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SEPARATION OF HIGH-ENERGY CHARGED PARTICLES BY RF DEFLECTION

1. The Procedure of Selective Deflection

We consider a sharply bunched high-energy beam of charged particles traveling in the z-direction. Let this beam have two different kinds of charged particles, the wanted and the unwanted. They have the same charge and the same momentum, but slightly different velocities. The wanted particles are assumed to travel slightly faster than the unwanted particles. It is sufficient to consider just this case; the opposite case can be treated in a similar manner. Let this beam pass through a microwave field of suitable configuration and of the correct frequency and phase. The wanted particles are to be synchronized with the field and to experience a cumulative deflection in one transverse direction, say, the x-direction. The unwanted particles are to remain practically undeflected, because they will gradually lag behind the field in phase and will experience both positive and negative deflections in any transverse direction. If the field is of sufficient strength and length, then, evidently, the wanted particles (if they live long enough) can be completely separated from the unwanted ones.

2. Notations

L = total length of beam separator.

P = total microwave power in megawatts.

λ = free space wavelength of microwave field.

Σ = area of aperture ($\cong 2 \times 2$ cm^2).

p_z = z-component of particle momentum in units of Mev/c; $p_z \cong$ constant along the whole length of the separator.

v_1 = speed of unwanted particles.

γ_1 = $(1 - v_1^2/c^2)^{-1/2} = (1 - \beta_1{}^2)^{-1/2}$.

v_2 = speed of the wanted particles; $v_1 < v_2 \cong c$; $\gamma_2 \cong \infty$.

σ = area of the beam.

$\delta\Omega$ = solid angle subtended by the beam in steradians.

$\Phi_\perp$ = $\delta x \delta y \delta p_x \delta p_y / p_z^2$ = transverse phase space of the beam in units of cm^2-steradian; $\Phi_\perp = \sigma \times \delta\Omega$.

$\Phi_{\perp 1}(0)$ = initial phase space of unwanted particles.

$\Phi_{\perp 2}(0)$ = initial phase space of wanted particles. If the wanted and unwanted particles are thoroughly mixed initially, then $\Phi_{\perp 2}(0) = \Phi_{\perp 1}(0) = \Phi_{\perp}(0)$.

$\Phi_{\perp 1}(L)$ = final phase space of unwanted particles. If no unwanted particles are lost to the separator walls, then according to Lionville's theorem $\Phi_{\perp 1}(L) = \Phi_{\perp 1}(0)$.

$\Phi_{\perp 2}(L)$ = final phase space of wanted particles. If no wanted particles are lost to the separator walls, $\Phi_{\perp 2}(L) = \Phi_{\perp 2}(0)$. If, furthermore, the wanted particles are completely separated from the unwanted particles, then $\Phi_{\perp}(L) = \Phi_{\perp 1}(L) + \Phi_{\perp 2}(L) = \Phi_{\perp 1}(L) + \Phi_{\perp 2}(0) \leqslant \Phi_{\perp 1}(0) + \Phi_{\perp 2}(0)$.

θ = phase angle of a particle (referred to the phase of the field at which the deflection in the desired direction, i.e., x-direction, would be maximum.)

Δx = total increment of x due to deflection by the microwave field.

Δp_x = total increment of p_x due to the microwave field.

3. Initial Conditions of the Beam

The initial solid angle $\delta\Omega_o$ subtended by the beam should be as small as practicable. If no supplementary focusing field is contemplated, we must demand that

$$\sqrt{\delta\Omega_o} \times L \lesssim \sqrt{\Sigma}$$

Otherwise, part of the wanted particles will be lost to the separator walls and the separation of the remaining part of the wanted particles from the unwanted ones may be very difficult. Thus, generally speaking, the admissable magnitude of the initial phase space $\Phi_{\perp}(0)$ depends on the strength of the supplementary focusing field.

For practical reasons we shall take σ_o (initial area of beam) = 1 cm² and $\Phi_{\perp}(0) = 10^{-5}$ cm² --steradian. Then $\delta\Omega_o = 10^{-5}$ ster, and $\sqrt{\delta\Omega_o} \times L >> \sqrt{\Sigma}$ = 2 cm for the kind of separator lengths ($L \gtrsim 40$ m) to be considered here. Hence, we shall assume that the focusing field is sufficiently strong so that in the absence of the separator field the area of the beam will always be smaller than the area of the aperture. In other words, we shall assume that

$$\sigma_o < (\sigma_1)_{max} \lesssim \Sigma$$

where $(\sigma_1)_{max}$ is the maximum area of the unwanted beam.

4. Approximate Formulas

Based on the experimental value obtained by Phillips[1], we have for the increment of the transverse momentum (in units of Mev/c) in the direction of maximum deflection

$$(\Delta p_x)_2 = \int_{z=0}^{L} dp_{x_2} \cong 0.80\sqrt{PL/\lambda}\ \cos\ \theta_2 \tag{1}$$

where θ_2 is the phase angle of the wanted particle, θ_2 being independent of z.

Similarly, we have

$$(\Delta x)_2 = \int_{z=0}^{L} \frac{(p_x)_2}{p_z} dz \cong \int_0^L \frac{0.80}{p_z} \sqrt{\frac{P}{L/\lambda}}\ \frac{z}{\lambda} \cos\ \theta_2 dz = \frac{0.80}{2p_z}\sqrt{PL/\lambda} \cdot L \cos\ \theta_2 \tag{2}$$

Also,

$$\left\{(\Delta p_x)_2 - (\Delta p_x)_1\right\} / p_z \cong \int_{z=0}^{L} \frac{0.80}{p_z} \sqrt{\frac{P}{L/\lambda}} \cdot \left\{\cos\ \theta_2 - \cos\left(\theta_2 - \frac{\pi z}{\lambda \gamma_1^2}\right)\right\} \frac{dz}{\lambda}$$

$$= \frac{(\Delta p_x)_2}{p_z} \cdot \left\{1 - \frac{\sin(\pi L/\lambda\gamma_1^2)}{\pi L/\lambda\gamma_1^2} \cdot \frac{\cos\left(\frac{\pi L}{2\lambda\gamma_1^2} - \theta_2\right)}{\cos\ \theta_2}\right\} \tag{3}$$

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[1] P. R. Phillips, "The Separation of High-Energy Particle Beams by Microwave Techniques," Ph.D. Thesis, Stanford University, Stanford, California, November 1960 (unpublished).

$$(\Delta x)_2 - (\Delta x)_1 = \int_{z=0}^{L} \left\{(p_x)_2 - (p_x)_1\right\} \frac{dz}{p_z}$$

$$\cong \int_0^L \frac{0.80}{p_z} \sqrt{\frac{P}{L/\lambda}} \cdot \frac{z}{\lambda} \left\{\cos\, \theta_2 - \frac{\sin(\pi z/2\lambda\gamma_1{}^2)}{\pi z/2\lambda\gamma_1{}^2} \cdot \cos\left(\frac{\pi z}{2\lambda\gamma_1{}^2} - \theta_2\right)\right\} dz$$

$$= (\Delta x)_2 \cdot \left[1 - \frac{2}{\pi L/\lambda\gamma_1{}^2} \cdot \left\{\frac{1}{\pi L/\lambda\gamma_1{}^2} \cdot \left(1 - \cos\frac{\pi L}{\lambda\gamma_1{}^2}\right) + \tan\, \theta_2 \left(1 - \frac{1}{\pi L/\lambda\gamma_1{}^2} \cdot \sin\frac{\pi L}{\lambda\gamma_1{}^2}\right)\right\}\right] \tag{4}$$

5. Criterion of Separability

If we have

$$(\sigma_1)_{max} \lesssim \Sigma \tag{5a}$$

and

$$(\Delta x)_2 - (\Delta x)_1 \gtrsim \sqrt{\Sigma} \tag{5b}$$

it is clear that we can arrange things so that $\Phi_{\perp 1}(L) = 0$. For example, we can bend the axis of the separator slightly and gradually in the direction that the wanted particles are deflected. With this arrangement the unwanted particles will gradually be lost to the separator walls, and σ_1, used above, may be understood to be the "virtual" area of the unwanted beam calculated as if the separator apertures were large enough to let all the unwanted particles pass. Thus, if we neglect the possible contamination of the wanted beam caused by the stray particles resulting from nuclear collisions in the separator walls, we may have $\Phi_{\perp}(L) = \Phi_{\perp 2}(L) \leqq \Phi_{\perp 2}(0)$. This means that the separation of particles may be complete.

In general, the phase space of the separated part of the wanted particles is $\left\{\Phi_{\perp}(L) - \Phi_{\perp 1}(L)\right\}$, and

$$\frac{\Phi_\perp(L) - \Phi_{\perp_1}(L)}{\Phi_\perp(0)} \cong \text{Inf.}\left\{\frac{(\Delta x)_2 - (\Delta x)_1 - \sqrt{\sigma_1(L)}}{\sqrt{\sigma_o}}\ ;\ 1\right\} \tag{6}$$

In this approximate equation "Inf." means "the smaller of," and $\sigma_1(L)$ is the virtual area of the unwanted beam at $z = L$.

6. <u>Special Cases</u>

(a)

$$\theta_2 = 0,\ L = \lambda\gamma_1^2 :\quad (\Delta p_x)_1 = 0 \tag{7a}$$

$$(\Delta x)_2 - (\Delta x)_1 = (\Delta x)_2 \cdot \left(1 - \frac{4}{\pi^2}\right) \tag{7b}$$

(b)

$$-\frac{\pi}{2} < \theta_2 \leqq 0\ ,\ \frac{L}{\lambda\gamma_1^2} = 1 - \frac{2}{\pi}\left|\theta_2\right| :\quad (\Delta p_x)_1 = 0 \tag{8a}$$

$$(\Delta x)_2 - (\Delta x)_1 = (\Delta x)_2 \cdot \left[1 - \frac{2}{\pi L/\lambda\gamma_1^2} \cdot \left\{\frac{2}{\pi L/\lambda\gamma_1^2} - \cot\frac{\pi L}{2\lambda\gamma_1^2}\right\}\right] \tag{8b}$$

(c)

$$-\frac{\pi}{2} < \theta_2 \leqq 0\ ,\quad 1 \gg \frac{L}{\lambda\gamma_1^2} \neq 1 - \frac{2}{\pi}\left|\theta_2\right| :$$

$$(\Delta p_x)_2 - (\Delta p_x)_1 \cong (\Delta p_x)_2 \cdot \frac{\pi L}{\lambda\gamma_1^2} \cdot \left(-\frac{1}{2}\tan\theta_2 + \frac{1}{3!}\frac{\pi L}{\lambda\gamma_1^2}\right) \tag{9a}$$

$$(\Delta x)_2 - (\Delta x)_1 \cong (\Delta x)_2 \cdot \left(\frac{\pi L}{\lambda\gamma_1^2}\right)^2 \cdot \left(-\frac{1}{3}\tan\theta_2 + \frac{1}{12}\right). \tag{9b}$$

(d)

$$-\frac{\pi}{2} < \theta_2 < 0 \ , \ 1 \gg \frac{L}{\lambda\gamma_1^2} = 1 - \frac{2}{\pi}\left|\theta_2\right| :$$

$$(\Delta p_x)_1 = 0 \tag{10a}$$

$$(\Delta x)_2 - (\Delta x)_1 \cong (\Delta x)_2 \cdot \left\{ \frac{2}{3} - \frac{1}{45}\left(\frac{\pi L}{2\lambda\gamma_1^2}\right)^2 \right\} \tag{10b}$$

7. Operating Phase Angle

From the foregoing results it is clear that

$$\frac{L}{\lambda\gamma_1^2} = 1 - \frac{2}{\pi}\left|\theta_2\right| \ , \ (\theta_2 \leqq 0) \ , \tag{11}$$

should be quite close to the optimum condition for high-energy beam separators. In the limit of small $L/\lambda\gamma_1^2$, i.e., $\theta_2 \to -\pi/2$, we have

$$(\Delta p_x)_2 - (\Delta p_x)_1 \cong 0.80\sqrt{PL/\lambda}\ \cos\left(\frac{\pi}{2} - \frac{\pi L}{2\lambda\gamma_1^2}\right)$$

$$\cong 0.80\sqrt{PL/\lambda} \cdot (\pi L/2\lambda\gamma_1^2). \tag{12a}$$

$$(\Delta x)_2 - (\Delta x)_1 \cong \frac{2}{3} \cdot \frac{0.80}{2p_z}\sqrt{PL/\lambda} \cdot L \cos\left(\frac{\pi}{2} - \frac{\pi L}{2\lambda\gamma_1^2}\right)$$

$$\cong 0.42x\sqrt{\frac{PL}{\lambda}} \cdot \frac{L^2}{\lambda\gamma_1^2} \cdot \frac{1}{p_z} \tag{12b}$$

8. Numerical Examples

Let us take $\lambda = 10.5$ cm, $p_z = 2 \times 10^4$ Mev/c, $(\delta x)_o = (\delta y)_o = \sqrt{\sigma_o} = 1$ cm, and

$$\frac{(\delta p_x)_o}{p_z} = \frac{(\delta p_y)_o}{p_z} = \sqrt{\delta\Omega_o} = 10^{-5/2} .$$

For different values of γ_1 we can choose suitable values of θ_2, L, and P, and calculate $\left\{(\Delta x)_2 - (\Delta x)_1\right\}/(\delta x)_o$ and $\left\{(\Delta p_x)_2 - (\Delta p_x)_1\right\}/(\delta p_x)_o$. These results are tabulated below.

γ_1	θ_2	$\frac{L}{\lambda\gamma_1^2}$	L	P	$\frac{(\Delta x)_2 - (\Delta x)_1}{(\delta x)_o}$	$\frac{(\Delta p_x)_2 - (\Delta p_x)_1}{(\delta p_x)_o}$
20	0	1	42 m	10 Mw	3.2	0.80
40	$-5\pi/16$	3/8	63 m	10 Mw	3.6	0.55
80	$-7\pi/16$	1/8	84 m	40 Mw	3.9	0.43
160	$-61\pi/128$	3/64	126 m	80 Mw	3.8	0.29
320	$-63\pi/128$	1/64	168 m	160 Mw	2.8	0.16

Here we may note that at an energy of 20 Bev muons ($\mu^{\pm}$) have $\gamma \cong 200$. The only lighter charged particles ($\gamma > 200$) presently known to exist are the electrons ($e^{\pm}$).

9. Supplementary Remarks

We may define a deflection impedance (r_x) in ohms/meter, for a separator in an obvious manner, namely,

$$r_x = \left|\frac{1}{k}\frac{\partial E_z}{\partial x}\right|^2 \Big/ (\text{power loss per unit length}). \tag{13}$$

In this equation $k = 2\pi/\lambda$, and $|\partial E_z/\partial x|$ is the maximum amplitude of $\partial E_z/\partial x$ evaluated on the axis ($x = 0$ and $y = 0$). In terms of r_x Eq. (12b) becomes

$$(\Delta x)_2 - (\Delta x)_1 = \text{const.} \times \frac{e}{c} \cdot \frac{\sqrt{r_x PL}}{p_z} \cdot \frac{L^2}{\lambda\gamma_1^{\,2}} \tag{14}$$

Since $r_x \sim \lambda^{-1/2}$, we can write the law of separator scaling as follows:

$$p_z^{-1} p^{1/2} L^{5/2} \gamma_1^{-2} \lambda^{-5/4} = \text{const.} \tag{15}$$

In this note we have neglected the possible contamination of stray particles and also the debunching effect of the separator field. It is believed that these effects should not significantly alter our qualitative results.