Covariant markovian random fields in four space-time dimensions with nonlinear electromagnetic interaction

by

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Abstract

We construct covariant random vector fields over 4-dimensional space-time as solutions of a system of first order coupled stochastic partial differential equations, best interpreted as equations for quaternionic valued random fields. The fields are covariant under the proper Euclidean transformations. We give necessary and sufficient conditions in terms of a given source of the infinitely divisible type, for the fields to be covariant also under reflections. In the case of a Gaussian white noise source the fields are Euclidean free electromagnetic potential fields and have the global Markov property. The fields with Poisson white noise source can be used as approximation of the Gaussian fields, with better support properties.

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1. Introduction

Local relativistic quantum field theory was invented more than sixty years ago to provide a synthesis of quantum mechanics and the (special) theory of relativity. In recent years models of local interacting relativistic quantum fields of scalar, vector or gauge type have been constructed in space-times of dimension less than 4, see e.g. [AHK1], [AFHKL], [DST], [GJ], [JLM].

In the case of 4-dimensional space times only partial results are known, see e.g. [DST], [AFHKL].
In the present paper we exhamine the possibility of constructing a four space-time dimensional theory describing quantum fields of the electromagnetic type, with a formal action which is not necessarily of the canonical type "kinetic energy minus potential term", but rather kinetic energy minus a term involving a nonlinear function of suitable linear combinations of derivatives of the field.
There is some relation of such models with those of non linear electromagnetic field theories, like Born-Infeld theory [BI]. Such nonlinear electromagnetic field theories had been introduced as approximations to Maxwell fields and our models can also be looked upon in the same spirit (and we prove indeed a result in this sense). Let us also remark that very recently the interest of Born-Infeld's action has been reactivated by the discovery that it describes heuristically the full effective self-interaction of vector fields in the Abelian limit in open bosonic strings (and superstrings), see e.g. [FT], [CLNY], [CF].
Our models exploit in an essential way the 4-dimensionality of the physical space-time, which permits to identify it, as a vector space, with the space $\mathbb{H}$ of quaternions[1)].
The fields are given as solutions of a system of coupled stochastic first order partial differential equations, having a natural formulation in terms of quaternions. The possibility of writing such equation relies on the isomorphism $SO(4) \cong (SU(2) \times SU(2))/\mathbb{Z}_2$. The Euclidean vector generalized random fields $\{A_r(x)\ ,\ x \in \mathbb{R}^4 \cong \mathbb{H}\ ,\ r = 0,1,2,3\}$, identified with quaternion fields $A(x)$, satisfy stochastic partial differential equations of the form $\partial A(x) = F(x)$, with $F(x)$ a quaternionic-valued infinitely divisible field (see e.g. [Kl], [Ku], [Su]) with suitable transformation properties under the proper Euclidean group $SO(4)\wedge\mathbb{R}^4$, ∂ being the basic 1-order quaternionic differential operator with unit coefficients[2)].
We discuss the transformation properties of A under reflections as well as Markovian properties of the fields. In the case of F being Gaussian white noise A is the free electromagnetic Euclidean potential field. We exhibit a way to approximate the latter field by fields A_p defined by taking F to be a Poisson type white noise.
We also point out that the fields A can be obtained as continuum limits of corresponding lattice fields, which makes appear their action as being heuristically given by
$\int f\left(|\text{div } A|\,,\,|\vec{E}-\vec{B}|\right)\,dx$, with $A(x) = \left(A_0(x)\,,\,\vec{A}(x)\right)$, $x = (x_0, \vec{x}) \in \mathbb{R}\times\mathbb{R}^3$,
$\vec{E} \equiv \frac{\partial}{\partial x_0}\vec{A} - \text{grad}_{\vec{x}} A_0\ ,\ \vec{B} \equiv \text{rot}_{\vec{x}}\, A$, for suitable real valued functions f on $\mathbb{R}$.
Let us also remark that the present work is connected with previous work (see e.g. [AHKH1-3], [AHK 6], [AHKHK], [Ka] and references therein) in which Markov and quantum fields associated to 1-codimensional hypersurfaces, instead of points, in $\mathbb{R}^d$ were constructed. For $d = 2$ such "cosurface fields" can be identified, on closed contours, with

quantum gauge fields; for $d = 4$ they include free electromagnetic fields and more generally 3-forms with values in the Lie algebra of compact semisimple Lie groups, providing (by duality) a natural extension of electromagnetic fields to "coloured fields" (this relies on the realization of $I\!R^4$ and the Lie algebra $u(2)$ of $U(2)$ as the space of quaternions [AHK3]). The constructed cosurface can also be connected to vector fields, using again the 4-dimensionality of space-time, and these fields satisfy the stochastic partial differential equation discussed ([AHK2]).

We finally remark that the present paper extends the work of [AHK4] and makes precise the point first overlooked in [AHK4a] (but shortly remarked in [AHK4b]) that A is not time reflection invariant in the non Gaussian case.

2. A covariant quaternionic partial differential equation

We shall consider a covariant partial differential equation over $I\!R^4$. This type of equations can only be considered over $I\!R^1$, $I\!R^2$ and $I\!R^4$ and their existence is tied to that of the associative fields of real, complex resp. quaternionic numbers. In this paper we shall consider the physical situation with underlying space $I\!R^4$, and the equation is best formulated using quaternions, as in [AHK2-4]. Let $I\!H$ be the field of quaternionic numbers and $\{1, i, j, k\}$ be its canonical basis.
As a real vector space $I\!H$ is isomorphic to $I\!R^4$ by

$$I\!H \ni x_0 1 + x_1 i + x_2 j + x_3 k \longleftrightarrow (x_0, x_1, x_2, x_3) \in I\!R^4 \ .$$

We regard $I\!R$ as being inbedded in $I\!H$ by identifying $t \in I\!R$ with $t1 \in I\!H$, then $I\!H$ forms a real associative algebra with the identity 1 under the multiplication rules : $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$.
There is a distinct automorphism of $I\!H$ called the conjugation :

$$x = x_0 + x_1 i + x_2 j + x_3 k \longrightarrow \bar{x} = x_0 - x_1 i - x_2 j - x_3 k \ .$$

As in the case of $\mathbb{C}$ we write

$$\begin{aligned} Re\ x :&= \frac{1}{2}(x + \bar{x}) = x_0 \\ Im\ x :&= \frac{1}{2}(x - \bar{x}) = x_1 i + x_2 j + x_3 k \ . \end{aligned}$$

Later we also use the notation $\vec{x}$ for $Im\ x$. We see that the square root of the nonnegative quantity $x\bar{x} = \bar{x}x$ is equal to $|x|$, the $I\!R^4$-norm of x, under the above mentioned isomorphism $I\!H \cong I\!R^4$, and moreover

$$x \cdot y := \frac{1}{4}\left(|x+y|^2 - |x-y|^2\right) = Re\ x\bar{y} = Re\ \bar{x}y \ .$$

$Sp(1) := \{a \in I\!H \ ; \ |a| = 1\}$ is a subgroup of the multiplicative group $I\!H^\times := I\!H \setminus \{0\}$ and it is isomorphic to $SU(2)$. By $I\!H \ni x \longmapsto axb^{-1} \in I\!H$ for $a, b \in Sp(1)$ we have a surjective homomorphism $Sp(1) \times Sp(1) \longrightarrow SO(4)$, whose kernel is $\{(1,1)\ ,\ (-1,-1)\} \cong \mathbb{Z}_2$, and hence $[Sp(1) \times Sp(1)]/\mathbb{Z}_2 \cong SO(4)$.
We consider the following two distinct $Sp(1) \times Sp(1)$ actions on $I\!R^4$-valued functions on $I\!R^4$: identifying $I\!R^4$ with $I\!H$, the first one is given by

$$A(x) \longrightarrow a\, A\left(a^{-1}(x-y)\, b\right) b^{-1} \qquad x, y \in I\!R^4\ ,\ (a,b) \in Sp(1) \times Sp(1) \qquad (i)$$

and A obeying this rules is called a <u>covariant 4-vector field</u>.
The second one is given by

$$A(x) \longrightarrow b\, A\left(a^{-1}(x-y)\, b\right) b^{-1} \qquad x, y \in I\!R^4\ ,\ (a,b) \in Sp(1) \times Sp(1) \qquad (ii)$$

and A obeying this rule is called a <u>covariant scalar 3-vector field</u>.
We define a bilinear form by

$$\langle \xi, A\rangle := \int_{\mathbb{R}^4} \xi(x)\cdot A(x)\, dx = Re \int_{\mathbb{R}^4} \xi(x)\cdot \overline{A(x)}\, dx \qquad \xi \in {C^\infty}_0(\mathbb{R}^4)$$

and extend this as the distributional pairing in the natural way. Note that $\langle\cdot,\cdot\rangle$ is invariant under $Sp(1)\times Sp(1)$ actions (i) and (ii).
Let

$$\partial := \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3} \text{ and } \bar{\partial} := \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\,,$$

then $\partial\bar{\partial} = \bar{\partial}\partial = \Delta$, the Laplacian in $\mathbb{R}^4$. Consider two variables $x, x' \in \mathbb{R}^4$ related by $x' = a^{-1}xb$ for some $(a,b) \in Sp(1)\times Sp(1)$ and define ∂' and $\overline{\partial'}$ in the same way as ∂ and $\bar{\partial}$. Then it is easily seen that $\overline{\partial'} = a^{-1}\bar{\partial}b$ and $\partial' = b^{-1}\partial a$. Therefore, if A is a covariant 4-vector field, then $F = \partial A$ is a covariant scalar 3-vector field. This is well understood, if we introduce a 1-form $\sigma := \sum_{i=0}^{3} A_i\, dx_i$, the orientation adapted to $\{1, i, j, k\}$ and the associated Hodge duals. In fact, identifying anti-self dual 2-forms with 3-vector fields, we have $({}^*d^*\sigma\ ,\ d\sigma - {}^*\, d\sigma) = (F_0\ ,\ \vec{F})$.
We note that the equation $\partial A = F$ is not covariant under reflections, since $\vec{F}$ corresponds to an anti-self dual 2-form.
We denote by g Green′ s function to $-\Delta$, i.e., $g(x) = \dfrac{1}{2\pi^2|x|^2}$ and set

$$S(x) := -\bar{\partial}\, g(x) = \frac{x}{\pi^2|x|^4} \quad , \quad \bar{S}(x) := \overline{S(x)}\,,$$

then we see that

$$\partial S(x) = -\partial\bar{\partial}\, g(x) = -\Delta\, g(x) = \delta(x) \quad , \quad \bar{\partial}\bar{S}(x) = \delta(x)\,,$$

where δ is the Dirac distribution. In order to give a precise meaning to the inverse of ∂ (resp $\bar{\partial}$) we introduce the following space

$$\mathcal{I} := \left\{ \varphi \in C^\infty(\mathbb{R}^4, \mathbb{H})\,;\ \lim_{|x|\to+\infty} \varphi(x) = 0\ ,\ \bar{\partial}\varphi \in \mathcal{S} \right\}$$

($\mathcal{S} \equiv \mathcal{S}(\mathbb{R}^4, \mathbb{H})$ is the Schwartz test space of rapidly decreasing test functions).
It is easily seen that $\mathcal{I} \ni \varphi \longrightarrow \bar{\partial}\varphi \in \mathcal{S}$ is bijective and the inverse map is given by $\mathcal{S} \ni \xi \longrightarrow \bar{S} * \xi \in \mathcal{I}$, where

$$\bar{S} * \xi(x) = \int_{\mathbb{R}^4} \bar{S}(x-y)\, \xi(y)\, dy\ .$$

Using this isomorphism we introduce a locally convex topology on $\mathcal{I}$. Note that the injection $\iota : \mathcal{S} \hookrightarrow \mathcal{I}$ is not dense and hence $\iota^* : \mathcal{I}' \longrightarrow \mathcal{S}'$ is not injective, since $\{\bar{\partial}\varphi\ ;\ \varphi \in \mathcal{S}\}$ is not dense in $\mathcal{S}$.
We have the

Theorem 1 : Let A be a covariant 4-vector field and F be a covariant scalar 3-vector field. Then the elliptic 1-order partial differential equation

$$\partial A(x) = F(x)$$

is covariant. If F belongs to $\mathcal{I}'$, $\partial A = F$ has a unique solution in $\mathcal{S}'$ given by $A = S * F$.

We interpret A as a classical electromagnetic Euclidean potential (in the Feynman gauge) and $E_i := \partial_0 A_i - \partial_i A_0$, $i = 1,2,3$, resp. $B_i = \partial_j A_k - \partial_k A_j$ (i,j,k) cyclic permutation of $(1,2,3)$ as electro resp. magnetic fields. $\vec{F}$ corresponds to $E_i - B_i$.

3. Random fields as solutions of a quaternionic partial differential equation with random source

We shall now consider the equation $\partial A = F$ in sect. 2, in the case where F is a generalized random field over $\mathbb{R}^4$ with values in $\mathbb{H}$. We assume that $\{F(x)\}$ and $\{bF(a^{-1}(x-y)b)b^{-1}\}$ have the same finite dimensional distributions for all $((a,b),y) \in Sp(1) \times Sp(1) \times \mathbb{R}^4$ and we call such F an invariant scalar 3-vector (generalized) random field. From the result in sect. 2 we see that the $\mathbb{H}$-valued (generalized) random field A related to F by the equation $\partial A = F$ is invariant, in the sense of law, under proper Euclidean transformations. We shall call such A an invariant 4-vector (generalized) random field (or also, for short, as in the title, a covariant random field). We have:

Theorem 2 : If F is an invariant scalar 3-vector generalized random field realized as a $\mathcal{I}'$-valued random variable, then $\partial A = F$ has a unique solution $A = S * F$ realized as an $\mathcal{S}'$-valued random variable. A is an invariant 4-vector random field. ■

In what follows we further assume that F is independent at every point, i.e., if we restrict its characteristic functional to $\mathcal{S}$, then taking translation invariance into account we have

$$C_F(\varphi) := E\left[e^{\sqrt{-1}\langle \varphi, F\rangle}\right] = \exp\left(-\int_{\mathbb{R}^4} \psi(\varphi(x))\,dx\right) \qquad \varphi \in \mathcal{S}$$

with ψ a continuous negative definite function on $\mathbb{H}$. Because of its $Sp(1)$ adjoint invariance, $\psi(b\lambda b^{-1}) = \psi(\lambda)$, ψ has the following Lévy-Khinchine representation :

$$\begin{aligned}\psi(\lambda) = &-\sqrt{-1}\,\lambda_0\beta + \frac{\sigma_0}{2}\lambda_0^2 + \frac{\sigma}{2}|\vec{\lambda}|^2 \\ &+ \int \left(1 + \sqrt{-1}\,\lambda\cdot\alpha\,\chi_{(0,1)}(|\alpha|) - e^{\sqrt{-1}\,\lambda\cdot\alpha}\right)\nu(d\alpha), \quad \lambda \in \mathbb{H}\end{aligned}$$

with $Sp(1)$ adjoint invariant Lévy measure ν $\left(\nu(b\,d\alpha\,b^{-1}) = \nu(d\alpha)\right)$ and $\beta \in \mathbb{R}$, $\sigma_0, \sigma \geq 0$. We call ψ resp. $(\beta, \sigma_0, \sigma, \nu)$ the Lévy characteristics of F. If it is possible to extend the domain of $C_F(\cdot)$ to $\mathcal{I}$, then F is realizable as a $\mathcal{I}'$-valued random variable. To this end we assume that $\psi(\lambda) = O\left(|\lambda|^{\frac{4}{3}+\varepsilon}\right)$ as $\lambda \to 0$ for some $\varepsilon > 0$. Indeed under this assumption the characteristic function $C_F(\cdot)$ defined on $\mathcal{I}$ is uniquely determined by $\exp\left(-\int_{\mathbb{R}^4} \psi(\varphi(x))\,dx\right)$, $\varphi \in \mathcal{S}$, and consequently the $\mathcal{I}'$-valued random variable F is uniquely characterized by the $\mathcal{S}'$-valued random variable $\iota^* \circ F$. For we see from Sobolevś inequality that

$$\|\bar{S} * \xi(x)\|_{L^p} \leq \left\{\int \left| \int \frac{|\xi(y)|}{\pi^2 |x-y|^3}\,dy \right|^p dx \right\}^{\frac{1}{p}} \leq C_p \|\xi\|_{L^q} \,, \quad p > \frac{4}{3}\,,\ \frac{1}{q} = \frac{1}{p} + \frac{1}{4}\,.$$

Theorem 3 : Let F be a translation invariant $\mathbb{H}$-valued generalized random field over $\mathbb{R}^4$ independent at every point with Lévy characteristic ψ. Then F has the properties as in Theorem 2 and A solving $\partial A = F$ has a distribution given by :

$$C_A(\xi) := E\left[e^{\sqrt{-1}\langle \xi, A\rangle}\right] = \exp\left(-\int_{\mathbb{R}^4} \psi(-\bar{S} * \xi(x))\,dx\right) \,,\ \xi \in \mathcal{S}\,.$$

In particular A is an invariant 4-vector generalized random field. If the Lévy measure ν has the p-th moment, i.e. $\int |\alpha|^p \nu(dp) < \infty$ for $p = 2, 3, \ldots$, then A also has the p-th moment

$$E\left[\langle \xi_1, A\rangle \ldots \langle \xi_p, A\rangle\right] \quad, \xi_1, \ldots, \xi_p \in \mathcal{S}\,,$$

as a continuous linear functional on $\mathcal{S}^{\otimes p}$.

We call such F with the properties in Theorem 3 an <u>invariant scalar 3-vector generalized random field of the infinitely divisible type</u>.

Concerning reflection invariance of A, the situation is utterly changed according to whether F is Gaussian distributed or not. By a reflection we mean the following $\mathbb{Z}_2$-action on covariant 4-vector fields :

$$\rho : \quad A(x) \longrightarrow -\overline{A(-\bar{x})} \quad , \quad x \in \mathbb{R}^4\,.$$

If A is ρ-invariant as well, then A is invariant under full Euclidean transformations. Before going into general cases, we first note that A is invariant under the reflection ρ when $\psi(\lambda)$ depends only on $\lambda_0 = Re\ \lambda$, i.e., $\sigma = 0$ and ν is supported by $\mathbb{R}\backslash\{0\} = \{\alpha \in \mathbb{H}^\times\,;\, Im\ \alpha = 0\}$. Indeed since the operators $g * \cdot$ and div $\cdot$ commute with ρ, we have

$$\begin{aligned} -Re\ \bar{S} * \rho\xi(x) &= Re\ \partial g * \rho\xi(x) = g * Re\ \partial\left(\rho\xi(x)\right) \\ &= g * \text{ div } \rho\xi(x) = -Re\ \bar{S} * \xi(-\bar{x}) \end{aligned}$$

for $x \in \mathbb{R}^4$, $\xi \in \mathcal{S}$, and therefore

$$\begin{aligned} C_A(\rho\xi) &= \exp\left\{-\int \psi\left(-\bar{S} * \rho\xi(x)\right)dx\right\} \\ &= \exp\left\{-\int \psi\left(-\bar{S} * \xi(-\bar{x})\right)dx\right\} = C_A(\xi) \end{aligned}$$

We shall now discuss the case of a pure Poisson source, i.e., $\sigma_0 = \sigma = 0$ and $\nu \neq 0$. We assume that ν has a compact support in $\mathbb{H}^\times$ and $\beta = 0$ for simplicity. Let $\{(\alpha_i, x_i)\}_{i=1}^{\infty}$ be the Poisson point process on $\mathbb{H}^\times \times \mathbb{R}^4$ with Lévy measure $\nu(d\alpha)\otimes dx$, then F is realizable as an $\mathbb{H}$-valued random measure over $\mathbb{R}^4$ by using $\{(\alpha_i, x_i)\}_{i=1}^{\infty}$:

$$F(x) = \sum_{i=1}^{\infty} \alpha_i\, \delta_{\{x_i\}}(x)$$

and therefore A solving $\partial A = F$ one has the following representation :

$$A(x) = S * F(x) = \sum_{i=1}^{\infty} \frac{x - x_i}{\pi^2 |x - x_i|^4}\, \alpha_i \quad .$$

We now see what happens if we perform the reflection ρ. Because of the invariance of the Lévy measure $\nu(d\alpha)\otimes dx$ the law of $\{(\bar{\alpha}_i, -\bar{x}_i)\}_{i=1}^{\infty}$ is equal to that of $\{(\alpha_i, x_i)\}_{i=1}^{\infty}$, so that we have

$$\rho A(x) \equiv -\overline{A(-\bar{x})} = -\sum_{i=1}^{\infty} \bar{\alpha}_i\, \frac{\overline{(-\bar{x} - x_i)}}{\pi^2 |-\bar{x} - x_i|^4} = \sum_{i=1}^{\infty} \bar{\alpha}_i\, \frac{x - (-\bar{x}_i)}{\pi^2 |x - (-\bar{x}_i)|^4} \overset{\mathrm{d}}{=} \sum_{i=1}^{\infty} \alpha_i\, \frac{x - x_i}{\pi^2 |x - x_i|^4}\,,$$

where $\overset{\mathrm{d}}{=}$ stands for the law equivalence. Suppose that $\rho A \overset{\mathrm{d}}{=} A$, then $\partial\rho A \overset{\mathrm{d}}{=} \partial A = F$. However this does not hold unless $Im\ \alpha_i = 0\ \ \forall i$ a.s., since it follows from the definition of S that

$$\begin{aligned}\partial\rho A(x) &\overset{\mathrm{d}}{=} \partial\left(\sum_{i=1}^{\infty} \alpha_i\, \frac{x - x_i}{\pi^2 |x - x_i|^4}\right) \\ &= \partial\sum_{i=1}^{\infty}\left(\alpha_i \frac{x - x_i}{\pi^2 |x - x_i|^4} - \frac{x - x_i}{\pi^2 |x - x_i|^4}\, \alpha_i\right) + \sum_{i=1}^{\infty} \alpha_i \delta_{\{x_i\}}(x)\end{aligned}$$

and hence there are subsets of $\mathcal{S}'$ which have zero measure for $\partial(\rho A)$ and measure 1 for F. Hence A in the pure Poisson case is not ρ-invariant, unless $Im\ \alpha_i = 0\ \ \forall i$ a.s. .
Let us look as a contrast to the case F Gaussian, i.e. $\nu = 0$, and see how one recovers the reflection invariance. We have

$$\begin{aligned}\int_{\mathbb{R}^4} |\bar{S} * \xi(x)|^2\, dx &= \langle \partial g * \xi, \partial g * \xi\rangle = -\langle g * \xi, \bar{\partial}\partial g * \xi\rangle \\ &= \langle g * \xi, \xi\rangle \\ &= \int g(x - y)\, \xi(x)\cdot\xi(y)\, dx dy\end{aligned}$$

and thus we have

$$\int_{\mathbb{R}^4} |\bar{S} * \rho\xi(x)|^2 dx = \int_{\mathbb{R}^4} |\bar{S} * \xi(x)|^2 dx \quad , \ \xi \in \mathcal{S}\,.$$

Combining this with the fact $Re\ \bar{S} * \rho\xi(x) = Re\ \bar{S} * \xi(-\bar{x})$ we get

$$\int_{\mathbb{R}^4} |Im\ \bar{S} * \rho\xi(x)|^2 dx = \int_{\mathbb{R}^4} |Im\ \bar{S} * \xi(x)|^2 dx \quad , \ \xi \in \mathcal{S}\,.$$

This implies $C_A(\rho\xi) = C_A(\xi),\quad \xi \in \mathcal{S}$, as far as ν vanishes. This fact is a striking contrast to the case of a pure Poisson source. We summarize these results in the following

Theorem 4 : Let F be an invariant scalar 3-vector generalized random field of the infinitely divisible type. Then A solving $\partial A = F$ is ρ-invariant iff the Lévy measure ν associated with F is supported by $I\!R\backslash\{0\}$.

From the point of view of the Euclidean field theory, A for $\beta = 0$, $\sigma_0 = \sigma = 1$, $\nu = 0$ corresponds to the free electromagnetic potential field in the Feynman gauge and A for $\beta = 0$, $\sigma_0 = 0$, $\sigma = 1$, $\nu = 0$ corresponds to that in the Coulomb gauge.

4. Some further properties of the constructed random fields

It is well known that the passage from Euclidean fields to relativistic fields is possible in general situations where the Osterwalder-Schrader (reflection) positivity (see e.g. [GJ]) holds under time reversal ρ. In the following we shall first see how we can construct from A an Osterwalder-Schrader ($\equiv$ O.S.) positive field by taking into account the gauge invariance of the underlying equation $\partial A = F$ (if A is a solution, then $A + \bar{\partial}\chi$ with χ harmonic, i.e. $\Delta\chi = 0$, also solves $\partial A = F$) in the Gaussian case, $\nu = 0$ in the notation of section 3. Let $\mathcal{S}_{df}$ be the subspace of $\mathcal{S}$ consisting of all ξ with $Re\ \partial\xi = \operatorname{div}\xi = 0$. We note that $\mathcal{S}_{df}$ is a Euclidean invariant test function space and $\{\langle\xi, A\rangle\,;\ \xi \in \mathcal{S}_{df}\}$ is a family of gauge invariant random variables, which we shall call the Euclidean transversal field with the gauge potential A. Since $Re\ \bar{S} * \xi = Re\ g * \partial\xi = 0$ for $\xi \in \mathcal{S}_{df}$, the covariance functional of the transversal field is equal to

$$\sigma \int |\bar{S} * \xi(x)|^2 dx = \sigma \int g(x-y)\xi(x) \cdot \xi(y)\, dx dy \equiv \sigma(\xi, \xi)\ ,$$

whatever the parameter σ_0 is. Suppose that $\xi \in \mathcal{S}_{df}$ has its support in $I\!R_+ \times I\!R^3$. By using the partial Fourier transformation, we have

$$(\xi, \rho\xi) = \int_{I\!R^3} \int_0^\infty e^{-|\vec{k}|t} \widehat{\xi(t,\cdot)}(\vec{k}) dt \cdot \int_{-\infty}^0 e^{|\vec{k}|s} \widehat{\rho\xi(s,\cdot)}(-\vec{k}) ds \frac{d\vec{k}}{2|\vec{k}|}\ .$$

Next we apply the integration by parts formula to the dt-integral, then, since $\operatorname{div}\xi = 0$, it follows that

$$\int_0^\infty e^{-|\vec{k}|t} \widehat{\xi_0(t,\cdot)}(\vec{k}) dt = -\sqrt{-1} \int_0^\infty e^{-|\vec{k}|t} |\vec{k}|^{-1} \vec{k} \cdot \widehat{\vec{\xi}(t,\cdot)}(\vec{k}) dt$$

and similary

$$\int_{-\infty}^0 e^{|\vec{k}|s} \widehat{(\rho\xi)_0(s,\cdot)}(-\vec{k}) ds = -\sqrt{-1} \int_0^\infty e^{-|\vec{k}|s} |\vec{k}|^{-1} \vec{k} \cdot \widehat{\vec{\xi}(s,\cdot)}(-\vec{k}) ds\ .$$

Hence we obtain the positivity

$$(\xi, \rho\xi) = \int_{I\!R^3} \left\{ \left| \int_0^\infty e^{-|\vec{k}|t} \widehat{\vec{\xi}(t,\cdot)}(\vec{k}) dt \right|^2 - \left| \int_0^\infty e^{-|\vec{k}|t} |\vec{k}|^{-1} \vec{k} \cdot \widehat{\vec{\xi}(t,\cdot)}(\vec{k}) dt \right|^2 \right\} \frac{d\vec{k}}{2|\vec{k}|} \geq 0\ .$$

This implies the O.S.-positivity of the transversal field : $\forall z_i \in \mathbb{C}$, $i = 1, \ldots, n$

$$\sum_{i,j=1}^{n} z_i \bar{z}_j E\left[e^{\sqrt{-1}\langle \xi_i, A\rangle} e^{-\sqrt{-1}\langle \xi_j, \rho A\rangle}\right] \geq 0 \quad , \quad \xi_j \in \mathcal{S}_{df}\,,\ \mathrm{supp}\,[\xi_j] \subset \mathbb{R}_+ \times \mathbb{R}^3\,.$$

As usual as we obtain the physical Hilbert space $\mathcal{H}$ spanned by $\left\{e^{\sqrt{-1}\langle \xi, A\rangle}\,;\ \xi \in \mathcal{S}_{df}\,,\ \mathrm{supp}\,[\xi] \subset \mathbb{R}_+ \times \mathbb{R}^3\right\}$ with the inner product naturally introduced by the O.S.-positive condition and the symmetric contraction semigroup acting on $\mathcal{H}$, which is determined by

$$E\left[e^{\sqrt{-1}\langle \xi, \tau_{-t} A\rangle} e^{-\sqrt{-1}\langle \xi, \rho A\rangle}\right]\,,$$

where $\{\tau_t\}$ is the shift along the x_0-axis : $\tau_t A(x) = A(x_0 - t, \vec{x})$.
The negative H of the generator of the above semigroup is the physical energy operator. Using this operator we can construct relativistic potential fields as operator-valued distributions (with test functions in $\mathcal{S}_{df}$). These fields can be identified with the electromagnetic free potential fields.

Remark 1 : It is possible to show that above Euclidean transversal electromagnetic potential fields $\{\langle \xi, A\rangle\}$ have the Markov property with respect to arbitrary open subsets of $\mathbb{R}^4$, in the sense that, extending $\langle \xi, A\rangle$ to all $\xi \in S'$ with $(\xi,\xi) < \infty$ and denoting by $\sum_\Lambda \equiv \sigma\left(\langle \xi, A\rangle; \xi \in \mathcal{S}', (\xi,\xi) < \infty, \mathrm{div}\,\xi = 0, \mathrm{supp}[\xi] \subset \Lambda\right) \vee \mathcal{N}$ the σ-algebra generated by the fields in the Borel region Λ, and the zero measure sets $\mathcal{N}$ (with respect to the measure associated with A), then for any open $D \subset \mathbb{R}^4$, $\sum_{\bar{D}}$ is conditionally independent of $\sum_{D^c}$ given $\sum_{\partial D}$, where $\bar{D}$ is the closure of D, $D^c \equiv \mathbb{R}^4 - D$, ∂D is the boundary of D.
This is proven using the Fock space or Wiener-chaos decomposition of $\mathcal{H}$. For some related discussions see e.g. [Lö] and references therein.
It is also known that the Markov property holds even for $\{\langle \xi, A\rangle\}$, with ξ not restricted to be in $\mathcal{S}_{df}$ ("non transversal fields "), provided one takes the Feynman gauge $\beta = 0$, $\sigma_0 = \sigma = 1$, $\nu = 0$. In fact this holds also for any Gaussian field defined by

$$E\left(e^{\sqrt{-1}\langle \xi, A\rangle}\right) = e^{-\frac{1}{2}(\xi,\xi)_c}\,,\ \xi \in \mathcal{S}\,,\ 0 \leq c < 1\,,$$

with

$$(\xi,\xi)_c \equiv \int_{\mathbb{R}^4} \left[|\hat{\xi}(k)|^2 - c\frac{|k \cdot \hat{\xi}(k)|^2}{|k|^2}\right] |k|^{-2}\, dk$$

(with $\hat{\xi}$ the Fourier transform of ξ and c the ratio σ_o/σ in the notation of section 3). $c = 1$ corresponds to Coulomb gauge, where one only has the Markov property when restricting ξ to be in $\mathcal{S}_{df}$.
That even in the non purely Gaussian case $\nu \neq 0$ one should still have Markovian properties is suggested by the fact that ∂ is a first order partial differential operator. However this is not yet fully mathematically settled. One difficulty is due to the bad spectral properties of ∂^{-1} ($\partial A = F$ being a "zero mass" equation).
In related positive mass equations it is possible to prove the 0-Markov property in the sense of Kusuoka [K], see [I]. Let us also remark that Surgailis has discussed related problems in the case where $\mathbb{R}^4$ is replaced by $\mathbb{R}^2$, see [Su2].

Remark 2 : As remarked in [AHK3], it is possible to associate to the quaternionic valued field A in the general case, a component wise 3-form $\omega = (\omega_\mu\, ,\, \mu = 0,1,2,3)$.
In fact let $a_0 \equiv A\, , a_1 \equiv -iA\, , a_2 = -jA\, , a_3 = -kA$. Then $a \equiv \sum_\mu a_\mu dx_\mu$ is also a quaternionic valued 1-form. We have $\sum_{\mu=0}^{3} \partial_\mu\, a_\mu = \partial A = F$.
Let ω be the Hodge dual of a, then $d\omega = F$, in the sense that $d\omega_\mu = F_\mu\, ,\, \mu = 0,1,2,3$, where ω_μ is looked upon as a 3-form over $I\!R^4$ and F_μ is looked upon as a 4-form over $I\!R^4$. $d\omega = F$ can be written as $\omega(\partial B) = F(B)$, for any measurable $B \subset I\!R^4$, where by definition $\omega(\partial B) = \int_B d\omega$ (in analogy with the corresponding formulae which hold when ω and B are smooth). ω is then a Markov Euclidean invariant cosurface in the sense of [AHKH1], a stochastic integral in the sense of [AHKH2]. In [AHK3] the relation $d\omega = F$ is extended to the case where the ω_μ are 3-forms with values in a Lie algebra g containing that of $U(2)$. ω is then a g-valued Markov cosurface.

Remark 3 : All considerations of this section, with invariance properties suitable reinterpreted, can also be made for the case where the region $I\!R^4$ on which the fields are defined is replaced by an open domain B with boundary ∂B. Let in fact $(F\, ,\, P_B)$ be the generalized random field defined by
$E_{P_B}(e^{\sqrt{-1}\langle\xi,F\rangle}) = \exp\left(-\int_B \psi\,(\xi(x))\, dx\right)$, with ψ as in Theor. 2. Let ∂_B be defined by closure in $L^2(dx)$ from ∂ on $C^\infty{}_0(B\, ;\, I\!R^4)$. Let S_B be the fundamental solution to ∂_B. S_B has the same local behavior as S. The analogue of Theor. 2 holds then with S replaced by S_B, yielding a solution of the equation $\partial_B\, A = F$. A is rotation invariant if B is rotation invariant.
Let μ_B be the probability measure giving the distribution of the field A. (A, μ_B) is a locally Markov field in the sense of [AHK7], [Ne]. (A, μ_B) converges weakly as $B \uparrow I\!R^4$ to (A, μ), with (A, μ) given by Theor. 2.

Remark 4 : It is possible to discuss a "lattice approximation" of the field A constructed in Sect. 3.
Let $\delta > 0\, ,\quad Z\!\!Z^4_\delta \equiv \{\delta n\, ,\, n \in Z\!\!Z^4\}\, ,\quad \Lambda_\delta \equiv \Lambda \cap Z\!\!Z^4_\delta$ for any bounded subset Λ of $I\!R^4$.
Let $P_{\Lambda_\delta}(\cdot)$ be the probability measure on $I\!H^{\Lambda_\delta}$ given by

$$dP_{\Lambda_\delta}(F) \equiv (Z_\delta)^{-|\Lambda_\delta|} \exp\left(-W_\delta(F)\right) \prod_{x\in\Lambda_\delta} dF(x)\ ,$$

with

$$W_\delta(F) \equiv \sum_{x\in\Lambda_\delta} \delta^4\, f_\delta\left(F_0(x)\, ,\, |\vec{F}(x)|\right)\ ,$$

with f_δ a positive function on $I\!R^4$ s.t. $f_\delta(\gamma) = f_\delta(|\gamma_0|, |\vec{\gamma}|)\ \forall \gamma \in I\!R^4$,
$Z_\delta \equiv \int_{I\!R^4} e^{-\delta^4 f_\delta(\gamma)} d\gamma < \infty$ and

$$\lim_{\delta\downarrow 0} \delta^{-4} \left\{\ln\left[\int_{I\!R^4} e^{-\sqrt{-1}\delta^4 \lambda\gamma}\, e^{-\delta^4 f_\delta(\gamma)}\, d\gamma / Z_\delta\right]\right\}$$

exists for all $\lambda \in I\!R^4$ and is a Lévy-Khinchine function having the same properties as the function ψ entering Theor. 2.

Remark 5 : An example is given by the convolution semigroup $\{P_t\}$, $t \geq 0$ of probability densities associated with ψ. Namely we choose $f_\delta(\gamma) \equiv -\delta^{-4} \log\left\{P_{\delta^4}(\delta^4\gamma)\delta^{16}\right\}$, then $\int e^{\sqrt{-1}\lambda\cdot\gamma}\, e^{-\delta^4 f_\delta(\gamma)}\, d\gamma = e^{-\delta^4\psi(\delta^{-4}\lambda)}$.
Let us define

$$d\mu_\delta(A) \equiv K_\delta e^{-W_\delta(\partial_\delta A)} \prod_{x\in\Lambda_\delta} dA(x)$$

with $A(x) : \Lambda_\delta \to I\!H$, K_δ a constant making $d\mu_\delta$ into a probability measure and ∂_δ a discrete version of ∂. It is possible to show that (A, μ_δ) converges weakly as $\delta \downarrow 0$ to the continuum limit (A, μ) described in Sect. 3.
Finally we remark that the field (A, μ) constructed from a Lévy characteristic $\psi = \psi_p$ of Poisson type can approximate the free electromagnetic Euclidean field arbitrary well.
In fact let us choose the Lévy characteristic ν_r of A to be in s.t., e.g. for r_0 , $r > 0$:

$$\nu_r\left(|\alpha_0|\,,\,|\vec{\alpha}|\right) = 3\,\left[\delta_{r_0}(\alpha_0) + \delta_{-r_0}(\alpha_0)\right]\,\rho(|\vec{\alpha}|)/(8\pi r^4)\,,$$

with ρ the restriction of Lebesgue measure to $|\vec{\alpha}| = r$.
Then

$$-\psi(\lambda) = 3\left[\cos\lambda_0 r_0 \int_{|\vec{\alpha}|=r} e^{\sqrt{-1}|\vec{\lambda}| r\cos\theta} r^2(\sin\theta)\,d\theta d\varphi - 4\pi r^2\right] / (4\pi r^4) \underset{r_0, r\downarrow 0}{\longrightarrow} -\frac{\lambda^2}{2}\,.$$

Calling μ_r the probability measure given by ψ, we have that (μ_r, A) converges in this case for $r \to 0$ weakly to the free Euclidean electromagnetic potential field. This can be used to study interactions with matter, see [AIW].
Exploiting the support properties of A one can study local perturbations of the field (μ_p, A).
Let v be a $I\!R$-valued Borel measurable function on $I\!R^4$ s.t., for $|\lambda| \to \infty$,

$$v(\lambda) = v\left(|\lambda|\right) = O\left(|\lambda|^\alpha\right)\,,\; \alpha < \frac{4}{3}\,,$$

v bounded on compacts.
Let μ_B be as in Remark 3, with $\psi = \psi_p$. Then

$$\int_B v\left(|A(x)|\right)\, dx \in L^1(\mu_B)\,,$$

for any $B \subset I\!R^4$ bounded measurable. Thus if in addition v is bounded from below, then

$$d\mu_B{}^v(A) \equiv Z_B^{-1}\, e^{-\int_B v(|A(x)|)\,dx}\, d\mu_B(A)\,,$$

with Z_B the normalizing constant, is a well defined probability measure. $(A, \mu_B{}^v)$ is locally Markov.
For v suitable, e.g. $v \geq 0$ one gets weak limits points as $B \nearrow I\!R^4$.
In this way we can create new locally Markov random fields, covariant under the proper Euclidean group.

Footnotes

[1]) This is similar to the association of $\mathbb{R}^2$ with complex numbers. Our use of quaternions is different from one done in a large literature involving quaternionic (and octonionic) Hilbert spaces for the study of elementary particle models (see e.g. [A] and references therein). In fact our use is more similar to the one done in relation with classical electromagnetic fields, starting with Maxwell. Our approach has been partly announced in [AHK2–4]. On the basis of this announcement Osipov [O] has given an extension, renouncing of course associativity, to 8-space-time dimensions by using octonions.

[2]) Euclidean (generalized) random fields as solutions of stochastic differential equations have been discussed before in [AHK2,3,4]. For lower space-time dimension or Gaussian fields (free fields) see [AHK1,5], [Ca], [GuL], [GuR], [Ha], [JLM], [Rö], [Su2] and references therein.

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