx 0. \tag{3.8}$$

This is the level-matching condition. The string ground state, $|0\rangle_{(T)}$, is the eigenstate of M_0 and then satisfies (3.8) if $a = \tilde{a}$.
Thus the string ground state in a given operator order must be the mass eigenstate corresponding to the minimum eigenvalue and then the eigenstate of M_0.[3]

[1] The *hermitian average* of operator A is defined as $\frac{1}{2}(A + A^\dagger)$.

[2] In the tensionful string theory, avoiding the Lorentz anomaly determines the critical dimension and the ordering constants, $D = 26$ and $a = \tilde{a} = 1$.

[3] The string ground state does not have to satisfy the constraint because it is enough to discover the physical state satisfying the constraint among the string ground state or excited states. The string ground state is just the eigenstate of M_0 to obtain the state satisfying the constraint.

Tensioness String : $T = 0$

Now we change the topic to the case of $T = 0$. Because the tensionless string theory has no dimensionful parameter such as tension T, we cannot make the combination such as $\alpha^I_{(T)n}$ or $\tilde{\alpha}^I_{(T)n}$. However we can refer to the tensionless limit of the tensionful string theory, $T \to 0$.

For the sufficiently large T, the left- and right-moving modes in (3.5) are

$$\alpha^I_{(T)n} \sim \frac{1}{\sqrt{2T}} P^I_n \ , \ \tilde{\alpha}^I_{(T)n} \sim \frac{1}{\sqrt{2T}} P^I_{-n}. \tag{3.9}$$

By collecting both positive modes, the state which is annihilated by $P_{I,n}$ for $n \neq 0$ is one of the natural candidates for the ground state in the case of the tensionless string theory. We decide to represent it as $|0\rangle_P$.

$$P_{I,n}|0\rangle_P = 0 \ \text{ for } n \neq 0 \tag{3.10}$$

In the case of the tensionless string theory, the mass square is written only with P-mode.

$$\mathcal{M}^2_{classical} = \sum_{n \neq 0} P_n \cdot P_{-n} \tag{3.11}$$

What is the operator order in which $|0\rangle_P$ becomes the minimum eigenstate of the mass square operator ? One candidate is the Reference order, in which all P-modes are to the right of X-modes, called R-order for short [40]. Because the R-ordered mass square operator is the same as the r.h.s. in (3.11), it is clear that $|0\rangle_P$ is the eigenstate of the R-ordered $\mathcal{M}^2$. Furthermore, because the eigenvalue of the R-ordered $\mathcal{M}^2$ is positive or zero from its structure,[4] $|0\rangle_P$ corresponds to the minimum eigenvalue.[5] The constraint is considered in the next section.

However there is a caution in the R-order. Because most of generators in the R-order are not hermitian, we have to *hermitianize* them. The generators of the translation which consist only of P-modes do not need the *hermitianization*, but other generators need usually.

We use the next *hermitianization*. In the R-order, the hermitian version of the R-ordered generator is

$$\mathcal{G} = \frac{1}{2}\left(\mathcal{G}_R + (\mathcal{G}_R)^\dagger\right) = \mathcal{G}_R - i g_R + \cdots , \tag{3.12}$$

where $\mathcal{G}_R$ and g_R are R-ordered operators.

Because $\mathcal{M}^2$ consists only of P-modes, we can choose other operator orders for the same ground state, $|0\rangle_P$. For example, the Weyl order [6] is the order in which $|0\rangle_P$ becomes the eigenstate of $\mathcal{M}^2$. The other candidate is the Hermitian R-order in which all operators

[4] The mass square operator consists of the product of a operator and its hermitian conjugate such as $A^\dagger A$.

[5] In the following chapters we see this minimality from the explicit eigenstates.

[6] In the Weyl order, all operators are totally symmetrized.

are ordered so that each term has its hermitian partner such as the second formula in (3.12). The generators and commutators of lower degree than 3 have the same ordering ambiguity in these operator orders, but the generators and commutators of higher degree such as $\mathcal{K}^-$ and $[\mathcal{J}^{-I}, \mathcal{K}^K]$ have the differences.[7] Such differences affect the investigation of the anomaly in the spacetime conformal symmetry.[8]

Above we have considered the operator order corresponding to $|0\rangle_P$, which is the limit of the string ground state for the tensionful string theory. Of course, other ground states are possible. From the structure of the mass square operator, $\mathcal{M}^2$, the state annihilated by positive modes of X and P is the eigenvalue of $\mathcal{M}^2$. We represent it as $|0\rangle_{XP}$, which satisfy

$$X_n^I|0\rangle_{XP} = P_{I,n}|0\rangle_{XP} = 0 \quad \text{for } n > 0. \tag{3.13}$$

The operator order corresponding to $|0\rangle_{XP}$ is the "XP-normal order", in which all positive modes of X and P are to the right of negative ones [36]. There is no ambiguity in the definition of the XP-normal order because any two positive modes (or negative modes) commute mutually. The positive P-modes in mass square operator are always to the right end without causing extra terms. Thus we find that $|0\rangle_{XP}$ is the mass eigenstate corresponding to zero eigenvalue. The constraint is considered in the next section.

The other candidates are the orders in which linear-combinational modes of X and P are normally ordered in the same way as the case of the usual tensionful string theory. Here we call them the combinational normal order collectively, or the C-normal order for short. One of the C-normal order is the normal order for the "left"- and "right"-moving modes which are combined with an artificial dimensionful parameter $\omega > 0$ instead of tension T.[9] We call this the ωLR-normal order. The fundamental modes are obtained from the case of $T = \omega^2$ in (3.5). In this order, the constraint is separated into the part of α-mode and that of $\tilde{\alpha}$-mode in the same way as the case of the tensionful string theory and then becomes the level-matching condition.
Because $P_{I,n} \propto \omega$, if $P_{I,n}$ solved with α and $\tilde{\alpha}$, we find $\mathcal{M}^2 \propto \omega^2$. The power of ω indicates the mass-dimension. Therefore we can always know the power of ω from the mass dimension. Hence we choose $\omega = 1$ and then omit the subscript (ω^2) in α and $\tilde{\alpha}$ below.
The mass square operator in the LR-normal order is

$$\mathcal{M}^2 = \sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + \alpha_n \cdot \tilde{\alpha}_n + \alpha_{-n} \cdot \tilde{\alpha}_{-n}) - b, \tag{3.14}$$

where b is the ordering constant. Unfortunately, the third and fourth terms in (3.14) cause the problem. Especially, because of the 4th term, the state which is annihilated by positive modes is not eigenstate of $\mathcal{M}^2$ in (3.14). Therefore we cannot use such a state as the ground state.

[7] The detail is in the later chapter.

[8] Because there is no difference between the R-order case, the Weyl order case and the Hermitian R-order case in $[\mathcal{J}^{-I}, \mathcal{J}^{-J}]$, the check of the Lorentz anomaly in these operator orders is done similarly.

[9] "Left" and "right" are artificial and irrelevant to the direction of oscillation on a string.

Instead of the LR-normal order, we consider another C-normal order and the ground state. We redefine the fundamental modes as $a_n^I = \alpha_n^I$ and $b_n^I = i\tilde{\alpha}_{-n}^I$. a_n^I and b_n^I satisfy the standard commutation relations, $[a_n^I, a_m^J] = [b_n^I, b_m^J] = n\delta^{I,J}\delta_{n,-m}$. In this case, the mass square operator is

$$\mathcal{M}^2 = \sum_{n=1}^{\infty}(a_{-n}\cdot a_n - b_{-n}\cdot b_n - ib_{-n}\cdot a_n - ia_{-n}\cdot b_n) - c, \tag{3.15}$$

where c is the constant of the ordering ambiguity. The state annihilated by all positive modes of a and b is the eigenstate of $\mathcal{M}^2$. Other C-normal orders are similarly considered. More discussion is given in next section.

3.3 Preparation and Consideration in Each Operator Order

For calculations of commutators in the following chapters, the important operators and their commutators are prepared. In this section we show the ordered version of the Fourier modes of $P_+(\sigma)$ and $X^-(\sigma)$ written by X_n^I and $P_{I,n}$, the ordered mass square operator and the ordered constraint operator in each order.

$$\begin{aligned} L_n &= \sum_m :P_m\cdot P_{n-m}:, \\ M_n &= -i\sum_m m :X_m\cdot P_{n-m}:, \end{aligned} \tag{3.16}$$

$$\mathcal{M}^2 = -2p_+p_- - p\cdot p = L_0 - p\cdot p = \sum_{n\neq 0} :P_n\cdot P_{-n}: \ , \tag{3.17}$$

$$M_0 = -i\sum_n n :X_n\cdot P_{-n}: \tag{3.18}$$

Then the commutators of them are given.

The results in this section are collected in appendix B.

3.3.1 R-order case

In this subsection we consider the case of the R-order. First the Fourier modes of $P_+(\sigma)$, L_n, is

$$L_n = \sum_m P_m\cdot P_{n-m}\ , \quad L_{-n} = (L_n)^\dagger. \tag{3.19}$$

Because P-modes commute mutually, there is no ordering ambiguity. From zero mode of (3.18), we obtain

$$\mathcal{M}^2 = L_0 - p\cdot p = \sum_{n\neq 0} P_n\cdot P_{-n}. \tag{3.20}$$

As seen in the last section, the ground state, $|0\rangle_P$, is the eigenstate of (3.20) corresponding to zero eigenvalue, namely the massless eigenstate.
Next the Fourier modes of $\bar{X}^-(\sigma)$, M_n for $n \neq 0$, are

$$M_n = -i\sum_m mX_m \cdot P_{n-m} \ , \quad M_{-n} = (M_n)^\dagger . \tag{3.21}$$

Because X^I_m and $P_{I,n-m}$ commute mutually for $n \neq 0$, there is no ordering ambiguity.
On the other hand, because X^I_m and $P_{I,-m}$ do not commute mutually, M_0 has a constant of the ordering ambiguity. We move the ordering constant to the r.h.s. of the constraint to obtain

$$M_0 = \frac{-i}{2}\sum_m m(X_m \cdot P_{-m} - P_m \cdot X_{-m}) = -i\sum_m mX_m \cdot P_{-m} \approx a, \tag{3.22}$$

where we used $\sum_m m = 0$ in the second equal sign. We expect that a is determined by the requirement of the spacetime symmetry. From (3.22), we find that $|0\rangle_P$ is the eigenstate of M_0 corresponding to zero eigenvalue.
Furthermore useful commutators of these operators are

$$\begin{aligned} &[X^I_n, M_m] = (n+m)X^I_{n+m} \ , \ [P_{I,n}, M_m] = nP_{I,n+m} \ , \\ &[X^I_n, L_m] = 2iP_{I,n+m} \ , \ [P_{I,n}, L_m] = 0 \ , \end{aligned} \tag{3.23}$$

$$[L_n, L_m] = 0 \ , \ [L_n, M_m] = (n-m)L_{n+m} \ , \ [M_n, M_m] = (n-m)M_{n+m} \ . \tag{3.24}$$

The commutators in (3.24) satisfy the 2d Galiean conformal algebra (GCA) without the central extension term.[10]

Hermitian R-order or Weyl order

The operators and commutators given above are unchanged in the cases of the Hermitian R-order or the Weyl order. As seen in the later chapter, the difference arises in operators of higher degree.

3.3.2 XP-normal order case

In this subsection we consider the case of the XP-normal order. First the Fourier modes of $P_+(\sigma)$ are

$$L_n = \sum_{m=0}^{n} P_m \cdot P_{n-m} + 2\sum_{m=1}^{\infty} P_{-m} \cdot P_{n+m} \ , \ L_{-n} = (L_n)^\dagger \tag{3.25}$$

for $n > 0$. There is no ordering ambiguity in the same way as the case of the R-order. From zero mode of (3.25), we obtain

$$\mathcal{M}^2 = 2\sum_{n=1}^{\infty} P_{-n} \cdot P_n . \tag{3.26}$$

[10] 2d GCA are obtained also from the tensionless limit of the Virasoro algebra [56, 57].

From this representation, we find that the string ground state, $|0\rangle_{XP}$, is massless eigenstate.
Next the Fourier modes of $\bar{X}^-(\sigma)$ are

$$\begin{aligned} M_n &= -i\sum_{m=1}^{n} mP_{n-m}\cdot X_m - i\sum_{m=1}^{\infty}((n+m)P_{-m}\cdot X_{n+m} - mX_{-m}\cdot P_{n+m})\ , \\ M_{-n} &= (M_n)^\dagger, \end{aligned} \tag{3.27}$$

for $n > 0$.
Because M_0 can contain the constant coming from the ordering ambiguity, we move it to the r.h.s. of the constraint to obtain

$$M_0 = -i\sum_{m=1}^{\infty} m(P_{-m}\cdot X_m - X_{-m}\cdot P_m) \approx a. \tag{3.28}$$

From this, we find that $|0\rangle_{XP}$ is the eigenstate of M_0 corresponding to zero eigenvalue.
Furthermore useful commutators of these operators are

$$\begin{aligned} &[X_n^I, M_m] = (n+m)X_{n+m}^I\ ,\ [P_{I,n}, M_m] = nP_{I,n+m}\ , \\ &[X_n^I, L_m] = 2iP_{I,n+m}\ ,\ [P_{I,n}, L_m] = 0\ , \end{aligned} \tag{3.29}$$

$$\begin{aligned} &[L_n, L_m] = 0\ ,\ [L_n, M_m] = (n-m)L_{n+m}\ , \\ &[M_n, M_m] = (n-m)M_{n+m} + \frac{D-2}{6}(n^3-n)\delta_{n,-m}\ . \end{aligned} \tag{3.30}$$

Here we note that there is the central extension in the last line.
As seen in the next next chapter, in the case of $D > 3$, the requirement of the Lorentz symmetry determines D and a.

3.3.3 C-normal order case

Here we give the representations of the important operators and commutators in the linear-combinational normal order and then see the C-normal order cause various problems: the complicated mass square operator and the inevitability of negative norms. Therefore we decide to exclude the C-normal orders in the following chapters.

Concretely, we consider the following linear-combinational modes.[11]

$$\begin{aligned} &a_n^I = \alpha_n^I = -i\frac{1}{\sqrt{2}}nX_n^I + \frac{1}{\sqrt{2}}P_n^I\ ,\ (a_n^I)^\dagger = a_{-n}^I \\ &b_n^I = i\tilde{\alpha}_{-n}^I = -\frac{1}{\sqrt{2}}nX_n^I + i\frac{1}{\sqrt{2}}P_n^I\ ,\ (b_n^I)^\dagger = -b_{-n}^I \\ &[a_n^I, a_m^J] = [b_n^I, b_m^J] = n\delta^{I,J}\delta_{n,-m}\ ,\ [a_n^I, b_m^J] = 0. \end{aligned} \tag{3.31}$$

Here we note that the abnormal hermitian property of b-mode. X_n^I and P_n^I are solved with a- and b-modes as

$$\begin{aligned} X_n^I &= \frac{i}{\sqrt{2}n}(a_n^I + ib_{-n}^I) \\ P_n^I &= \frac{1}{\sqrt{2}}(a_n^I - ib_{-n}^I)\ ,\ p^I = \sqrt{2}a_0^I = -i\sqrt{2}b_0^I. \end{aligned} \tag{3.32}$$

[11] As well as the last section, we set $\omega = 1$.

By using (3.32), we get L_n and M_n for all n as follows:

$$\begin{aligned} L_n &= A_n - B_n + C_n - c\delta_{n,0} \ , \ L_{-n} = (L_n)^\dagger \\ M_n &= A_n + B_n \ , \ M_{-n} = (M_n)^\dagger , \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} A_n &= \frac{1}{2}\sum_{m=0}^{n} a_m \cdot a_{n-m} + \sum_{m=1}^{\infty} a_{-m} \cdot a_{n+m} \ , \ A_{-n} = (A_n)^\dagger \\ B_n &= \frac{1}{2}\sum_{m=0}^{n} b_m \cdot b_{n-m} + \sum_{m=1}^{\infty} b_{-m} \cdot b_{n+m} \ , \ B_{-n} = (B_n)^\dagger \\ C_n &= -i\sum_{m=0}^{n} \alpha_m \cdot b_{n-m} - i\sum_{m=1}^{\infty} (a_{-m} \cdot b_{n+m} + b_{-m} \cdot a_{n+m}) \ , \ C_{-n} = (C_n)^\dagger . \end{aligned} \tag{3.34}$$

As well as M_0, L_0 also has the ordering constant c unlike the former two cases because the fundamental modes are a and b, not X and P. The ordering constant in M_0 is moved the r.h.s. of the constraint.
The commutators of a-mode (or b-mode) and these operators are

$$\begin{aligned} \left[a_n^I, A_m\right] &= n a_{n+m}^I \ , \ \left[a_n^I, C_m\right] = -in b_{n+m}^I \\ \left[b_n^I, B_m\right] &= n b_{n+m}^I \ , \ \left[b_n^I, C_m\right] = -in a_{n+m}^I , \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} &[L_n, L_m] = 0 \ , \ [L_n, M_m] = (n-m)(L_{n+m} - c\delta_{n,-m}) \\ &[M_n, M_m] = (n-m)M_{n+m} + \frac{D-2}{6}(n^3 - n)\delta_{n,-m} . \end{aligned} \tag{3.36}$$

In the last line of (3.36), there is the central extension term.
From zero mode in (3.33), we obtain the representations of $\mathcal{M}^2$ and M_0. The mass square operator is represented as

$$\mathcal{M}^2 = A_0 - B_0 + C_0 - p \cdot p = \sum_{n=1}^{\infty} (a_{-n} \cdot a_n - b_{-n} \cdot b_n - ib_{-n} \cdot a_n - ia_{-n} \cdot b_n) - c \tag{3.37}$$

and M_0 is

$$M_0 = A_0 + B_0 = \sum_{m=1}^{\infty} a_{-m} \cdot a_m + \sum_{m=1}^{\infty} b_{-m} \cdot b_m \approx a, \tag{3.38}$$

where the ordering constant, a, was moved to the r.h.s. of the constraint. Although M_0 separates into the part of a-mode and the part of b-mode, two parts have the same sign unlike the LR-normal order or the normal order in a usual tensionful string theory. Therefore the constraint in (3.38) is a strong condition. Because the ground state is the eigenstate corresponding to $M_0 = 0$, there is no physical state for negative a or non-integer a. Therefore a must be zero or positive integer.
When a equals to positive integer k, the states at level k satisfy the constraint. By using

the next commutators of $\mathcal{M}^2$ and a-mode or b-mode, we find $(a^I_{-k} - ib^I_{-k})|0\rangle_C$ as the simple eigenstate of $\mathcal{M}^2$.

$$[\mathcal{M}^2, a^I_{-n}] = n(a^I_{-n} - b^I_{-n}) \ , \ [\mathcal{M}^2, b^I_{-n}] = -in(a^I_{-n} - b^I_{-n}). \tag{3.39}$$

However, because of $(b^I_n)^\dagger = -b^I_{-n}$, the norm of $(a^I_{-k} - ib^I_{-k})|0\rangle_C$ is zero. Although we create more complicated eigenstates for $k > 1$, their norms are zero or negative. The ground state $|0\rangle_C$ is only the mass eigenstate with positive norm. The cause of such negative norm states is the hermitian property of b-mode, $(b^I_n)^\dagger = -b^I_{-n}$. If we restrict states excited only by a to avoid negative norm states, we cannot find any mass eigenstates in excited states.

The LR-normal order, which is the original C-normal order, has the problem that the state annihilated by positive modes is not the mass eigenstate. Therefore we have considered a^I_n and b^I_n above as other combinations. However we have found above that the excited states in new C-order cause the problem. Though the unique case in which we can avoid the problem is $a = 0$, the case of $a = 0$ in the C-normal order is very similar to the case of $a = 0$ in the XP-normal order. Therefore the C-normal order is excluded from the discussion in the following chapters.

3.4 Operator Orders used in the Following Chapters

Here we collect the operator orders and the string ground states used in the following chapters below.

Operator Orders used in the Following Chapters

Order name	Ordering rule	ground state
R-order	all P-modes are to the right of X-modes	$\|0\rangle_P$
Hermitian R-order*	the *hermitian average* of the R-order	$\|0\rangle_P$
XP-normal order	all positive modes of X & P are to the right of negatives	$\|0\rangle_{XP}$

$*$: Weyl order is similar.

Chapter 4

Quantum 3D Tensionless String in Light-cone Gauge with Poincaré Symmetry

In this chapter we consider a quantum 3-dim. tensionless closed bosonic string with the Poincaré symmetry in the light-cone gauge.

For the preparation or the reference, the generators of the Poincaré symmetry in three dimensions and the light-cone gauge are shown below. From (2.59) and (2.60),

$$
\begin{aligned}
&\mathcal{P}_- = p_- \ , \ \mathcal{P} = p \ , \ \mathcal{P}_+ = -\frac{:L_0:}{2p_-} \ , \\
&\mathcal{J}^+ = \mathcal{J}^{+2} = -xp_- \ , \ \mathcal{J} = -\mathcal{J}^{+-} = x^- p_- - \frac{i}{2} \ , \\
&\mathcal{J}^- = -\mathcal{J}^{-2} = -x^- p - \frac{1}{2p_-} x :L_0: + \frac{i}{2p_-} p + \frac{i}{p_-} \sum_{n\neq 0} \frac{1}{n} :M_n P_{-n}: - \frac{1}{2p_-} \sum_{n\neq 0} :X_n L_{-n}: .
\end{aligned}
\tag{4.1}
$$

where the part sandwiched by : are ordered.

Because the dangerous commutator for the Lorentz anomaly vanishes trivially in three dimensions, $[\mathcal{J}^-, \mathcal{J}^-] = 0$, the ordering constant a in the constraint (3.21) is not determined by the requirement of the Lorentz symmetry in any operator order.[1] Therefore there are many possibilities in the mass spectrum according to the value of a. We investigate the mass spectrum[2] to consider the natural choice of the ordering constant.
In the Reference order, its hermitian version and the Weyl order, we find the continuous spectrum in addition to zero. Furthermore in these orders we find non-separable eigenstates, which are not factorized with each Fourier-mode, as well as separable ones. From such informations, we consider the natural choice of the ordering constant. In the XP-normal order, we investigate the norm of mass eigenstates to consider the natural choice of the ordering constant.

[1] This specialty of three dimensions is the same as the case of the 3-dim. tensionful string theory [50–52].
[2] The spectrum given in this chapter would be also a help to the case of higher dimensions.

4.1 Mass Spectrum in R-order

First we consider the mass spectrum in the R-order.[3] The string ground state is $|0\rangle_P$, which is annihilated by non-zero P-modes.
The general states are created by acting $\{X_n\}_{n\neq 0}$ on $|0\rangle_P$. The fundamental elements of general states are

$$X_{n_1}X_{n_2}\cdots X_{n_l}|0\rangle_P, \tag{4.2}$$

where $n_i \neq 0$ $(i=1,2,\cdots,l)$. More generally the states acted the function of $\{X_n\}_{n\neq 0}$ are possible.
To be the eigenstate of $\mathcal{M}^2$ and to satisfy the constraint $M_0 = a$, the function of $\{X_n\}_{n\neq 0}$ are restricted greatly. Furthermore it is also desirable to be the function consists of the positive integer power of $\{X_n\}_{n\neq 0}$.

To find the concrete eigenstates, we use the X-representation for non-zero mode. From the commutation relation, $[X_n, P_m] = i\delta_{n,-m}$, P_n is written in the X-rep. as

$$P_n = -i\frac{\partial}{\partial X_{-n}} \tag{4.3}$$

Then we close the states by the bra-state which is the eigenstate of $\{X_n\}_{n\neq 0}$, $\langle\{X_x\}|$, to consider the wave functions.

4.1.1 Mass square operator $\mathcal{M}^2$ and constraint operator M_0

First the mass square operator in the X-rep. is

$$\mathcal{M}^2 = 2\sum_{n>0} P_n P_{-n} = -2\sum_{n>0}\frac{\partial}{\partial X_{-n}}\frac{\partial}{\partial X_n}. \tag{4.4}$$

This reduce two Xs from the function acting on $|0\rangle_P$. Therefore most of eigenfunctions of (4.4) are infinite series of $\{X_n\}_{n\neq 0}$. Of course, we expect such series consists of the positive integer power of $\{X_n\}_{n\neq 0}$.
From the spacetime conformal symmetry, we expect the eigenvalue of $\mathcal{M}^2$ is zero or positive continuous.[4] In fact, we can know the continuity of the mass spectrum from the representation of (4.4). If a mass eigenfunction corresponding to eigenvalue M^2 is given by $\psi_M(\{X_n\})$, we do the scale transformation as $X_n \to \lambda X_n$ for all n to obtain a mass eigenfunction corresponding to the eigenvalue $\lambda^2 M^2$ as $\psi_M(\{\lambda X_n\}) = \psi_{\lambda M}(\{X_n\})$.

Next the constraint in the X-rep. is

$$M_0 = -i\sum_n nX_nP_{-n} = -\sum_n nX_n\frac{\partial}{\partial X_n} \approx a. \tag{4.5}$$

For example, for the state in (4.2),

$$\sum_{i=1}^{l} n_i = a. \tag{4.6}$$

[3]Because the string ground state and the representation of the mass square operator are the same as in the Hermitian R-order and the Weyl order, the mass spectra in these orders is the same.

[4]In the later subsection, we will see for normalizable eigenfunctions that there is no negative eigenvalue.

If we want the states consisting of the positive integer power of $\{X_n\}_{n\neq 0}$, a of integer must be chosen.

4.1.2 Mass eigenfunction for separable type

Now we investigate a mass eigenfunction. To simplify the leading of eigenfunctions, we use the polar representation of X_n and X_{-n}.

$$X_n = r_n e^{i\theta_n} \ , \ X_{-n} = r_n e^{-i\theta_n} \tag{4.7}$$

for $n > 0$. Then $\mathcal{M}^2$ is written as

$$\mathcal{M}^2 = -\frac{1}{2}\sum_{n>0}\left[\frac{\partial^2}{\partial {r_n}^2} + \frac{1}{r_n}\frac{\partial}{\partial r_n} + \frac{1}{{r_n}^2}\frac{\partial^2}{\partial {\theta_n}^2}\right]. \tag{4.8}$$

Now (4.8) can be separated into the part of each n. Therefore we can separate variables to find solutions. The fundamental differential equation is

$$-\frac{1}{4}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\psi_m(r,\theta) = m^2\psi_m(r,\theta), \tag{4.9}$$

where $r \geq 0$. From the symmetry of the "angular" θ, we can assume $\psi_m(r,\theta) = \phi_{m,s}(r)e^{is\theta}$ for integer s. Then $\phi_{m,s}(r)$ satisfy the next differential equation.

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + 4m^2 - \frac{s^2}{r^2}\right]\phi_{m,s}(r) = 0. \tag{4.10}$$

Thus we obtain the ordinary differential equation. To get the solution of (4.10) concretely, we divide in two cases, $m^2 \neq 0$ and $m^2 = 0$.

Solutions in $m^2 \neq 0$ case

In the case of $m^2 \neq 0$, (4.10) is written as

$$\left[\frac{d^2}{d\hat{r}^2} + \frac{1}{\hat{r}}\frac{d}{d\hat{r}} + 1 - \frac{s^2}{\hat{r}^2}\right]\phi_{m,s}\left(\frac{\hat{r}}{2m}\right) = 0. \tag{4.11}$$

where $\hat{r} = 2mr$. The solution of this differential equation is given by the Bessel function $J_{|s|}(\hat{r})$.[5] Thus we obtain

$$\psi_m(r,\theta) = N_m J_{|s|}(2mr)e^{is\theta} \tag{4.12}$$

where N_m is a normalization constant. From the regularity of the Bessel function on the infinity, m must be real. Therefore $m^2 < 0$ is excluded. We restrict to $m^2 > 0$ below. Because the argument of the Bessel function in (4.12) is $2mr$, we obtain the eigenfunction corresponding to eigenvalue $\lambda^2 m^2$ by rescaling $r \to \lambda r$ in (4.12).

[5] For integer ν, $J_\nu(x) = (-1)^\nu J_{-\nu}(x)$.

The solution in (4.12) is written by the original variables, $X_n = r_n e^{i\theta_n}$ and $X_{-n} = r_n e^{-i\theta_n}$, as[6]

$$\begin{aligned}\psi_{m,\pm|s|}(r_n,\theta_n) &= N_m J_{|s|}(2m(X_n X_{-n})^{\frac{1}{2}}) \left(\frac{X_n}{X_{-n}}\right)^{\frac{s}{2}} \\ &= N_m (m X_{\pm n})^{|s|} \sum_{l=0}^{\infty} \frac{(-m^2)^l}{l!\ (l+|s|)!} (X_n X_{-n})^l, \end{aligned} \tag{4.13}$$

where $\pm$ is the sign of s and we used

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\ \Gamma(l+\nu+1)} \left(\frac{z}{2}\right)^{2l}. \tag{4.14}$$

From (4.13), we find that the eigenfunction is the positive integer power of X_n and X_{-n} as we expected.

Orthogonality and Normalization for $m > 0$

Here we show the orthogonality of above eigenfunctions and give the normalization of eigenfunctions. According to appendix C, the inner product of two wave functions, $\psi_1(r,\theta)$ and $\psi_2(r,\theta)$, is

$$(\psi_1, \psi_2) \equiv \int_0^\infty dr \int_0^{2\pi} d\theta\ r\ \psi_1^*(r,\theta)\psi_2(r,\theta). \tag{4.15}$$

For two eigenfunctions written by the Bessel function, we get

$$(\psi_{m,s}, \psi_{m',s'}) = \frac{\pi}{2} \frac{|N_m|^2}{m} \delta(m-m')\delta_{s,s'} \tag{4.16}$$

(4.16) means the orthogonality of two eigenfunctions corresponding to different eigenvalues. If we set $N_m = \sqrt{\frac{2m}{\pi}}$, we obtain $(\psi_{m,s}, \psi_{m',s'}) = \delta(m-m')\delta_{s,s'}$.
The detail is in appendix C.

Solution in $m^2 = 0$ case

In the case of $m^2 = 0$, (4.10) is written as

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{s^2}{r^2}\right] \phi_{0,s}(r) = 0. \tag{4.17}$$

The solution for $s \neq 0$ is $r^{|s|}$ or $r^{-|s|}$, and the solution for $s = 0$ is constant or $\log r$. Because of the regularity on $r = 0$, $r^{|s|}$ is chosen for $s \neq 0$ and constant solution is chosen for $s = 0$. The detail of such choice and the orthogonality is in appendix C.
These solutions are written by the original variables, X_n and X_{-n}, as

$$\psi_{0,\pm|s|}(r,\theta) \propto r^{|s|} e^{\pm i|s|\theta} = (X_{\pm n})^{|s|}. \tag{4.18}$$

This is the positive integer power of X_n or X_{-n} and the structure estimated easily in the original differential equation (4.4).

[6] We dropped n in s_n or m_n to avoid the confused representation.

Total mass eigenfunction for separable type

The eigenfunctions of (4.10) for each $n > 0$ are given above. The separable total mass eigenfunction is given by the product of them.

$$\Psi = \prod_{n>0} \psi_{m_n,s_n}(r_n, \theta_n) \tag{4.19}$$

Its eigenvalue is the summation over n,

$$\mathcal{M}^2 = 2\sum_{n>0}(m_n)^2. \tag{4.20}$$

As mentioned before, the eigenvalue of $\mathcal{M}^2$ is zero or continuous. The separable total mass eigenfunction corresponding to $\mathcal{M}^2 = 0$ is given by the product of eigenfunctions (4.18) corresponding to $m_n = 0$ for all $n > 0$. The separable total mass eigenfunction corresponding to $\mathcal{M}^2 > 0$ is given by using the eigenfunction corresponding to $m_n^2 > 0$ for at least one of n.[7]

4.1.3 Physical mass eigenfunction

The mass eigenfunction has been investigated above. The physical states or wave functions must satisfy the constraint $M_0 = a$. In the polar coordinate, r_n and θ_n, the constraint is

$$M_0 = -\sum_{n>0} n\left[X_n\frac{\partial}{\partial X_n} - X_{-n}\frac{\partial}{\partial X_{-n}}\right] = i\sum_{n>0} n\frac{\partial}{\partial \theta_n}. \tag{4.21}$$

For (4.19), the constraint is

$$M_0 = -\sum_{n>0} n s_n = a. \tag{4.22}$$

Because s_n is integer, a must be integer. This is the requirement from the mass spectrum, not from the spacetime symmetry. Furthermore the choice of integer a is separated into two cases, of zero and non-zero integer. The ground state $|0\rangle_P$ is physical in the first case, but not in the second case. On the other hand, the states excited by one X-mode are not physical in the first case, but in the second case.

We summarize the choice of a and the possible physical mass eigenstates in the next table.

Choice of a and Physical mass eigenstates.

a	$\|0\rangle_P$	$X_{-i}\|0\rangle_P$	higher
0	$\circ$	$\times$	$\circ$
k (non-zero integer)	$\times$	$\circ$ (for $k = i$)	$\circ$
non-integer	$\times$	$\times$	$\times$

$\circ$ = "physical states can exist", $\times$ = "unphysical"

[7] For other n's, we can choose $m_n^2 = 0$.

Thus $a = 0$ is special case. In fact, we must choose $a = 0$ to avoid the Lorentz anomaly for the tensionless string in $D > 3$ in some operator orders, and to avoid the spacetime conformal anomaly for the tensionless string in $D = 3$ in the Hermitian R-order.
Because the case of $a = 0$ is special and important, we consider particularly the case of $a = 0$ below. The case of non-zero integer a is similarly discussed.

4.1.4 More physical mass eigenfunction for non-separable type and Λ-action

So far we have considered the mass eigenfunction which can be factorized into each n solution. But there sre also non-separable mass eigenfunctions.
If the number of variables is finite and a good symmetry exists, all we have to consider is the separable eigenfunction. For example, we can know the general solution of the 3d Schrödinger equation with the rotational symmetry from separable solutions expanded by the spherical harmonics. However, because the number of variables is infinite, such a way is too difficult.

Here we verify the existence of non-separable mass eigenfunctions. Then we find that non-separable eigenfunctions are given by acting the operator Λ on separable ones.
First we consider the massless case and next the massive case.

Massless case : $a = 0$

First of all we give separable massless states. We must choose $m_n = 0$ for all n to get separable massless states. From (4.18) and (4.22), the simplest massless state for $a = 0$ is the string ground state $|0\rangle_P$ and other separable ones are

$$X_{n_1} X_{n_2} \cdots X_{n_l} |0\rangle_P, \tag{4.23}$$

where $\sum_{i=1}^{l} n_i = 0$ and $n_i + n_j \neq 0$ for $\forall i, j$.[8] From this, one of the second simplest separable massless state is

$$X_2 X_{-1} X_{-1} |0\rangle_P. \tag{4.24}$$

The monomial state of degree 1 of X is massless, but does not satisfy the constraint for $a = 0$. The monomial state of degree 2 can satisfy the constraint such as $X_n X_{-n}|0\rangle_P$, but is not mass eigenstate.

Next we give examples of the non-separable massless state.

$$\begin{aligned} &(X_1 X_{-1} - X_2 X_{-2})|0\rangle_P \ , \\ &((X_1 X_{-1})^2 + (X_2 X_{-2}^2) - 4(X_1 X_{-1})(X_2 X_{-2}))|0\rangle_P. \end{aligned} \tag{4.25}$$

They are polynomial. The non-separable massless state of higher degree must be polynomial similarly.
It is difficult to find the non-separable massless state of higher degree directly. Therefore we consider the indirect way.

[8] Although we may order n_i's as $n_1 \leq n_2 \cdots \leq n_l$, it is not always necessary.

In three dimensions, there is another Poincaré invariant in addition to the mass square operator, $\mathcal{M}^2 = -\mathcal{P}^2$. It is Λ defined as

$$\mathcal{J}^\mu \mathcal{P}_\mu \equiv \Lambda = -\frac{1}{2} \sum_{\substack{n\neq 0 \\ n+m\neq 0}} \sum_{m\neq 0} \left(1 + \frac{m}{n} + \frac{n}{m}\right) X_{n+m} \frac{\partial}{\partial X_n} \frac{\partial}{\partial X_m}. \tag{4.26}$$

Two Poincaré invariants commute mutually and also with M_0. Therefore the action Λ on a mass eigenstate create another mass eigenstate corresponding to the same eigenvalue.[9] In fact, two states in (4.25) are created by the action of Λ on another massless state as follows:

$$\begin{aligned} \Lambda X_2 X_{-1} X_{-1} |0\rangle_P &= 3(X_1 X_{-1} - X_2 X_{-2})|0\rangle_P \\ \Lambda X_2 X_2 X_{-1} X_{-1} X_{-1} X_{-1} |0\rangle_P &\sim 6(2(X_1 X_{-1}) - 3(X_2 X_{-2})) X_2 X_{-1} X_{-1} |0\rangle_P \\ \Lambda\Lambda X_2 X_2 X_{-1} X_{-1} X_{-1} X_{-1} |0\rangle_P &\sim 54((X_1 X_{-1})^2 + (X_2 X_{-2})^2 - 4(X_1 X_{-1})(X_2 X_{-2}))|0\rangle_P, \end{aligned} \tag{4.27}$$

where $\sim$ in the second and third line means omitting separable massless states. The r.h.s. in the first line is the simplest non-separable massless state and the r.h.s. in the third line is the fundamental non-separable massless state of degree 4. Because the fundamental element is $(X_n X_{-n})$, we find that any non-separable massless state of degree 3 does not exist.
Thus we can create a non-separable massless state from a separable massless state by acting Λ.

The inner product of these non-separable massless states and massive ones is considered in appendix C.

Massive case : $a = 0$

We expect the existence of non-separable massive states. Although it is difficult to find them directly, we can find them by acting Λ on other massive eigenfunctions. We consider the case of $\mathcal{M}^2 = 2m^2$ as example. We use $\hat{r}_1 = 2m\sqrt{X_1 X_{-1}}$ and determine the normalization of $N_m = 1$ for the simplicity to obtain

$$\begin{aligned} \Lambda \psi_{m,0}(X_1, X_{-1})|0\rangle_P &= \Lambda J_0(\hat{r}_1)|0\rangle_P = -\frac{3}{2} m^2 \left(X_2 \psi_{m,-2}(X_1, X_{-1})\right)|0\rangle_P \\ \Lambda X_4 X_{-2} X_{-2} \psi_{m,0}(X_1, X_{-1})|0\rangle_P &\sim 3(X_2 X_{-2} - X_4 X_{-4}) \psi_{m,0}(X_1, X_{-1})|0\rangle_P \\ \Lambda X_3 X_{-2} \psi_{m,-1}|0\rangle_P &\sim \frac{7}{6m} \left[m^2 (2X_2 X_{-2} - 3X_3 X_{-3}) J_0(\hat{r}_1) + \frac{1}{2} \hat{r}_1 J_1(\hat{r}_1) \right] |0\rangle_P, \end{aligned} \tag{4.28}$$

where $\sim$ in the second and third lines means omitting separable massive states. In the above calculation, we have used various relations for the Bessel function [10]. The r.h.s. of the first line is a separable massive state again. The r.h.s of the second line is the

[9] Because $\mathcal{M}^2$ and Λ commute mutually, we may diagonalize simultaneously and search the simultaneous eigenfunction of both. However, it is difficult and not practical. We use another operator in the spacetime conformal symmetry to characterize the massless eigenstate. Therefore we use Λ as the tool to find another eigenfunction.

[10] For example, we use the definitional differential equation, $J''_\nu(z) + \frac{1}{z} J'_\nu(z) + (1 - \frac{\nu^2}{z^2}) J_\nu(z) = 0$, the recursion relation, $J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z)$, and $\frac{d^k J_n(z)}{dz^k} = \frac{1}{2^k} \sum_{r=0}^{k} \frac{(-1)^r \; k!}{r! \; (k-r)!} J_{n-k+2r}(z)$.

product of a separable eigenstate corresponding to non-zero eigenvalue of (4.10) and non-separable eigenstate corresponding to zero eigenvalue. The r.h.s. of the third line is the non-separable massive state.
Thus we can get a non-separable massive state from a separable massive state by acting Λ.

4.1.5 Characterization of massless states

So far we have considered the eigenstate of $\mathcal{M}^2$ and M_0. The physical state is characterized by eigenvalues of $\mathcal{M}^2$ and M_0.
From the explicit solution such as (4.23) and (4.25), we expect that the massless state is characterized by another operator which counts the number of X-action. The operator Δ which counts the number of X-action is defied by using a part of the dilatation generator as

$$\Delta \equiv i\tilde{\mathcal{D}}_R = \sum_n X_n \frac{\partial}{\partial X_n}. \tag{4.29}$$

where $\tilde{\mathcal{D}}_R$ is the non-zero mode part of $\mathcal{D}_R$. Δ satisfy

$$[\Delta, X_n] = X_n \ , \quad \text{for } n \neq 0. \tag{4.30}$$

$|0\rangle_P$ has $\Delta = 0$ and (4.23) has $\Delta = l$. Moreover the first line in (4.25) has $\Delta = 2$ and the second line has $\Delta = 4$.[11] Thus Δ characterize massless states.

In fact, because the commutator of Δ and $\mathcal{M}^2$ is

$$[\Delta, \mathcal{M}^2] = -2\mathcal{M}^2, \tag{4.31}$$

Δ is good characterization for massless states with $\mathcal{M}^2 = 0$. Therefore we can use Δ instead of Λ.[12]

4.2 Mass Spectrum in XP-normal Order

The string ground state in the XP-normal order is $|0\rangle_{XP}$, which is annihilated by positive X- and P-modes. The excited states are created by acting $\{X_{-n},\ P_{-n}\}_{n>0}$ on $|0\rangle_{XP}$. If the action of the negative exponent of X- and P-modes is prohibited, the general excited states are

$$X_{-m_1} \cdots X_{-m_k} P_{-n_1} \cdots P_{-n_l} |0\rangle_P, \tag{4.32}$$

where $m_i > 0$ $(i = 1, 2, \cdots, k)$ and $n_j > 0$ $(j = 1, 2, \cdots, l)$. The level of (4.32) is $\sum_{i=1}^k m_i + \sum_{j=1}^l n_j$.
The constraint $M_0 = a$ determines the level. By using the freedom of choosing the ordering constant a, we investigate values of a which give the interesting physics.

First of all we give possible values as a. Then we find mass eigenfunctions and mass eigenvalues for some a's. Finally we extract the interesting value of a.

[11] Δ which counts X is opposite sign of the scaling dimension which counts P.
[12] Because of $[\Delta, \Lambda] = -\Lambda$, we cannot use both Δ and Λ generally.

4.2.1 Constraint

The constraint in the XP-normal order is

$$M_0 = i\sum_{n=1}^{\infty} n(X_{-n}P_n - P_{-n}X_n) \approx a. \tag{4.33}$$

M_0 gives the level of states. The level of the ground state is zero and the level of the finitely excited states is positive integer. Therefore a must be zero or positive integer. For example, $M_0 = 0$ for $|0\rangle_{XP}$ and $M_0 = \sum_{i=1}^{k} m_i + \sum_{j=1}^{l} n_j$ for (4.32). More explicitly, the state which satisfy the constraint for $a = 0$ is only the string ground state $|0\rangle_{XP}$, for $a = 1$ are $P_{-1}|0\rangle_{XP} \equiv |P_{-1}\rangle_{XP}$ and $X_{-1}|0\rangle_{XP} \equiv |X_{-1}\rangle_{XP}$, and for $a = 2$ are the states created by acting $P_{-2},\ P_{-1}P_{-1},\ P_{-1}X_{-1},\ X_{-1}X_{-1},\ X_{-2}$ on $|0\rangle_{XP}$. The case of larger a is similarly discussed.

4.2.2 Action of mass square operator on physical states

Here we consider the action of the mass square operator for some values of a. The mass square operator in the XP-normal order is

$$\mathcal{M}^2 = 2\sum_{n=1}^{\infty} P_{-n}P_n. \tag{4.34}$$

The action of this operator on states reduces one X and then adds one P. Therefore we easily find that states created only by $\{P_{-n}\}_{n>0}$ are massless eigenstates. We verify below that mass eigenstates are only such massless states.

$a = 0$ case

The state which satisfies the constraint for $a = 0$ is only the string ground state $|0\rangle_{XP}$. This is the mass eigenstate corresponding to zero eigenvalue, massless state. The case of $a = 0$ is point-like particle because there is no stringy excitation.

$a = 1$ case

The states which satisfy the constraint for $a = 1$ are $|P_{-1}\rangle_{XP}$ and $i|X_{-1}\rangle_{XP}$. We act mass square operator on them to obtain $\mathcal{M}^2|P_{-1}\rangle_{XP} = 0$ and $\mathcal{M}^2 i|X_{-1}\rangle_{XP} = 2|P_{-1}\rangle_{XP}$. This action is represented as

$$\mathcal{M}^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}. \tag{4.35}$$

From this, we find that $|P_{-1}\rangle_{XP}$ is the mass eigenstate corresponding to zero eigenvalue, massless state. On the other hand, $|X_{-1}\rangle_{XP}$ is not a mass eigenstate.
(4.35) is the representation which can not be diagonalized. Because the mass square operator is originally hermitian, it is strange. The cause of this strangeness is that we represent $\mathcal{M}^2$ for the states restricted by the positive integer power of X_{-n} or P_{-m}. Therefore we firstly investigate the eigenstates of (4.34) without restriction and then should restrict the states which consist of the action of the positive integer powers of X_{-n}

or P_{-m}.

For the simplicity, we consider only the action of X_{-1} and P_{-1}. The general state which satisfy the constraint $M_0 = a = 1$ is represented as

$$P_{-1} F\left(\frac{X_{-1}}{P_{-1}}\right) |0\rangle_{XP}, \tag{4.36}$$

where F is a function of $\frac{X_{-1}}{P_{-1}}$. We act the mass square operator on (4.36) to get

$$\mathcal{M}^2 P_{-1} F\left(\frac{X_{-1}}{P_{-1}}\right) |0\rangle_{XP} = -2iP_{-1}F'\left(\frac{X_{-1}}{P_{-1}}\right) |0\rangle_{XP}, \tag{4.37}$$

where $'$ means differential with respect to the argument, $\frac{X_{-1}}{P_{-1}}$. If $F'(z) \propto F(z)$, we obtain a mass eigenstate. The solution of $F'(z) \propto F(z)$ is an exponential function. Concretely,

$$P_{-1} e^{c\frac{X_{-1}}{P_{-1}}} |0\rangle_{XP} = \left(P_{-1} + cX_{-1} + \frac{c^2}{2}\frac{X_{-1}X_{-1}}{P_{-1}} + \cdots\right) |0\rangle_{XP} \tag{4.38}$$

is a mass eigenstate corresponding to $\mathcal{M}^2 = -2ic$. For each c, there is an eigenstate. From (4.38), the state in which all terms consist of the zero or positive integer power is only the case of $c = 0$. The state for $c = 0$ is $|P_{-1}\rangle_{XP}$, which is one of the state at level 1 given above. Because another state at level 1 given above, $|X_{-1}\rangle_{XP}$, is a part of other eigenstates with $c \neq 0$ such as (4.38), we should not extract only $|X_{-1}\rangle_{XP}$. The case of allowing the action of other modes as well as X_{-1} and P_{-1} is similarly discussed.

Thus we find that the mass eigenstate satisfying the constraint for $a = 1$ is only $|P_{-1}\rangle_{XP}$.

However, it is not practical to extract only the states consisting of the positive integer power after searching the general mass eigenstate. Therefore we firstly give the states consisting of the positive integer power and then extract mass eigenstates and exclude others.

$a = 2$ **case**

The state which satisfy the constraint for $a = 2$ and consist of the positive integer powers are $|P_{-2}\rangle_{XP}$, $|P_{-1}P_{-1}\rangle_{XP}$, $i|P_{-1}X_{-1}\rangle_{XP}$, $-|X_{-1}X_{-1}\rangle_{XP}$ and $i|X_{-2}\rangle_{XP}$. The action of the mass square operator (4.34) is represented as

$$\mathcal{M}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.39}$$

From this, the first and second states are massless eigenstates. On the other hand, the last three states are not eigenstates. Therefore we exclude the last three states.

Thus we obtain $|P_{-2}\rangle_{XP}$ and $|P_{-1}P_{-1}\rangle_{XP}$ as mass eigenstate which satisfy the constraint for $a = 2$.

Here note that in three dimensions we can not interpret these eigenstates as a dilaton or a graviton because of the specialty of the irreducible representation for 3d Poincaré group [54].

$a = k$ **: positive integer case**

More generally, the mass eigenstates which satisfy the constraint for $a = k$ are

$$P_{-n_1} \cdots P_{-n_l}|0\rangle_P \quad \text{with} \quad \sum_{j=1}^{l} n_j = k, \tag{4.40}$$

where k is positive integer. The their mass eigenvalues are zero, massless.

4.2.3 Inner product and norm

Above we have investigated states consisting of the positive integer power of X_{-n} and P_{-m} and then have found that mass eigenstates are created by the action of P_{-m} on the string ground state $|0\rangle_{XP}$. Here we consider the norm of such mass eigenstates.
The ground state $|0\rangle_{XP}$ is annihilated by positive modes of X and P. The bra-state conjugate of this, ${}_{XP}\langle 0|$, is the state annihilated by negative modes of X and P. This is consistent with the commutation relation, $[X_n, P_m] = i\delta_{n,-m}$.
The norm of the string ground state is defined as

$$\| \, |0\rangle_{XP} \, \|^2 = {}_{XP}\langle 0 \mid 0\rangle_{XP} = 1, \tag{4.41}$$

where we normalized this to 1.
The norm of the general state $F(\{X_{-n}\}, \{P_{-n}\})|0\rangle_{XP} \equiv |F(\{X_{-n}\}, \{P_{-n}\})\rangle_{XP}$ is defined with the inner product. The inner product of $|F\rangle_{XP}$ and $|G\rangle_{XP}$ is defined as

$$\begin{aligned}\Big(|F\rangle_{XP}, |G\rangle_{XP}\Big) &\equiv {}_{XP}\langle 0| \big(F(\{X_{-n}\}, \{P_{-n}\})\big)^\dagger \, G(\{X_{-n}\}, \{P_{-n}\})|0\rangle_{XP} \\ &= {}_{XP}\langle 0|F^*(\{X_n\}, \{P_n\})G(\{X_{-n}\}, \{P_{-n}\})|0\rangle_{XP}.\end{aligned} \tag{4.42}$$

From this, the norm of $|F\rangle_{XP}$ is

$$\| \, |F\rangle_{XP} \, \|^2 = {}_{XP}\langle 0|F^*(\{X_n\}, \{P_n\})F(\{X_{-n}\}, \{P_{-n}\})|0\rangle_{XP} \tag{4.43}$$

Now we consider the norm of mass eigenstates. For lower level, we obtain

$$\begin{aligned}\| \, |P_{-1}\rangle_{XP} \, \|^2 &= {}_{XP}\langle 0|P_1 P_{-1}|0\rangle_{XP} = {}_{XP}\langle 0|P_{-1} P_1|0\rangle_{XP} = 0 \\ \| \, |P_{-1}P_{-2}\rangle_{XP} \, \|^2 &= {}_{XP}\langle 0|P_2 P_1 P_{-1} P_{-2}|0\rangle_{XP} = {}_{XP}\langle 0|P_{-1}P_{-2}P_2 P_1|0\rangle_{XP} = 0\end{aligned} \tag{4.44}$$

Similarly, for higher level, the norm of mass eigenstates is zero because two P-modes commute mutually. The physical interpretation of such zero norm is unclear.[13]

The fact that the unique positive norm state is the string ground state $|0\rangle_{XP}$, which is not excited, probably indicates that the tensionless string in the XP-normal order must be point-like. In the following chapter we will consider the requirement of the spacetime conformal symmetry to see that this is a natural interpretation.

[13] To avoid zero norm of mass eigenstates, we may define the different inner product as follows:

$$\Big(|F\rangle_{XP}, |G\rangle_{XP}\Big) \equiv {}_{XP}\langle 0|G^*(\{iP_n\}, \{-iX_n\})F(\{X_{-n}\}, \{P_{-n}\})|0\rangle_{XP}. \tag{4.45}$$

However, this inner product changes the meaning of the hermitian conjugate, $(A|F\rangle, |G\rangle) \neq (|F\rangle, A^\dagger|G\rangle)$. Therefore we must reconstruct the foundation of quantum theory. That is no longer the "quantum" theory we know.

Chapter 5

Quantum Higher Dimensional Tensionless String in Light-cone Gauge with Poincaré Symmetry ($D > 3$)

In this chapter we consider a quantum D-dimensional tensionless closed bosonic string with the Poincaré symmetry in the light-cone gauge. In $D > 3$, the avoidance of the Lorentz anomaly give the restriction. In the Reference order, the Hermitian R-order and the Weyl order, the ordering constant is determined to be zero, but the dimension is free. On the other hand, in the XP-normal order, the critical dimension is 26 and the ordering constant is also determined. Furthermore we give the mass spectra in theories without the anomaly.

5.1 R-order Case

In this section we consider the anomaly and mass spectrum of the tensionless string in the light-con gauge in the R-order. For the Poincaré symmetry, the cases of the Hermitian R-order and the Weyl order are similar.

5.1.1 Generators in R-order

In the R-order, some generators are not hermitian. Therefore we must *hermitianize* them such as (B.6). From (B.1) and (B.6), the hermitian generators of the Poincaré symmetry in the R-order are

$$\begin{aligned}
\mathcal{P}_- =& p_- \ , \quad \mathcal{P}_I = p_I \ , \quad \mathcal{P}_+ = -\frac{L_0}{2p_-} \ , \\
\mathcal{J}^{+I} =& -x^I p_- \ , \quad \mathcal{J}^{+-} = -x^- p_- + \frac{i}{2} \ , \quad \mathcal{J}^{IJ} = \sum_n (X_n^I P_{-n}^J - X_n^J P_{-n}^I) \ , \\
\mathcal{J}^{-I} =& x^- p^I - \frac{i}{2p_-} p^I - \frac{i}{p_-} \sum_{n \neq 0} \frac{1}{n} M_n P_{-n}^I + \frac{1}{2p_-} \sum_n X_n^I L_{-n} \ ,
\end{aligned} \tag{5.1}$$

where L_n and M_n are R-ordered operators.

5.1.2 Lorentz anomaly and constraint

Most commutators in the Poincaré symmetry are easily calculated, but the infamous dangerous commutator $[\mathcal{J}^{-I}, \mathcal{J}^{-J}]$ needs long calculation [36]. From the calculation in appendix E, we obtain

$$[\mathcal{J}^{-I},\ \mathcal{J}^{-J}] = -\frac{1}{p_-^2}\sum_{n\neq 0}\frac{1}{n}\left(P_n^I P_{-n}^J - P_n^J P_{-n}^I\right) M_0. \tag{5.2}$$

This is the result obtained also from the classical calculation, because there is no quantum effect in $[\mathcal{J}^{-I}, \mathcal{J}^{-J}]$ in the R-order as we can see from the consideration in next chapter. The constraint on physical states is $M_0 \approx a$. Therefore there is no anomaly in the Poincaré symmetry if and only if $a = 0$. The dimension D is free.

5.1.3 Mass square operator and constraint

In the R-order, we use the X-representation for the non-zero mode to consider the wave function. From the commutation relation $[X_n^I, P_{J,m}] = i\delta_J^I \delta_{n,-m}$, $P_{I,n}$ in the X-rep. is

$$P_{I,n} = -i\frac{\partial}{\partial X_{-n}^I} \tag{5.3}$$

By using this, the mass square operator and the constraint are written as

$$\mathcal{M}^2 = \sum_{n\neq 0} P_{-n}\cdot P_n = -2\sum_{n>0}\frac{\partial}{\partial X_n}\cdot\frac{\partial}{\partial X_{-n}} \tag{5.4}$$

$$M_0 = -\sum_{n\neq 0} nX_n\cdot\frac{\partial}{\partial X_n} = \sum_{n>0} n\left(-X_n\cdot\frac{\partial}{\partial X_n} + X_{-n}\cdot\frac{\partial}{\partial X_{-n}}\right) \approx 0. \tag{5.5}$$

Below we investigate the eigenfunction of $\mathcal{M}^2$ which satisfy the constraint.

5.1.4 Separable mass eigenfunction

In this subsection we consider the separable mass eigenfunction in the R-order according to [36].

First we consider the eigenfunctions for each part of $n > 0$ in (5.4). The case of $D > 3$ is a little more complicated than the case of $D = 3$. We replace X_n^I and X_{-n}^I with y^I and y_I^* respectively to consider the next fundamental differential equation.

$$\frac{\partial}{\partial y}\cdot\frac{\partial}{\partial y^*}\psi(y, y^*) = -m^2\psi(y, y^*). \tag{5.6}$$

From the irreducible representation of $SO(D-2)$, we consider the solution of (5.6) such that

$$\psi^{I_1\ldots I_q}_{J_1\ldots J_{\bar{q}},m}(y, y^*) = N\left(y^{I_1}\cdots y^{I_q} y^*_{J_1}\cdots y^*_{J_{\bar{q}}}\right)_{tl}\psi_m(r) \tag{5.7}$$

where $r = \sqrt{y \cdot y^*}$, N is the normalization constant and $(\cdots)_{tl}$ means the traceless condition,

$$\delta^{J_{\bar{a}}}_{I_a} \psi^{I_1 \ldots I_q}_{J_1 \ldots J_{\bar{q}}, m} = 0 \quad (a \in [1, \ldots, q], \ \bar{a} \in [1, \ldots, \bar{q}]) \tag{5.8}$$

Now we consider the action of $\frac{\partial}{\partial y} \cdot \frac{\partial}{\partial y^*}$ on $\psi^{I_1 \ldots I_q}_{J_1 \ldots J_{\bar{q}}, m}$. If $\psi_m(r)$ satisfy the next differential equation, $\psi^{I_1 \ldots I_q}_{J_1 \ldots J_{\bar{q}}, m}$ satisfy (5.6).

$$\psi''_m(r) + \frac{2(D-3+h)+1}{r} \psi'_m(r) + 4m^2 \psi_m(r) = 0, \tag{5.9}$$

where $h \equiv q + \bar{q}$. The solution of (5.9) for $m > 0$ is written by the Bessel function as

$$\psi_{m,h}(r) = N_m r^{3-D-h} J_{D-3+h}(2mr), \tag{5.10}$$

where $N_m$is the normalization constant. The solution of (5.9) for $m = 0$ is

$$\psi_{0,h}(r) = \text{constant}. \tag{5.11}$$

Note that the solution excluded in the case of $h = 0$ is not logarithm unlike in three dimensions. Thus we find that $\psi^{I_1 \ldots I_q}_{J_1 \ldots J_{\bar{q}}, m}$ is a solution of (5.6).

The inner product of two eigenfunctions corresponding to non-zero eigenvalue, $\psi^{I_1 \ldots I_q}_{J_1 \ldots J_{\bar{q}}, m}(y, y^*)$ and $\phi^{K_1 \ldots K_q}_{L_1 \ldots L_{\bar{q}}, m'}(y, y^*)$, is

$$\left(\psi^{I_1 \ldots I_q}_{J_1 \ldots J_{\bar{q}}, m}, \phi^{K_1 \ldots K_{q'}}_{L_1 \ldots L_{\bar{q}'}, m)}\right) \propto \delta_{q,q'} \delta_{\bar{q},\bar{q}'} \left[\delta^{(K_1}_{(I_1} \cdots \delta^{K_q)}_{I_q)} \delta^{(J_1}_{(L_1} \cdots \delta^{J_q)}_{L_q)}\right]_{t.l.} \delta(m - m'). \tag{5.12}$$

The derivation of (5.12) and the orthogonality of eigenfunctions such as (5.10) or (5.11) are discussed in the same way as the case of $D = 3$. The detail is in [36].

Next we consider the constraint in (5.5). The fundamental element of M_0 is

$$-y \cdot \frac{\partial}{\partial y} + y^* \cdot \frac{\partial}{\partial y^*}. \tag{5.13}$$

The eigenvalue of (5.13) corresponding to $\psi^{I_1 \ldots I_q}_{J_1 \ldots J_{\bar{q}}, m}$ is $-q + \bar{q} \equiv -s$. Therefore the constraint imposed on states is

$$M_0 = -\sum_{n>0} n s_n = 0. \tag{5.14}$$

The total mass eigenfunction which satisfies the constraint (5.14) is given by the product over $n > 0$ such as

$$\Psi(\{X^I_n\}) = \prod_{n>0} \left[\left(X^{I_1}_n \cdots X^{I_{q_n}}_n X_{J_1,-n} \cdots X_{J_{\bar{q}_n},-n}\right)_{tl} \psi_{m_n,h_n}(2m_n r_n)\right], \tag{5.15}$$

where $\sum_{n=1}^{\infty} n s_n = 0$ and we used

$$r_n = \sqrt{X_n \cdot X_{-n}} \ , \ h_n = q_n + \bar{q}_n \ , \ s_n = q_n - \bar{q}_n. \tag{5.16}$$

The mass eigenvalue of (5.15) is

$$\mathcal{M}^2 = 2\sum_{n>0} m_n^2. \tag{5.17}$$

The inner product of total eigenfunctions is defined by the product of each n's inner product.

Thus we obtain the separable total mass eigenfunction corresponding to (5.17). Then we find that the case of $D = 3$ is the special one.
Below we consider the massless state to find the interesting state which does not exist in three dimensions. The massive state is discussed similarly.

Massless states

The traceless condition (5.8) in $D > 3$ is weaker than the case of $D = 3$. Therefore we find the solution with new combination which can not be obtained in the case of $D = 3$. Here we show examples of massless states with new combination. Although we cannot obtain the separable massless state of degree 2 in $D = 3$, we obtain the following massless states satisfying the traceless condition in $D > 3$.

$$\begin{aligned} &\left(X_n^I X_{-n}^J + X_n^J X_{-n}^I - \frac{2}{D-2}\delta^{I,J}(X_n \cdot X_{-n})\right)|0\rangle_P \\ &(X_n^I X_{-n}^J - X_n^J X_{-n}^I)|0\rangle_P \end{aligned} \tag{5.18}$$

They are states with spin 2 rep. and spin 1 rep. of $SO(D-2)$. Therefore they correspond to 1 graviton and 1 Kalb-Ramond respectively. By considering the multiple action of the operators in (5.18) for distinct n's, we get "multi-particle"[1] massless states. This fact is interesting because the usual tensionful string does not have such a state.
Furthermore we can similarly create massless states with higher spin. The appearance of the massless higher spin state is also interesting.

5.1.5 Non-separable mass eigenfunction

The non-separable mass eigenfunction is expected in $D > 3$, as well as $D = 3$. We consider the operator $\bar{\Lambda}^I$ corresponding to Λ in three dimensions. $\bar{\Lambda}^I$ is defined as

$$\bar{\Lambda}^I \equiv \sum_{n\neq 0}\Big[-\frac{1}{2}X_n^I \bar{L}_{-n} + \frac{i}{n}\bar{M}_n P_{-n}^I\Big], \tag{5.19}$$

where the barred operators are defined by removing zero mode p^J from the unbarred ones. $\bar{\Lambda}^I$ commute with $\mathcal{M}^2$.

$$[\mathcal{M}^2, \bar{\Lambda}^I] = 0 \tag{5.20}$$

Therefore, by acting $\bar{\Lambda}^I$ on a mass eigenfunction, we can obtain another mass eigenfunction. In particular, it is possible to obtain the non-separable mass eigenfunction, as well as the case of $D = 3$. We have to decompose such a eigenfunction into the irreducible representation, though.

[1]It is unclear whether these excitation for distinct n means the identical particle or not.

5.1.6 Characterization of massless states

As well as the case of $D = 3$, we define the operator counting the number of X-action as

$$\Delta \equiv \sum_n X_n \cdot \frac{\partial}{\partial X_n} \tag{5.21}$$

Δ satisfy the following commutation relations:

$$[\Delta, \mathcal{M}^2] = -2\mathcal{M}^2, \tag{5.22}$$

$$[\Delta, X_n] = X_n. \tag{5.23}$$

Therefore Δ is a good operator which characterizes massless states.

5.2 XP-normal Order Case

In this section we consider the anomaly and the mass spectrum in the XP-normal order.

5.2.1 Generators in XP-normal order

From (B.13), the generators of the Poincaré symmetry in the XP-normal order are

$$\begin{aligned}
\mathcal{P}_{-,N} =& p_- \ , \quad \mathcal{P}_{I,N} = p_I \ , \quad \mathcal{P}_{+,N} = p_+ = -\frac{L_0}{2p_-}, \\
\mathcal{J}_N^{+I} =& - x^I p_- \ , \quad \mathcal{J}_N^{+-} = -x^- p_- + \frac{i}{2}, \\
\mathcal{J}_N^{IJ} =& x^I p^J - x^J p^I + \sum_{n=1}^{\infty} \Big[(X_{-n}^I P_n^J - X_{-n}^J P_n^I) + h.c. \Big], \\
\mathcal{J}_N^{-I} =& x^- p^I + \frac{1}{2p_-} x^I L_0 - \frac{i}{2p_-} p^I \\
&+ \frac{1}{2p_-} \sum_{n=1}^{\infty} (X_{-n}^I L_n + L_{-n} X_n^I) - \frac{i}{p_-} \sum_{n=1}^{\infty} \frac{1}{n} (P_{-n}^I M_n - M_{-n} P_n^I)
\end{aligned} \tag{5.24}$$

where L_n and M_n are XP-normal ordered ones, (3.24) and (3.26).

5.2.2 Lorentz anomaly

Most commutators of the Poincaré symmetry are easily calculated, but the infamous dangerous commutator $[\mathcal{J}_N^{-I}, \mathcal{J}_N^{-J}]$ needs long calculation. From the calculation in appendix E, we obtain

$$[\mathcal{J}_N^{-I}, \ \mathcal{J}_N^{-J}] = \frac{1}{p_-^2} \sum_{n=1}^{\infty} \left[\left(\frac{D-2}{6} - 4 \right) n + \left(2M_0 - \frac{D-2}{6} \right) \frac{1}{n} \right] \left(P_{-n}^I P_n^J - P_{-n}^J P_n^I \right). \tag{5.25}$$

From this, the critical dimension is $D = 26$ and the constraint is $M_0 \approx a = 2$. The fact that this critical dimension is the same as one of the usual tensionful bosonic string is interesting.

5.2.3 Mass spectrum

Because of $M_0 \approx 2$, the mass spectrum in the XP-normal order is strongly restricted. From (3.28),

$$M_0 = -i \sum_{m=1}^{\infty} m(P_{-m} \cdot X_m - X_{-m} \cdot P_m) \approx a = 2. \tag{5.26}$$

Therefore, among states consisting of the positive integer powers of operators, states which satisfy the constraint are

$$\begin{aligned} &P^I_{-2}|0\rangle_{XP}\ ,\ P^I_{-1}P^J_{-1}|0\rangle_{XP}\ ,\ (P^I_{-1}X^J_{-1} - P^J_{-1}X^I_{-1})|0\rangle_{XP}\ , \\ &(P^I_{-1}X^J_{-1} + P^J_{-1}X^I_{-1})|0\rangle_{XP}\ ,\ X^I_{-1}X^J_{-1}|0\rangle_{XP}\ ,\ X^I_{-2}|0\rangle_{XP}. \end{aligned} \tag{5.27}$$

From (3.26), the mass square operator in the XP-normal order is

$$\mathcal{M}^2 = 2\sum_{n=1}^{\infty} P_{-n} \cdot P_n, \tag{5.28}$$

and its representation for (5.26) is

$$\mathcal{M}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.29}$$

The states in the first line of (5.27) are massless eigenstates. On the other hand, the states in the second line of (5.27) are not mass eigenstates, but are a part of other mass eigenstates. Therefore they are excluded in the same way as the discussion in the case of $D = 3$.
The first and second states in the first line are the same type as the states obtained in $D = 3$ and their norms is zero. The third state in the first line is new type and its norm is negative. The interpretation of these zero or negative norm states is unclear.

To avoid the zero or negative norm states, we might interpret the dangerous commutator (5.25) as a new additional constraint. In that case, the commutators of new constraint and generators (or other constraints) create additional constraints one after another. In the XP-normal order, all additional constraints are satisfied only by the string ground state. It means that the tensionless string in the XP-normal order is point-like. Also in the following chapters, we will see the fact is natural.

Chapter 6

Operator Orders and Structure of Commutators

Some commutators in the spacetime conformal symmetry can be anomalous. Therefore we take care in the calculation of them.
Some commutators contain terms of higher degree with respect to Fourier modes, X_n and P_n. For example, $[\mathcal{J}^{-I}, \mathcal{K}^{K}]$ contains operators of degree 4 and $[\mathcal{J}^{-I}, \mathcal{K}^{-}]$ contains operators of degree 5. Furthermore, because of divergent terms, we need the regularization in some operator orders. Therefore the calculation of commutators in the spacetime conformal symmetry is very complicated.

Before we consider the spacetime conformal symmetry for the tensionless string theory, we investigate the relation between the choice of operator orders and the structure of commutators.

The effect of quantization in the calculation of commutators arises from terms which are exchanged with other operators several times to be ordered correctly. Such quantum effect's terms, which is differences from classical results, can be anomalous. The fact that the commutator of two Hermitian operators is anti-Hermite (or imaginary unit i times Hermitian operator) restricts the structure of such quantum effect's terms in commutators.

In this chapter we show some examples to consider the structure of quantum effect's terms which arise when a certain hermitian operator is ordered in a given operator order.[1] If we know how quantum effects appear in commutators, we can determine the structure of commutators of generators to some extent to help the concrete calculation of their commutators. Furthermore we give the possible structure of some dangerous commutators in the Hermitian R-order and the XP-normal order, as examples.

6.1 Example 1 : Coordinate and Momentum

First we consider the coordinate x and the momentum p which satisfy the usual commutation relation $[x, p] = i$ as fundamental operators. We compare some hermitian combinations with Weyl ordered ones. $(A)_W$ indicates Weyl ordered version of operator A.

[1] The case of Grassmannian operator is similarly discussed.

Cubic example

Cubic operators of the product of two x and one p case are xxp, xpx and pxx. Hermitian combinations of them are xpx, $\frac{1}{2}(xxp + pxx)$ and $(xxp)_W \equiv \frac{1}{3}(xxp + xpx + pxx)$. The first and second combinations cause no extra term when they are Weyl-ordered.

$$xpx = \frac{1}{2}(xxp + pxx) = (xxp)_W. \tag{6.1}$$

Operators of the product of one x and two p are discussed similarly.

Quartic example

Hermitian combinations of quartic operators which are the product of three x and one p are $\frac{1}{2}(xxxp + pxxx)$, $\frac{1}{2}(xxpx + xpxx)$, $(xxxp)_W \equiv \frac{1}{4}(xxxp + xxpx + xpxx + pxxx)$ and so on. The first and second combinations are Weyl-ordered as

$$\frac{1}{2}(xxxp + pxxx) = \frac{1}{2}(xxpx + xpxx) = (xxxp)_W. \tag{6.2}$$

There is no extra term between two combinations and the Weyl-ordered operator. Operators of the product of one x and three p are discussed similarly.

Hermitian combinations of operators which are the product of two x and two p are $xppx$, $pxxp$, $\frac{1}{2}(xpxp+pxpx)$, $\frac{1}{2}(xxpp+ppxx)$, $(xxpp)_W \equiv \frac{1}{6}(xxpp+xpxp+xppx+pxxp+pxpx + ppxx)$ and so on. These are Weyl-ordered as

$$\begin{aligned} xppx = pxxp &= (xxpp)_W + \frac{1}{2} \\ \frac{1}{2}(xpxp + pxpx) &= (xxpp)_W \\ \frac{1}{2}(xxpp + ppxx) &= (xxpp)_W - \frac{1}{2}. \end{aligned} \tag{6.3}$$

In this case, there are not quadratic but constant quantum effect's terms. In the last line, there is an extra term between the Hermitian R-ordered operator and the Weyl-ordered one. Note that all these quantum effect's terms are in the degree which is 4 degrees lower than the highest degree.

Similarly, in the higher degree case, quantum effect's terms can appear in terms which is 4 degrees lower than the highest degree.

6.2 Example 2 : Creation and Annihilation Operators

Next we consider the creation operator $a^\dagger$ and the annihilation operator a which are Hermitian conjugate with each other and satisfy the usual commutation relation, $[a, a^\dagger] = 1$.
Here we reorder some Hermitian combinations in the normal order.

Cubic example

The example of cubic operator is

$$aa^\dagger a + a^\dagger a a^\dagger = a^\dagger a a + a^\dagger a^\dagger a + a + a^\dagger. \tag{6.4}$$

The quantum effect's terms are linear, which is 2 degrees lower than the highest degree, unlike the case of x and p. The reason is that a and $a^\dagger$ are not self Hermitian conjugate and are complex combinations of x and p, $a \leftrightarrow \frac{1}{\sqrt{2}}(p - ix),\ a^\dagger \leftrightarrow \frac{1}{\sqrt{2}}(p + ix)$.

Quartic example

The examples of quartic operator which is the product of two a and two $a^\dagger$ are

$$\begin{aligned} a^\dagger a a^\dagger a =& a^\dagger a^\dagger a a + a^\dagger a, \\ a a^\dagger a a^\dagger =& a^\dagger a^\dagger a a + 3a^\dagger a + 1, \\ a a a^\dagger a^\dagger =& a^\dagger a^\dagger a a + 4a^\dagger a + 2, \\ \frac{1}{2}(a a^\dagger a^\dagger a + a^\dagger a a a^\dagger) =& a^\dagger a^\dagger a a + 2a^\dagger a. \end{aligned} \tag{6.5}$$

Thus the quantum effect's terms are quadratic terms, which is 2 degrees lower than the highest degree, and constant terms, which is 4 degrees lower than the highest degree.

Similarly, in the higher degree case, quantum effect's terms can appear in the degree which is 2 degrees lower than the highest degree.

6.3 Example 3 : Fourier Modes

In this section we consider Fourier modes of coordinate and momentum for a closed string, X_n and P_n. For simplicity, we omit the index of the spacetime. X_n and P_n satisfy the usual commutation relation and the reality condition, $[X_n, P_m] = i\delta_{n,-m}$ and $(X_n)^\dagger = X_{-n}$ and $(P_n)^\dagger = P_{-n}$.
We consider quartic operators as example. Because the "dangerous" commutators such as $[\mathcal{J}^{-I}, \mathcal{J}^{-J}]$ and $[\mathcal{J}^{-I}, \mathcal{K}^K]$ are quartic,[2] it is important for finding the structure of "anomalous" terms in these dangerous commutators to consider the quartic operator. Here we concretely consider the latter case which can contain a constant quantum effect's term as well as quadratic terms. The former case is simpler.

First we compare some Hermitian combinations with Hermitian R-ordered one, $X_n X_m P_l P_{-n-m-l} + P_{n+m+l} P_{-l} X_{-m} X_{-n}$.
For example,

$$\begin{aligned} & X_n P_l X_m P_{-n-m-l} + P_{n+m+l} X_{-m} P_{-l} X_{-n} \\ & \quad = X_n X_m P_l P_{-n-m-l} + P_{n+m+l} P_{-l} X_{-m} X_{-n} - i\delta_{m,-l}(X_n P_{-n} - P_n X_{-n}), \end{aligned} \tag{6.6}$$

[2]In the highest degree, the former contain quartic terms of the product of one X-mode and three P-modes and the latter contain quartic terms of the product of two X-modes and two P-modes.

$$
\begin{aligned}
&X_nP_lP_{-n-m-l}X_m + X_{-m}P_{n+m+l}P_{-l}X_{-n} \\
&\quad = X_nX_mP_lP_{-n-m-l} + P_{n+m+l}P_{-l}X_{-m}X_{-n} - i(\delta_{n,-l}+\delta_{m,-l})(X_nP_{-n}-P_nX_{-n}).
\end{aligned} \tag{6.7}
$$

Thus quantum effect's terms appear in the second degree, which is 2 degrees lower than the highest. If there are indexed-inverted partners, terms of the second degree make a commutator. Thus we obtain only the constant quantum effect's term, which is 4 degrees lower than the highest.

Next, more generally, we consider the next operator which has the same structure as generators or commutators in the spacetime symmetry for the tensionless string theory.

$$
\sum_n\sum_m\sum_l f_{n,m,l}X_nX_mP_lP_{-n-m-l} \tag{6.8}
$$

where the summation symbol $\sum_n$ indicates $\sum_{n=-\infty}^{\infty}$ and the coefficient has the symmetry which invert the sign of the Fourier mode index, $f_{n,m,l} = f_{-n,-m,-l}$, so called mode flipping symmetry or world sheet parity symmetry.[3]
(6.8) in the Hermitian R-order is

$$
\begin{aligned}
&\frac{1}{2}\sum_n\sum_m\sum_l f_{n,m,l}(X_nX_mP_lP_{-n-m-l} + P_{n+m+l}P_{-l}X_{-m}X_{-n}) \\
&\quad = \frac{1}{2}\sum_n\sum_m\sum_l f_{n,m,l}(X_nX_mP_lP_{-n-m-l} + P_{-n-m-l}P_lX_mX_n).
\end{aligned} \tag{6.9}
$$

We compare other Hermitian combinations with (6.9). If the operator order preserves the mode flipping symmetry, a quantum effect's term in Hermitian operator of degree 4 is only constatnt. We explicitly show this below.

6.3.1 Operator order with mode flipping symmetry

We reorder some Hermitian combinations preserving the mode flipping symmetry in the Hermitian R-order and compare them with (6.9).

v.s. Sandwich order

For example, the Sandwich ordered-(6.8) is reordered in the Hermitian R-order as

$$
\begin{aligned}
&\frac{1}{2}\sum_n\sum_m\sum_l f_{n,m,l}(X_nP_lP_{-n-m-l}X_m + X_{-m}P_{n+m+l}P_{-l}X_{-n}) \\
&\quad = (6.9) - \frac{1}{2}i\sum_n\sum_m(f_{n,m,-n}+f_{n,m,-m})(X_nP_{-n}-P_nX_{-n}) \\
&\quad = (6.9) + \frac{1}{2}\sum_n\sum_m(f_{n,m,-n}+f_{n,m,-m}).
\end{aligned} \tag{6.10}
$$

The quantum effect's term is constant, which in 4 degrees lower than the highest degree.

[3]We do not use the other symmetry for the coefficient such as $f_{n,m,l} = f_{m,n,l}$ and $f_{n,m,l} = f_{n,m,-n-m-l}$.

v.s. Weyl order

The Weyl ordered-(6.8) is reordered in the Hermitian R-order as

$$\begin{aligned}
&\sum_n \sum_m \sum_l f_{n,m,l} (X_n X_m P_l P_{-n-m-l})_W \\
&\quad \equiv \frac{1}{24} \sum_n \sum_m \sum_l f_{n,m,l} (X_n X_m P_l P_{-n-m-l} + (\text{totally symmetric})) \\
&\quad = (6.9) + \frac{1}{4} \sum_n \sum_m (f_{n,m,-n} + f_{n,m,-m})
\end{aligned} \tag{6.11}$$

and the quantum effect's term is constant, which is 4 degrees lower than the highest degree.
Thus we find that quantum differences of a Hermitian operator between some two orders preserving the mode flipping symmetry is 4 degrees lower than the highest degree.

From these two example, we can know the structure of $[\mathcal{J}^{-I}, \mathcal{K}^K]$ in a operator order preserving the mode flipping symmetry. For example, in the Hermitian R-order, we find the next structure by using (B.1) and (B.2).

$$[\mathcal{J}^{-I}, \mathcal{K}^K] = i(\text{ordered classical part}) + i\delta^{I,K} \frac{C}{p_-}, \tag{6.12}$$

where we included the term with M_0 at the right end in the first part and C is constant. We need some regularization if the constant C diverges. The detail of the regularization is considered in appendix D.

6.3.2 Operator order without mode flipping symmetry : XP-normal order

As an example of the order which breaks the mode flipping symmetry, we consider the XP-normal order.[4] We ignore zero mode.[5] In the XP-normal order, all summation symbols are unified into the sum over the half range, $\sum_{n>0}$.

Now we reorder (6.9) into the XP-normal order.
We divide the summation into four region where coefficients equal. The part of $(n, m, l) = (+++), (---)$ with the coefficient $f_{n,m,l} = f_{-n,-m,-l}$ $(n > 0, m > 0, l > 0)$ in (6.9) is written as

$$\begin{aligned}
&\frac{1}{2}(X_n X_m P_l P_{-n-m-l} + h.c.) + \frac{1}{2}(X_{-n} X_{-m} P_{-l} P_{n+m+l} + h.c.) \\
&\quad = P_{-n-m-l} P_l X_m X_n + X_{-n} X_{-m} P_{-l} P_{n+m+l}.
\end{aligned} \tag{6.13}$$

Note that all operators commute with each other. There is no extra term.
The part of $(n, m, l) = (++-), (--+)$ with the coefficient $f_{n,m,-l} = f_{-n,-m,l}$ $(n > 0, m >$

[4] All positive mode of X and P are to the right of negative ones.

[5] We assume that the summation of (6.8) is restricted over $n \neq 0, m \neq 0, l \neq 0, n + m + l \neq 0$, or that the coefficient $f_{n,m,l}$ is restricted.

$0, l > 0$) in (6.9) is written as

$$
\begin{aligned}
&\frac{1}{2}(X_n X_m P_{-l} P_{-n-m+l} + h.c.) + \frac{1}{2}(X_{-n} X_{-m} P_l P_{n+m-l} + h.c.) \\
&= P_{-l} P_{-n-m+l} X_m X_n + X_{-n} X_{-m} P_{n+m-l} P_l \\
&\quad - \frac{i}{2}(\delta_{n,l} + \delta_{m,l})(X_{-n} P_n + X_{-m} P_m - P_{-n} X_n - P_{-m} X_m) - (\delta_{n,l} + \delta_{m,l}).
\end{aligned}
\tag{6.14}
$$

The part of $(n, m, l) = (+-+), (-+-)$ with the coefficient $f_{n,-m,l} = f_{-n,m,-l}$ $(n > 0, m > 0, l > 0)$ in (6.9) is written as

$$
\begin{aligned}
&\frac{1}{2}(X_n X_{-m} P_l P_{-n+m-l} + h.c.) + \frac{1}{2}(X_{-n} X_m P_{-l} P_{n-m+l} + h.c.) \\
&= X_{-m} P_{-n+m-l} P_l X_n + X_{-n} P_{-l} P_{n-m+l} X_m \\
&\quad + \frac{i}{2}\delta_{m,l}(X_{-n} P_n + X_{-m} P_m - P_{-n} X_n - P_{-m} X_m).
\end{aligned}
\tag{6.15}
$$

Finally we exchange n and m in (6.15) to obtain the part of $(n, m, l) = (-++), (+--)$ with the coefficient $f_{-n,m,l} = f_{n,-m,-l}$ $(n > 0, m > 0, l > 0)$ in (6.9).
We unify the above result to find that the quantum effect's terms in XP-normal order are quadratic and constant,

$$
\begin{aligned}
&-\frac{i}{2}\sum_{n>0}\sum_{m>0}(f_{n,m,-n} + f_{n,m,-m} - f_{n,-m,m} - f_{-n,m,n})(X_{-n} P_n + X_{-m} P_m - P_{-n} X_n - P_{-m} X_m) \\
&-\sum_{n>0}\sum_{m>0}(f_{n,m,-n} + f_{n,m-m}).
\end{aligned}
\tag{6.16}
$$

From this, we can know the structure of $[\mathcal{J}^{-I}, \mathcal{K}^K]$ in the XP-normal order. By using (B.13), we find the possible structure in the XP-normal order as

$$
\begin{aligned}
[\mathcal{J}_N^{-I}, \mathcal{K}_N^K] =& i(\text{ordered classical part}) + \delta^{I,K}\frac{1}{p_-}((X \cdot P)\text{-quadratic}) \\
&+ \frac{1}{p_-}(X^I P^K\text{-quadratic}) + \frac{1}{p_-}(X^K P^I\text{-quadratic}) + i\delta^{I,K}\frac{\tilde{C}}{p_-},
\end{aligned}
\tag{6.17}
$$

where we included the term with M_0 at the right end in the first part and $\tilde{C}$ is constant and the last part arises from the operator ordering of zero-mode terms. If the third or 4th part exists in the case of $I \neq K$, $[\mathcal{J}^{-I}, \mathcal{K}^K]$ has off-diagonal part and then becomes anomalous.

6.3.3 Comment on other dangerous commutators

Other important dangerous commutators are $[\mathcal{J}^{-I}, \mathcal{J}^{-J}]$ in $D > 3$ and $[\mathcal{J}^-, \mathcal{K}^-]$ in $D = 3$. $XPPP$-quartic operators appears in the halfway of the calculation of $[\mathcal{J}^{-I}, \mathcal{J}^{-J}]$ and $XXPPP$-quintic operators appears in the halfway of the calculation of $[\mathcal{J}^-, \mathcal{K}^-]$.

From the consideration in the last two subsection, the structure of $[\mathcal{J}^{-I}, \mathcal{J}^{-J}]$ in $D > 3$ and in the R-order is

$$
[\mathcal{J}^{-I}, \mathcal{J}^{-J}] = i(\text{ordered classical part}) \tag{6.18}
$$

where we included the term with M_0 at the right end in the first part. This is the same as (5.2) calculated explicitly.
Similarly, the structure of $[\mathcal{J}^{-I}, \mathcal{J}^{-J}]$ in $D > 3$ and in the XP-normal order is

$$[\mathcal{J}_N^{-I}, \mathcal{J}_N^{-J}] = i(\text{ordered classical part}) + \frac{1}{p_-^2}((P^I P^J - P^J P^I)\text{-quadratic}) \tag{6.19}$$

where we included the term with M_0 at the right end in the first part. This is the same as (5.25) calculated explicitly.

Furthermore, the structure of $[\mathcal{J}^-, \mathcal{K}^-]$ in $D = 3$ and in the Hermitian R-order is

$$[\mathcal{J}^-, \mathcal{K}^-] = i(\text{ordered classical part}) + iC' \frac{p}{p_-^2} \tag{6.20}$$

where we included the term with M_0 at the right end in the first part and C' is constant. From (6.20), we find that there is no anomalous term of degree 3. This is calculated concretely in the next chapter. Because of the algebraic requirement, in the calculation in the next chapter we must use redefined $\tilde{\mathcal{K}}^-$ instead of $\mathcal{K}^-$.
Similarly, the structure of $[\mathcal{J}^-, \mathcal{K}^-]$ in $D = 3$ and in the XP-normal order is

$$[\mathcal{J}_N^-, \mathcal{K}_N^-] = i(\text{ordered classical part}) + \frac{1}{p_-^2}(XPP\text{-cubic}) + i\tilde{C}' \frac{p}{p_-^2} \tag{6.21}$$

where we included the term with M_0 at the right end in the first part and $\tilde{C}'$ is constant. If exists, the second part in the r.h.s. of (6.21) becomes an anomalous part of degree 3. This is calculated directly in the next chapter. Because of the algebraic requirement, in the next chapter we must use redefined $\tilde{\mathcal{K}}^-$ instead of $\mathcal{K}^-$.

Thus we obtain the structure of dangerous commutators, such as (6.12) and (6.17)-(6.21). However we would not find coefficients of the quantum effect's terms in dangerous commutators unless we calculate concretely. Therefore, in the following chapter, we calculate dangerous commutators in the spacetime conformal symmetry concretely with the help of the information about the structure of commutators obtained in this chapter.

Chapter 7

Quantum 3D Tensionless String in Light-cone Gauge with Spacetime Conformal Symmetry

In chapter 4 and 5, we have investigated the condition to avoid the Lorentz anomaly in the tensionless string theory and the mass spectrum under such conditions in various dimensions and operator orders. As seen in chapter 4, the ordering constant, a, is not determined in three dimensions, unlike in $D > 3$. Thanks to the freedom of a, we obtain the great possibility for the mass spectrum of the tensionless string theory. However, from another point of view, we can say that the theory has an ambiguity. Removing this ambiguity is one of the interesting problem.

The classical tensionless string theory has the spacetime conformal symmetry. Therefore, it is natural to expect that the quantum tensionless string theory also has it. However the quantum theory can cause the anomaly. Hence we must verify whether the anomaly exists in the quantum theory and find what the condition to avoid the anomaly is.

In [38, 40], the existence of the anomaly in the traceless part of $[\mathcal{J}^{-I}, \mathcal{K}^K]$ is shown. In three dimensions,[1] because the number of the transverse direction is only one, the traceless part does not exist and then there is no anomaly in $[\mathcal{J}^{-I}, \mathcal{K}^K]$. However, the difference in the trace part exists. Therefore the trace part must be absorbed in a new definition for $\mathcal{K}^-$, namely $\tilde{\mathcal{K}}^-$. Furthermore we must verify commutation relations of $\tilde{\mathcal{K}}^-$ and other generators. According to appendix A, the commutators which we must calculate in three dimensional conformal group are the definition, $\tilde{\mathcal{K}}^- \equiv -i[\mathcal{J}^-, \mathcal{K}]$, and the dangerous commutator, $[\mathcal{J}^-, \hat{\mathcal{K}}^-]$.

In this chapter we calculate the dangerous commutators in the spacetime conformal symmetry for the quantum 3-dim. tensionless string theory. First we calculate them in the Hermitian R-order, and next in the XP-normal order. In the case of the Hermitian R-order, particularly, we can avoid the spacetime conformal anomaly under the condition of $M_0 = a = 0$.

[1] In two dimensions and the light-cone gauge, because of $\bar{X}^- = P_+ = 0$, there is no transverse variable which should be quantized. The dynamical variables are x^- and p_-. By using these two operators, for example, we obtain the finite closed algebra consisting of $l_{-1} \propto p_-$, $l_0 \propto x^- p_- = -\mathcal{J}^{+-}$ and $l_{+1} \propto x^- x^- p_- = \mathcal{K}^-$. Then, by using $l_n \propto (x^-)^{n+1} p_-$ for integer n, we obtain the infinite algebra, Virasoro algebra. Furthermore, we add $w_{n,m} \propto (x^-)^n (p_-)^m$ to get the higher algebra. The detail is not discussed in this thesis.

7.1 Hermitian R-order Case

In the Poincaré symmetry, there is no difference between the cases of the pure R-order, the Hermitian R-order and the Weyl order. On the other hand, in the spacetime conformal symmetry, because commutators of higher degree exist, there are differences in the quantum effect, which appears in terms of lower degree. Furthermore, we need some regularization because of inevitable divergent terms.
As seen in appendix D, the R-order breaks the hermitian property in the cut-off regularization[2]. Therefore we consider the case of the Hermitian R-order below.[3]

7.1.1 Generators of dilatation and special conformal transformation

All generators in the Hermitian R-order is ordered as follows.

$$\mathcal{G} \equiv \frac{1}{2}\left(\mathcal{G}_R + (\mathcal{G}_R)^\dagger\right), \tag{7.1}$$

where $\mathcal{G}_R$ is the R-ordered generator. We use $\mathcal{G}$ in the calculation of commutators, instead of $\mathcal{G}_R$.
Here we give $\mathcal{G}_R$ for the rest of generators in the spacetime conformal symmetry. From (B.2), $\mathcal{G}_R$s for the dilatation and the special conformal transformation are

$$\begin{aligned}
\mathcal{D}_R =& x^- p_- + \sum_n X_n P_{-n} \ , \\
\mathcal{K}^+_R =& -\frac{1}{2}\sum_n X_n X_{-n} p_- \ , \\
\mathcal{K}_R =& x x^- p_- + \frac{1}{2}\sum_n\sum_m X_n X_m P_{-n-m} + i\sum_{n\neq 0}\frac{1}{n}X_n M_{-n} \ , \\
\mathcal{K}^-_R =& x^- x^- p_- + x^-\sum_n X_n P_{-n} + \frac{1}{4p_-}\sum_n\sum_m X_n X_m L_{-n-m} \\
& -\frac{i}{p_-}\sum_n\sum_{m\neq 0}\left(\frac{n}{m^2}+\frac{1}{m}\right)X_n M_m P_{-n-m},
\end{aligned} \tag{7.2}$$

where $\frac{1}{p_-}\sum_{m\neq 0}\frac{1}{m^2}M_m M_{-m}$ in the classical $\mathcal{K}^-$ has been ordered correctly.

We can easily check that the commutation relations of generators except for $\mathcal{K}^-$ is correct. However, $\mathcal{K}^-$ and some commutators with it must are modified.

7.1.2 Definition of $\tilde{\mathcal{K}}^-$

First we define $\tilde{\mathcal{K}}^-$ by

$$\tilde{\mathcal{K}}^- \equiv -i[\mathcal{J}^-, \mathcal{K}]. \tag{7.3}$$

[2]This is seen as some kind of anomaly.
[3]The Weyl order case is similarly discussed.

Because of the algebraic requirement, we must use this instead of $\mathcal{K}^-$ in commutators of the spacetime conformal symmetry.

According to the discussion in chapter 6, the structure of $[\mathcal{J}^-, \mathcal{K}]$ is (6.12).
Firstly, we consider the structure of the r.h.s. in (7.3) again in detail. From the number of spacetime index, we find that the r.h.s. in (7.3) consists of the part with x^- and the part proportional to $\frac{1}{p_-}$. The part with x^- is easily calculated. Then we find that it is the same as the part with x^- in $\mathcal{K}^-$.
On the other hand, the part proportional to $\frac{1}{p_-}$ consists of $XXPP$-quartic, XP-quadratic and a constant. The last two parts are the quantum effect's terms, which arise in exchanging operators to get the correctly ordered result. Because $XXPP$-quartic terms are in the highest degree, we can know them by the classical calculation. After a long calculation, we obtain the same result as $XXPP$-quartic terms in $\mathcal{K}^-$ and terms with M_0 at the right end. As seen in chapter 6, in the Hermitian R-order, XP-quadratic terms make commutators and then become constant.

Thus we obtain the next:[4]

$$\tilde{\mathcal{K}}^- = \mathcal{K}^- + \frac{C}{p_-} + \frac{2i}{p_-} \sum_{n \neq 0} \frac{1}{n} X_n P_{-n} M_0. \tag{7.4}$$

The first term is the original generator defined as (7.1). The second term is the gap by quantum effect and C is generally divergent constant. The last term is related to the choice of the ordering constant, through the constraint, $M_0 \approx a$.
Because there is the contribution from $[X_n, P_{-n}] = i$ for non-zero n to C, C is a divergent constant even in the Hermitian R-order. Therefore we need some regularization.

Now we want to know whether the anomaly unrelated to the choice of the regularization exists in the spacetime conformal symmetry. Therefore we assume the existence of a good regularization and then use $\tilde{\mathcal{K}}^-$, as it is in (7.4). The calculation with the concrete regularization is given in appendix E.

7.1.3 Commutation relations with $\tilde{\mathcal{K}}^-$

If all commutators with $\tilde{\mathcal{K}}^-$ is desirable, we can maintain that the 3-dim. tensionless string theory has the spacetime conformal symmetry. From the relatively easy calculation, we can verify the following commutation relations.

$$\begin{aligned}
&[\mathcal{P}, \tilde{\mathcal{K}}^-] = -i(\mathcal{D} + \mathcal{J}) \ , \ [\mathcal{P}, \tilde{\mathcal{K}}^-] = i\mathcal{J}^- \ , \ [\mathcal{P}_+, \tilde{\mathcal{K}}^-] = 0 \\
&[\mathcal{J}^+, \tilde{\mathcal{K}}^-] = i\mathcal{K} \ , \ [\mathcal{J}, \tilde{\mathcal{K}}^-] = -i\tilde{\mathcal{K}}^- \\
&[\mathcal{D}, \tilde{\mathcal{K}}^-] = -i\tilde{\mathcal{K}}^- \ , \ [\mathcal{K}^+, \tilde{\mathcal{K}}^-] = 0
\end{aligned} \tag{7.5}$$

The rest of commutation relations, $[\mathcal{J}^-, \tilde{\mathcal{K}}^-] = 0$ and $[\mathcal{K}, \tilde{\mathcal{K}}^-] = 0$, must be calculated in detail because they are dangerous commutators. By using the Jacobi identity[5], the

[4] The last term in (7.4) is obtained from $[\frac{i}{p_-} \sum_{l \neq 0} \frac{1}{l} M_l P_{-l}, \ i \sum_{m \neq 0} \frac{1}{m} X_m M_{-m}]$ in $[\mathcal{J}^-, \mathcal{K}]$.

[5] In the calculation with the regularization, we will find the extra terms which vanish in the limit of a regulator. Here we have to check that commutators of such extra terms vanish in the limit of a regulator. Therefore we must take care of using the Jacobi identity in the calculation with the regularization. In appendix E, we consider commutators of such extra terms.

second dangerous commutator is calculated as follows:

$$\begin{aligned}\big[\mathcal{K},\tilde{\mathcal{K}}^-\big] &= \big[i[\mathcal{J}^-,\mathcal{K}^+],\tilde{\mathcal{K}}^-\big] \\ &= i\big[\mathcal{J}^-,[\mathcal{K}^+,\tilde{\mathcal{K}}^-]\big] + i\big[[\mathcal{J}^-,\tilde{\mathcal{K}}^-],\mathcal{K}^+\big].\end{aligned} \tag{7.6}$$

The first term in the second line vanishes because of $[\mathcal{K}^+,\tilde{\mathcal{K}}^-]=0$ in (7.5). The vanishing of the second term depends on $[\mathcal{J}^-,\tilde{\mathcal{K}}^-]$. Therefore all we have to calculate is the first dangerous commutator, $[\mathcal{J}^-,\tilde{\mathcal{K}}^-]$.

7.1.4 Dangerous commutator : $[\mathcal{J}^-,\tilde{\mathcal{K}}^-]$

Here we calculate the dangerous commutator, $[\mathcal{J}^-,\tilde{\mathcal{K}}^-]$, which may be anomalous. Firstly we see the structure of this commutator. Then we calculate explicitly. The calculation with the cut-off regularization is given in appendix E.

Structure of $[\mathcal{J}^-,\tilde{\mathcal{K}}^-]$

From the number of the spacetime index, $[\mathcal{J}^-,\tilde{\mathcal{K}}^-]$ consists of the part proportional to x^-x^-, the part proportional to $\big(x^-\frac{1}{p_-}+\frac{1}{p_-}x^-\big)$ [6] and the part proportional to $\frac{1}{p_-^2}$.

Furthermore, from the mass dimension, the first part can contain only x^-x^-p. It is easily calculated and then we find its cancellation.
The second part can consist of XPP-cubic terms and p-linear terms. XPP-cubic terms in the second part are ordered classical results. On the other hand, p-linear terms are forbidden because they are hermitian.
The third part consists of $XXPPP$-quintic terms, XPP-cubic terms and p-linear terms. $XXPPP$-quintic terms in the third part are ordered classical results. As seen in chapter 6, XPP-cubic terms in the third part make a commutator $[X_n,P_m]=i\delta_{n,-m}$ in the Hermitian R-order to be contained in p-linear term.

Thus we find the structure of $[\mathcal{J}^-,\tilde{\mathcal{K}}^-]$ in the Hermitian R-order.

$$\begin{aligned}[\mathcal{J}^-,\ \tilde{\mathcal{K}}^-] &= ix^-x^-p\times(\text{const.}) + i\Big(x^-\frac{1}{p_-}+\frac{1}{p_-}x^-\Big)(\text{cubic term}: XPP+h.c.) \\ &\qquad + i\frac{1}{p_-^2}(\text{quintic term}: XXPPP+h.c.) + iC'\frac{p}{p_-^2} \\ &= i(\text{ordered classical part}) + iC'\frac{p}{p_-^2},\end{aligned} \tag{7.7}$$

where C' is a (divergent) constant and we contained terms with M_0 at the right end in the ordered classical part. The second term in the last line is a quantum effect's term.

Because there is no term with the square of M_0, the constraint, $M_0\approx a$, does not mix the first term and the second term in the last line of (7.7). Therefore they must be zero separately.

[6] Although we may consider $x^-\frac{1}{p_-}$ and $\frac{1}{p_-}x^-$ separately, the hermitian combination is only $\big(x^-\frac{1}{p_-}+\frac{1}{p_-}x^-\big)$. Their differences from the hermitian combination are $\mp\frac{i}{2p_-^2}$. Then they are included in the part proportional to $\frac{1}{p_-^2}$.

Calculation : Classical part of $[\mathcal{J}^-, \tilde{\mathcal{K}}^-]$

First we calculate the classical part of $[\mathcal{J}^-, \tilde{\mathcal{K}}^-]$ in (7.7). Among $\tilde{\mathcal{K}}^-$ defined in (7.4), terms which contribute to the classical part are $\mathcal{K}^-$ and the third term with M_0.

Because we want the classical information, we can use R-ordered generators.

$$\begin{aligned}\Big[\mathcal{J}^-,\ \mathcal{K}^- + \frac{2i}{p_-}\sum_{n\neq 0}\frac{1}{n}X_nP_{-n}M_0\Big] \simeq \Big[\mathcal{J}_R^-,\ \mathcal{K}_R^-\Big] + \Big[\mathcal{J}_R^-,\ \frac{2i}{p_-}\sum_{n\neq 0}\frac{1}{n}X_nP_{-n}M_0\Big] \\ \Rightarrow i(\text{ordered classical part}),\end{aligned} \tag{7.8}$$

where $\simeq$ means the extraction of the classical part.

Firstly, we consider the first commutator in (7.8). Because the calculation is long, we separate $\mathcal{J}_R^-$ into three parts as below, and then calculate the contribution from each part.

$$\mathcal{J}_R^- = -x^-p - \frac{1}{2p_-}\sum_l X_lL_{-l} + \frac{i}{p_-}\sum_{l\neq 0}\frac{1}{l}M_lP_{-l} \tag{7.9}$$

The commutator of the first part in (7.9) and $\mathcal{K}_R^-$ is classically

$$\begin{aligned}\big[-x^-p,\ \mathcal{K}_R^-\big] \simeq{}& i\Big(\frac{1}{4}\sum_n\sum_m X_nX_mL_{-n-m} - i\sum_n\sum_{m\neq 0}\frac{n+m}{m^2}X_nM_mP_{-n-m}\Big)\frac{p}{p_-^2} \\ &+\frac{i}{2}x^-\frac{1}{p_-}\sum_n X_nL_{-n} + x^-\frac{1}{p_-}\sum_{m\neq 0}\frac{1}{m}M_mP_{-m}.\end{aligned} \tag{7.10}$$

The commutator of the second part in (7.9) and $\mathcal{K}_R^-$ is classically

$$\begin{aligned}\Big[-\frac{1}{2p_-}\sum_l X_lL_{-l},\ \mathcal{K}_R^-\Big] \simeq{}& -\frac{i}{2}x^-\frac{1}{p_-}\sum_n X_nL_{-n} - \frac{i}{4p_-^2}\sum_n\sum_m\sum_l X_nX_mP_lL_{-n-m-l} \\ &- \frac{1}{2p_-^2}\sum_n\sum_{m\neq 0}\frac{1}{m}X_nM_mL_{-n-m}.\end{aligned} \tag{7.11}$$

The commutator of the third part in (7.9) and $\mathcal{K}_R^-$ is classically

$$\begin{aligned}&\Big[\frac{i}{p_-}\sum_{l\neq 0}\frac{1}{l}M_lP_{-l},\ \mathcal{K}_R^-\Big] \\ &\simeq -x^-\frac{1}{p_-}\sum_{m\neq 0}\frac{1}{m}M_mP_{-m} + \frac{i}{4p_-^2}\sum_n\sum_m\sum_l X_nX_mP_lL_{-n-m-l} \\ &\quad - i\Big(\frac{1}{4}\sum_n\sum_m X_nX_mL_{-n-m} - i\sum_n\sum_{m\neq 0}\frac{n+m}{m^2}X_nM_mP_{-n-m}\Big)\frac{p}{p_-^2} \\ &\quad + \frac{1}{2p_-^2}\sum_n\sum_{m\neq 0}\frac{1}{m}X_nM_mL_{-n-m} + \frac{2}{p_-^2}\sum_n\sum_{m\neq 0}\frac{n}{m^2}X_nM_mP_{-m}P_{-n}.\end{aligned} \tag{7.12}$$

We collect (7.10)-(7.12) to obtain

$$[\mathcal{J}_R^-,\ \hat{\mathcal{K}}_R^-] \simeq \frac{2}{p_-^2}\sum_n\sum_{m\neq 0}\frac{n}{m^2}X_nM_mP_{-m}P_{-n} \simeq \frac{2i}{p_-^2}\sum_{m\neq 0}\frac{1}{m^2}M_mP_{-m}M_0, \tag{7.13}$$

where we used $M_0 = -i\sum_n nX_nP_{-n}$ and then moved it to the right end. In the calculation of the quantum part, we must not forget the contribution which arises in moving M_0 to the right end.

Next we consider the commutator of the second term in (7.8). By using the fact that M_0 commutes with all generators, we obtain

$$\begin{aligned}&\Big[\mathcal{J}^-,\ \frac{2i}{p_-}\sum_{n\neq 0}\frac{1}{n}X_nP_{-n}M_0\Big]\\&=\Big(-\frac{6}{p_-^2}\sum_{n\neq 0}\frac{1}{n}X_nP_{-n}p+\frac{1}{p_-^2}\sum_{n\neq 0}\frac{1}{n}X_nL_{-n}+\frac{2i}{p_-^2}\sum_{n\neq 0}\frac{1}{n^2}M_nP_{-n}\Big)\cdot M_0.\end{aligned}\tag{7.14}$$

This contains M_0 at the right end again.

We collect (7.13) and (7.14) to find that the classical part in (7.7) consists only of terms with $M_0 \approx a$ at the right end. Therefore we determine the ordering constant such that $a = 0$.[7]

Calculation : Quantum effect part of $[\mathcal{J}^-, \tilde{\mathcal{K}}^-]$

Next we calculate the quantum effect part of $[\mathcal{J}^-, \tilde{\mathcal{K}}^-]$, that is the second part in the last line of (7.7). This part contains p-zero mode. So we can know the cancellation of this part from the commutator of x and $[\mathcal{J}^-, \tilde{\mathcal{K}}^-]$.
By using the Jacobi identity, we deform it as follows.

$$\begin{aligned}\big[x,\ [\mathcal{J}^-,\ \tilde{\mathcal{K}}^-]\big] &= \big[[x,\mathcal{J}^-],\ \tilde{\mathcal{K}}^-\big]+\big[\mathcal{J}^-,\ [x,\tilde{\mathcal{K}}^-]\big]\\&=\frac{1}{2}\Big(\big[[x,\mathcal{J}^-],\ \mathcal{K}_R^-\big]+\big[\mathcal{J}^-,\ [x,\mathcal{K}_R^-]\big]+h.c.\Big)\\&\quad+\Big[[x,\mathcal{J}^-],\ \frac{C}{p_-}+\frac{2i}{p_-}\sum_{n\neq 0}\frac{1}{n}X_nP_{-n}M_0\Big].\end{aligned}\tag{7.15}$$

Below, we calculate each part.

The first commutator in the second line of (7.15) is calculated as follows.

$$\begin{aligned}\big[[x,\mathcal{J}^-],\ \mathcal{K}_R^-\big] =& x^-x^- + 2x^-x\frac{p}{p_-}+\frac{i}{2}x\frac{p}{p_-^2}-\frac{i}{2}x^-\frac{1}{p_-}+\frac{1}{p_-^2}\\&+x^-\mathcal{D}_R\frac{1}{p_-}+2x\mathcal{D}_R\frac{p}{p_-^2}-\frac{i}{2}\mathcal{D}_R\frac{1}{p_-^2}+x\mathcal{J}^-\frac{1}{p_-}-\mathcal{K}_R\frac{p}{p_-^2}-\mathcal{K}_R^-\frac{1}{p_-},\end{aligned}\tag{7.16}$$

where we used

$$\begin{aligned}\mathcal{J}^- &= \mathcal{J}_R^- + \frac{i}{2p_-}p,\\ [x,\mathcal{J}^-] &= (-i)\Big(x^- + x\frac{p}{p_-}-\frac{i}{2p_-}\Big).\end{aligned}\tag{7.17}$$

[7]Note that the classical part and the quantum effect part must vanish separately.

The second commutator in the second line of (7.15) is calculated as follows.

$$\begin{aligned}\big[\mathcal{J}^-,[x,\mathcal{K}_R^-]\big] &= -x^-x^- - 2x^-x\frac{p}{p_-} - ix\frac{p}{p_-^2} + \frac{i}{2}x^-\frac{1}{p_-} - \frac{1}{p_-^2} \\ &\quad - x^-\mathcal{D}_R\frac{1}{p_-} - 2x\mathcal{D}_R\frac{p}{p_-^2} + \frac{i}{2}\mathcal{D}_R\frac{1}{p_-^2} - x\mathcal{J}^-\frac{1}{p_-} + \mathcal{K}_R\frac{p}{p_-^2} - i\big[\mathcal{J}^-,\mathcal{K}_R\big]\frac{1}{p_-},\end{aligned} \tag{7.18}$$

where we used the following relations:

$$[x,\mathcal{K}_R^-] = i\Big(x^-x + x\mathcal{D}_R\frac{1}{p_-} - \mathcal{K}_R\frac{1}{p_-}\Big). \tag{7.19}$$

We collect (7.16) and (7.18) to obtain

$$\big[[x,\mathcal{J}^-],\ \mathcal{K}_R^-\big] + \big[\mathcal{J}^-,[x,\mathcal{K}_R^-]\big] = -\frac{i}{2}x\frac{p}{p_-^2} - \mathcal{K}_R^-\frac{1}{p_-} - i\big[\mathcal{J}^-,\mathcal{K}_R\big]\frac{1}{p_-}. \tag{7.20}$$

Then we find that the second line of (7.15) is as follows:

$$\begin{aligned}&\frac{1}{2}\left(\big[[x,\mathcal{J}^-],\ \mathcal{K}_R^-\big] + \big[\mathcal{J}^-,[x,\mathcal{K}_R^-]\big] + h.c.\right) \\ &= \frac{1}{4p_-^2} + (\tilde{\mathcal{K}}^- - \mathcal{K}^-)\frac{1}{p_-} - \frac{1}{2}\Big[\frac{1}{p_-},(\mathcal{K}_R^-)^\dagger\Big] + \frac{i}{2}\Big[\frac{1}{p_-},[\mathcal{J}^-,\mathcal{K}_R^-]^\dagger\Big].\end{aligned} \tag{7.21}$$

The third term in the r.h.s. of (7.21) is calculated as

$$-\frac{1}{2}\Big[\frac{1}{p_-},(\mathcal{K}_R^-)^\dagger\Big] = -\frac{1}{2}\Big[\frac{1}{p_-},(\mathcal{D}_R)^\dagger x^-\Big] = -\frac{i}{2}\Big(\frac{1}{p_-}x^- + (\mathcal{D}_R)^\dagger\frac{1}{p_-}\Big) \tag{7.22}$$

and the 4th term in the r.h.s. of (7.21) is calculated as

$$\begin{aligned}\frac{i}{2}\Big[\frac{1}{p_-},[\mathcal{J}^-,\mathcal{K}_R^-]^\dagger\Big] &= -\frac{i}{2}\Big[\frac{1}{p_-},[\mathcal{J}^-,\mathcal{K}_R^-]\Big]^\dagger \\ &= \frac{i}{2}\Big(\Big[\mathcal{K}_R^-,\Big[\frac{1}{p_-},\mathcal{J}^-\Big]\Big] + \Big[\mathcal{J}^-,\Big[\mathcal{K}_R^-,\frac{1}{p_-}\Big]\Big]\Big)^\dagger \\ &= \frac{i}{2}\Big(\Big[\mathcal{K}_R^-,-i\frac{p}{p_-^2}\Big] + \Big[\mathcal{J}^-,-i\frac{x}{p_-}\Big]\Big)^\dagger \\ &= \frac{i}{2}\Big(\frac{1}{p_-}x^- + (\mathcal{D}_R)^\dagger\frac{1}{p_-}\Big) - \frac{1}{4p_-^2}.\end{aligned} \tag{7.23}$$

Thus we get

$$\begin{aligned}&\frac{1}{2}\left(\big[[x,\mathcal{J}^-],\ \mathcal{K}_R^-\big] + \big[\mathcal{J}^-,[x,\mathcal{K}_R^-]\big] + h.c.\right) \\ &= (\tilde{\mathcal{K}}^- - \mathcal{K}^-)\frac{1}{p_-} = \frac{C}{p_-^2} + \frac{2i}{p_-^2}\sum_{n\neq 0}\frac{1}{n}X_nP_{-n}M_0.\end{aligned} \tag{7.24}$$

The last line in (7.15) is calculated as

$$\left[[x,\mathcal{J}^-],\ \frac{C}{p_-}+\frac{2i}{p_-}\sum_{n\neq 0}\frac{1}{n}X_nP_{-n}M_0\right]=-\frac{C}{p_-^2}-\frac{2i}{p_-^2}\sum_{n\neq 0}\frac{1}{n}X_nP_{-n}M_0. \tag{7.25}$$

We collect (7.24) and (7.25) to get

$$\left[x,\ [\mathcal{J}^-,\hat{\mathcal{K}}^-]\right]=0. \tag{7.26}$$

Thus we find that p-zero mode in $[\mathcal{J}^-,\hat{\mathcal{K}}^-]$, including terms with p of higher degree, is canceled. Therefore there is no anomalous term in $[\mathcal{J}^-,\ \hat{\mathcal{K}}^-]$.

In this section, in order to know whether the anomaly irrelevant to an explicit regularization exists, we have calculated the dangerous commutator without the regularization. In appendix E, we will calculate the dangerous commutator explicitly with the cut-off regularization and without using the Jacobi identity. There we will calculate regularized-C in (7.4) concretely and then verify the cancellation of C' in (7.7).

7.2 XP-normal Order Case

Here we investigate the anomaly in the spacetime conformal symmetry for the tensionless string in the XP-normal order. The commutators we should calculate are $[\mathcal{J}^-,\mathcal{K}]$ and $[\mathcal{J}^-,\ \hat{\mathcal{K}}^-]$, as well as in the case of the Hermitian R-order. The first commutator defines $\hat{\mathcal{K}}^-$ and the second one can be anomalous.
Because divergent terms do not appear in the XP-normal order, we do not need any regularization.

7.2.1 Generators of dilatation and special conformal transformation

In the normal order, the non-zero mode part of generators is automatically hermitian but the zero mode part is not hermitian. Therefore we *hermitianize* the zero mode part.

The important Lorentz generator, $\mathcal{J}_N^-$, in the XP-normal order is written as

$$\mathcal{J}_N^-=-x^-p-\frac{1}{2p_-}xL_0+\frac{i}{2p_-}p+\frac{\Lambda}{p_-}, \tag{7.27}$$

where $L_0=pp+\mathcal{M}^2=pp+2\sum_{n>0}P_{-n}P_n$ and Λ is defined as

$$\Lambda\equiv\sum_{l=1}^{\infty}\Big[-\frac{1}{2}(X_{-l}L_l+L_{-l}X_l)+\frac{i}{l}(P_{-l}M_l-M_{-l}P_l)\Big]. \tag{7.28}$$

Here we note that Λ is hermitian and p zero mode in Λ is canceled.
The important generators among the dilatation and the special conformal transformation are

$$
\begin{aligned}
\mathcal{K}_N =& xx^- p_- + \frac{1}{2}xxp - ix + x\sum_{n=1}^{\infty}(X_{-n}P_n + P_{-n}X_n) + \sum_{n=1}^{\infty} X_{-n}X_n p \\
&+ \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[(X_{-n}X_{-m}P_{n+m} + 2X_{-n}P_{-m}X_{n+m}) + h.c.\Big] \\
&- i\sum_{m=1}^{\infty}\frac{1}{m}(X_{-m}M_m - M_{-m}X_m),
\end{aligned} \tag{7.29}
$$

and

$$
\begin{aligned}
\mathcal{K}_N^- =& x^-\mathcal{D}_N - \frac{i}{2}x^- + \frac{1}{8p_-}(xxL_0 + L_0 xx) - \frac{1}{p_-}x\Lambda \\
&+ \frac{1}{2p_-}\sum_{n=1}^{\infty} X_{-n}L_0X_n + \frac{1}{4p_-}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[(X_{-n}X_{-m}L_{n+m} + 2X_{-n}L_{-m}X_{n+m}) + h.c.\Big] \\
&+ \frac{i}{p_-}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[\Big(\frac{n+m}{m^2}X_{-n}M_{-m}P_{n+m} + \frac{n}{m^2}X_{-n-m}M_mP_n - \frac{m}{(n+m)^2}X_{-n}P_{-m}M_{n+m}\Big) + h.c.\Big].
\end{aligned} \tag{7.30}
$$

For the algebraic requirement, we must use $\tilde{\mathcal{K}}_N^-$ defined below, instead of $\mathcal{K}_N^-$.

$$
\tilde{\mathcal{K}}_N^- \equiv -i[\mathcal{J}_N^-, \mathcal{K}_N] = \mathcal{K}_N^- + \frac{C}{p_-} + \delta\mathcal{K}_N^-. \tag{7.31}
$$

The first term in the r.h.s. of (7.31) is the original XP-normal ordered generator in (7.30). The second term is the quantum effect term arising only from zero modes. The third term consists of the quantum effect term of degree 2 and terms with M_0 at the right end, such as

$$
\delta\mathcal{K}_N^- = \frac{i}{p_-}\sum_{n=1}^{\infty}(X_{-n}P_n - P_{-n}X_n)(g_{(n)} + h_{(n)}M_0), \tag{7.32}
$$

where $g_{(n)}$ and $h_{(n)}$ is coefficients dependent on Fourier-mode index, n. In three dimensions, we can absorb all quantum effect terms in the definition of $\delta\mathcal{K}^-$. However, as seen in the next chapter, terms of the second degree like (7.32) become anomalous in $D > 3$.

Now we calculate $\tilde{\mathcal{K}}_N^-$ defined in (7.31) and the commutators of $\tilde{\mathcal{K}}_N^-$ in detail because the commutators of generators except for $\tilde{\mathcal{K}}_N^-$ is correct. All we have to calculate in detail are $[\mathcal{J}_N^-, \mathcal{K}_N]$ and $[\mathcal{J}_N^-, \tilde{\mathcal{K}}_N^-]$, as well as the case of the Hermitian R-order. We can check relatively easily that the rest commutators of $\tilde{\mathcal{K}}_N^-$ is correct.

7.2.2 Calculation of $\mathcal{K}_N^-$

We calculate $[\mathcal{J}_N^-, \mathcal{K}_N]$ explicitly to determine C and $\delta\mathcal{K}_N^-$ in (7.31). Because the calculation is lengthy, the contribution to $\mathcal{K}_N^-$ obtained by the classical calculation is omitted.

The contribution from the commutator of the first three terms in (7.27) and $\mathcal{K}_N$ is

$$\Big[-x^- p - \frac{1}{2p_-} x L_0 + \frac{i}{2p_-} p,\ \mathcal{K}_N\Big] \sim \frac{i}{4p_-} - \frac{1}{2p_-}\sum_{n=1}^{\infty}(X_{-n}P_n - P_{-n}X_n), \tag{7.33}$$

where $\sim$ means the extraction of the contribution to C and $\delta\mathcal{K}_N^-$. There is only the contribution to $i\frac{C}{p_-}$ in (7.33).

The contribution from the commutator of the last terms in (7.27) is lengthy. Therefore, firstly we calculate the contribution from each line of $\mathcal{K}_N$ in (7.29) and next collect them to obtain $[\frac{\Lambda}{p_-}, \mathcal{K}_N]$.
In the calculation, we may use the help of the following relations.

$$\begin{aligned}
&[\Lambda, x] = [\Lambda, p] = 0 \ ,\\
&[\Lambda, X_n] = \frac{1}{n}\bar{M}_n + in\Big[\sum_{l=1}^{n-1}\frac{1}{l}X_{n-l}P_l + \sum_{l=1}^{\infty}\Big(\frac{1}{n+l}X_{-l}P_{n+l} - \frac{1}{l}P_{-l}X_{n+l}\Big)\Big] \ ,\\
&[\Lambda, P_n] = -\frac{i}{2}\bar{L}_n + in\Big[\sum_{l=1}^{n-1}\frac{1}{l}P_{n-l}P_l + \sum_{l=1}^{\infty}\Big(\frac{1}{n+l} - \frac{1}{l}\Big)P_{-l}P_{n+l}\Big] \ ,\\
&[\Lambda, \bar{M}_n] = iP_n\Big(2M_0 + \frac{1}{6}(n^2-1) - n(n+1)\Big) + n\bar{L}_0X_n \\
&\qquad + \frac{1}{2}n\Big[\sum_{l=1}^{n-1}\bar{L}_{n-l}X_l + \sum_{l=1}^{\infty}(X_{-l}\bar{L}_{n+l} + \bar{L}_{-l}X_{n+l})\Big]\\
&\qquad + in\Big[\sum_{l=1}^{n-1}\frac{1}{l}\bar{M}_{n-l}P_l + \sum_{l=1}^{\infty}\Big(\frac{1}{n+l}\bar{M}_{-l}P_{n+l} - \frac{1}{l}P_{-l}\bar{M}_{n+l}\Big)\Big] \ ,\\
&[\Lambda, X_{-n}] = -[\Lambda, X_n]^\dagger \ , \ [\Lambda, P_{-n}] = -[\Lambda, P_n]^\dagger \ , \ [\Lambda, M_{-n}] = -[\Lambda, M_n]^\dagger,
\end{aligned} \tag{7.34}$$

where the barred operators are defined by removing to zero mode from unbarred ones. After the calculation, we find that there is no contribution from the first and second lines of $\mathcal{K}_N$ in (7.29).

$$\begin{aligned}
\Big[\frac{\Lambda}{p_-}, (\text{First line in (7.29)})\Big] &\sim 0 \ ,\\
\Big[\frac{\Lambda}{p_-}, (\text{Second line in (7.29)})\Big] &\sim 0.
\end{aligned} \tag{7.35}$$

Because x- or p-zero modes remain in the first commutator in (7.35), it does not contribute to $\frac{C}{p_-}$ and $\delta\mathcal{K}_N^-$. The contribution from the second commutator is canceled.

The contribution from the third line of $\mathcal{K}_N$ in (7.29) is [8]

$$\Big[\frac{\Lambda}{p_-}, (\text{Third line in (7.29)})\Big] \sim \frac{1}{p_-}\sum_{n=1}^{\infty}(X_{-n}P_n - P_{-n}X_n)f_{(n)}, \tag{7.36}$$

where

$$f_{(n)} = \big(\frac{1}{6} - 3\big)n + \frac{1}{2} + \big(2M_0 - \frac{1}{6}\big)\frac{1}{n} - 2\sum_{m=1}^{n-1}\frac{n(n-m)}{m^2}. \tag{7.37}$$

[8] We must not forget the contribution from terms arising in moving M_0 to the right end.

The sum of the last term in $f_{(n)}$ arises when we order $\frac{2i}{p_-}\sum_{m>0}\frac{1}{m^2}\bar{M}_{-m}\bar{M}_m$ correctly.

Thus we collect (7.33) and (7.35)-(7.36) to find

$$\tilde{\mathcal{K}}_N^- = \mathcal{K}_N^- + \frac{1}{4p_-} - \frac{i}{p_-}\sum_{n=1}^{\infty}(X_{-n}P_n - P_{-n}X_n)f_{(n)}. \tag{7.38}$$

7.2.3 Anomalous terms in $[\mathcal{J}_N^-, \tilde{\mathcal{K}}_N^-]$

Here we calculate the anomalous terms in $[\mathcal{J}_N^-, \tilde{\mathcal{K}}_N^-]$ by using $\tilde{\mathcal{K}}_N^-$ obtained above.[9]

In chapter 6, we have investigate the structure of $[\mathcal{J}_N^-, \tilde{\mathcal{K}}_N^-]$. By using the mode expansion in (6.21), terms with M_0 at the right end and quantum effect terms are separated into the following parts.

$$\begin{aligned} &i\frac{p}{p_-^2} \times (\text{const.}),\\ &\frac{p}{p_-^2}\sum_{n=1}^{\infty}(X_{-n}P_n - h.c.)k_{(n)}, \end{aligned} \tag{7.39}$$

$$\begin{aligned} &\frac{1}{p_-^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(X_{-n-m}P_mP_n - h.c.)g_{(n,m)},\\ &\frac{1}{p_-^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(X_{-n}P_{-m}P_{n+m} - h.c.)h_{(n,m)}, \end{aligned} \tag{7.40}$$

where $g_{(n,m)}$ and $h_{(n,m)}$ are coefficients which depend of n and m and contain $M_0 \approx a$.

The first line in (7.39) has the same structure as in the Hermitian R-order and the second line in (7.39) is new one. Both structures in (7.39) contain zero mode, p. Therefore, after the similar calculation to the case of the Hermitian R-order in which we have considered the commutator of x and $[\mathcal{J}^-, \tilde{\mathcal{K}}^-]$, we will find the absence of them.

Two structures in (7.40) are caused by the fact that the XP-normal order breaks the mode flipping symmetry of coefficients. The anomaly in $[\mathcal{J}_N^-, \tilde{\mathcal{K}}_N^-]$ vanishes if and only if all $g_{(n,m)}$ and $h_{(n,m)}$ equal to zero. Because there is only one freedom in the choice of M_0, this dangerous commutator is anomalous unless the profound reason exists. We extract the lowest part of $(n, m) = (1, 1)$ in two structures of (7.40) to verify the existence of the anomaly which is removed by the choice of M_0.

Because of the lengthy calculation, we separate $\tilde{\mathcal{K}}_N^-$ into $\mathcal{K}_N^- + \frac{1}{4p_-}$ and $\delta\mathcal{K}^-$. First we separate the commutator of $\mathcal{J}_N^-$ and $\mathcal{K}_N^- + \frac{1}{4p_-}$ into two parts as

$$\left[\mathcal{J}_N^-, \mathcal{K}_N^- + \frac{1}{4p_-}\right] = \left[-x^-p - \frac{1}{2p_-}xL_0 + \frac{i}{2p_-}p,\ \mathcal{K}_N^- + \frac{1}{4p_-}\right] + \left[\frac{\Lambda}{p_-},\ \mathcal{K}_N^- + \frac{1}{4p_-}\right]. \tag{7.41}$$

The contribution from the first part in the r.h.s. of (7.41) is obtained as [10]

$$\left[-x^-p - \frac{1}{2p_-}xL_0 + \frac{i}{2p_-}p,\ \mathcal{K}_N\right] \sim \frac{3}{4p_-^2}(X_{-2}P_1P_1 - h.c.) + \frac{9}{4p_-^2}(X_{-1}P_{-1}P_2 - h.c.) \tag{7.42}$$

[9] $[\mathcal{K}_N, \tilde{\mathcal{K}}_N^-]$ is obtained from $[\mathcal{J}_N^-, \tilde{\mathcal{K}}_N^-]$. Therefore we will find it is anomalous. The rest of commutators with $\tilde{\mathcal{K}}_N^-$ is correct.

[10] Note that $\mathcal{P}_{+N} = -\frac{L_0}{2p_-}$ and $[\mathcal{P}_{+N}, \mathcal{K}_N^-] = 0$.

where $\sim$ means the extraction of the contribution to $(X_{-2}P_1P_1-h.c.)$ or $(X_{-1}P_{-1}P_2-h.c.)$. The contribution from the second part in the r.h.s. of (7.41) is calculated for each line of (7.29) separately as follows.

$$
\begin{aligned}
\Big[\frac{\Lambda}{p_-},\ \text{(First line in (7.29))}\Big] &\sim \frac{9}{4p_-^2}(X_{-2}P_1P_1-h.c.)-\frac{3}{4p_-^2}(X_{-1}P_{-1}P_2-h.c.)\\
\Big[\frac{\Lambda}{p_-},\ \text{(Second line in (7.29))}\Big] &\sim -\frac{3}{4p_-^2}(X_{-2}P_1P_1-h.c.)+\frac{3}{2p_-^2}(X_{-1}P_{-1}P_2-h.c.)\\
\Big[\frac{\Lambda}{p_-},\ \text{(Third line in (7.29))}\Big] &\sim \frac{1}{p_-^2}(X_{-2}P_1P_1-h.c.)[4-2M_0]\\
&\quad+\frac{1}{p_-^2}(X_{-1}P_{-1}P_2-h.c.)\Big[\frac{25}{4}-\frac{7}{2}M_0\Big].
\end{aligned}
\tag{7.43}
$$

Then we collect them to obtain

$$
\begin{aligned}
\Big[\frac{\Lambda}{p_-},\ \mathcal{K}_N^-+\frac{1}{4p_-}\Big] \sim&\frac{1}{p_-^2}(X_{-2}P_1P_1-h.c.)\Big[\frac{11}{2}-2M_0\Big]\\
&+\frac{1}{p_-^2}(X_{-1}P_{-1}P_2-h.c.)\Big[7-\frac{7}{2}M_0\Big].
\end{aligned}
\tag{7.44}
$$

From (7.42) and (7.44), we find

$$
\begin{aligned}
\Big[\mathcal{J}^-,\ \mathcal{K}_N^-+\frac{1}{4p_-}\Big] \sim&\frac{1}{p_-^2}(X_{-2}P_1P_1-h.c.)\Big[\frac{25}{4}-2M_0\Big]\\
&+\frac{1}{p_-^2}(X_{-1}P_{-1}P_2-h.c.)\Big[\frac{37}{4}-\frac{7}{2}M_0\Big]
\end{aligned}
\tag{7.45}
$$

Next we calculate the contribution from the commutator of $\mathcal{J}_N^-$ and $\delta\mathcal{K}_N^-$. The result is

$$
\begin{aligned}
\Big[\mathcal{J}^-,\ \delta\mathcal{K}_N^-\Big] \sim&\frac{1}{p_-^2}\big((X_{-2}P_1P_1-h.c.)+(X_{-1}P_{-1}P_2-h.c.)\big)\Big[\frac{3}{2}f_{(2)}-3f_{(1)}\Big]\\
=&\frac{1}{p_-^2}\big((X_{-2}P_1P_1-h.c.)+(X_{-1}P_{-1}P_2-h.c.)\big)\Big[-\frac{51}{8}-\frac{9}{2}M_0\Big].
\end{aligned}
\tag{7.46}
$$

Note that the coincidence of coefficients of $(X_{-2}P_1P_1-h.c.)$ and $(X_{-1}P_{-1}P_2-h.c.)$ is the specialty in $n=m=1$. [11]

Thus we collect (7.45) and (7.46) to obtain

$$
\begin{aligned}
[\mathcal{J}^-,\ \tilde{\mathcal{K}}_N^-] \sim&\frac{1}{p_-^2}(X_{-2}P_1P_1-h.c.)\Big[-\frac{1}{8}-\frac{13}{2}M_0\Big]\\
&+\frac{1}{p_-^2}(X_{-1}P_{-1}P_2-h.c.)\Big[\frac{23}{8}-8M_0\Big].
\end{aligned}
\tag{7.47}
$$

From this, we find that there is no choice of M_0 such that two coefficients vanish.

[11] $\frac{m}{n+m}+\frac{n}{m}=\frac{1}{2}\left(1+\frac{n}{m}+\frac{m}{n}\right)$ for $n=m=1$.

In this way, we find that the dangerous commutator, $[\mathcal{J}^-,\ \tilde{\mathcal{K}}_N^-]$, does not vanish. Therefore there is the spacetime conformal anomaly in the 3-dim. tensionless string theory in the XP-normal order.

If we interpret anomalous terms in $[\mathcal{J}_N^-, \tilde{\mathcal{K}}_N^-]$ as a new constraint condition, commutators of such a constraint and generators produce other new constraints, as we see in [38, 40]. Under all constraint conditions, only the string ground state $|0\rangle_{XP}$ survives. It is point-like, not stringy. Because there is no anomalous term of $i\frac{p}{p_-^2}$ in the dangerous commutator, the theory restricted to the point-like state has the spacetime conformal symmetry. It means that the tensionless string in the XP-normal order is point-like. This fact is natural from the consideration of the mass spectrum in chapter 4. There we have found that the mass eigenstate with positive norm is only the string ground state, which satisfy $M_0 = 0$.

Chapter 8

Calculation of Dangerous Commutators in Spacetime Conformal Symmetry in $D > 3$

In [38, 40], the existence of the spacetime conformal anomaly for a light-cone tensionless closed bosonic string in the R-order is shown. R-ordered generators must be regularized to avoid divergences. In [38, 40], the dangerous commutator $[\mathcal{J}^{-I}, \mathcal{K}^K]$ is considered with the generic regularization and without Fourier expansion. Then it is verified that the anomaly can appear in the traceless part of $[\mathcal{J}^{-I}, \mathcal{K}^K]$ with respect to transverse indexes as follows:

$$[\mathcal{J}^{-I}, \mathcal{K}^K] + i\delta^{I,K}\tilde{\mathcal{K}}^- \propto \frac{1}{\epsilon p_-} \oint d\sigma [\bar{X}^K(\sigma)\bar{P}^I(\sigma) + (H.C.)]_{t.l.} \equiv \frac{1}{\epsilon p_-} L^{IK}_{t.l.}, \tag{8.1}$$

where ϵ is regulator[1] and $H.C.$ is the part compensated to preserve the hermitian property. The form of $H.C.$ depends on the operator order.

From the consideration in chapter 6, the possible structure of $[\mathcal{J}^{-I}, \mathcal{K}^K]$ in the general operator order of X^I and P^J is

$$\begin{aligned}[\mathcal{J}^{-I}, \mathcal{K}^K] =& i(\text{ordered classical part}) + \delta^{I,K}\frac{1}{p_-}((X \cdot P)\text{-quadratic}) + \delta^{I,K}\frac{i}{p_-} \times (\text{const.}) \\ &+ \frac{1}{p_-}(X^I P^K\text{-quadratic}) + \frac{1}{p_-}(X^K P^I\text{-quadratic})\end{aligned} \tag{8.2}$$

where we included the term with M_0 at the right end in the first part. From this, we find that the traceless part comes from the second line and then that whether such a anomalous term vanishes depends on the operator order greatly.

8.1 Hermitian R-order Case

Because the Reference order breaks the anti-hermitian property in commutators of higher degree,[2] we consider the Hermitian R-order instead of the R-order. The case of the Weyl

[1] The regularized commutation relation at the same point is $[X^I(\sigma), P_J(\sigma)]_{reg} = i\delta_\epsilon(0) \sim \frac{i}{\epsilon}$.

[2] The check of breaking the anti-hermitian property for the R-order is done with the cut-off regularization in appendix E.

order is similarly discussed.
In $D>3$ and in the Hermitian R-order (and the Weyl order), the constraint is $M_0 \approx 0$ from (5.2). Therefore the first ordered classical part in (8.2) is $\mathcal{K}^-$. In the following calculations, we drop terms with M_0 at the right end.
Furthermore, in the Hermitian R-order, which preserve the mode flipping symmetry for coefficients of Fourier series, all quadratic terms in the r.h.s. of (8.2) make the commutator and then are absorbed into the third part in the r.h.s of (8.2). Therefore there is no traceless part in (8.2), such as (6.12). The difference in the trace part of (8.2) is absorbed into the redefinition of $\mathcal{K}^-$. From (8.2) for $I=K$, we obtain

$$\tilde{\mathcal{K}}^- \equiv \mathcal{K}^- + \frac{C}{p_-} \approx i[\mathcal{J}^{-I}, \mathcal{K}^I], \tag{8.3}$$

where C is (divergent) constant and I is one of the transverse direction, not summed and $\approx$ means the dropping terms with M_0 at the right end. On the other hand, from (8.2) for $I \neq K$, we obtain

$$[\mathcal{J}^{-I}, \mathcal{K}^K] \approx 0 \tag{8.4}$$

where $I \neq K$.

Below we assume that some good regularization is used.[3]

Most of commutators with $\tilde{\mathcal{K}}^-$ give the desirable result. The dangerous commutators with $\tilde{\mathcal{K}}^-$ are $[\mathcal{J}^{-J}, \tilde{\mathcal{K}}^-]$ and $[\mathcal{K}^K, \tilde{\mathcal{K}}^-]$.[4]
The first dangerous commutator, $[\mathcal{J}^{-J}, \tilde{\mathcal{K}}^-]$, is calculated by using the Jacobi identity as

$$\begin{aligned}[\mathcal{J}^{-J}, \tilde{\mathcal{K}}^-] &\approx [\mathcal{J}^{-J}, i[\mathcal{J}^{-I}, \mathcal{K}^I]] \\ &= -i[\mathcal{K}^I, [\mathcal{J}^{-J}, \mathcal{J}^{-I}]] - i[\mathcal{J}^{-I}, [\mathcal{K}^I, \mathcal{J}^{-J}]] \approx 0,\end{aligned} \tag{8.5}$$

where we used I different from J in the definition of $\tilde{\mathcal{K}}^-$. The first term in the second line contains a dangerous commutator for the Lorentz symmetry and the second term contains (8.4). Therefore, under the condition of $M_0 = 0$, $[\mathcal{J}^{-J}, \tilde{\mathcal{K}}^-]$ vanishes.[5]

---

[3] With the cut-off regularization, (8.2) in the Hermitian R-order is

$$[\mathcal{J}^{-I}_{(N)}, \mathcal{K}^K_{(N)}] \approx i\delta^{I,K}\Big(\mathcal{K}^-_{(N)} + \frac{C_{(N)}}{p_-} + \delta\mathcal{K}^-_{(N)}\Big) + i\delta(O^{IK}_{(N)})_{t.l.},$$

where $\delta(O^{IK}_{(N)})_{t.l.}$ is a traceless part consisting of operators of degree 4. $\delta\mathcal{K}^-_{(N)}$ and $\delta(O^{IK}_{(N)})_{t.l.}$ vanish in the limit of $N \to \infty$. We decide to use $\tilde{\mathcal{K}}^- \equiv \mathcal{K}^-_{(N)} + \frac{C_{(N)}}{p_-} + \delta\mathcal{K}^-_{(N)}$ in the rest of commutators. Furthermore we must check that commutators of extra terms which vanish in the limit of $N \to \infty$, such as $\delta\mathcal{K}^-_{(N)}$, $\delta(O^{IK}_{(N)})_{t.l.}$ or their commutators, also vanishes in the same limit. The check should be discussed in the same way, as well as the three dimensional case in appendix E. Even though $\delta(O^{IK}_{(N)})_{t.l.} \to 0$, it is not good that there is the traceless part in $[\mathcal{J}^{-I}_{(N)}, \mathcal{K}^K_{(N)}]$. Therefore we may need better regularization instead of the cut-off regularization.

[4] Although $[\mathcal{K}^I, \mathcal{K}^J]$ is also dangerous, this is calculated in the same way as $[\mathcal{J}^{-I}, \mathcal{J}^{-J}]$. Then we find that this vanishes under $M_0 = 0$.

[5] In the calculation with the cut-off regularization, we must check the dropping of contributions from commutators of extra terms such as $\delta\mathcal{K}^-_{(N)}$ or $\delta(O^{IK}_{(N)})_{t.l.}$.

Similarly, the second dangerous commutator $[\mathcal{K}^K, \tilde{\mathcal{K}}^-]$ is calculated by using the Jacobi identity as

$$\begin{aligned}[\mathcal{K}^K, \tilde{\mathcal{K}}^-] &\approx [\mathcal{K}^K, i[\mathcal{J}^{-I}, \mathcal{K}^I]] \\ &= -i[\mathcal{K}^I, [\mathcal{K}^K, \mathcal{J}^{-I}]] - i[\mathcal{J}^{-I}, [\mathcal{K}^I, \mathcal{K}^K]] \approx 0,\end{aligned} \tag{8.6}$$

where we used I different from K in the definition of $\tilde{\mathcal{K}}^-$. The first term in the second line contains (8.4) and the second term contains $[\mathcal{K}^I, \mathcal{K}^K]$, which vanish under the condition of $M_0 = 0$. Therefore, under $M_0 = 0$, $[\mathcal{J}^{-J}, \tilde{\mathcal{K}}^-]$ vanishes.

Thus we find that the anomaly at least unrelated to the regularization does not exist in the Hermitian R-order. The calculation with the cut-off regularization should be done in the same way as the case of $D = 3$ in appendix E. However, it is not good that there is the traceless part in $[\mathcal{J}^{-I}_{(N)}, \mathcal{K}^K_{(N)}]$, even though it is dropped in the limit of the cut-off scale. . Therefore we may need a better regularization than the cut-off regularization.
In any way, there is the difference between the results in the light-cone gauge quantization and the BRST quantization formalism in $D > 3$ as well as $D = 3$. It is probably the reason of such a difference that the mode flipping symmetry for coefficients of Fourier series is broken in the BRST formalism because of the fermionic ghost, which must be ordered in the normal order.

Because the mass spectrum in $M_0 = 0$ has been already investigated in chapter 5, it is omitted here.

8.2 XP-normal Order Case

In chapter 7, we have verified the existence of the spacetime conformal anomaly for the 3-dim. tensionless string in the XP-normal order. There we have found the anomaly in $[\mathcal{J}^-_N, \tilde{\mathcal{K}}^-_N]$. In $D > 3$, terms in the second line of (8.2) can exist. However, because these terms possibly vanish, we must calculate the dangerous commutator $[\mathcal{J}^{-I}_N, \mathcal{K}^K_N]$ explicitly to verify the concrete coefficient of these terms.

Before the explicit calculation, we investigate the expected result of $[\mathcal{J}^{-I}_N, \mathcal{K}^K_N]$ in a little more detail. As seen in chapter 5, in the case of the XP-normal order, the requirement of Lorentz symmetry determines the critical dimension and the ordering constant in the constraint such that $D = 26$ and $M_0 \approx a = 2$.
When we calculate $[\mathcal{J}^{-I}, \mathcal{K}^K]$ under these conditions, by using the Jacobi identity, we obtain

$$[\mathcal{J}^{-I}, \mathcal{K}^K] \approx [\mathcal{J}^{-K}, \mathcal{K}^I] \tag{8.7}$$

where $\approx$ means the dropping of terms which vanish under $D = 26$ and $M_0 = 2$. From this, we expect that $[\mathcal{J}^{-I}, \mathcal{K}^K]$ consists of the symmetric part in exchanging of I and K and the part which vanishes under $D = 26$ and $M_0 = 2$.

8.2.1 Preparation for explicit calculation

Here we give the explicit representation of important generators and some useful commutation relations.

In XP-normal ordered generators, non-zero mode is normal-ordered but zero mode is not. Therefore we decouple zero mode.
First, the important Lorentz generator, $\mathcal{J}_N^{-I}$, is written as

$$
\begin{aligned}
\mathcal{J}_N^{-I} =& x^- p^I + \left(\frac{1}{2p_-}x^I p\cdot p - \frac{i}{2p_-}p^I\right) + \frac{1}{2p_-}x^I \bar{L}_0 \\
&+ \frac{p_J}{p_-}\sum_{l=1}^{\infty}\left[(X_{-l}^I P_l^J + P_{-l}^J X_l^I) - (X_{-l}^J P_l^I + P_{-l}^I X_l^J)\right] - \frac{\bar{\Lambda}^I}{p_-},
\end{aligned}
\tag{8.8}
$$

where barred operators are defined by removing zero mode from unbarred ones and we used

$$
\bar{\Lambda}^I \equiv \sum_{l=1}^{\infty}\left[-\frac{1}{2}(X_{-l}^I \bar{L}_l + \bar{L}_{-l}X_l^I) + \frac{i}{l}(P_{-l}^I \bar{M}_l - \bar{M}_{-l}P_l^I)\right]. \tag{8.9}
$$

Here we note that commutation relations of $\bar{L}_n$ or $\bar{M}_n$ is the same as the relations of unbarred ones, (3.29), because unbarred ones does not contain any x zero mode.
In addition, the important generator of the special conformal transformation, $\mathcal{K}_N^K$, is

$$
\begin{aligned}
\mathcal{K}_N^K =& x^K x^- p_- + x^K(x\cdot p) - \frac{1}{2}(x\cdot x)p^K \\
&+ x^K\sum_{n=1}^{\infty}(X_{-n}\cdot P_n + h.c.) - p^K\sum_{n=1}^{\infty}X_{-n}\cdot X_n + x_L\sum_{n=1}^{\infty}\Big((X_{-n}^K P_n^L - X_{-n}^L P_n^K) + h.c.\Big) \\
&+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[\big(X_{-n}^K(X_{-m}\cdot P_{n+m}) + X_{-n}^K(P_{-m}\cdot X_{n+m}) + X_{-n-m}^K(P_m\cdot X_n)\big) + h.c.\Big] \\
&- \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[\big((X_{-n}\cdot X_{-m})P_{n+m}^K + 2(X_{-n-m}\cdot X_m)P_n^K\big) + h.c.\Big] \\
&- i\sum_{n=1}^{\infty}\frac{1}{n}(X_{-n}^K \bar{M}_n - \bar{M}_{-n}X_n^K),
\end{aligned}
\tag{8.10}
$$

where $h.c.$ indicates hermitian conjugate of the adjacent part. Because we do not need $\mathcal{K}^-$ in the calculation in the next subsection, we omit the representation of $\mathcal{K}^-$ in $D > 3$.

Next we give some useful commutation relations below. For $n > 0$,

$$
\begin{aligned}
[\bar{\Lambda}^I, X_n^J] =& \delta^{I,J}\frac{1}{n}\bar{M}_n + i\sum_{l=1}^{n-1}\Big(P_{n-l}^J X_l^I + \frac{n-l}{l}X_{n-l}^J P_l^I\Big) \\
&+ i\sum_{l=1}^{\infty}\Big(X_{-l}^I P_{n+l}^J + P_{-l}^J X_{n+l}^I - \frac{n+l}{l}P_{-l}^I X_{n+l}^J - \frac{l}{n+l}X_{-l}^J P_{n+l}^I\Big), \\
[\bar{\Lambda}^I, P_n^J] =& -\delta^{I,J}\frac{i}{2}\bar{L}_n + in\sum_{l=1}^{n-1}\frac{1}{l}P_{n-l}^J P_l^I + in\sum_{l=1}^{\infty}\Big(\frac{1}{l}P_{-l}^I P_{n+l}^J - \frac{1}{n+l}P_{-l}^J P_{n+l}^I\Big), \\
[\bar{\Lambda}^I, X_{-n}^J] =& -[\bar{\Lambda}^I, X_n^J]^\dagger, \quad [\bar{\Lambda}^I, P_{-n}^J] = -[\bar{\Lambda}^I, P_n^J]^\dagger.
\end{aligned}
\tag{8.11}
$$

and

$$
\begin{aligned}
\left[\bar{\Lambda}^I, \bar{M}_n\right] =& inP^I_n\left(-(n+1)+\left[2M_0\frac{1}{n}+\frac{D-2}{6}\Big(n-\frac{1}{n}\Big)\right]\right)\\
&+\frac{1}{2}n\left[\sum_{l=1}^{n-1}\bar{L}_{n-l}X^I_l+\sum_{l=1}^{\infty}(\bar{L}_{-l}X^I_{n+l}+X^I_{-l}\bar{L}_{n+l})\right]\\
&+in\left[\sum_{l=1}^{n-1}\frac{1}{l}\bar{M}_{n-l}P^I_l+\sum_{l=1}^{\infty}\Big(\frac{1}{n+l}\bar{M}_{-l}P^I_{n+l}-\frac{1}{l}P^I_{-l}\bar{M}_{n+l}\Big)\right]\ ,\\
\left[\bar{\Lambda}^I, \bar{M}_{-n}\right] =& -\left[\bar{\Lambda}^I, \bar{M}_n\right]^\dagger.
\end{aligned}
\tag{8.12}
$$

8.2.2 Calculation of $[\mathcal{J}^{-I}, \mathcal{K}^K]$ for $I \neq K$ and anomaly

Here we calculate $[\mathcal{J}^{-I}, \mathcal{K}^K]$ for $I \neq K$ to get the explicit structure of the anomaly in the spacetime conformal symmetry. Because the calculation is too lengthy, we omit the ordered classical part except for terms with M_0 at the right end and then show only the anomalous part. The contribution to the anomalous part in $[\mathcal{J}^{-I}, \mathcal{K}^K]$ comes from terms with M_0 at the right end and quantum effect's terms, which arise from extra exchanging operators to get the correctly ordered result.[6]

We firstly calculate the contribution from each part of $\mathcal{J}^{-I}$ in (8.8) and then collect the results. The commutator of the first term in (8.8) is

$$
[x^-p^I,\ \mathcal{K}^K] = ix^-x^Ip^K - ix^-\sum_{n=1}^{\infty}\Big[(X^I_{-n}P^K_n - X^K_{-n}P^I_n) + h.c.\Big] \sim 0,
\tag{8.13}
$$

where $\sim$ means extracting the contribution to the anomalous part. This has no contribution to the anomaly.

Next the commutator of the second part in (8.8) is

$$
\begin{aligned}
\Big[\frac{1}{2p_-}x^Ip\cdot p - \frac{i}{2p_-}p^I,\ \mathcal{K}^K\Big] =& -ix^-x^Ip^K + \frac{i}{2p_-}\Big[x^I(x\cdot p)p^K + h.c.\Big]\\
&-\frac{i}{2p_-}(x^Ip_J + p_Jx^I)\sum_{n=1}^{\infty}\Big[(X^J_{-n}P^K_n - X^K_{-n}P^J_n) + h.c.\Big]\\
\sim& \, 0.
\end{aligned}
\tag{8.14}
$$

Then there is no contribute from this to the anomaly.

Next the commutator of the third part in (8.8) is

$$
\Big[\frac{1}{2p_-}x^I\bar{L}_0,\ \mathcal{K}^K\Big] = \frac{i}{p_-}x^Ip^K\sum_{n=1}^{\infty}(X_{-n}\cdot P_n + h.c.) + \frac{i}{p_-}x^I\Lambda^K \sim 0.
\tag{8.15}
$$

Then there is no contribute from this to the anomalous part.

There are the contributions from the commutators of 4th and 5th parts in (8.8) to the anomalous part. Though it is lengthy. Because the terms with odd degree of x^I and p^I

[6] Because the central extension terms in the commutation relation of $\bar{M}_n$ in (B.23) also arises from extra exchanging operator to get the correctly ordered result, they are the quantum effect's terms.

does not contribute to the anomalous part, we find that the contribution from the 4th part to the anomalous part comes from commutators with the first and second lines in (8.10). From a short calculation, we find that the commutator of the 4th part in (8.8) and the first lines in (8.10) does not contribute to the anomalous part. Thus we obtain

$$\begin{aligned}&\Big[(\text{4th term of (8.8)}),\ \mathcal{K}^K\Big] \sim \Big[(\text{4th term of (8.8)}),\ (\text{2nd line of (8.10)})\Big]\\&\sim -\frac{D-2}{2p_-}\sum_{n=1}^{\infty}\Big(X^I_{-n}P^K_n - h.c.\Big) - \frac{D-6}{2p_-}\sum_{n=1}^{\infty}\Big(X^K_{-n}P^I_n - h.c.\Big).\end{aligned} \tag{8.16}$$

Similarly, the contribution from the 5th part in (8.8) to the anomalous part comes from commutators with the 3rd, 4th and 5th lines in (8.10). We collect them to obtain

$$\begin{aligned}&\Big[-\frac{\bar{\Lambda}^I}{p_-},\ \mathcal{K}^K\Big] \sim \Big[-\frac{\bar{\Lambda}^I}{p_-},\ (\text{3rd, 4th and 5th line of (8.10)})\Big]\\&\sim -\frac{2}{p_-}\sum_{n=1}^{\infty}(n-1)\Big(X^I_{-n}P^K_n - h.c.\Big) + \frac{D-1}{2p_-}\sum_{n=1}^{\infty}(n-1)\Big(X^K_{-n}P^I_n - h.c.\Big)\\&\quad + \frac{D-4}{2p_-}\sum_{n=1}^{\infty}(n-1)\Big(X^I_{-n}P^K_n - h.c.\Big) - \frac{1}{2p_-}\sum_{n=1}^{\infty}(n-1)\Big(X^K_{-n}P^I_n - h.c.\Big)\\&\quad + \frac{1}{p_-}\sum_{n=1}^{\infty}(n-1)\Big(X^I_{-n}P^K_n - h.c.\Big)\\&\quad - \frac{1}{p_-}\sum_{n=1}^{\infty}\Big(X^K_{-n}P^I_n - h.c.\Big)\left[\Big(\frac{D-2}{6}-2\Big)n + \Big(2M_0 - \frac{D-2}{6}\Big)\frac{1}{n}\right]\\&\sim \frac{D-6}{2p_-}\sum_{n=1}^{\infty}(n-1)\Big(X^I_{-n}P^K_n - h.c.\Big) + \frac{D-2}{2p_-}\sum_{n=1}^{\infty}(n-1)\Big(X^K_{-n}P^I_n - h.c.\Big)\\&\quad - \frac{1}{p_-}\sum_{n=1}^{\infty}\Big(X^K_{-n}P^I_n - h.c.\Big)\left[\Big(\frac{D-2}{6}-2\Big)n + \Big(2M_0 - \frac{D-2}{6}\Big)\frac{1}{n}\right].\end{aligned} \tag{8.17}$$

We collect (8.13)-(8.17) to get

$$\begin{aligned}\Big[\mathcal{J}^{-I},\ \mathcal{K}^K\Big] \sim& \frac{1}{p_-}\sum_{n=1}^{\infty}\Big[(X^I_{-n}P^K_n + X^K_{-n}P^I_n) - h.c.\Big]\left[\frac{D-6}{2}n - (D-4)\right]\\&- \frac{1}{p_-}\sum_{n=1}^{\infty}\Big(X^K_{-n}P^I_n - h.c.\Big)\left[\Big(\frac{D-2}{6}-4\Big)n + \Big(2M_0 - \frac{D-2}{6}\Big)\frac{1}{n}\right]\end{aligned} \tag{8.18}$$

As expected from (8.7), the first line in the r.h.s. of (8.18) is symmetric in exchanging I and K and the second line vanishes under $D = 26$ and $M_0 = 2$. Therefore the first line is anomalous.

Thus we have gotten explicitly the anomaly of the spacetime conformal symmetry for the tensionless string in the XP-normal order. If we interpret anomalous terms in $[\mathcal{J}^-_N, \tilde{\mathcal{K}}^-_N]$ as a new constraint condition to avoid the anomaly, in the same way as the three dimensional case, only the string ground state $|0\rangle_{XP}$ survives. Therefore the tensionless string theory in the XP-normal order is point-like.

Chapter 9

Discussion of Other Types of String : Supersymmetric and Open

So far, we have considered the closed and bosonic type among the tensionless string theories. In this chapter we discuss the other types of string, especially a superstring and a open string.

9.1 Tensionless String with Supersymmetry

For the simplicity, we consider the tensionless closed string theory with the spacetime supersymmetry in three dimensions to compare it with the bosonic case. Then we give the short discussion of the anomaly in the spacetime (super)conformal symmetry.
For the tensionful case, there are two ways which construct the tensionful string theory with the spacetime supersymmetry, the Green-Schwarz (GS) formalism and the Ramond-Neveu-Schwarz (RNS) formalism. In three dimensions, the GS formalism is also possible because of the Majorana spinor.
These formalisms can apply on the tensionless case. In this section we use the GS formalism to consider the superstring theory. The tensionless superstring theory has the fermionic gauge symmetry, κ-symmetry as the local symmetry and the spacetime supersymmetry as the global one.
The tensionless spinning string by the RNS formalism is similarly discussed.

9.1.1 Fermionic field in generators for tensionless superstring theory

The action of a tensionless superstring is obtained with the next replacements from the bosonic action.

$$\begin{aligned} \dot{\mathbf{X}}^\mu &\to \mathbf{\Pi}^\mu_\tau \equiv \dot{\mathbf{X}}^\mu + i\bar{\Theta}^a \mathbf{\Gamma}^\mu \dot{\Theta}^a \ , \\ (\mathbf{X}^\mu)' &\to \mathbf{\Pi}^\mu_\sigma \equiv (\mathbf{X}^\mu)' + i\bar{\Theta}^a \mathbf{\Gamma}^\mu (\Theta^a)' \ , \end{aligned} \tag{9.1}$$

where Θ^a is a two-component Majorana fermion and $a = 1, \cdots, \mathcal{N}$ and $\mathcal{N}$ is the number of SUSY.
Thus the action of a tensionless superstring is written as the action on the phase space

with constraints:

$$S[\mathbf{X},\mathbf{P},\Theta;V,U]=\int d\tau\oint\frac{d\sigma}{2\pi}\left\{\mathbf{\Pi}^{\mu}_{\tau}\mathbf{P}_{\mu}-\frac{1}{2}V\mathbf{P}^2-U\mathbf{\Pi}^{\mu}_{\sigma}\mathbf{P}_{\mu}\right\},\tag{9.2}$$

where $\bar{\Theta}\equiv\Theta^t\mathbf{\Gamma}^0$ and we used the next representation for the gamma matrix: $\mathbf{\Gamma}^0=i\sigma^2$, $\mathbf{\Gamma}^1=\sigma^1$, $\mathbf{\Gamma}^2=\sigma^3$. Here note that we do not need the Wess-Zumino term, which is proportional to the tension, T. The action of the null spinning string on the phase space is in [45].

The action in (9.2) has the next gauge symmetry and the κ-symmetry as the local symmetry.

$$\begin{cases}\delta\mathbf{X}^{\mu}=\alpha\Big[\mathbf{P}^{\mu}-\frac{i}{V}\bar{\Theta}^b\mathbf{\Gamma}^{\mu}(\dot{\Theta}-U\Theta')^b\Big]+\beta\mathbf{X}'^{\mu}\\ \delta\mathbf{P}_{\mu}=(\beta\mathbf{P}_{\mu})'\\ \delta\Theta^a=\frac{\alpha}{V}(\dot{\Theta}-U\Theta')^a+\beta\Theta'^a\\ \delta V=\dot{\alpha}+U'\alpha-U\alpha'+V'\beta-V\beta'\\ \delta U=\dot{\beta}+U'\beta-U\beta'.\end{cases}\tag{9.3}$$

$$\begin{cases}\delta\mathbf{X}^{\mu}=-i\bar{\Theta}^b\mathbf{\Gamma}^{\mu}(\not{\mathbf{P}}\kappa)^b\\ \delta\mathbf{P}_{\mu}=0\\ \delta\Theta^a=(\not{\mathbf{P}}\kappa)^a\\ \delta V=-4i\bar{\kappa}^b(\dot{\Theta}-U\Theta')^b\\ \delta U=0.\end{cases}\tag{9.4}$$

Furthermore, as we expect, the action in (9.2) has also the spacetime superconformal symmetry as the global symmetry. The explicit representations for the generators of the spacetime conformal symmetry are

$$\begin{aligned}\mathcal{P}_{\mu}&=\oint\frac{d\sigma}{2\pi}\mathbf{P}_{\mu},\\ \mathcal{J}^{\rho}&=\oint\frac{d\sigma}{2\pi}\Big[\epsilon^{\rho\mu\nu}\mathbf{X}_{\mu}\mathbf{P}_{\nu}+\frac{i}{2}(\bar{\Theta}^a\Theta^a)\mathbf{P}^{\rho}\Big],\\ \mathcal{D}&=\oint\frac{d\sigma}{2\pi}\mathbf{X}^{\mu}\mathbf{P}_{\mu},\\ \mathcal{K}^{\mu}&=\oint\frac{d\sigma}{2\pi}\Big[\mathbf{X}^{\mu}(\mathbf{X}^{\nu}\mathbf{P}_{\nu})-\frac{1}{2}\mathbf{X}^2\mathbf{P}^{\mu}+\frac{1}{8}(\bar{\Theta}^a\Theta^b)(\bar{\Theta}^b\Theta^a)\mathbf{P}^{\mu}-\frac{i}{4}\epsilon^{\mu\nu\rho}\mathbf{X}_{\nu}\mathbf{P}_{\rho}(\bar{\Theta}^a\Theta^a)\Big],\end{aligned}\tag{9.5}$$

and the representations of the additional generators of the spacetime superconformal symmetry are

$$\begin{aligned}\mathcal{Q}^a&=\oint\frac{d\sigma}{2\pi}\not{\mathbf{P}}\Theta^a,\\ \mathcal{S}^a&=\oint\frac{d\sigma}{2\pi}\Big[\mathbf{X}_{\mu}\mathbf{\Gamma}^{\mu}\not{\mathbf{P}}\Theta^a-\frac{i}{2}\not{\mathbf{P}}(\bar{\Theta}^b\Theta^b)\Theta^a\Big].\end{aligned}\tag{9.6}$$

Of course, all the above generators are invariant undeer the gauge transformations in (9.3) and (9.4).[1]

[1]To check the gauge invariance, we use the Fierz identity and the following properties of the Majorana spinors: $\bar{\Psi}\Phi=\bar{\Phi}\Psi$, $\bar{\Psi}\mathbf{\Gamma}^{\mu}\Phi=\bar{\Phi}\mathbf{\Gamma}^{\mu}\Psi$ and so on.

9.1.2 Light-cone gauge

Next we choose the light-cone guage to fix the gauge symmetries in (9.3) and (9.4). In the light-cone guage, the bosonic variables are fixed by

$$X^+(\tau,\sigma) = \tau \ , \quad P_-(\tau,\sigma) = p_-(\tau) \neq 0 \ , \tag{9.7}$$

and the fermionic variable is fixed by

$$\Gamma^+\Theta = 0. \tag{9.8}$$

In our representation of the gamma matrix,

$$\mathbf{\Gamma}^+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}. \tag{9.9}$$

Therefore Θ has only the upper-component. Because of the scale transformation of Θ, we should write Θ as

$$\Theta = \sqrt{\frac{1}{2\sqrt{2}p_-}}\begin{pmatrix} \theta \\ 0 \end{pmatrix}. \tag{9.10}$$

This leads to $(\bar{\Theta}^b\Theta^a) = 0$. Therefore the generators of the spacetime superconformal symmetry become simple. Especially, the bosonic part, (9.5), has the same representations as the tensionless bosonic string.

We repeat the same steps as ones in chapter 2 to solve the constraints of the Lagrangian in the light-cone guage. Most of the constraints are solved by the following relations:

$$\begin{aligned} P_+ &= -\frac{1}{2p_-}PP \\ p_-(\bar{X}^-)' &= -\Big(\bar{X}'\cdot P + \frac{i}{2}\theta\bar{\theta}'\Big) + \oint\frac{d\sigma}{2\pi}\Big(\bar{X}'\cdot\bar{P} + \frac{i}{2}\bar{\theta}\bar{\theta}'\Big). \end{aligned} \tag{9.11}$$

The first line is exactly the same as the bosonic case. Therefore P_+ and H are written by L_n in (2.49). Especially, the mass square operator is unchanged and does not contain any fermionic field.
The second line is solved with the mode expansion:

$$\bar{X}^-(\sigma) = -\frac{1}{p_-}\sum_{m\neq0}\frac{i}{m}N_m e^{im\sigma}, \tag{9.12}$$

$$N_m \equiv M_m + \frac{1}{2}\sum_n\big(\frac{m}{2} - n\big)\theta_n\theta_{m-n}. \tag{9.13}$$

where M_n is the same bosonic part as (2.52) and the second term is the fermionic part. Thus the contribution from the fermionic field to generators of the spacetime conformal symmetry comes only through X^-.

After constraints except for one are solved, we obtain the next Lagrangian of the tensionless superstring in the light-cone gauge with only one constraint.[2]

$$L = \dot{x} \cdot p + \dot{x}^- p_- + \frac{i}{2}\theta_0^a \dot{\theta}_0^a + \sum_{n \neq 0} \Big[\dot{X}_n \cdot P_{-n} + \frac{i}{2}\theta_n^a \dot{\theta}_{-n}^a \Big] - H, \tag{9.14}$$

The unique constraint is

$$N_0 \equiv M_0 - \frac{1}{2}\sum_n n\theta_n^a \theta_{-n}^a \approx a' \ , \tag{9.15}$$

where a' is the ordering constant. Here note the fermionic contribution.

9.1.3 Dangerous commutators in quantum tensionless superstring theory

The quantization of the tensionless superstring is done by the next commutation relations and anti-commutation relation.

$$[x^-, p_-] = i \ , \ [X_n, P_m] = i\delta_{n,-m} \ , \ \{\theta_n^a, \theta_m^b\} = \delta_{n,-m}\delta^{a,b} \tag{9.16}$$

From the last relation, the ground state for the fermionic part is usually the state annihilated by positive (or negative) modes of θ_n. Therefore the natural operator order corresponding to it is the normal order.

In three dimensions, there is no Lorentz anomaly in the super Poincaré symmetry as well as the bosonic case. We can easily check that the commutators and anti-commutators with fermionic generators are correct. Thus the quantum tensionless superstring theory preserve the super Poincaré symmetry. Therefore the mass spectrum is meaningful. Because $\mathcal{M}^2$ is the same as the bosonic case, we can use the results in chapter 4 and 5. However the value of N_0 changes according to the fermionic excitation level. Therefore we fill find more various mass eigenstates than the bosonic case.

Next we consider the spacetime superconformal symmetry. In the bosonic case of chapter 7 and 8, we have found the absence of the anomaly in the Hermitian R-order, where the mode flipping symmetry is preserved, and the concrete form of the anomaly in the XP-normal order, where the mode flipping symmetry is broken.
In the supersymmetric case, if we choose the Hermitian R-order for the bosonic modes as well as the bosonic case, the mixing of the different operator order causes the anomaly in the dangerous commutator such as $[\mathcal{J}^-, \mathcal{K}^-]$. If we choose the XP-normal order for the bosonic modes, we must investigate the possibility of the cancellation of anomalous terms in dangerous commutators for various numbers of SUSY. It is the future work.

9.2 Tensionless Open String

The open string has the end points. Therefore we have to investigate the boundary conditions. For the simplicity, we consider the bosonic case.

[2]We use the following results: $\Pi_\tau^+ = 1$, $\Pi_\tau^- = \dot{X}^- + \frac{i}{2p_-}\theta\dot{\theta}$, $\Pi_\tau^2 = \dot{X}$ and $\Pi_\sigma^+ = 0$, $\Pi_\sigma^- = (X^-)' + \frac{i}{2p_-}\theta\theta'$, $\Pi_\sigma^2 = X'$.

In order to compare the usual tensionful string, firstly, we consider the next action with tension, T.

$$S = \int d\tau \int_0^1 d\sigma \Big[\dot{\mathbf{X}}^\mu \mathbf{P}_\mu - \frac{V}{2}\big(\mathbf{P}^2 + T^2(\mathbf{X}')^2\big) - U(\mathbf{X}^\mu)'\mathbf{P}_\mu \Big], \tag{9.17}$$

where $\sigma \in [0,1]$. The boundary term of the variation of this action is

$$\delta_b S = \int d\tau \int_0^1 d\sigma \Big[-VT^2 \partial_\sigma \mathbf{X}_\mu \delta \mathbf{X}^\mu - U\mathbf{P}_\mu \delta \mathbf{X}^\mu \Big]. \tag{9.18}$$

The boundary values of U is chosen as $U|_{\sigma=0} = U|_{\sigma=1} = 0$, which are consistent with the field equations in the light-cone guage. Therefore the second term of the r.h.s. in (9.18) vanishes regardless of tensionful or tensionless.
In the tensionful case, the first term of the r.h.s. in (9.18) requires the Neumann boundary condition, $\partial_\sigma \mathbf{X}^\mu|_{\sigma=0} = \partial_\sigma \mathbf{X}^\mu|_{\sigma=1} = 0$, the Dirichlet boundary condition, $\mathbf{X}^\mu|_{\sigma=0} = \mathbf{X}^\mu|_{\sigma=1} = 0$, or the periodic boundary condition. For the open string theory with the Poincaré symmetry, the Neumann boundary condition is chosen. In the case of Neumann boundary condition, we can expand $\mathbf{X}^\mu(\sigma)$ by the cosine function.
On the other hand, in the tensionless case, we do not need the boundary condition for $\mathbf{X}^\mu(\sigma)$. Therefore there is no evidence to decompose $\mathbf{X}^\mu(\sigma)$ into modes. Unless $\mathbf{X}^\mu(\sigma)$ is expanded into some modes, we cannot use most of techniques in this thesis.

To compare with the tensionful case, we assume the Neumann boundary condition for the tensionless open string. In the light-cone guage, we obtain the next expansions.

$$X^I(\sigma) = x^I + 2\sum_{n>0} X_n^I \cos(\pi\sigma) = \sum_n X_n^I e^{in\pi\sigma} \ , \quad X_{-n}^I = X_n^I \tag{9.19}$$

$$P^I(\sigma) = p^I + 2\sum_{n>0} P_n^I \cos(\pi\sigma) = \sum_n P_n^I e^{in\pi\sigma} \ , \quad P_{-n}^I = P_n^I, \tag{9.20}$$

where the eigenmode is cosine. Because fields are expanded by cosine, the delta function is represented as $\delta(\sigma-\sigma') = 1 + 2\sum_{n>0}\cos(n\pi\sigma)\cos(n\pi\sigma') = 1 + \sum_{n>0}\big[\cos(n\pi(\sigma+\sigma')) + \cos(n\pi(\sigma+\sigma'))\big]$. From this, the quantization condition for modes is

$$[X_n^I, P_m^J] = \frac{i}{2}\delta^{I,J}(\delta_{n,-m} + \delta_{n,m}) \tag{9.21}$$

In the tensionful case, we combine X-modes and P-modes to make $\alpha_n^I = -i\frac{\omega}{\sqrt{2}} n X_n^I + \frac{1}{\sqrt{2}\omega} P_n^I$, for $T = \omega^2$. α-modes satisfy the usual commutation relation of a harmonic oscillator, $[\alpha_n^I, \alpha_m^J] = n\delta_{n,-m}\delta^{I,J}$. Therefore we can make use of the results in the tensionful closed string theory.
However, in the tensionless case, as seen in chapter 3, such combination is not useful because the mass square operator is written by P-modes. It is natural to use X-modes and P-modes, which satisfy (9.21), as the fundamental modes. We find that most of the commutators in the spacetime symmetry is correct even in the case of (9.21). However the dangerous commutator, $[\mathcal{J}^{-I}, \mathcal{K}^-]$, is greatly different. The structure of this commutator is

$$[\mathcal{J}^-, \ \tilde{\mathcal{K}}^-] = i(\text{ordered classical part}) + iC'\frac{p}{p_-^2} + \frac{i}{p_-^2}\sum_{n>0} C'_{(n)} P_{2n}. \tag{9.22}$$

The linear term of P-even modes in the last part is a different point from the closed case. Although we omit the detail calculation here, $C'_{(n)}$ contains divergences of $\sum_{n>0} 1$-type and $\sum_{n>0} n^2$-type roughly. Because of the independence of two divergences, we cannot renormalize both by the one ordering ambiguity. Therefore extra terms in (9.22) cause the anomaly even in the Hermitian R-order.
For further investigation, we may need other methods without the mode expansion. However, because it is beyond this thesis, we do not discuss anymore. It is the future work.

Chapter 10

Summary and Outlook

In this thesis based on [1], we have studied the tensionless closed bosonic string theory in the light-cone gauge in various dimensions and operator orders.
The main products of our study in this thesis are as follows:

- Construction of non-separable mass eigenfunctions in (Hermitian) R-order
- Avoidance of spacetime conformal anomaly in Hermitian R-order
- Concrete calculation of spacetime conformal anomaly in XP-normal order

The first product has been obtained in chapter 4 and 5. There, in the R-order, we have constructed the non-separable eigenfunctions of the mass square operator, $\mathcal{M}^2$, as well as the separable ones.
We obtain the separable mass eigenfunctions relatively easily because of the factorization into eigenfunctions of the ordinary differential equation for each Fourier-mode in (4.10) or (5.9). On the other hand, in general, it is difficult to find the non-separable mass eigenfunction because we must solve the (infinite dimensional) partial differential equation. However, in three dimensions, we have constructed the non-separable mass eigenfunctions by acting another Poincaré invariance, Λ, on the separable mass eigenfunctions. There it is essential that $\mathcal{M}^2$ and Λ commute mutually. Similarly, in the higher dimensions, the non-separable mass eigenfunctions are constructed by acting $\bar{\Lambda}^I$, which commutes with $\mathcal{M}^2$, on the separable eigenfunctions. Furthermore, for the massless states, we have given the additional characterization.
The investigation of such non-separable mass eigenfunctions is very important to extract the information of the helicity.

The second product has been obtained mainly from chapter 6, 7 and 8. In chapter 6, we have considered the structure of (dangerous) commutators in each operator order. By using the information about the structure of dangerous commutators considered in chapter 6, we have calculated the dangerous commutators of the spacetime conformal symmetry in the Hermitian R-order, where the mode flipping symmetry corresponding to the world sheet parity is preserved. Against the critical dimension of the conformal string in the BRST formalism or the R-ordered calculation in the light-cone gauge without Fourier expansion, we have found that there is no spacetime conformal anomaly unrelated to the regularization in the Hermitian R-order (and the Weyl order), by the calculation with Fourier expansion. Furthermore, in appendix, we have calculated the dangerous commutators in the Hermitian R-order in three dimensions with the cut-off regularization

to obtain the set of the desirable commutation relations for the spacetime conformal symmetry in the limit of the cut-off scale, $N \to \infty$.
The results obtained is a new example for the differences between the results in the BSRT formalism and in the light-cone gauge quantization, such as the critical dimension of the tensionful string theory in the BRST quantization formalism and the avoidance of the Lorentz anomaly in the 3-dim. light-cone gauge quantization. Such differences between the results in the BSRT formalism and in the light-cone gauge quantization are strange and interesting and then need the further studies.

The third product has been obtained mainly from chapter 6, 7 and 8. By using the information about the structure of dangerous commutators considered in chapter 6, we have calculated concretely the dangerous commutators in three and higher dimensional spacetime conformal symmetry in the XP-normal order, where the string ground state breaks the mode flipping symmetry. Then we have verified the existence of the spacetime conformal anomaly for the tensionless string theory in the XP-normal order. Furthermore we have found that the tensionless string theory in the XP-normal order becomes point-like when we interpret the anomalous commutators as additional constraints to avoid the anomaly. It is similar to other works [38, 40, 49].

The outlooks for the future works are roughly divided as follows:

- Further study of mass spectrum in R-order
- Other types of tensionless string in light-cone gauge
- Reproduction of results in 3-dim. theories in light-cone gauge by a covariant method or Reason for differences between results in BSRT and in light-cone gauge
- Interaction of tensionless string
- Relation between tensionless string theory and higher spin gauge theory
- Application of duality

Firstly, we must study more the mass spectrum of the tensionless string in the R-order, especially the massless spectrum. In this thesis as well as other studies, we consider the eigenfunctions only of the mass square operator in the tensionless string theory. However there is another Poincaré invariance corresponding to spin, Λ, in three dimensions. Although we have used it to construct the non-separable mass eigenfunctions, it is possible in principle to diagonalize simultaneously $\mathcal{M}^2$ and Λ. It is interesting to find states with various helicities like the case of the 3-dim. tensionful string in the light-con gauge [50–53, 63–65].

Next, as considered in the last chapter, other types of the tensionless string in the light-cone gauge are interesting. The spacetime conformal anomalies of these theories in the light-cone gauge are not satisfyingly investigated yet, against many studies of the Lorentz anomaly [34, 36, 37].

Next, it is important to reproduce the results obtained in the light-cone gauge by some covariant method. At present, the special results in three dimensional theories comes only from the light-cone gauge quantization. It is not understood well whether the BSRT formalism and the light-cone gauge quantization give really the same physics. Therefore it is important to reproduce a result obtained in one method by other method(s). If we cannot reproduce, we must consider its reason.

In this thesis, we have not consider the interaction between tensionless strings. Although there are abundant spectrum even in a single string, there will be new interesting feature or some restriction in the interacting theory. Then the interacting theory probably tell us the relation between tensionless string theories and higher spin gauge theories.

Finally we hope that the further study of the tensionless string leads to the applications of the duality and the understanding of the relation between tensionless string theories and higher spin gauge theories and then contributes to the development of the elementary particle physics and other fields of physics.

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Appendix A

Generators and Commutation Relations of Spacetime Symmetry

A classical string has the Poincaré symmetry as the global symmetry. We expect that a quantum string also has the Poincaré symmetry. However the quantization of the string can induce the Lorentz anomaly. Then we choose the critical dimension and the ordering constant properly to avoid the anomaly.
Similarly we expect that a quantum tensionless string to have the space-time conformal symmetry.
In this appendix, we give the commutation relations for the Poincaré group and the spacetime conformal group in the light-cone base as well as the orthogonal base. Then we indicate dangerous commutators, which can induce anomalies for string theories in light-cone gauge.

A.1 Poincaré Group

Here we give commutation relations of the Poincaré group in general dimensions D and three dimensions.

The generators of the Poincaré group are the translations, $\mathcal{P}_\mu$, and the Lorentz transformation, $\mathcal{J}^{\mu\nu}$, such that

$$\begin{aligned}
&[\mathcal{P}^\mu, \mathcal{P}^\nu] = 0 \ , \quad [\mathcal{P}^\mu, \mathcal{J}^{\rho\sigma}] = i\left(\eta^{\mu\sigma}\mathcal{P}^\rho - \eta^{\mu\rho}\mathcal{P}^\sigma\right), \\
&[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i\left(\eta^{\nu\sigma}\mathcal{J}^{\mu\rho} - \eta^{\nu\rho}\mathcal{J}^{\mu\sigma} - \eta^{\mu\sigma}\mathcal{J}^{\nu\rho} + \eta^{\mu\rho}\mathcal{J}^{\nu\sigma}\right).
\end{aligned} \tag{A.1}$$

In light-cone coordinate we find the following commutation relations:

$$\begin{aligned}
&[\mathcal{P}^\pm, \mathcal{J}^{+-}] = \pm i\mathcal{P}^\pm \ , \quad [\mathcal{P}^\pm, \mathcal{J}^{\mp I}] = -i\mathcal{P}_I, \\
&[\mathcal{P}^I, \mathcal{J}^{\pm J}] = i\delta^{IJ}\mathcal{P}^\pm \ , \quad [\mathcal{P}^I, \mathcal{J}^{JK}] = i\left(\delta^{IK}\mathcal{P}^J - \delta^{IJ}\mathcal{P}^K\right), \\
&[\mathcal{J}^{+-}, \mathcal{J}^{\pm I}] = \mp i\mathcal{J}^{\pm I} \ , \quad [\mathcal{J}^{+I}, \mathcal{J}^{-K}] = i\left(\mathcal{J}^{IK} + \delta^{IK}\mathcal{J}^{+-}\right), \\
&[\mathcal{J}^{\pm I}, \mathcal{J}^{KL}] = i\left(\delta^{IL}\mathcal{J}^{\pm K} - \delta^{IK}\mathcal{J}^{\pm L}\right), \\
&[\mathcal{J}^{IJ}, \mathcal{J}^{KL}] = i\left(\delta^{JL}\mathcal{J}^{IK} - \delta^{JK}\mathcal{J}^{IL} - \delta^{IL}\mathcal{J}^{JK} + \delta^{IK}\mathcal{J}^{JL}\right),
\end{aligned} \tag{A.2}$$

and other commutators vanish. Here $I, J, K, L = 2, \cdots, D-1$ and we used $\eta_{+-} = \eta^{+-} = 1$ and $\eta_{IJ} = \delta_{IJ}$.

The infamous dangerous commutator for quantum strings in the light-cone gauge is

$$[\mathcal{J}^{-I}, \mathcal{J}^{-J}] \stackrel{?}{=} 0. \tag{A.3}$$

This is not always zero for a quantum tensionful string in $D > 3$. To avoid the anomaly, we must choose $D = 26$ and determine the operator ordering constant [27,28].
For a quantum tensionless string in $D > 3$ and in the Reference order,[1] this commutator is proportional to the constraint [36]. Therefore we can avoid the Lorentz anomaly.
For a quantum tensionless string in $D > 3$ and in the XP-normal order,[2] we obtain $D = 26$ and the definite ordering constant [36].
In $D \leq 3$, this commutator for both tensionful and tensionless strings is trivially zero.

$D = 3$

In three dimensions Lorentz generators are rewritten as the vector, $\mathcal{J}^{\pm} = \pm\mathcal{J}^{\pm 2}$, $\mathcal{J} = -\mathcal{J}^{+-}$. Hence the commutation relations in (A.1) become simple as follows:

$$[\mathcal{P}^{\mu}, \mathcal{P}^{\nu}] = 0\ , \quad [\mathcal{J}^{\mu}, \mathcal{P}^{\nu}] = i\epsilon^{\mu\nu\rho}\mathcal{P}_{\rho}, \quad [\mathcal{J}^{\mu}, \mathcal{J}^{\nu}] = i\epsilon^{\mu\nu\rho}\mathcal{J}_{\rho}. \tag{A.4}$$

Furthermore we can make two Poincaré invariants which commute with each other,

$$\begin{aligned} M^2 &= -\mathcal{P}^2, \\ \Lambda &= \mathcal{J}^{\mu}\mathcal{P}_{\mu}. \end{aligned} \tag{A.5}$$

Unitary irreducible representations of Poincaré group are labeled by the value of these two Casimirs and irereducible representations with $M^2 \geq 0$ are only physical. When $M^2 > 0$ we define the relativistic helicity by

$$s = \frac{\Lambda}{M}. \tag{A.6}$$

Furthermore we call $|s|$ "spin". If the Lorentz group is $SO(1,2)$, its double cover $SL(2;\mathbb{R})$ or its universal cover, s is respectively integer, half-integer or real number including the rational and the irrational number [54].

In the light-cone base, the commutation relations of the 3-dim. Poincaré group are

$$\begin{aligned} &[\mathcal{J}^{\pm}, \mathcal{P}^{\mp}] = \pm i\mathcal{P}\ , \quad [\mathcal{J}, \mathcal{P}^{\pm}] = \pm i\mathcal{P}^{\pm}\ , \quad [\mathcal{J}^{\pm}, \mathcal{P}] = \mp i\mathcal{P}^{\pm}, \\ &[\mathcal{J}, \mathcal{J}^{\pm}] = \pm i\mathcal{J}^{\pm}\ , \quad [\mathcal{J}^{+}, \mathcal{J}^{-}] = i\mathcal{J}, \end{aligned} \tag{A.7}$$

and other commutators vanish. In three dimensions $[\mathcal{J}^{-}, \mathcal{J}^{-}]$ corresponding to a dangerous commutator in (A.3) vanishes trivially because there is only one transverse direction in three dimensions [**?**,50–52]. Therefore the 3-dim. string theories in the light-cone gauge have no Lorentz anomaly and preserves the Poincaré symmetry.

[1] In the Reference order, all momentum modes are to the right of coordinate modes.

[2] In the XP-normal order, all positive modes of coordinates and momentums are to the right of negative ones.

A.2 Conformal Group

First we give commutation relations of the spacetime conformal group in general dimensions and next in three dimensions.
The generators of the space-time conformal group are the dilatation, $\mathcal{D}$, and the special conformal transformation, $\mathcal{K}^\mu$, in addition to $\mathcal{P}_\mu$ and $\mathcal{J}^{\mu\nu}$. Additional commutation relations are

$$\begin{aligned}
&[\mathcal{D},\mathcal{P}^\mu]=i\mathcal{P}^\mu\ ,\quad [\mathcal{D},\mathcal{J}^{\mu\nu}]=0\ ,\quad [\mathcal{D},\mathcal{K}^\mu]=-i\mathcal{K}^\mu\ ,\quad [\mathcal{K}^\mu,\mathcal{K}^\nu]=0,\\
&[\mathcal{K}^\mu,\mathcal{P}^\nu]=i\left(\eta^{\mu\nu}\mathcal{D}+\mathcal{J}^{\mu\nu}\right)\ ,\quad [\mathcal{K}^\mu,\mathcal{J}^{\rho\sigma}]=i\left(\eta^{\mu\sigma}\mathcal{K}^\rho-\eta^{\mu\rho}\mathcal{K}^\sigma\right).
\end{aligned}\tag{A.8}$$

In light-cone base,

$$\begin{aligned}
&[\mathcal{D},\mathcal{P}^\pm]=i\mathcal{P}^\pm\ ,\quad [\mathcal{D},\mathcal{P}^I]=i\mathcal{P}^I\ ,\quad [\mathcal{D},\mathcal{K}^\pm]=-i\mathcal{K}^\pm\ ,\quad [\mathcal{D},\mathcal{K}^I]=-i\mathcal{K}^I,\\
&[\mathcal{K}^\pm,\mathcal{P}^\mp]=i\left(\mathcal{D}\pm\mathcal{J}^{+-}\right)\ ,\quad [\mathcal{K}^\pm,\mathcal{P}^I]=-[\mathcal{K}^I,\mathcal{P}^\pm]=i\mathcal{J}^{\pm I},\\
&[\mathcal{K}^I,\mathcal{P}^J]=i\left(\delta^{IJ}D+\mathcal{J}^{IJ}\right),\\
&[\mathcal{K}^\pm,\mathcal{J}^{+-}]=\pm i\mathcal{K}^\pm\ ,\quad [\mathcal{K}^\pm,\mathcal{J}^{\mp I}]=-i\mathcal{K}^I\ ,\quad [\mathcal{K}^I,\mathcal{J}^{\pm J}]=i\delta^{IJ}\mathcal{K}^\pm,\\
&[\mathcal{K}^I,\mathcal{J}^{JK}]=i\left(\delta^{IL}\mathcal{K}^K-\delta^{IK}\mathcal{K}^L\right),
\end{aligned}\tag{A.9}$$

and other commutators vanish.
The quantum tensionless string theory in general dimensions D have a dangerous commutator as well as the case of the Lorentz anomaly for quantum tensionful string theories. The first dangerous commutator in the tensionless string theory is

$$[\mathcal{K}^I,\mathcal{J}^{-K}]\overset{?}{=}i\delta^{IK}\mathcal{K}^-.\tag{A.10}$$

The right hand side of this have the traceless part with respect to I,J in a certain operator order. The difference of trace part is absorbed in the redefinition of $\mathcal{K}^-$. However the traceless part can become anomalous in some operator orders [38, 40]. Moreover this anomaly produces more anomalies from commutators of $\mathcal{K}^-$ and other generators. The other dangerous commutators are $[\mathcal{J}^{-I},\mathcal{K}^-]$, $[\mathcal{K}^+,\mathcal{K}^-]$ and $[\mathcal{K}^I,\mathcal{K}^-]$.
If there is no traceless part in the right hand side of (A.10), we use $\mathcal{K}^-=i[\mathcal{J}^{-J},\mathcal{K}^J]$ for $J\neq I$ and the Jacobi identity to find that these dangerous commutators are not anomalous.

$D=3$

In three dimensions the commutation relations (A.8) become simple a little as follows:

$$\begin{aligned}
&[\mathcal{D},\mathcal{P}^\mu]=i\mathcal{P}^\mu\ ,\quad [\mathcal{D},\mathcal{J}^{\mu}]=0\ ,\quad [\mathcal{D},\mathcal{K}^\mu]=-i\mathcal{K}^\mu\ ,\quad [\mathcal{K}^\mu,\mathcal{K}^\nu]=0,\\
&[\mathcal{K}^\mu,\mathcal{P}^\nu]=i\left(\eta^{\mu\nu}D-\epsilon^{\mu\nu\rho}\mathcal{J}_\rho\right)\ ,\quad [\mathcal{K}^\mu,\mathcal{J}^{\nu}]=i\epsilon^{\mu\nu\rho}\mathcal{K}_\rho.
\end{aligned}\tag{A.11}$$

In light-cone base,

$$\begin{aligned}
&[\mathcal{D},\mathcal{P}^\pm]=i\mathcal{P}^\pm\ ,\quad [\mathcal{D},\mathcal{P}]=i\mathcal{P}\ ,\quad [\mathcal{D},\mathcal{K}^\pm]=-i\mathcal{K}^\pm\ ,\quad [\mathcal{D},\mathcal{K}]=-i\mathcal{K},\\
&[\mathcal{K}^\pm,\mathcal{P}^\mp]=i\left(\mathcal{D}\mp\mathcal{J}\right)\ ,\quad [\mathcal{K}^\pm,\mathcal{P}]=-[\mathcal{K},\mathcal{P}^\pm]=\pm i\mathcal{J}^\pm\ ,\quad [\mathcal{K},\mathcal{P}]=i\mathcal{D},\\
&[\mathcal{K}^\pm,\mathcal{J}^\mp]=\pm i\mathcal{K}\ ,\quad [\mathcal{K}^\pm,\mathcal{J}]=\mp i\mathcal{K}^\pm\ ,\quad [\mathcal{K},\mathcal{J}^\pm]=\pm i\mathcal{K}^\pm,
\end{aligned}\tag{A.12}$$

and other commutators vanish.
In three dimensions the commutator $[\mathcal{K}, \mathcal{J}^-]$, corresponding to a dangerous commutator in (A.10), has no traceless part [1]. This commutator give us the redefinition of $\mathcal{K}^-$ as

$$\tilde{\mathcal{K}}^- \equiv -i[\mathcal{J}^-, \mathcal{K}]. \tag{A.13}$$

Therefore there is no anomaly in (A.10).
However, in the calculation of commutators, we must use $\tilde{\mathcal{K}}^-$ instead of $\mathcal{K}^-$. We can check easily that most of commutators in the spacetime conformal symmetry are correct. However there are the following dangerous commutators [1].

$$\begin{aligned} [\mathcal{J}^-, \hat{\mathcal{K}}^-] &\stackrel{?}{=} 0 \ , \\ [\mathcal{K}, \hat{\mathcal{K}}^-] &\stackrel{?}{=} 0. \end{aligned} \tag{A.14}$$

The check of the first relation in (A.14) is one of the main contents in this thesis. Though, the calculation of this is very complicated and lengthy.
If the first relation in (2.14) is correct, we use the Jacobi identity to find that the second relation is not anomalous.

$$\begin{aligned} [\mathcal{K}, \hat{\mathcal{K}}^-] &= [i[\mathcal{J}^-, \mathcal{K}^+],\ \hat{\mathcal{K}}^-] \\ &= -i[[\hat{\mathcal{K}}^-, \mathcal{J}^-],\ \mathcal{K}^+] - i[[\mathcal{K}^+, \hat{\mathcal{K}}^-],\ \mathcal{J}^-] \\ &= i[[\mathcal{J}^-, \hat{\mathcal{K}}^-],\ \mathcal{K}^+] \to 0. \end{aligned} \tag{A.15}$$

A.3 Dangerous Commutators

Here we summarize dangerous commutators which we must calculate in $D = 3$ and $D > 3$ on the next table.

Dangerous commutators which we must calculate

Spacetime dimension	Lorentz anomaly	Spacetime conformal anomaly
$D = 3$	nothing	$\tilde{\mathcal{K}}^- \equiv -i[\mathcal{J}^-, \mathcal{K}]$, $[\mathcal{J}^-, \hat{\mathcal{K}}^-]$
$D > 3$	$[\mathcal{J}^{-I}, \mathcal{J}^{-J}]$	$[\mathcal{J}^{-I}, \mathcal{K}^K]$

Appendix B

List of Generators and Commutators

In this appendix the generators of the spacetime conformal symmetry and some important commutation relations in the tensionless string theory are collected in each dimension and operator order. If necessary, the list in this appendix should be referred.

B.1 R-order Case

B.1.1 Generators

Here we represent the generators of the spacetime conformal symmetry in general dimensions and in the Reference order, called R-order for short. Then we give the alternative representation of generators in three dimensions and important operators.
The R-ordered generators of the Poincaré symmetry are

$$\begin{aligned}
&\mathcal{P}_{-,R} = p_- \ , \quad \mathcal{P}_{I,R} = p_I \ , \quad \mathcal{P}_{+,R} = p_+ = -\frac{L_0}{2p_-}, \\
&\mathcal{J}_R^{+I} = -x^I p_- \ , \quad \mathcal{J}_R^{+-} = -x^- p_- \ , \quad \mathcal{J}_R^{IJ} = \sum_n (X_n^I P_{-n}^J - X_n^J P_{-n}^I), \\
&\mathcal{J}_R^{-I} = x^- p^I + \frac{1}{2p_-}\sum_n X_n^I L_{-n} - \frac{i}{p_-}\sum_{n\neq 0}\frac{1}{n} M_n P_{-n}^I,
\end{aligned} \tag{B.1}$$

and the additional R-ordered generators of the spacetime conformal symmetry are

$$\begin{aligned}
&\mathcal{D}_R = x^- p_- + \sum_n X_n \cdot P_{-n}, \\
&\mathcal{K}_R^+ = -\frac{1}{2}\sum_n X_n \cdot X_{-n} p_-, \\
&\mathcal{K}_R^I = x^I x^- p_- + \sum_n \sum_m \left[X_n^I X_m^J P_{J,-n-m} - \frac{1}{2} X_n^J X_{J,m} P_{-n-m}^I\right] + i\sum_{m\neq 0}\frac{1}{m} X_m^I M_{-m}, \\
&\mathcal{K}_R^- = x^- x^- p_- + x^- \sum_n X_n P_{-n} \\
&\qquad + \frac{1}{4p_-}\sum_n\sum_m X_n^J X_{J,m} L_{-n-m} - \frac{i}{p_-}\sum_n\sum_{m\neq 0}\frac{n+m}{m^2} X^J M_m P_{J,-n-m}.
\end{aligned} \tag{B.2}$$

Here we used

$$L_n = \sum_m P_m \cdot P_{n-m} \ , \ M_n = -i \sum_m m X_m \cdot P_{n-m}. \tag{B.3}$$

From the zero mode of above operators, we find the following important operators.

$$\text{Mass} \ : \ \mathcal{M}^2 \equiv -\mathcal{P}^2 = L_0 - p \cdot p = \sum_{m \neq 0} P_m \cdot P_{-m}, \tag{B.4}$$

$$\text{Constraint} \ : \ M_0 = -i \sum_m m X_m \cdot P_{-m}. \tag{B.5}$$

Because many R-ordered generators in (B.2) are not hermitian, we must *hermitianize* them as

$$\mathcal{G} = \frac{1}{2}(\mathcal{G}_R + h.c.). \tag{B.6}$$

In the R-order or the Weyl order, a hermitian generator in (B.6) must be reordered correctly. On the other hand, in the Hermitian R-order, we use the combination of (B.6) as a generator.

Specialty of $D = 3$

In $D = 3$, we represent Lorentz generators as vector:

$$\begin{aligned} &\mathcal{J}_R^+ = \mathcal{J}_R^{+2} \ , \ \mathcal{J}_R = -\mathcal{J}_R^{+-}, \\ &\mathcal{J}_R^- = -\mathcal{J}_R^{-2} = -x^- p - \frac{1}{2p_-} \sum_n X_n L_{-n} + \frac{i}{p_-} \sum_{n \neq 0} \frac{1}{n} M_n P_{-n}, \end{aligned} \tag{B.7}$$

where $\mathcal{J}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}\mathcal{J}_{\nu\rho}$, $\epsilon^{+-2} = 1$ and $\eta^{+-} = 1$. Furthermore the generator of the next special conformal translation is simpler than that of $D > 3$.

$$\mathcal{K}_R = \mathcal{K}_R^{I=2} = x x^- p_- + \frac{1}{2} \sum_n \sum_m X_n X_m P_{-n-m} + i \sum_{m \neq 0} \frac{1}{m} X_m M_{-m} \tag{B.8}$$

The important Poincaré invariant corresponding to "spin" in three dimensions is represented as

$$\Lambda \equiv \mathcal{J}^\mu \mathcal{P}_\mu = \sum_{n \neq 0} \Big[-\frac{1}{2} X_n L_{-n} + \frac{i}{n} M_n P_{-n} \Big]. \tag{B.9}$$

B.1.2 Useful commutation relations

The useful commutation relations in general dimensions are

$$\begin{aligned} &[X_n^I, L_m] = 2i P_{I,n+m} \ , \ [P_{I,n}, L_m] = 0 \ , \\ &[X_n^I, M_m] = (n+m) X_{n+m}^I \ , \ [P_{I,n}, M_m] = n P_{I,n+m} \ , \\ &[L_n, L_m] = 0 \ , \ [L_n, M_m] = (n-m) L_{n+m} \ , \ [M_n, M_m] = (n-m) M_{n+m}. \end{aligned} \tag{B.10}$$

and the important relation for arbitrary operators $\mathcal{O}$ taken the average with respect to string coordinate σ is

$$[M_0, \mathcal{O}] = 0. \tag{B.11}$$

Specialty of $D = 3$

The important commutation relation of Λ are

$$\begin{aligned}&[\Lambda, x] = [\Lambda, p] = 0\ ,\\ &[\Lambda, \mathcal{M}^2] = 0.\end{aligned} \tag{B.12}$$

From the first line of this, we find that Λ has no zero mode, x and p. And from the second line, we find that Λ commutes with another Poincaré invariant, $\mathcal{M}^2$.

B.2 XP-normal Order Case

B.2.1 Generators

Here we represent the generators of the spacetime conformal symmetry in general dimensions and in the XP-normal order. Then we give the alternative representation of generators in three dimensions and important operators.
In the XP-normal order, the non-zero mode part of generators are hermitian automatically, but the zero mode part is not. Therefore we *hermitianize* terms with zero modes.

The XP-normal ordered generators of the Poincaré symmetry are

$$\begin{aligned}
&\mathcal{P}_{-,N} = p_- \ , \quad \mathcal{P}_{I,N} = p_I \ , \quad \mathcal{P}_{+,N} = p_+ = -\frac{L_0}{2p_-},\\
&\mathcal{J}_N^{+I} = -x^I p_- \ , \quad \mathcal{J}_N^{+-} = -x^- p_- + \frac{i}{2},\\
&\mathcal{J}_N^{IJ} = x^I p^J - x^J p^I + \sum_{n=1}^{\infty}\Big[(X_{-n}^I P_n^J - X_{-n}^J P_n^I) + h.c.\Big],\\
&\mathcal{J}_N^{-I} = x^- p^I + \Big(\frac{1}{2p_-}x^I L_0 - \frac{i}{2p_-}p^I\Big) + \sum_{l=1}^{\infty}\big[(X_{-l}^I P_l^J - X_{-l}^J P_l^I) + h.c.\big]\frac{p_J}{p_-} - \frac{1}{p_-}\bar{\Lambda}^I,
\end{aligned} \tag{B.13}$$

and the additional XP-normal ordered generators of the spacetime conformal symmetry[1]

$$\begin{aligned}
&\mathcal{D}_N = x^- p_- + x\cdot p + \sum_{n=1}^{\infty}(X_{-n}\cdot P_n + P_{-n}\cdot X_n) - i\frac{D-1}{2},\\
&\mathcal{K}_N^+ = -\frac{1}{2}(x\cdot x)p_- - \sum_{n=1}^{\infty} X_{-n}\cdot X_n p_-,\\
&\mathcal{K}_N^K = x^K x^- p_- + x^K(x\cdot p) - \frac{1}{2}(x\cdot x)p^K - i\frac{D-1}{2}x^K\\
&\quad + x^K\sum_{n=1}^{\infty}(X_{-n}\cdot P_n + h.c.) - p^K\sum_{n=1}^{\infty}X_{-n}\cdot X_n + x_L\sum_{n=1}^{\infty}\Big((X_{-n}^K P_n^L - X_{-n}^L P_n^K) + h.c.\Big)\\
&\quad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[\Big(X_{-n}^K(X_{-m}\cdot P_{n+m}) + X_{-n}^K(P_{-m}\cdot X_{n+m}) + X_{-n-m}^K(P_m\cdot X_n)\Big) + h.c.\Big]\\
&\quad - \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[\Big((X_{-n}\cdot X_{-m})P_{n+m}^K + 2(X_{-n-m}\cdot X_m)P_n^K\Big) + h.c.\Big] - i\sum_{n=1}^{\infty}\frac{1}{n}(X_{-n}^K\bar{M}_n - h.c.),
\end{aligned} \tag{B.14}$$

[1] Here we omit $\mathcal{K}_N^-$ which is not used in $D > 3$. $\mathcal{K}_N^-$ in three dimensions is given subsequently.

where barred operators are defined by removing of x- and p-zero mode from unbarred ones, and Λ^I is

$$\bar{\Lambda}^I \equiv \sum_{l=1}^{\infty}\left[-\frac{1}{2}(X^I_{-l}\bar{L}_l + \bar{L}_{-l}X^I_l) + \frac{i}{l}(P^I_{-l}\bar{M}_l - \bar{M}_{-l}P^I_l)\right]. \tag{B.15}$$

These barred operators are

$$\begin{aligned}
\bar{L}_n &= \sum_{l=1}^{n-1} P_l \cdot P_{n-l} + 2\sum_{l=1}^{\infty} P_{-l}\cdot P_{n+l},\\
\bar{M}_n &= -i\sum_{l=1}^{n-1} lX_l\cdot P_{n-l} - i\sum_{l=1}^{\infty}\big((n+l)P_{-l}\cdot X_{n+l} - lX_{-l}\cdot P_{n+l}\big),\\
\bar{L}_{-n} &= (\bar{L}_n)^\dagger\ ,\ \ \bar{M}_{-n} = (\bar{M}_n)^\dagger.
\end{aligned} \tag{B.16}$$

for $n \geq 0$ and from zero mode of these operators we obtain

$$\mathcal{M}^2 = \bar{L}_0 = 2\sum_{m=1}^{\infty} P_{-m}\cdot P_m, \tag{B.17}$$

$$\bar{M}_0 = i\sum_{m=1}^{\infty} m(X_{-m}\cdot P_m - P_{-m}\cdot X_m) = M_0 \tag{B.18}$$

Specialty of $D = 3$

In $D = 3$, we represent Lorentz generators as vector:

$$\begin{aligned}
&\mathcal{J}^+_N = \mathcal{J}^{+2}_N\ ,\ \ \mathcal{J}_N = -\mathcal{J}^{+-}_N,\\
&\mathcal{J}^-_N = -\mathcal{J}^{-2}_N = -x^- p - \frac{1}{2p_-}xL_0 + \frac{i}{2p_-}p + \frac{\Lambda}{p_-}
\end{aligned} \tag{B.19}$$

where $\mathcal{J}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}\mathcal{J}_{\nu\rho}$, $\epsilon^{+-2} = 1$ and $\eta^{+-} = 1$.
And generators of the special conformal translation, $\mathcal{K}_N$ and $\mathcal{K}^-$ are

$$\begin{aligned}
\mathcal{K}_N =& xx^-p_- + \frac{1}{2}xxp - ix + x\sum_{n=1}^{\infty}(X_{-n}P_n + P_{-n}X_n) - \sum_{n=1}^{\infty}X_{-n}X_np\\
&+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[(X_{-n}X_{-m}P_{n+m} + 2X_{-n}P_{-m}X_{n+m}) + h.c.\Big] - i\sum_{m=1}^{\infty}\frac{1}{m}(X_{-m}\bar{M}_m - h.c.),
\end{aligned} \tag{B.20}$$

$$\begin{aligned}
\mathcal{K}^-_N =& x^-\mathcal{D}_N - \frac{i}{2}x^- + \frac{1}{8p_-}(xxL_0 + L_0xx) - \frac{1}{p_-}x\Lambda\\
&+\frac{1}{2p_-}\sum_{n=1}^{\infty}X_{-n}L_0X_n + \frac{1}{4p_-}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[(X_{-n}X_{-m}L_{n+m} + 2X_{-n}L_{-m}X_{n+m}) + h.c.\Big]\\
&+\frac{i}{p_-}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[\Big(\frac{n+m}{m^2}X_{-n}M_{-m}P_{n+m} + \frac{n}{m^2}X_{-n-m}M_mP_n - \frac{m}{(n+m)^2}X_{-n}P_{-m}M_{n+m}\Big) + h.c.\Big].
\end{aligned} \tag{B.21}$$

where we separated x in $\mathcal{K}^-_N$, but not p because $\mathcal{K}^-_N$ is too lengthy.
In three dimensions, because the helicity operator have originally no zero mode, $\Lambda = \bar{\Lambda}$.

B.2.2 Useful commutation relations

The useful commutation relations in general dimensions are

$$
\begin{aligned}
&[X^I_n, \bar{L}_m] = 2iP_{I,n+m}\ ,\ [P_{I,n}, \bar{L}_m] = 0,\\
&[X^I_n, \bar{M}_m] = (n+m)X^I_{n+m}\ ,\ [P_{I,n}, \bar{M}_m] = nP_{I,n+m}\ \ \text{for } n+m \neq 0,\\
&[X^I_n, \bar{M}_{-n}] = [P^I_n, \bar{M}_{-n}] = 0,
\end{aligned}
\tag{B.22}
$$

$$
\begin{aligned}
&[\bar{L}_n, \bar{L}_m] = 0\ ,\ [\bar{L}_n, \bar{M}_m] = (n-m)\bar{L}_{n+m}\ ,\\
&[\bar{M}_n, \bar{M}_m] = (n-m)\bar{M}_{n+m} + \frac{D-2}{6}(n^3-n)\delta_{n,-m}.
\end{aligned}
\tag{B.23}
$$

Note that the commutation relation in the last line has the central extension term. Because L_n and M_n have no x-zero mode, barred ones satisfy the same commutation relations.
As well as R-order case, for arbitrary operators $\mathcal{O}$ taken the average with respect to σ,

$$
[M_0, \mathcal{O}] = 0. \tag{B.24}
$$

Furthermore we give the following relations with Λ^I,

$$
\begin{aligned}
\big[\bar{\Lambda}^I, x^J\big] =&\big[\bar{\Lambda}^I, p^J\big] = \big[\bar{\Lambda}^I, \mathcal{M}^2\big] = 0,\\
\big[\bar{\Lambda}^I, X^J_n\big] =&\delta^{I,J}\frac{1}{n}\bar{M}_n + i\sum_{l=1}^{n-1}\Big(P^J_{n-l}X^I_l + \frac{n-l}{l}X^J_{n-l}P^I_l\Big)\\
&+ i\sum_{l=1}^{\infty}\Big(X^I_{-l}P^J_{n+l} + P^J_{-l}X^I_{n+l} - \frac{n+l}{l}P^I_{-l}X^J_{n+l} - \frac{l}{n+l}X^J_{-l}P^I_{n+l}\Big),\\
\big[\bar{\Lambda}^I, P^J_n\big] =& -\delta^{I,J}\frac{i}{2}\bar{L}_n + in\sum_{l=1}^{n-1}\frac{1}{l}P^J_{n-l}P^I_l + in\sum_{l=1}^{\infty}\Big(\frac{1}{l}P^I_{-l}P^J_{n+l} - \frac{1}{n+l}P^J_{-l}P^I_{n+l}\Big),\\
\big[\bar{\Lambda}^I, \bar{M}_n\big] =&inP^I_n\left(-(n+1) + \left[2\bar{M}_0\frac{1}{n} + \frac{D-2}{6}\Big(n - \frac{1}{n}\Big)\right]\right)\\
&+ \frac{1}{2}n\left[\sum_{l=1}^{n-1}\bar{L}_{n-l}X^I_l + \sum_{l=1}^{\infty}(\bar{L}_{-l}X^I_{n+l} + X^I_{-l}\bar{L}_{n+l})\right]\\
&+ in\left[\sum_{l=1}^{n-1}\frac{1}{l}\bar{M}_{n-l}P^I_l + \sum_{l=1}^{\infty}\Big(\frac{1}{n+l}\bar{M}_{-l}P^I_{n+l} - \frac{1}{l}P^I_{-l}\bar{M}_{n+l}\Big)\right],\\
\big[\bar{\Lambda}^I, X^J_{-n}\big] =& -[\bar{\Lambda}^I, X^J_n]^\dagger\ ,\ \big[\bar{\Lambda}^I, P^J_{-n}\big] = -[\bar{\Lambda}^I, P^J_n]^\dagger\ ,\ \big[\bar{\Lambda}^I, \bar{M}_{-n}\big] = -\big[\bar{\Lambda}^I, \bar{M}_n\big]^\dagger.
\end{aligned}
\tag{B.25}
$$

Appendix C

Inner Product and Normalization of Mass Eigenfunctions

In chapter 4 and chapter 5, we have investigated in the R-order eigenfunctions of mass squared operator, $\mathcal{M}^2$. In this appendix we give the detail of the inner product and the normalization of mass eigenfunctions in the R-order.
Firstly we consider each n's factor of the separable wave function. For the simplicity, we consider the case of three dimensions in chapter 4. The case of the higher dimensions is in [36].
The inner product of two wave functions, $\psi_1(r, \theta)$ and $\psi_2(r, \theta)$, is defined as

$$(\psi_1, \psi_2) \equiv \int_0^\infty dr \int_0^{2\pi} d\theta \; r \; \psi_1^*(r, \theta)\psi_2(r, \theta). \tag{C.1}$$

The inner product of two total wave functions are defined by the product of the inner product for each n.
Because of the symmetry of θ direction, we assume that the eigenfunction of (4.9) is the product as $\psi_{m,s}(r, \theta) = \phi_{m,s}(r)e^{is\theta}$. Here $\phi_{m,s}(r)$ satisfy the next differential equation in (4.10).

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + 4m^2 - \frac{s^2}{r^2}\right]\phi_{m,s}(r) = 0, \tag{C.2}$$

where m is zero or positive continuous and s is integer.

Orthogonality for two eigenfunctions corresponding to $m > 0$ and $m' > 0$

First we check the orthogonality of eigenfunctions corresponding to non-zero eigenvalue. From (4.12), the eigenfunctions corresponding to non-zero eigenvalue are given by the Bessel function: $\psi_{m,s}(r, \theta) = J_s(2mr)e^{is\theta}$. From (C.1), the inner product of two eigenfunctions, $\psi_{m,s}$ and $\psi_{m',s'}$, is

$$(\psi_{m,s}, \psi_{m',s'}) = 2\pi\delta_{s,s'}N_m^* N_{m'} \int_0^\infty dr r J_s(2mr)J_s(2m'r). \tag{C.3}$$

The integral in (C.3) can be calculated by using the next relation.

$$\int_0^y dx x J_l(ax)J_l(bx) = \frac{y}{a^2-b^2}\left[aJ_{l+1}(ay)J_l(by) - bJ_l(ay)J_{l+1}(by)\right]. \tag{C.4}$$

We consider the limit of $y \to \infty$ in (C.4) to obtain the next result for $a > 0$ and $b > 0$.

$$\begin{aligned}\int_0^\infty dx x J_l(ax)J_l(bx) &= \lim_{y\to\infty} \frac{y}{a^2-b^2}\left[aJ_{l+1}(ay)J_l(by) - bJ_l(ay)J_{l+1}(by)\right] \\ &= \frac{1}{\pi}\frac{1}{\sqrt{ab}}\lim_{y\to\infty}\left[\frac{\sin(a-b)y}{a-b} - (-1)^l\frac{\cos(a+b)y}{a+b}\right] \\ &= \frac{1}{a}\delta(a-b),\end{aligned} \tag{C.5}$$

where we used $J_l(0) = 0$ for $l > 0$ and the asymptotic form

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(x - \frac{2\nu+1}{4}\pi\right) + \mathcal{O}(x^{-1})\right] \quad \text{as } x \to \infty \tag{C.6}$$

and the delta function defined by the *Sinc* function as[1]

$$\lim_{y\to\infty}\frac{\sin(yx)}{\pi x} \equiv \delta(x). \tag{C.7}$$

Thus we obtain

$$(\psi_{m,s}, \psi_{m',s'}) = \frac{\pi}{2}\delta_{s,s'}\frac{|N_m|^2}{m}\delta(m-m'). \tag{C.8}$$

If we choose $|N_m| = \sqrt{\frac{2m}{\pi}}$, we obtain $(\psi_{m,s}, \psi_{m',s'}) = \delta_{s,s'}\delta(m-m')$.

$\psi_{0,s'}$ and Orthogonality

Next we consider the choice of the solution for $m' = 0$ and the orthogonality between $\psi_{m,s}$ and $\psi_{0,s'}$. For $\psi_{0,s'}$, we consider the $s' = 0$ case and the $s' \neq 0$ case separately.
From the general solution of (C.2) for $m = 0$ and $s = 0$, we obtain

$$\psi_{0,0}(r,\theta) = A\log r + B, \tag{C.9}$$

where A and B are constants. Because of the regularity on $r = 0$, the first term is suspected to be problematic.
The inner product of $\psi_{m,s}$ and $\psi_{0,0}$ is

$$\begin{aligned}(\psi_{m,s}, \psi_{0,0}) &= 2\pi\delta_{s,0}N_m\int_0^\infty dr\ r\ (A\log r + B)J_0(2mr) \\ &= 2\pi\delta_{s,0}N_m\Big[-\frac{A}{(2m)^2} + \lim_{\Lambda\to\infty}(A\log\Lambda + B)\sqrt{\frac{2}{\pi}}\frac{\Lambda^{\frac{1}{2}}}{(2m)^{\frac{3}{2}}}\cos\Big(2m\Lambda - \frac{3}{4}\pi\Big)\Big]\end{aligned} \tag{C.10}$$

[1]The delta function defined in (C.7) and the dropping of the second term in the second line of (C.5) make sense only on the finite support.

where we use the asymptotic form in (C.6) and the following results

$$\int_0^y dx\ x\ J_0(x) = yJ_1(y) \tag{C.11}$$

$$\int_0^y dx\ x\log x\ J_0(x) = y\log y J_1(y) + J_0(y) - 1. \tag{C.12}$$

For $m \neq 0$, the second term of the last line in (C.10) is seen as zero because of the same reason as that of the dropping of the second term in the second line of (C.5). Therefore the r.h.s. of (C.10) vanishes if and only if $A = 0$. Thus the solution for $s' = 0$ is a constant: $\psi_{0,0} = \text{const}$.

Next we consider the $s' > 0$ case. The $s' < 0$ case is also discussed similarly. From the general solution of (C.2) for $m = 0$ and $s' > 0$, we obtain

$$\psi_{0,s'}(r,\theta) = (Ar^{-s'} + Br^{s'})e^{is'\theta}, \tag{C.13}$$

where A and B are constants. Because of the regularity on $r = 0$, the first term is suspected to be problematic.
The inner product of $\psi_{m,s}$ and $r^{-s'}e^{is'\theta}$ for $s' > 0$ is

$$\begin{aligned}(\psi_{m,s},\ r^{-s'}e^{is'\theta}) &= 2\pi\delta_{s,s'}N_m\int_0^\infty dr\ r\ r^{-s}J_s(2mr)\\ &= 2\pi\delta_{s,s'}N_m\frac{m^{s-2}}{2(s-1)!},\end{aligned} \tag{C.14}$$

where we use the asymptotic form (C.6) and the next result for positive integer l

$$\int_0^y dx x^{-l+1}J_l(x) = -y^{-l+1}J_{l-1}(y) + \frac{1}{2^{l-1}(l-1)!} \underset{y\to\infty}{\longrightarrow} \frac{1}{2^{l-1}(l-1)!}. \tag{C.15}$$

Because of (C.14), we must set $A = 0$ in solution (C.13).
On the other hand, the inner product of $\psi_{m,s}$ and $r^{s'}e^{is'\theta}$ for $s' > 0$ is

$$\begin{aligned}(\psi_{m,s},\ r^{s'}e^{is'\theta}) &= 2\pi\delta_{s,0}N_m\int_0^\infty dr\ r\ r^{s}J_s(2mr)\\ &= 2\pi\delta_{s,s'}N_m\sqrt{\frac{2}{\pi}}\lim_{\Lambda\to\infty}\left[\frac{\Lambda^{s+\frac{1}{2}}}{(2m)^{\frac{3}{2}}}\cos\left(2m\Lambda - \frac{2s+3}{4}\pi\right) + \cdots\right],\end{aligned} \tag{C.16}$$

where we use the asymptotic form (C.6) and the next result for positive integer l

$$\int_0^y dx x^{l+1}J_l(x) = y^{l+1}J_{l-1}(y). \tag{C.17}$$

Although (C.16) is not determined strictly, we assume that (C.16) vanishes because of the similar reason to that of the dropping of the second term in the second line of (C.5). Thus the solution for $s > 0$ is $\psi_{0,s} = r^s e^{is\theta}$. In the same way, we find that the solution for $s < 0$ is $\psi_{0,s} = r^{-s}e^{is\theta}$.
If we collect these three cases, $s = 0$, $s > 0$ and $s < 0$, we obtain

$$\psi_{0,s}(r,\theta) = Br^{|s|}e^{is\theta}. \tag{C.18}$$

Comment on inner product of total eigenfunctions

The inner product of two total wave functions are defined by the product of the inner product for each n. If a massless eigenfunction is separable (or monomial type), the inner product of eigenfunctions is straightforward, as seen above. If a massless eigenfunction is non-separable (or polynomial type), the integrands for some ns are different from (C.10) or (C.16). They have more exponents of r than that of (C.10) or (C.16). After we integrate partially, we will get the result like (C.10) or (C.16).
Note that the inner product of a separable eigenfunction and a non-separable one corresponding to the same positive mass does not always vanish.

Appendix D

Cut-off Regularization in R-order and Hermitian R-order

In some operator orders such as the R-order and the Hermitian R-order, some Hermitian generators of the spacetime symmetry have divergent terms [38, 40]. Therefore we have to regularize them for explicit calculations. For example, the dilatation in the R-order is

$$\mathcal{D}_R = x^- p_- + \sum_n X_n \cdot P_{-n}. \tag{D.1}$$

Because this is not hermitian, we must *hermitianize* this. If we reorder the hermitian generator into the R-order, we find the divergent term as follows:

$$\mathcal{D} = \frac{1}{2}\Big(\mathcal{D}_R + (\mathcal{D}_R)^\dagger\Big) = \mathcal{D}_R - \frac{i}{2}\Big[1 + \sum_n (D-2)\Big]. \tag{D.2}$$

In the Hermitian R-order where the ordered result is the second formula, the regularization is not necessary in the dilatation. However, as seen in (6.12) or (7.2), generators of higher degree such as $\tilde{\mathcal{K}}^-$ defined by the commutator can contain the divergent term from the quantum effect, $\frac{C}{p_-}$. Therefore we need some regularization.

In this appendix, we regularize divergences with the cut-off regularization, which drops higher Fourier modes. The advantages of the cut-off regularization are that infinite series become the sum of finite terms and that all commutators of generators except for $\tilde{\mathcal{K}}^-$ satisfy the desirable relations in the spacetime conformal group.[1]

For the simplicity, we consider the case of three dimensions below. The higher dimensional case is similarly discussed.

[1] Of course, other regularizations exist. A example is using the approximate delta function. This regularization smear the delta function of the commutation relation $[X(\sigma), P(\sigma')] = 2\pi i \delta(\sigma - \sigma')$. Explicit example is $[X_n, P_m]_\epsilon = \frac{i}{|n|^\epsilon}\delta_{n,-m}$ in Fourier mode, where the regulator ϵ measures the scale of smearing. In this regularization, the divergent series may be represented by zeta functions. However all the commutator have the difference in $\mathcal{O}(\epsilon)$ and multiple commutators are too complicated.

D.1 Cutoff Regularization

Here we define the cut-off regularization. We drop Fourier modes with index which is larger than N as follows:

$$X_n = P_n = 0 \ \text{ for } |n| > N, \tag{D.3}$$

where N is some large positive integer. From (D.3), the Fourier mode of P_+ and X^- are

$$\begin{aligned} L_n^{(N)} &\equiv \sum_{\substack{|m|\le N \\ |n-m|\le N}} P_m P_{n-m}, \\ M_n^{(N)} &\equiv -i \sum_{\substack{|m|\le N \\ |n-m|\le N}} m X_x P_{n-m}, \end{aligned} \tag{D.4}$$

where $_{(N)}$ in the superscript indicates the regularized operators. By the definition, we find

$$L_n^{(N)} = M_n^{(N)} = 0 \ \text{ for } |n| > 2N. \tag{D.5}$$

Furthermore the fundamental commutation relations are for $|n| \le N$ and $|n+m| \le N$

$$\begin{aligned} &[X_n, L_m^{(N)}] = 2iP_{n+m}, \ [P_n, L_m^{(N)}] = 0, \\ &[X_n, M_m^{(N)}] = (n+m)X_{n+m}, \ [P_n, M_m^{(N)}] = nP_{n+m}, \end{aligned} \tag{D.6}$$

and otherwise vanish. "$|n+m| \le N$" is a new restriction for the summation. Therefore $L_n^{(N)}$ and $M_m^{(N)}$ with finite N do not satisfy 2D Galilean conformal algebra (GCA)[2] and the algebra of these is not even closed. However we expect that the limit of $N \to \infty$ recovers the original algebra. The detail is given in the next section.

The regularized generators are defined by the restriction in (D.3) and indicated by $_{(N)}$ in the subscript. For example,

$$\begin{aligned} \mathcal{J}^- &= -x^- p - \frac{1}{2p_-}\sum_l X_l L_{-l} + \frac{i}{p_-}\sum_{l\neq 0}\frac{1}{l} M_l P_{-l} + \frac{i}{2p_-} p \\ \to \mathcal{J}^-_{(N)} &= -x^- p - \frac{1}{2p_-}\sum_{|l|\le N} X_l L_{-l}^{(N)} + \frac{i}{p_-}\sum_{0<|l|\le N}\frac{1}{l} M_l^{(N)} P_{-l} + \frac{i}{2p_-} p. \end{aligned} \tag{D.7}$$

D.2 Limit of $N \to \infty$

In this section we consider the important point in taking the limit of the regulator N. First we consider the commutators of $L_n^{(N)}$ and $M_m^{(N)}$ defined in (D.4). The commutators of original L_n and M_n satisfy 2d GCA without any regularization. Therefore the limit of the commutation relations of $L_n^{(N)}$ and $M_m^{(N)}$ must recover 2d GCA.

[2] 2D GCA is referred to [55–57]

In the R-order or the Hermitian R-order[3], the commutation relations of regularized operators are

$$\begin{aligned}
\left[L_n^{(N)}, L_m^{(N)}\right] =& 0 \\
\left[L_n^{(N)}, M_m^{(N)}\right] =& (n-m)L_{n+m}^{(N)} - 2 \sum_{\substack{|l|\le N, |l-n|>N \\ |n+m-l|\le N}} (n-l)P_l P_{n+m-l} \\
\left[M_n^{(N)}, M_m^{(N)}\right] =& (n-m)M_{n+m}^{(N)} - i \sum_{\substack{|l|\le N, |l-n|>N \\ |n+m-l|\le N}} l(l-n)X_l P_{n+m-l} \\
& + i \sum_{\substack{|l|\le N, |l-m|>N \\ |n+m-l|\le N}} l(l-m)X_l P_{n+m-l}.
\end{aligned} \tag{D.8}$$

The first commutator is good, but the second term of the second commutator and the second and third terms of the third commutator are extra. However, because they are ordered correctly, we assume that they have a good convergence property and drop them in limit of $N \to \infty$. Thus we drop extra terms in the limit of $N \to \infty$ to recover 2d GCA. Dropping the extra terms correctly ordered corresponds to the shift of the index of summation in series of ordered operators.[4] Therefore, when we take the limit of $N \to \infty$ in the cut-off regularization, we must drop the extra terms correctly ordered.

R-order and Hermitian R-order

Here we show that the above dropping in the R-order breaks the Hermitian property. For example, we consider the commutator of the following two hermitian operators in the R-order case and the Hermitian R-order case,

$$\frac{1}{2}\sum_{\substack{|n|\le N \\ |n+m|\le N}}\sum_{|m|\le N}(X_n X_m P_{-n-m} + h.c.) = \sum_{\substack{|n|\le N \\ |n+m|\le N}}\sum_{|m|\le N} X_n X_m P_{-n-m} - i(2N+1)x \ , \tag{D.9}$$

$$\frac{1}{2}\sum_{\substack{|k|\le N \\ |k+l|\le N}}\sum_{|l|\le N}(X_{k+l} P_{-k} P_{-l} + h.c.) = \sum_{\substack{|k|\le N \\ |k+l|\le N}}\sum_{|l|\le N} X_{k+l} P_{-k} P_{-l} - i(2N+1)p. \tag{D.10}$$

Here, in order to obtain terms of lower degree correctly, we write all the range of summation explicitly as well as extra ranges.
First the commutator of above two hermitian operators in the R-order is

$$\begin{aligned}
&\Bigg[\sum_{\substack{|n|\le N \\ |n+m|\le N}}\sum_{|m|\le N} X_n X_m P_{-n-m} - i(2N+1)x \ , \ \sum_{\substack{|k|\le N \\ |k+l|\le N}}\sum_{|l|\le N} X_{k+l} P_{-k} P_{-l} - i(2N+1)p\Bigg] \\
&= 3i \sum_{\substack{|n|\le N \\ |n+m+l|\le N}}\sum_{|m|\le N}\sum_{|l|\le N} X_n X_m P_l P_{-n-m-l} + 2\sum_{|l|\le N}\big[3(2N+1)-|l|\big]X_l P_{-l} - i(2N+1)^2 \\
&\quad -4i \sum_{\substack{|n|\le N \\ |n+m+l|\le N, \ |m+l|>N}}\sum_{|m|\le N}\sum_{|l|\le N} X_n X_m P_l P_{-n-m-l} + i \sum_{\substack{|n|\le N \\ |n+m+l|\le N, \ |n+m|>N}}\sum_{|m|\le N}\sum_{|l|\le N} X_n X_m P_l P_{-n-m-l},
\end{aligned} \tag{D.11}$$

[3]These orders are equivalent to the Weyl order in lower degree.

[4]The shift of the index of summation in series of ordered operators is done when we derive the Virasoro algebra for the usual tensionful string theory.

Note that all together the right hand side becomes anti-hermitian. The last line consists of the R-ordered extra terms. However it is not anti-hermitian. Therefore dropping the extra terms correctly ordered in the limit of $N \to \infty$ breaks the anti-hermitian property in the equation. Such a breaking causes the anomaly unrelated to one which we would like to know. If such breaking is accepted, anomalous quadratic terms such as the second part in the r.h.s. of (D.11) arise in $[\mathcal{J}^{-I}, \mathcal{K}^K]$. This is probably the cause of the anomaly in the traceless part of $[\mathcal{J}^{-I}, \mathcal{K}^K]$ shown in [38,40].

Next we reorder the commutator (F.12) in the Hermitian R-order, instead of the R-order.

$$\begin{aligned}
&\Bigg[\sum_{|n|\le N} \sum_{\substack{|m|\le N \\ |n+m|\le N}} X_n X_m P_{-n-m} - i(2N+1)x \ , \ \sum_{|k|\le N} \sum_{\substack{|l|\le N \\ |k+l|\le N}} X_{k+l} P_{-k} P_{-l} - i(2N+1)p \Bigg] \\
&= \frac{3i}{2} \sum_{|n|\le N} \sum_{|m|\le N} \sum_{\substack{|l|\le N \\ |n+m+l|\le N}} (X_n X_m P_l P_{-n-m-l} + h.c.) + i(7N^2 + 7N + 1) \qquad \text{(D.12)} \\
&\quad -2i \sum_{|n|\le N} \sum_{|m|\le N} \sum_{\substack{|l|\le N \\ |n+m+l|\le N,\ |m+l|>N}} (X_n X_m P_l P_{-n-m-l} + h.c.) + \frac{i}{2} \sum_{|n|\le N} \sum_{|m|\le N} \sum_{\substack{|l|\le N \\ |n+m+l|\le N,\ |n+m|>N}} (X_n X_m P_l P_{-n-m-l} + h.c.).
\end{aligned}$$

Because of hermitian combinations, dropping the extra terms in the last line of (D.13) does not break the anti-hermitian property in the equation. After taking the limit, the quantum effect appears only in constant, which is 4 lower degree than the highest one. Though, because such a quantum effect's constant may cause the anomaly in commutators of the spacetime conformal symmetry, we must calculate concretely such dangerous commutators.

In this way, when we consider operators of higher degree such as the special conformal transformation and their commutators, the Hermitian R-order is supposed to be appropriate. Therefore, we should choose the Hermitian R-order instead of the R-order. In the Hermitian R-order, all generators and commutators are ordered in the form with the hermitian partner as follows.

$$\mathcal{G} \equiv \frac{1}{2}\big(\mathcal{G}_R + (\mathcal{G}_R)^\dagger\big), \qquad \text{(D.13)}$$

Appendix E

Calculations

In this appendix we give the detail of the calculations which have been omitted in the main body of this thesis.

E.1 Lorentz Anomaly in $D > 3$ and in R-order

Here we calculate the dangerous commutator of the Poincaré symmetry, $[\mathcal{J}^{-I},\ \mathcal{J}^{-J}]$, in $D > 3$ and in the R-order and then obtain (5.2). We use the next representation of $\mathcal{J}^{-}$.

$$\mathcal{J}^{-I} = (x^{-}p^{I} - \frac{i}{2p_{-}}p^{I}) + \frac{1}{2p_{-}}\sum_{n} X_{n}^{I}L_{-n} - \frac{i}{p_{-}}\sum_{n\neq 0}\frac{1}{n}M_{n}P_{-n}^{I}. \tag{E.1}$$

Because the calculation of $[\mathcal{J}^{-I},\ \mathcal{J}^{-J}]$ is lengthy, we divide the commutator into three parts of $\mathcal{J}^{-}$.
The first part is

$$\begin{aligned}\left[x^{-}p^{I} - \frac{i}{2p_{-}}p^{I},\ \mathcal{J}^{-J}\right] =& -\frac{1}{p_{-}^{2}}\sum_{n\neq 0}\frac{1}{n}M_{n}P_{-n}^{J}p^{I} - \frac{i}{2p_{-}^{2}}\sum_{n} X_{n}^{J}L_{-n}p^{I} \\ &+ \delta^{IJ}\left(-\frac{i}{2}x^{-}\frac{1}{p_{-}}L_{0} - \frac{1}{4p_{-}^{2}}L_{0}\right).\end{aligned} \tag{E.2}$$

The second part is

$$\begin{aligned}\left[\frac{1}{2p_{-}}\sum_{n} X_{n}^{I}L_{-n},\ \mathcal{J}^{-J}\right] =& \frac{i}{2p_{-}^{2}}\sum_{n}\sum_{m} X_{n}^{J}P_{m}^{I}L_{-n-m} \\ &+ \delta^{IJ}\left(\frac{i}{2}x^{-}\frac{1}{p_{-}}L_{0} + \frac{1}{4p_{-}^{2}}L_{0} + \frac{1}{2p_{-}^{2}}\sum_{n\neq 0}\frac{1}{n}M_{n}L_{-n}\right).\end{aligned} \tag{E.3}$$

The third part is

$$\begin{aligned}\left[-\frac{i}{p_-}\sum_{n\neq 0}\frac{1}{n}M_nP^I_{-n},\ \mathcal{J}^{-J}\right] =&\frac{1}{p_-^2}\sum_{n\neq 0}\frac{1}{n}M_nP^J_{-n}p^I+\frac{i}{2p_-^2}\sum_n X^J_nL_{-n}p^I\\ &-\frac{i}{2p_-^2}\sum_n\sum_m X^J_nP^I_mL_{-n-m}-\delta^{IJ}\frac{1}{2p_-^2}\sum_{n\neq 0}\frac{1}{n}M_nL_{-n}\\ &-M_0\frac{1}{p_-^2}\sum_{n\neq 0}\frac{1}{n}\left(P^I_nP^J_{-n}-P^J_nP^I_{-n}\right).\end{aligned}\tag{E.4}$$

We collect (E.2)-(E.4) to get the next result.

$$\begin{aligned}[\mathcal{J}^{-I},\ \mathcal{J}^{-J}] &= \left[x^-p^I-\frac{i}{2p_-}p^I-\frac{i}{p_-}\sum_{n\neq 0}\frac{1}{n}M_nP^I_{-n}+\frac{1}{2p_-}\sum_n X^I_nL_{-n},\ \mathcal{J}^{-J}\right]\\ &=-\frac{1}{p_-^2}\sum_{n\neq 0}\frac{1}{n}\left(P^I_nP^J_{-n}-P^J_nP^I_{-n}\right)M_0\end{aligned}\tag{E.5}$$

where we set M_0 to the right end. Thus we obtain (5.2).
The constraint on the physical state is $M_0\approx a$. If we choose $a=0$, the Lorentz anomaly for the tensionless string in the light-cone gauge vanishes in all higher dimensions as well as $D=3$, where the dangerous commutator is zero trivially. Therefore there is no critical dimension for the Poincaré symmetry in the light-cone tensionless string theory in the R-order. The cases of the Hermitian R-order and the Weyl order are the same.

E.2 Lorentz Anomaly in $D>3$ and in XP-normal Order

Here we calculate the dangerous commutator of the Poincaré symmetry, $[\mathcal{J}^{-I},\ \mathcal{J}^{-J}]$, in $D>3$ and in the XP-normal order and then obtain (5.25). We use the next representation of $\mathcal{J}^-$.

$$\begin{aligned}\mathcal{J}^{-I}=&x^-p^I+\left(\frac{1}{2p_-}x^IL_0-\frac{i}{2p_-}p^I\right)\\ &+\frac{1}{2p_-}\sum_{n=1}^{\infty}(X^I_{-n}L_n+L_{-n}X^I_n)+\frac{i}{p_-}\sum_{n=1}^{\infty}\frac{1}{n}(M_{-n}P^I_n-P^I_{-n}M_n).\end{aligned}\tag{E.6}$$

Because the calculation of $[\mathcal{J}^{-I},\ \mathcal{J}^{-J}]$ is lengthy, we divide the commutator into four parts of $\mathcal{J}^-$.
The first part is

$$\left[x^-p^I,\ \mathcal{J}^{-J}\right]=-\delta^{I,J}\frac{i}{2}x^-\frac{1}{p_-}L_0-i(\mathcal{J}^{-J}-x^-p^J)\frac{p^I}{p_-}.\tag{E.7}$$

The second part is

$$\begin{aligned}
&\left[\frac{1}{2p_-}x^I L_0 - \frac{i}{2p_-}p^I,\ \mathcal{J}^{-J}\right] \\
&= \delta^{I,J}\frac{i}{2p_-}x^- \frac{1}{p_-}L_0 + \frac{i}{2p_-^2}x^J L_0 p^I + \frac{1}{2p_-^2}p^I p^J - \frac{1}{p_-^2}\sum_{n=1}^{\infty}(P^I_{-n}P^J_n - P^J_{-n}P^I_n) \\
&\quad + \frac{i}{2p_-^2}\sum_{n=1}^{\infty}(P^I_{-n}L_0 X^J_n + X^J_{-n}L_0 P^I_n - P^J_{-n}L_0 X^I_n - X^I_{-n}L_0 P^J_n).
\end{aligned} \tag{E.8}$$

The third part is

$$\begin{aligned}
&\left[\frac{1}{2p_-}\sum_{n=1}^{\infty}(X^I_{-n}L_n + L_{-n}X^I_n),\ \mathcal{J}^{-J}\right] \\
&= \frac{i}{2p_-^2}\sum_{n=1}^{\infty}(X^J_{-n}L_n + L_{-n}X^J_{-n})p^I + \frac{i}{2p_-^2}x^J\sum_{n=1}^{\infty}(P^I_{-n}L_n + L_{-n}P^I_n) \\
&\quad - \delta^{I,J}\frac{1}{2p_-^2}\sum_{n=1}^{\infty}\frac{1}{n}(M_{-n}L_n - L_{-n}M_n) + \frac{i}{2p_-^2}\sum_{n=1}^{\infty}(P^J_{-n}L_0 X^I_n + X^I_{-n}L_0 P^J_n) \\
&\quad + \frac{i}{2p_-^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[(X^J_{-n-m}L_n P^I_m + X^J_{-n}L_{-m}P^I_{n+m} + X^J_{-n}P^I_{-m}L_{n+m}) + h.c.\right] \\
&\quad - \frac{3}{2p_-^2}\sum_{n=1}^{\infty}(n-1)(P^I_{-n}P^J_n - P^J_{-n}P^I_n) + \frac{1}{p_-^2}\sum_{n=1}^{\infty}P^J_{-n}P^I_n,
\end{aligned} \tag{E.9}$$

where $h.c.$ in 4th line indicates the Hermitian conjugate of the adjacent bracket.
The last part is lengthy and we must take care of the center term of $[M_n, M_m]$. The result of calculation is

$$\begin{aligned}
&\left[\frac{i}{p_-}\sum_{n=1}^{\infty}\frac{1}{n}(M_{-n}P^I_n - P^I_{-n}M_n),\ \mathcal{J}^{-J}\right] \\
&= -\frac{1}{p_-^2}\sum_{n=1}^{\infty}\frac{1}{n}(M_{-n}P^J_n - P^J_{-n}M_n)p^I - \frac{i}{2p_-^2}x^J\sum_{n=1}^{\infty}(P^I_{-n}L_n + L_{-n}P^I_n) \\
&\quad + \delta^{I,J}\frac{1}{2p_-^2}\sum_{n=1}^{\infty}\frac{1}{n}(M_{-n}L_n - L_{-n}M_n) - \frac{i}{2p_-^2}\sum_{n=1}^{\infty}(P^I_{-n}L_0 X^J_n + X^J_{-n}L_0 P^I_n) \\
&\quad - \frac{i}{2p_-^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[(X^J_{-n-m}L_n P^I_m + X^J_{-n}L_{-m}P^I_{n+m} + X^J_{-n}P^I_{-m}L_{n+m}) + h.c.\right] \\
&\quad - \frac{1}{2p_-^2}\sum_{n=1}^{\infty}(n-1)(P^I_{-n}P^J_n - P^J_{-n}P^I_n) - \frac{1}{p_-^2}\sum_{n=1}^{\infty}P^I_{-n}P^J_n \\
&\quad + \frac{1}{p_-^2}\sum_{n=1}^{\infty}\left[\left(\frac{D-2}{6} - 2\right)n + \left(2M_0 - \frac{D-2}{6}\right)\frac{1}{n}\right](P^I_{-n}P^J_n - P^J_{-n}P^I_n).
\end{aligned} \tag{E.10}$$

We collect (E.7)-(E.10) to obtain

$$[\mathcal{J}^{-I},\ \mathcal{J}^{-J}] = \frac{1}{p_-^2}\sum_{n=1}^{\infty}\left[\left(\frac{D-2}{6} - 4\right)n + \left(2M_0 - \frac{D-2}{6}\right)\frac{1}{n}\right](P^I_{-n}P^J_n - P^J_{-n}P^I_n) \tag{E.11}$$

This is anti-symmetric for exchanging I and J as a matter of course. Thus we obtain (5.25).
From (E.11), the critical dimension and the ordering constraint for the Poincaré symmetry in the tensionless string theory in the XP-normal order are determined as $D = 26$ and $M_0 \approx a = 2$.

E.3 Commutators of 3-dim. Spacetime Conformal Symmetry in Cut-off Regularization

In this section we calculate dangerous commutators of the spacetime conformal symmetry in three dimensions with the cut-off regularization to investigate whether anomaly exist or not. The cut-off regularization is defined as

$$X_n = P_n = 0, \quad \text{for } |n| > N. \tag{E.12}$$

The regularized operators or generators are defined by restricting only X_n and P_m in them and are indicated by ${}_{(N)}$ at superscript or subscript, such as $M_n^{(N)}$, $L_n^{(N)}$ and $\mathcal{K}^{\mu}_{(N)}$. Note that some commutators have extra restriction in the range of sum.

E.3.1 Definition of $\tilde{\mathcal{K}}^{-}_{(N)}$

First we calculate $\tilde{\mathcal{K}}^{-}_{(N)} \equiv -i[\mathcal{J}^{-}_{(N)}, \mathcal{K}_{(N)}]$ and then find the effect of the cut-off regularization.
Because of

$$[\mathcal{J}^{-}_{(N)}, \mathcal{K}_{(N)}] = \frac{1}{2}\left([\mathcal{J}^{-}_{(N)}, \mathcal{K}_{(N),R}] - h.c.\right), \tag{E.13}$$

it is enough to calculate the commutator of the following two generators.

$$\begin{aligned}\mathcal{J}^{-}_{(N)} =& - x^- p + \left(-\frac{1}{2p_-} x L_0^{(N)} + i\frac{p}{2p_-}\right) + \frac{\Lambda^{(N)}}{p_-} \ , \\ \Lambda^{(N)} \equiv& \sum_{l\neq 0}\left[-\frac{1}{2} X_l L_{-l}^{(N)} + \frac{i}{l} M_l^{(N)} P_{-l}\right] ,\end{aligned} \tag{E.14}$$

$$\mathcal{K}_{(N),R} = x x^- p_- + \frac{1}{2}\sum_n \sum_m X_n X_m P_{-n-m} + i\sum_{m\neq 0}\frac{1}{m} X_m M_{-m}^{(N)}. \tag{E.15}$$

As we find in another appendix, we must take care so that the limit of $N \to \infty$ does not break the hermitian property. Therefore we use the Hermitian R-order where all terms always have the hermitian conjugate partner of themselves.

Because the calculation is lengthy, we divide the commutator $[\mathcal{J}^{-}_{(N)}, \mathcal{K}_{(N)}]$ into three hermitian parts of $\mathcal{J}^{-}_{(N)}$ in (E.14). The first part is

$$\left[-x^- p,\ \mathcal{K}_{(N)}\right] = \frac{i}{2}\Bigg(\Big(x^- x^- p_- + x^- \sum_{l\neq 0} X_l P_{-l}\Big) + h.c.\Bigg). \tag{E.16}$$

The second part is

$$\left[-\frac{1}{2p_-}xL_0^{(N)} + i\frac{p}{2p_-},\ \mathcal{K}_{(N)}\right]$$
$$= \frac{i}{2}\left(\left(x^- xp + \frac{1}{4p_-}\sum_n X_n X_{-n} L_0^{(N)} - x\frac{\Lambda^{(N)}}{p_-}\right) + h.c.\right) + \frac{i}{4p_-}. \tag{E.17}$$

The third part is

$$\left[\frac{\Lambda^{(N)}}{p_-},\ \mathcal{K}_{(N)}\right]$$
$$= \frac{i}{2}\left(\left(\frac{1}{4p_-}\sum_{\substack{n \\ n+m\neq 0}}\sum_m X_n X_m L^{(N)}_{-n-m} - \frac{i}{p_-}\sum_n\sum_{m\neq 0}\frac{n+m}{m^2}X_n M_m^{(N)} P_{-n-m} + x\frac{\Lambda^{(N)}}{p_-}\right) + h.c.\right)$$
$$+ \frac{i}{p_-}\left(-\frac{3}{4}N^2 - \frac{1}{4}N - \frac{1}{2}\sum_{\substack{|n|\leq N \\ |n+m|\leq N}}\sum_{0<|m|\leq N}\frac{n^2}{m^2}\right) - \frac{2}{p_-}\sum_{n\neq 0}\frac{1}{n}X_n P_{-n} M_0^{(N)} + \delta\mathcal{K}^-_{(N)}, \tag{E.18}$$

where $\delta\mathcal{K}^-_{(N)}$ is the effect of the cut-off regularization and its concrete form is

$$\begin{aligned}\delta\mathcal{K}^-_{(N)} =& \frac{1}{2p_-}\sum_{\substack{n \\ |n+m|>N}}\sum_m\sum_l\left(\frac{1}{4} + \frac{n}{2m} + \frac{m^2}{nl} + \frac{n+m}{l}\right)(X_n X_m P_l P_{-n-m-l} + h.c.) \\ &- \frac{1}{2p_-}\sum_n\sum_{\substack{m \\ |m+l|>N}}\sum_l\left(\frac{l}{m+l} + \frac{m^2}{nl} + \frac{n+m}{l} - \frac{nm}{(m+l)^2}\right)(X_n X_m P_l P_{-n-m-l} + h.c.).\end{aligned} \tag{E.19}$$

We collect (E.16)-(E.18) to get

$$[\mathcal{J}^-_{(N)}, \mathcal{K}_{(N)}] = i\left(\mathcal{K}^-_{(N)} + \frac{C_{(N)}}{p_-} + \delta\mathcal{K}^-_{(N)} + \frac{2i}{p_-}\sum_{n\neq 0}\frac{1}{n}X_n P_{-n} M_0^{(N)}\right) \equiv i\tilde{\mathcal{K}}^-_{(N)}, \tag{E.20}$$

where $\mathcal{K}^-_{(N)}$ is the ordered generator without extra terms,

$$\begin{aligned}\mathcal{K}^-_{(N)} &= \frac{1}{2}\left(\mathcal{K}^-_{(N),R} + h.c.\right) \\ \mathcal{K}^-_{(N),R} &= x^-\mathcal{D}_{(N),R} + \frac{1}{4p_-}\sum_n\sum_m X_n X_m L^{(N)}_{-n-m} - \frac{i}{p_-}\sum_n\sum_{m\neq 0}\frac{n+m}{m^2}X_n M_m^{(N)} P_{-n-m},\end{aligned} \tag{E.21}$$

and

$$C_{(N)} = -\frac{3}{4}N^2 - \frac{1}{4}N + \frac{1}{4} - \frac{1}{2}\sum_{\substack{|n|\leq N \\ |n+m|\leq N}}\sum_{0<|m|\leq N}\frac{n^2}{m^2}. \tag{E.22}$$

Here we emphasize that $\delta\mathcal{K}^-_{(N)}$ is obviously hermitian and does not contain any zero mode, x and p. Although $\delta\mathcal{K}^-_{(N)}$, which is ordered correctly in the Hermitian R-order, is dropped in the limit of $N \to \infty$, we must verify whether the commutators of this and other generators becomes consist of only terms dropped in the limit.

E.3.2 Dangerous part in $[\mathcal{J}^-_{(N)}, \tilde{\mathcal{K}}^-_{(N)}]$

From the consideration in chapter 6, the structure of $[\mathcal{J}^-_{(N)}, \tilde{\mathcal{K}}^-_{(N)}]$ in the Hermitian R-order is

$$[\mathcal{J}^-_{(N)}, \tilde{\mathcal{K}}^-_{(N)}] = i(\text{ordered classical}) + i\frac{1}{p_-}(\text{XPP-cubic}) \times M_0^{(N)} + i\frac{p}{p_-^2} \times (\text{const.}). \quad (\text{E.23})$$

The first term of (E.23) corresponds to the part which is zero classically in the calculation without the regularization. Therefore it consists of the terms which are dropped in the limit of $N \to \infty$.[1] The second part[2] is related to the choice of the ordering constant, $M_0^{(N)} \to M_0 \approx a$. The third part consists of terms of the lowest degree, which arises from only the quantum calculation. The anomalous part consists of the second and third parts.

Here we calculate the anomalous part directly without considering the commutator with x unlike the case of the main text. Then we verify that there is no contribution to the anomalous part from the commutator of $\delta\mathcal{K}^-_{(N)}$.
Because of the lengthy results, we divide $[\mathcal{J}^-_{(N)}, \tilde{\mathcal{K}}^-_{(N)}]$ into the commutators of each part of $\tilde{\mathcal{K}}^-_{(N)}$ defined by (E.20) and then give only the results of the anomalous part, $\propto i\frac{p}{p_-^2}$.[3]

First, we consider

$$[\mathcal{J}^-_{(N)}, \mathcal{K}^-_{(N)}] = \frac{1}{2}\left([\mathcal{J}^-_{(N)}, \mathcal{K}^-_{(N),R}] - h.c.\right). \quad (\text{E.24})$$

and then see the contribution to the anomalous part.[4]
Because the calculation of (E.24) is lengthy, we divide $\mathcal{J}^-_{(N)}$ into three hermitian parts in (E.14).
We calculate the contribution from the commutator of the first part of $\mathcal{J}^-_{(N)}$ in detail.

$$\begin{aligned}
[-x^- p, \mathcal{K}^-_{(N),R}] =& \frac{i}{4p_-^2}\sum_n\sum_m X_n X_m L^{(N)}_{-n-m} p - \frac{i}{p_-^2}\sum_n\sum_{m\neq 0}\frac{n+m}{m^2} X_n M^{(N)}_m P_{-n-m} p \\
&+ ix^-\frac{1}{p_-}\left(\frac{1}{2}xL_0^{(N)} - \Lambda^{(N)}\right) \\
=& \frac{i}{4p_-^2}\sum_n\sum_m X_n X_m L^{(N)}_{-n-m} p - \frac{i}{p_-^2}\sum_n\sum_{m\neq 0}\frac{n+m}{m^2} X_n M^{(N)}_m P_{-n-m} p \\
&+ \frac{i}{2}\left(x^-\frac{1}{p_-} + \frac{1}{p_-}x^-\right)\left(\frac{1}{2}xL_0^{(N)} - \Lambda^{(N)}\right) + \frac{1}{2p_-^2}\left(\frac{1}{2}xL_0^{(N)} - \Lambda^{(N)}\right)
\end{aligned} \quad (\text{E.25})$$

From this, we obtain

$$\frac{1}{2}\left([-x^- p, \mathcal{K}^-_{(N),R}] - h.c.\right) \sim \frac{1}{2}\left(\frac{1}{2p_-^2}\left(\frac{1}{2}xL_0^{(N)} - \Lambda^{(N)}\right) - h.c.\right) \sim i\frac{p}{4p_-^2}, \quad (\text{E.26})$$

[1]Because of the classical part, we do not have to mind the divergence.

[2]We can know the second part from the classical calculation as well as quantum one.

[3]The first term of (E.23) is the part which is dropped in the limit of $N \to \infty$.

[4]In the calculation of (E.24), to obtain the Hermitian R-ordered version of terms proportional to $x^-\frac{1}{p_-}$, we deform such terms by using $x^-\frac{1}{p_-} = \frac{1}{2}\left(x^-\frac{1}{p_-} + \frac{1}{p_-}\right) - \frac{i}{2p_-}$.

where $\sim$ indicates extracting of the anomalous part, $\propto i\frac{p}{p_-^2}$.
Similarly, the contributions of other parts are calculated. The results are below.
The contribution from the commutator of the second part of $\mathcal{J}^-_{(N)}$ is

$$\frac{1}{2}\Big(\Big[-\frac{1}{2p_-}xL_0^{(N)}+i\frac{p}{2p_-},\ \mathcal{K}^-_{(N),R}\Big]-h.c.\Big)\sim -i\frac{p}{2p_-^2}. \tag{E.27}$$

The contribution from the commutator of the third part of $\mathcal{J}^-_{(N)}$ is

$$\begin{aligned}&\frac{1}{2}\Big(\Big[\frac{\Lambda^{(N)}}{p_-},\ \mathcal{K}^-_{(N),R}\Big]-h.c.\Big)\\ &\sim\frac{2i}{p_-^2}\sum_{n\neq 0}\frac{1}{n^2}M_n^{(N)}P_{-n}M_0^{(N)}+i\left(\frac{3}{4}N^2+\frac{1}{4}N+\frac{1}{2}\sum_{|n|\le N}\sum_{\substack{0<|m|\le N\\ |n+m|\le N}}\frac{n^2}{m^2}\right)\frac{p}{p_-^2}\end{aligned} \tag{E.28}$$

where we added the contribution from the term arisen when we move $M_0^{(N)}$ to the right end.
We collect (E.26)-(E.28) to obtain

$$\begin{aligned}&[\mathcal{J}^-_{(N)},\mathcal{K}^-_{(N)}]\\ &\sim\frac{2i}{p_-^2}\sum_{n\neq 0}\frac{1}{n^2}M_n^{(N)}P_{-n}M_0^{(N)}+i\left(\frac{3}{4}N^2+\frac{1}{4}N-\frac{1}{4}+\frac{1}{2}\sum_{|n|\le N}\sum_{\substack{0<|m|\le N\\ |n+m|\le N}}\frac{n^2}{m^2}\right)\frac{p}{p_-^2}\\ &=\frac{2i}{p_-^2}\sum_{n\neq 0}\frac{1}{n^2}M_n^{(N)}P_{-n}M_0^{(N)}-iC_{(N)}\frac{p}{p_-^2}.\end{aligned} \tag{E.29}$$

Next the commutator of $\frac{C_{(N)}}{p_-}$ is

$$\Big[\mathcal{J}^-_{(N)},\frac{C_{(N)}}{p_-}\Big]=\Big[-x^-p,\frac{C_{(N)}}{p_-}\Big]=iC_{(N)}\frac{p}{p_-^2}. \tag{E.30}$$

This is canceled by (E.29).

Next we consider the contribution from the commutator with $\delta\mathcal{K}^-_{(N)}$ to the anomalous part. Because of the extra restriction on the range of the summation in (E.19), $\delta\mathcal{K}^-_{(N)}$ does not contain any zero mode. Therefore the contribution to $i\frac{p}{p_-^2}$ comes from the commutators of terms with p in $\mathcal{J}^-_{(N)}$ and $\delta\mathcal{K}^-_{(N)}$.
The terms with p in $\mathcal{J}^-_{(N)}$ are

$$\mathcal{J}^-_{(N)}|_p=-x^-p-\frac{1}{2p_-}xpp+\frac{i}{2p_-}p. \tag{E.31}$$

Then we obtain

$$\Big[\mathcal{J}^-_{(N)},\delta\mathcal{K}^-_{(N)}\Big]\sim\Big[\mathcal{J}^-_{(N)}|_p,\delta\mathcal{K}^-_{(N)}\Big]=-\Big[x^-,\delta\mathcal{K}^-_{(N)}\Big]p=i\delta\mathcal{K}^-_{(N)}\frac{p}{p_-}. \tag{E.32}$$

This is dropped in the limit of $N\to\infty$ and has no contribution to the anomalous part.

Finally the commutator of the last term in (E.20) is again the term with $M_0^{(N)}$ at the right end because $M_0^{(N)}$ commutes with all regularized generators.

$$\begin{aligned}
&\Big[\mathcal{J}^-_{(N)},\ \frac{2i}{p_-}\sum_{n\neq 0}\frac{1}{n}X_nP_{-n}M_0^{(N)}\Big] \\
&= \Big(-\frac{6}{p_-^2}\sum_{n\neq 0}\frac{1}{n}X_nP_{-n}p+\frac{1}{p_-^2}\sum_{n\neq 0}\frac{1}{n}X_nL_{-n}+\frac{2i}{p_-^2}\sum_{n\neq 0}\frac{1}{n^2}M_nP_{-n}\Big)\cdot M_0^{(N)}
\end{aligned} \tag{E.33}$$

All the terms with $M_0^{(N)}$ at the right end in $[\mathcal{J}^-_{(N)},\tilde{\mathcal{K}}^-_{(N)}]$ vanishes if and only if we choose $a=0$.[5]

Thus the anomalous part in $[\mathcal{J}^-_{(N)},\tilde{\mathcal{K}}^-_{(N)}]$ vanishes if and only if $a=0$.

E.3.3 The rest commutators with $\tilde{\mathcal{K}}^-_{(N)}$

Here we consider the rest of commutators with $\tilde{\mathcal{K}}^-_{(N)}$. We can easily verify that most of them give expected results even for finite N,

$$\begin{aligned}
&[\mathcal{P}_{-(N)},\tilde{\mathcal{K}}^-_{(N)}]=-i(\mathcal{D}_{(N)}+\mathcal{J}_{(N)})\ ,\ [\mathcal{P}_{(N)},\tilde{\mathcal{K}}^-_{(N)}]=i\mathcal{J}^-_{(N)}\ ,\ [\mathcal{P}_{+(N)},\tilde{\mathcal{K}}^-_{(N)}]=0 \\
&[\mathcal{J}^+_{(N)},\tilde{\mathcal{K}}^-_{(N)}]=i\mathcal{K}_{(N)}\ ,\ [\mathcal{J}_{(N)},\tilde{\mathcal{K}}^-_{(N)}]=-i\tilde{\mathcal{K}}^-_{(N)} \\
&[\mathcal{D}_{(N)},\tilde{\mathcal{K}}^-_{(N)}]=-i\tilde{\mathcal{K}}^-_{(N)}\ ,\ [\mathcal{K}^+_{(N)},\tilde{\mathcal{K}}^-_{(N)}]=0,
\end{aligned} \tag{E.34}$$

Although the last commutator which should be calculated, $[\mathcal{K}_{(N)},\tilde{\mathcal{K}}^-_{(N)}]$, is complicated, the calculation of this is parallel to that of $[\mathcal{J}^-_{(N)},\tilde{\mathcal{K}}^-_{(N)}]$. This consists of the classical ordered part, the terms with $M_0^{(N)}$ at the right end and the term of the lowest degree, $i\frac{x}{p_-}$ such as (E.23). The commutator of $\delta\mathcal{K}^-_{(N)}$ does not contribute to the anomalous part in the same way as the last subsection. So we get $[\mathcal{K}_{(N)},\tilde{\mathcal{K}}^-_{(N)}]\to 0$ in the limit of $N\to\infty$ and $a=0$.

Thus we obtain the set of commutators which give the desirable results for the space-time conformal symmetry in the limit of $N\to\infty$.

[5] In the cut-off regularization, the constraint is $M_0^{(N)}\to M_0\approx a=0$.

Appendix F

Tensionful 3D String in Light-cone Gauge

Three dimensional string theories have the specialty of three dimensions that the "dangerous" commutator is zero trivially, $[\mathcal{J}^-, \mathcal{J}^-] = 0$, and therefore preserves the Poincaré symmetry. In this appendix, we give the short review about the three dimensional string theory with tension $T \neq 0$ in light-cone gauge [50–52][1]. This chapter will be useful to remember the usual string with tension.

F.1 Classical Theory

The starting point is the next action which is equivalent to the Nambu-Goto action for a tensionful string.

$$S[\mathbf{X}, \mathbf{P}; V, U] = \int d\tau \oint \frac{d\sigma}{2\pi} \left\{ \dot{\mathbf{X}}^\mu \mathbf{P}_\mu - \frac{1}{2} V \left[\mathbf{P}^2 + (T\mathbf{X}')^2 \right] - U \mathbf{X}'^\mu \mathbf{P}_\mu \right\}. \tag{F.1}$$

This action has the next gauge symmetry which corresponds to the world sheet diffeomorphism,

$$\begin{cases} \delta \mathbf{X}^\mu = \alpha \mathbf{P}^\mu + \beta \mathbf{X}'^\mu, \\ \delta \mathbf{P}_\mu = T^2 (\alpha \mathbf{X}')' + (\beta \mathbf{P}_\mu)', \\ \delta V = \dot{\alpha} + U'\alpha - U\alpha' + V'\beta - V\beta', \\ \delta U = \dot{\beta} + U'\beta - U\beta' + T^2(\alpha V' - \alpha' V). \end{cases} \tag{F.2}$$

We choose the light-cone gauge as

$$X^+ = \tau, \;\; P_- = p_-(\tau) \neq 0, \tag{F.3}$$

to fix the gauge symmetry in (F.2). We solve some constraints and field equations with the Fourier mode to obtain the Lagrangian as

$$L = \dot{x}p + \dot{x}^- p_- - i \sum_{n=1}^{\infty} \frac{1}{n} \Big[\dot{\alpha}_{-n} \alpha_n + \dot{\tilde{\alpha}}_{-n} \tilde{\alpha}_n \Big] - H + u \sum_{n=1}^{\infty} \Big[\alpha_{-n}\alpha_n - \tilde{\alpha}_{-n}\tilde{\alpha}_n \Big], \tag{F.4}$$

[1] Their study have become the start of this paper's study.

where the Fourier modes are defined by

$$\begin{aligned} P - TX' &= \sqrt{2T}\sum_n \alpha_n e^{in\sigma} \ , \ \alpha_{-n} = (\alpha_n)^* \\ P + TX' &= \sqrt{2T}\sum_n \tilde{\alpha}_n e^{-in\sigma} \ , \ \tilde{\alpha}_{-n} = (\tilde{\alpha}_n)^*, \end{aligned} \tag{F.5}$$

and the Hamiltonian and the mass square operator are

$$\begin{aligned} H &\equiv -p_+ = \frac{1}{2p_-}(p^2 + \mathcal{M}^2) \ , \\ \mathcal{M}^2 &= -\mathcal{P}^2 = 2T\sum_{n=1}\big[\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n\big]. \end{aligned} \tag{F.6}$$

Most of constraints are solved as

$$X^- = x^- - \frac{1}{p_-}\sum_{n\neq 0}\frac{i}{n}\big[L_n^\perp e^{in\sigma} + \tilde{L}_n^\perp e^{-in\sigma}\big] \tag{F.7}$$

$$P_+ = -\frac{T}{p_-}\sum_n\big[L_n^\perp e^{in\sigma} + \tilde{L}_n^\perp e^{-in\sigma}\big] \tag{F.8}$$

$$L_n^\perp = \frac{1}{2}\sum_m \alpha_m\alpha_{n-m} \ , \ \tilde{L}_n^\perp = \frac{1}{2}\sum_m \tilde{\alpha}_m\tilde{\alpha}_{n-m}, \tag{F.9}$$

but only the next constraint remains unsolved.

$$\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n - \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\tilde{\alpha}_n = 0. \tag{F.10}$$

This is the level matching condition. We impose this condition on physical states.

F.2 Quantum Theory

The commutation relations of dynamical variables are

$$\begin{aligned} &[x^-, p_-] = i \ , \ [X(\sigma), P(\sigma')] = 2\pi\delta(\sigma - \sigma') \\ \leftrightarrow\ &[x^-, p_-] = [x, p] = i \ , \ [\alpha_n, \alpha_m] = [\tilde{\alpha}_n, \tilde{\alpha}_m] = n\delta_{n,-m} \ , \ [\alpha_n, \tilde{\alpha}_m] = 0. \end{aligned} \tag{F.11}$$

In the normal order for the left- and right-moving modes, the mass operator in (F.6) and the constraint in (F.10) are represented as

$$\mathcal{M}^2 = 2T\big(N + \tilde{N} - a\big) \approx 2T\big(2N - a\big), \tag{F.12}$$

$$(N - \tilde{N})|\mathrm{phys}\rangle = 0, \tag{F.13}$$

where N and $\tilde{N}$ are the level numbers of left- and right-moving mode respectively,

$$N = \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n \ , \ \tilde{N} = \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\tilde{\alpha}_n, \tag{F.14}$$

and a is the constant of the ambiguity coming from the operator ordering. Because the "dangerous" commutator vanishes trivially in three dimensions, the ordering constant, a, is not determined.
In the normal order,

$$
\begin{aligned}
L_n^\perp &= \frac{1}{2}\sum_{m=0}^{n} \alpha_m \alpha_{n-m} + \sum_{m=1}^{\infty} \alpha_{-m}\alpha_{n+m} \ , \ L_{-n}^\perp = (L_n^\perp)^\dagger \\
\tilde{L}_n^\perp &= \frac{1}{2}\sum_{m=0}^{n} \tilde{\alpha}_m \alpha_{n-m} + \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}\tilde{\alpha}_{n+m} \ , \ \tilde{L}_{-n}^\perp = (\tilde{L}_n^\perp)^\dagger
\end{aligned}
\tag{F.15}
$$

for $n \geq 0$. These operators satisfy the famous Virasoro algebra with the central extension term severally. However we don't have to calculate them because of the trivial vanishing of $[\mathcal{J}^-, \mathcal{J}^-] = 0$.

Mass operator in (F.12) has the same structure as the harmonic oscillator. Therefore the string ground state is the tensor product, $|0\rangle \equiv |0\rangle \equiv |0\rangle_+ \otimes |0\rangle_-$. Here the state $|0\rangle_+$ is annihilated by positive α-mode and the state $|0\rangle_-$ is annihilated by positive $\tilde{\alpha}$-mode. That is

$$
\alpha_n |0\rangle_+ = \tilde{\alpha}_n |0\rangle_- = 0 \ \text{ for } n > 0. \tag{F.16}
$$

The excited states are created by acting negative modes on the string ground state. Furthermore the physical states must satisfy the level-matching condition $N = \tilde{N}$ and their mass eigenvalues are given by level numbers such as (F.12). Here note that in the case of $a > 0$ tachyonic states can exist.

F.3 Helicity Spectrum

The above results are the same as the usual tensionful string in higher dimensions. In three dimensions, there is another Poincaré invariant,

$$
\Lambda \equiv \mathcal{J}^\mu \mathcal{P}_\mu = p_- \oint \frac{d\sigma}{2\pi}\left[\bar{X}\bar{P}_+ - \bar{X}^-\bar{P}\right] = \sqrt{2T}(\lambda + \tilde{\lambda}) \tag{F.17}
$$

$$
\lambda = \sum_{n=1}^{\infty} \frac{i}{n}(\alpha_{-n}L_n^\perp - L_{-n}^\perp \alpha_n) \ , \ \tilde{\lambda} = \sum_{n=1}^{\infty} \frac{i}{n}(\tilde{\alpha}_{-n}\tilde{L}_n^\perp - \tilde{L}_{-n}^\perp \tilde{\alpha}_n) \tag{F.18}
$$

where we note that $p = \sqrt{2T}\alpha_0 = \sqrt{2T}\tilde{\alpha}_0$ in λ and $\tilde{\lambda}$ is canceled.
Because Λ commutes with the mass square operator, $\mathcal{M}^2$, we can diagonalize them simultaneously. If the mass eigenvalue is non-zero, the helicity is defined by

$$
s \equiv \frac{\Lambda}{\mathcal{M}}. \tag{F.19}
$$

Now we give eigenvalues of Λ at some lower levels. Because the left-moving part and the right-moving part split, it is enough to calculate one hand of the left-moving or the right-moving.

Level 0

The state at level 0 is only the string ground state, $|0\rangle = |1\rangle_+ \otimes |1\rangle_-$. This has $\mathcal{M}^2 = -2Ta$ and $\Lambda = 0$. Therefore $s = 0$.

Level 1

The staet at level 1 is only $\alpha_{-1}\tilde{\alpha}_{-1}|0\rangle \equiv |1\rangle_+ \otimes |1\rangle_-$. This has $\mathcal{M}^2 = 2T(2-a)$ and $\Lambda = 0$. Therefore $s = 0$.

Level 2

The states of the left-moving part at level 2 are $|2\rangle_+ \equiv \frac{1}{\sqrt{2}}\alpha_{-2}|0\rangle_+$ and $|1,1\rangle_+ \equiv \frac{1}{\sqrt{2}}\alpha_{-1}\alpha_{-1}|0\rangle_+$. In this base, λ is represented as

$$\lambda = \frac{3}{2}i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{F.20}$$

The eigenvalues of this are $\pm\frac{3}{2}$. We collect both moving parts to obtain eigenvalues of Λ as $\Lambda = \sqrt{2T}(0,0,3,-3)$. Because the mass square eigenvalue is $\mathcal{M}^2 = 2T(4-a)$ at level 2, helicity s_2 at level 2 is

$$s_2 = \left(0, 0, \pm\frac{3}{\sqrt{4-a}}\right). \tag{F.21}$$

From this, we find that s_2 can become rational or irrational number as well as integer and half integer according to a. Therefore we can find the anyons in the spectrum for the tensionful string theory in the 3-dim. light-cone gauge.

Level 3

The states of the left-moving part at level 3 are $|3\rangle_+ \equiv \frac{1}{\sqrt{3}}\alpha_{-3}|0\rangle_+$, $|2,1\rangle_+ \equiv \frac{1}{\sqrt{2}}\alpha_{-2}\alpha_{-1}|0\rangle_+$ and $|1,1,1\rangle_+ \equiv \frac{1}{\sqrt{6}}\alpha_{-1}\alpha_{-1}\alpha_{-1}|0\rangle_+$.
In this base, λ is represented as

$$\lambda = \frac{i}{2\sqrt{3}} \begin{pmatrix} 0 & 7\sqrt{2} & 0 \\ -7\sqrt{2} & 0 & 9 \\ 0 & -9 & 0 \end{pmatrix}. \tag{F.22}$$

and the eigenvalues are $0, \pm\sqrt{\frac{179}{12}}$. Because the mass square eigenvalue is $\mathcal{M}^2 = 2T(6-a)$ at level 3, we collect both moving parts to obtain the helicity, s_3, at level 3.

$$s_3 = \left(0, 0, 0, \pm\sqrt{\frac{179}{12(6-a)}}, \pm\sqrt{\frac{179}{12(6-a)}}, \pm\sqrt{\frac{179}{3(6-a)}}\right) \tag{F.23}$$

At level 3, we find that the helicity can become rational or irrational number as well as integer and half integer according to a. Therefore we can find the anyons in the spectrum for the tensionful string theory in the 3-dim. light-cone gauge.

At higher level, irrational helicities appear infinitely. All of these irrational helicity, anyons, cannot be removed by the choice of a. The existence of such anyons are expected from the representation theory of three dimensional Poincaré group [54, 58–60]. Anyons with various spin values (in three dimensions) are studied in various points of view [61–67].

F.4 Some Comments

In three dimensions, we cannot determine a for the tensionful bosonic string because of no dangerous commutator. Therefore we get various spectrum of mass and helicity according to a. For example, for $a > 4$, there are non-scalar tachyons.
Above all, $a = 0$ case is special. In $a = 0$ case, there is no tachyon and then the string ground state is massless.[2]
There is a questions of how a should be determined. In the main body of this thesis, we find that the requirement of the spacetime conformal symmetry give $a = 0$ for a tensionless string. For a string with tension, one of the answers of this question is superstring.[3] Because of the supersymmetry, there is no zero-point energy, $a = 0$. The detail is in [51, 52].

[2]A massive scalar exists at level 1 and states with fermionic helicity, $s_2 = \left(0, 0, \pm\frac{3}{2}\right)$, exist at level 2. Though irrational helicities at higher level are inevitable.

[3]In [53], the evidence of the equivalent between RNS formalism and GS formalism for the 3-dim. tensionful string theory with the spacetime supersymmetry are investigated.

Bibliography

[1] K. M. , ”Quantum 3D Tensionless String in Light-cone Gauge,” JHEP **1402** (2014) 038 [arXiv:1303.7202 [hep-th]].

[2] I.R. Klebanov, A.M. Polyakov, ”AdS Dual of the Critical O(N) Vector Model,” Phys. Lett. B **550** (2002) 213-219 [arXiv:hep-th/0210114].

[3] Simone Giombi, Xi Yin, ”Higher Spin Gauge Theory and Holography: The Three-Point Functions,” JHEP **1009** (2010) 115 [arXiv:0912.3462[hep-th]].

[4] Simone Giombi and Xi Yin, ”On Higher Spin Gauge Theory and the Critical $O(N)$ Model,” Phys. Rev. D **85** (2012) 086005 [arXiv:1105.4011 [hep-th]].

[5] Chi-Ming Chang and Xi Yin, ”Higher Spin Gravity with Matter in AdS_3 and Its CFT Dual,” JHEP **1210** (2012) 024 [arXiv:1106.2580[hep-th].

[6] Simone Giombi, Shiraz Minwalla, Shiroman Prakash, Sandip P. Trivedi, Spenta R. Wadia and Xi Yin, ”Chern-Simons Theory with Vector Fermion Matter,” Eur. Phys. J. C **72** (2012) 2112 [arXiv:1110.4386[hep-th]].

[7] Chi-Ming Chang, Shiraz Minwalla, Tarun Sharma and Xi Yin, ”ABJ Triality: from Higher Spin Fields to Strings,” J. Phys. A **46** (2013) 214009 [arXiv:1207.4485 [hep-th]].

[8] A. Campoleoni, S. Fredenhagen, S. Pfenninger, and S. Theisen, ”Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields,” JHEP **1011** (2010) 007 [arXiv:1008.4744[hep-th]].

[9] Andrea Campoleoni, Stefan Fredenhagen and Stefan Pfenninger, ”Asymptotic W-symmetries in three-dimensional higher-spin gauge theories,” JHEP **1109** (2011) 113 [arXiv:1107.0290 [hep-th]].

[10] Matthias R Gaberdiel and Rajesh Gopakumar, ”An AdS_3 Dual for Minimal Model CFTs,” Phys. Rev. D **83** (2011) 066007 [arXiv:1011.2986 [hep-th]].

[11] Matthias R. Gaberdiel and Thomas Hartman, ”Symmetries of Holographic Minimal Models,” JHEP **1105** (2011) 031 [arXiv:1101.2910 [hep-th]].

[12] Matthias R. Gaberdiel and Rajesh Gopakumar, Thomas Hartman and Suvrat Raju, ”Partition Functions of Holographic Minimal Models,” JHEP **1108** (2011) 077 [arXiv:1106.1897 [hep-th]].

[13] Martin Ammon, Michael Gutperle, Per Kraus and Eric Perlmutter, "Spacetime Geometry in Higher Spin Gravity," JHEP **1110** (2011) 053 [arXiv:1106.4788 [hep-th]].

[14] Alfredo Pereza, David Tempoa and Ricardo Troncosoa, "Higher spin gravity in 3D: Black holes, global charges and thermodynamics," Phys. Lett. B **726** (2013) 444-449 [arXiv:1207.2844 [hep-th]].

[15] Ippei Fujisawa, Kenta Nakagawa and Ryuichi Nakayama, "AdS/CFT for 3D Higher-Spin Gravity Coupled to Matter Fields," Class. Quant. Grav. **31** (2014) 065006 [arXiv:1311.4714 [hep-th]].

[16] Andrea Campoleoni, Marc Henneaux, "Asymptotic symmetries of three-dimensional higher-spin gravity: the metric approach," arXiv:1412.6774 [hep-th].

[17] Hong Lu, C.N. Pope, K. Thielemans, X.J. Wang and K.W. Xu, "Quantizing higher spin string theories," Int. J. Mod. Phys. A **10** (1995) 2123-2142 [arXiv:hep-th/9410005].

[18] Amihay Hanany and Igor R. Klebanov, "On tensionless strings in (3+1)-dimensions," Nucl. Phys. B **482** (1996) 105-118 [arXiv:hep-th/9606136].

[19] Giulio Bonelli, "On the tensionless limit of bosonic strings, infinite symmetries and higher spins," Nucl.Phys. B **669** (2003) 159-172 [arXiv:hep-th/0305155].

[20] G.K. Savvidy, "Tensionless strings: Physical Fock space and higher spin fields," Int. J. Mod. Phys. A **19** (2004) 3171-3194 [arXiv:hep-th/0310085].

[21] A. Sagnotti, "Notes on Strings and Higher Spins," J. Phys. A **46** (2013) 214006 [arXiv:1112.4285 [hep-th]].

[22] K. Surya Kiran, Chethan Krishnan, Ayush Saurabh and Joan Simon, "Strings vs Spins on the Null Orbifold," JHEP **1412** (2014) 002 [arXiv:1408.3296 [hep-th]].

[23] C. Fronsdal, "Massless fields with integer spin," Phys. Rev. D **18** (1978) 3624.

[24] Mikhail Vasiliev, "Higher-Spin Gauge Theories in Four, Three and Two Dimensions," Int. J. Mod. Phys. D **5** (1996) 763-797 [arXiv:hep-th/9611024].

[25] Mikhail Vasiliev, "Higher Spin Gauge Theories: Star-Product and AdS Space," In *Shifman, M.A. (ed.): The many faces of the superworld* 533-610 [arXiv:hep-th/9910096].

[26] M.A. Vasiliev, "Higher-Spin Theory and Space-Time Metamorphoses," Lect. Notes Phys. 892 (2015) 227-264 [arXiv:1404.1948 [hep-th]].

[27] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 1 : Introduction," Cambridge, UK: Univ. Pr. (1987).

[28] J. Polchinski, "String theory. Vol. 1: An introduction to bosonic string," Cambridge, UK: Univ. Pr. (1988).

[29] Barton Zwiebach, "A First Course in STRING THEORY," Cambridge Univ. Pr. (2004).

[30] Mitsuhiro Kato and Kaku Ogawa, "Covariant Quantization of String Based on BRS Invariance," Nucl.Phys. B **212** (1983) 443.

[31] Nobuyoshi Ohta, "Covariant quantization of superstrings based on Becchi-Rouet-Stora invariance," Phys.Rev. D **33** (1986) 1681.

[32] Stephen Hwang, "Covariant Quantization of the String in Dimensions D ¡= 26 Using a BRS Formulation," Phys.Rev. D **28** (1983) 2614.

[33] A. Karlhede and U. Lindstrom, "The Classical Bosonic String in the Zero Tension Limit," Class. Quant. Grav. **3** (1986) L73-L75.

[34] F. Lizzi, B. Rai, G. Sparano and A. Srivastava, "Quantization Of The Null String And Absence Of Critical Dimensions," Phys.Lett. B **182** (1986) 326.

[35] J. Gamboa, C. Ramirez and M. Ruiz-Altaba, "Quantum Null (super)strings," Phys.Lett. B **225** (1989) 335-339.

[36] J. Gamboa, Cupatitzio Ramirez and M. Ruiz-Altaba, "Null spinning strings," Nuc. Phys. B **338** (1990) 143-187.

[37] H.Gustafsson, U.Lindstrom, P.Saltsidis, B.Sundborg and R.v.Unge, "Hamiltonian BRST Quantization of the Conformal String," Nucl.Phys. B **440** (1995) 495-520 [arXiv:hep-th/9410143].

[38] J. Isberg, U. Lindstrom and B. Sundborg, "Space-Time Symmetries of Quantized Tensionless Strings," Phys. Lett. B **293** (1992) 321-326 [arXiv:hep-th/9207005].

[39] Ulf Lindstrom, "The Zero tension limit of strings and superstrings," arXiv:hep-th/9303173.

[40] J. Isberg and U. Lindstrom, B. Sundborg and G. Theodoridis, "Classical and Quantized Tensionless Strings," Nucl. Phys. B **411** (1994) 122-156 [arXiv:hep-th/9307108].

[41] J. Barcelos-Neto1 and M. Ruiz-Altaba, "Superstrings with zero tension," Phys. Let. B **228** (1989) 193-199.

[42] U. Lindstrom, B. Sundborg and G. Theodoridis, "The zero tension limit of the superstring," Phys. Let. B **253** (1991) 319-323.

[43] U. Lindstrom, B. Sundborg and G. Theodoridis, "The zero tension limit of the spinning string," Phys. Lett. B **258** (1991) 331-334.

[44] P. Saltsidis, "Hamiltonian BRST quantization of the conformal spinning string," Nucl.Phys. B **446** (1995) 286-298 [arXiv:hep-th/9503062].

[45] J. Barcelos-Neto, C. Ramirez and M. Ruiz-Altaba, "Phase Space Lagrangians For Null Spinning Strings," Z.Phys. C **47** (1990) 241-246.

[46] U. Lindstrom and M. Roček, "D=2 null superspaces," Phys. Lett. B **271** (1991) 79-84.

[47] Chao-Zheng Zha and Yu. Cai, "Quantization of null spinning strings with U(1) gauge symmetry," J.Phys. A24 (1991) 3427-3434.

[48] P. Saltsidis, "Tensionless p-branes with manifest conformal invariance," Phys. Lett. B **401** (1997) 21-29 [arXiv:hep-th/9702081].

[49] P. Saltsidis, "The Mass Spectrum of the 2-dimensional Conformal String," Phys.Lett. B **396** (1997) 107-114 [arXiv:hep-th/9609169].

[50] Luca Mezincescu and Paul K. Townsend, "Anyons from Strings," Phys. Rev. Lett. **105**, 191601,2010 [arXiv:1008.2334 [hep-th]].

[51] L. Mezincescu and P. K. Townsend, "Quantum 3D Superstrings," Phys.Rev. D **84** (2011) 106006 [arXiv:1106.1374 [hep-th]].

[52] L. Mezincescu and P. K. Townsend, "3D strings and other anyonic things," Fortsch.Phys. **60** (2012) 1076-1079 [arXiv:1111.3384 [hep-th]].

[53] L. Mezincescu, A. J. Routh and P. K. Townsend, "Equivalence of 3D Spinning String and Superstring," JHEP **1307** (2013) 024 [arXiv:1305.5049 [hep-th]].

[54] Birne Binegar, "Relativistic field theories in three dimensions," J. Math. Phys. **23** (1982) 1511-1517.

[55] F. Lizzi, "The Zero Tension Limit of The Virasoro Algebra and the Central Extension," Mod.Phys.Lett. A **9** (1994) 1495-1500 [arXiv:hep-th/9404148].

[56] Arjun Bagchi, Rajesh Gopakumar, Ipsita Mandal, and Akitsugu Miwa, "GCA in 2d," JHEP **1008** (2010) 004 [arXiv:0912.1090 [hep-th]].

[57] Arjun Bagchi, "Tensionless Strings and Galilean Conformal Algebra," JHEP **1305** (2013) 141 [arXiv:1303.0291[hep-th]].

[58] S. Daser and R. Jackiw, "Statistics without spin. Massless D=3 systems," Phys. Lett. B **263** (1991) 431-436.

[59] Stefano Forte, "Quantum mechanics and field theory with fractional spin and statistics," Rev. Mod. Phys. **64** (1992) 193.

[60] Jens Mund, "The Spin-Statistics Theorem for Anyons and Plektons in d=2+1," Commun. Math. Phys. **286** (2009) 1159-1180 [arXiv:0801.3621 [math-ph]].

[61] R. Jackiw and V.P. Nair, "Relativistic wave equations for anyons," Phys. Rev. D **43** (1991) 1933-1942.

[62] V.P. Nair, "Relativistic wave equation for anyons," Int. J. Mod. Phys. B **5** (1991) 1685-1692.

[63] Luca Mezincescu and Paul K. Townsend, "The Quantum 3D Superparticle," SIGMA **7** (2011) 005 [arXiv:1011.5049 [hep-th]].

[64] Luca Mezincescu and Paul K. Townsend, "Semionic Supersymmetric Solitons," J. Phys. A **43** (2010) 465401 [arXiv:1008.2775 [hep-th]].

[65] Luca Mezincescu, Alasdair J. Routh and Paul K. Townsend, "All Superparticles are BPS," J. Phys. A **47** (2014) 175401 [arXiv:1401.5116 [hep-th]].

[66] Nicolas Boulanger, Per Sundell and Mauricio Valenzuela, "A Higher-Spin Chern-Simons Theory of Anyons," arXiv:1311.4589 [hep-th].

[67] Nicolas Boulanger, Per Sundell, and Mauricio Valenzuela, "Three-dimensional fractional-spin gravity," JHEP **1402** (2014) 052 [arXiv:1312.5700 [hep-th]].