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Simple Factorization of the Jarlskog Invariant for Neutrino Oscillations in Matter

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For neutrino propagation in matter, we show that the Jarlskog invariant, which controls the size of true CP violation in neutrino oscillation appearance experiments, factorizes into three pieces: the vacuum Jarlskog invariant times two simple two-flavor matter resonance factors that control the matter effects for the solar and atmospheric resonances independently. If the effective matter potential and the effective atmospheric Δm^2 are chosen carefully for these two resonance factors, then the fractional corrections to this factorization are an impressive 0.04% or smaller.

I. INTRODUCTION

The discovery of an invariant, the Jarlskog invariant [1], that controls the size of CP violation in both quark and neutrino sectors was a monumental step in the understanding of flavor physics. For neutrinos, using the standard parameterization of the PMNS matrix, this invariant is given by

$$J \equiv s_{23}c_{23}s_{13}c_{13}^2 s_{12}c_{12}\sin\delta\,, \tag{1}$$

where we use the usual notation, $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, and δ is the CP-violating phase. The CP-violating part of the vacuum neutrino oscillation probability in the appearance channels, e.g. $\nu_\mu \to \nu_e$, is given by

$$8J\sin\Delta_{31}\sin\Delta_{32}\sin\Delta_{21}\,, \tag{2}$$

where the kinematic phases are given by $\Delta_{jk} = \Delta m_{jk}^2 L/4E_\nu$ with $\Delta m_{jk}^2 = m_j^2 - m_k^2$ for an experiment of baseline L and neutrino energy E_ν.

For neutrinos propagating in matter, like the currently running NOvA [2] and T2K [3] experiments and the upcoming DUNE [4] and T2HK(K) [5, 6], the part of the appearance oscillation probability that depends on the intrinsic CP violation is given by

$$8\widehat{J}\sin\widehat{\Delta}_{31}\sin\widehat{\Delta}_{32}\sin\widehat{\Delta}_{21}\,, \tag{3}$$

where $\widehat{x}$ is the matter value for the vacuum variable x. The Jarlskog invariant in matter, $\widehat{J}$, is given by same expression as eq. 1, but with the mixing angles and phase replaced by their matter values. Both θ_{12} and θ_{13} have a strong dependence on density of the matter and the energy of the neutrino through the Wolfenstein matter potential [7], a, given by

$$a \equiv 2\sqrt{2}G_F N_e E_\nu\,, \tag{4}$$

where G_F is the Fermi constant, N_e is the number density of electrons and E_ν is the neutrino energy in the matter rest frame.

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II. THE RESULT

While the exact expressions for the mixing angles in matter are extremely complicated [8], it is possible to approximate, at the 0.04% level, the Jarlskog invariant in matter as simply

$$\widehat{J} \approx \frac{J}{\mathcal{S}_{12}\mathcal{S}_{13}}\,, \tag{5}$$

where

$$\mathcal{S}_{12} = \sqrt{1 - 2\cos 2\theta_{12}(c_{13}^2 a/\Delta m_{21}^2) + (c_{13}^2 a/\Delta m_{21}^2)^2}\,,$$
$$\mathcal{S}_{13} = \sqrt{1 - 2\cos 2\theta_{13}(a/\Delta m_{ee}^2) + (a/\Delta m_{ee}^2)^2}\,. \tag{6}$$

The $\mathcal{S}$ factors are the two-flavor resonance factors for solar (1-2) and atmospheric (1-3) resonances respectively. To achieve fractional precision of $\mathcal{O}(s_{13}^2 \frac{\Delta m_{21}^2}{\Delta m_{ee}^2}) \sim 0.04\%$ for this factorization, the following are crucial:

- for the solar (1-2) resonance factor, $\mathcal{S}_{12}$, the effective matter potential is $c_{13}^2 a$, not just a,
- for the atmospheric (1-3) resonance factor, $\mathcal{S}_{13}$, the effective Δm^2 is $\Delta m_{ee}^2 \equiv c_{12}^2\Delta m_{31}^2 + s_{12}^2\Delta m_{32}^2$ [9, 10], not Δm_{31}^2 or Δm_{32}^2.

With only one of these choices, the fractional error is 2-3%, $\mathcal{O}(s_{13}^2)$ and/or $\mathcal{O}(\frac{\Delta m_{21}^2}{\Delta m_{ee}^2})$, see [11]. But with both of these choices the fractional uncertainty is an impressive 0.04% which is $\mathcal{O}(s_{13}^2 \frac{\Delta m_{21}^2}{\Delta m_{ee}^2})$ or better for all values of the matter potential.

In fig. 1, we have plotted the fractional error to the approximation in eq. 5 as a function of the matter potential for both neutrinos and anti-neutrinos and find that the expression is precise to the 0.04% level or better. We also show the fractional error for the expression derived in ref. [11] which is the same as eqs. 5 and 6 except that the c_{13}^2 term in $\mathcal{S}_{12}$ is not included. This expression leads to ∼2% precision or better and is consistently one or more orders of magnitude worse than eqs. 5 and 6.

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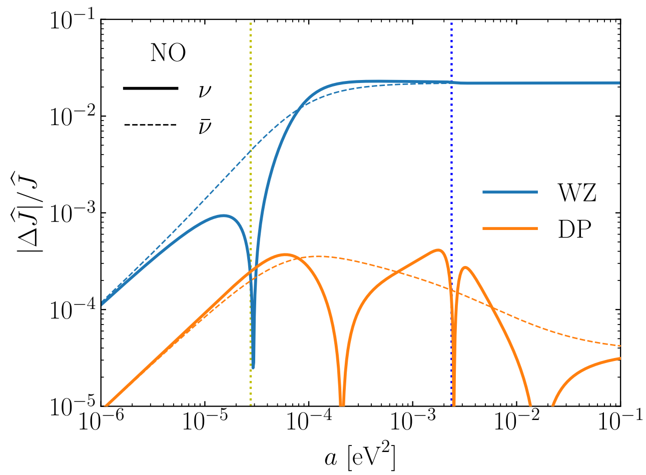

FIG. 1. The fractional error in $\widehat{J}$ compares two approximate expressions with the exact expression from [8]. The orange curves labeled DP are calculated using our approximate expression given in eqs. 5 and 6. The blue curves labeled WZ are calculated using the approximate expression from ref. [11] which is the same as ours without the c_{13}^2 terms in $\mathcal{S}_{12}$. The solid curves are for neutrinos and the dashed curves for anti-neutrinos. The yellow and blue vertical lines are at the solar, $\cos 2\theta_{12}\Delta m^2_{21}/c^2_{13}$, and atmospheric, $\cos 2\theta_{13}\Delta m^2_{ee}$, resonances respectively. The downward spikes occur where the exact and approximate expressions cross. The normal mass ordering (NO) is assumed.

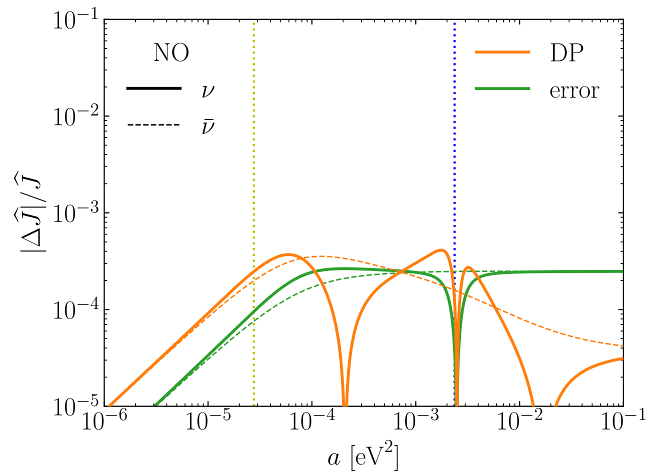

FIG. 2. The orange curves are the same as in fig. 1. The green curves are the analytic approximation of the error using DMP [14] shown in eq. 10 divided by $\prod_{i>j}\widehat{\Delta m^2}_{ij}$ which makes it a fractional error.

We have numerically verified that c_{13}^2 is the optimal correction, Δm^2_{ee} is the optimal atmospheric mass splitting, and that these results are generally independent of the mass ordering.

III. ERROR ESTIMATE

In order to better understand the precision of eqs. 5 and 6, we have estimated the error in this expression. By using the exact Naumov-Harrison-Scott identity [12, 13],

$$\widehat{J}\widehat{\Delta m^2}_{32}\widehat{\Delta m^2}_{31}\widehat{\Delta m^2}_{21} = J\Delta m^2_{32}\Delta m^2_{31}\Delta m^2_{21}\,, \qquad (7)$$

we can express our approximate expression in terms of the exact matter eigenvalues,

$$\widehat{\Delta m^2}_{32}\widehat{\Delta m^2}_{31}\widehat{\Delta m^2}_{21} \approx \mathcal{S}_{12}\mathcal{S}_{13}\Delta m^2_{32}\Delta m^2_{31}\Delta m^2_{21}\,. \quad (8)$$

While the exact eigenvalues have a very complicated analytic form [8] due to the presence of the $\cos(\frac{1}{3}\cos^{-1}\cdots)$ terms, they can be extremely well approximated using the DMP approach [14][1]. Using the expressions from DMP we have calculated the difference of the square of the left and right hand sides of eq. 8 as a power series in s_{13}^2 and $\epsilon \equiv \Delta m^2_{21}/\Delta m^2_{ee}$. We find that the zeroth order term (in ϵ and s_{13}^2) and the terms proportional to ϵ or s^2_{13} are all zero, confirming our earlier error estimate of $\mathcal{O}(\epsilon s^2_{13})$. The first non-zero term correcting eq. 8 is then[2]

$$\left(\prod_{i>j}\widehat{\Delta m^2}_{ij}\right)^2 - \mathcal{S}^2_{12}\mathcal{S}^2_{13}\left(\prod_{i>j}\Delta m^2_{ij}\right)^2$$
$$\approx -2\epsilon s^2_{13}\cos 2\theta_{12}\, a^2(a-\Delta m^2_{ee})^2(\Delta m^2_{ee})^2\,. \quad (9)$$

By propagating the error from the product of $\widehat{\Delta m^2}$'s squared to the error in $\widehat{J}$ via the Naumov-Harrison-Scott identity, we find that the fractional error in $\widehat{J}$ is approximately given by

$$\frac{\Delta\widehat{J}}{\widehat{J}} \approx \frac{\epsilon s^2_{13}\cos 2\theta_{12}\, a(a-\Delta m^2_{ee})\Delta m^2_{ee}}{\prod_{i>j}\widehat{\Delta m^2}_{ij}}\,, \qquad (10)$$

up to an overall sign. We plot eq. 10 (note that using either the exact expression for the denominator or the approximate expression from DMP is indistinguishable) in fig. 2. Also shown for comparison is the exact fractional error of eqs. 5 and 6 as in fig. 1.

This error estimate gets the magnitude of the error correct as well as the general features: the error goes to zero for small a and peaks at the level of 0.04%. In addition, the error goes to zero at $a = \Delta m^2_{ee}$ for neutrinos but not for anti-neutrinos as it is supposed to.

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[1] The eigenvalues in DMP [14] are accurate to better than $\mathcal{O}(\epsilon^2)$ while the mixing angles are only correct to $\mathcal{O}(\epsilon)$ [15] hence the importance of using the Naumov-Harrison-Scott identity.

[2] At the atmospheric (1-3) resonance both sides of eq. 8 are proportional to s_{13}, this causes additional large fractional errors unless $s^2_{13} \approx \epsilon$.

IV. DISCUSSION

In light of the Naumov-Harrison-Scott identity, it isn't surprising that $\widehat{J}/J$ has a form that looks like the inverse of the matter-corrections to the Δm^2's. It may not be obvious, however, why $\widehat{J}$ is well-approximated by only two such expressions instead of all three. The reason is because for nearly any value of a, there is always one $\widehat{\Delta m^2}$ that is essentially constant. In the NO this is $\widehat{\Delta m^2}_{32}$ for anti-neutrinos and $\widehat{\Delta m^2}_{\ell 1}$ for neutrinos where $\ell = 3$ below the atmospheric resonance and $\ell = 2$ above the atmospheric resonance. As such having two $\mathcal{S}_{ij}$ terms is to be expected.

In addition, while the presence of the c_{13}^2 term breaks an otherwise relatively symmetric definition of $\mathcal{S}_{12}$ and $\mathcal{S}_{13}$, this can be understood using the DMP [14] expressions. In that formalism the (1-2) sector is handled second and thus contains a small (1-3) correction since the (1-3) sector was handled first.

For completeness, in addition to our error estimate calculated in section III, we performed several numerical checks to confirm that eqs. 5 and 6 represents the optimal compact expression for the CP-violating term in matter. We considered various alternative formulations of $\mathcal{S}_{12}$ and $\mathcal{S}_{13}$ such as those where $\Delta m_{ee}^2 \to \Delta m_{3\ell}^2$ for $\ell = 1, 2$ was used, or where the c_{13}^2 term in $\mathcal{S}_{12}$ was allowed to float freely. The c_{13}^2 correction is clearly the optimal value, and changing the definition of the atmospheric mass splitting away from Δm_{ee}^2 either resulted in no change or made the expressions worse.

V. CONCLUSIONS

In this paper we have shown that the Jarlskog invariant for neutrino oscillations in matter can be factorized in to the vacuum Jarlskog invariant times two simple matter resonance factors which when carefully chosen make the fractional precision of this factorization at the $\mathcal{O}(s_{13}^2 \frac{\Delta m_{21}^2}{\Delta m_{ee}^2}) \sim 0.04\%$. Using compact expressions for the eigenvalues in matter we calculated the error estimate which has a simple form as well and is quite accurate. To achieve this precision one needs to use the θ_{13} corrected value for the matter potential for the solar (1-2) sector, ac_{13}^2 as well as the effective Δm_{ee}^2 instead of Δm_{31}^2 or Δm_{32}^2 for the atmospheric (1-3) sector. This precision factorization of the Jarlskog invariant in matter further enhances our understanding of neutrinos in matter relevant for the current T2K and NOvA experiments and the upcoming DUNE and T2HK/K experiments.

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