MANY-PARTICLE STRUCTURE OF GREEN'S FUNCTIONS

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My talk will be in two parts. First, I shall describe in some detail the general idea and the method of structure analysis of Green's functions [1]. Then the current results will be sketched comparatively briefly, since they are still far from having the form of simple statements, e.g., of analyticity, about observable quantities. There are many steps necessary.

Axiomatic field theory consists in studying the implications of essentially the following three postulates :

I. Relativistic invariance.

II. Vanishing of the commutator or anticommutator of any two local field quantities for space-like distances :

$$[\Psi_1(x), \Psi_2(y)]_\pm = 0 \ \text{ if } \ (x-y)^2 < 0 \tag{1}$$

III. Existence of a unique vacuum and of positive energy states only, with positive norm, and of eigenstates corresponding to discrete eigenvalues (specific to the theory) of the total mass operator. Also irreducibility of the ingoing (and, as a consequence of TCP, also of the outgoing) fields with those masses.

In the following we discuss, in order to keep the formalism as transparent as possible, a theory with a scalar hermitian irreducible field $A(x)$ and with one kind of neutral spinless particles of mass m only. The generalization of the method and all the results to more complicated cases (even with so-called " composite particles ") is straightforward.

The most manageable c-number formulation of the axioms seems to be in terms of Green's functions. Let $J(x)$ be a real function of x. The operator [2]

$$\mathcal{T}\{J\} \equiv T \exp\left[i \int dx A(x) J(x)\right]$$

defined e.g. by its formal power series expansion in J, is the generating functional of time-ordered operator products, whose vacuum expectation values are called Feynman amplitudes (τ-functions). One easily proves

$$\mathcal{T}^+\{J\}\mathcal{T}\{J\} = \mathcal{T}\{J\}\mathcal{T}^+\{J\} = 1. \tag{2}$$

Clearly, we have

$$\frac{\delta^n \mathcal{T}\{J\}}{\delta J(x_1)\dots\delta J(x_n)} \equiv \mathcal{T}_{x_1\dots x_n} =$$

$$= i^n T\left\{A(x_1)\dots A(x_n) \exp\left[i\int dx A(x)J(x)\right]\right\}. \tag{3}$$

The convenient, but perhaps not crucial, assumption that in every physically sensible local field theory there should (besides many other local fields) also exist an irreducible set of local fields with their time-ordered products can be argued as follows (in essence an old argument) : Assume the existence of a local Lagrangian density. The common opinion that the commutator condition, Eq. (1), means that two measurements, or local disturbances, do not interfere with each other if separated by a space-like interval tacitly assumes (cf. the discussion of the measuring process by Bohr and Rosenfeld) the possibility of adding a classical source term to the Lagrangian :

$$\mathcal{L}(A) \rightarrow \mathcal{L}(A) + AJ$$

With the help of e.g. Schwinger's functional differential equation one proves that under this change, scattering amplitudes undergo the change

$$\langle\beta^{\text{out}}|\alpha^{\text{in}}\rangle \rightarrow \langle\beta^{\text{out}}|\mathcal{T}\{J\}|\alpha^{\text{in}}\rangle$$

to be calculated in terms of undisturbed system, provided the source is coupled to the renormalized field. The physically suggestive postulate that for infinitely differentiable and for $J(x)$ decreasing sufficiently fast at infinity (in space and time) this change should be finite and uniquely determined by the source leads to the existence of $\mathcal{T}\{J\}$ and of time-ordered products as distributions. It is very satisfying that this holds for all terms of the respective (renormalized) perturbation-theoretic expansions.

The usual asymptotic condition [2], written symbolically as

$$A(x) \to A_{\text{out, in}}(x) \qquad \text{if } x_0 \to \pm\infty$$

for the operator $\mathcal{T}\{J\}$ is equivalent to the formula

$$S\mathcal{T}\{J\} = :\exp\left[\int du A_{\text{in}}(u) K_u \frac{\delta}{\delta J(u)}\right]:\langle\mathcal{T}\{J\}\rangle \qquad (4)$$

where $\rangle$ is the vacuum, S the scattering operator

$$A_{\text{out}}(x) = S^+ A_{\text{in}}(x) S$$

and K_u the Klein-Gordon operator

$$K_u \equiv \frac{\partial^2}{\partial u_0{}^2} - \Delta_u + m^2.$$

From Eq. (1) and Eq. (2) one finds

$$\mathcal{T}_{xy} = \theta(x_0 - y_0)\mathcal{T}_x \mathcal{T}^+ \mathcal{T}_y + \theta(y_0 - x_0)\mathcal{T}_y \mathcal{T}^+ \mathcal{T}_x$$

or

$$\mathcal{T}^+ \mathcal{T}_{xy} = -\theta(x_0 - y_0)\mathcal{T}_x{}^+ \mathcal{T}_y - \theta(y_0 - x_0)\mathcal{T}_y{}^+ \mathcal{T}_x\,. \qquad (5)$$

Insertion of Eq. (4) into Eq. (5) gives, upon taking all possible matrix elements, an infinite system of coupled nonlinear integral equations, with arbitrarily many terms each, for τ-functions. The occurrence of the step function $\theta(x_0) = \frac{1}{2}(1 + \text{sign}\, x_0)$ in the nonlinear equations (together with other reasons) makes their analysis cumbersome.

A more suitable system of equations is obtained from Eq. (5) for the Hermitian functional [3]

$$R_x\{J\} \equiv -i\mathcal{T}^+ \mathcal{T}_x = i\mathcal{T}_x{}^+ \mathcal{T}\,. \qquad (6)$$

One finds [4]

$$\frac{\delta R_x\{J\}}{\delta J(y)} \equiv R_{x,y} = i\theta(x_0 - y_0)[R_x, R_y] \qquad (7)$$

and from Eq. (4) and Eq. (6)

$$R_x\{J\} = :\exp\left[\int du A_{\text{in}}(u) K_u \frac{\delta}{\delta J(u)}\right]:\langle R_x\{J\}\rangle\,. \qquad (8)$$

The expansion coefficients of R_x in

$$R_x\{J\} = A(x) + \\ + \sum_{n=1}^{\infty} \frac{1}{n!} \int \dots \int dy_1 \dots dy_n R(x, y_1, \dots, y_n) J(y_1) \dots J(y_n) \qquad (9)$$

are the retarded multiple commutators

$$R(x, y_1, \dots, y_n) = \\ = i^n \sum_{\text{perm}} \theta(x_0 - y_{10})\theta(y_{10} - y_{20}) \dots \theta(y_{n-1,0} - y_{n0}) \times \\ \times [\dots[[A(x), A(y_1)], A(y_2)], \dots, A(y_n)]$$

whose vacuum expectation values are the Lorentz-invariant real r-functions. Eq. (7) can also be written as the retardedness condition

$$R_{x,y} = 0 \qquad \text{unless } x_0 > y_0 \quad (7a)$$

and the " unitarity condition "

$$R_{x,y} - R_{y,x} = i[R_x, R_y]. \qquad (7b)$$

The vacuum expectation values of Eqs. (*7a*, *b*), evaluated with Eq. (8), give, upon expansion in powers of J, the retardedness condition and an infinite system of nonlinear integral equations for r-functions, respectively. It is easy to prove the equivalence of these conditions for r-functions to the axiomatic requirements, formulated before, specialized for the present model.

We are going to show that the Bethe-Salpeter structure (here, because of some generalization, called many-particle structure) of r-functions, known from perturbation theory, is necessary for Eq. (*7b*) to be solved, and sufficient to allow us to give to Eq. (*7b*) a most interesting and also practically useful " reduced " form.

Before writing out Eq. (*7b*), we need some preparations. With Eq. (*7a*) and

$$[A_{\text{in}}(y), R_x] = i\int dy' \Delta(y - y') K_{y'} R_{x,y'} \qquad (10)$$

from Eq. (8) it is not difficult to prove that the functions

$$\tilde{r}(p_1, \dots, p_n) \equiv \\ \equiv \int \dots \int dy_1 \dots dy_n e^{i(x-y_1)p_1 + \dots + i(x-y_n)p_n} r(x, y_1, \dots, y_n) \qquad (11)$$

are, for $n > 1$, singular like

$$\text{const.} \times \left[m^2 - (p_1 + \ldots + p_n + i\varepsilon)^2\right]^{-1}$$

with respect to the coordinate x, and like

$$\text{const.} \times \left[m^2 - (p_\nu + i\varepsilon)^2\right]^{-1}$$

with respect to y_ν, with ε an infinitesimal positive timelike vector. Perturbation theory suggests, however, not to amputate the corresponding Δ_{Ret} but instead use

$$\Delta'_{\text{Ret}}(x-y) = r(x, y) = i\theta(x_0 - y_0)\langle[A(x), A(y)]\rangle$$

functions. This amputation, a division by $\tilde{\Delta}'_{\text{Ret}}(p)$ in momentum space, is unique even when $\tilde{\Delta}'_{\text{Ret}}(p)$ has (under our assumptions, necessarily real) " CDD " zeros [5] if the boundary condition that also the amputated functions should be retarded is imposed. Indicating such amputation by barring the respective index, defining

$$R'_x \equiv R_x - A_{\text{in}}(x) - \int dy \Delta'_{\text{Ret}}(x-y) J(y) \qquad (12)$$

(therefore, R'_x is built up, as in Eq. (8), only by r-functions with at least three coordinates, i.e. $n \geqslant 2$ in Eq. (11)), and using Eq. (10), we obtain for the vacuum expectation value of Eq. (7*b*) the form (sufficient because of Eq. (8)) :

$$\begin{aligned} &\Delta'_{\text{Ret}}(x-u)\langle R'_{\bar{u},\bar{v}}\rangle\left[\Delta'_{\text{Ret}}(v-y) + \Delta(v-y)\right] - \\ &\quad - \left[\Delta'_{\text{Ret}}(u-x) + \Delta(u-x)\right]\langle R'_{\bar{v},\bar{u}}\rangle \Delta'_{\text{Ret}}(y-v) + \\ &\quad + \Delta'_{\text{Ret}}(x-y) - \Delta'_{\text{Ret}}(y-x) + \Delta(x-y) - \\ &\quad - i\Delta'_{\text{Ret}}(x-u)\langle[R'_{\bar{u}}, R'_{\bar{v}}]\rangle \Delta'_{\text{Ret}}(y-v) = 0 \qquad (13) \end{aligned}$$

where the commutator is to be evaluated with Eq. (8), and repeated coordinates are integrated over.

The case $J = 0$ of Eq. (13) is the starting equation of an earlier investigation [6] of the one-particle propagation function. The main result was that the proper self energy part can be calculated from its absorptive part with not more than one subtraction (in contrast to the perturbation theoretical two), and that the amputated vertex (in our model, with four coordinates) has to vanish at infinite off-shell momentum, with three momenta on the mass shell with equal frequency sign. Furthermore, at zeros of $\tilde{\Delta}'_{\text{Ret}}(p)$, which are at most of first order, the unamputated vertices, with all momenta but one on the mass shell, as before, vanish (in the limiting case of first order). This suggests that $\tilde{\Delta}'_{\text{Ret}}(p)$ is a genuine factor, especialy since in some cases the analyticity of the vertex at the point in question can be proven. We conjecture that $\tilde{\Delta}'_{\text{Ret}}(p)$ is a factor, in this sense, at all coordinates off the mass shell, as it is in perturbation theory.

Let functionals $\langle R^i_{x, y \ldots z}\rangle$ be defined by the equations

$$\langle R'_{x, y\ldots z}\rangle = \langle R^i_{x, y\ldots z}\rangle + \langle R'_{x, u}\rangle\langle R^i_{\bar{u}, y\ldots z}\rangle \qquad (14a)$$

which can be solved for these functionals by iteration (rigorously, since it terminates for any finite order of J). From Eq. (14*a*)

$$\langle R'_{x, y\ldots z}\rangle = \langle R^i_{x, y\ldots z}\rangle + \langle R^i_{x, \bar{u}}\rangle\langle R'_{u, y\ldots z}\rangle \qquad (14b)$$

easily follows. These functionals are called " one-particle irreducible between x and the set $y \ldots z$ ". One can prove [1] that the corresponding r^i-functions, in contrast to the r-functions, are not one-particle singular (i.e. like $\text{const.} \times \left[m^2 - (p + i\varepsilon)^2\right]^{-1}$) with respect to the momentum p that is a partial sum of the momenta to the latest and some other coordinates, but not to any of the coordinates $y \ldots z$. (Note that there is a distinction in Eq. (14*a*, *b*) between the coordinates $y \ldots z$ displayed and those introduced by further functional differentiation.) Intuitively, the irreducible functions, if expanded perturbation-theoretically, are described by *double graphs* [7] that cannot be separated into two parts by cutting only one line such that the two distinguished coordinate groups are on different parts.

Eqs. (14*a*, *b*) are now substituted into Eq. (13) in such a way that in all terms the one-particle reducibility next to x is exhibited. Thereupon, using Eq. (13) once again, one obtains

$$\begin{aligned} &\Delta'_{\text{Ret}}(x-u)\Big[\langle R^i_{\bar{u},\bar{v}}\rangle - \langle R^i_{\bar{v},\bar{u}}\rangle - i\langle[R^i_{\bar{u}}, R^i_{\bar{v}}]\rangle\big|_{J=0} - \\ &\quad - \langle R^i_{\bar{u},\bar{r}}\rangle \Delta(r-s)\langle R^i_{\bar{v},\bar{s}}\rangle\big|_{J\neq 0}\Big]\langle R_{y,v}\rangle + \\ &\quad + \Big[\Delta'_{\text{Ret}}(x-u) - \Delta'_{\text{Ret}}(u-x) + \Delta(x-u) - \\ &\quad - i\langle[R'_x, R'_u]\rangle\big|_{J\neq 0}\Big]\langle R_{y,\bar{u}}\rangle = 0 \qquad (15) \end{aligned}$$

where the irreducible commutator is to be evaluated with Eq. (8) such that all contraction coordinates are distinguished coordinates in the sense of Eq. (14*a*, *b*).

The last term in the first square bracket precisely cancels the one-particle intermediate state contributions to the commutator. $J=0$ gives the equation mentioned before. For $J \neq 0$, the amputation of $\Delta'_{\text{Ret}}(x-u)$ and removal of $\langle R_{y,v}\rangle$ are not in general possible if $\tilde{\Delta}'_{\text{Ret}}(p)$ has zeros, since the retarded boundary condition is not available here. On the basis of the earlier conjecture about singularity-free amputability of $\tilde{\Delta}'_{\text{Ret}}(p)$, however, one shows that the first square bracket in Eq. (15) must vanish.

In the " one-particle reduced " nonlinear system thus obtained not only all one-particle singularities (as proved independently) and one-particle intermediate states have been eliminated, but also many nonsingular terms have been cancelled. For that reason structure analysis goes beyond an identification and elimination of singularities. This becomes fully obvious in the more interesting but less simple two-particle case to be discussed now.

In order to find the successful " ansatz " analogous to Eq. (14*a*, *b*) for this case, one uses perturbation theory as a source of inspiration (and only as such). Inspection of double graphs [8, 1] suggests the definition for generalized retarded functions :

$$r(x_1 \ldots x_k, y_1 \ldots y_l) = \langle R(x_1 \ldots x_k, y_1 \ldots y_l)\rangle$$

with

$$R(x_1 \ldots x_k, y_1 \ldots y_l) \equiv \frac{\delta^l}{\delta J(y_1) \ldots \delta J(y_l)} R_{x_1 \ldots x_k}\{J\}_{|J=0}$$

and

$$R_{x_1 \ldots x_k}\{J\} \equiv$$

$$\equiv \frac{\delta^k}{\delta \bar{J}(x_1) \ldots \delta \bar{J}(x_k)} \mathcal{T}^+\left\{J + i\frac{\bar{J}}{2}\right\} \mathcal{T}\left\{J - i\frac{\bar{J}}{2}\right\}_{|\bar{J}=0} =$$

$$= \frac{\delta^k}{\delta \bar{J}(x_1) \ldots \delta \bar{J}(x_k)} (\bar{\text{T}} \exp\left[\tfrac{1}{2} \textstyle\int dz \bar{J}(z) R_z\{J\}\right] \times$$

$$\times \text{T} \exp\left[\tfrac{1}{2} \textstyle\int du \bar{J}(u) R_u\{J\}\right])_{|\bar{J}=0} . \tag{16}$$

These relativistically invariant real functions correspond to double graphs into which more than one retarded line enters, as obtained by implementing many-particle cuts in ordinary double graphs. $\mathrm{r}(x_1 \ldots x_k, y_1 \ldots y_l)$ vanishes unless each y_λ is earlier than some x_k. From Eq. (8) and Eq. (16) we obtain

$$R_{x_1 \cdots x_k}\{J\} =$$

$$= :\exp\left[\int du A_{\text{in}}(u) K_u \frac{\delta}{\delta J(u)}\right] : \langle R_{x_1 \cdots x_k}\{J\}\rangle . \tag{17}$$

Perturbation theory now suggests the ansatz

$$F = (1 + FE)F_i \tag{18a}$$

where F is the two-by-two matrix with the elements (for merely technical reasons, $x_0 > y_0$ and $z_0 > u_0$ is always understood, also in the integrations)

$$F_{11} = \langle R^c_{\bar{x}\bar{y},\, \bar{z}\bar{u}}\rangle \qquad F_{12} = \langle R^c_{\bar{x}\bar{y}\bar{u},\, \bar{z}}\rangle$$
$$F_{21} = \langle R_{\bar{x},\, \bar{y}\bar{z}\bar{u}}\rangle \qquad F_{22} = \langle R^c_{\bar{x}\bar{u},\, \bar{y}\bar{z}}\rangle , \tag{19}$$

where the superscript c means " connected part ", e.g.

$$\langle R^c{}_{xy,zu}\rangle = \langle R_{xy,zu}\rangle - \langle R_{x,z}\rangle\langle R_{y,u}\rangle - \langle R_{x,u}\rangle\langle R_{y,z}\rangle ,$$

and from now on barring a coordinate means amputation of the full one particle reducibility, not of Δ'_{Ret} alone. F_i is the same matrix as Eq. (19), but with R or R^c replaced by R^i (always connected). E is the matrix with the elements

$$E_{11} = \langle R_{x,z}\rangle\langle R_{y,u}\rangle + \langle R_{x,u}\rangle\langle R_{y,z}\rangle$$
$$E_{12} = \langle R_{x,z}\rangle\langle R_{yu}\rangle + \langle R_{xu}\rangle\langle R_{y,z}\rangle$$
$$E_{21} = 0$$
$$E_{22} = \langle R_{x,z}\rangle\langle R_{u,y}\rangle$$

which, when applied to F or F_i from the right or left, reverses the amputation. x, y are the left, z, u the right hand side coordinates of F etc.

First set $J=0$ in Eq. (18*a*). Then it can in principle be solved for the irreducible functions by the Fredholm method. Of course, we do not know enough about r-functions to find out if the classical method applies. Here we have to rely provisionally upon the assumption that it does, and that the ambiguity due to the possible vanishing of Fredholm denominators is resolved by imposing the condition that the irreducible functions have the same retardedness properties as the corresponding original functions. (The formal iteration solution of (18*a*) has this property.) The uniqueness of F_i from Eq. (18*a*) implies to within interchange of integrations)

$$F = F_i(1 + EF) \tag{18b}$$

It is satisfying that the two forms (18*a*), (18*b*) of the " retarded inhomogeneous Bethe-Salpeter equation " imply each other, because there is no reason to prefer one of them to the other. The case $J \neq 0$ can easily be reduced to the former problem.

However, to give these equations a sense, we have to find out what they imply for the irreducible functions, on the basis of Eq. (13) or Eq. (15). (Note that one- and two-particle reducibilities can be eliminated in an arbitrary order, they never interfere.) First, on the basis of the foregoing discussion, Eq. (18*a*, *b*) are immediately generalized to an arbitrary number of distinguished coordinates instead of the pair z, u, similar to Eq. (14*a*, *b*).

Actually, now one must deal with four systems instead of one. Define the matrix $\widetilde{F}$ with the elements

$$\widetilde{F}_{11} = \langle R^c_{\overline{yz},\overline{xu}} \rangle \qquad \widetilde{F}_{12} = \langle R^c_{\overline{yzu},\bar{x}} \rangle$$
$$\widetilde{F}_{21} = \langle R_{\bar{z},\overline{xyu}} \rangle \qquad \widetilde{F}_{22} = \langle R^c_{\overline{zu},\overline{xy}} \rangle \tag{20}$$

and correspondingly $\widetilde{F}_i$ and $\widetilde{E}$. Then we have

$$\widetilde{F} = (1+\widetilde{F}\widetilde{E})\widetilde{F}_i \tag{21a}$$

from Eq. (18*b*) and

$$\widetilde{F} = \widetilde{F}_i(1+\widetilde{E}\widetilde{F}) \tag{21b}$$

from Eq. (18*a*). Of the four systems, easily derivable from the definitions of Eq. (16) etc., I give only the first, relevant for the 11-element of the matrix $F - \widetilde{F}$:

$$\langle R_{xy,zu} \rangle - \langle R_{yz,xu} \rangle - i\langle [R_{xy}, R_{z,u}] \rangle - i\langle [R_{xy,u}, R_z] \rangle -$$
$$- \tfrac{1}{2}\langle \{R_x, R_{z,yu}\} \rangle - \tfrac{1}{2}\langle \{R_{x,u}, R_{z,y}\} \rangle = 0\,. \tag{22}$$

The others look very similar. Insertion of Eq. (18*b*) and Eq. (21*a*) into these systems and eight more,

$$\langle R_{xy,zu} \rangle - i\langle [R_{xy,z}, R_u] \rangle - \tfrac{1}{2}\langle \{R_{x,z}, R_{u,y}\} \rangle = 0 \tag{23}$$

and seven very similar ones, gives the result :

The solution F of Eq. (18) (unique for given F_i) satisfies Eq. (22) etc. and Eq. (23) etc. if and only if the irreducible functionals F_i satisfy the same (amputated) systems, with all brackets replaced by " irreducible brackets ", and the right hand sides being replaced by functionals x_1^i to x_{12}^i that have to satisfy twelve linear homogeneous equations.

Irreducible brackets are here defined by integral equations of the type (we take the simplest example) :

$$\langle (R_{\bar{x},\bar{u}} R_{\bar{z},\bar{y}})^c \rangle = \langle (R_{\bar{x},\bar{u}} R_{\bar{z},\bar{y}})^i \rangle +$$
$$+ \langle (R_{\bar{x},\bar{r}} R_{\bar{s},\bar{y}})^i \rangle \langle (R_{r,\bar{u}} R_{z,s})^c \rangle \tag{24}$$

which can be solved by iteration that terminates for any finite order of J and any finite momentum transferred between the pair x, u (and further coordinates introduced by differentiation) and y, z (same). One can prove [1] that the irreducible brackets do not permit a two-particle cut (consisting of two one-particle cuts in the earlier sense) that separates x, y and z, u from each other (note that in the example Eq. (24) such cuts would be entirely nonsingular) and do not have two-particle intermediate states for such a separation.

It seems difficult to discuss the terms x_1^i to x_{12}^i on the basis of the homogeneous equations which they must satisfy. These terms are the analogs of the square brackets in Eq. (15). One class of solutions (suggested to me by S. Mandelstam), however, can be found : the two-particle generalizations of the Wigner R-functions that were the solution in the one-particle case, exemplified by an unstable particle in perturbation theory. With the help of a perturbation theoretically verified conjecture, similar to the earlier one, the functions with more than four coordinates of this class, which perhaps exhausts all cases of physical interest, become simple and quite tractable.

Though the actual calculations would, with the present technique, become prohibitively lengthy, there is no doubt about how the generalizations of these structure decompositions to higher particle numbers, or to multiple decompositions (between several coordinate groups) look. In each case in the absorptive part of the irreducible functions (or functionals) intermediate states with the particle number for which irreductibility is imposed do not appear, nor anywhere reducibilities (as defined by the Bethe-Salpeter equation) in the same number of particle lines, whether or not they can give rise to a singularity. This latter point, together with the elimination of the whole continuum to the particle number in question, shows that structure analysis goes quite beyond an elimination of threshold singularities only.

If the conjecture about the occurrence of " CDD " R-functions is correct, the foregoing picture is modified in a very simple way that can be read off from perturbation theory. The absence of the characteristic singularities for the irreducible functions has so far been proven only for the one-particle case, as mentioned earlier, but is not as essential as is the absence of certain intermediate states in the absorptive parts. (In simple cases, this already suffices to exclude those singularities.)

How can such results be utilized for problems of current interest such as analytic and other properties of elastic scattering amplitudes? We have gained little if we had only reduced the original problem to a more complicated one with higher mass thresholds. This reduction step, however, is useful if a less precise knowledge of the irreducible functions, obtained by a rather incomplete analysis of their nonlinear system, already gives new information about the original functions due to their decompositions. This is indeed the case: assume four functions $\mathrm{r}_i(\overline{xy}, \overline{zu})$, $\mathrm{r}_i(\overline{xyu}, \overline{z})$, $\mathrm{r}_i(\overline{x}, \overline{yzu})$, and $\mathrm{r}_i(\overline{xu}, \overline{yz})$ to be given such that they have the *linear* properties following from the eight systems Eq. (23) etc. (written for irreducible functions as explained earlier), and their absorptive parts have the *linear* properties following from Eq. (22) etc. in the same sense. (" Linear properties " means here: the function consists of two, respectively four, terms with as well the retardedness properties (if any) following from the two factors in the bracket, as the momentum-space support resulting by insertion of intermediate states between the factors, under omission of two-particle intermediate states for partitions x, y to z, u. Also, commutators and anticommutators of the same factors must have the correct momentum-space support relationship.) Then the four functions in F, obtained uniquely from e.g. Eq. (18*a*), satisfy the original systems Eq. (22) etc. and Eq. (23) etc., with the brackets given uniquely by Eq. (24) etc. This means: these functions have the same " linear properties " as before, but with the specification that in the elastic region (masses $4m^2$ to $9m^2$ or $16m^2$, depending on selection rules) the brackets to the partition x, y to z, u (like the third term in Eq. (22)) are given correctly by the four-point functions under discussion, as required by Eq. (8), (17) and picking out two-particle intermediate states. This has the consequence that in this region " unitarity " (even off-shell) is satisfied, besides fulfillment of the " linear conditions " everywhere.

To satisfy the known properties of the scattering amplitudes, upon reduction of the foregoing functions to the mass shell, momentum transfer analyticity also has to be secured. This is done by imposing on the irreducible functions entering Eq. (18*a*) four more conditions, the first of which is derived from the system ($x_0 > y_0$, $z_0 > u_0$)

$$\langle R_{xy,zu}\rangle - \tfrac{1}{2}\langle\{R_{x,z}, R_{y,u}\}\rangle - \tfrac{1}{2}\langle\{R_{x,zu}, R_y\}\rangle - \\ - \tfrac{1}{2}\langle\{R_x, R_{y,zu}\}\rangle - \tfrac{1}{2}\langle\{R_{x,u}, R_{y.z}\}\rangle = 0$$

in the usual way.

All these conditions for the irreducible functions certainly have no fewer solutions than Wightman's " linear program " for the four-point function and have, in any case, the multitude of perturbation theoretical ones. Thus, the linear problem, as well as the nonlinear " unitarity " problem up to a higher mass threshold, are solved in terms of the linear problem with the higher mass threshold. The foregoing discussion shows that this is the general solution up to the not yet entirely solved " R-function problem ".

TCP invariance and crossing symmetry are not yet satisfied by our solution. The first can be satisfied by also discussing " advanced functions " and setting up systems very similar to those discussed before, that contain retarded and advanced functions simultaneously. To satisfy crossing symmetry, however, we have not yet found a better way than by an iterative procedure, to be explained here for the simpler case of the ordinary Bethe-Salpeter equation

$$G = (1+GE)G_i \qquad (25)$$

Assume an irreducible and crossing-symmetric kernel G_i^0 is given. (To find the most general such kernel is easy, once the solution of the "linear" problem is known). Then solve

$$G_i = G_i^0 + (G - G_i)^{\mathrm{cr}} \qquad (26)$$

where cr means " crossed ", for instance by iteration:

$$G_i^{n+1} = G_i^0 + (G^n - G_i^n)^{\mathrm{cr}}$$

where G^n is the solution of Eq. (25) with the kernel G_i^n. The convergence of this process can be made plausible if a hard core is built up, since then the wave function (solution G'') is insensitive to a change of the short-range forces.

Comparison of the method presented here with the nonlinear integral-equation method of Chew and Mandelstam shows that no nonlinear equations need be solved here (apart from Eq. (26) for which the iteration solution seems to be safe), but instead one needs to solve the comparatively simple Bethe-Salpeter equation which (up to crossing symmetry) is the natural solution of the unitarity problem. The CDD-ambiguity is exhibited even before an equation has been written down. The disadvantage of the present method is that it uses off-shell quantities. However, these off-shell functions have to be analyzed anyway, at least in the axiomatic scheme.

Unfortunately, the complete solution of the " linear problem " might not be known even after some lapse of time. It is hoped that the methods explained here will lead to new results. Of special interest are the consequences of the structure decompositions for analytical continuations of the original functions, possibily evaluated on the mass shell. In simple cases, it is known that the irreducible functions no longer have the singularities characteristic to the particle number considered, for instance the one-nucleon pole for meson-nucleon scattering. It can be expected that now cuts can also be removed, though our only rudimentary knowledge about most functions makes proofs difficult. However, the fact that in rigorous proofs always the lowest masses make trouble and have here been separately dealt with gives hope for technical improvements.

It is comparatively simple to extend structure analysis to the generalized retarded functions of Steinmann, Ruelle, Araki, and Burgoyne, which seem to be going to play an important role in rigorous proofs.

Structure-analytical considerations give so far only moderate support to the assumption that perturbation theory cannot mislead as far as analytical properties are concerned. Vertices are here replaced by higher-and-higher irreducible ones, and never by points, and the " linear problem " in any stage admits also non-perturbation theoretical singularities. A careful analysis will be needed to ascertain that there is no "residual effect " on the analytic properties of the original functions. In addition perturbation theory (at least if understood in Landau's [8]) sense) does not lead to the CDD singularities of the irreducible functions. Of course, we understand them very little in local field theory.

LIST OF REFERENCES

1. Symanzik, K.J. Math. Phys. **1**, p. 249, 1960.
2. Lehmann, H., Symanzik, K., and Zimmermann, W. Nuovo Cimento **1**, p. 205, 1955.
3. Lehmann, H., Symanzik, K., and Zimmermann, W. Nuovo Cimento **6**, p. 319, 1957.
4. Glaser, V., Lehmann, H., and Zimmermann, W. Nuovo Cimento **6**, p. 1122, 1957.
5. Castillejo, L., Dalitz, R. H., and Dyson, F. J. Phys. Rev. **101**, p. 453, 1956.
6. Lehmann, H., Symanzik, K., and Zimmermann, W. Nuovo Cimento **2**, p. 425, 1955.
7. Dyson, F.J. Phys. Rev. **82**, p. 428, 1951.
8. Landau, L.D. Nuclear Phys. **13**, p. 181, 1959.

DISCUSSION

BREIT: Would it be possible to use these CDD singularities as a definition of an unstable particle?

SIMANZIK: I do not think so. The characteristics of the function in which one inserts a CDD pole into the denominator are not those of an unstable particle because unstable particles correspond to poles in other Riemann sheets. These functions just have zeros along the real axis. I have not investigated this problem in full, but at first sight there seems to be no connection. In perturbation theory these " particles " show up in the bare Hamiltonian, if you wish, but they do not necessarily produce the dynamical characteristics of an unstable particle.

LEHMANN: I would like to make one remark on this point. I agree that this has not been investigated in relativistic theories, but I would say that, in simple models like the Lee model and the charged scalar theory without recoil, I think the zeros on the real axis and the poles in the second Riemann sheet are really closely related.

SYMANZIK: In the more general case here, I do not think the connection is so simple, but I do not know.

LEHMANN: There is the question of how many solutions we may expect from these equations which do not follow from a definite Hamiltonian but from general principles. Sometimes the view has been expressed that there are only a few solutions for given stable particles and the idea is more or less that they correspond to the renormalizable Lagrangians, but I think that in connection with these ambiguities this view has very little support. Even the statement that there are only a few renormalizable Lagrangians is, I believe, wrong because you can certainly add many fields which correspond to particles with unstable masses and couple them all with trilinear terms. You would then have many solutions corresponding to as many parameters as you please.

SYMANZIK: But actually, all these additions should be required to satisfy the unitarity conditions which are extremely strong ones. There are characteristic differences between what is permitted in perturbation theory and what is permitted in reality.

LEHMANN: Sure, but is there reason to believe that the presence of an arbitrary number of unstable particles conflicts with the unitarity?

SYMANZIK: I believe that there are such indications.

LEHMANN: You mean there are conditions, but it is not clear how strong they are and what they really lead to later on.

SYMANZIK: Yes.

OPPENHEIMER: It would seem not so much that one has a small number of solutions as that one is well along on a program of understanding in physical terms all the singularities which can appear, and this question of more or less arbitrary zeros will have to be worked out a little more before this program is reasonably complete. In this program, uniqueness is not what one is seeking but, rather, some connection with the physics and the analytic behavior of these r-functions.

SYMANZIK: Yes. The questions of the arbitrariness of the x_1 to x_{12} for the problem here has no relevance. One can try to find the most general solution of the unitary equations in a certain sector of momentum space without listing the most general solutions of the linear homogeneous system. The interesting point here for me is that one can have everything together—unitarity and analytic properties in one. One does not have to separate them as one does in the Chew-Mandelstam approach. Unfortunately, I forgot to mention one thing, namely, crossing symmetry. These solutions I presented here are not crossing symmetric. Crossing symmetry can be preserved in an infinite iteration process for which I refer to the paper.

TAYLOR: I did not fully understand the separation between the linear and nonlinear problems. You discussed the satisfaction of the nonlinear problem for the nonlinear equations in which you had subtracted the singularities from the one- and two-particle states and you said that these nonlinear equations can then be satisfied essentially up to where inelastic processes come in. Is this a difficult thing to satisfy or is this automatically satisfied in principle by removing one- or two-particles singularities?

SYMANZIK: The trick is to exhibit the low mass structure explicitly and to remove everything that has to do with low masses from the problem. For the remaining problem, all masses are essentially higher; at least, the characteristics of them are higher. Technically this problem is precisely the four-point function problem of Wightman, which is known to have a multitude of solutions. Even if one finds too broad a class of solutions from a superficial analysis of the new system of equations, the original functions would satisfy many more conditions, namely, the non-linear system in the low energy parts. The proof will be given in a paper that should come out in the fall.

BREIT: I did not understand just what you meant by the Bethe-Salpeter equation for one particle. Does this mean a complete description of one particle?

SYMANZIK: The physical one-particle problem is entirely contained in the one-particle propagation function and the analysis of that function presented here does not give anything new beyond what is already known. The one-particle structure is not the structure of one particle but the structure with respect to one-particle connecting lines in graphs. The characteristic singularities of these functions can be proven axiomatically. Those irreducible functions defined here as well as their analytic continuation do not have these characteristic singularities. One can prove that the irreducible functions do not even have complex singularities. This is standard knowledge in the meson-nucleon scattering problem, as Zimmerman first observed.

OPPENHEIMER: Would you expect any great difficulty in continuing this program of removing the higher intermediate states?

SYMANZIK: It would be rather complicated to do it analytically. If one carries this out in general terms, it will not be necessary to do everything analytically, since I think much can be obtained simply by formal manipulations. All of this has been done without ever writing down any integral explicitly. Nevertheless, it has content at the end.

BJORKEN: What is the outlook with regard to removing the singularities in the irreducible kernels with respect to, say, the momentum transfer variable?

SYMANZIK: One simply has to apply the analysis twice with respect to those separations as is done here. If one does this with the momentum transfer singularities, for instance, the pole in nucleon-nucleon scattering, they will be removed. One would be able to show that the pole is removed if one could show that there are no singularities around the pole. This procedure would yield a detailed characteristic of the function in the complex domain. Everything which is done here refers to real values of t. One can say something about the parts that are infinite and something about the parts that are finite, but this does not cover physical needs as one would like to make statements about the behavior in the complex domain.

BJORKEN: Does the nonlinear system hold in that direction for the irreducible parts as well as for the complete r-function?

SYMANZIK: Yes; with respect to those separations, the ordinary system holds. This is the best I can say here without being too specific.

THE USE OF PERTURBATION METHODS IN DISPERSION THEORY (*)

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I. INTRODUCTION

In this talk I will outline some proofs of dispersion relations for every order in perturbation theory. After this I will indicate some further topics that can be studied by perturbation methods.

The following dispersion relations (DR) have now been proved in perturbation theory:

Single-variable DR

(*a*) Vertex parts
(*b*) Forward scattering
(*c*) Non-forward scattering (in a limited range)
(*d*) External-mass DR
(*e*) Internal-mass DR

(*) This work was done under the auspices of the U.S. Atomic Energy Commission.
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