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ELECTROMAGNETIC FORM FACTORS FOR COMPOSITE PARTICLES AT LARGE MOMENTUM TRANSFER*

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ABSTRACT

Experimental information now exists on the behavior of electromagnetic form factors $F(q)$ at very high momentum transfer q. This has led to speculations on the asymptotic behavior of the form factors as $q \rightarrow \infty$. In this paper we consider non-relativistic Schrödinger models of composite systems and correlate the asymptotic behavior of $F(q)$ with the nature of the forces in the limit of zero interparticle separation. Conditions for fall-off more rapid than powers of $\frac{1}{q}$ are analyzed, and comparisons with proton and deuteron data are presented.

I. INTRODUCTION

The extension of experimental information on the electromagnetic form factors[1] $F(q)$ to very large values of q and to very small values of $F(q)$ has spurred increasing interest in the asymptotic behavior of electromagnetic form factors for large momentum transfers $q \to \infty$. Analyses of the observations for both electron-proton and electron-deuteron elastic scattering show that the $F(q)$ continue to decrease very rapidly with increasing q and that this rate of decrease is sufficiently rapid to defy attempts at simple models. In the case of the proton[2] the fall-off is at least as fast as $\sim \frac{1}{q^4}$; to reproduce this rate theoretically in terms of the usual narrow resonance model requires a carefully arranged cancellation between the contributions of two or more states in the spectral representation of $F(q)$. In the case of the deuteron for which $F(q)$ drops to less than 10^{-2} $F(0)$ for $q \sim 1$ BeV/c an elaborate parametrization of the forces and of the deuteron wave function is required[3] as the inter-nucleon separation $r \to 0$.

Wu and Yang[4] have conjectured on the basis of general statistical arguments that the nucleon form factors approach asymptotically

$$F(q) \underset{q \to \infty}{\sim} e^{-aq} \qquad (1)$$

with $a = (0.6 \text{ GeV/c})^{-1}$.

This behavior is correlated with the Orear[5] fit to elastic high energy nucleon-nucleon scattering at large momentum transfers

$$\left(\frac{d\sigma}{d\Omega}\right) \sim se^{-4aq_\perp} \tag{2}$$

where $q_\perp \equiv q \sin\theta$ is the transverse momentum transfer. Wu and Yang go further to suggest that Eq. (2) is a law for all strong interaction processes. This suggestion has stimulated further more elaborate studies based on the assumption of scale, or dilation, invariance of high energy processes,[6] as well as on more detailed statistical analyses of multiple production processes[7] which also predict a form factor behavior as in Eq. (1).

The exponential form of Eq. (1) has one appealing theoretical feature: within a polynomial factor, it is the maximal rate of decrease consistant with polynomial boundedness of F(t) in the complex $t = q_o^2 - \underline{q}^2$ plane.[8] Without polynomial boundedness it is impossible to write a dispersion relation for F(t) with a finite number of subtraction constants (a circumstance too unattractive to entertain seriously).

Figs. 1 and 2 show the latest measurements of proton and deuteron form factors. For the proton a universal form factor is assumed for t large and negative with $G_M(t) = \mu_p G_E(t)$; $\mu_p \equiv 2.79$. In relating the deuteron cross section to a form factor the isoscalar nucleon charge form factor is divided out.

Thus far only very weak bounds on asymptotic form factor behavior have been constructed within the framework of relativistic local quantum field theory.[9] With the aim of gaining some insight into the physical significance of the asymptotic behavior of F(q) we shall work within the framework of non-relativistic potential theory. The recent studies of Serber,[10] Bertocchi, Fubini and Furlan,[11] and Tiktopolous[12] have related the decrease of the nucleon-nucleon scattering amplitude at large momentum transfers to the nature of the inter-nucleon force as $r \to 0$. Since all partial waves contribute to the scattering amplitude which is a function of energy as well as of momentum transfer this is a much more complex problem than is the study of F(q) which involves the momentum transfer only.

Our primary interest in this paper is to correlate the asymptotic behavior of the electromagnetic form factor of a compound system with the nature of the forces in the limit of zero inter-particle separations. We shall group the form factors into three broad categories according to their rate of decrease:

(a) with a power law: $F(q) \sim \frac{1}{q^p}$; $p > 0$

(b) exponentially: $F(q) \sim e^{-aq} P(q) \quad a > 0$ (3)

(c) or with a fractional exponential:

$$F(q) \sim e^{-aq^{p/p+1}} P(q) \quad a, p > 0$$

where P(q) denotes a polynomial in $1/q$ and/or an oscillatory factor in q.

As mentioned above the data as well as theoretical reasons permit us to ignore the possibility of a more rapid decrease than the exponential one. In order to emphasize that the differences between power, exponential, and fractional exponential decreases in $F(q)$ are not just a purely mathematical nicety but are indeed of experimental relevance we show in Fig. 3 the three alternatives adjusted to go through two fixed points.

Section II is devoted to the analysis of the asymptotic form factor behavior for a non-relativistic spinless "deuteron"--i.e. a spherically symmetric bound state of two spinless particles of equal mass, one electrically neutral and the other charged, interacting via a central static potential in the Schrödinger equation.

In Section III we generalize the discussion of Section II to a bound system of three particles--the motivation for this being a quark model for the proton structure as well as the existence of He^3, H^3 and of heavier nuclear targets.

Finally in Section IV we discuss a new way of presenting the form factor data in order to make more readily apparent the "approach to asymptotic behavior."

II. NON-RELATIVISTIC POTENTIAL THEORY FOR TWO-PARTICLES

In this section we discuss the deuteron form factor at large momentum transfer in a potential theory model. For simplicity we assume that the deuteron is an S-state formed of two scalar nucleons obeying a Schrödinger equation and bound by a static, central potential $V(r)$. The charge form factor is defined as the Fourier transform of the charge density; i.e.

$$\begin{aligned} F(q) &= \int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, |\psi(\vec{r})|^2 \\ &= \frac{4\pi}{q}\,\mathrm{Im}\int_0^\infty r\,dr\, e^{iqr}\,\psi^2(r) = \frac{4\pi}{q}\int_{-\infty}^{\infty} r \sin qr\, \psi^2(r)\,\theta(r)\,dr \end{aligned} \tag{4}$$

where

$$\theta(r) = \begin{cases} 1 & r > 0 \\ 0 & r < 0 \end{cases} \tag{5}$$

and $\psi^*(r) = \psi(r)$ since it represents a stationary S state solution with no flow of current.

The behavior of the form factor at large momentum transfer $q \to \infty$ is determined by the behavior of the wave function at its singularities. We assume that the potential is infinitely differentiable for real positive r except possibly at the origin. In this case the asymptotic behavior of the form factor is determined by the nature of any possible singularities in the wave function at $r = 0$. Since $\psi(r)$ must satisfy the boundary condition of square integrability which rules out infinities as strong as or stronger than $1/r$ at $r = 0$, the only types of singularities that are possible at the origin are essential singularities of the type

$e^{-A/r^{\alpha}}$ with A and $\alpha > 0$ or discontinuities of the form r^{β} with $\beta > -1$. We want to correlate the singularities of $\psi(r)$ for $r \to 0$ with the nature of the singularities of the potential at the origin and in this way relate the asymptotic behavior of $F(q)$ as $q \to \infty$ with the inter-nucleon forces.

First in order to illustrate the influence of the singularities of $\psi(r)$ on the form factor we derive several general results from the form of Eqs. (4) and (5):

A. If the product of factors $r\,\psi^2(r)$ exists as an ordinary function for $r \geq 0$ and is well behaved at infinity, then asymptotically[13] for large q

$$F(q^2) \sim \frac{g(0)}{q} - \frac{g''(0)}{q^3} + \frac{g^{(IV)}(0)}{q^5} + \ldots \qquad (6)$$

where $g(r) \equiv r\,\psi^2(r)$. $F(q)$ will decrease faster than $\frac{1}{q^p}$, for any p as $q \to \infty$ if and only if all the even derivatives of $g(r)$ vanish at $r = 0$.

One example of such behavior is a solution $\psi(r)$ which has an essential singularity at the origin so that $\psi(r)$ and all its derivatives vanish at $r = 0$. Another example is a solution such that $\psi(r)$ can be analytically continued in a neighborhood of $r = 0$ with definite parity, i.e. $\psi(-r) = \pm\psi(+r)$. We can then drop the $\theta(r)$ in Eq. (5) and note that the even derivatives of $r\psi^2(r)$ vanish for $r = 0$.

B. If we make the stronger assumption that $\psi(r)$ can be continued as a function of the complex variable r into the complex r plane with definite parity $\psi(r) = \pm\psi(-r)$ and with a strip of analyticity extending along the real r axis from $-\infty$ to $+\infty$ of minimum width $-a \leq \mathrm{Im}\ r \leq +a$ as illustrated in Fig. 4, then the form factor falls off at least as fast as e^{-aq}.

This theorem is readily verified with the aid of Cauchy's theorem. We can once again drop the θ function since by assumption $\psi^2(r) = \psi^2(-r)$ and write

$$F(q) = \frac{2\pi}{q} \operatorname{Im} \int_{-\infty}^{\infty} dr e^{iqr} \, r \, \psi^2(r) \quad . \tag{7}$$

Displacing the contour as shown in Fig. 5 we have

$$\begin{aligned} F(q^2) = \operatorname{Im} \frac{2\pi}{iq} \lim_{R\to\infty} \Bigg\{ & e^{-iqR} \int_0^a dy\, e^{-qy} (-R + iy) \left|\psi(-R + iy)\right|^2 \\ & + e^{-qa} \int_{-R}^{R} dx\, e^{iqx} (x + ia) \left|\psi(x + ia)\right|^2 \\ & + e^{iqR} \int_a^0 dy\, e^{-qy} (R + iy) \left|\psi(R + iy)\right|^2 \Bigg\} \quad . \end{aligned} \tag{8}$$

Since $\psi(r)$ is assumed to represent a bound state wave function the first and third terms vanish as $R \to \infty$. By the assumed analyticity in the strip, $\psi(x + ia)$ is analytic for all x, and the integral in the second term goes to zero as $q \to \infty$ by the Riemann-Lebesgue lemma. This establishes our claim.

We now turn to the physically interesting problem of relating the behavior of $\psi(r)$ as $r \to 0$ to the properties of the potential $V(r)$ as $r \to 0$. Almost any $V(r)$ will lead to form factors with a power fall off as $q \to \infty$. For example the Hulthén potential[14]

$$2M\,V(r) = -\gamma \frac{1}{1 - e^{-\delta r}}$$

leads to a bound solution $\frac{e^{-\alpha r} - e^{-\beta r}}{r}$ where $\alpha = \frac{1}{2}(\frac{\gamma}{\delta} + \delta)$ and $\beta = \frac{1}{2}(\frac{\gamma}{\delta} - \delta)$ and to a form factor

$$F(q) \propto \frac{1}{q}\left[\arc\tan\frac{q}{2\alpha} + \arc\tan\frac{q}{2\beta} - 2\arc\tan\frac{q}{\alpha+\beta}\right] \propto \frac{1}{q^4} \text{ as } q \to \infty .$$

The failure to meet conditions A or B above for faster than a power fall off may be traced in $\psi(r)$, or $V(r)$, to this fact: if we continue $\psi(r)$ into the complex plane with a definite parity we must write

$$\psi^2\left(\sqrt{r^2}\right) = \frac{1}{r^2}\left(e^{-2\alpha\sqrt{r^2}} + e^{-2\beta\sqrt{r^2}} - 2e^{-(\alpha+\beta)\sqrt{r^2}}\right)$$

and thus have a singularity at the origin. Alternatively we may make the continuation simply by writing

$$\psi^2(r) = \frac{1}{r^2}\left(e^{-2\alpha r} + e^{-2\beta r} - 2e^{-(\alpha+\beta)r}\right)$$

and since we do not have a definite parity, i.e., $\psi^2(r) \neq \pm\,\psi^2(-r)$, the step function $\theta(r)$ in Eq. (5) introduces singularities into $F(q)$ and prevents us from invoking the Riemann-Lebesgue lemma.

From the form of the radial Schrodinger equation ($\hbar = 1$; M = nucleon mass; $E_B > 0$ is the positive binding energy and $u(r) = r\,\psi(r)$ is the radial solution)

$$\frac{1}{M}\frac{d^2u(r)}{dr^2} = \left[E_B + V(r)\right]u(r) \tag{9}$$

we see that if $\psi(r)$ is to satisfy condition B above for exponential form factor decrease, the potential must have the analytic properties of the right hand side of the equation:

$$V(r) = \frac{1}{Mu(r)} \frac{d^2u(r)}{dr^2} - E_B \quad . \tag{10}$$

V(r) must thus be of even parity when analytically continued for complex r, and must be analytic in the complex r plane along the real axis within a strip of minimum width $-a \leq \mathrm{Im}\ r \leq +a$ as in Fig. 4 for $\psi(r)$. (The potential $V(r) = \frac{V_o}{r^2}$ can also lead to exponential fall-off for certain discrete values of V_o.)

Although such a potential would lead to the behavior conjectured by Wu and Yang it is at variance with our usual physical notions. When working within the framework of potential models a more singular repulsive core behavior for the potential as $r \to 0$ is indicated by experimental comparisons. We turn then to potentials increasing at the origin as a power

$$V(r) \underset{r \to 0}{\approx} \frac{1}{Mr_o^2} \left(\frac{r_o}{r}\right)^{2(1+p)} \quad ; \quad p > 0 ; \quad r_o > 0 \quad . \tag{11}$$

In this case the potential dominates the behavior of the Schrodinger equation at the origin and we may write for the radial equation

$$\frac{d^2u(r)}{dr^2} \approx MV(r)\ u(r) \ ; \quad r \to 0 \quad .$$

The square integrable solution to this equation near the origin is

$$u(r) \approx C\, e^{-\frac{1}{p(r/r_o)^p}} \tag{12}$$

and the form factor becomes (asymptotically for $q \to \infty$)

$$F(q) \underset{q\to\infty}{\sim} \frac{4\pi}{q}|C'|^2 \, \mathrm{Im} \int_o^\infty \frac{dr}{r}\, e^{iqr - \left[\frac{2}{p}\frac{1}{(\frac{r}{r_o})^p}\right]} \quad . \tag{13}$$

We can evaluate the asymptotic value of this integral as $q \to \infty$ by the method of steepest descents. Let

$$r = \left(\frac{2r_o{}^p}{q}\right)^{\frac{1}{1+p}} s \ .$$

Then we have

$$F(q) \approx \frac{4\pi}{q}\,|C'|^2 \ \mathrm{Im} \int_o^\infty \frac{ds}{s} \exp\left[\left(2(qr_o)^p\right)^{\frac{1}{1+p}} \left(is - \frac{1}{ps^p}\right)\right]$$

$$\approx \frac{4\pi}{q}\,|C'|^2 \ \mathrm{Im}\exp\left[i\left(2(qr_o)^p\right)^{\frac{1}{1+p}} \frac{p+1}{p}\, s_o\right]$$

$$\times \int_o^\infty \frac{ds}{s_o} \ \exp\left[-\left(2(qr_o)^p\right)^{\frac{1}{1+p}} \frac{(s-s_o)^2}{2}\ \frac{(p+1)}{s_o{}^{p+2}}\right] \tag{14}$$

where s_o is the saddle point through which we detour the contour along a path of constant phase. s_o satisfies

$$i = \frac{-1}{s_o^{p+1}} . \tag{15}$$

In general Eq. (15) has more than 1 root, possibly infinitely many. We choose the root such that the following rules are satisfied:

i) The contour will be deformed into the first quadrant so that the integrand remains bounded at ∞ for real, positive q.

ii) In the case that p is not integral, we want the root to be on the sheet corresponding to $1/(1)^{1+p} = 1$ so that we may deform our exponent continuously from the real axis to the saddle point without passing through cuts.

Thus we have

$$s_o = e^{\frac{i\pi}{2(1+p)}} . \tag{16}$$

With this choice the remaining integral is finite and the asymptotic behavior of F(q) is

$$F(q) \approx \frac{|C''|}{q^{1+\frac{p}{2(1+p)}}} \, \text{Im} \exp\left[\left(2(r_o q)^p\right)^{\frac{1}{1+p}} \frac{p+1}{p} \quad e^{\frac{i\pi}{2}\left(\frac{2+p}{1+p}\right)} + \frac{i\pi}{4}\,\frac{p}{1+p}\right]$$

$$\approx \frac{|C''|}{q^{1+\frac{p}{2(1+p)}}} \exp\left[-\frac{p+1}{p}\sin\left(\frac{\pi}{2}\,\frac{1}{1+p}\right)\left(2(qr_o)^p\right)^{\frac{1}{1+p}}\right]$$

$$X \sin\left[\frac{\pi}{4}\,\frac{p}{1+p} + \frac{p+1}{p}\sin\left(\frac{\pi}{2}\,\frac{2+p}{1+p}\right)\left(2(r_o q)^p\right)^{\frac{1}{1+p}}\right] . \tag{17}$$

For the special case $p = 1$ corresponding to $V \propto \frac{1}{r^4}$ we have

$$F(q) \approx \frac{|C''|}{q^{5/4}} e^{-2\sqrt{q_o r_o}} \sin\left[\frac{\pi}{8} + 2(q_o r_o)^{1/2}\right] \tag{18}$$

which agrees with the exact evaluation.[15] These solutions are examples of case A discussed earlier wherein solutions of the type in Eq. (12) with essential singularities at the origin such that $\psi(0)$ and all its derivatives vanish lead to faster than a power fall-off of the form factor.

These fractional exponential fall-offs are of especial interest since the strength of the singularity in $V(r)$ at the origin determines the fractional power of q in the exponent of $F(q)$.

Naïvely one might expect to be able to derive restrictions on the parameter p by continuing our result analytically into the region of timelike q^2 and assuming polynomial boundedness of the form factor for deuteron pair production; however, since our result is only an asymptotic form, its continuation would not have to be bounded even if the true result were.

The form factor in Eq. (17) has three unknown parameters appearing in it: the magnitude of the normalization constant C'', the "range" of the repulsive potential r_o, and the power of the singularity p as defined by Eq. (11). We have thus far suppressed the appearance of a fourth parameter by insisting on the form of Eq. (12) for the solution as $r \to 0$. In fact $u(r)$ in Eq. (12) can be multiplied by an arbitrary power[16] r^n to read

$$u_{(n)}(r) = Ce^{-\frac{1}{p}\left(\frac{r_o}{r}\right)^p} r^n \tag{19}$$

which also satisfies Eqs. (10) and (11) for $p > 0$ as $r \to 0$. The added polynomial r^{2n} thereby introduced into Eqs. (13) and (14) can be replaced by its value at the saddle point. The form factor is found to be

$$F_n(q) \simeq \frac{|C_n|}{q^{1 + \frac{p+4n}{2(1+p)}}} \exp - \left[\frac{p+1}{p} \sin\left(\frac{\pi}{2} \frac{1}{1+p}\right)\left(2(qr_o)^p\right)^{\frac{1}{1+p}} \right] \mathrm{X} \tag{20}$$

$$\mathrm{X} \quad \sin\left[\frac{\pi}{4} \frac{p}{1+p} + \frac{n\pi}{1+p} + \frac{p+1}{p} \sin\left(\frac{\pi}{2} \frac{2+p}{1+p}\right)\left(2(r_o q)^p\right)^{\frac{1}{1+p}} \right] .$$

In the asymptotic region this additional polynomial factor should not conceal the dominant qualitative feature of a fractional exponential decay modulated by the sin factor.

Practical implications of these results will be explored in more detail in Section IV.

III. THE THREE BODY CASE

We now consider the asymptotic behavior of the electromagnetic form factor of three point particles, only one of which is charged, under the assumption that the forces are two body forces of the type considered in the previous section. We also assume scalar particles and use non-relativistic quantum mechanics as before. For simplicity we consider the equal mass case, although different masses give essentially the same result.

The three body Schrödinger equation is

$$\left\{ -\frac{1}{2M}\left(\nabla^2_{r_1} + \nabla^2_{r_2} + \nabla^2_{r_3}\right) + V(|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|) + V(|\underset{\sim}{r}_1 - \underset{\sim}{r}_3|) + V(|\underset{\sim}{r}_2 - \underset{\sim}{r}_3|) \right\} \psi = E\psi \qquad (21)$$

where r_1, r_2, and r_3 represent the position vectors of particles 1, 2, and 3, respectively. Index (1) refers to the charged particle. Defining

$$\underset{\sim}{R} = \underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3, \quad \underset{\sim}{r} = \frac{\underset{\sim}{r}_3 - \underset{\sim}{r}_2}{2}, \quad \underset{\sim}{s} = 1/2\,(2\underset{\sim}{r}_1 - \underset{\sim}{r}_2 - \underset{\sim}{r}_3),$$

and eliminating the center of mass coordinate R, we rewrite Eq. (21) as

$$\left\{ -\frac{3}{4M}\nabla^2_s - \frac{1}{4M}\nabla^2_r + V(|\underset{\sim}{s} - \underset{\sim}{r}|) + V(|\underset{\sim}{s} + \underset{\sim}{r}|) + V(|2\underset{\sim}{r}|) \right\} \psi = E\psi \quad . \qquad (22)$$

We shall not solve Eq. (22). Rather in order to search for any differences in the form factor behavior between the two and three body cases we shall confine our attention to two body potentials that increase as $1/d^4$ as the separation between each pair of particles $d \to 0$.

Furthermore since this case is not exactly soluble, we assume that the asymptotic behavior of the wave function is of the form:

$$\psi = e^{-a\left(\frac{1}{2r} + \frac{1}{|\underset{\sim}{s} - \underset{\sim}{r}|} + \frac{1}{|\underset{\sim}{s} + \underset{\sim}{r}|}\right)} f(\underset{\sim}{r},\underset{\sim}{s}) \tag{23}$$

where f(r,s) is some less singular function. This corresponds to a two body potential $V \propto \frac{1}{r^4}$ plus three-body terms of the form

$$\frac{a^2}{(|\underset{\sim}{r} + \underset{\sim}{s}|)^2(|\underset{\sim}{r} - \underset{\sim}{s}|)^2}$$

in addition to less singular terms. The three-body terms are as singular as the two-body terms only in the limit $r = s = 0$; therefore there is at least some justification for assuming that this sort of wave function has about the right form to solve the most singular part of the problem. In any case, we shall attempt to find the corresponding asymptotic behavior of the form factor in order to obtain some insight into the actual behavior in the three-body case.

Since we have a bound state we may assume that for large separations the function f takes the form

$$\begin{aligned} f &\approx e^{-2\mu r - \mu|\underset{\sim}{r} - \underset{\sim}{s}| - \mu|\underset{\sim}{r} + \underset{\sim}{s}|} \\ &\equiv e^{-2\varphi(\underset{\sim}{r},\underset{\sim}{s})} \quad . \end{aligned} \tag{24}$$

We approximate f by this form everywhere. We must evaluate the following integral for the form factor

$$F = \int d^3s\, e^{i\frac{2}{3}\underset{\sim}{q}\underset{\sim}{s}}\, d^3r\, e^{-2a\left\{\frac{1}{2r} + \frac{1}{|\underset{\sim}{s} - \underset{\sim}{r}|} + \frac{1}{|\underset{\sim}{s} + \underset{\sim}{r}|}\right\} - 2\mu\varphi(\underset{\sim}{s},\underset{\sim}{r})} \tag{25}$$

in the limit $q \to \infty$.

We can immediately perform the integrals over three of the angles. This gives

$$F(q) = - \mathrm{Re}\, \frac{1}{q}\frac{d}{dq}\left(\frac{3}{2}\right)^2 (2\pi)^2 \int_0^\infty ds \int_0^\infty dr r^2 \int_{-1}^{1} dz \exp\left[iq\frac{2}{3}s \right.$$
$$\left. - 2a\left\{\frac{1}{2r} + \frac{1}{\sqrt{r^2 - 2szr + s^2}} + \frac{1}{\sqrt{r^2 + 2szr + s^2}}\right\} - \eta(r,z,s)\right] \qquad (26)$$

where $\eta(r,z,s) \equiv 2\mu\varphi(\underset{\sim}{r},\underset{\sim}{s})$, $z = \frac{\underset{\sim}{r}\cdot\underset{\sim}{s}}{r\ s}$.

We can now use theorem A of Section II to show that in this case again the repulsive core potential gives rise to a form factor which decreases more rapidly than any power of q as $q \to \infty$. To do this we write

$$F(q) \sim \frac{1}{q}\int_0^\infty ds \sin\frac{2}{3} qs \left[\int_{-1}^{1} dz \int_0^\infty dr r^2 s \right.$$
$$\left. \times\, e^{-2a\left(1/2r - \frac{1}{\sqrt{r^2 + 2srz + s^2}} - \frac{1}{\sqrt{r^2 - 2srz + s^2}}\right) - \eta(r,s,z)}\right] \qquad (27)$$

and observe that the function in brackets (corresponding to g in the theorem) is an odd function of s and that all of its derivatives exist at s = 0, since

$$\int_0^\infty dr \frac{r^2}{r^p}\ e^{-5a/r}\ e^{-8\mu r}$$

exists for all p. Thus for all n, the $g^{(n)}(0)$ exist, and $g^{(2n)}(0) = 0$ so that the theorem indeed applies.

Having assured ourselves that we have faster-than-power fall-off, we now turn to the problem of computing the actual asymptotic behavior.

None of the three integrals in F(q) can be performed explicitly. We dismiss the z integration by noting that it is over a finite integral $(-1 \leq z \leq +1)$ and that integrand is positive definite and smoothly varying in this interval. Hence, in the interest of simplifying the algebra and with no further justification, we set the z integrand equal to its maximum (i.e., $z = 0$) and write

$$F(q) \alpha \frac{1}{q} \frac{d}{dq} \mathrm{Re} \int_0^\infty drr^2 \, e^{-a/r} \int_0^\infty ds \, e^{i\frac{2}{3}qs - \frac{4a}{\sqrt{r^2 + s^2}} - \eta} \quad . \tag{28}$$

Here we have made use of the absolute convergence insured by η to exchange orders of integration.

The s integration can now be performed by the saddle point method yielding the asymptotically leading term for $q \to \infty$. First we scale the coordinates: let $k = \frac{8}{3} aq$ then

$$s \to s \sqrt{\frac{2a}{\frac{2}{3} q}} = \frac{4as}{\sqrt{k}} \; ; \quad r \to \frac{4ar}{\sqrt{k}}$$

so that

$$F(q) \alpha - \frac{1}{k} \frac{d}{dk} \frac{1}{k^2} \mathrm{Re} \int_0^\infty drr^2 e^{- k/4r} \int_0^\infty ds \, e^{\sqrt{k} \;\; is - \left[\frac{1}{r^2 + s^2}\right] - \frac{4a\eta}{\sqrt{k}}} \quad . \tag{29}$$

For fixed r, the s integral is in the standard form for evaluation by steepest descents as done in the last section in Eqs. (14)-(17). However, the subsequent r integration extends over the entire range

$0 \leq r \leq \infty$, and it is not apparent from the form of Eq. (29) that the variable r does not itself grow $\sim\sqrt{k}$ or larger and so spoil the asymptotic approximation. One way to verify that this region of $r \gtrsim \sqrt{k}$ does not affect the leading term in the asymptotic series for F(q) given by the method of steepest descent is as follows:

We can write the s integral as

$$\mathrm{Re} \int_0^\infty e^{\sqrt{k}\left[is - \frac{1}{\sqrt{r^2 + s^2}}\right] - \frac{4a\eta}{\sqrt{k}}} ds$$

$$= \frac{1}{2} e^{-\sqrt{k}r} \int_{-\infty}^\infty e^{\sqrt{k}\left[is - \frac{1}{\sqrt{s^2 + 2irs}}\right] - \frac{4a\eta(s + ir, r, o)}{\sqrt{k}}} ds \tag{30}$$

where we have used the fact that the exponent has a strip of analyticity $-ir \leq \mathrm{Im}s \leq ir$ to translate the contour as in theorem B of Section II. The Riemann-Lebesgue lemma implies that the integral in Eq. (30) vanishes as k approaches infinity. For $r \geq \sqrt{k}$ the exponential factor outside the integral is less than e^{-k}. Hence the region $r \geq k$ can be ignored in comparison with the other region to which we now turn.

The saddle point of the exponent of the s integral in Eq. (29) occurs at s_o where

$$is_o = (r^2 + s_o^2)^{\frac{3}{2}} \quad . \tag{31}$$

There are three roots of this equation consistent with the branch $\sqrt{s^2} = s$ for s real and positive. They are:

$$s_o^{(1)} = -i \left\{(A + B)\right\}^{\frac{3}{2}} \tag{32}$$

where the positive square root is taken;

$$s_o^{(2)} = -i\left\{-\frac{1}{2}(A+B) + i\frac{\sqrt{3}}{2}(A-B)\right\}^{\frac{3}{2}} \tag{33}$$

where the root lying in the upper half plane is taken, and

$$s_o^{(3)} = -i\left\{-\frac{1}{2}(A+B) - i\frac{\sqrt{3}}{2}(A-B)\right\}^{\frac{3}{2}} \tag{34}$$

where the root in the lower half plane is taken.

A and B are given by

$$\begin{aligned} A &= \left\{\sqrt{\frac{1}{27} + \left(\frac{r^2}{2}\right)^2} + \frac{r^2}{2}\right\}^{\frac{1}{3}} \\ B &= -\left\{\sqrt{\frac{1}{27} + \left(\frac{r^2}{2}\right)^2} - \frac{r^2}{2}\right\}^{\frac{1}{3}} \end{aligned} \tag{35}$$

The positions of the roots are shown in Fig. 6. We must choose the root in the first quadrant, $s^{(2)}$, for the reasons given in Section II. Then the form factor is

$$F(q) \sim \frac{1}{k}\frac{d}{dk}\frac{1}{k^{9/4}} \operatorname{Re}\int_0^\infty r^2\, dr\, e^{\sqrt{k}\left[is_o + i\frac{\left(r^2 + s_o^2\right)}{s_o} - \frac{1}{4r} - \frac{4a\eta(s_o, r, o)}{k}\right]} G(r) \tag{36}$$

where

$$G(r) \equiv \left\{\left[\frac{d^2}{ds^2}\left(is - \frac{1}{\sqrt{r^2 + s^2}}\right)\right]_{s = s_o(r)}\right\}^{-1/2} .$$

is a slowly varying function of r, and s_o is given by Eqs. (33) and (35).

In principle, we could once more resort to the method of steepest descents to find the asymptotic form for the remaining integral. However, in practice, the algebra is too complicated to enable us to locate the saddle point exactly. Therefore, we restrict ourselves to the easier task of finding the location approximately which suffices for our purposes.

Recalling from our discussion of Eq. (30) that the region of large r is unimportant, we expand the exact expression for s_o and for the exponential factor in Eq. (36) for $r < 1$. We get

$$s_o \underset{r<1}{\approx} \frac{1}{\sqrt{2}} \left[(1 + i) - \frac{3}{4} (1 - i) r^2\right] \tag{37}$$

and the exponential factor

$$\begin{aligned} w(s_o, r) &\equiv i\, s_o + \frac{i(r^2 + s_o^2)}{s_o} - \frac{1}{4r} \\ &\approx -\frac{1}{4r} - \sqrt{2}\,(1 - i) - \frac{(1 + i)}{2\sqrt{2}}\, r^2 \quad . \end{aligned} \tag{38}$$

Using this approximation, we find the only relevant saddle point to be at

$$r_o = 4^{-\frac{1}{3}} e^{-i\frac{\pi}{12}} = (0.61) - (0.16)\, i \quad . \tag{39}$$

This gives

$$F(q) \propto \frac{1}{K} \frac{d}{dk} \operatorname{Re} P(k)\, e^{w(s_o,\, r_o)\sqrt{k}} \tag{40}$$

where $$k = \frac{8}{3} aq \tag{41}$$

and $$w(s_o, r_o) \approx - 1.98 + (1.26) i$$

or $$F(q) \propto \frac{1}{k} \frac{d}{dk} e^{-1.98 \sqrt{k}} \operatorname{Re}\left(P(k)\, e^{-i(1.26) \sqrt{k}}\right) \quad . \tag{42}$$

We have confirmed this result by using a computer (Burroughs B5500) to plot the real and imaginary parts of $w(s_o, r)$ and then locating the saddle point. The saddle point is at $r_o = .651 - .126\, i$, and $w(s_o, r_o)$ is found to be $-1.979 + 1.285\, i$ which is in good agreement with Eq. (41).

IV. CONCLUSIONS

In conclusion we may attempt to assess the value of constructing asymptotic forms such as Eqs. (19) and (20) for comparison with elastic electron deuteron scattering data. At what values of q do we arrive at the asymptotic region, for example?

A priori we can give no answer. In practice, we look at the data in search of clear evidence that one or another of the asymptotic behaviors is emerging. This is not easy to do and requires a very broad range of accurate data. From Fig. 3 it appears that power, exponential, and fractional exponential fall offs have grossly different appearances. The situation is much less clear if we remove the arbitrary restriction that the curves go through two specific points in common and demand instead that we construct the best fit to the data available. The latest proton form factor data can be plotted over several decades of values and as is seen in the three graphs of Fig. 7 is fully compatible with all three asymptotic decay laws of $\frac{1}{q^4}$, e^{-aq}, and $e^{-b\sqrt{q}}$.

For the deuteron the detailed functional forms of elaborate theoretical models have no difficulty in reproducing accurately the data over more than two decades. Now for the first time the simple exponential form e^{-aq} appears to be inadequate. The latest precision data of Friedman, Hartmann, and Kendall, when added to all earlier work on the deuteron form factor, shows positive curvature in a logarithmic scale in Fig. 2.

The form of Eq. (20) is free of all details of the deuteron bound state and depends only on the nature of the assumed repulsive core singularity as $r \to 0$. For it to be of any real value we must be able to demonstrate that the function

$$\frac{\ell n \left[F(q)/\sin\left(A_{p,n} q^{p/p+1} + \varphi_{p,n}\right)\right]}{q^{p/p+1}} \tag{43}$$

$$\left(\text{where } A_{p,n} = \frac{p+1}{p} \sin\left(\frac{\pi}{2}\,\frac{2+p}{1+p}\right) \left(2r_o\right)^{1/1+p}, \quad \text{and } \varphi_{p,n} = \frac{\pi}{4}\,\frac{p+4n}{1+p}\right)$$

approaches a horizontal straight line for some values of $p > 0$, n, and r_o. We expect that the length r_o should be approximately 0.3f to 0.4f, corresponding to a repulsive core that is neither too large to interfere with low energy deuteron parameters nor too small to prevent the data from being in the asymptotic region of $q\, r_o > 1$ for $q \sim 4f^{-1}$ to $5f^{-1}$. The normalization constant of Eq. (20) does not appear in Eq. (43). The polynomial power n will be important in collaboration with the parameters r_o and p in locating the nodes of F at the appropriate q values for experimental agreement. Thus far there is no evidence for the oscillating behavior of the sine factor but further extension of observations to larger q values is needed. At present the nodes can be avoided successfully as apparent from the dotted curve of Fig. 7c. In any event it will be of great interest to see whether nodes appear in the nucleon or deuteron form factor as the momentum transfer increases.

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FOOTNOTES AND REFERENCES

1. In this paper we take $\hbar = c = 1$, and $q = |\underset{\sim}{q}|$ is the magnitude of the three momentum transfer in the center of mass system.

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16. In particular, $n = (p + 1)/2$ removes the most singular term in $[-\nabla^2 + MV(r)]\, u(r)$ for $V = \frac{1}{r_o^2}\left(\frac{r_o}{r}\right)^{2(1 + p)}$.

FIGURE CAPTIONS

Fig. 1 - The proton magnetic form factor from recent experiments at DESY. The works referred are respectively those of H. J. Behrend, F. W. Brasse, J. Engler, H. Hultschig, S. Galster, G. Hartwig, H. Schopper, and E. Ganszuage, submitted to XIII International Conference on High Energy Physics, Berkeley (1966); W. Albrecht, H. J. Behrend, F. W. Brasse, W. Flauger, H. Hultschig, and K. G. Steffen, submitted to XIII International Conference on High Energy Physics, Berkeley (1966); W. Bartel, B. Dudelzak, H. Krehbiel, J. M. McElroy, U. Meyer-Berkhout, R. J. Morrison, H. Nguyen-Ngoc, W. Schmidt, and G. Weber, submitted to XIII International Conference on High Energy Physics, Berkeley (1966); and Phys. Rev. Letters 17, 608 (1966).

Fig. 2 - The deuteron form factor. The points are due to: J. A. McIntyre and S. Dhar, Phys. Rev. 106, 1074 (1957); J. I. Friedman, H. W. Kendall, and P. A. M. Gram, Phys. Rev. 120, 992 (1960); E. F. Erickson, Nucleon Structure: Proceedings of the International Conference at Stanford University (1963); D. J. Drickey and L. N. Hand, Phys. Rev. Letters 9, 521 (1962); D. Benaksas, D. Drickey, and D. Frerejacque, Phys. Rev. Letters 13, 353 (1964); C. D. Buchanan and M. R. Yearian, Proceedings of the International Symposium on Electron and Photon Interactions at High Energies, Deutsche Physikalische Gesellschaft e.V. Hamburg (1966); and J. Friedman, G. Hartmann, H. Kendall, and Leon Van Speybroeck, submitted to XIII International Conference on High Energy Physics, Berkeley (1966), and private communication.

Fig. 3 - Logarithmic plots of e^{-aq}, $e^{-b\sqrt{q}}$ and q^{-c} with a, b, and c chosen so that the values of the functions coincide at 1 and at 5.

Fig. 4 - Infinite strip of analyticity of width 2a about the real axis in the complex r plane.

Fig. 5 - Displaced integration contour discussed below Eq. (7).

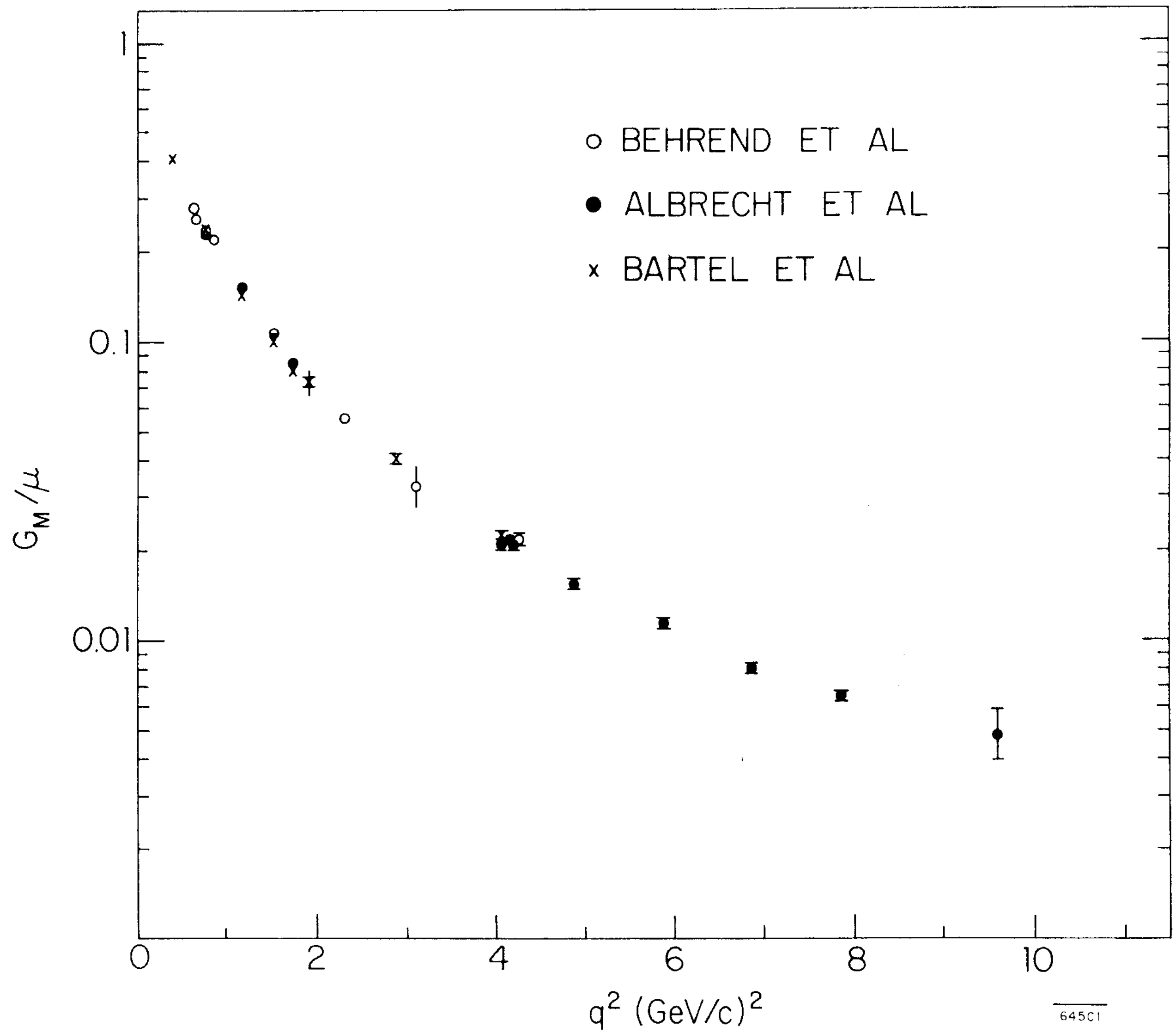
Fig. 6 - The trajectories of the three roots s_o of Eq. (31) as a function of increasing, real r. Each trajectory starts at r = 0. The dashed lines represent Re $s_o = \pm$ Im s_o.

Fig. 7 - Best fits to the proton form factor of the type discussed

a. Fit to a double pole (plotted vs. q^2).

b. Fit to e^{-aq} (plotted vs. q).

c. Fit to $e^{-\sqrt{q/q_0}}$ (plotted vs. $\sqrt{q}$). Also shown are dashed lines representing different fits with the oscillatory factor shown.



BEHREND ET AL
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1
0.1
0.01
G_M/μ
0
2
4
6
8
10
q^2 (GeV/c)^2
645C1

FIG. 1

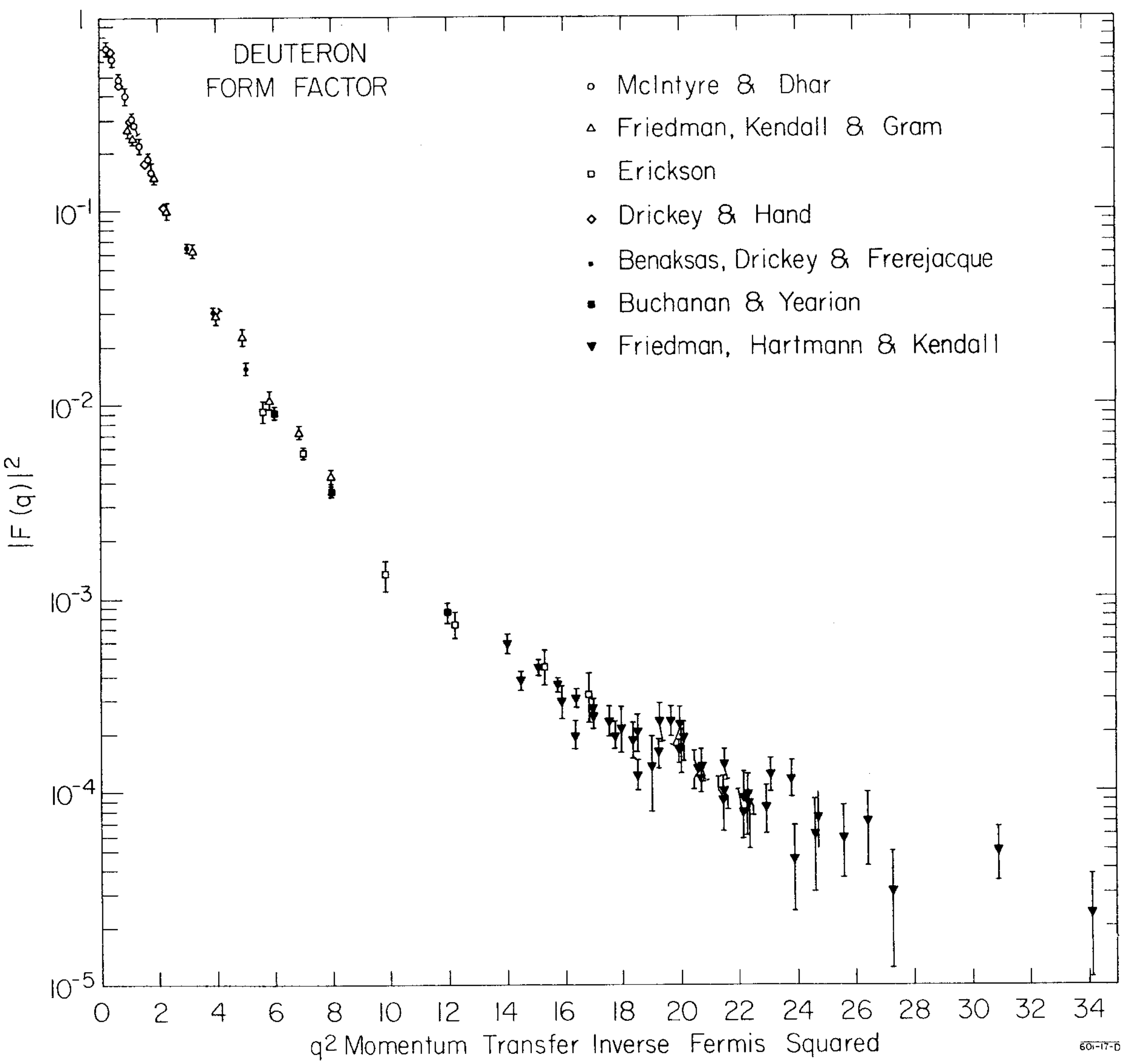
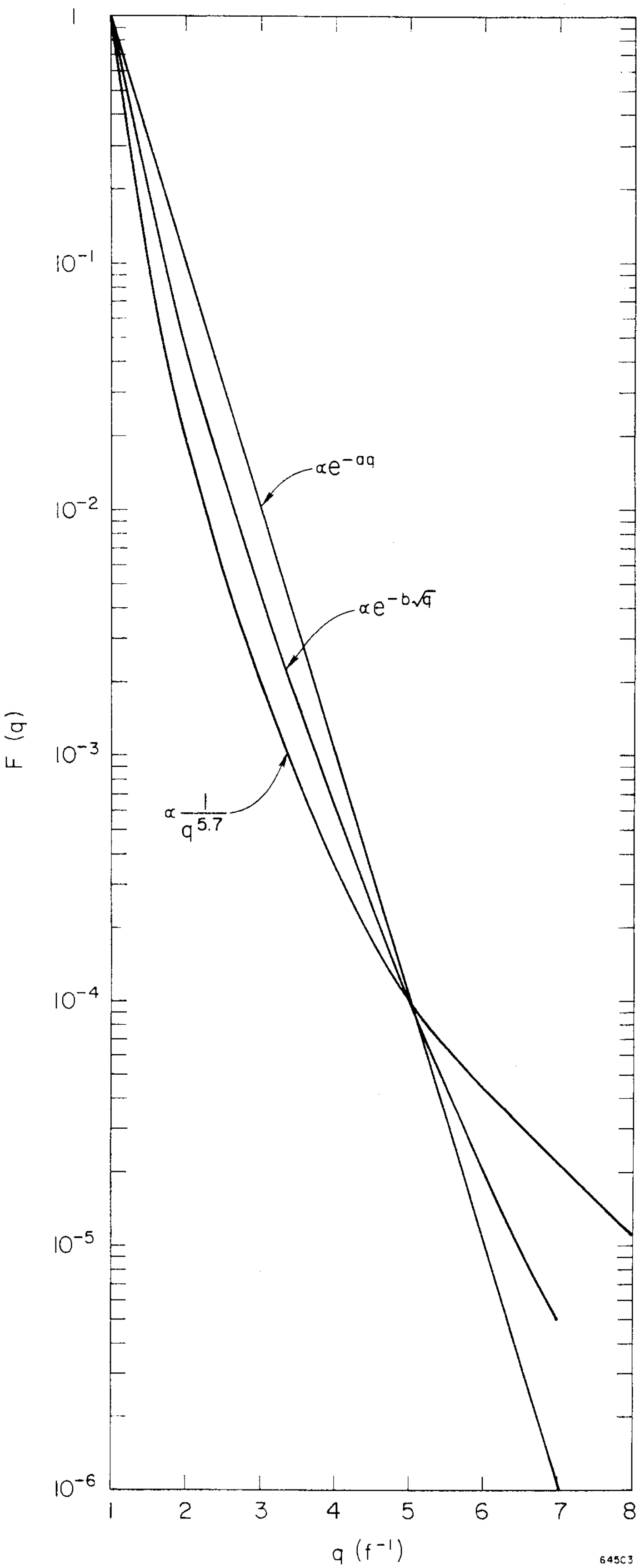

DEUTERON
FORM FACTOR
McIntyre & Dhar
Friedman, Kendall & Gram
Erickson
Drickey & Hand
Benaksas, Drickey & Frerejacque
Buchanan & Yearian
Friedman, Hartmann & Kendall
$|F(q)|^2$
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34
q^2 Momentum Transfer Inverse Fermis Squared
601-17-B

FIG. 2



1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
F (q)
αe^{-aq}
$\alpha e^{-b\sqrt{q}}$
$\alpha \frac{1}{q^{5.7}}$
1 2 3 4 5 6 7 8
q (f^{-1})
645C3

FIG. 3

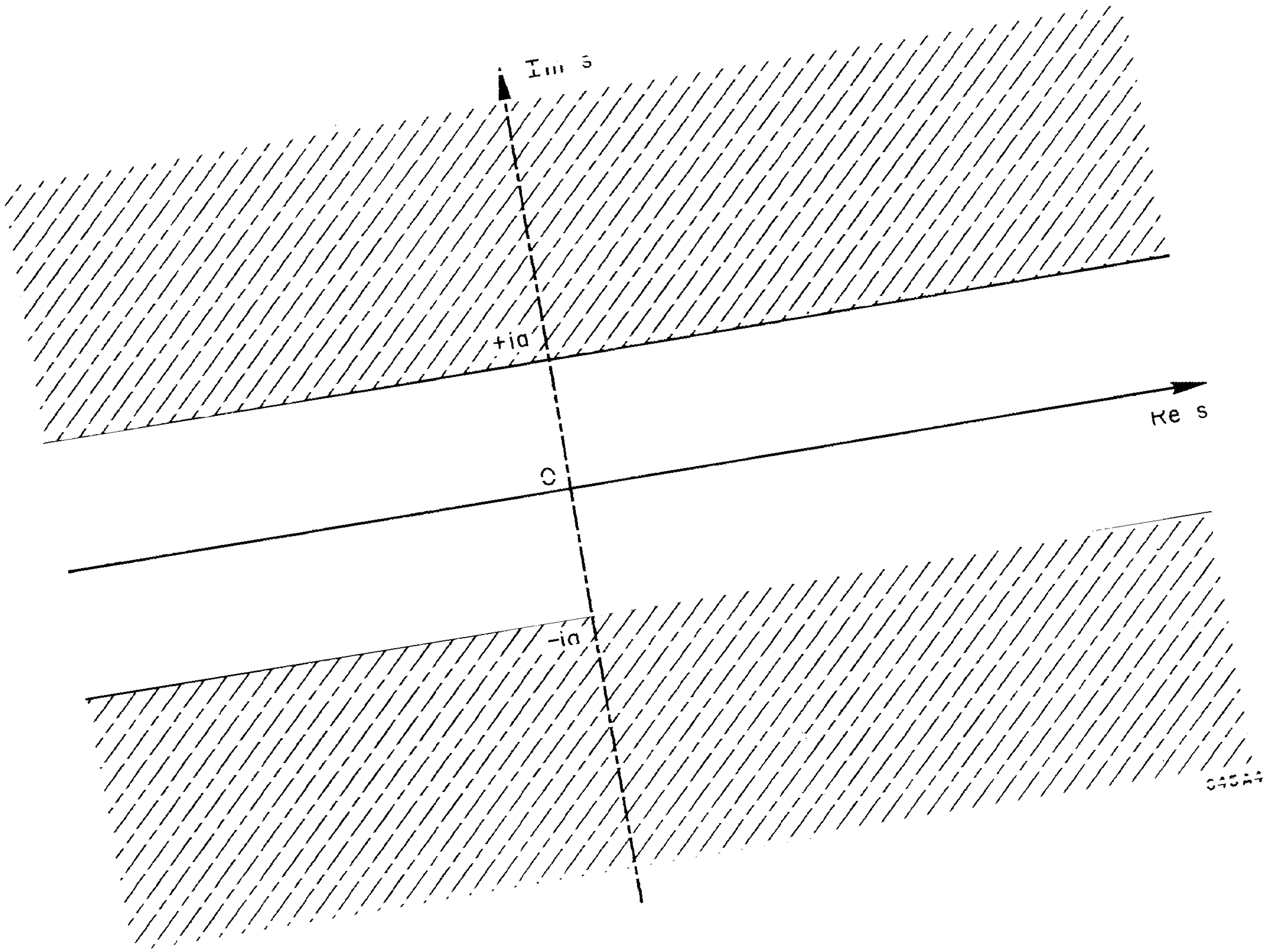


FIG. 4

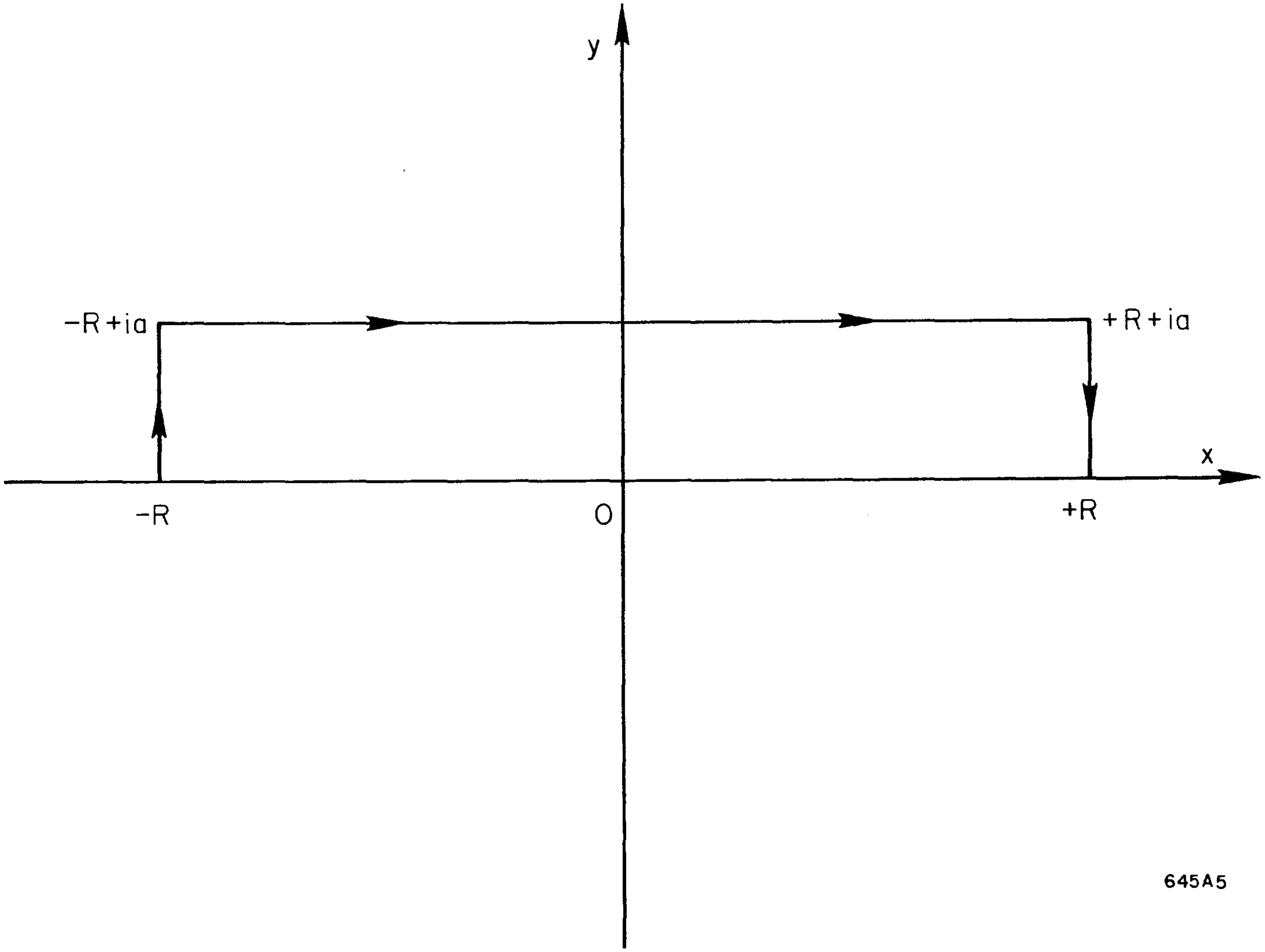


FIG. 5

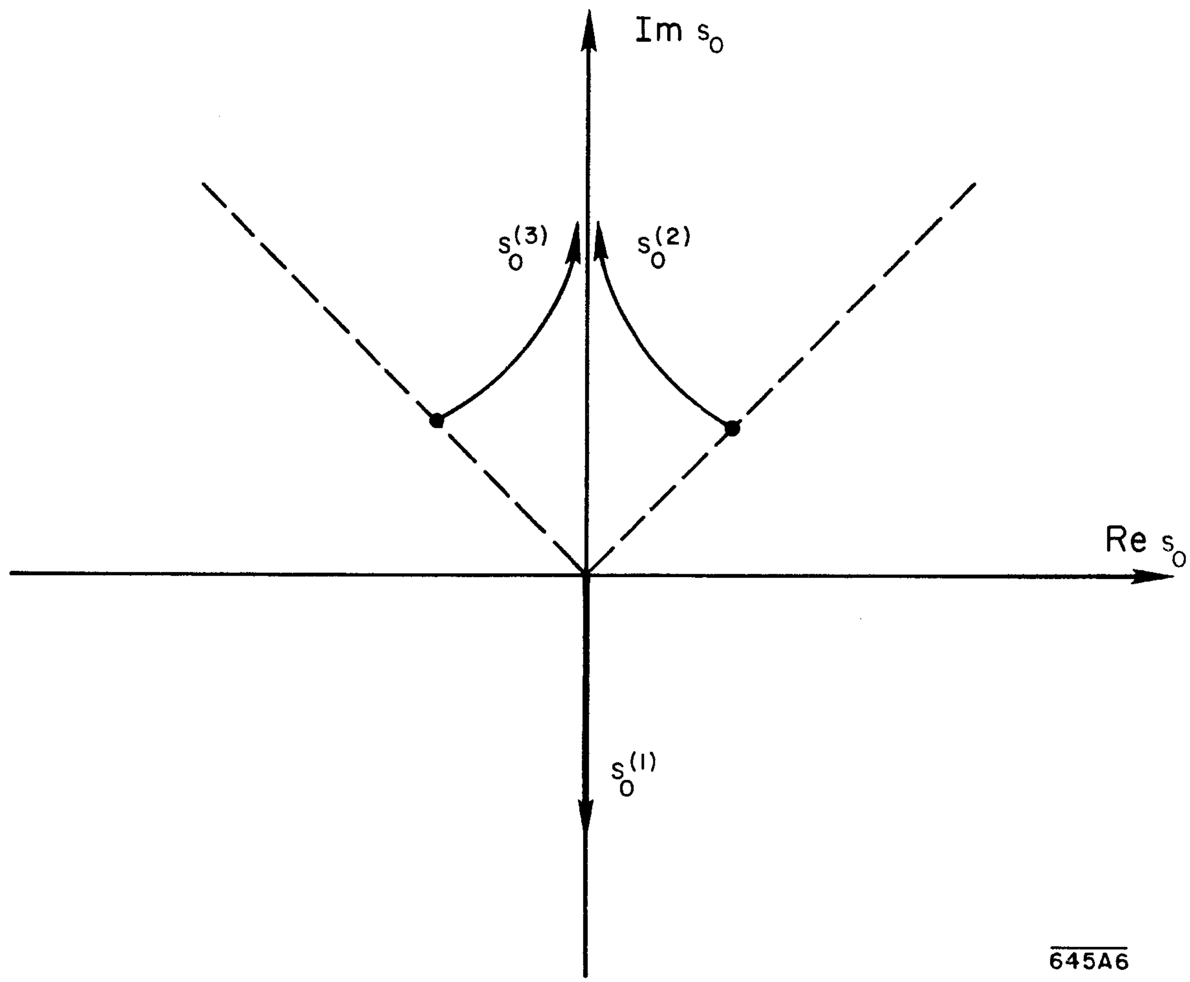
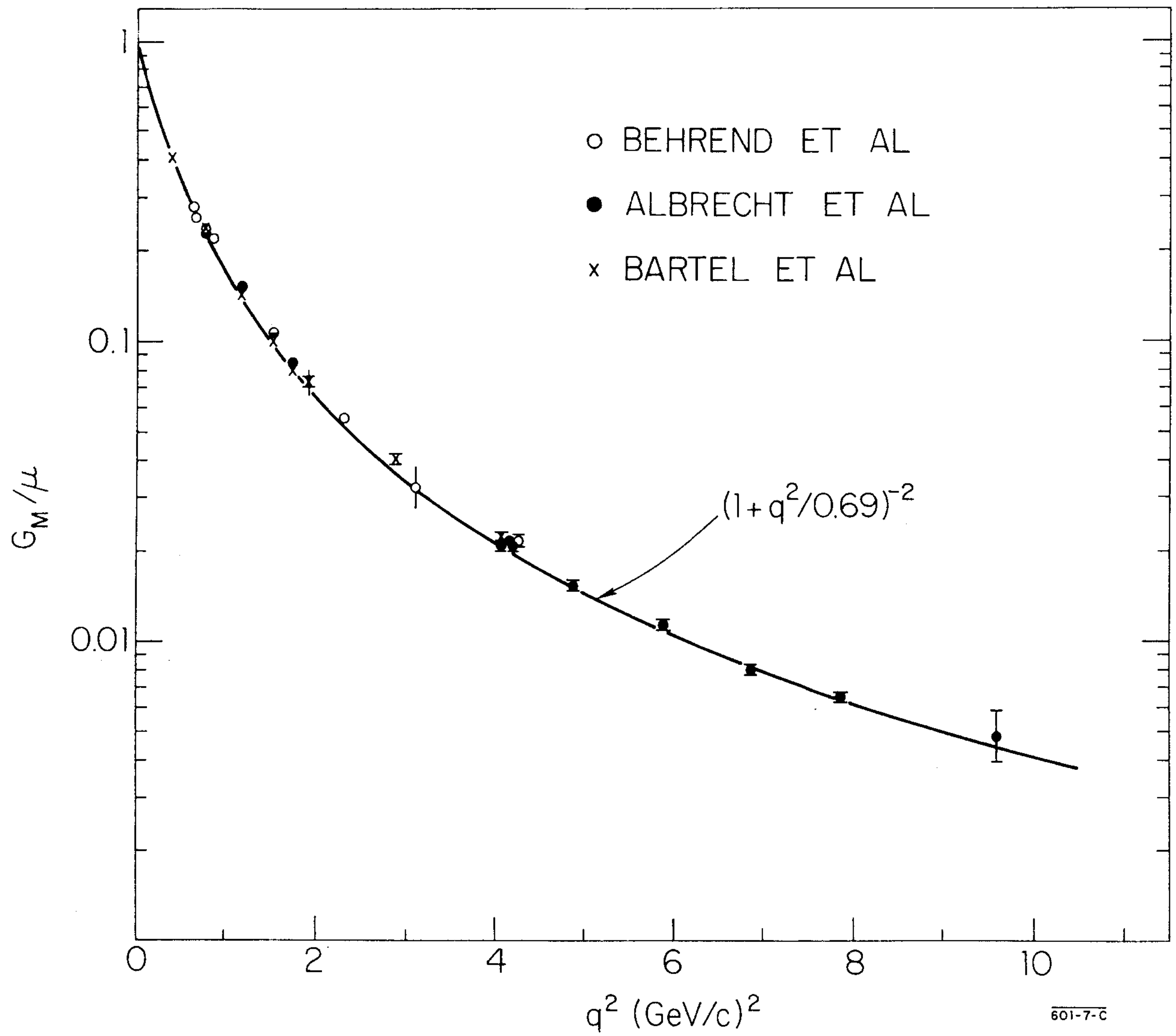
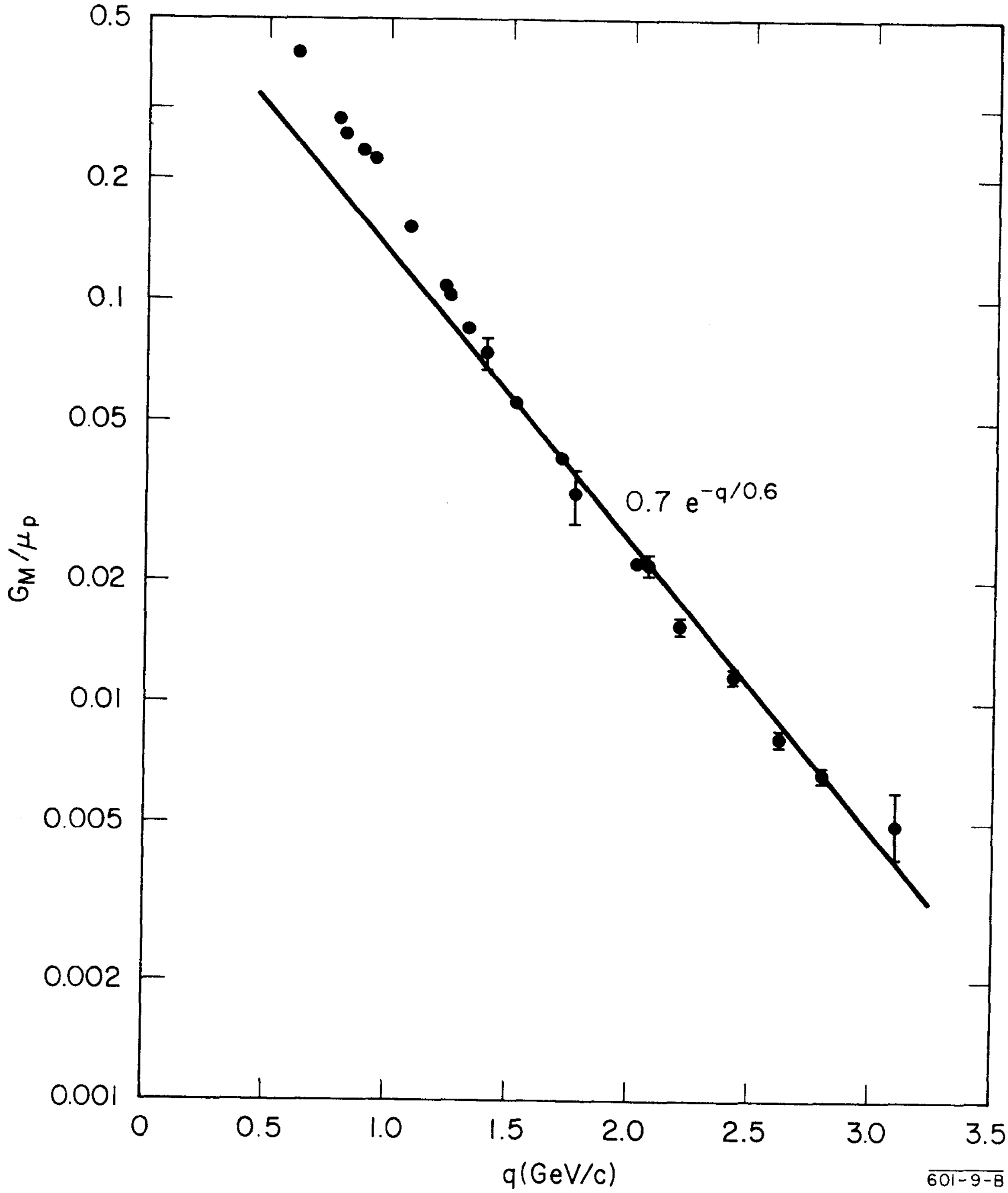


FIG. 6



○ BEHREND ET AL
● ALBRECHT ET AL
x BARTEL ET AL
$(1+q^2/0.69)^{-2}$
G_M/μ
1
0.1
0.01
0
2
4
6
8
10
q^2 (GeV/c)^2
601-7-C

FIG. 7a



0.5
0.2
0.1
0.05
0.02
0.01
0.005
0.002
0.001
G_M/μ_p
$0.7\ e^{-q/0.6}$
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
q(GeV/c)
601-9-B

FIG. 7b

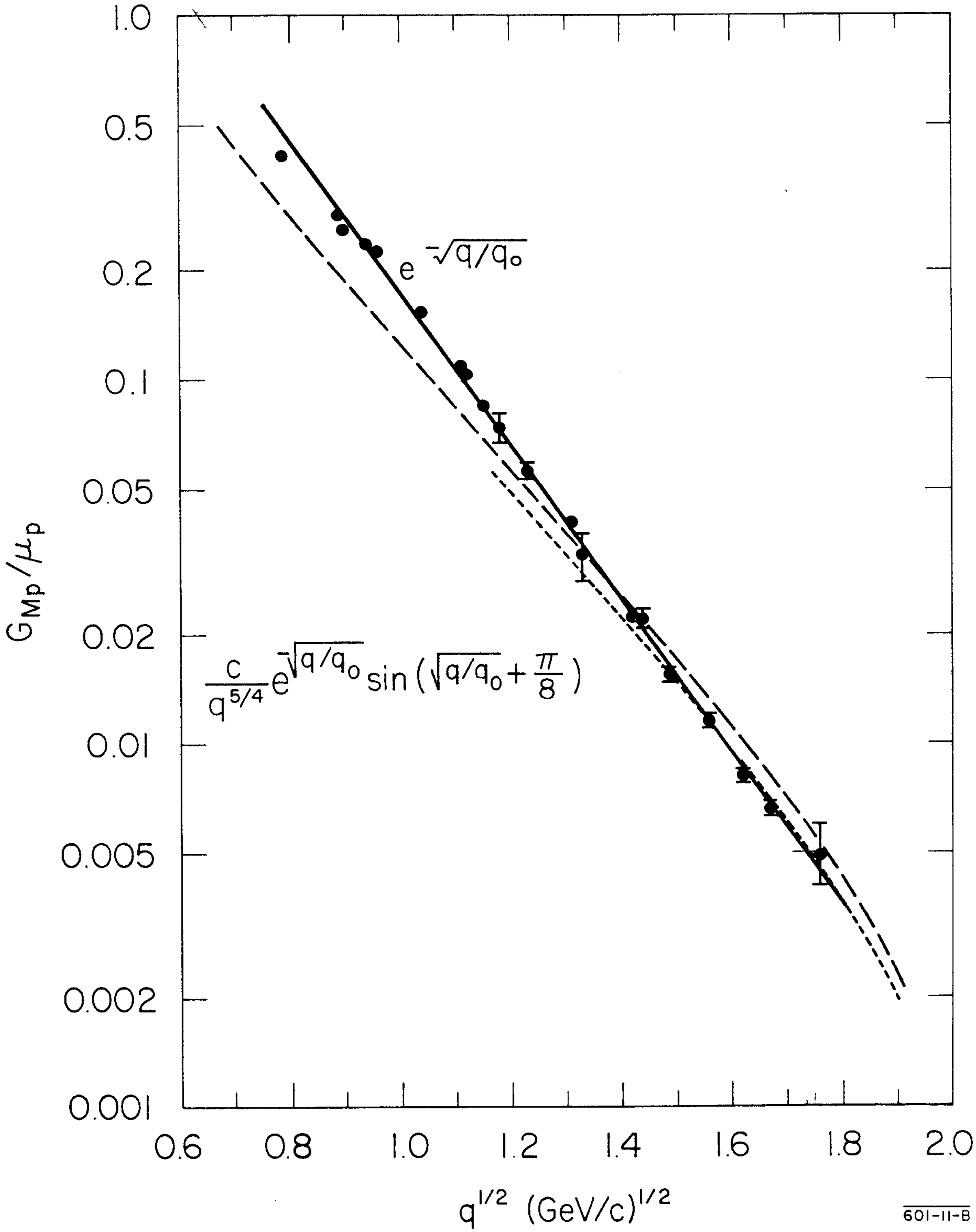

1.0
0.5
0.2
0.1
0.05
0.02
0.01
0.005
0.002
0.001
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
G_{Mp}/μ_p
$q^{1/2}$ (GeV/c)$^{1/2}$
$e^{-\sqrt{q/q_0}}$
$\frac{c}{q^{5/4}} e^{-\sqrt{q/q_0}} \sin(\sqrt{q/q_0} + \frac{\pi}{8})$
601-11-B

FIG. 7c