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Lecture notes: Functional Renormalisation Group and Asymptotically Safe Quantum Gravity

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These lecture notes give an introduction to the functional renormalisation group and asymptotically safe quantum gravity. We cover the basics of generating functionals in quantum field theories and derive the Wetterich equation. The anharmonic oscillator is presented as a simple example and compared to perturbation theory with resummation methods. A special focus is set on the implementation of symmetries in the functional renormalisation group. The second half of these notes is dedicated to quantum gravity, explaining the failure of perturbative gravity and how these issues are addressed in the non-perturbative approach. After detailing the implementation of the diffeomorphism symmetry, we present a simple computation in the Einstein-Hilbert truncation and give a short outlook on the state-of-the-art in the field. These notes are based on lectures given at the *XV Modave Summer School in Mathematical Physics*.

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1. Introduction

These lecture notes present an introduction to the functional renormalisation group (FRG) and its application to asymptotically safe quantum gravity. The renormalisation group (RG) deals with the physics of scales and was pioneered by Wilson [1–4]. At different scales, different degrees of freedom dominate a physical system. For example, at small length scales, which is equivalent to high energies and large momenta, quarks and gluons are the dominant degrees of freedom of quantum chromodynamics (QCD). On the other hand, at large length scales, the dominant degrees of freedom are hadrons and, in principle, no knowledge about quarks and gluons is needed to accurately describe the physics at these scales. The RG is the perfect tool to describe these changes of physics over the scales. A guiding question throughout these lecture notes is how the fundamental degrees of freedom at the microscopic scale evolve into macroscopic physics.

The FRG is a functional, non-perturbative RG equation derived by Christof Wetterich in 1993 [5] and is therefore also called the Wetterich equation. Other functional RG equations have been used before like the Wegner-Houghton equation [6] or the Polchinski equation [7]. Compared to those, the Wetterich equation implements the RG process on the quantum effective action, i.e., it aims at computing the generating functional of the one-particle irreducible (1PI) n-point functions. The Wetterich equation is easily applied to various systems and it offers many different systematic approximation schemes. This has led to successful applications in solid-state physics, QCD, quantum gravity and many other fields.

With the FRG we aim at exploring quantum field theories at strong coupling where perturbation theory is failing. The FRG allows us to describe physics beyond perturbation theory. Standard examples of non-perturbative physics are functions that involve e^{-1/x^2}. In these functions, each Taylor coefficient around $x = 0$ is identically vanishing. They are closely related to instanton contributions. Loosely speaking, the FRG captures these contributions since we do not expand around $x = 0$. The FRG also naturally yields results that are all order in the coupling. Typically they appear in the shape of $\sim 1/(1+Cg)$, where g is the coupling and C is a constant or a function of other couplings.

Since the FRG aims at exploring non-perturbative physics, it has to be compared to the lattice, which is the standard tool in non-perturbative physics. Compared to the lattice, the FRG can provide an analytic and thus more fundamental understanding of many physical mechanisms. More importantly, the FRG can be applied in areas where the lattice fails. For example in QCD, a central goal is to obtain the full phase diagram of the theory in terms of the temperature T and the chemical potential μ. We live at low temperature and low density where QCD is confining and the quarks and gluons form hadrons. At high temperatures, which we can probe at collider experiments, we find the quark-gluon plasma, where the quarks and gluons are almost non-interacting due to the asymptotic freedom of the theory. The corresponding phase transition is a crossover at zero chemical potential while it is a first-order phase transition at zero temperature. In between, there is a critical point where the phase transition is second-order. The existence and the location of this critical point is an up-to-date research topic within theory and experiment. From the theoretical side, the lattice has been the most powerful tool to explore the phase diagram. However, the lattice only works as long as μ/T is small due to the infamous sign problem. At non-vanishing chemical potential, the action becomes complex and cannot be used as a probability measure for

Monte-Carlo algorithms anymore. In consequence, lattice simulations become exponentially more expensive with an increasing chemical potential, effectively restricting the accessible region to small μ/T. The FRG is providing a complementary non-perturbative tool to explore the phase diagram of QCD. The FRG does not have a sign problem and thus should be able to describe the phase diagram at a finite chemical potential.

Asymptotically safe quantum gravity is the main area of FRG application that we discuss in these lecture notes. Quantum gravity is a very particular example since ordinary perturbation theory fails, as we will discuss in Sec. 3.1. As a consequence, a plethora of different theories have emerged, for example, string theory [8–13], loop quantum gravity and spin foams [14–21], causal and euclidean dynamical triangulations [22–26], Hořava-Lifshitz gravity [27–30], and causal sets [31–33]. The asymptotic safety scenario conjectures the existence of an interacting fixed point of the RG flow, which allows the non-perturbative quantisation of gravity. The FRG is here a unique non-perturbative tool for this computation within the framework of quantum field theories and without discretising spacetime. Lattice approaches like causal and euclidean dynamical triangulations are probing the same theory and the methods should complement each other in the future.

These notes do not fulfil the purpose of a review and I apologise for any omission of references. For reviews about the FRG see [34–38] and about asymptotic safety see [39–47]. For other lecture notes about the functional renormalisation group, I recommend [48, 49]. I included some ideas of these lecture notes here. For lecture notes about asymptotically safe quantum gravity, I recommend [50].

2. The Functional Renormalisation Group

Throughout these notes, we assume that the QFT has already been Wick rotated and consequently we work with Euclidean signature. The physical information of the theory has to be extracted afterwards by analytic continuation. The derivations in this section are fully general, for all examples, we resort to a scalar theory for simplicity. We start with a recap of generating functionals in QFT, which is a necessary basis for the derivation of the Wetterich equation in the subsequent section.

2.1 Generating functionals

The Euclidean generating functional of correlation functions is given by

$$\mathscr{Z}[J] = \frac{1}{\mathscr{N}} \int \mathscr{D}\varphi \exp\left\{ -S[\varphi] + \int_x J(x)\, \varphi(x) \right\} . \tag{2.1}$$

Here $\mathscr{N}$ is a normalisation factor, $J(x)$ are the sources and $\int_x = \int \mathrm{d}^4 x$. The path integral contains divergences as usual. These divergences need to be regularised and renormalised. In (2.1) we are assuming that this has already been done, e.g., by a cutoff regularisation and thus the generating functional is finite. By taking functional derivatives of (2.1) with respect to the source we can generate the n-point correlation functions

$$\langle \varphi(x_1) \dots \varphi(x_n) \rangle_J = \frac{1}{\mathscr{Z}[J]} \frac{\delta^n \mathscr{Z}[J]}{\delta J(x_1) \dots \delta J(x_n)}$$

n	n-point correlation function			interpretation
0	$\langle 1 \rangle$	$=$	1	normalisation
1	$\langle \varphi(x) \rangle$	$=$	$\phi(x)$	mean field
2	$\langle \varphi(x_1)\varphi(x_2) \rangle$	$=$	$\langle \varphi(x_1)\varphi(x_2)\rangle_\mathrm{c} + \phi(x_1)\phi(x_2)$	propagator
3	$\langle \varphi(x_1)\varphi(x_2)\varphi(x_3) \rangle$	$=$	$\langle \varphi(x_1)\varphi(x_2)\varphi(x_3)\rangle_\mathrm{c} + \dots$	three-point vertex
$\vdots$	$\vdots$		$\vdots$	$\vdots$

Table 1: The n-point correlation functions of a real scalar field theory. The prescription $\langle \cdots \rangle_\mathrm{c}$ denotes connected correlation functions. The table is taken from [49].

$$= \frac{\int \mathscr{D}\varphi\, \varphi(x_1)\cdots\varphi(x_n) \exp\{-S[\varphi] + \int_x J(x)\,\varphi(x)\}}{\int \mathscr{D}\varphi \exp\{-S[\varphi] + \int_x J(x)\,\varphi(x)\}}\,. \tag{2.2}$$

In the second line of (2.2) we display the path integral representation of the correlation function. This representation is not needed for major part of these lecture notes. In Tab. 1 we display the lowest-order correlation functions. There we have distinguished between the connected and not-connected part of the n-point functions. At the level of the propagator, the correlation function contains a connected and a disconnected part, where the latter is described by the mean field. This illustrates that the generating functional does not store information most efficiently: the information of the mean field is already contained in the one-point function and it does not need to be computed again. Consequently, the generating functional of the connected n-point functions, also known as Schwinger functional, is introduced

$$\mathscr{W}[J] = \ln \mathscr{Z}[J]\,. \tag{2.3}$$

The connected n-point functions are generated with functional derivatives with respect to the source

$$\frac{\delta^n \mathscr{W}[J]}{\delta J(x_1)\dots\delta J(x_n)} \equiv \mathscr{W}^{(n)}[J] = \langle \varphi(x_1)\dots\varphi(x_n)\rangle_{J,\mathrm{c}}\,. \tag{2.4}$$

Here we have introduced the notation $\mathscr{W}^{(n)}$ for n functional derivatives. How can we see that the Schwinger functional generates only connected correlation functions? As an example, we look at the propagator

$$\begin{aligned}
\frac{\delta^2 \mathscr{W}[J]}{\delta J(x_1)\delta J(x_2)} &= \frac{\delta^2 \ln \mathscr{Z}[J]}{\delta J(x_1)\delta J(x_2)} = \frac{\delta}{\delta J(x_1)} \frac{1}{\mathscr{Z}[J]} \frac{\delta \mathscr{Z}[J]}{\delta J(x_2)} \\
&= \frac{1}{\mathscr{Z}[J]} \frac{\delta^2 \mathscr{Z}[J]}{\delta J(x_1)\delta J(x_2)} - \frac{1}{\mathscr{Z}[J]^2} \frac{\delta \mathscr{Z}[J]}{\delta J(x_1)} \frac{\delta \mathscr{Z}[J]}{\delta J(x_2)} \\
&= \langle \varphi(x_1)\varphi(x_2)\rangle_J - \langle \varphi(x_1)\rangle_J \langle \varphi(x_2)\rangle_J \\
&= \langle \varphi(x_1)\varphi(x_2)\rangle_{J,\mathrm{c}} \equiv G(x_1,x_2)\,.
\end{aligned} \tag{2.5}$$

Here we have defined the full quantum propagator $G(x_1,x_2)$. Similarly, it is easy to show that also the higher $\mathscr{W}^{(n)}$ correspond to the connected part of the n-point functions.

An even more efficient way to store the information of a quantum theory is the effective action, which is the Legendre transformation of the Schwinger functional with respect to the mean field

$$\Gamma[\phi] = \sup_J \left\{ \int_x J(x)\,\phi(x) - \mathscr{W}[J] \right\} = \int_x J_{\text{sup}}(x)\,\phi(x) - \mathscr{W}[J_{\text{sup}}]\,. \tag{2.6}$$

In the second expression we have picked out a configuration of sources, which maximises the Legendre transform. This supremum of the source is a function of the mean field $J_{\text{sup}}[\phi]$. The effective action generates one-particle irreducible (1PI) n-point functions. 1PI means that the corresponding Feynman diagram cannot be cut into two diagrams by the cut of a single internal line. These 1PI correlation functions are generated from the effective action by functional differentiation with respect to the mean field

$$\Gamma^{(n)}[\phi] \equiv \frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)} = \langle \varphi(x_1)\cdots\varphi(x_n)\rangle_{\text{1PI}}\,. \tag{2.7}$$

The effective action is the quantum analog of the classical action in the sense that it encodes the full quantum physics at tree level.

So far we have only claimed that the effective action generates 1PI diagrams. We illustrate this property again in an inductive way. Let us start with the fact that conjugate variable of the source in (2.6) is indeed the mean field

$$\phi(x) = \left.\frac{\delta\mathscr{W}[J]}{\delta J(x)}\right|_{J_{\text{sup}}} = \frac{1}{\mathscr{Z}[J]}\left.\frac{\delta\mathscr{Z}[J]}{\delta J(x)}\right|_{J_{\text{sup}}} = \langle\varphi(x)\rangle_{J_{\text{sup}}}\,. \tag{2.8}$$

By taking one derivative of the effective action with respect to the mean field, we obtain the quantum equation of motion

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = J_{\text{sup}}(x) + \sup_J \left\{ \int_y \frac{\delta J(y)}{\delta\phi(x)} \underbrace{\left(\phi(y) - \frac{\delta\mathscr{W}[J]}{\delta J(y)}\right)}_{=0} \right\} = J_{\text{sup}}(x)\,. \tag{2.9}$$

We turn now to the two-point function, where we will find that the quantum propagator is the inverse of the 1PI two-point function. Anticipating that result, we compute

$$\begin{aligned}\int_y \frac{\delta^2\mathscr{W}}{\delta J(x_1)\delta J(y)}\frac{\delta^2\Gamma}{\delta\phi(y)\delta\phi(x_2)} &= \int_y \frac{\delta}{\delta J(x_1)}\left[\frac{\delta\mathscr{W}}{\delta J(y)}\right]\frac{\delta}{\delta\phi(y)}\left[\frac{\delta\Gamma}{\delta\phi(x_2)}\right] \\ &= \int_y \frac{\delta\phi(y)}{\delta J(x_1)}\frac{\delta J(x_2)}{\delta\phi(y)} = \delta(x_1-x_2)\,.\end{aligned} \tag{2.10}$$

This proves the relation

$$\mathscr{W}^{(2)}(x_1,x_2) = G(x_1,x_2) = \left(\Gamma^{(2)}(x_1,x_2)\right)^{-1}\,. \tag{2.11}$$

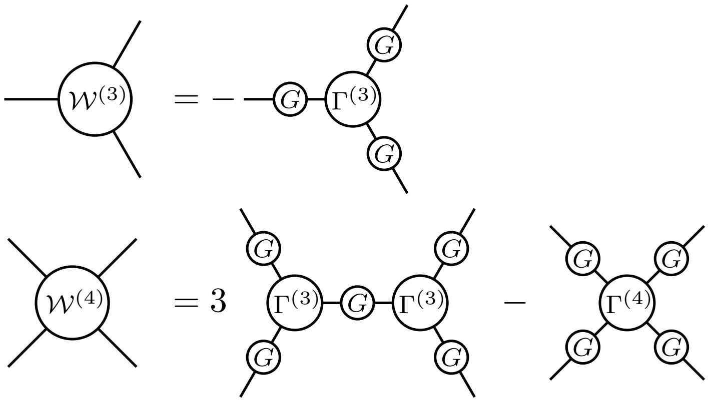


Figure 1: Diagrammatic representation of $\mathscr{W}^{(3)}$ and $\mathscr{W}^{(4)}$ in terms of G, $\Gamma^{(3)}$, and $\Gamma^{(4)}$. The first diagram in the second equation summarised the s, t, and u channel, indicated by the factor 3.

For the higher n-point functions, we need the relation between a derivative with respect to the source and with respect to the mean field

$$\frac{\delta}{\delta J(x)} = \int_y \frac{\delta\phi(y)}{\delta J(x)} \frac{\delta}{\delta\phi(y)} = \int_y \frac{\delta \mathscr{W}^{(1)}(y)}{\delta J(x)} \frac{\delta}{\delta\phi(y)} = \int_y G(x,y) \frac{\delta}{\delta\phi(y)} \,. \tag{2.12}$$

This relation allows us to derive

$$\langle \varphi(x_1)\cdots\varphi(x_n)\rangle_{\text{c}} \equiv \mathscr{W}^{(n)} = \prod_{i=1}^{n-1} \left(\int_{x_i'} G(x_i,x_i') \frac{\delta}{\delta\phi(x_i')} \right) \phi(x_n)\,. \tag{2.13}$$

In (2.13) it is important to notice that the propagator is still a function of the mean field ϕ and that derivatives of the propagator generate the 1PI three-point function

$$\frac{\delta}{\delta\phi(x_1)} G(x_2,x_3) = \frac{\delta}{\delta\phi(x_1)} \left(\Gamma^{(2)}(x_2,x_3)\right)^{-1} = -\int_{y_1,y_2} G(x_2,y_1)\,\Gamma^{(3)}(x_1,y_1,y_2)\,G(x_3,y_2)\,. \tag{2.14}$$

Evaluating (2.13) leads to the explicit representations of the connected correlation functions in terms of 1PI correlation functions. For $n=3$ this leads to

$$\mathscr{W}^{(3)} = -\int_{y_1,y_2,y_3} G(x_1,y_1)G(x_2,y_2)G(x_3,y_3)\Gamma^{(3)}(y_1,y_2,y_3)\,. \tag{2.15}$$

This relation is displayed diagrammatically in Fig. 1, together with the corresponding one for $n=4$. From this diagrammatic representation, it becomes clear that the $\Gamma^{(n)}$ are amputated correlation function and it becomes apparent that $\Gamma^{(3)}$ and $\Gamma^{(4)}$ are indeed the 1PI correlation function as claimed before. This can be shown order by order for the higher n-point function.

The effective action is a very powerful object and so far we have discussed its derivation from the generating functional. For a computation in terms of a path integral, we consider its exponential and the relation to the generating functional (2.1). We obtain

$$e^{-\Gamma[\phi]} = e^{-\int_x J_{\text{sup}}(x)\phi(x) + \mathscr{W}[J_{\text{sup}}]}$$

$$= e^{-\int_x \frac{\delta\Gamma[\phi]}{\delta\phi(x)}\phi(x)} \int \mathscr{D}\varphi\, e^{-S[\varphi]+\int_x J_{\text{sup}}(x)\varphi(x)}$$
$$= \int \mathscr{D}\varphi' e^{-S[\phi+\varphi']+\int_x \varphi'(x)\frac{\delta\Gamma[\phi]}{\delta\phi(x)}} . \tag{2.16}$$

In the last line we performed a shift of the integration variable $\varphi \to \varphi' + \phi$. This is a path integral, where the integrand depends on $\delta\Gamma/\delta\phi$. It can only be solved for very simple cases. The representation in (2.16) is nonetheless useful as it allows to discuss the symmetries of a theory on the quantum level. A systematic approximation scheme of (2.16) is the vertex expansion

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_1,\dots,x_n} \Gamma^{(n)}[\phi = 0](x_1,\dots,x_n)\phi(x_1)\cdots\phi(x_n)\,, \tag{2.17}$$

which we also use later in the FRG context. Inserting (2.17) into (2.16) and comparing the field monomials leads to an infinite tower of integro-differential equations known as Dyson-Schwinger equations [51–53]. This tower can be truncated to a finite amount of equations and, for example, the application to QCD is well developed, see, e.g., [54–57]. We go a different path and combine RG techniques with these functional methods.

2.2 Derivation of the Wetterich equation

The Wetterich equation [5, 58, 59] is a functional differential equation that interpolates from the classical action to the quantum effective action. In comparison to (2.16), where all quantum fluctuations are integrated out at once, the Wetterich equation integrates out these quantum fluctuations shell by shell, in the spirit of a Wilsonian RG. For the derivation, we follow the same steps as in the last section. We start from the generating function, where we suppress IR modes below a scale k. We derive a flow equation for the scale-dependent generating function in the scale parameter k. Subsequently, we switch to the Schwinger functional and eventually transform to the quantum effective action. We denote the generating functional with suppressed IR modes by

$$\mathscr{Z}_k[J] = \frac{1}{\mathscr{N}} \int \mathscr{D}\varphi_{p^2\geq k^2} \exp\left\{-S[\varphi] + \int_x J(x)\,\varphi(x)\right\}, \tag{2.18}$$

where the subscript $p^2 \geq k^2$ indicates that we only include momentum modes above the scale k. For $k \to 0$, we get back the full generating functional $\mathscr{Z}_{k=0} = \mathscr{Z}$. Such a restriction of the path integral does not preserve the symmetries of most QFTs. We come back to this issue in Sec. 2.7.

The full suppression of the IR modes leads to the Wegner-Houghton equation [6]. A more general approach is to introduce a function that smoothly suppresses these modes. Thus, we define

$$\int \mathscr{D}\varphi_{p^2\geq k^2} = \int \mathscr{D}\varphi \, \exp\{-\Delta S_k[\phi]\}\,, \tag{2.19}$$

where

$$\Delta S_k[\phi] = \frac{1}{2}\int_p \phi(p) R_k(p^2)\phi(-p)\,. \tag{2.20}$$

Here we have defined $\int_p = \int \mathrm{d}^4p/(2\pi)^4$ and R_k is the regulator function that suppresses modes with $p^2 \lesssim k^2$ but leaves modes with $p^2 \gtrsim k^2$ unaffected. This can be viewed as a momentum-dependent mass term. The regulator function is required to have three properties

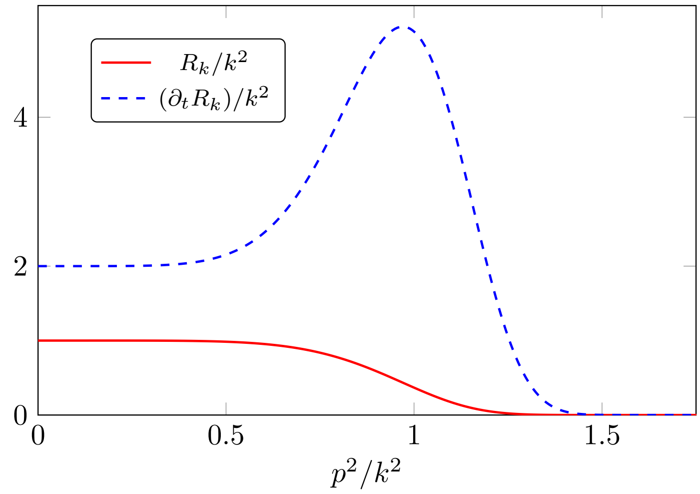


Figure 2: Example of the regulator function $R_k(p^2)$ (red solid curve) and its scale derivative $\partial_t R_k(p^2)$ (blue dashed curve). We plot (2.26) with $n = 3$. The regulator suppresses IR modes for $p^2 \lesssim k^2$ due to its mass-like behaviour. The derivative of the regulator is peaked around $p^2 \approx k^2$ and in this way implements the integration of momentum modes around that scale. It also implements the UV finiteness, since it is vanishing for $p^2/k^2 \to \infty$.

- Suppression of IR modes:

$$\lim_{p^2 \to 0} R_k(p^2) > 0\,. \tag{2.21}$$

- Physical limit to ensure that $\mathscr{Z}_{k=0} = \mathscr{Z}$:

$$\lim_{k \to 0} R_k(p^2) = 0\,. \tag{2.22}$$

- UV-limit to ensure that $\Gamma_{k=\Lambda} = S$:

$$\lim_{k \to \Lambda \to \infty} R_k(p^2) \to \infty\,. \tag{2.23}$$

A frequent parameterisation of the regulator is

$$R_k(p^2) = p^2 r(p^2/k^2)\,, \tag{2.24}$$

where r is the dimensionless shape function of the regulator. A common choice for the shape function is the Litim-type regulator [60, 61]

$$r_{\text{Litim}}(x) = \left(\frac{1}{x} - 1\right)\Theta(1-x)\,. \tag{2.25}$$

This shape function has the advantage that it often provides analytical flow equations. For numerical purposes, the Litim-type shape function is less advantageous since it is not smooth. The exponential shape function is an example for a smooth shape function

$$r_{\exp}(x) = \frac{e^{-x^{2n}}}{x}\,. \tag{2.26}$$

In Fig. 2, we display an example of a regulator, where we have chosen (2.26) with $n = 3$. There we introduced the dimensionless scale derivative $\partial_t = k\,\partial_k$. The parameter t is called the RG time and is defined by $t = \ln \frac{k}{k_0}$, where k_0 is some reference scale. Often the initial scale is used as reference scale, $k_0 = \Lambda$.

Let us now turn back to the scale-dependent generating functional $\mathscr{Z}_k$. We are interested in a flow equation for $\mathscr{Z}_k$ and thus we take a derivative of (2.18) with respect to the RG time t. The term ΔS_k is the only term that is scale-dependent and thus

$$\begin{aligned}\partial_t \mathscr{Z}_k[J] &= \frac{1}{\mathscr{N}} \int \mathscr{D}\varphi \left(-\partial_t \Delta S_k[\varphi]\right) \exp\left\{-S[\varphi] - \Delta S_k[\varphi] + \int_x J(x)\,\varphi(x)\right\} \\ &= -\left\langle \partial_t \Delta S_k[\varphi]\right\rangle \mathscr{Z}_k[J]\,. \end{aligned} \tag{2.27}$$

Another convenient way to express this flow equation is to replace the field by a derivative with respect to the source, $\varphi = \delta/\delta J$. Then we obtain

$$\partial_t \mathscr{Z}_k[J] = -\left(\partial_t \Delta S_k\left[\frac{\delta}{\delta J}\right]\right) \mathscr{Z}_k[J] = -\frac{1}{2}\int_p \frac{\delta^2 \mathscr{Z}_k[J]}{\delta J(p)\delta J(-p)}\partial_t R_k(p^2)\,. \tag{2.28}$$

This is already a useful formulation of the flow equation for the generating functional. As we can see, this is an integro-differential equation, the flow of $\mathscr{Z}_k$ depends on $\mathscr{Z}_k^{(2)}$. Importantly, we do not need to solve a path integral to obtain $\mathscr{Z} = \mathscr{Z}_{k=0}$.

We now switch to a flow equation for the Schwinger functional $\mathscr{W}_k = \ln \mathscr{Z}_k$. Again, the full Schwinger functional is obtained in the limit $k \to 0$, $\mathscr{W}_{k=0} = \mathscr{W}$. We multiply (2.28) with $1/\mathscr{Z}_k$ and use

$$\begin{aligned}\frac{\delta^2 \mathscr{W}_k}{\delta J(x_1)\delta J(x_2)} &= \frac{1}{\mathscr{Z}_k}\frac{\delta^2 \mathscr{Z}_k}{\delta J(x_1)\delta J(x_2)} - \frac{1}{\mathscr{Z}_k^2}\frac{\delta \mathscr{Z}_k}{\delta J(x_1)}\frac{\delta \mathscr{Z}_k}{\delta J(x_2)} \\ &= \frac{1}{\mathscr{Z}_k}\frac{\delta^2 \mathscr{Z}_k}{\delta J(x_1)\delta J(x_2)} - \frac{\delta \mathscr{W}_k}{\delta J(x_1)}\frac{\delta \mathscr{W}_k}{\delta J(x_2)}\,, \end{aligned} \tag{2.29}$$

as well as $\partial_t \mathscr{W}_k = \frac{1}{\mathscr{Z}_k}\partial_t \mathscr{Z}_k$. The flow equation is then given by

$$\partial_t \mathscr{W}_k[J] = -\frac{1}{2}\int_p \left[\frac{\delta^2 \mathscr{W}_k}{\delta J(p)\delta J(-p)} + \frac{\delta \mathscr{W}_k}{\delta J(p)}\frac{\delta \mathscr{W}_k}{\delta J(-p)}\right]\partial_t R_k(p^2)\,. \tag{2.30}$$

The Polchinski equation [7] is a flow equation for the Schwinger functional and it can be obtained from (2.30) by amputating the legs from the connected correlation functions. We turn now to the flow equation for the scale-dependent effective action Γ_k. For this we use a modified Legendre transform compared to (2.6)

$$\Gamma_k[\phi] = \sup_J \left\{\int_x J(x)\phi(x) - \mathscr{W}_k[J] - \Delta S_k[\phi]\right\}. \tag{2.31}$$

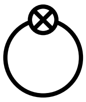
It is a choice to include the term ΔS_k into the Legendre transform. We only need to guarantee that for $k = 0$ the original Legendre transform (2.6) is restored, which is indeed the case since $\Delta S_{k=0} = 0$. We will see that the choice to include ΔS_k in the Legendre transform results in a simpler flow equation. Eq. (2.31) implies that $\Gamma_k + \Delta S_k$ is the Legendre transform of $\mathscr{W}_k$. Thus, the relations (2.8) and (2.9) are modified and now read

$$\frac{\delta(\Gamma_k + \Delta S_k)}{\delta\phi(x)} = J_{\text{sup}}[\phi(x)]\,, \qquad\qquad \frac{\delta \mathscr{W}_k}{\delta J(x)}\bigg|_{J_{\text{sup}}} = \phi(x)\,. \tag{2.32}$$

Consequently, also the relation to the quantum propagator G_k, see (2.11), is modified

$$G_k(p,-p) = \frac{\delta^2 \mathscr{W}_k}{\delta J(p)\delta J(-p)} = \left(\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\phi(p)\delta\phi(-p)}\right)^{-1} = \frac{1}{\Gamma_k^{(2)} + R_k}(p,-p)\,. \tag{2.33}$$

We take now a scale derivative of (2.31) and use (2.30), (2.32), and (2.33). The flow of the scale-dependent effective action is then given by

$$\begin{aligned}
\partial_t \Gamma_k[\phi] &= -\partial_t \mathscr{W}_k[J] - \partial_t \Delta S_k[\phi] + \int_x \partial_t J(x) \underbrace{\left[\phi(x) - \frac{\delta \mathscr{W}_k[J]}{\delta J(x)}\right]}_{=0}\Bigg|_{J=J_{\text{sup}}[\phi]} \\
&= \frac{1}{2}\int_p \left[G_k(p,-p) + \phi(p)\phi(-p)\right]\partial_t R_k(p^2) - \partial_t \Delta S_k \\
&= \frac{1}{2}\int_p \frac{1}{\Gamma_k^{(2)} + R_k}(p,-p)\partial_t R_k(p^2) \\
&= \frac{1}{2}\text{STr}\left[\frac{1}{\Gamma_k^{(2)} + R_k}\partial_t R_k\right] =
\end{aligned}$$

$$. \tag{2.34}$$

This is the Wetterich equation in its most compact form. In the last step, we have generalised our derivation and introduced the super trace, STr. The super trace sums over all discrete indices, such as Lorentz and gauge indices, and integrates over continuous indices, such as space or momentum. It further includes a minus sign for Grassmann valued fields, such as fermions or ghosts. In (2.34), we have also introduced a diagrammatic representation of the Wetterich equation. The solid line stands for the quantum propagator and the cross represents a regulator insertion. We use this diagrammatic notation also later in these notes. Note that all quantities in this equation are fully dressed, i.e., all quantities are formulated in term of the scale-dependent effective action and not in terms of the bare action.

2.3 Properties of the Wetterich equation

It is now time to discuss the properties of the Wetterich equation.

- By construction, the limits of the scale-dependent effective action are given by the bare action in the UV, $\Gamma_{k=\Lambda} = S$, and by the quantum effective action in the IR, $\Gamma_{k=0} = \Gamma$. The latter follows straightforwardly from the definition of Γ_k, (2.31), together with property (2.22) of the regulator. For the former, strictly speaking, only holds up to regulator terms. One uses that the regulator is divergent in the UV, (2.23), and this makes a saddlepoint approximation in (2.31) exact.

- The properties of the regulator guarantee the correct limits of Γ_k, but the regulator serves more purposes. The derivative of the regulator is peaked around $p^2 \approx k^2$, see Fig. 2. This implements that momentum shells around $p^2 \approx k^2$ are integrated out. Furthermore, the Wetterich equation is inherently finite due to the regulator in the UV as well as in the IR

$$\frac{1}{\Gamma_k^{(2)} + R_k} \qquad \longleftrightarrow \qquad \text{IR finiteness,}$$
$$\partial_t R_k \qquad \longleftrightarrow \qquad \text{UV finiteness.}$$

- We can interpret a solution to (2.34) as trajectory between the bare action and the quantum effective action in theory space. The theory space is the infinite-dimensional space of all couplings. The couplings are the prefactors of all operators that are compatible with the symmetry of the theory. We display a sketch of the theory space in Fig. 3.

- The Wetterich equation depends explicitly on the regulator, which we have just introduced as a tool to integrate out fluctuations. We have certain restrictions on the choice of the regulator, see (2.21), (2.22), and (2.23), but besides that, we are free to choose any function. This dependence on the regulator vanishes by construction for $k = 0$, where we end up with the quantum effective action. We can visualise this in theory space: different regulators correspond to different trajectories but the endpoint remains the same, see Fig. 3. The regulator dependence of the trajectories reflects the scheme dependence of non-universal quantities that is present in any QFT. Unfortunately, the regulator-independence does not hold in practical computations. If we truncate the coupling space and employ approximation, then different regulators lead to different quantum effective actions. It is an important task to find regulators that allow for a fast convergence, which is known as optimisation of the regulator function [60–62].

- It is worth to compare (2.34) with (2.16). In (2.16), the quantum effective action was given by in terms of a path integral, while (2.34) has reduced the task to a functional differential equation. Furthermore, it is easier to implement systematic approximation schemes to (2.34).

- We can expand (2.34) in loop orders and by that retain perturbation theory. We expand the scale-dependent effective action with $\Gamma_k = S + \Gamma_{k,\text{1-loop}}$. The Wetterich equation is 1-loop on the right-hand side and thus at 1-loop order we get

$$\partial_t \Gamma_{k,\text{1-loop}} = \frac{1}{2}\text{Tr}\left[\frac{1}{S^{(2)} + R_k}\partial_t R_k\right] = \partial_t \frac{1}{2}\text{Tr}\left[\ln(S^{(2)} + R_k)\right] , \tag{2.35}$$

  where we used in the last step that $S^{(2)}$ is not k dependent. Now it follows that

$$\Gamma_{\text{1-loop}} = \frac{1}{2}\text{Tr}\ln S^{(2)} + \text{const.}\,, \tag{2.36}$$

  which is the standard formula for the 1-loop effective action.

The flow equations for the 1PI n-point correlation functions are generated from (2.34) by functional derivatives with respect to the mean field. It is important to remember that all n-point functions in

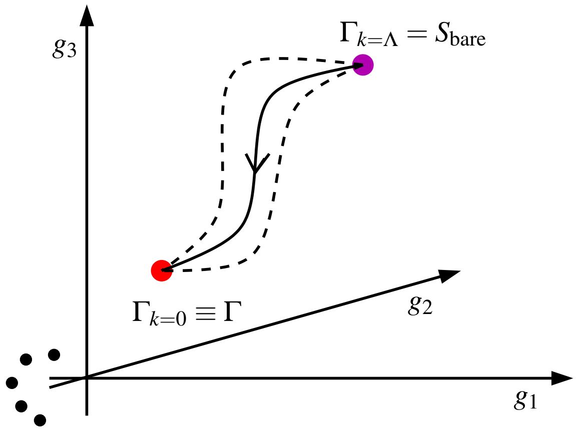
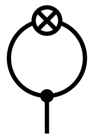
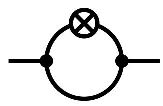
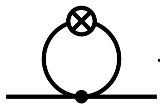


Figure 3: Sketch of the theory space, which is the infinite-dimensional space of all couplings compatible with the symmetry of the theory. The bare action (purple dot) and the quantum effective action (red dot) are beginning and the end point of a trajectory, which is a solution to the Wetterich equation (2.34). Different choices of regulators correspond to different trajectories, but they all lead to the same quantum effective action as long as no truncation error is made. The figure is taken and adapted from [48].

(2.34) still carry field dependence, for instance, $G_k[\phi] = (\Gamma_k^{(2)}[\phi] + R_k)^{-1}$. Consequently, a field derivative acts on the propagator with $\delta/\delta\phi\, G_k = -\Gamma_k^{(3)} G_k \Gamma_k^{(3)}$. The flow equation for the one-point function is given by

$$\partial_t \Gamma_k^{(1)} = -\frac{1}{2} \text{Tr}\Big[G_k \Gamma_k^{(3)} G_k \partial_t R_k\Big] =$$  $$, \tag{2.37}$$

while the flow equation for the two-point function reads

$$\partial_t \Gamma_k^{(2)} = -\frac{1}{2} \text{Tr}\Big[G_k (\Gamma_k^{(4)} - 2\Gamma_k^{(3)} G_k \Gamma_k^{(3)}) G_k \partial_t R_k\Big] =$$  $$-\frac{1}{2}$$  $$. \tag{2.38}$$

In these equations, we can observe an important hierarchy. The flow of the effective action (2.34) depends on the two-point function, but not on higher n-point functions. The flow of the one-point function (2.37) depends on the two- and three-point function, and the flow of the two-point function (2.38) depends on the two-, three- and four-point function. In summary, $\partial_t \Gamma_k^{(n)}$ depends on $\Gamma_k^{(m)}$ with $m = 2, \dots, n+2$. This suggests a systematic expansion scheme of the Wetterich equation: the vertex expansion. The vertex expansion of the scale-dependent effective reads

$$\Gamma_k[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_1,\dots,x_n} \Gamma_k^{(n)}[\phi = 0](x_1, \dots, x_n)\, \phi(x_1) \cdots \phi(x_n)\,. \tag{2.39}$$

Plugging this ansatz into (2.34) leads to an infinite tower of coupled differential equations, with the first ones precisely given by (2.37) and (2.38). However, an important difference is that the $\Gamma_k^{(n)}$

are now evaluated at vanishing field $\phi = 0$. We can now cut the vertex expansion (2.39) for certain n_{max} and systematically improve the truncation by increasing n_{max}.

Remarkably, all flow equations, (2.34), (2.37), and (2.38), are non-perturbative one-loop equations. The key point is that all quantities on the right-hand side of the Wetterich equation are fully dressed, i.e., we do not use the bare propagator and vertices on the right-hand side. At the scale where we initialise the flow $k = \Lambda$, the scale-dependent effective action is the bare action $\Gamma_{k=\Lambda} = S$ and thus at that scale, the full quantum propagator equals the bare propagator. As we successively lower the RG scale, we feed back the changes of the propagator and the vertices at each RG step. In this way, the flow iteratively collects loop contributions to all orders.

2.4 Beta functions and fixed points

The Wetterich equation is not only a tool to compute the quantum effective action, it also allows for the computation of beta functions. We expand the scale-dependent effective action in operators with scale-dependent couplings

$$\Gamma_k[\phi] = \sum_i \bar{g}_i(k)\mathscr{O}_i(\phi)\,. \tag{2.40}$$

The $\bar{g}_i(k)$ are precisely the couplings that span the infinite-dimensional theory space, depicted in Fig. 3. The scale derivative of the couplings result in the respective beta functions $k\,\partial_k\,\bar{g}_i(k) = \partial_t\,\bar{g}_i(k) = \beta_{\bar{g}_i}$. By taking the scale derivative of (2.40), we obtain

$$\partial_t\Gamma_k[\phi] = \sum_i \beta_{\bar{g}_i}\mathscr{O}_i(\phi)\,, \tag{2.41}$$

and thus the flow of the scale-dependent effective action can be seen as beta functional. Since the effective action is dimensionless, the couplings $\bar{g}_i$ have the inverse dimension of the operator $\mathscr{O}_i$, which we denote by $d_{\mathscr{O}_i} = -d_{\bar{g}_i}$. In (2.40) we include operators of all mass dimensions, not only marginal operators. We make the couplings $\bar{g}_i$ dimensionless with appropriate powers of the RG scale k

$$g_i = \bar{g}_i\,k^{-d_{\bar{g}_i}}\,. \tag{2.42}$$

The beta functions of the dimensionless couplings g_i are then given by

$$\beta_{g_i} = -d_{g_i}g_i + k^{-d_{\bar{g}_i}}\beta_{\bar{g}_i}\,. \tag{2.43}$$

We see that the beta functions have a canonical part, which stems from the mass dimension of the coupling, and a part that comes from quantum fluctuations.

Fixed points of the renormalisation group flow are of particular interest since a theory becomes scale invariant there. They are defined by the vanishing of the beta functions of all dimensionless couplings

$$\beta_{g_i}(\vec{g}^*) = 0\,, \quad \text{for all } i\,. \tag{2.44}$$

Here, $\vec{g}^*$ are the values of the fixed point. In theory space, fixed points are points where trajectories of the flow equation begin or end. At these fixed points, the theory becomes quantum scale invariant. There is an important distinction: theories can have a classical scale invariance, which is often

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broken by the quantum fluctuations. Here, the classical theory is not necessarily scale invariant but it becomes scale invariant due to the quantum fluctuations. The most famous example for a fixed point is $g^* = 0$ in QCD, which is responsible for the asymptotic freedom of the theory. At this fixed point, the couplings are vanishing and we called it a Gaußian fixed point. If the couplings are non-vanishing, we call the fixed point non-Gaußian or interacting. Examples for interacting fixed points are the Banks-Zaks fixed point [63, 64], the Wilson-Fisher fixed point [65] and a fixed point in the Veneziano limit of scalar-gauge-Yukawa models, also known as Litim-Sannino model [66, 67].

Fixed points can either serve as a start point of an RG trajectory, in which case we call it an ultraviolet (UV) fixed point, or they can serve as an endpoint of an RG trajectory, in which case it is an infrared (IR) fixed point. Some fixed points are can only serve as UV fixed point, some only as IR fixed point, some can be both. In the examples above, the Banks-Zaks and the Wilson-Fisher fixed points are IR, while the Litim-Sannino fixed point is UV. This property is determined by the flow in the vicinity of the fixed point. Consequently, we characterise a fixed point with the linearised beta functions around the fixed point. The linearised beta functions are given by

$$\beta_{g_i}(\vec{g}) = \underbrace{\beta_{g_i}(\vec{g}^*)}_{=0} - \sum_j B_{ij}(\vec{g}^*)(g_j - g_j^*) + \mathscr{O}\big((g_j - g_j^*)^2\big) \,, \tag{2.45}$$

where we have introduced the stability matrix B_{ij} defined by

$$B_{ij}(\vec{g}) = -\frac{\partial \beta_{g_i}(\vec{g})}{\partial g_j} \,. \tag{2.46}$$

We call the eigenvalues of the stability matrix θ_j and corresponding eigenvectors $\vec{b}_j$. Let us look at the flow in one eigendirection of the stability matrix, i.e, $\vec{g} = \vec{g}^* + c(t)\vec{b}_j$. The linearised flow equation (2.45) reduces to $\partial_t c(t)\vec{b}_j = -\theta_j c(t)\vec{b}_j$ (no summation over j) and thus $c(t) \sim e^{-\theta_j t}$. Now the sign of θ_j becomes important. If θ_j is positive and we run towards the UV ($t \to \infty$), then $c(t)$ is decreasing and we flow into the fixed point. Consequently, this is called a UV attractive direction of the fixed point. If θ_j is negative, then $c(t)$ increases for $t \to \infty$ and we flow away from the fixed point. This is called a UV repulsive direction of the fixed point. If we run towards the IR ($t \to -\infty$), then the roles are precisely opposite. For positive θ_j, we flow away from the fixed point and the direction is IR repulsive and for negative θ_j, we flow into the fixed point and the direction is IR attractive. If $\theta_j = 0$, we call the direction marginal. In this case, the higher-order contributions in (2.45) have to be considered.

In the discussion so far, we implicitly assumed that the θ_j are real, however, they do not have to be. Assuming that the beta functions are real, then the eigenvalues can show up as complex conjugate pairs. For the discussion above, the real part of these eigenvalues determines whether the direction is attractive or repulsive. The imaginary part is responsible for a spiralling of the flow around the fixed point. We will see later examples of this in the quantum gravity part. To summarise, the real part of the eigenvalues determines whether a given direction of a fixed point is attractive or repulsive in the following way

$$\Re(\theta_j) > 0 \qquad \longleftrightarrow \qquad \text{UV attractive/IR repulsive,}$$
$$\Re(\theta_j) < 0 \qquad \longleftrightarrow \qquad \text{UV repulsive/IR attractive,}$$

$$\Re(\theta_j) = 0 \qquad \longleftrightarrow \qquad \text{marginal.} \tag{2.47}$$

For a Gaußian fixed point, the eigendirections align exactly with the couplings and the eigenvalues are given by the mass dimension of the operator. For a non-Gaußian fixed point, the situation is more complicated since the eigendirections can have overlap with many couplings and even have imaginary parts. Nonetheless, it is useful to separate the eigenvalues in a canonical part from the mass dimension of the operator and an anomalous part which stems from the quantum fluctuations. The size of the anomalous part can be used as a measure how non-perturbative the fixed point is.

The critical exponents of a fixed point are crucial information about a fixed point. Indeed, the values of a non-Gaußian fixed point are non-universal, while the critical exponents are. We illustrate this on a simple one-dimensional example. We consider the beta function of a coupling g, with mass dimension d_g and a fixed-point value g^*. We write the beta function as

$$\beta_g = d_g\, g + \beta_1\, g^2 + \beta_2\, g^3 + \dots\,, \tag{2.48}$$

and perform a transformation of the coupling according to

$$\tilde{g} = f(g) = g + f_1\, g^2 + f_2\, g^3 + \dots\,. \tag{2.49}$$

This can be interpreted as a change of the renormalisation scheme. The beta function of the new coupling $\tilde{g}$ is now given by

$$\beta_{\tilde{g}} = \partial_t f(g) = f'(g)\partial_t g = f'(g)\beta_g\,. \tag{2.50}$$

Here we see already that $\beta_{\tilde{g}}$ is vanishing if we evaluate it at g^* and consequently the new fixed-point value is given by $\tilde{g}^* = f(g^*)$. Thus fixed-point values change under simple redefinitions of the coupling. Let us continue and express (2.50) as a function of $\tilde{g}$ and expand in $\tilde{g}$. The resulting beta function is

$$\begin{aligned}\beta_{\tilde{g}} &= d_g\, \tilde{g} + (\beta_1 + d_g f_1)\, \tilde{g}^2 + \left(\beta_2 - 2d_g f_1^2 + 2d_g f_2\right) \tilde{g}^3 \\ &\quad + \left(\beta_3 - f_1\beta_2 + (f_2 - f_1^2)\beta_1 + 5d_g f_1^3 - 8d_g f_1 f_2 + 3d_g f_3\right) \tilde{g}^4 + \mathcal{O}(\tilde{g}^5) \\ &\overset{d_g=0}{=} \beta_1\, \tilde{g}^2 + \beta_2\, \tilde{g}^3 + \left(\beta_3 - f_1\beta_2 + (f_2 - f_1^2)\beta_1\right) \tilde{g}^4 + \mathcal{O}(\tilde{g}^5)\,. \end{aligned} \tag{2.51}$$

We see that for $d_g = 0$ the first two coefficients of the beta function remain unchanged and only the higher-order coefficients are affected by the scheme change. This is known as two-loop universality of the beta function and it holds for marginal couplings in mass-independent renormalisation schemes. The FRG is not a mass-independent renormalisation scheme and thus one does not necessarily obtain the universal two-loop coefficient [68]. As illustrated in (2.51), for non-marginal couplings, i.e., couplings with $d_g \neq 0$, not even the one-loop coefficient is universal. Also contributions from non-marginal couplings to the running of marginal couplings are not universal.

Let us have a look at the critical exponents under the same scheme transformation. The derivative of the beta function with respect to the coupling can be expressed as

$$\begin{aligned}\frac{\partial \beta_{\tilde{g}}}{\partial \tilde{g}} &= \frac{\partial g}{\partial \tilde{g}} \frac{\partial}{\partial g} f'(g)\beta_g = \frac{1}{f'(g)} \left(f'(g) \frac{\partial \beta_g}{\partial g} + f''(g)\beta_g \right) \\ &= \frac{\partial \beta_g}{\partial g} + \frac{f''(g)}{f'(g)}\beta_g \overset{g=g^*}{=} \frac{\partial \beta_g}{\partial g}\,. \end{aligned} \tag{2.52}$$

At the fixed point, the stability matrix and the critical exponents remain unchanged and are thus universal.

2.5 Choice of the bare action

At this point, we pause for a second and think how in practical computations the bare action at the initial scale is chosen. In this context, it is useful to distinguish between a fundamental and a non-fundamental QFT. A fundamental QFT allows for a description on all energy scales, i.e., it allows to push the initial scale towards infinity $\Lambda \to \infty$. The best known example in this category is QCD. The theory becomes asymptotically free in the UV due to the Gaußian fixed point. In the last section, we have explained that fixed points can also be interacting, which generalises asymptotic freedom to asymptotic safety. Asymptotic safety can show up in a perturbative way as in the Litim-Sannino model [66, 67] or in a non-perturbative way, as in quantum gravity [69], gauge theories at large N_f [70–75] or other matter models [76]. It is very challenging to prove the existence of a non-perturbative fixed point and thus we just state that these theories might be asymptotically safe. The existence of a UV fixed point makes a theory fundamental. If we want to describe such a theory, the UV fixed point dictates the initial conditions and we can send the initial scale to infinity $\Lambda \to \infty$. The remaining freedom is how to flow away from the fixed point, which is described by the UV attractive directions of the fixed point as explained in the last section, see (2.47).

Most theories do not have a UV fixed point and are thus not fundamental. These theories typically have a Landau pole at finite energy. For example in the Standard Model, the $U(1)$ hypercharge coupling is diverging around 10^{40} GeV, assuming no additional particle content. Scalar and $U(1)$ theories by themselves are already not fundamental. If we insist and force the cutoff towards infinity, the theory becomes free at all scales. This is known as the triviality problem, see [77–81] for scalar theories and [82–84] for $U(1)$. In non-fundamental theories, the chosen initial scale is typically motivated by physics. It is the scale where we understand well the dominating degrees of freedom or where we want to parameterise new physics. For example, the Standard Model couplings are well described by perturbation theory up to the Planck scale. We can initialise the flow at the Planck scale with the initial values provided by perturbation theory [85] and if we set all higher-dimensional operators to zero, we obtain just the same IR physics as in perturbation theory. But we can also parameterise new physics by including some higher-dimensional operators at a scale Λ, see [86–91] for works in that direction. Here, the FRG works similar to an effective field theory with the difference that the higher-dimensional operator does not have to be small and we are not restricted to polynomial higher-dimensional operators.

We have introduced the RG scale k just as a tool to interpolate between the bare action and the quantum effective action. Thus, this scale is a priori unphysical, just as beta functions are unphysical. The physics is contained in the quantum effective action at $k = 0$. However, this does not imply that it is impossible to extract physical information at finite k. Also in perturbation theory, the first-order correction to the Coulomb potential can be extracted from a scale identification of the running of the electric charge with the distance. Similarly, the scale k can often be identified with a physical scale such as distance or momentum if the system has only one physical scale. However, one has to be careful with artefacts from the regulator.

2.6 Example: the anharmonic oscillator

We consider the quantum mechanical anharmonic oscillator as a simple example. We emphasise in particular the comparison of the FRG with ordinary perturbation theory. The idea for this example was taken from [48], where a lot of details are very nicely discussed. Here we also include the comparison to resummed perturbation theory.

The action for the anharmonic oscillator is given by

$$S = \int \mathrm{d}\tau \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \frac{1}{24}\lambda x^4 \right) . \tag{2.53}$$

The dot indicates a derivative with respect to τ. We consider only the case with $\omega^2 > 0$ and $\lambda > 0$. The symmetry-breaking case with $\omega^2 < 0$ was investigated with the FRG in [92, 93]. We are interested in the ground state energy in the non-perturbative regime, i.e., for large $\lambda \gg 1$. We display the computation with the FRG and with perturbation theory.

FRG We have to make an ansatz for the scale-dependent effective action. We choose the truncation

$$\Gamma_k[x] = \int \mathrm{d}\tau \left(\frac{1}{2}\dot{x}^2 + V_k(x) \right) . \tag{2.54}$$

In a larger truncation we could add, for example, an x-dependent wave-function renormalisation to the kinetic term $\frac{1}{2}Z_k(x)\dot{x}^2$, or higher-derivative terms. We do not include these terms since we are interested in the ground-state energy and the contribution from these terms is subleading. From (2.54), we compute the second derivative with respect to x, which is given by

$$\frac{\delta\Gamma_k[x]}{\delta x(\tau_1)\delta x(\tau_2)} = \left(-\partial_{\tau_1}^2 + V_k''(x)\right)\delta(\tau_1 - \tau_2) . \tag{2.55}$$

We furthermore perform a Fourier transformation from τ to p and obtain

$$\Gamma_k^{(2)}[x] = p^2 + V_k''(x) . \tag{2.56}$$

Now we need to choose the regulator function. As explained in (2.25), the Limit-type regulator is advantageous since it allows for analytic flow equations. For the anharmonic oscillator, it is even an optimised regulator [60, 61]. It is given by

$$R_k = (k^2 - p^2)\Theta(k^2 - p^2) . \tag{2.57}$$

From this we can compute the scale derivative of the regulator

$$\partial_t R_k = 2k^2\Theta(k^2 - p^2) + 2k^2(k^2 - p^2)\delta(k^2 - p^2) = 2k^2\Theta(k^2 - p^2) , \tag{2.58}$$

where we have used that $x\,\delta(x) = 0$. From (2.56) and (2.57), we obtain the full propagator

$$G_k = \frac{1}{\Gamma_k^{(2)} + R_k} = \frac{1}{p^2 + (k^2 - p^2)\Theta(k^2 - p^2) + V''(x)} = \begin{cases} \frac{1}{k^2 + V''(x)} & \text{for } p^2 \le k^2 \\ \frac{1}{p^2 + V''(x)} & \text{for } p^2 \ge k^2 . \end{cases} \tag{2.59}$$

We have now all ingredients to write down the full flow equation. We are only interested in the ground-state energy and thus we consider only vanishing external momentum. This leads us directly to a flow equation for the effective potential

$$\partial_t V_k(x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \frac{2k^2\Theta(k^2 - p^2)}{k^2 + V_k''(x)} = \frac{1}{\pi}\frac{k^3}{k^2 + V''(x)} . \tag{2.60}$$

The initial condition of this flow equation at $k=\Lambda$ is given by the potential in the bare action (2.53). This flow equation can already be solved with numerical methods, for example with a grid in the potential or with pseudo-spectral methods [91, 94, 95]. It is more instructive and suffices for our purposes to expand the potential in a polynomial

$$V_k(x) = \tilde{E}_k + \frac{1}{2}w_k^2 x^2 + \frac{1}{24}\lambda_k x^4 + \dots . \tag{2.61}$$

In comparison to the bare action (2.53), the couplings ω_k and λ_k are now scale dependent. The flow equation evolves them from their classical value, $\omega_\Lambda = \omega$ and $\lambda_\Lambda = \lambda$, to their values ω_0 and λ_0 where all quantum fluctuations are taken into account. The parameter $\tilde{E}_k$ is not the desired ground-state energy but shifted by an unphysical constant due to the regulator. One can avoid this unphysical shift by adapting the implementation of the regulator [96]. Here, we subtract the unphysical constant by demanding that the ground-state energy should be zero for $w=\lambda=0$. In other words, the initial condition $\tilde{E}_\Lambda$ depends on Λ and we have to adjust it with counterterms such that $\tilde{E}_0 = 0$ for $w=\lambda=0$. From (2.60) and (2.61), we obtain a flow equation for $\tilde{E}_k$ by setting $x=0$

$$\partial_k V_k(x=0) = \partial_k \tilde{E}_k = \frac{1}{\pi}\frac{k^2}{k^2+w_k^2} . \tag{2.62}$$

We shift $\tilde{E}_k$ to E_k, such that $E_k = 0$ for $w=\lambda=0$ with the initial condition $E_\Lambda = 0$. The shifted flow is given by

$$\partial_k E_k = \frac{1}{\pi}\left(\frac{k^2}{k^2+w_k^2} - 1\right) . \tag{2.63}$$

From (2.60) and (2.61), we can easily obtain the other flow equations

$$\partial_k w_k^2 = -\frac{1}{\pi}\frac{k^2}{\left(k^2+w_k^2\right)^2}\lambda_k , \tag{2.64}$$

$$\partial_k \lambda_k = \frac{6}{\pi}\frac{k^2}{\left(k^2+w_k^2\right)^3}\lambda_k^2 . \tag{2.65}$$

The flow of the ground-state energy (2.63) depends only on ω_k but not λ_k. Diagrammatically it corresponds to a bubble without external legs as in (2.34). The flow of the frequency (2.64) depends now also on λ_k, but it does not depend on higher-order operators, that we have neglected in (2.61). Diagrammatically it corresponds to the second diagram in (2.38). The flow of λ_k (2.65) would depend on higher-order terms $\sim x^6$. This is an important hierarchy that justifies the polynomial expansion for the computation of the ground-state energy. A term $\sim x^6$ only enters in $\partial_k\lambda_k$, which in turn enters in $\partial_k\omega_k$, which is the only quantity that influences $\partial_k E_k$.

We can now integrate numerically (2.64) and (2.65) for different values of the initial condition ω and λ. We plug the resulting functions, ω_k and λ_k, into (2.63), which we also integrate numerically and obtain the ground-state energy E_0. The results are displayed in Fig. 4 and Fig. 5. We discuss them after introducing the results from perturbation theory.

For analytic results, we need to expand the flow equations in a perturbation series. Generally, I would not recommended this procedure since the FRG computations are more tedious than ordinary perturbation theory. Also, the validity of the FRG results reduces to small values of λ since we expand perturbatively. Nonetheless, it is an instructive computation since all integrals can be performed analytically. For this computation we drop the scale dependence of the quartic coupling $\lambda_k = \lambda$ and expand the frequency in powers of λ, $w_k^2 = w_{0.k}^2 + w_{1,k}^2\lambda + w_{2,k}^2\lambda^2$. We plug this into (2.64), expand again in λ and integrate down from ∞ to some scale k. We use the result to integrate (2.63) from $k = \infty$ to $k = 0$, which gives us the desired ground state energy. The boundary conditions are given by $w_{0,k=\infty} = w$ and $w_{i,k=\infty} = 0$. The flow of w_k^2 is vanishing at $\mathcal{O}(\lambda^0)$ and hence $w_{0,k} = w$. At order $\mathcal{O}(\lambda)$ and $\mathcal{O}(\lambda^2)$ we have

$$\partial_t w_{1,k}^2 = -\frac{1}{\pi}\frac{k^2}{(k^2+w^2)^2}\,, \qquad \partial_t w_{2,k}^2 = \frac{2}{\pi}\frac{k^2 w_{1,k}^2}{(k^2+w^2)^2}\,. \tag{2.66}$$

The corresponding integrals can be solved analytically, for example, with Mathematica. We plug these results into (2.63) and integrate them down, which again can be done analytically. The result is

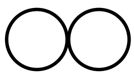
$$\begin{aligned} E_0 &= -\int_0^\infty \mathrm{d}k\,\partial_k E_k = \int_0^\infty \mathrm{d}k\,\frac{1}{\pi}\left(1-\frac{k^2}{k^2+w_k^2}\right) \\ &= \frac{1}{2}w + \frac{3}{4}\left(\frac{\lambda}{24w^3}\right)w - \frac{3}{16\pi}(8\pi^2+29)\left(\frac{\lambda}{24w^3}\right)^2 w + \dots\,. \end{aligned} \tag{2.67}$$

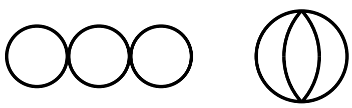
Let us compare this result to ordinary perturbation theory

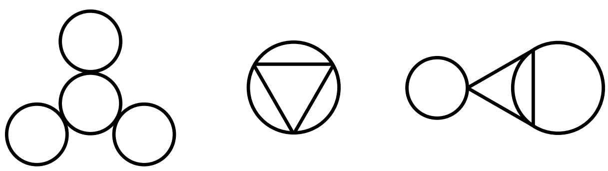
$$E_{0,\mathrm{PT}} = \frac{1}{2}w + \frac{3}{4}\left(\frac{\lambda}{24w^3}\right)w - \frac{21}{8}\left(\frac{\lambda}{24w^3}\right)^2 w + \dots\,. \tag{2.68}$$

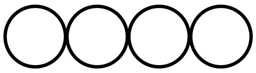
The one-loop coefficient is universal and thus the got the right answer. The two-loop coefficient in (2.67) depends on the choice of the regulator. With the Litim regulator (2.57) we obtained a coefficient that is $\sim 20\%$ too small. An interesting feature is that the FRG two-loop result is a combination of a rational and an irrational number, while all coefficients in perturbation theory remain rational. This is due to the regulator: with a different regulator, the coefficient can become rational. In summary, the FRG has given us a good estimate of the perturbative two-loop coefficient. However, it is most powerful with a numerically integration of the flow equations as we will see soon.

Perturbation theory The standard approach in perturbation theory is the computation in terms of Feynman diagrams. The lowest loop orders are given by

$\mathcal{O}(\lambda^1)$:  (2.69a)

$\mathcal{O}(\lambda^2)$:  (2.69b)

$\mathcal{O}(\lambda^3)$: 

 (2.69c)

All momentum integrals can be solved easily and we obtain the result in (2.68). The computation of higher loop orders via Feynman diagrams is very limited. The number of Feynman diagrams at each order grows factorially and also the momentum integrals become more and more challenging to solve. The growing number of Feynman diagrams reflects an important property of the perturbation series: it is an asymptotic series, i.e., the radius of convergence is zero. This implies that the estimate for the ground state energy with an increasing order in perturbation theory gets worse for large λ while it improves for small λ. The optimal order in perturbation theory for a fixed λ is roughly given by $N \approx 1/\lambda$. Fortunately, this does not imply that we cannot get good estimates of the ground-state energy for large λ, it only means that we have to employ resummation methods like Borel resummation or Padé methods.

With Feynman diagrams we cannot get to high-orders in the perturbation series. However, in this computation we are not doing QFT, we are only doing quantum mechanics and thus we can get help from the Schrödinger equation. The Schrödinger equation for the anharmonic oscillator is given by

$$\left(-\partial_x^2 + \frac{1}{2}x^2 + \frac{1}{24}\lambda x^4\right)\phi(x) = E(\lambda)\phi(x)\,. \tag{2.70}$$

With an appropriate ansatz, one can obtain a difference equation, which allows to determine the coefficients of the perturbative series recursively. This was first derived by Bender and Wu [97]. The recursive equation is given by

$$2jB_{i,j} = (j+1)(2j+1)B_{i,j+1} + B_{i-1,j-2} - \sum_{p=1}^{i-1} B_{i-p,1}B_{p,j}\,, \tag{2.71}$$

with the initial conditions $B_{0,0} = 1$, $B_{i,0} = B_{0,i} = 0$, and $B_{i,j} =$ if $j > 2i$. Then, $a_n = (-1)^{n+1}B_{n,1}$ are the coefficients of the series

$$E_0 = \sum_n a_n \left(\frac{\lambda}{24w^3}\right)^n w\,. \tag{2.72}$$

With this recursive equation one can easily obtain the first 100 coefficients on a computer within a few minutes.

In terms of resummation methods, we use Padé approximants in combination with a Borel transform. Padé approximants provide analytic continuations of truncated series expansions and are widely used in physical applications [98]. The Borel transform turns the asymptotic series into a convergent series. Often these transformations are supplemented with a conformal map, which further improves the accuracy [99–102]. In principle, one can apply the Padé approximant without the Borel transform, but then the results are less accurate. The Padé-Borel transformation is given by the following steps:

i) We first compute the Borel transform of the perturbation series, given by

$$E_0(\lambda) = \sum_{n=0}^{N} A_n \lambda^n \qquad \longrightarrow \qquad \mathscr{B}[E_0](z) = \sum_{n=0}^{N} \frac{A_n}{n!} z^n\,. \tag{2.73}$$

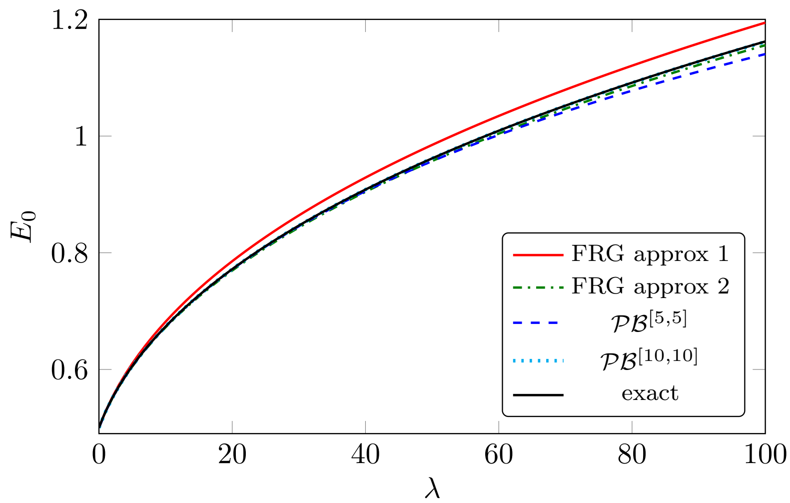


Figure 4: The energy of the ground state of the anharmonic oscillator as a function of λ for $\omega = 1$. FRG approx 1 is based on a numerical integration of (2.63) and (2.64), while FRG approx 2 additionally includes (2.65). We compare this to the Padé approximants of the Borel transform $\mathscr{PB}^{[n,m]}$, where n is the degree of the polynomial of the numerator, m of the denominator, and $n+m$ is the order of the perturbation series needed for this approximant. The exact solution stems from a well-converged numerical diagonalisation of Hamilton operator in terms of ladder operators.

While the original perturbation series has zero radius of convergence, its Borel transform has a finite radius of convergence.

ii) We apply a Padé approximant to the perturbation series in the Borel plane. The conversion of a perturbation series of order N to a Padé approximant is given by

$$\mathscr{B}[E_0](z) = \sum_{n=0}^{N} \frac{A_n}{n!} z^n \qquad \longrightarrow \qquad \mathscr{P}^{[R,S]}(z) = \frac{P_R(z)}{Q_S(z)} \,. \tag{2.74}$$

The conversion is algorithmic and leads to a ratio of two polynomials P_R and Q_S, of order R and S respectively, where $R+S=N$. It is available as built-in function in Maple and Mathematica.

iii) We now have to invert the Borel transform. The inverse of a Borel transform is nothing else but a Laplace transform

$$\mathscr{P}^{[R,S]}(z) = \frac{P_R(z)}{Q_S(z)} \qquad \longrightarrow \qquad \mathscr{PB}^{[R,S]} = \int_0^\infty \mathrm{d}z\, e^{-\lambda}\, \mathscr{P}^{[R,S]}(\lambda z) \,. \tag{2.75}$$

It might be necessary to perform the integral numerically.

We denote this whole procedure by $\mathscr{PB}^{[R,S]}$. With these transformations, we obtain access to results at large coupling $\lambda \gg 1$.

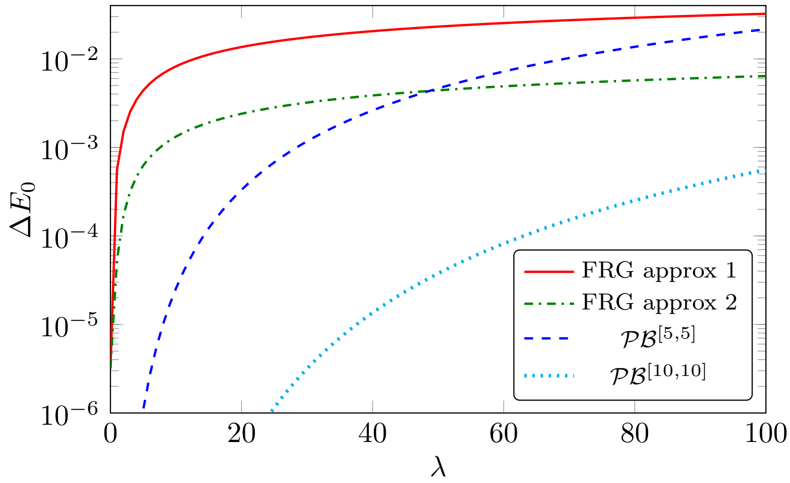


Figure 5: The error of the ground-state energy of the anharmonic oscillator as a function of λ for $\omega = 1$. The labels are identical to Fig. 4. All approximations have small errors for the displayed region. Remarkably, the error of the FRG approximations is growing slower than the error of the Padé-Borel approximants.

Comparison We compare now the results from FRG and perturbation theory. We look at the ground-state energy as a function of λ for $\omega = 1$. In Fig. 4, we display FRG results in two approximations: approximation 1 is based on the numerical integration of (2.63) and (2.64), while approximation 2 also includes (2.65). From perturbation theory, we include the diagonal Padé-Borel transforms $\mathscr{PB}^{[5,5]}$ and $\mathscr{PB}^{[10,10]}$. We also provide a reliable estimate of the exact result, which stems from a well-converged numerical diagonalisation of the Hamilton operator in terms of ladder operators. The errors with respect to this well-converged solution are displayed in Fig. 5. Remarkably, all results have a very small error in the displayed region. The error of the FRG approximation 1 never exceeds 3% and the FRG approximation 2 is even below 0.6%. The Padé-Borel transform stemming from 10 perturbative coefficients is maximally 2% off, while the error of the transform from 20 coefficients is even below 0.06%. There is one very remarkable feature: the error of the FRG is growing slower with λ than the error of the Padé-Borel transform. This means that for some large λ, the FRG approximation is better than a Padé-Borel transform. This fact can be understood: in the FRG computation we never expand around $\lambda = 0$. The error increases nonetheless because the system get more strongly coupled and higher-order terms contribute more significantly. But in comparison, the Padé-Borel transform is based on the perturbative expansion around $\lambda = 0$ and thus a stronger growth ot the error with λ is expected. In summary, the FRG presents itself as a very useful tool, in particular, at strong coupling and in theories where the computation of many terms in the perturbation series is tedious.

2.7 Symmetries with the FRG

In this section, we discuss how symmetries are handled in the FRG framework. This section takes many ideas from [48]. In most theories, we have an underlying classical symmetry, for example, a (non-)Abelian gauge, a diffeomorphism, a $\mathscr{Z}_2$ or an $O(N)$ symmetry. Whether this symmetry is broken upon quantisation depends on the anomaly associated to the symmetry. We assume that the theory is anomaly free and preserved upon quantisation.

It is important how the regularisation scheme handles the symmetry of the theory. For example, dimensional regularisation typically preserves a gauge symmetry, while a cutoff regularisation breaks it. For this reason, dimensional regularisation is often the preferred regularisation scheme. This does not imply that one cannot use a cutoff regularisation in gauge theories. Indeed, the FRG is inherently cutoff regularised and we show in this section that symmetries can be properly treated with the FRG. The upshot is that symmetries are apparently broken, but we can formulate symmetry identities that encode the symmetry. These identities allow us to restore the symmetry at $k = 0$.

We start with the assumption that the QFT is invariant under a symmetry transformation, for example, $O(N)$ or $SU(N)$. With $\mathscr{G}$ we denote the generator of an infinitesimal version of this symmetry transformation. This means that $\mathscr{G}\phi$ is linear in ϕ. For a global $O(N)$ symmetry, $\mathscr{G}$ is given by

$$\mathscr{G}^a_{O(N)} = -f^{abc} \int_x \phi^b(x) \frac{\delta}{\delta\phi^c(x)} . \tag{2.76}$$

The generator takes the same form without the spacetime integral for the local version of the symmetry. A gauge transformation in Yang-Mills theory is generated by

$$\mathscr{G}^a_{SU(N)} = -\mathscr{D}^{ab}_\mu(x) \frac{\delta}{\delta A^b_\mu(x)} = -\left(\partial_\mu \delta^{ab} - g f^{abc} A^c_\mu(x)\right) \frac{\delta}{\delta A^b_\mu(x)} . \tag{2.77}$$

We apply now a generic generator to the path integral representation in (2.1)

$$0 = \frac{1}{\mathscr{Z}[J]} \int \mathscr{D}\varphi \, \mathscr{G} \, e^{-S[\varphi]+\int_x J(x)\varphi(x)} , \tag{2.78}$$

where we have assumed that the measure is invariant under the symmetry transformation. We make this equation more explicit by applying the generator to the exponential

$$\begin{aligned} 0 &= \frac{1}{\mathscr{Z}[J]} \int \mathscr{D}\varphi \left(-(\mathscr{G}S[\varphi]) + \int_x J(x)(\mathscr{G}\varphi(x)) \right) e^{-S[\varphi]+\int_x J(x)\varphi(x)} \\ &= -\langle \mathscr{G}S \rangle_J + \left\langle \int_x J(x)(\mathscr{G}\varphi(x)) \right\rangle_J . \end{aligned} \tag{2.79}$$

We are interested in the effect of the symmetry transformation on the effective action and consequently we evaluate the equation at the supremum of the source $J[\phi]$

$$0 = -\langle \mathscr{G}S \rangle_{J[\phi]} + \int_x J[\phi] \, \langle \mathscr{G}\varphi \rangle_{J[\phi]} = -\langle \mathscr{G}S \rangle_{J[\phi]} + \int_x \frac{\delta\Gamma}{\delta\phi(x)} \mathscr{G}\phi(x) . \tag{2.80}$$

Here we have used that $\langle\varphi\rangle_{J[\phi]} = \phi$ and $J[\phi] = \frac{\delta\Gamma}{\delta\phi}$, see (2.8) and (2.9). This leads us to

$$\mathscr{G}\Gamma[\phi] = \langle \mathscr{G}S \rangle_{J[\phi]} . \tag{2.81}$$

The effective action is invariant under a symmetry if the bare action and the measure is invariant under that symmetry. This simple statement becomes more involved when we consider gauge fixing

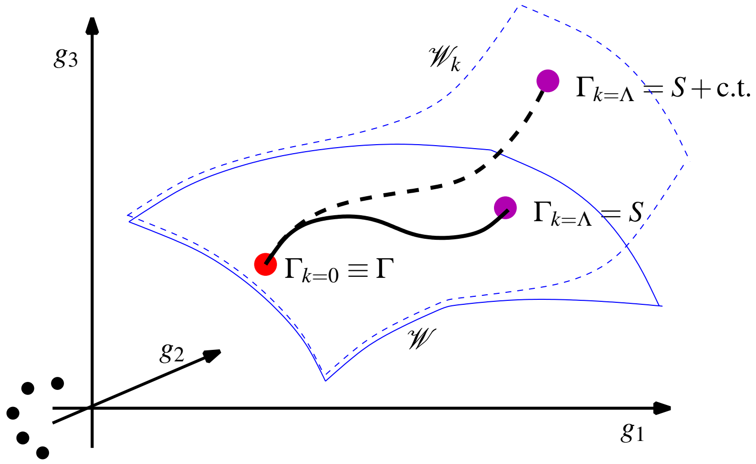


Figure 6: Sketch of the (modified) Ward identity as a hypersurface in theory space. The symmetry-breaking terms have to be carefully adjusted to lie within the surface of the modified Ward identity. An error of the flow due to the truncation can lead to a deviation of the flow trajectory from the surface of the modified Ward identity. The figure is taken and adapted from [48].

and regularisation. Let us assume that we are dealing with a symmetry that requires a gauge-fixing action S_{gf} with the corresponding ghost action S_{gh}. Both are breaking the explicit gauge invariance, but instead, they are invariant under a BRST symmetry [103, 104]. Now, (2.81) becomes the Ward-Takahashi identity [105, 106]

$$\mathscr{W} = \mathscr{G}\Gamma[\phi] - \big\langle \mathscr{G}\big(S_{\text{gf}} + S_{\text{gh}}\big)\big\rangle_{J[\phi]} = 0\,. \tag{2.82}$$

This Ward-Takahashi identity encodes the gauge invariance of the effective action. It also dictates which kind of terms are allowed in the effective action. In the case of Yang-Mills theory, it forbids, for example, a mass term for the gauge boson, i.e., the term $\int m_A^2 A_\mu A^\mu$ is not compatible with (2.82). In the case of QED, it forbids (sub-)diagrams with an odd number of external photon legs.

What happens to the Ward-Takahashi identity if we add a regulator? In the best case, we can choose a regulator that is invariant under the respective symmetry. In case of an $O(N)$ symmetry this is possible with

$$\Delta S_k = \frac{1}{2}\int_p \varphi^a(-p)\delta^{ab} R_k(p)\varphi^b(p)\,. \tag{2.83}$$

In case of $SU(N)$ or gravity, this is not possible. The regulator is a mass-like term, which is quadratic in the fields, and we just stated that in Yang-Mills theory the Ward-Takahashi identity (2.82) forbids such a term. In such cases, the regulator introduces a new source of symmetry breaking. We can compute the effect on the scale-dependent effective action

$$0 = \frac{1}{\mathscr{Z}[J]}\int \mathscr{D}\varphi\, \mathscr{G}\, e^{-S[\varphi]+\Delta S_k[\varphi]+\int_x J(x)\varphi(x)} = -\langle \mathscr{G}(S+\Delta S_k\rangle_J + \left\langle \int_x J(x)(\mathscr{G}\varphi(x))\right\rangle_J\,. \tag{2.84}$$

We again evaluate this equation at the supremum of the source $J[\phi]$ and use $\delta_\phi(\Gamma_k + \Delta S_k) = J_{\text{sup}}$, see (2.32). This leads us to the modified Ward-Takahashi identity

$$\mathscr{W}_k = \mathscr{G}\Gamma_k + \mathscr{G}\Delta S_k - \langle \mathscr{G}\,(S+\Delta S_k)\rangle_{J[\phi]} = 0\,. \tag{2.85}$$

For a gauge theory with gauge-fixing and ghost action this implies

$$\mathscr{W}_k = \mathscr{G}\Gamma_k + \mathscr{G}\Delta S_k - \left\langle \mathscr{G}\left(S_{\text{gf}} + S_{\text{gh}} + \Delta S_k\right)\right\rangle_{J[\phi]} = 0\,. \tag{2.86}$$

The modified Ward-Takahashi identity (mWI) reduces by construction to the standard Ward-Takahashi identity at $k = 0$ since $\Delta S_{k=0} = 0$. Then, the flow of the mWI is proportional to itself, $\partial_t \mathscr{W}_k \propto \mathscr{W}_k$. This means that if the mWI is fulfilled at some scale k, it is also fulfilled at all other scales. This statement has to be taken with a grain of salt: in any practical computation, we have to work in a truncation and usually the error due to the truncation violates the property $\partial_t \mathscr{W}_k \propto \mathscr{W}_k$.

We can imagine the (modified) Ward identity $\mathscr{W}_k$ as hypersurface in theory space. This is schematically displayed in Fig. 6. To be more precise, the modified Ward identity is a hypersurface for each fixed k and for $k \to 0$ it is identical to the standard Ward identity. In Fig. 6, we display $\mathscr{W}_k$ 'flowing' from $k = \Lambda$ to $k = 0$. An error due to the truncation leads to a deviation of the flow from this hypersurface and we have to control that the error remains small. The starting point of the flow has to be adjusted such that it lies within $\mathscr{W}_k$. We illustrate that at the example of the gluon mass parameter: The standard Ward identity (2.82) forbids a mass parameter of the type $\int m_A^2 A_\mu A^\mu$. However, the regulator is precisely a mass parameter of this type and thus the modified Ward identity (2.86) does not forbid such a mass term. In fact, the modified Ward identity tells us precisely what the value of the mass parameter has to be. By solving $\mathscr{W}_k$, one finds that $m_{A,k}^2 \sim g^2 k^2$, which also implies $m_{A,k}^2 \to 0$ for $k \to 0$. Instead, with the naive choice $m_{A,\Lambda}^2 = 0$, one would violate the modified Ward identity and one would *not* find $m_{A,k}^2 \to 0$ for $k \to 0$. For more details on the gluon mass parameter, see, for example, [107]. This illustrates that the parameter at the initial scale has to carefully chosen to fulfil the modified Ward identity.

3. Asymptotically Safe Quantum Gravity

We turn now to asymptotically safe quantum gravity. The development of the FRG was crucial for this approach to quantum gravity, since it offers the necessary non-perturbative computational tool without discretising spacetime.

3.1 Basics and perturbative quantum gravity

In asymptotically safe quantum gravity we make two basic assumptions:

- Diffeomorphism invariance is the fundamental symmetry of spacetime.
- The metric carries the fundamental degrees of freedom.

In other words, we would like to make sense of the path integral

$$Z[J] = \int \mathscr{D} g_{\mu\nu} \exp\left\{ -S_{\text{gravity}}[g_{\mu\nu}] + \int_x J^{\mu\nu}(x)\, g_{\mu\nu}(x) \right\}, \tag{3.1}$$

for some suitable measure that is renormalised and selects exactly one configuration from each diffeomorphism equivalence class. With the FRG, the latter is implemented with a gauge-fixing condition. For the moment we assume that there is a gauge fixing, but we postpone the technical details to Sec. 3.3. The action S_{gravity} contains the diffeomorphism invariant operators. The basic quantity to build these operators and to describe the curvature of spacetime is the Riemann tensor $R^\rho{}_{\sigma\mu\nu}$. Expressed in terms of the metric connection, it reads

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma} \,. \tag{3.2}$$

If we demand that the metric connection is torsion free then the unique solution is the Levi-Civita connection

$$\Gamma^\sigma{}_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) . \tag{3.3}$$

From the Riemann tensor and the metric, we can build all curvature invariants including the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R, which are defined by

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \,, \qquad\qquad R = g^{\mu\nu} R_{\mu\nu} \,. \tag{3.4}$$

It is often useful to decompose the Riemann tensor in terms of those two invariants and the Weyl tensor $C_{\mu\nu\rho\sigma}$, which is the fully traceless part of the Riemann tensor, i.e., $C^\lambda{}_{\mu\lambda\nu} = 0$. The decomposition in d spacetime dimensions is given by

$$\begin{aligned} R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} &- \frac{1}{d-2}\left(R_{\mu\sigma}\, g_{\nu\rho} - R_{\mu\rho}\, g_{\nu\sigma} + R_{\nu\rho}\, g_{\mu\sigma} - R_{\nu\sigma}\, g_{\mu\rho}\right) \\ &- \frac{R}{(d-1)(d-2)}\left(g_{\mu\rho}\, g_{\nu\sigma} + g_{\mu\sigma}\, g_{\nu\rho}\right) . \end{aligned} \tag{3.5}$$

We can see from (3.2) and (3.3) that the Riemann tensor and all other invariants contain two derivatives and thus have mass dimension 2 independent on the spacetime dimension. Let us now order the curvature invariants according to their mass dimension

$$\begin{aligned} &\mathscr{O}(R^0): && \Lambda \,, \\ &\mathscr{O}(R^1): && R \,, \\ &\mathscr{O}(R^2): && R^2, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \,, \\ &\mathscr{O}(R^3): && \ldots . \end{aligned} \tag{3.6}$$

At lowest order we only have the cosmological constant Λ and at mass dimension two only the Ricci scalar R. At mass dimension four, more invariants show up of which three elements form a basis. The square of the Riemann tensor can be expressed in terms of the other invariants

$$R^2_{\mu\nu\rho\sigma} = C^2_{\mu\nu\rho\sigma} - \frac{2}{(d-1)(d-2)} R^2 + \frac{4}{d-2} R^2_{\mu\nu} \,. \tag{3.7}$$

There is one very interesting combination in four spacetime dimensions, the Gauß-Bonnet term, which is given by

$$E = R^2_{\mu\nu\rho\sigma} - 4 R^2_{\mu\nu} + R^2 \,. \tag{3.8}$$

This term is a topological invariant, meaning that all infinitesimal variations of the metric are vanishing, $\delta E/\delta g_{\mu\nu} = 0$.

At higher mass dimensions, the number of curvature invariants is growing immensely. This already hints at a problem that we encounter in quantum gravity: in principle we need to include all curvature invariants in the non-perturbative approach. This task poses very fast an impossible technical challenge due to the vast amount of curvature invariants. To properly include all curvature terms up to mass dimension four is already non-trivial.

Before we discuss the non-perturbative approach to quantum gravity, it is important to understand the failure of perturbative quantum gravity. In the perturbative setup, we have to discuss two distinct cases, since we can either quantise the Einstein-Hilbert action or a higher-derivative action. We start with the Einstein-Hilbert action in d spacetime dimensions

$$S_{\text{EH}} = \frac{1}{16\pi G_{\text{N}}} \int \mathrm{d}^d x \sqrt{g}\,(R - 2\Lambda)\,, \tag{3.9}$$

where $\sqrt{g} = \sqrt{\det g_{\mu\nu}}$. In natural units, the Newton coupling relates to the Planck scale with $G_{\text{N}} = 1/M_{\text{Pl}}^2$. The Planck scale indicates where quantum gravity fluctuations become important, $M_{\text{Pl}} = 1.22 \cdot 10^{19}$ GeV. The Newton coupling has the mass dimension $[G_{\text{N}}] = 2 - d$. In $d = 2$ spacetime dimensions, it is marginal and the theory is perturbatively renormalisable. However, gravity is trivial in $d = 2$, in the sense that the Einstein-Hilbert action is topological. In $d = 2 + \varepsilon$, it can be shown that the theory has a perturbative asymptotically safe fixed point, which merges with the Gaußian fixed point for $\varepsilon \to 0$ [69, 108–112]. This interesting property has triggered speculations, whether this fixed point persists until $d = 4$.

In $d = 4$, the Newton coupling has a negative mass dimension $[G_{\text{N}}] = -2$. This is the first hint that the Einstein-Hilbert action is perturbatively non-renormalisable in $d = 4$. Simply speaking, there are up to n factors of Newton couplings in an n-loop diagram. In order to compensate for the mass dimension of the Newton coupling, the only available quantity are momenta (except for the cosmological constant). If these higher-order momentum terms appear in the divergent part of the diagrams, then these divergences need to be reabsorbed in counterterms. This is only possible by higher-order curvature invariants. If this mechanism takes place, then an infinite amount of counterterms is needed, since at each loop order new higher-order curvature invariants appear. This is precisely the failure of perturbative quantum gravity from the Einstein-Hilbert action: the theory has no predictivity since infinitely many measurements are necessary to fix all the counterterms.

There is the small caveat, that we assumed that the higher-order momentum terms appear in the divergent part of the diagrams. This is a reasonable assumption and one would need miraculous cancellations at each loop order. Nonetheless, actual computations are needed to back up this assumption. At one-loop order [113], the invariants R^2 and $R_{\mu\nu}^2$ indeed appear in the divergent parts of the diagrams. However, for $\Lambda = 0$ and without matter content, these invariants vanish on-shell, which allows for a field redefinition of the metric that absorbs these divergences. Only $R_{\mu\nu\rho\sigma}$, or equivalently $C_{\mu\nu\rho\sigma}$, does not vanish on-shell for $\Lambda = 0$, but such a term can always be absorbed by the Gauß-Bonnet invariant in $d = 4$, see (3.8). In other words, pure quantum gravity with $\Lambda = 0$ is doing fine at one-loop order. This may be a purely theoretical statement since a theory of quantum gravity without matter is not able to describe our Universe. In any case, the situation becomes worse at two-loop order [114–116]. There the divergences reveal that the infamous Goroff-Sagnotti counter term is contributing

$$S_{\mathrm{GS}} = \frac{1}{\varepsilon}\frac{209}{2880}\frac{1}{(4\pi)^4}\int \mathrm{d}^4x\sqrt{g}\,C_{\mu\nu}{}^{\kappa\lambda}C_{\kappa\lambda}{}^{\rho\sigma}C_{\rho\sigma}{}^{\mu\nu}\,, \tag{3.10}$$

This term does not vanish on-shell and has to be added as a counter term to the Einstein-Hilbert action. This is interpreted as the onset of infinitely many counterterms that need to be introduced and thus the theory is not predictive.

Next we consider a higher-derivative action as a starting point

$$S_{\mathrm{HD}} = S_{\mathrm{EH}} + \int \mathrm{d}^4x\sqrt{g}\left(\frac{1}{2\lambda}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \frac{w}{3\lambda}R^2 + \frac{\theta}{\lambda}E\right)\,. \tag{3.11}$$

Here we have decided to write the action in terms of the square of the Weyl tensor (3.5), the square of the Ricci tensor, and the Gauß-Bonnet term (3.8). All couplings are dimensionless in $d=4$, $[\lambda]=[w]=[\theta]=0$. This gives a strong indication, that the theory is perturbatively renormalisable. Compared to the Einstein-Hilbert action, the loop diagrams do not introduce higher powers of momenta and all divergences can be absorbed into already existing terms of the action. The contributions stemming from the Einstein-Hilbert part in (3.11) are subleading for large momenta.

The parameterisation in (3.11) is very useful: the square of the Weyl tensor contributes only to the transverse-traceless mode of the graviton propagator, while the square of the Ricci tensor contributes only to the scalar mode[1]. The Gauß-Bonnet term does not contribute at all to any graviton n-point function, since it is a topological invariant, $\delta E/\delta g_{\mu\nu}=0$. The transverse-traceless mode of the graviton propagator has now the shape

$$G_{\mathrm{tt}} \sim \frac{1}{p^2+\frac{a}{M_{\mathrm{Pl}}^2}p^4} = \frac{1}{p^2} - \frac{1}{p^2+\frac{M_{\mathrm{Pl}}^2}{a}}\,. \tag{3.12}$$

We make an important observation: the coefficient in front of the second term is negative, i.e., it is a ghost state spoiling unitarity. Even more severe, if the coupling of the higher-derivative term has the wrong sign, then the ghost state also becomes tachyonic [117–119]. Typically any theory with higher-order time derivatives features these ghost states at the classical level. They are known as Ostrogradsky instabilities [120], see also [121, 122].

This does not yet seal the fate of perturbative quantum gravity, precisely because these are classical instabilities. In (3.12), the bare graviton propagator is given and radiative correction could change the shape of the propagator and cure the instabilities. In other words, unitarity is only spoiled if the full quantum propagator contains a ghost state. We take a look at the beta function to investigate this possibility. The one-loop beta functions are universal and not scheme dependent since the couplings are dimensionless. They are given by [123–126]

$$(4\pi)^2\beta_\lambda = -\frac{133}{10}\lambda^2\,, \tag{3.13a}$$

$$(4\pi)^2\beta_\omega = -\frac{25+1098\omega+200\omega^2}{60}\lambda\,, \tag{3.13b}$$

$$(4\pi)^2\beta_\theta = \frac{2}{90}(56-171\theta)\lambda\,. \tag{3.13c}$$

[1] We discuss the decomposition of the graviton in (3.36).

Eq. (3.13a) shows us that λ has an attractive Gaußian UV fixed point. Eq. (3.13b) is vanishing for $\omega_1^* = -5.47$ and $\omega_2^* = -0.0229$, which are two non-Gaußian fixed points. From (3.13c), we read off that $\theta^* = 0.327$ is a fixed point. The most important information is the asymptotic freedom of λ. This causes that all higher derivative terms become parametrically enhanced towards the UV, see (3.11). This implies that all other operators have their canonical critical exponents, independent on whether they have themselves a Gaußian or non-Gaußian UV fixed point. In consequence, the ghost state displayed in (3.12) carries through to the quantum propagator and spoils unitarity at this Gaußian fixed point. There are various attempts to safe the unitarity of the theory, for example, by a different prescription of the propagator [127–129]. This comes at the cost of losing microcausality.

In summary, perturbative quantum gravity is either non-renormalisable or non-unitary, which are are both unsatisfying prospects. In the following sections, we explore this picture with a non-perturbative quantisation.

3.2 The idea of asymptotic safety

The conjecture of the asymptotic safety scenario is the existence of a non-Gaußian UV fixed point, which renders the UV behaviour of quantum gravity finite. The idea was brought up by Weinberg in 1979 [130]. There have been several work in $2+\varepsilon$ dimensions [108–111] and in the large-N expansion [131–133]. The pioneering work in four spacetime dimensions with the use of the FRG stems from Reuter in 1996 [69] and thus the fixed point is often called the Reuter fixed point.

A fixed point is defined as the root of the beta functions of the dimensionless couplings. We use dimensionless couplings since those appear in the computation of cross-sections. We define the dimensionless couplings by a rescaling with appropriate powers of the RG scale, for example $G_\text{N} = g/k^2$ in case of the Newton coupling. The beta function of the dimensionless Newton coupling has a part, which stems from the canonical mass dimension, and a part, which stems from quantum fluctuations

$$\beta_g \equiv \partial_t g = \partial_t G_\text{N} k^2 = 2g + k^2 \partial_t G_\text{N}\,, \tag{3.14}$$

see also the discussion in Sec. 2.4. The canonical running indicates that dimensionless Newton coupling grows towards the UV without quantum fluctuations. The quantum fluctuations need to be strong to compensate for the canonical running. We have the following picture in mind: Below the Planck scale, all gravitational quantum fluctuations are strongly suppressed. Thus the dimensionful Newton coupling is a constant, while the dimensionless Newton coupling rises quadratically with the RG scale. Around the Planck scale, we have a transition period. Above the Planck, we reach the asymptotically safe regime where the quantum fluctuations balance the canonical running and the dimensionless Newton coupling becomes a constant. In turn, the dimensionful Newton coupling goes to zero with k^{-2}. This is displayed schematically in Fig. 7, where we plot the dimensionless Newton coupling as $g(k) = k^2/(M_\text{Pl}^2 + k^2/g^*)$. We denote the fixed-point value of the Newton coupling by g^*, which we set to $g^* = 1$ in Fig. 7. The picture for the other gravitational couplings, such as the cosmological constant, is in straight analogy to that.

Predictivity In the perturbative quantisation of the Einstein-Hilbert action, we had to introduce infinitely many counterterms and thus the theory was not predictive. How does this issue translate

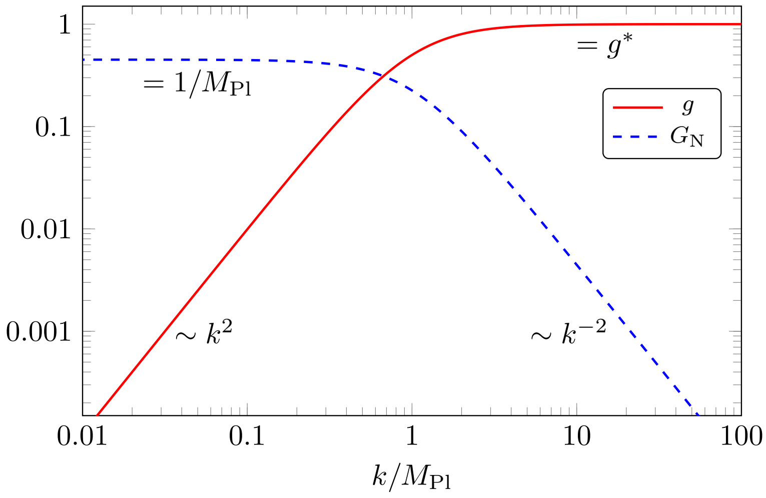


Figure 7: Sketch of the dimensionless and dimensionful Newton coupling, g and $G_{\rm N}$. The dimensionful Newton coupling is a constant in the IR with its known value and becomes quadratically weaker above the Planck scale. The dimensionless Newton coupling instead rises quadratically until the Planck scale and takes its fixed point value shortly above the Planck scale.

to the non-perturbative picture? Here we have to count the number of UV attractive directions of the non-perturbative fixed point. The UV attractive directions are characterised by the linearised beta functions around the fixed point as discussed in Sec. 2.4. All UV attractive directions together span the critical hypersurface that includes all trajectories that are attracted by the UV fixed point. The dimension of the critical hypersurface corresponds to the number of required measurements to fully determine the theory. The theory is predictive if this hypersurface is finite-dimensional. We display the critical hypersurface schematically in Fig. 8.

The dimension of the critical hypersurface was investigated in many truncations of asymptotically safe quantum gravity. Usually two to four UV attractive directions are found, mostly associated with $\sqrt{g}$, $\sqrt{g}R$, $\sqrt{g}R^2$, and $\sqrt{g}R_{\mu\nu}^2$. In particular, the first three operators are often found to be UV relevant. More UV relevant directions appear in gravity-matter systems, for example the operator $R\phi^2$ is a candidate since it is canonically marginal.

Most of these truncations are finite-dimensional and hence the demand that the UV critical hypersurface is finite-dimensional is trivially fulfilled. Nonetheless, there is an important lesson: the canonical mass dimension seems to remain a good guiding principle at the non-Gaußian fixed point. We do not find operators that are canonically highly irrelevant but then become relevant due to quantum fluctuation. This was demonstrated by several works that expanded the effective action to high orders in the Ricci scalar or Ricci tensor [135–139]. This is by no means a trivial statement, in particular, if the fixed point is highly non-perturbative. There are other hints that the Reuter fixed point might be in the semi-non-perturbative region: they stem from ’effective universality’ of different avatars of the Newton coupling [140, 141]. Lastly, the infamous Goroff-Sagnotti counter term was included in [142] and turned out to be asymptotically safe and UV irrelevant.

Unitarity In quadratic gravity at the asymptotically free fixed point suffers from the lack of

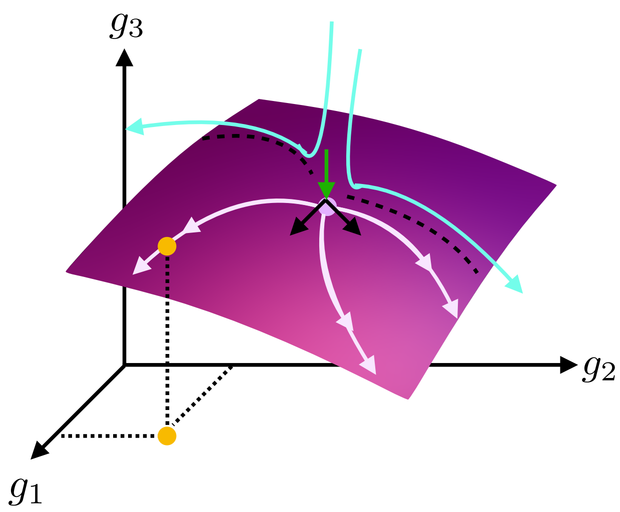


Figure 8: Sketch of the critical hypersurface (purple) in theory space. The dimension of the critical hypersurface determines the number of measurements needed to fully determine the theory. Trajectories close to the critical hypersurface flow towards the surface in the IR. The figure is taken from [134].

unitarity. Unitarity is one of the most difficult properties to access in asymptotically safe quantum gravity. The graviton propagator contains all momentum powers since all operators compatible with the symmetry have to be included. Thus one can naively think that this triggers ghost states in the spirit of (3.12). As discussed above, unitarity is not a property of the bare propagator but of the spectral function of the full quantum propagator. This is very hard to access and first attempts have been made [122, 143–147]. So far no definite statement can be made whether asymptotically safe quantum gravity is unitary or not, and it remains one of the most intriguing questions of this approach. For a more detailed discussion on this topic see [148].

Program of asymptotic safety The (ideal) asymptotic safety program is given by the following items (which is a subjective list by the author):

1. Find a fixed point that is well converged. This means that the UV relevant operators should not change upon inclusion of new diffeomorphism invariant operators. This fixed point must have phenomenological viable trajectories to the IR.

2. Determine the number of UV attractive operators at the fixed point and thus check if the theory is predictive.

3. Find a trajectory that matches the known IR physics. This includes that the matter couplings of the Standard Model are compatible with the asymptotically safe fixed point [149–156].

4. Analytically continue the full quantum n-point functions to obtain the spectral functions and cross-sections. This gives insight if the theory is unitarity or not. For this task it is necessary to keep the momentum dependence of the n-point function [140, 157–165]. The analytic continuation itself is a very delicate task. We do not go into details here and simply refer to the literature [166–169].

5. Applications: find solutions to the quantum equations of motions that describe black holes [170–181] or the evolution of the universe including inflation [43, 182–192].

This lists illustrates that finding a UV fixed point is only the first of many step. The asymptotic safety program has already achieved partial results on all of the above points as indicated by the references.

3.3 Diffeomorphism symmetry and gauge fixing

After this general idea of asymptotically safe quantum gravity, we now detailed the technical implementation of diffeomorphism invariance and the corresponding gauge fixing. Diffeomorphism transformations are generated by the Lie derivative $\mathscr{L}_\omega$ with respect to a vector field ω_μ. For example, the Lie derivative acts on a scalar field ϕ by

$$\mathscr{L}_\omega \phi = \omega^\mu \partial_\mu \phi = \omega^\mu \nabla_\mu \phi \,, \tag{3.15a}$$

while it acts on vectors and two-tensors by

$$\mathscr{L}_\omega A_\mu = \omega^\rho \partial_\rho A_\mu + A_\rho \partial_\mu \omega^\rho = \omega^\rho \nabla_\rho A_\mu + A_\rho \nabla_\mu \omega^\rho \,, \tag{3.15b}$$

$$\mathscr{L}_\omega T_{\mu\nu} = \omega^\rho \partial_\rho T_{\mu\nu} + T_{\mu\rho}\partial_\nu \omega^\rho + T_{\rho\nu}\partial_\mu \omega^\rho = \omega^\rho \nabla_\rho T_{\mu\nu} + T_{\mu\rho}\nabla_\nu \omega^\rho + T_{\rho\nu}\nabla_\mu \omega^\rho \,. \tag{3.15c}$$

In all cases, the Lie derivative can be either represented as partial derivative or as covariant derivative. The Lie derivative of the metic is thus

$$\mathscr{L}_\omega g_{\mu\nu} = g_{\mu\rho}\nabla_\nu \omega^\rho + g_{\rho\nu}\nabla_\mu \omega^\rho \,, \tag{3.16}$$

where we used metric compatibility to get rid of one term. Any gravity action, like the Einstein-Hilbert action (3.9) or the higher-derivative action (3.11), is invariant under the transformation

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + \mathscr{L}_\omega g_{\mu\nu} \,. \tag{3.17}$$

In quantum gravity, it is necessary to introduce a background metric $\bar{g}_{\mu\nu}$ and a corresponding fluctuation field $h_{\mu\nu}$ about that background. Common choices are the linear split

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,, \tag{3.18}$$

or the exponential split [110, 112, 166, 193, 194]

$$g_{\mu\nu} = \bar{g}_{\mu\rho}(e^h)^\rho{}_\nu \,. \tag{3.19}$$

Here, $(e^h)^\rho{}_\nu = \delta^\rho{}_\nu + h^\rho{}_\nu + \frac{1}{2}h^\rho{}_\sigma h^\sigma{}_\nu + \dots$. The path integral in (3.1) is now performed over $h_{\mu\nu}$. Depending on the split, the measure term is modified, which can be taken into account by the Jacobian of the transformation. Strictly speaking, we should distinguish between the field we integrate over $\hat{h}_{\mu\nu}$ and its expectation value $\langle \hat{h}_{\mu\nu} \rangle = h_{\mu\nu}$. We will not make this distinction to keep the notation light.

The split of the metric into a background and a fluctuation is necessary, such that we can define a proper gauge fixing and a local coarse-graining procedure. How does the split affect the diffeomorphism transformation in (3.17)? One can now construct two independent diffeomorphism transformation that both leave the action invariant. One is the quantum diffeomorphism transformation, which for the linear split reads

$$\begin{aligned} h_{\mu\nu} &\longrightarrow h_{\mu\nu} + \mathscr{L}_\omega(\bar{g}_{\mu\nu} + h_{\mu\nu})\,, \\ \bar{g}_{\mu\nu} &\longrightarrow \bar{g}_{\mu\nu} \end{aligned} \tag{3.20}$$

and the other is the background diffeomorphism transformation, in case of the linear split

$$\begin{aligned} h_{\mu\nu} &\longrightarrow h_{\mu\nu} + \mathscr{L}_\omega h_{\mu\nu}\,, \\ \bar{g}_{\mu\nu} &\longrightarrow \bar{g}_{\mu\nu} + \mathscr{L}_\omega \bar{g}_{\mu\nu}\,. \end{aligned} \tag{3.21}$$

We will see soon that the quantum diffeomorphism symmetry is going to be broken by the gauge fixing, which turns it into a BRST symmetry [103, 104], and then further broken by the regulator. The background diffeomorphism symmetry, on the other hand, is always going to be preserved. In Sec. 2.7, we learned how symmetries are treated in the FRG: the broken symmetry is encoded in a symmetry identity and restored at $k = 0$. The identities corresponding to the BRST or quantum diffeomorphism symmetry are called (modified) Slavnov-Taylor identities [195, 196]. These identities encode physical diffeomorphism invariance. We detail them at the end of this section after we have introduced the gauge fixing and the BRST transformations.

The metric split also introduces a new symmetry, the split symmetry, given by the transformation

$$g(\bar{g}, h) \longrightarrow g(\bar{g} + \delta\bar{g}, h + \delta h) = g(\bar{g}, h)\,. \tag{3.22}$$

We have $\delta\bar{g} = -\delta h$ with the linear split. This split symmetry is going to be broken by the gauge fixing and the regulator. Looking back to Sec. 2.7, we first ignore the regulator contribution and use (2.82) with $\mathscr{G} = \delta\bar{g} + \delta h$. This leads us to the Nielsen identity [197, 198] for the linear metric split

$$\mathrm{NI} = \frac{\delta\Gamma}{\delta\bar{g}_{\mu\nu}} - \frac{\delta\Gamma}{\delta h_{\mu\nu}} - \left\langle \left[\frac{\delta}{\delta\bar{g}_{\mu\nu}} - \frac{\delta}{\delta h_{\mu\nu}}\right](S_{\mathrm{gf}} + S_{\mathrm{gh}}) \right\rangle = 0\,. \tag{3.23}$$

This identity encodes the background independence of the effective action. At finite k, the regulator contribution turns the Nielsen identity into a modified Nielsen identity. We use (2.86) and rewrite the regulator part to obtain

$$\mathrm{mNI} = \mathrm{NI}_k - \frac{1}{2}\mathrm{Tr}\left[\frac{1}{\sqrt{\bar{g}}}\frac{\delta\sqrt{\bar{g}}R_k}{\delta\bar{g}_{\mu\nu}} G_k\right] = 0\,. \tag{3.24}$$

Here, R_k is the regulator and $G_k = (\Gamma_k^{(2)} + R_k)^{-1}$ is the full quantum propagator. The index k at NI indicates the use of (3.23) with Γ_k instead of Γ. The (modified) Nielsen identity tells us, that the (scale-dependent) effective action cannot be written as a function of the full metric, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ in case of the linear split. This result seems unsettling at first but we expected it from the discussion in Sec. 2.7. We should interpret in the following way: although we write the scale-dependent effective action as a function of $\bar{g}$ and h separately, the modified Nielsen identity tells us how these two fields are related. In this manner, the modified Nielsen identity reduces the dependence on two fields to one field and in the end encodes the background independence of the theory.

Let us go back to the gauge fixing. The graviton two-point function is not invertible on-shell, just as the gauge two-point function in a not-gauge-fixed (non-)Abelian gauge theory. The gauge fixing removes the redundant degrees of freedom and makes the two-point function invertible on-shell. The gauge-fixing action is given by

$$S_{\text{gf}} = \frac{1}{2\alpha}\int \mathrm{d}^4x\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}F_\mu F_\nu\,, \tag{3.25}$$

where F_μ is a gauge fixing condition linear in the fluctuation field. This is analogous to QED and QCD, where, for example, the gauge-fixing condition is $F = \partial_\mu A^\mu$ in Lorentz gauge. The gauge fixing parameter α describes how sharply the gauge-fixing condition is implemented. The limit $\alpha \to 0$ is called the Landau limit and is often preferred since it implements the gauge-fixing condition with a delta function and thus fully disentangles the transverse and longitudinal parts of the field. Furthermore, if one considers α as a quantity with an RG running, then $\alpha \to 0$ is a fixed point of the RG flow [199].

In gravity, the gauge fixing condition F_μ is a one-parameter family, if one restricts the gauge-fixing condition to be linear in the derivative. Higher-derivative gauge fixing can be useful, in particular in higher-derivative gravity. The gauge-fixing condition is

$$F_\mu = \bar{\nabla}^\alpha h_{\alpha\mu} - \frac{1+\beta}{4}\bar{\nabla}_\mu h^\nu{}_\nu\,, \tag{3.26}$$

with the gauge fixing parameter β. The parameter β can be chosen arbitrarily except for $\beta = 3$ (in $d = 4$ spacetime dimensions) where the gauge-fixing condition becomes incomplete. Common choices are $\beta = 1$, which corresponds to harmonic gauge, and $\beta = -1$, which was denoted the 'physical gauge' [200, 201]. The ghost action corresponding to the gauge-fixing condition is given by

$$S_{\text{gh}} = \int \mathrm{d}^4x\sqrt{\bar{g}}\,\bar{c}^\mu \mathscr{M}_{\mu\nu}c^\nu\,, \tag{3.27}$$

where $\mathscr{M}_{\mu\nu}$ is the Faddeev-Popov operator. The latter is computed as

$$\mathscr{M}_{\mu\nu} = \frac{\partial F_\mu}{\partial h_{\alpha\beta}}\frac{\partial}{\partial \omega^\nu}\mathscr{L}_\omega g_{\alpha\beta} = \bar{\nabla}^\rho\left(g_{\mu\nu}\nabla_\rho + g_{\rho\nu}\nabla_\mu\right) - \frac{1+\beta}{2}\bar{g}^{\sigma\rho}\bar{\nabla}_\mu g_{\nu\rho}\nabla_\rho\,. \tag{3.28}$$

Here the bar indicates that the covariant derivatives are constructed with respect to the background metric $\bar{g}_{\mu\nu}$. This equation is linear in $h_{\mu\nu}$, which is apparent from is representation in terms of the Lie derivative. The ghost field c_μ is a four-vector and thus the ghost action removes 8 degrees of freedom in total. Tn the end, from the 10 degrees of freedom of $h_{\mu\nu}$, two remain, which are the two polarisations of the graviton.

The gauge fixing turns the quantum diffeomorphism symmetry (3.20) into a BRST symmetry. The BRST transformation $\mathfrak{s}$ is given by

$$\begin{aligned} \mathfrak{s}h_{\mu\nu} &= \mathscr{L}_c(\bar{g}_{\mu\nu}+h_{\mu\nu}) = \bar{\nabla}_\mu c_\nu + \bar{\nabla}_\nu c_\mu + \mathscr{L}_c h_{\mu\nu}\,, & \mathfrak{s}\bar{g}_{\mu\nu} &= 0\,, \\ \mathfrak{s}c_\mu &= c_\rho\bar{\nabla}^\rho c_\mu\,, & \mathfrak{s}\bar{c}_\mu &= -\frac{F_\mu}{\alpha}\,. \end{aligned} \tag{3.29}$$

Here, F_μ transforms trivially under the BRST transformation, $\mathfrak{s}F_\mu = 0$. The gauge-fixed action is invariant under this transformation, $\mathfrak{s}(S + S_{\text{gf}} + S_{\text{gh}}) = 0$. Note, that, restricted to $\bar{g}_{\mu\nu}$ and $h_{\mu\nu}$, this transformation is identical to the quantum diffeomorphism symmetry (3.20). Furthermore, $\mathfrak{s}$ is a nilpotent operator with $\mathfrak{s}^2 = 0$.

On the level of the effective action, the BRST invariance is encoded in the Slavnov-Taylor identities. For the Slavnov-Taylor identities, it is convenient to include a source term for the BRST variations of the field in the generating functional. We write this source term as $Q^a \mathfrak{s}\phi_a$ and ϕ_a is understood to include all fields in (3.29). The Slavnov-Taylor identity is then given by

$$\text{STI} = \frac{\delta\Gamma}{\delta Q^a}\frac{\delta\Gamma}{\delta\phi_a} = 0\,. \tag{3.30}$$

This is the symmetry identity at $k = 0$ without regulator contribution. At finite k the Slavnov-Taylor identity turns into a modified Slavnov-Taylor identity, which reads

$$\text{mSTI} = \text{STI}_k - 2R_k^{ab}\frac{\delta^2\Gamma_k}{\delta Q^b\delta\phi_c}G_{k,ca} = 0\,. \tag{3.31}$$

Note, that the indices a, b, c sum over the different fields but also include the Lorentz or gauge indices of the fields summed over. This identity encodes the quantum diffeomorphism invariance of the scale-dependent effective action. Together with the modified Nielsen identity (3.24), it encodes the full symmetry constraints of the theory. For further information and implementation of these symmetry identities, see [35, 69, 140, 202–214].

3.4 Wetterich equation and background-field approximation

So far we have not written down the Wetterich equation for quantum gravity. As a first step, we specify the regulator function (2.20) to the field content in gravity. In pure gravity have the fluctuation field of the graviton $h_{\mu\nu}$ and the ghost fields c_μ and $\bar{c}_\mu$. This can be easily augmented with matters fields. For pure gravity, the regulator reads

$$\Delta S_k = \frac{1}{2}\int \mathrm{d}^4x\sqrt{\bar{g}}\, h_{\mu\nu}R_{k,h}[\bar{g}]^{\mu\nu\rho\sigma}h_{\rho\sigma} + \int \mathrm{d}^4x\sqrt{\bar{g}}\,\bar{c}_\mu R_{k,c}[\bar{g}]^{\mu\nu}c_\nu\,. \tag{3.32}$$

As discussed in the last section, the regulator does not respect neither the split symmetry (3.22) nor the quantum diffeomorphism invariance (3.20), but it is background diffeomorphism invariant (3.21). Without a background metric $\bar{g}_{\mu\nu}$, we would have been able to write down a bilinear in the metric field.

The graviton and the ghost regulator have four and two open indices, respectively, and we are free to choose the tensor structure of the regulator, as long as it implements a proper regularisation. For example, a simple choice in momentum space is $R_{k,h}^{\mu\nu\rho\sigma} = \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}p^2 r(p^2/k^2)$, with some shape function r. Often, the regulator is chosen proportional to the respective two-point function

$$R_{k,h}^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{\bar{g}(x)\bar{g}(y)}}\frac{\delta^2\Gamma_k[\bar{g},\phi=0]}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(y)}\,, \qquad R_{k,c}^{\mu\nu} = \frac{1}{\sqrt{\bar{g}(x)\bar{g}(y)}}\frac{\delta^2\Gamma_k[\bar{g},\phi=0]}{\delta c_\mu(x)\delta\bar{c}_\nu(y)}\,. \tag{3.33}$$

The graviton two-point function depends on momentum, curvature invariants, and couplings, such as the cosmological constant. One can chose to include curvature invariants and the cosmological constant into the regulator. In [215], this was categorised with type I, type II, and type III regulator. The type I regulator contains only the bare Laplacian, i.e., only the momentum. The type II regulator contains also curvature terms, while the type III regulator includes everything. Usually, regulators of type I and II are preferred.

Now, we are ready to adapt the Wetterich equation (2.34) to quantum gravity

$$\partial_t \Gamma_k[\bar{g},\phi] = \frac{1}{2}\mathrm{Tr}\left[\frac{1}{\Gamma_k^{(0,2)}+R_k}\partial_t R_k\right]_{hh} - \mathrm{Tr}\left[\frac{1}{\Gamma_k^{(0,2)}+R_k}\partial_t R_k\right]_{\bar{c}c} . \tag{3.34}$$

Here we have introduced the multi-field $\phi = (h_{\mu\nu}, c_\mu, \bar{c}_\mu)$ that includes all fluctuations fields. It is important to note that it is the second derivative of the fluctuation field that enters on the right-hand side of the Wetterich equation. Setting up the Wetterich equation requires to invert the graviton two-point function. This inversion on general curved backgrounds is non-trivial. Is it therefore important to set up a tensor basis, which is for example the transverse-traceless York decomposition [216] or the Stelle decomposition [117]. The decompositions consist of a separation of the trace and the longitudinal parts of the fluctuation field

$$h_{\mu\nu} = h^{\mathrm{tt}}_{\mu\nu} + h^{\mathrm{L}}_{\mu\nu} + \frac{1}{d}\bar{g}_{\mu\nu} h . \tag{3.35}$$

Here h is the trace $h = \bar{g}^{\mu\nu} h_{\mu\nu}$ and $h^{\mathrm{tt}}_{\mu\nu}$ is transverse $\bar{\nabla}^\mu h^{\mathrm{tt}}_{\mu\nu} = 0$ and traceless $\bar{g}^{\mu\nu} h^{\mathrm{tt}}_{\mu\nu} = 0$. The longitudinal part is decomposed further by introducing a transverse vector field ξ and a corresponding scalar σ. The final result for the fluctuation field reads

$$h_{\mu\nu} = h^{\mathrm{tt}}_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \left(\bar{\nabla}_\mu \bar{\nabla}_\nu - \frac{1}{d}\bar{g}_{\mu\nu}\bar{\Delta}\right)\sigma + \frac{1}{d}\bar{g}_{\mu\nu} h . \tag{3.36}$$

Here the bar indicates that the covariant derivatives are constructed with respect to the background metric $\bar{g}_{\mu\nu}$ and $\bar{\Delta} = -\bar{\nabla}^2$ is the Laplacian. We have split the unconstrained spin-2 field $h_{\mu\nu}$ into constrained fields of spin-2 $h^{\mathrm{tt}}_{\mu\nu}$, spin-1 ξ and spin-0 σ and h. The graviton two-point function is now written in terms of a matrix, which simplifies the inversion

$$\Gamma^{(0,2)}_{hh} = \begin{pmatrix} \Gamma^{(0,2)}_{h^{\mathrm{tt}}h^{\mathrm{tt}}} & 0 & 0 & 0 \\ 0 & \Gamma^{(0,2)}_{\xi\xi} & 0 & 0 \\ 0 & 0 & \Gamma^{(0,2)}_{hh} & \frac{1}{2}\Gamma^{(0,2)}_{h\sigma} \\ 0 & 0 & \frac{1}{2}\Gamma^{(0,2)}_{\sigma h} & \Gamma^{(0,2)}_{\sigma\sigma} \end{pmatrix} . \tag{3.37}$$

In case of the flat background, the inversion is now trivial. For a non-flat background one has to pay attention since curvature invariants and derivatives are not always commuting.

A very common approximation to the Wetterich equation is the background field approximation. It corresponds to an ansatz for the scale-dependent effective action of the type

$$\Gamma_k[\bar{g},\phi] = \Gamma_k[g] + S_{\mathrm{gf}}[\bar{g},\phi] + S_{\mathrm{gh}}[\bar{g},\phi] . \tag{3.38}$$

In other words, the effective action depends only on one metric field $g_{\mu\nu}$ and the difference between background metric and fluctuation field is only resolved in the ghost and gauge-fixing parts. The background-field approximation reduces the dependence on the background metric and the fluctuation field down to only one field. Thus the resulting scale-dependent action is seemingly diffeomorphism invariance. However, this is only a seeming property since the approximation neither fulfils the modified Nielsen identity (3.24) nor the modified Slavnov-Taylor identity (3.31), and in consequence, the approximation is neither background independent nor quantum diffeomorphism invariant. Nonetheless, it is a very useful approximation as it tries to project the flow on the physically important degrees of freedom and dramatically decreases the size of the theory space, which simplifies the computation significantly. In the next section, we employ the background field approximation on the Einstein-Hilbert truncation.

3.5 Einstein-Hilbert truncation

We now present a simple example of a quantum gravity computation with the FRG. This computation is similar to the original computation by Reuter [69]. We use the Einstein-Hilbert truncation in the spin-2 approximation, which means that we only include contributions from the transverse-traceless mode of the graviton, see (3.36). In many cases, this is a sufficient approximation that contains the most important physical features. The transverse-traceless mode is gauge-independent and invertible by itself. Thus, we do neither need to specify the gauge-fixing condition nor to introduce ghost fields. Furthermore, we employ the background-field approximation (3.38). The Wetterich equation simplifies to

$$\partial_t \Gamma_k[g] = \frac{1}{2} \mathrm{Tr} \left[\frac{1}{\Gamma^{(2)}_{k,tt}[g] + R_{k,tt}} \partial_t R_{k,tt} \right] , \tag{3.39}$$

with the Einstein-Hilbert ansatz

$$\Gamma_k[g] = 2\kappa^2 Z_k \int \mathrm{d}^4 x \sqrt{g} \left[-R + 2\Lambda_k \right] . \tag{3.40}$$

Here, we have defined

$$\kappa^2 = (32\pi G_\mathrm{N})^{-1} , \qquad\qquad G_{\mathrm{N},k} = G_\mathrm{N} Z_k^{-1} . \tag{3.41}$$

The scale dependence of the Newton coupling $G_{\mathrm{N},k}$ is encoded in the wave-function renormalisation Z_k. G_N is the bare Newton coupling. We choose the regulator to be proportional to the two-point function without including the cosmological constant and curvature terms

$$R_{k,\mu\nu\rho\sigma}[\Delta] = \Gamma^{(2)}_{k,\mu\nu\rho\sigma}\Big|_{\Lambda_k = R = 0} \cdot r\left(\frac{\Delta}{k^2}\right) , \tag{3.42}$$

where $\Delta = -\nabla^2$ is the Laplacian. For the shape function r, we take a Litim-type cutoff

$$r(x) = \left(\frac{1}{x} - 1\right) \Theta(1-x) . \tag{3.43}$$

With these choices and approximations, we have fully determined our computation.

We first evaluate the left-hand side of the Wetterich equation (3.39). We apply a scale derivative to (3.40), which acts on the Newton coupling $G_{\mathrm{N},k}$ (or, equivalently, on Z_k) and the cosmological constant Λ_k. This yields

$$\partial_t \Gamma^\mathrm{grav}_k = 2\kappa^2 \int \mathrm{d}^4 x \sqrt{g} \left[-(\partial_t Z_k) R + 2 (\partial_t Z_k \Lambda_k) \right]$$

$$= 2Z_k\kappa^2 \int \mathrm{d}^4x\sqrt{g}\left[\eta_g R + 2\left(k^2\partial_t\lambda_k + 2\Lambda_k - \eta_g\Lambda_k\right)\right] . \tag{3.44}$$

Here, we have introduced the anomalous dimension $\eta_g = -(\partial_t Z_k)/Z_k = -\partial_t \ln Z_k$ as well as the dimensionless cosmological constant $\lambda_k = \Lambda_k k^{-2}$. We have one term proportional to $\sqrt{g}R$ and one proportional to $\sqrt{g}$, which we will also encounter on the right-hand side of the Wetterich equation together with higher-order curvature terms. The comparison of left- and right-hand side then leads to expressions for the anomalous dimension η_g and the beta function for the cosmological constant $\beta_\lambda = \partial_t\lambda_k$. The beta function for the dimensionless Newton coupling, $g_k = G_{\mathrm{N},k}k^2 = G_\mathrm{N}k^2/Z_k$, follows directly from the anomalous dimension

$$\beta_g = \partial_t g_k = (2+\eta_g)\, g_k\,. \tag{3.45}$$

We turn to the right-hand side of the Wetterich equation (3.39). We first compute the graviton propagator $G_k = (\Gamma_k^{(2)} + R_k)^{-1}$ from the Einstein-Hilbert truncation (3.40). The Einstein-Hilbert ansatz for the graviton two-point function $\Gamma_k^{(2)}$ is obtained by two functional derivatives of (3.40) with respect to the metric field $g_{\mu\nu}$. Without the background-field approximation, we would take the derivatives with respect to the fluctuation field $h_{\mu\nu}$. The two-point function has four open indices and depends explicitly on the curvature invariants R, $R_{\mu\nu}$, and $R_{\mu\nu\rho\sigma}$. We can simplify the situation by exploiting the freedom of choosing the background metric before applying the background-field approximation, i.e., before setting the fluctuation to zero. We choose a maximally symmetric space

$$R_{\mu\nu} = \frac{1}{d}g_{\mu\nu}R\,, \qquad\qquad R_{\mu\nu\rho\sigma} = \frac{1}{d(d-1)}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R\,. \tag{3.46}$$

As mentioned before, we restrict ourselves to the spin-2 approximation, i.e., we are only interested in the transverse-traceless part. With these simplifications, the second variation of the effective action can be computed as

$$\delta^2\Gamma_{k,tt} = \frac{1}{2}\kappa^2 Z_k \int \mathrm{d}^4x\sqrt{g}\,\delta g^{tt}_{\mu\nu}\left[\Delta - 2\Lambda_k + \frac{2}{3}R\right]\delta g^{tt,\mu\nu}\,, \tag{3.47}$$

and consequently

$$\Gamma^{(2)}_{k,tt} = \frac{Z_k}{32\pi}\left(\Delta - 2\Lambda_k + \frac{2}{3}R\right)\delta^{(4)}(x_1 - x_2)\,. \tag{3.48}$$

From now on, we leave the delta function implicit. The propagator of the transverse-traceless spin-2 mode is thus given by

$$G_{k,tt} = \frac{32\pi}{Z_k}\frac{1}{\Delta(1+r(\Delta)) - 2\Lambda_k + \frac{2}{3}R}\,, \tag{3.49}$$

and the scale derivativ of the regulator reads

$$\partial_t R_{k,tt} = \frac{Z_k}{32\pi}\Delta\left(\partial_t r\left(\frac{\Delta}{k^2}\right) - \eta_g r\left(\frac{\Delta}{k^2}\right)\right)\,. \tag{3.50}$$

We now have to consider the traces on the right-hand side of (3.34). These are traces over the Laplacian in curved space and they can be evaluated with heat-kernel techniques or spectral methods. We give a very short description of these techniques in App. A. The result of these traces includes term proportional to $\sqrt{g}$ and $\sqrt{g}R$, which are the ones we are interested in. The comparison of these terms with the left-hand side (3.44) yields the beta functions for the Newton coupling and the cosmological constant as discussed above. The traces also yield higher-order curvature terms, which signal that the flow creates all terms compatible with the symmetry. We neglect those terms here, but one should include them in an improved truncation. We plug (3.49) and (3.50) into (3.39) and now we need to evaluate the trace over the Laplacian

$$\begin{aligned}\mathrm{Tr}\left[\frac{1}{\Gamma^{(2)}_{k,tt}+R_k}\partial_t R_k\right] &= \mathrm{Tr}\left[\frac{\Delta(\partial_t r(\frac{\Delta}{k^2})-\eta_g r(\frac{\Delta}{k^2}))}{\Delta(1+r(\frac{\Delta}{k^2}))-2\Lambda_k+\frac{2}{3}R}\right] \\ &= \mathrm{Tr}\left[\frac{z(-2zr'(z)-\eta_g r(z))}{z(1+r(z))-2\lambda_k}\right]-\frac{2}{3}\frac{R}{k^2}\mathrm{Tr}\left[\frac{z(-2zr'(z)-\eta_g r(z))}{(z(1+r(z))-2\lambda_k)^2}\right].\end{aligned} \tag{3.51}$$

Here we have introduced the dimensionless Laplacian $z=\Delta/k^2$ and also switched to the dimensionless version of the cosmological constant $\lambda_k=\Lambda_k/k^2$. With the heat-kernel formula (A.4) from App. A we obtain for the first term in (3.51)

$$\begin{aligned}\mathrm{Tr}\left[\frac{z(-2zr'(z)-\eta_g r(z))}{z(1+r(z))-2\lambda_k}\right] &= \frac{1}{(4\pi)^2}\Bigg(B_0(z)\,Q_2\left[\frac{z(-2zr'(z)-\eta_g r(z))}{z(1+r(z))-2\lambda_k}\right] \\ &\quad + B_2(z)\,Q_1\left[\frac{z(-2zr'(z)-\eta_g r(z))}{z(1+r(z))-2\lambda_k}\right]+\mathscr{O}(R^2)\Bigg) \\ &= \frac{1}{(4\pi)^2}\int \mathrm{d}^4x\sqrt{g}\left[5\,\Phi^1_2(-2\lambda_k)-\frac{5}{6}\frac{R}{k^2}\Phi^1_1(-2\lambda_k)\right].\end{aligned} \tag{3.52}$$

In the last step we have expressed the result in terms of the threshold functions Φ^p_n, which are defined by

$$\Phi^p_n(\omega)=\frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}z z^{n-1}\frac{z(-2zr(z)-\eta_g r(z))}{(z(1+r(z))+\omega)^p}=\frac{1}{\Gamma(n)}\frac{1}{(1+\omega)^p}\left(\frac{2}{n}-\frac{\eta_g}{n(n+1)}\right). \tag{3.53}$$

In the last step, we have evaluated the threshold functions for the Litim-type cutoff (3.43). The use of threshold functions allows us to keep the result independent of the shape function and thus it is easier scan over different shape functions. With the Litim-type cutoff, (3.52) becomes

$$5\,\Phi^1_2(-2\lambda_k)-\frac{5}{6}\frac{R}{k^2}\Phi^1_1(-2\lambda_k)=5\frac{1-\frac{\eta_g}{6}}{1-2\lambda_k}-\frac{5}{6}\frac{R}{k^2}\frac{2-\frac{\eta_g}{2}}{1-2\lambda_k}. \tag{3.54}$$

In straight analogy, we evaluate the second term in (3.51)

$$\begin{aligned}\mathrm{Tr}\left[\frac{z(-2zr'(z)-\eta_g r(z))}{(z(1+r(z))-2\lambda_k)^2}\right] &= \frac{1}{(4\pi)^2}B_0(z)\,Q_2\left[\frac{z(-2zr'(z)-\eta_g r(z))}{(z(1+r(z))-2\lambda_k)^2}\right]+\mathscr{O}(R) \\ &= \frac{5}{(4\pi)^2}\int \mathrm{d}^4x\sqrt{g}\,\Phi^2_2(-2\lambda_k)\end{aligned}$$

$$= \frac{5}{(4\pi)^2}\int \mathrm{d}^4x\sqrt{g}\,\frac{1-\frac{\eta_g}{6}}{(1-2\lambda_k)^2}\,. \tag{3.55}$$

Note, that this term is multiplied with $-\frac{2}{3}R$ in (3.51) and thus we do not compute the term of order R. We are now ready to compare the terms $\int_x\sqrt{g}$ and $\int_x\sqrt{g}R$ from the left-hand and the right-hand side. Let us start with $\int_x\sqrt{g}R$, where we collect (3.44) on the left-hand side and (3.52), (3.55) on the right-hand side. This gives us the anomalous dimension

$$\eta_g = -\frac{5}{6\pi}g_k\left(2\frac{1-\frac{1}{6}\eta_g}{(1-2\lambda_k)^2}+\frac{1-\frac{1}{4}\eta_g}{1-2\lambda_k}\right)\,. \tag{3.56}$$

The anomalous dimension η_g depends on itself: it also enters on the right-hand side in the loop contributions. This feature reflects that the Wetterich equation is a one-loop equation, but takes the dressed propagators and vertices as input. We solve (3.56) for η_g

$$\eta_g = \frac{-\frac{5g_k}{6\pi(1-2\lambda_k)}-\frac{5g_k}{3\pi(1-2\lambda_k)^2}}{1-\frac{5g_k}{24\pi(1-2\lambda_k)}-\frac{5g_k}{18\pi(1-2\lambda_k)^2}} = \frac{60g_k(2\lambda_k-3)}{5g_k(6\lambda_k-7)+72\pi(1-2\lambda_k)^2}\,, \tag{3.57}$$

and now it becomes manifest that η_g receives all contributions from all orders in perturbation theory. By comparing the terms proportional to $\int_x\sqrt{g}$ on the left-hand side, (3.44), and the right-hand side, (3.52), we obtain the flow equation for the cosmological constant

$$\partial_t\lambda_k = -4\lambda_k + \frac{\lambda_k}{g_k}\partial_t g_k + \frac{5}{4\pi}g_k\frac{1-\frac{1}{6}\eta_g}{1-2\lambda_k}\,, \tag{3.58}$$

which also depends on η_g and consequently also receives contribution from all orders in perturbation theory. Together with the beta function of the Newton coupling, which is fully determined by the anomalous dimension,

$$\partial_t g_k = (2+\eta_g)\,g_k\,. \tag{3.59}$$

we have all beta function ready. Already in this simple truncation we find a non-Gaußian fixed point at the coupling values

$$(g^*,\lambda^*) = (0.86, 0.18)\,. \tag{3.60}$$

The critical exponents, which are the eigenvalues of the stability matrix, see (2.46), are given by

$$\theta_{1,2} = 2.9\pm 2.6\,i\,. \tag{3.61}$$

The real part of both critical exponents is positive, which implies that the fixed point is fully UV attractive. The critical exponents have an imaginary part, which causes a spiralling of the RG flow around the fixed point. The full phase diagram is depicted in Fig. 9, where we can clearly observe this behaviour of the RG flow. Also the Gaußian fixed point $(g^*,\lambda^*) = (0,0)$ is a fixed point of the beta functions. There is exactly one trajectory, which connects these two fixed points and it is marked in green in Fig. 9. On this trajectory, the dimensionless Newton coupling runs for small k with k^2 and thus the dimensionful Newton coupling is a constant, just as depicted in

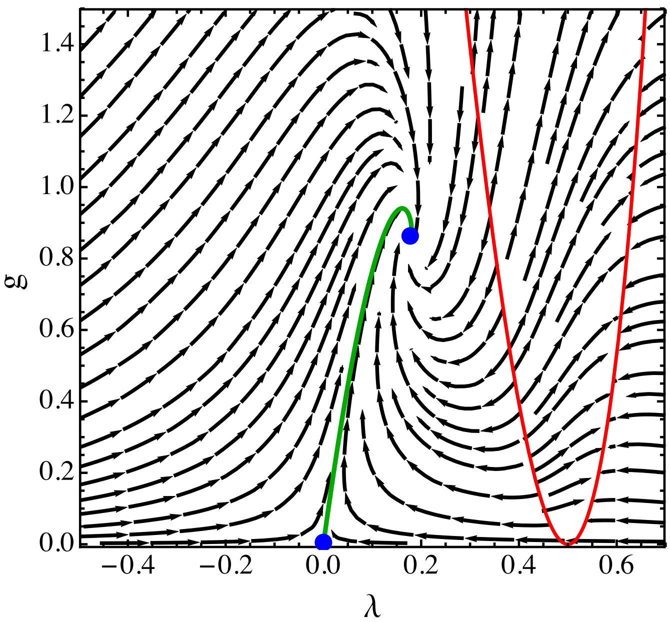


Figure 9: Phase diagram of quantum gravity in the spin-2 approximation of the Einstein-Hilbert truncation, i.e., by the beta functions given in (3.57), (3.58), and (3.59). The Gaußian and the UV fixed point (3.60) are indicated with a blue dot. The green line is the trajectory that connects these two fixed points. The red line indicates where the anomalous dimension is diverging (3.62), which is thus a singularity in the beta functions. The UV fixed point is fully attractive and the trajectories have a spiralling behaviour, which is described by the positive sign and the imaginary part of the critical exponents (3.61).

Fig. 7. The dimensionful and the dimensionless cosmological constant go to zero on this trajectory. On trajectories that go to $\lambda \to -\infty$ and $g \to 0$, we find that the dimensionless Newton coupling also runs for small k with k^2 but the dimensionless cosmological constant runs as k^2 and thus the dimensionful cosmological constant becomes a negative constant. For a positive non-vanishing cosmological constant, we need to find a trajectory that runs to the non-trivial IR fixed point at $\lambda = \frac{1}{2}$ and $g = 0$, which is difficult due to singularities in the beta functions. For work in this direction, see [146, 157, 158, 161, 217, 218]. The phase diagram has cuts where the beta functions are diverging. These are marked with red lines in Fig. 9. The anomalous dimension η_g (3.57) is diverging at the values

$$g_{\text{sing}} = -\frac{72\pi\left(4\lambda^2 - 4\lambda + 1\right)}{5(6\lambda - 7)}\,. \tag{3.62}$$

For $g \to 0$, the singularity goes to $\lambda = \frac{1}{2}$ as expected from the pole of the propagator.

3.6 Outlook

In the last section, we have obtained the phase diagram of quantum gravity in a very simple truncation that only includes the cosmological constant and the Newton coupling. Already in this simple truncation, we found an attractive UV fixed point and a trajectory that connects it to a classical IR regime. A lot of progress has happened since this truncation was implemented for the first time in [69]. Even within the Einstein-Hilbert truncation, a lot of research had been done. This ranged from investigations of gauge and parameterisation dependence [194, 219–222] and regulator dependence [223–226] to a better inclusion of the ghost sector via ghost anomalous dimensions [227–229]. Beyond Einstein-Hilbert, a lot of progress has been made in the inclusion of higher-dimensional operators and truncations that include a function $f(R)$ [126, 135–139, 215, 230–257]. This progress has led to more trust in the existence of the UV fixed point and to the strong hint that the UV critical hypersurface is indeed finite-dimensional.

There are several issues, which we have discussed far too little in these lecture notes. The first is how we can fulfil the symmetry identities of quantum gravity. Most of the above-listed works use the background-field approximation (3.38), which has the nice property of being seemingly diffeomorphism invariant. However, the background-field approximation violates the symmetry identities, namely the mSTI (3.31) and the mNI (3.24). For a true diffeomorphism invariant theory, one has to go beyond the background-field approximation. One approach is to tackle this issue is to disentangle contributions from the background metric and the fluctuation field. This implies that one has to solve a theory with two independent metric fields and then afterwards solve the mNI at one scale k. This approach has recently received a lot of attention and high-orders in the fluctuation field expansion have been computed [141, 157–159, 161, 205, 206, 251, 258–263].

The second issue is the interplay of matter and gravity. Asymptotically safe quantum gravity is in the very fortunate position that the inclusion of matter fields can be implemented in a standard QFT way. Thus the impact of matter fields on the running of the gravitational couplings was investigated already very early on. The key question of these investigations is for which matter content the UV fixed point exists. Several works have pointed out potential bounds on the matter content, while other works argued against it. So far, no conclusive result was obtained and it remains an interesting topic of research [140, 150–153, 162, 252, 256, 257, 263–295].

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A. Heat-kernel techniques

We want to evaluate the trace of a function that depends on the Laplace operator on a curved background. The function can depend on couplings and background curvature but we assume that it does not depend on covariant derivatives. In general, such a trace is defined as sum/integral over

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the eigenvalues of the Laplace operator,

$$\mathrm{Tr} f(\Delta) = N \sum\!\!\!\!\!\!\int_{\ell} \rho(\ell) f(\lambda(\ell)) \,. \tag{A.1}$$

Here, $\lambda(\ell)$ are the spectral values, $\rho(\ell)$ are the multiplicities or the spectral density and N is some normalisation factor. A simple example is the flat background where (A.1) turns into a standard momentum integral. For non-flat backgrounds a standard example is the four-sphere with constant background curvature $r = \frac{\bar{R}}{k^2} > 0$. There the spectrum of the scalar Laplacian is discrete and the spectral values and the multiplicities are given by

$$\lambda(\ell) = \frac{\ell(3+\ell)}{12} r \,, \qquad\qquad \rho(\ell) = \frac{(2\ell+3)(\ell+2)!}{6\ell!} \,, \tag{A.2}$$

where ℓ takes integer values $\ell \geq 0$. The normalisation N is the inverse volume of the four sphere $N = V^{-1} = \frac{k^4 r^2}{384\pi^2}$. In summary, we can evaluate (A.1) for constant positive curvature in terms of an infinite series

$$\mathrm{Tr} f(\Delta) = \frac{k^4 r^2}{384\pi^2} \sum_{\ell=0}^{\infty} \frac{(2\ell+3)(\ell+2)!}{6\ell!} f\left(\frac{\ell(3+\ell)}{12} r\right) \,, \tag{A.3}$$

which is called a spectral sum. When the curvature is negative similar formulas hold, but the spectrum of the Laplacian is continuous. The resulting integrals are called spectral integrals.

The heat-kernel method can be understood as curvature expansion about the flat background. The idea is to separately treat the dependence of the spectrum of the operator and the dependence on the function f. The master formula for the heat-kernel techniques is

$$\mathrm{Tr} f(\Delta) = \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{n=0}^{\infty} B_{2n}(\Delta) Q_{\frac{d}{2}-n}[f(t)] \,, \tag{A.4}$$

where the B_n are the heat-kernel coefficients of the Laplace operator Δ and the Q_n are defined by

$$Q_n[f(x)] = \frac{1}{\Gamma(n)} \int \mathrm{d}x \, x^{n-1} f(x) \,. \tag{A.5}$$

A very common derivation of this equation uses the Laplace transformation

$$f(\Delta) = \int_0^\infty \mathrm{d}s \, e^{-s\Delta} \tilde{f}(s) \,. \tag{A.6}$$

Applying this to (A.1) leads to

$$\mathrm{Tr} f(\Delta) = \int_0^\infty \mathrm{d}s \, \tilde{f}(s) \mathrm{Tr} \, e^{-s\Delta} \,, \tag{A.7}$$

where the last term is precisely the trace of the heat kernel. For this trace the expansion is well established

$$\mathrm{Tr} \, e^{-s\Delta} = \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{n=0}^{\infty} s^{\frac{n-d}{2}} B_n(\Delta) \,, \tag{A.8}$$

Table 2: Heat kernel coefficients for transverse-traceless tensors (TT), transverse vectors (TV) and scalars (S) on S^4.

	TT	TV	S
$\mathrm{tr}\, b_0$	5	3	1
$\mathrm{tr}\, b_2$	$-\frac{5}{6}R$	$\frac{1}{4}R$	$\frac{1}{6}R$

and the coefficients B_n are again the heat-kernel coefficients as in (A.4). Using this in (A.7) results in

$$\begin{aligned} \mathrm{Tr} f(\Delta) &= \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{n=0}^{\infty} B_n(\Delta) \int_0^\infty \mathrm{d}s\, s^{\frac{n-d}{2}} \tilde{f}(s) \\ &= \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{n=0}^{\infty} \frac{1}{\Gamma(\frac{d-n}{2})} B_n(\Delta) \int_0^\infty \mathrm{d}t\, t^{\frac{d-n}{2}-1} f(t) \\ &= \frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{n=0}^{\infty} B_n(\Delta) Q_{\frac{d-n}{2}}[f(t)]\,. \end{aligned} \tag{A.9}$$

Here we have used the relation $\int_s s^{-x} \tilde{f}(x) = \frac{1}{\Gamma(x)} \int_z z^{x-1} f(z)$ and further used the definition of Q_n, see (A.5). Lastly, we use that the odd coefficients B_{2n+1} are vanishing and we arrive precisely at (A.4).

These reformulations have brought us the advantage that we have separated the dependences of the function f and of the Laplace operator Δ. The $Q_n[f]$ are in general easy to determine either numerically or even analytically. We are left with determining the heat-kernel coefficients $B_k(\Delta)$, which was already done for many different Laplacians in the literature [296–298]. We do not go into details here and simply state the results, if we choose the sphere as background, where the results tremendously simplify. The heat-kernel coefficients are often expressed as

$$B_n(\Delta) = \int \mathrm{d}^d x \sqrt{g}\, \mathrm{tr}\, b_n(\Delta)\,, \tag{A.10}$$

and we display the coefficients b_n for the sphere in Tab. 2.

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