On quantum properties of kinks and solitons

J. Mateos Guilarte[1]

[1] *Departamento de Física Fundamental and IUFFyM. Universidad de Salamanca, SPAIN.*

28/04/2008

Abstract

The goal of my talk is to describe recent developments in the topic of quantum corrections to the energy of classical solitons. The basics in this matter were solidly founded in the mid seventies of the XX century by Faddeev and Korepin in Sankt-Petersburg and Dashen-Hasslacher and Neveu in Princeton. Nevertheless, a new conceptual approach based in heat kernel/zeta function regularization methods proved to be fruitful in 2002 both in the supersymmetric (Bordag, Goldhaber, van Nieuwenhuizen, Vassilevich, Leipzig/Stony Brook) and bosonic (Izquierdo, Fuertes, Guilarte, Leon, Oviedo/Salamanca) cases. This year Vassilevich (Sao Paulo) extended the method to compute the one-loop mass shift to non-commutative kinks, both purely bosonic and supersymmetric. I will also comment on a subtle connection between this subject and supersymmetric quantum mechanics.

Contents

1 Perturbation theory of the (1+1)-dimensional $\lambda(\phi)^4_2$ model

1.1 The $\lambda(\phi^4)$-model on a line

- The action, the bosonic field, metric convention, dimensions, and the field equations

$$S = \int dy^2 \left\{ \frac{1}{2}\frac{\partial\chi}{\partial y^\mu}\frac{\partial\chi}{\partial y_\mu} - \frac{\lambda}{4}(\chi^2(y_0,y) - \frac{m^2}{\lambda})^2 \right\}$$

$$\chi(y_0,y): \quad \mathbb{R}^{1,1} \longrightarrow \quad \mathbb{R}$$

$$g_{\mu\nu} = \mathrm{diag}(1,-1) \qquad , \qquad [\hbar] = ML \quad , \quad [\chi] = M^{\frac{1}{2}}L^{\frac{1}{2}} \quad , \quad [\lambda] = M^{-1}L^{-3}$$

In terms of non-dimensional space-time coordinates and fields

$$y^\mu \to y^\mu = \frac{\sqrt{2}}{m}\cdot x^\mu \qquad ; \qquad \chi(y^\mu) \to \psi(y^\mu) = \frac{m}{\sqrt{\lambda}}\cdot\phi(x^\mu) \qquad ,$$

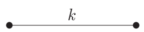
$$S = \frac{m^2}{\lambda}\int dx^2 \left\{ \frac{1}{2}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x_\mu} - \frac{1}{2}(\phi^2(x_0,x) - 1)^2 \right\}$$

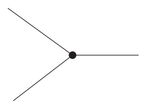
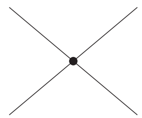
$$\frac{\partial^2\phi}{\partial x_0^2}(x_0,x) - \frac{\partial^2\phi}{\partial x^2}(x_0,x) = 2\phi(x_0,x)(1 - \phi^2(x_0,x)) \qquad .$$

1.2 Particle spectrum and perturbation theory

• Shift of the field from the homogeneous solution $\phi(x^\mu) = 1 + H(x^\mu) \quad \Longrightarrow$ Higgs mechanism, spontaneous symmetry breaking of $\phi \;\to\; -\phi$ and Feynman rules

$$S = \frac{m^2}{\lambda} \int d^2x \left\{ \left[\frac{1}{2} \partial_\mu H \partial^\mu H - 2H^2(x^\mu) \right] - \left[2H^3(x^\mu) + \frac{1}{2} H^4(x^\mu) \right] \right\} \qquad ,$$

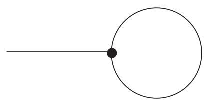
Particle	*Field*	*Propagator*	*Diagram*
Higgs	$H(x^\mu)$	$\frac{i\lambda\hbar}{m^2(k_0^2 - k^2 - 4 + i\varepsilon)}$	

Vertex	*Weight*	*Vertex*	*Weight*
	$-12i\frac{m^2}{\hbar\lambda}$		$-12i\frac{m^2}{\hbar\lambda}$

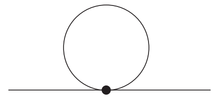
1.3 Mass renormalization in (1+1)-dimensional QFT

One-loop ultraviolet divergent graphs: Higgs tadpole and self-energy :

$$-6i \cdot I(4) = -6i \cdot \int \frac{d^2k}{(2\pi)^2} \cdot \frac{i}{(k_0^2 - k^2 - 4 + i\varepsilon)} =$$



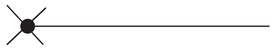
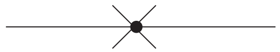
$$-6i \cdot I(4) = -6i \cdot \int \frac{dk}{4\pi} \cdot \frac{1}{\sqrt{k^2+4}} = -6i \cdot \frac{\sqrt{2}}{mL} \cdot \frac{1}{2} \sum_n \frac{1}{\sqrt{\frac{n^2}{R^2}+4}} =$$



Combinatorial factor: $\frac{1}{2}$

- Lagrangian density of counter-terms: $\mathcal{L}_{C.T.} = 3\hbar \left(\phi^2(x) - 1\right) \cdot I(4)$

Table 1: One-loop counter-terms

Diagram	*Weight*
	$6iI(4)$
	$6iI(4)$

1.4 Zero point vacuum energy

- General solution of the linearized field equations

$$\frac{\partial^2 \delta H}{\partial x_0^2}(x_0, x) - \frac{\partial^2 \delta H}{\partial x^2}(x_0, x) + 4\delta H(x_0, x) = 0$$

$$\delta H(x_0,x) = \frac{\sqrt{\lambda}}{m}\cdot\sqrt{\frac{\sqrt{2}\hbar}{mL}}\sum_k \frac{1}{\sqrt{2\omega(k)}}\left\{a(k)e^{-ik_0x_0+ikx} + a^*(k)e^{ik_0x_0-ikx}\right\} \qquad ,$$

$$k_0 = \omega(k) = \sqrt{k^2+4} \quad , \quad k_0^2 - k^2 - 4 = 0 \qquad , \qquad K_0 e^{ikx} = \omega^2(k)e^{ikx} \qquad , \qquad K_0 = -\frac{d^2}{dx^2} + 4$$

- Normalization interval, Periodic Boundary Conditions and spectral density of K_0

$$I = [-\frac{mL}{2\sqrt{2}}, \frac{mL}{2\sqrt{2}}] \qquad , \qquad k\frac{mL}{\sqrt{2}} = 2\pi n \quad , n \in \mathbb{Z} \qquad , \qquad \rho_{K_0}(k) = \frac{dn}{dk} = \frac{1}{2\pi}\frac{mL}{\sqrt{2}}$$

- Classical and quantum free Hamiltonian

$$\begin{aligned} H^{(2)} &= \frac{m^3}{\sqrt{2}\lambda}\int dx \left\{\frac{1}{2}\frac{\partial\delta H}{\partial x_0}\cdot\frac{\partial\delta H}{\partial x_0} + \frac{1}{2}\frac{\partial\delta H}{\partial x}\cdot\frac{\partial\delta H}{\partial x} + \delta H(x_0,x)\cdot\delta H(x_0,x)\right\} \\ &= \sum_k \hbar\frac{m}{2\sqrt{2}}\omega(k)\left(a^*(k)a(k) + a(k)a^*(k)\right) \qquad , \end{aligned}$$

$$\hat{H}_0^{(2)} = \sum_k \hbar\frac{m}{\sqrt{2}}\omega(k)\left(\hat{a}^\dagger(k)\hat{a}(k) + \frac{1}{2}\right) \qquad , \qquad [\hat{a}(k), \hat{a}^\dagger(q)] = \delta_{kq}$$

- Vacuum energy

$$\Delta E_0 = \frac{\hbar m}{2\sqrt{2}}\sum_k \omega(k) = \frac{\hbar m}{2\sqrt{2}}\mathrm{Tr}K_0^{\frac{1}{2}} \qquad .$$

2 $\lambda(\phi^4)_2$ kinks

2.1 Extended classical lumps

• Topology of the configuration space

$$\mathcal{C} = \{\phi(x) \in \text{Maps}(\mathbb{R},\mathbb{R}) / E[\phi] < +\infty\}$$

$\mathcal{C} = \mathcal{C}_{++} \sqcup \mathcal{C}_{--} \sqcup \mathcal{C}_{+-} \sqcup \mathcal{C}_{-+}$

• Finite energy and asymptotic conditions

$$E = \frac{m^3}{\sqrt{2}\lambda} \int dx \left\{ \frac{1}{2} \frac{d\phi}{dx} \cdot \frac{d\phi}{dx} + \frac{1}{2}(1-\phi^2)^2 \right\}$$

$$\lim_{x\to\pm\infty} \frac{d\phi}{dx} = 0 \qquad , \qquad \lim_{x\to\pm\infty} \phi(x) = \begin{cases} \phi_+ = +1 \\ \phi_- = -1 \end{cases}$$

• Bogomolny splitting and first-order equations

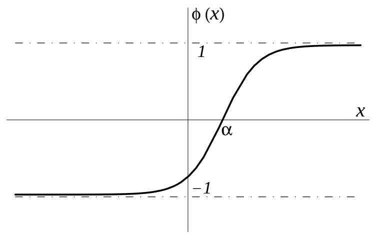
$$E \;=\; \frac{m^3}{\sqrt{2}\lambda} \int dx \, \frac{1}{2} \left(\frac{d\phi}{dx} \mp (1-\phi^2) \right)^2 \pm \frac{m^3}{\sqrt{2}\lambda} \cdot \left(\phi - \frac{\phi^3}{3} \right) \Big|_{\phi(-\infty)}^{\phi(\infty)}$$

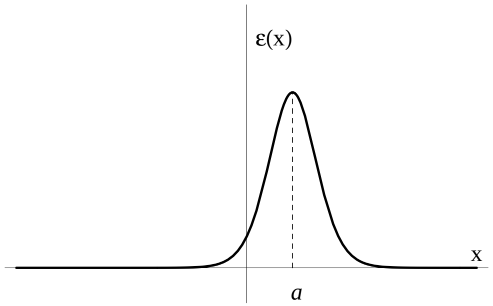
$$\frac{d\phi}{dx} = \pm(1-\phi^2)$$

• Finite energy static solutions

$$E(\phi_\pm) = 0 \qquad , \qquad \phi_\pm = \pm 1 \qquad , \qquad E(\phi_K) = \frac{4m^3}{3\sqrt{2}\lambda}$$

$$\phi_K(x) = \pm\tanh(x-a) \quad , \quad \phi_K(x_0, x) = \pm\tanh\Big(\frac{x-a-vt}{\sqrt{1-v^2}}\Big) \qquad , \qquad \varepsilon_K(x) = \frac{1}{\cosh^2(x-a)}$$





2.2 Small fluctuations around kinks: Pösch-Teller Hamiltonians

• Small kink deformations still solve the first-order ODE if:

$$\phi(x) = \phi_K(x) + \delta\phi(x) \qquad , \qquad \delta\phi(x) \in \text{Ker}\, D$$

$$D\delta\phi(x) = \left(-\frac{d}{dx} + 2\phi_K(x)\right)\delta\phi(x) = \left(-\frac{d}{dx} + 2\text{tanh}x\right)\delta\phi(x) = 0 \qquad .$$

• Hidden SUSY Quantum Mechanics: because

$$K^- = D^\dagger D = -\frac{d^2}{dx^2} + 4 - \frac{2}{\cosh^2 x} \qquad , \qquad K = DD^\dagger = -\frac{d^2}{dx^2} + 4 - \frac{6}{\cosh^2 x}$$

$$Q = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \qquad , \qquad Q^\dagger = \begin{pmatrix} 0 & 0 \\ D^\dagger & 0 \end{pmatrix} \qquad , \qquad H = \begin{pmatrix} K & 0 \\ 0 & K^- \end{pmatrix}$$

• Shift of the Higgs field from the kink configuration: $\phi(x^\mu) = \phi_K(x) + H(x^\mu)$

$$\begin{aligned} S = & - \frac{4m^3}{3\sqrt{2}\lambda} \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} dx_0 \\ & + \frac{m^2}{\lambda} \int d^2x \left\{ \left[\frac{1}{2}\partial_\mu H \partial^\mu H - (2 - \frac{3}{\cosh^2 x}) H^2(x^\mu) \right] - \left[2\mathrm{tanh} x H^3(x^\mu) + \frac{1}{2} H^4(x^\mu) \right] \right\} \qquad . \end{aligned}$$

2.3 Kink Casimir energy

• Spectral resolution of the K-operator with Periodic Boundary Conditions in the interval $I = [-\frac{mL}{2\sqrt{2}}, \frac{mL}{2\sqrt{2}}]$:

$$K f_\varepsilon(x) = \left[-\frac{d^2}{dx^2} + 4 - \frac{6}{\cosh^2 x} \right] f_\varepsilon(x) = \varepsilon^2 f_\varepsilon(x) \qquad , \qquad k \in \mathbb{R}$$

Eigenvalues	Eigenfunctions
$\varepsilon^2 = 0$	$f_0(x) = \frac{1}{\cosh^2 x}$
$\varepsilon_3^2 = 3$	$f_3(x) = \frac{\sinh x}{\cosh^2 x}$
$\varepsilon^2 = k^2 + 4$	$f_\varepsilon(x) = e^{ikx}(3\tanh^2 x - 1 - 3ik\tanh x - k^2)$

- Kink phase shifts and spectral density

$$k\frac{mL}{\sqrt{2}} + \delta(k) = 2\pi n \qquad , \qquad \delta(k) = -2\arctan\frac{3k}{2-k^2}$$

$$k - \frac{n}{R} = \frac{1}{\pi R} \cdot \arctan\frac{3k}{2-k^2} \quad , \quad n \in \mathbb{Z} \qquad , \qquad \rho_K(k) = \frac{1}{2\pi}(\frac{mL}{\sqrt{2}} + \frac{d\delta(k)}{dk})$$

- General solution of the linearized field equation around the kink background: $\phi(x_0, x) = \phi_K(x) + \delta H(x_0, x)$

$$\frac{\partial^2 \delta H}{\partial x_0^2}(x_0, x) - \frac{\partial^2 \delta H}{\partial x^2}(x_0, x) + \left(4 - \frac{6}{\cosh^2 x}\right)\delta H(x_0, x) = \left[\frac{\partial^2}{\partial x_0^2} + K\right]\delta H(x_0, x) = 0$$

$$\delta H'(x_0, x) \; = \; \frac{\sqrt{\lambda}}{m} \cdot \sqrt{\frac{\sqrt{2}\hbar}{mL}}(\frac{1}{\sqrt{2\sqrt{3}}}A_3 e^{-i\sqrt{3}x_0} + \frac{1}{\sqrt{2\sqrt{3}}}A_3^* e^{i\sqrt{3}x_0})f_3(x) +$$

$$+\ \frac{\sqrt{\lambda}}{m}\cdot\sqrt{\frac{\sqrt{2}\hbar}{mL}}\sum_k\frac{1}{\sqrt{2\varepsilon(k)}}\left(A(k)e^{-i\varepsilon x_0}f_\varepsilon(x)+A^*(k)e^{i\varepsilon x_0}f^*_\varepsilon(x)\right)$$

• Classical and quantum free Hamiltonian for kink fluctuations

$$\begin{aligned} H^{(2)} &= \frac{2m^3}{\sqrt{2}\lambda}\int dx\left[\frac{\partial\delta H}{\partial x_0}\frac{\partial\delta H}{\partial x_0}+\ \delta H(x_0,x)K\delta H(x_0,x)\right]= \\ &= \frac{\hbar m}{2\sqrt{2}}\left\{\sqrt{3}(A_3^*A_3+A_3A_3^*)+\sum_k\varepsilon(k)(A^*(k)A(k)+A(k)A^*(k))\right\} \end{aligned} \quad ,$$

$$[\hat{A}_3,\hat{A}_3^\dagger]=1 \qquad , \qquad [\hat{A}(k),\hat{A}^\dagger(q)]=\delta_{kq}$$

$$\hat{H}^{(2)}=\hbar\frac{m}{\sqrt{2}}\left(\sqrt{3}(\hat{A}_3^\dagger\hat{A}_3+\frac{1}{2})+\sum_k\varepsilon(k)(\hat{A}^\dagger(k)\hat{A}(k)+\frac{1}{2})\right)$$

• Zero point kink energy

$$\Delta E(\phi_K)=\frac{\hbar m}{2\sqrt{2}}(\sqrt{3}+\sum_k\varepsilon(k))=\frac{\hbar m}{2\sqrt{2}}\mathrm{Tr}K^{\frac{1}{2}}$$

• One-loop kink mass

1. Kink classical mass $E(\phi_K)=\frac{4m^3}{3\sqrt{2}\lambda}$.

2. Kink Casimir energy

$$\Delta M_K^C = \Delta E(\phi_K) - \Delta E_0 = \frac{\hbar m}{2\sqrt{2}}\left(\mathrm{Tr} K^{\frac{1}{2}} - \mathrm{Tr} K_0^{\frac{1}{2}}\right) \qquad .$$

3. Counter-term contribution to the kink mass

$$\Delta M_K^R = -3\frac{\hbar m}{\sqrt{2}} \cdot I(4) \cdot \int dx \, \left(\phi_K^2(x) - \phi_\pm^2\right) = 6\frac{\hbar m}{\sqrt{2}} \cdot I(4) \qquad .$$

$$\Delta M_K = \Delta M_K^C + \Delta M_K^R \qquad , \qquad E_S(\phi_K) = E(\phi_K) + \Delta M_K \qquad .$$

CAVEAT:

Both summands in the one-loop mass shift $\Delta M_K = \Delta M_K^C + \Delta M_K^R$ are ultraviolet divergent

3 Heat kernel/zeta function regularization of kink masses

3.1 Zeta function regularization of kink and vacuum zero point energies

• Generalized zeta functions:

$$\Delta M_K^C(s) = \frac{\hbar}{2}(2\frac{\mu^2}{m^2})^s \mu \left(\zeta_K^*(s) - \zeta_{K_0}(s)\right) \qquad , \qquad [\mu] = L^{-1}$$

$$\zeta_K^*(s) = \frac{1}{\varepsilon_3^{2s}} + \sum_k \frac{1}{\varepsilon(k)^{2s}} \qquad , \qquad \zeta_{K_0}(s) = \sum_k \frac{1}{\omega(k)^{2s}}$$

$$\Delta M_K^C = \lim_{s\to -\frac{1}{2}} \Delta M_K^C(s) = \frac{\hbar m}{2\sqrt{2}} \left(\zeta_K^*(-\frac{1}{2}) - \zeta_{K_0}(-\frac{1}{2}) \right) \qquad ,$$

$\Delta M_K^C(s)$ is a meromorphic function of s with a pole at $s = -\frac{1}{2}$

3.2 Mass renormalization

- Regularization of ΔM_K^R in terms of zeta functions

$$I(4) = -\lim_{s\to -\frac{1}{2}} \frac{1}{\mu L} \cdot \frac{\Gamma(s+1)}{\Gamma(s)} \cdot \left(\frac{2\mu^2}{m^2} \right)^{s+1} \cdot \zeta_{K_0}(s+1) \qquad , \qquad [-\frac{mL}{2\sqrt{2}}, \frac{mL}{2\sqrt{2}}]$$

$$\Delta M_K^R(s) = -\frac{6\hbar}{L} \cdot \left(\frac{2\mu^2}{m^2} \right)^{s+\frac{1}{2}} \cdot \frac{\Gamma(s+1)}{\Gamma(s)} \cdot \zeta_{K_0}(s+1) \qquad , \qquad \Delta M_K^R = \lim_{s\to -\frac{1}{2}} \Delta M_K^R(s) = \frac{3\hbar}{L} \cdot \zeta_{K_0}(\frac{1}{2})$$

3.3 Dashen-Hasslacher-Neveu (DHN) exact formula

- Vacuum heat and generalized zeta functions

$$\mathrm{Tr} e^{-\beta K_0} = \frac{mL}{2\sqrt{2}\pi} \int_{-\infty}^{+\infty} dk\, e^{-\beta(k^2+4)} = \frac{mL}{\sqrt{8\pi\beta}} \cdot e^{-4\beta}$$

$$\zeta_{K_0}(s)=\frac{1}{\Gamma(s)}\int_0^\infty d\beta\,\beta^{s-1}\mathrm{Tr}e^{-\beta K_0}=\frac{mL}{\sqrt{8\pi}}\cdot\frac{1}{\Gamma(s)}\cdot\int_0^\infty d\beta\,\beta^{s-\frac{3}{2}}\,e^{-4\beta}=\frac{mL}{\sqrt{8\pi}}\cdot\frac{1}{2^{2s-1}}\cdot\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}$$

Poles of $\zeta_{K_0}(x)$: $s-\frac{1}{2}=0,-1,-2,\cdots,-n,\cdots$.

- Regularized zero point vacuum energy

$$\Delta E(\phi_\pm)=\lim_{s\to-\frac{1}{2}}\frac{\hbar}{2}\left(\frac{2\mu^2}{m^2}\right)^s\mu\cdot\zeta_{K_0}(s)=\lim_{s\to-\frac{1}{2}}\frac{\hbar}{2}\left(\frac{2\mu^2}{m^2}\right)^s\mu\cdot\frac{mL}{\sqrt{8\pi}}\cdot\frac{1}{2^{2s-1}}\cdot\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}$$

- Kink heat and generalized zeta functions

$$\begin{aligned}\mathrm{Tr}^*e^{-\beta K} &= e^{-3\beta}+\frac{mL}{2\sqrt{2}\pi}\int_{-\infty}^{+\infty}dk\,e^{-\beta(k^2+4)}+\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\,\frac{d\delta(k)}{dk}\,e^{-\beta(k^2+4)}\\ &= \frac{mL}{\sqrt{8\pi\beta}}\cdot e^{-4\beta}+e^{-3\beta}(1-\mathrm{Erfc}\sqrt{\beta})-\mathrm{Erfc}2\sqrt{\beta}\end{aligned}$$

$$\begin{aligned}\zeta_K^*(s) &= \zeta_{K_0}(s)+\frac{1}{\Gamma(s)}\left[\frac{1}{3^s}+\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\,\frac{d\delta}{dk}(k)\cdot\frac{1}{(k^2+4)^s}\right]\\ &= \zeta_{K_0}(s)+\frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\Gamma(s)}\left[\frac{2}{3^{s+\frac{1}{2}}}\cdot{}_2F_1[\frac{1}{2},s+\frac{1}{2},\frac{3}{2},-\frac{1}{3}]-\frac{1}{4^s}\frac{1}{s}\right]\end{aligned}\qquad,$$

$${}_2F_1[\frac{1}{2},s+\frac{1}{2},\frac{3}{2},-\frac{1}{3}]=\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(s+\frac{1}{2})}\cdot\sum_{l=0}^\infty\frac{(-1)^l}{3^l l}\cdot\frac{\Gamma(l+\frac{1}{2})\Gamma(s+l+\frac{1}{2})}{\Gamma(l+\frac{3}{2})}\qquad.$$

Poles of $\zeta_K(s)$: $s+l+\frac{1}{2}=0,-1,-2,\cdots,-n,\cdots$, $l\in\mathbb{N}_0$ plus the poles of $\zeta_{K_0}(s)$.

- Kink Casimir energy

$$\begin{aligned}
\Delta M_K^C &= \frac{\hbar m}{2\sqrt{2\pi}}\lim_{\varepsilon\to 0}\left(\frac{2\mu^2}{m^2}\right)^{\varepsilon}\frac{\Gamma(\varepsilon)}{\Gamma(-\frac{1}{2}+\varepsilon)}\left[\frac{2}{3^\varepsilon}\,{}_2F_1[\tfrac{1}{2},\varepsilon,\tfrac{3}{2},-\tfrac{1}{3}]-\frac{1}{(-\frac{1}{2}+\varepsilon)\,4^{-\frac{1}{2}+\varepsilon}}\right]\\
&= \frac{\hbar m}{2\sqrt{2\pi}}\lim_{\varepsilon\to 0}\left[-\frac{3}{\varepsilon}+2+\ln\frac{3}{4}-3\ln\frac{2\mu^2}{m^2}-{}_2F_1'[\tfrac{1}{2},0,\tfrac{3}{2},-\tfrac{1}{3}]+o(\varepsilon)\right]\\
&= -\frac{\hbar m}{2\sqrt{2\pi}}\lim_{\varepsilon\to 0}\left[\frac{3}{\varepsilon}+3\ln\frac{2\mu^2}{m^2}-\frac{\pi}{\sqrt{3}}\right]
\end{aligned}$$

I have used

$${}_2F_1[\frac{1}{2},0,\frac{3}{2};-\frac{1}{3}]=1 \qquad , \qquad {}_2F_1'[\frac{1}{2},0,\frac{3}{2};-\frac{1}{3}]=2-\frac{\pi}{\sqrt{3}}-\ln\frac{4}{3}$$

- Mass renormalization counter-term contribution

$$\begin{aligned}
\Delta M_K^R &= -\frac{6\hbar}{L}\lim_{s\to-\frac{1}{2}}\left(\frac{2\mu^2}{m^2}\right)^{s+\frac{1}{2}}\cdot\frac{mL}{\sqrt{8\pi}}\cdot\frac{1}{4^{s+\frac{1}{2}}}\cdot\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}\\
&= -\frac{3\hbar m}{\sqrt{2\pi}}\lim_{\varepsilon\to 0}\left(\frac{2\mu^2}{m^2}\right)^{\varepsilon}\frac{4^{-\varepsilon}\Gamma(\varepsilon)}{\Gamma(-\frac{1}{2}+\varepsilon)}\\
&= \frac{3\hbar m}{2\sqrt{2\pi}}\lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}+\ln\frac{2\mu^2}{m^2}-\ln 4+(\psi(1)-\psi(-\frac{1}{2}))+o(\varepsilon)\right]
\end{aligned}$$

$$= \frac{\hbar m}{2\sqrt{2}\pi} \lim_{\varepsilon \to 0} \left[\frac{3}{\varepsilon} + 3 \ln \frac{2\mu^2}{m^2} - 2(2+1) \right]$$

I have used that the digamma functions differ as: $\psi(1) - \psi(-\frac{1}{2}) = \ln 4 - 2$

- DHN formula

$$\Delta M_K = \Delta M_K^C + \Delta M_K^R = \frac{\hbar m}{2\sqrt{6}} - \frac{3\hbar m}{\pi\sqrt{2}}$$

4 Cahill-Comtet-Glauber formula

- Cahill-Comtet-Glauber (CCG) formula and bound states

$$\Delta M_K = -\frac{\hbar m}{\sqrt{2}\pi} \left(\sum_{i=0}^{1} 2(\mathrm{sin}\theta_i - \theta_i \mathrm{cos}\theta_i) \right) \qquad (1)$$

Angles are defined in terms of the eigenvalues of the bound states of K:

$$\theta_0 = \arccos(\frac{0}{2}) = \frac{\pi}{2} \quad , \quad \theta_1 = \arccos(\frac{\sqrt{3}}{2}) = \frac{\pi}{6} \quad , \quad \mathrm{sin}\theta_0 = 1 \quad , \quad \mathrm{sin}\theta_1 = \frac{1}{2}$$

$$\Delta M_K = -\frac{\hbar m}{\pi\sqrt{2}}(2+1-\sqrt{3}\ \frac{\pi}{6}) = \frac{\hbar m}{2\sqrt{6}} - \frac{3\hbar m}{\pi\sqrt{2}} = -0.471113\hbar m$$

5 High-temperature one-loop kink mass shift formula

5.1 High-temperature expansion of the heat function

- Kink Seeley coefficients

$$\begin{aligned}
\mathrm{Tr}e^{-\beta K} &= \frac{e^{-4\beta}}{\sqrt{4\pi}}\left(\frac{mL}{\sqrt{2\beta}} + \sqrt{4\pi}\left(e^{\beta}\mathrm{Erf}(\sqrt{\beta}) + e^{4\beta}\mathrm{Erf}(2\sqrt{\beta})\right)\right) \\
&= \frac{e^{-4\beta}}{\sqrt{4\pi}} \cdot \sum_{n=0}^{\infty} c_n(K)\beta^{n-1/2} \qquad , \qquad n \in \mathbb{Z}^+ \\
&= \frac{e^{-4\beta}}{\sqrt{4\pi}}\left(\frac{mL}{\sqrt{2\beta}} + 12\beta^{1/2} + 24\beta^{3/2} + \frac{176}{5}\beta^{5/2} + \frac{1376}{35}\beta^{7/2} + \frac{1216}{35}\beta^{9/2} + \mathcal{O}(\beta^{11/2})\right)
\end{aligned}$$

$$c_0(K) = \lim_{L\to\infty}\frac{mL}{\sqrt{2}} \qquad , \qquad c_n(K) = \frac{2^{n+1}(2^{2n-1}+1)}{(2n-1)!!} \ , \ n \geq 1$$

5.2 The K-heat equation kernel

• Vacuum heat equation kernel and high-temperature asymptotics for β small

$$\left(\frac{\partial}{\partial\beta}-\frac{\partial^2}{\partial x^2}+4\right)K_{K_0}(x,y;\beta)=0 \quad , \quad K_{K_0}(x,y;0)=\delta(x-y)$$

$$K_{K_0}(x,y;\beta)=\frac{e^{-4\beta}}{\sqrt{4\pi\beta}}\cdot e^{-\frac{(x-y)^2}{4\beta}} \qquad .$$

• K-heat equation kernel

$$K=-\frac{d^2}{dx^2}+4+V(x) \qquad , \qquad V(x)=6\phi_K^2(x)-6=-\frac{6}{\cosh^2 x} \qquad ,$$

$$\left(\frac{\partial}{\partial\beta}-\frac{\partial^2}{\partial x^2}+4+V(x)\right)K_K(x,y;\beta)=0 \quad , \quad K_K(x,y;0)=\delta(x-y)$$

• Transfer equation

$$K_K(x,y;\beta)=K_{K_0}(x,y;\beta)\cdot C_K(x,y;\beta)$$

$$\left(\frac{\partial}{\partial\beta}+\frac{x-y}{\beta}\frac{\partial}{\partial x}-\frac{\partial^2}{\partial x^2}+V(x)\right)C_K(x,y;\beta)=0 \quad , \quad C_K(x,y;0)=1 \qquad ,$$

• Power series in β expansion, recurrence relations, and high temperature asymptotics of the K-heat equation kernel

$$C_K(x,y;\beta)=\sum_{n=0}^{\infty} c_n(x,y;K)\,\beta^n \qquad , \qquad c_0(x,y;K)=1 \qquad ,$$

$$(n+1)c_{n+1}(x,y;K)+(x-y)\frac{\partial c_{n+1}}{\partial x}(x,y;K)+V(x)c_n(x,y;K)=\frac{\partial^2 c_n}{\partial x^2}(x,y;K) \qquad .$$

$$K_K(x,y;\beta)=\frac{e^{-4\beta}}{\sqrt{4\pi\beta}}\cdot e^{-\frac{(x-y)^2}{4\beta}}\cdot\sum_{n=0}^{\infty} c_n(x,y;K)\,\beta^n \qquad .$$

- From the heat kernel to the heat function. First step: $\lim_{y\to x}$

$$c_{n+1}(x,x;K)=\frac{1}{n+1}\left[{}^{(2)}C_n(x)-V(x)c_n(x,x;K)\right] \qquad , \qquad {}^{(k)}C_n(x)=\lim_{y\to x}\frac{\partial^k c_n(x,y;K)}{\partial x^k}$$

Infinite temperature conditions and recurrence relations among derivatives

$${}^{(k)}C_0(x)=\lim_{y\to x}\frac{\partial^k c_0}{\partial x^k}=\delta^{k0}\ , \quad {}^{(k)}C_n(x)=\frac{1}{n+k}\left[{}^{(k+2)}C_{n-1}(x)-\sum_{j=0}^{k}\left(\begin{array}{c}k\\ j\end{array}\right)\frac{d^jV(x)}{dx^j}\cdot{}^{(k-j)}C_{n-1}(x)\right] \qquad .$$

- From the heat kernel to the heat function. Second step: $\lim_{L\to\infty}\int_{-\frac{mL}{2\sqrt{2}}}^{\frac{mL}{2\sqrt{2}}}$

$$\mathrm{Tr}e^{-\beta K}=\frac{e^{-4\beta}}{\sqrt{4\pi\beta}}\cdot\sum_{n=0}^{\infty} c_n(K)\,\beta^n \qquad , \qquad c_n(K)=\lim_{L\to\infty}\int_{-\frac{mL}{2\sqrt{2}}}^{\frac{mL}{2\sqrt{2}}} dx\, c_n(x,x;K) \qquad .$$

$$c_0(K) = \lim_{L\to\infty} \frac{mL}{\sqrt{2}} \qquad , \qquad c_n(K) = \frac{2^{n+1}(1+2^{2n-1})}{(2n-1)!!} \,, \quad n \geq 1 \qquad .$$

5.3 Seeley densities and Korteweg-de Vries equation

• Iterative solution of the recurrence relations (abbreviated notation: $u_k = \frac{d^kV}{dx^k}(x)$, $u_k^n = \left(\frac{d^kV}{dx^k}(x)\right)^n$)

$$c_1(x,x) = u_0$$

$$c_2(x,x) = \frac{1}{2}u_0^2 + \frac{1}{6}u_2$$

$$c_3(x,x) = \frac{1}{6}u_0^3 + \frac{1}{6}u_2u_0 + \frac{1}{12}u_1^2 + \frac{1}{60}u_4$$

$$c_4(x,x) = \frac{1}{24}u_0^4 + \frac{1}{12}u_2u_0^2 + \frac{1}{12}u_1^2u_0 + \frac{1}{60}u_4u_0 + \frac{1}{40}u_2^2 + \frac{1}{30}u_1u_3 + \frac{1}{840}u_6$$

$$\begin{aligned} c_5(x,x) &= \frac{1}{120}u_0^5 + \frac{1}{36}u_2u_0^3 + \frac{1}{24}u_1^2u_0^2 + \frac{1}{120}u_4u_0^2 + \frac{1}{40}u_2^2u_0 + \frac{1}{30}u_1u_3u_0 + \frac{1}{840}u_6u_0 + \frac{11}{360}u_1^2u_2 \\ &+ \frac{23}{5040}u_3^2 + \frac{19}{2520}u_2u_4 + \frac{1}{280}u_1u_5 + \frac{1}{15120}u_8 \end{aligned}$$

$$
\begin{aligned}
c_6(x,x) &= \frac{1}{720}u_0^6 + \frac{1}{144}u_2u_0^4 + \frac{1}{72}u_1^2u_0^3 + \frac{1}{360}u_4u_0^3 + \frac{1}{80}u_2^2u_0^2 + \frac{1}{60}u_1u_3u_0^2 + \frac{11}{360}u_1^2u_2u_0 + \frac{1}{280}u_1u_5u_0 \\
&+ \frac{1}{288}u_1^4 + \frac{1}{15120}u_8u_0 + \frac{61}{15120}u_2^3 + \frac{43}{2520}u_1u_2u_3 + \frac{23}{5040}u_0u_3^2 + \frac{5}{1008}u_1^2u_4 + \frac{19}{2520}u_0u_2u_4 \\
&+ \frac{23}{30240}u_4^2 + \frac{19}{15120}u_3u_5 + \frac{1}{1680}u_0^2u_6 + \frac{11}{15120}u_2u_6 + \frac{1}{3780}u_1u_7 + \frac{1}{332640}u_{10}
\end{aligned}
$$

$$
\begin{aligned}
c_7(x,x) &= \frac{1}{5040}u_0^7 + \frac{1}{720}u_2u_0^5 + \frac{1}{288}u_1^2u_0^4 + \frac{1}{240}u_2^2u_0^3 + \frac{1}{180}u_1u_3u_0^3 + \frac{11}{720}u_1^2u_2u_0^2 + \frac{1}{560}u_1u_5u_0^2 \\
&+ \frac{1}{288}u_1^4u_0 + \frac{61}{15120}u_2^3u_0 + \frac{43}{2520}u_1u_2u_3u_0 + \frac{5}{1008}u_1^2u_4u_0 + \frac{1}{332640}u_{10}u_0 + \frac{23}{10080}u_3^2u_0^2 \\
&+ \frac{19}{5040}u_2u_4u_0^2 + \frac{1}{5040}u_6u_0^3 + \frac{83}{10080}u_1^2u_2^2 + \frac{1}{252}u_1^3u_3 + \frac{31}{10080}u_2u_3^2 + \frac{1}{280}u_1u_3u_4 + \frac{1}{1440}u_0^4u_4 \\
&+ \frac{5}{2016}u_2^2u_4 + \frac{23}{30240}u_0u_4^2 + \frac{1}{420}u_1u_2u_5 + \frac{19}{15120}u_0u_3u_5 + \frac{71}{665280}u_5^2 + \frac{1}{2016}u_1^2u_6 \\
&+ \frac{11}{15120}u_0u_2u_6 + \frac{61}{332640}u_4u_6 + \frac{1}{3780}u_0u_1u_7 + \frac{19}{166320}u_3u_7 + \frac{1}{30240}u_0^2u_8 + \frac{17}{332640}u_2u_8 \\
&+ \frac{1}{66528}u_1u_9 + \frac{1}{8648640}u_{12}
\end{aligned}
$$

$$
c_8(x,x) = \frac{1}{40320}u_0^8 + \frac{1}{960}u_2^2u_0^4 + \frac{1}{720}u_1u_3u_0^4 + \frac{1}{576}u_1^4u_0^2 + \frac{1}{252}u_1^3u_3u_0 + \frac{1}{280}u_1u_3u_4u_0 + \frac{1}{420}u_1u_2u_5u_0
$$

$$
\begin{aligned}
&+ \frac{31}{10080}u_2u_3^2u_0 + \frac{5}{2016}u_2^2u_4u_0 + \frac{1}{2016}u_1^2u_6u_0 + \frac{1}{8648640}u_{12}u_0 + \frac{23}{60480}u_4^2u_0^2 + \frac{19}{30240}u_3u_5u_0^2 \\
&+ \frac{11}{30240}u_2u_6u_0^2 + \frac{1}{7560}u_1u_7u_0^2 + \frac{11}{2160}u_1^2u_2u_0^3 + \frac{1}{90720}u_8u_0^3 + \frac{1}{7200}u_4u_0^5 + \frac{1}{1440}u_0^5u_1^2 \\
&+ \frac{1}{4320}u_0^6u_2 + \frac{17}{8640}u_1^4u_2 + \frac{83}{10080}u_0u_1^2u_2^2 + \frac{61}{30240}u_0^2u_2^3 + \frac{1261}{1814400}u_2^4 + \frac{43}{5040}u_0^2u_1u_2u_3 \\
&+ \frac{227}{37800}u_1u_2^2u_3 + \frac{23}{30240}u_0^3u_3^2 + \frac{659}{302400}u_1^2u_3^2 + \frac{5}{2016}u_0^2u_1^2u_4 + \frac{19}{15120}u_0^3u_2u_4 + \frac{527}{151200}u_1^2u_2u_4 \\
&+ \frac{7939}{9979200}u_3^2u_4 + \frac{6353}{9979200}u_2u_4^2 + \frac{1}{1680}u_0^3u_1u_5 + \frac{17}{30240}u_1^3u_5 + \frac{13}{12320}u_2u_3u_5 + \frac{3067}{4989300}u_1u_4u_5 \\
&+ \frac{71}{665280}u_0u_5^2 + \frac{1}{20160}u_0^4u_6 + \frac{3001}{9979200}u_2^2u_6 + \frac{13}{29700}u_1u_3u_6 + \frac{61}{332640}u_0u_4u_6 \\
&+ \frac{3433}{259459200}u_6^2 + \frac{109}{498960}u_1u_2u_7 + \frac{19}{166320}u_0u_3u_7 + \frac{1501}{64864800}u_5u_7 + \frac{71}{1995840}u_1^2u_8 \\
&+ \frac{17}{332640}u_0u_2u_8 + \frac{2003}{129729600}u_4u_8 + \frac{1}{66528}u_0u_1u_9 + \frac{5}{648648}u_3u_9 + \frac{1}{665280}u_0^2u_{10} \\
&+ \frac{73}{25945920}u_2u_{10} + \frac{1}{1441440}u_1u_{11} + \frac{1}{259459200}u_{14}
\end{aligned}
$$

- Invariants of the KdV equation. The one-parametric family of Schrödinger operators

$$H(\tau) = -\frac{d^2}{dx^2} + V(x,\tau)$$

is isospectral if

$$\frac{\partial V}{\partial \tau}(x,\tau) = \frac{\partial^3 V}{\partial x^3}(x,\tau) - 6V(x,\tau)\frac{\partial V}{\partial x}(x,\tau)$$

5.4 Mellin transform of the asymptotic expansion

• From the heat function to the generalized zeta function: the Mellin transform

$$\zeta_K^*(s) = \frac{1}{\Gamma(s)}\left[\frac{1}{\sqrt{4\pi}}\cdot\sum_{n=0}^{\infty} c_n(K)\cdot\int_0^1 d\beta\,\beta^{s+n-\frac{3}{2}}\cdot e^{-4\beta} + \int_1^{\infty} d\beta\,\beta^{s-1}\mathrm{Tr}^* e^{-\beta K} - \int_0^1 d\beta\,\beta^{s-1}\right]$$

$$B_K(s) = \frac{1}{\Gamma(s)}\int_1^{\infty} d\beta\,\beta^{s-1}\mathrm{Tr}^* e^{-\beta K}$$

entire function of s.

• The zero mode contribution $I = \frac{1}{\Gamma(s)}\cdot\int_0^1 d\beta\,\beta^{s-1}$ is is an improper integral if $\mathrm{Re}s < 0$ regularized as

$$\lim_{\varepsilon\to 0}\frac{1}{\Gamma(s)}\cdot\int_0^1 d\beta\,\beta^{s-1}e^{-\varepsilon\beta} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^s}\cdot\frac{\gamma[s,\varepsilon]}{\Gamma(s)}$$

$$\gamma[s,\varepsilon]_{\varepsilon\to 0} \cong \frac{1}{s}\cdot\varepsilon^s - \frac{1}{s+1}\cdot\varepsilon^{s+1} \qquad \Rightarrow \qquad I^R = \frac{1}{s\Gamma(s)} = \frac{1}{\Gamma(s+1)}$$

• Splitting the vacuum generalized zeta function

$$\zeta_{K_0}(s) \;=\; \frac{mL}{\sqrt{8\pi}}\cdot\frac{1}{\Gamma(s)}\cdot\left[\int_0^1 d\beta\,\beta^{s-\frac{3}{2}}e^{-4\beta} + \int_1^{\infty} d\beta\,\beta^{s-\frac{3}{2}}e^{-4\beta}\right]$$

$$= \frac{mL}{\sqrt{8\pi}} \cdot \frac{1}{\Gamma(s)} \cdot \frac{1}{4^{s-\frac{1}{2}}} \cdot \left[\gamma[s-\frac{1}{2},4] + \Gamma[s-\frac{1}{2},4] \right] \qquad .$$

$\gamma[s-\frac{1}{2},4]$ has poles at: $s-\frac{1}{2}=0,-1,-2,-3,\cdots$. $\Gamma[s-\frac{1}{2},4]$ is an entire function of s.

• Truncation of the asymptotic series giving the kink generalized zeta function

$$\zeta_K^*(s) = \frac{1}{\Gamma(s)} \left[\frac{1}{\sqrt{4\pi}} \cdot \sum_{n=0}^{N_0} c_n(K) \cdot \frac{\gamma[s+n-\frac{1}{2},4]}{4^{s+n-\frac{1}{2}}} + \frac{1}{\sqrt{4\pi}} \cdot \sum_{n=N_0+1}^{\infty} c_n(K) \cdot \frac{\gamma[s+n-\frac{1}{2},4]}{4^{s+n-\frac{1}{2}}} - \frac{1}{s} \right] + B_K(s)$$

$\gamma[s+n-\frac{1}{2},4]$ has poles at: $s+n-\frac{1}{2}=0,-1,-2,-3,\cdots$. A large but finite number N_0 is chosen to separate the contribution of the high-order coefficients.

$$b_K^{N_0}(s) = \frac{1}{\sqrt{4\pi}} \cdot \sum_{n=N_0+1}^{\infty} c_n(K) \cdot \frac{\gamma[s+n-\frac{1}{2},4]}{4^{s+n-\frac{1}{2}}}$$

is holomorphic, however, for $\mathrm{Re}s > -N_0 - 1$.

5.5 High-temperature one-loop kink mass correction

• Neglect the (very small) contribution of the entire functions and subtract the zero point vacuum energy

$$\Delta M_K^C \cong \frac{\hbar}{2} \cdot \lim_{s\to -\frac{1}{2}} \left(\frac{2\mu^2}{m^2} \right)^s \cdot \mu \cdot \frac{1}{\Gamma(s)} \cdot \left[\frac{1}{\sqrt{4\pi}} \sum_{n=1}^{N_0} c_n(K) \frac{\gamma[s+n-\frac{1}{2},4]}{4^{s+n-\frac{1}{2}}} - \frac{1}{s} \right] \qquad ,$$

zero-point vacuum energy renormalization takes care of the term coming from $c_0(K)$

- Mass renormalization

$$\Delta M_K^R = -\frac{\hbar\mu}{2\sqrt{4\pi}} \cdot c_1(K) \cdot \lim_{s\to-\frac{1}{2}} \left(\frac{2\mu^2}{m^2}\right)^{s+\frac{1}{2}} \cdot \frac{1}{4^{s+\frac{1}{2}}\Gamma(s)} \cdot \left[\gamma[s+\frac{1}{2},4] + \Gamma[s+\frac{1}{2},4]\right]$$

mass renormalization counter-term exactly cancels the $c_1(K)$ contribution

- High-temperature one-loop kink mass shift formula

$$\Delta M_K = -\frac{\hbar m}{4\sqrt{2\pi}} \cdot \left[\frac{1}{\sqrt{4\pi}} \cdot \sum_{n=2}^{N_0} c_n(K) \cdot \frac{\gamma[n-1,4]}{4^{n-1}} + 2\right] \qquad .$$

5.6 Mathematica calculations

- Numerical data

$$\Delta M_K \cong -0.199471\hbar m + D_{N_0}\hbar m \qquad , \qquad D_{N_0} = -\sum_{n=2}^{N_0} c_n(K) \frac{\gamma[n-1,4]}{8\sqrt{2}\pi\, 4^{n-1}}$$

n	$c_n(K)$	N_0	D_{N_0}
2	24.0000	2	-0.165717
3	35.2000	3	-0.221946
4	39.3143	4	-0.248281
5	34.7429	5	-0.261260
6	25.2306	6	-0.267436
7	15.5208	7	-0.270186
8	8.27702	8	-0.271317
9	3.89498	9	-0.271748
10	1.63998	10	-0.271900

n	$c_n(K)$	N_0	D_{N_0}
11	0.62475	11	-0.271949
12	0.217305	12	-0.271964
13	0.0695378	13	-0.271968
14	0.020603	14	-0.271969523040415
15	0.0056838	15	-0.27196980618685
16	0.00146678	16	-0.27196987296988
17	0.000355585	17	-0.271969887872730
18	0.0000812766	18	-0.271969891027559
19	0.0000175733	19	-0.27196989166267
20	$3.60478 \cdot 10^{-6}$	20	-0.27196989178453

$D_{10} = -0.271900\hbar m$ $\qquad$ $D_{20} = -0.271969\hbar m$

$\Delta M_K \cong -0.471371\hbar m$, $\Delta M_K \cong -0.471440\hbar m$

- Error $-0.0002580\hbar m$ $\qquad$ $-0.0003270\hbar m$

$$
\begin{aligned}
&\frac{\hbar m}{2}[B_K(-\tfrac{1}{2}) - B_{K_0}(-\tfrac{1}{2})] + \frac{3\hbar m}{\sqrt{2}} B_{K_0}(\tfrac{1}{2}) \\
&= \frac{\hbar m}{2\sqrt{2\pi}} \int_1^\infty d\beta \left(-\frac{e^{-3\beta}}{2\beta^{\frac{3}{2}}} + \frac{e^{-3\beta}\mathrm{Erfc}\sqrt{\beta}}{2\beta^{\frac{3}{2}}} + \frac{\mathrm{Erfc}\, 2\sqrt{\beta}}{2\beta^{\frac{3}{2}}} + \frac{3e^{-4\beta}}{\sqrt{\pi}\beta} \right) \\
&\approx 0.00032792\hbar m
\end{aligned}
$$

$-\frac{\hbar m}{4\sqrt{2\pi}} b_K^{N_0}(-\frac{1}{2}) \approx 10^{-4}\hbar m$.

6 One-loop mass shifts for non-commutative kinks

6.1 Non-commutative $(\phi)^4_2$-model

• The action and the Moyal product

$$S = \frac{m^2}{\lambda} \int d^2x \, \frac{1}{2} \left\{ \frac{\partial \phi}{\partial x^\mu} \cdot \frac{\partial \phi}{\partial x_\mu} - \left(\phi^2_\star(x^\mu) - 1\right)^2 \right\}$$

$$(f \star g)(x^\mu) = \left[\exp\left(\frac{i}{2} \Theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \cdot \frac{\partial}{\partial y^\mu} \right) f(x^\mu) g(y^\mu) \right]_{y^\mu = x^\mu} \quad , \quad \Theta^{\mu\nu} = -\Theta^{\nu\mu} \quad , \quad \Theta^{\mu\nu} = 2\theta \varepsilon^{\mu\nu} \quad , \quad \varepsilon^{01} = 1 = -\varepsilon^{10}$$

• Field equations

$$\Box\phi(x^\mu) - 2\phi(x^\mu) + 2\phi(x^\mu) \star \phi(x^\mu) \star \phi(x^\mu) = 0$$

6.2 Non-commutative kinks

• Static solutions

$$\phi_V^{\pm} = \pm 1 \qquad , \qquad \phi_K(x) = \pm \mathrm{tanh}(x - x_0) \qquad , \qquad E_K = \frac{m^3}{\lambda\sqrt{2}}$$

• Small fluctuations around kinks and linearized equations

$$\phi(t,x) = \phi_K(x) + \delta\phi(t,x) \quad , \quad \Box\delta\phi - 2\delta\phi + 2(\phi_K \star \phi_K \star \delta\phi + \phi_K \star \delta\phi \star \phi_K + \delta\phi \star \phi_K \star \phi_K) = 0$$

• Separation of space-time variables and the Moyal product

$$e^{i\omega t} \star f(x) = e^{i\omega t} f(x - \theta\omega) \qquad , \qquad f(x) \star e^{i\omega t} = e^{i\omega t} f(x + \theta\omega) \qquad , \qquad \delta\phi(t,x) = e^{i\omega t}\eta(x)$$

• Non-linear spectral problem: $K(\omega)\eta_\omega(x) = \omega^2\eta_\omega(x) = \left[-\frac{d^2}{dx^2} + V(x;\omega)\right]\eta_\omega(x)$

$$V(x;\omega) = 2\left(\mathrm{tanh}^2(x+\theta\omega) + \mathrm{tanh}(x+\theta\omega)\mathrm{tanh}(x-\theta\omega) + \mathrm{tanh}^2(x-\theta\omega) - 1\right)$$

6.3 Exotic SUSY Quantum Mechanics

• First-order differential operators, supercharges and SUSY Hamiltonian

$$\partial(\omega) = -\frac{d}{dx} + \tanh x_+(\omega) + \tanh x_-(\omega) \quad , \quad x_\pm(\omega) = x \pm \theta\omega \quad , \quad \partial^\dagger(\omega) = \frac{d}{dx} + \tanh x_+(\omega) + \tanh x_-(\omega)$$

$$Q(\omega) = \begin{pmatrix} 0 & \partial(\omega) \\ 0 & 0 \end{pmatrix} \qquad , \qquad Q^\dagger(\omega) = \begin{pmatrix} 0 & 0 \\ \partial^\dagger(\omega) & 0 \end{pmatrix}$$

$$Q^\dagger(\omega)Q(\omega) + Q(\omega)Q^\dagger(\omega) = H(\omega) = \begin{pmatrix} H^+(\omega) & 0 \\ 0 & H^-(\omega) \end{pmatrix}$$

$$H^+(\omega) = K(\omega) = H^-(\omega) - \frac{2}{\cosh^2 x_+(\omega)} - \frac{2}{\cosh^2 x_-(\omega)}$$

$$H^-(\omega) = -\frac{d^2}{dx^2} + 2(1 + \tanh x_+(\omega)\tanh x_-(\omega))$$

• Zero mode of $H^+(\omega)$

$$\partial^\dagger(\omega)\eta_0(x;\omega) = 0 \qquad \Leftrightarrow \qquad \eta_0(x;\omega) = \frac{1}{\cosh x_+(\omega)\cosh x_-(\omega)}$$

References

[1] R. Dashen, B. Hasslacher and A. Neveu, *"Non-perturbative methods and extended hadron models in field theory: 1, semiclassical functinal methods; 2, two-dimensional models and extended hadrons; 3, four-dimensional non-Abelian models"*, Phys. Rev. **D10** (1974) 4130.

[2] R. Dashen, B. Hasslacher and A. Neveu, *"Semi-classical bound states in an asymptotically free theory"*, Phys. Rev. **D12** (1975) 3424.

[3] L. D. Faddeev and V. E. Korepin, *"Quantum theory of solitons"*, Phys. Rep. **42C** (1978) 1-87.

[4] M. Bordag, A. Goldhaber, P. van Nieuwenhuizen and D. Vassilevich, *"Heat kernels and zeta-function regularization for the mass of the SUSY kink"*, Phys. Rev. **D66** (2002) 125014.

[5] A.A. Izquierdo, J.M. Guilarte, M.A.G. Leon, W. G. Fuertes, *"Generalized zeta functions and one-loop corrections to quantum kink masses"*, Nucl. Phys. **B 635** (2002) 525-557

[6] K. Cahill, A. Comtet, and R. Glauber, *"Mass fomulas for static solitons"*, Phys. Lett. **64B** (1976) 283-285.

[7] R. A. Konoplya, and D. V. Vassilevich, *"Quantum corrections to non-commutative kink"*, JHEP 0801 (2008) 068

[8] A.A. Izquierdo, J.M. Guilarte, M.A.G. Leon, W. G. Fuertes, *"Semi-classical mass of of quantum k-component topological kinks"*, Nucl. Phys. **B 638** (2002)378

[9] A.A. Izquierdo, J.M. Guilarte, M.A.G. Leon, W. G. Fuertes, *"One-loop corrections to classical masses of kink families"*, Nucl. Phys. **B 681** (2004) 163-194

[10] D. Vassilevich, *"Quantum corrections to the mass of the supersymmetric vortex"*, Phys. Rev. **D 68** (2003) 045005

[11] A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, *"Nonvanishing quantum corrections to the mass and central charge of the $N = 2$ vortex and BPS saturation"*, Nucl. Phys. **B 679** (2004) 382-394.

[12] A.A. Izquierdo, J.M. Guilarte, M.T. Mayado and W. G. Fuertes, *"Quantum corrections to the mass of self-dual vortices"*, Phys. Rev. **D70** (2004) 061702(R)

[13] A.A. Izquierdo, J.M. Guilarte, M.T.Mayado, W. G. Fuertes, *"Quantum oscillations of self-dual Abrikosov-Nielsen-Olesen vortices"*, Phys. Rev. **D71** (2005) 125010

[14] A.A. Izquierdo, J.M. Guilarte, M.T.Mayado, W. G. Fuertes, *"One-loop coorrection to the mass of self-dual semi-local planar topological solitons*, Nucl. Phys. **B797**[PM] (2008) 431