General Relativity and Gravitation (2011)
DOI 10.1007/s10714-009-0846-2

ESSAY AWARDED BY THE GRAVITY RESEARCH FOU

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On vacuum density, the initial singularity and dark energy

Abstract Standard cosmology poses a number of important questions. Apart from its singular origin, it possesses early and late accelerating phases required to account for observations. The vacuum energy has been considered as a possible way to resolve some of these questions. The vacuum energy density induced by free fields in an early de Sitter phase has earlier been estimated to be proportional to H^4, while more recently it has been suggested that the QCD condensate induces a term proportional to H at late times. These results have been employed in models which are non-singular and inflationary at early times and accelerating at late times. Here we cast these models in terms of scalar fields and study the corresponding spectrum of primordial perturbations. At early times the spectrum is found to be not scale-invariant, thus implying that slow-roll inflation is still required after the phase transition induced by the vacuum. At late times the corresponding scalar-field potential is harmonic, with a mass of the order of the Hubble scale, a result that may be understood in the light of the holographic conjecture.

Keyword Cosmology,

Second Award in the 2009 Essay Competition of the Gravity Research Foundation

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Recent years have witnessed a tremendous accumulation of high resolution data in cosmology. This has led to the so called standard model, which poses a number of important questions. Apart from having a singular origin, at which laws of physics break down, it also possesses accelerating phases at early and late times. This has resulted in numerous attempts at generalizing the gravitational and energy sectors of GR in order to account for these features. Interestingly all viable models have proved to be very close to the ΛCDM.

An important ingredient expected from quantum considerations is the vacuum energy, which has often been thought to be responsible for the origin of the cosmological parameter Λ. As is well known, it is not easy to obtain the vacuum density in curved spacetimes, even in the simple case of a scalar field [1; 2; 3]. An exception is the case of fields with conformal invariance in de Sitter spacetime [2; 3], for which the renormalized vacuum density has been found to be proportional to H^4, where H is the Hubble parameter. In a general FLRW background, some ambiguities can be fixed by imposing the conservation of the vacuum energy-momentum tensor [3]. This allows a general expression for the vacuum density to be derived, which was in fact originally used by Starobinsky in his pioneering inflationary model [4].

Recently, the de Sitter ansatz $\Lambda \propto H^4$ was employed in a quasi-de Sitter setting in order to construct a non-singular scenario, with a phase transition between a past-eternal de Sitter phase and a radiation-dominated epoch [5]. An underlying assumption was that a dynamical vacuum interacts with matter, since only the conservation of the total energy-momentum tensor is implied by Einstein equations. Thus in this model the decaying vacuum acts as a source of relativistic matter during the expansion.

On the other hand, it has been suggested that the QCD condensate induces a vacuum density proportional to H at late times [6; 7]. This has also been employed to construct models to account for the late acceleration of the universe.

In this paper we formulate these ansatz in terms of self-interacting scalar fields in a spatially flat FLRW spacetime.[1] In this framework the Lagrangian takes the usual form

$$\mathscr{L} = \sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right], \tag{1}$$

with the corresponding field equations given by

$$3H^2 = V + 2H'^2, \tag{2}$$

$$\dot{\phi} = -2H'. \tag{3}$$

Here the prime and the dot denote derivatives with respect to ϕ and the cosmological time t, respectively.

We begin with the early phase and consider a potential proportional to H^4, which has a maximum in the asymptotic de Sitter limit. The energy density and pressure of this field are given by $\rho = V + \dot{\phi}^2/2$ and $p = -V + \dot{\phi}^2/2$, respectively. Therefore, we can interpret the material content as formed by a vacuum term with density V and pressure $-V$, plus a stiff fluid with density and pressure given by $\dot{\phi}^2/2$. As the transition from the de Sitter phase takes place, with the field rolling

[1] This procedure can similarly be applied to other ansatz.

Fig. 1 The scalar field potential corresponding to Eq. (4) as a function of $\phi(\sigma = 1)$

Fig. 2 The Hubble parameter as a function of time ($\sigma = 1$)

down the potential, the energy stored in the vacuum term is transferred to the stiff component, and eventually converted into matter fields by a suitable coupling, as in the usual reheating mechanism.

With the ansatz

$$V = 3\sigma H^4, \tag{4}$$

where σ is a positive constant, it is straightforward, using Eqs. (2) and (3), to find the solution

$$H = \frac{2\,e^{\sqrt{3/2}\phi}}{1+\sigma e^{\sqrt{6}\phi}}. \tag{5}$$

The corresponding potential, with $\sigma = 1$, is shown in Fig. 1.

Also, using $\dot{H} = H'\dot{\phi}$ in Eqs. (2) and (3), we obtain an evolution equation for H,

$$\dot{H} + 3H^2 - 3\sigma H^4 = 0. \tag{6}$$

This equation has the equilibrium point $\sigma H^2 = 1$, which corresponds to a de Sitter universe with energy scale $\sigma^{-1/2}$ in Planck units. This solution is, however, unstable, as indicated by the solution

$$\tilde{t} = \frac{1}{\tilde{H}} - \tanh^{-1}\tilde{H}, \tag{7}$$

where we have introduced the re-scaled quantities $\tilde{H} = \sqrt{\sigma}H$ and $\tilde{t} = 3t/\sqrt{\sigma}$, and have conveniently chosen the integration constant.

The time evolution of $\tilde{H}$ is depicted in Fig. 2. As $t \to -\infty$ the solution approaches the de Sitter point, with $\sigma H^2 = 1$, as expected. The solution remains quasi-de Sitter during an indefinitely long time. Then, around $t = 0$, it goes through a phase transition, with H and V decaying very fast. It is worth noting that for $\sigma > 1$ we have $H < 1$ during the entire evolution, thus we never enter a trans-Planckian regime, where the semi-classical, one-loop derivation of the vacuum density would not be valid.

This model has a number of appealing features. In addition to being non-singular, the presence of a quasi-de Sitter phase solves some of the problems usually addressed by inflation, such as the horizon and flatness problems. However, the most important outcome of an inflationary phase is to produce a scale invariant spectrum of primordial perturbations. To be viable as an inflationary scenario, it is therefore important to check whether the above model can also produce such a scale-invariant spectrum.

The evolution equation for perturbations in the scalar field is given by [8]

$$\delta\phi^{**} + 2aH\delta\phi^{*} + (k^2 + a^2V'')\delta\phi = 0, \tag{8}$$

where a is the scale factor, k is the co-moving wave number of a given mode, and * denotes a derivative with respect to the conformal time η.

From (4), (5) and (3) it is possible to obtain

$$V'' = 18\sigma H^4(4-5\sigma H^2), \tag{9}$$

$$\varepsilon = 3(1-\sigma H^2), \tag{10}$$

with $\varepsilon = d(H^{-1})/dt$. In the quasi-de Sitter phase we have $\sigma H^2 \approx 1$ and, hence, $V'' \approx -18H^2$ and $\varepsilon \ll 1$. By using $Ha \approx -1/\eta$ and introducing $\delta\tilde{\phi} = a\delta\phi$, the perturbation Eq. (8) can be rewritten as

$$\delta\tilde{\phi}^{**} + \left(k^2 - \frac{20}{\eta^2} + \frac{30\varepsilon}{\eta^2}\right)\delta\tilde{\phi} = 0. \tag{11}$$

Neglecting the term in ε, the appropriately normalized solution has the form

$$\delta\tilde{\phi} \approx \frac{\sqrt{-\pi\eta}}{2} H^{(1)}_{\frac{9}{2}}[-k\eta], \tag{12}$$

where $H^{(1)}_n$ is the Hankel function of the first kind. For $k \to \infty$, this solution reduces to $e^{-ik\eta}/\sqrt{2k}$, as expected. On the other hand, for $k\eta = -1$, when the mode crosses the horizon, we have $|\delta\tilde{\phi}| \approx 10^2/\sqrt{2k}$.

The power spectrum of scalar perturbations in the metric is given by [8]

$$P_\Psi = \frac{4}{9}\left(\frac{aH}{\phi^*}\right)^2 |\delta\phi|^2_{k\eta=-1}. \tag{13}$$

Using (12) and $(aH/\phi^*)^2 = (2\varepsilon)^{-1}$, we have

$$k^3 P_\Psi \approx \frac{10^4 H^2}{9\varepsilon}, \tag{14}$$

with the right-hand side evaluated at the horizon crossing. Apart from the factor of 10^4, this is the same expression we find in slow-roll inflation. The power spectrum of tensor modes is the same, and hence the extra factor of 10^4 leads to a stronger suppression of primordial gravitational waves. Nevertheless, ε here has a strong dependence on k, which makes the spectrum scale-dependent. Indeed, from (10) it is possible to show that, for $\sigma H^2 \approx 1, \varepsilon \propto \eta^{-6}$. Therefore, at the horizon crossing we have $\varepsilon \propto k^6$, leading to a scalar spectral index $n-1 \approx -6$. This result follows from the fact that we do not have a slow-roll potential, since from (4) and (9) we have, for $\sigma H^2 \approx 1, V''/V \approx -6$. Therefore, this scenario cannot by itself produce a scale-invariant spectrum. To achieve this, the phase transition described here must be followed by a usual inflationary epoch.

Let us now discuss the potential role of the vacuum energy at late times. The free-field vacuum density, of order H^4, is presently too small to be identified with the dark energy. On the other hand, any contribution proportional to H^2 may be absorbed by the gravitational constant in Einstein equations [9]. It is usually believed that, apart from terms in H^2 and H^4, the only remaining contribution is a free constant, which may be absorbed by a cosmological constant. Nevertheless, it has recently been claimed that the introduction of interactions may change this picture [6; 7]. In particular, the QCD condensate leads to a vacuum density of order m^3H, where $m \approx 150\,\mathrm{MeV}$ is the energy scale of the QCD phase transition.[2] Although not conclusive, this result gives the correct order of magnitude for the cosmological term, $\Lambda \approx m^6$, and, as a byproduct, the Dirac large number coincidence $H \approx m^3$.

At present it is not known whether this linear relation is valid only in the de Sitter limit or can also hold in the dynamical regime near this limit. Assuming the latter, again in order to ensure the conservation of total energy, matter production would be required. This scenario was studied recently, by confronting its predictions with the current observations. A joint analysis of SNe Ia, baryonic acoustic oscillations and the position of the first peak of CMB anisotropy spectrum produced a good concordance [11]. However, the production of dark matter leads to a power suppression in the mass spectrum, which seems to rule out this scenario [12]. This difficulty may be overcome if we avoid matter production by associating the varying Λ with a quintessence field, as was done in the early-time limit above. A full study, which also includes matter, is in progress. Here we shall consider the simpler case where only the quintessence field is present, which is reasonable very near the de Sitter limit.

With the ansatz $V = 3\sigma H$, where now $\sigma \approx m^3$, Eqs. (2)–(3) have the solution

$$V = \frac{3\sigma\left(e^{\sqrt{3/2}\phi} + \sigma\right)^2}{4e^{\sqrt{3/2}\phi}}, \tag{15}$$

$$H = \frac{\sigma e^{3\sigma t}}{e^{3\sigma t} - 1}. \tag{16}$$

As can be seen, as $t \to \infty, H \to \sigma$ as expected.

The potential possesses a minimum at $\phi_0 = 2\ln\sigma/\sqrt{6}$, corresponding to the de Sitter limit. Around this point the potential can be expanded as

$$V \approx V_0 + \frac{M^2}{2}(\phi - \phi_0)^2, \tag{17}$$

where $V_0 = 3\sigma^2$ and $M = 3\sigma/2$. The mass term is worthy of note. In the de Sitter limit we have $\sigma = H = \sqrt{\Lambda/3}$. Therefore, the mass of the quintessence field is given by

$$M = \frac{3}{2}\sqrt{\Lambda/3}. \tag{18}$$

This is the mass expected for elementary degrees of freedom in the context of the holographic conjecture [13; 14; 15]. Indeed, recalling the observable mass in the universe, $E \approx \rho/H^3 \approx 1/\sqrt{\Lambda}$, and the entropy associated with the de Sitter horizon, $N \approx 1/H^2 \approx 1/\Lambda$, we obtain $M \approx E/N \approx \sqrt{\Lambda}$ for the mass of each degree of freedom.

In conclusion, we note that the vacuum energy, usually considered a problem in quantum field theories and cosmology, may actually shed new light on other

[2] A term linear in H also appears in models with a modified Friedmann equation, in the context of high-dimensional theories. See, for example, reference [10].

fundamental problems, such as the initial singularity and the nature of dark energy. At early times the free-field contribution to the vacuum density, proportional to H^4, leads to a non-singular scenario, which in this paper was modeled by a self-interacting scalar field, with the potential playing the role of the vacuum term. In this context, it was possible to show that the primordial transition from the quasi-de Sitter phase does not produce a scale-invariant spectrum of perturbations, for which a subsequent, usual inflationary epoch is required. On the other hand, at late times we have associated the vacuum term with a quintessence field with potential linear in H, as suggested by QCD results in the de Sitter space-time. The presence of such a field allows the conservation of the total energy without invoking matter production. Near the future de Sitter limit, the quintessence potential is shown to be harmonic, with a mass term in agreement with the holographic prescription.

Acknowledgments We wish to thank James Lidsey and Raul Abramo for very helpful discussions. Saulo Carneiro was partially supported by CAPES (Brazil).

References

1. L.H. Ford (1975) *Phys. Rev. D* **11** 3370
2. J.S. Dowker R. Critchley (1976) *Phys. Rev. D* **13** 3224
3. P.C.W. Davies (1977) *Phys. Lett. B* **68** 402
4. A.A. Starobinsky (1980) *Phys. Lett. B* **91** 99
5. S. Carneiro (2006) *Int. J. Mod. Phys. D* **15** 2241
6. R. Schutzhold (2002) *Phys. Rev. Lett.* **89** 081302
7. F.R. Klinkhamer G.E. Volovik (2009) *Phys. Rev. D* **79** 063527
8. S. Dodelson (2003) Modern Cosmology Academic Press New York
9. S.A. Fulling L. Parker (1974) *Phys. Rev. D* **10** 3905
10. Dvali, G., Turner, M.: astro-ph/0301510
11. S. Carneiro M.A. Dantas C. Pigozzo J.S. Alcaniz (2008) *Phys. Rev. D* **77** 083504
12. H.A. Borges S. Carneiro J.C. Fabris C. Pigozzo (2008) *Phys. Rev. D* **77** 043513
13. G.A. Mena Marugán S. Carneiro (2002) *Phys. Rev. D* **65** 087303
14. R. Tavakol G.F.R. Ellis (1999) *Phys. Lett. B* **469** 37
15. R. Bousso (2002) *Rev. Mod. Phys.* **74** 825