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Equivalent transformations and exact solutions to the generalized cylindrical KdV type of equation ☆

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Abstract

In this paper, by constructing equivalent transformations (ETs) of the generalized cylindrical KdV (cKdV) types of equations, we transform the variable-coefficient partial differential equations (vc-PDEs) into constant-coefficient PDEs (cc-PDEs) under some conditions. Particularly, the classical cKdV equation is transformed into the classical KdV equation accordingly, then the exact solutions to the vc-PDEs are provided in terms of the ETs. Thus, an effective approach to getting exact solutions to vc-PDEs is presented based on the solutions to cc-PDEs.

1. Introduction

The cylindrical Korteweg-de Vries (cKdV) equation arises in plasma physics and water waves, they are of great importance in mathematical physics, fluid mechanics and nonlinear wave theory, etc. In practice, many physical, mechanical and engineering models can be depicted by such

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variable-coefficient KdV (vc-KdV) types of equations [1–4]. In the present paper, we investigate the generalized cKdV type of equation as follows:

$$u_t + a(t)u + \beta u u_x + \gamma u_{xxx} = 0, \tag{1.1}$$

where $u = u(x,t)$ denotes the unknown function of x and time t, $a = a(t) \neq 0$ is a given analytic function, β and γ are arbitrary nonzero constants.

A rough description of this paper is as follows:

First, by using the following Clarkson-Kruskal (CK) type of transformation

$$u \equiv u(x,t) = A(x,t) + B(x,t)U(X,T), \tag{1.2}$$

we transform the variable-coefficient partial differential equation (vc-PDE) (1.1) into the nonlinear constant-coefficient partial differential equation (cc-PDE) as follows

$$u_t + \alpha u + \lambda u u_x + \mu u_{xxx} = 0, \tag{1.3}$$

where α, λ and μ are all constants, while $A = A(x,t)$, $B = B(x,t)$, $X = X(x,t)$ and $T = T(x,t)$ are functions of x and t to be determined in (1.2). Such a non-degenerate transformation (1.2) of variables from (x,t,u) to (X,T,U) is called equivalent transformation (ET) [5–9]. In particular, if $\alpha = 0$, then the nonlinear cc-PDE (1.3) becomes the following classical KdV equation

$$u_t + \lambda u u_x + \mu u_{xxx} = 0. \tag{1.4}$$

Second, we consider the exact solutions to the cc-PDEs (1.3) and (1.4). If the exact solutions to these cc-PDEs are obtained, then the exact solutions to vc-PDE (1.1) are provided through ET (1.2).

For providing more information to understand the physical process in applications, the exact solutions to the governing equations are required. As a matter of fact, many properties of nonlinear PDEs (NLPDEs) can be effectively studied by using their exact solutions [10–12]. For example, the exact traveling wave solutions [10] can depict the dynamic behavior of a nonlinear system. W. Hu et al. [12] studied the axial dynamic buckling properties of the nanotube by the structure-preserving method and numerical approach, these are all effective methods for studying NLPDEs in mathematical physics and physical applications. In addition, P. Clarkson and M. Kruskal [5] proposed a direct method for similarity reductions and exact solutions of NLPDEs, it is called the CK direct method sometimes. The main feature of this method is that it does not involve the group theory, and this method has been greatly extended in recent decades [1,5–9]. Moreover, it is known that the Lie group analysis is a systematic method for dealing with symmetries, similarity reductions and exact solutions to the NLPDEs [8,9,11,13–23]. Recently, we studied some vc-PDEs by Lie group analysis and CK direct method, the symmetries, exact solutions and other properties of the equations are provided [9,11,13,16,21–23]. In general, it is very difficult to tackle exact solutions to vc-PDEs. In [13], we give all of the point symmetries of the generalized cKdV type of equation by the complete group classification method, the exact solutions and conservation laws are considered, but the results are relatively few due to the complexity of the matter. While if the equivalent transformations can be given, then we have a new approach to getting exact solutions to the vc-PDEs, i.e., we can obtain the exact solutions to vc-PDEs in terms of the exact solutions to the corresponding cc-PDEs. This ET method has at least one obvious advantage: it can get more exact solutions to the vc-NLPDEs, because the exact solutions to cc-PDEs are more richer. Now, we summarize the contribution and novelty of the current paper as follows:

• We construct the ETs to transform the vc-PDEs into cc-PDEs.

• Through the ETs, the exact solutions to the vc-PDEs are provided based on the exact solutions to cc-PDEs.

The main purpose of this paper is to construct the ETs of the form (1.2) for the cKdV types of equations and tackle the exact solutions to the vc-PDEs in terms of the ET method. The rest of the paper is organized as follows: In Sect. 2, we construct the equivalent transformation of the generalized cKdV equation (1.1), and transform this vc-PDE into cc-PDE (1.3) under some constraint condition. As an important example, the classical cKdV equation is transformed into classical KdV equation through the ET accordingly, so the exact solutions to the classical cKdV equation can be obtained based on the exact solutions to the classical cc-KdV equation. Considering that the solutions to cc-KdV equation are given, so the solutions to the cKdV equation are provided by the ET given in the present paper. To our best knowledge, this is the first systematic work that applies solutions of the classical KdV equation to construct solutions of the cKdV equation completely. In Sect. 3, we give all of the point symmetries of cc-NLPDE (1.3), and the symmetry reductions and exact solutions to this equation are investigated, thus the exact solutions to the generalized cKdV equation (1.1) are presented. Finally, we will conclude our paper in Sect. 4.

2. Equivalent transformations of Eq. (1.1)

In this section, we develop the improved CK direct reduction method for investigating the relationship between the variable-coefficient equation (1.1) and its corresponding constant-coefficient counterpart (1.3).

Firstly, substituting (1.2) into Eq. (1.1), and requiring that $U = U(X,T)$ satisfies the same type of equation as $u = u(x,t)$ with the transformation $\{u,x,t\} \to \{U,X,T\}$. That is, requiring that $\{U,X,T\}$ satisfy Eq. (1.3) also, i.e.,

$$U_T + \alpha U + \lambda U U_X + \mu U_{XXX} = 0, \tag{2.1}$$

where α, λ and μ are all constants.

Then, through the CK reduction method, we get the following result:

Theorem 2.1. *For the arbitrary analytic function $a'(t) \neq 0$, that is, $a = a(t)$ is not a constant, in view of (1.2), we have*

$$A = \frac{x}{e^{F(t)}(\beta G(t) + c_1)} + c_2 e^{-F(t)-\beta H(t)}, \quad B = \frac{\gamma\lambda}{\beta\mu} c_3^2 e^{-2\beta H(t)},$$
$$X = c_3 e^{-\beta H(t)} x - \beta c_2 c_3 \int e^{-F(t)-2\beta H(t)} dt + c_4, \quad T = \frac{\gamma}{\mu} c_3 \int e^{-3\beta H(t)} dt + c_5, \tag{2.2}$$

where $F = F(t) = \int a(t)dt$, $G = G(t) = \int e^{-F(t)} dt$ and $H = H(t) = \int \frac{dt}{e^{F(t)}(\beta G(t)+c_1)}$, while c_i $(i = 1, 2, ..., 5)$ are arbitrary constants, and $c_3 \neq 0$. □

Thus, if we obtain the exact solution to Eq. (2.1), then the exact solution to Eq. (1.1) can be given through the transformation as follows

$$u(x,t) = \frac{x}{e^{F(t)}(\beta G(t) + c_1)} + c_2 e^{-F(t)-\beta H(t)} + \frac{\gamma\lambda}{\beta\mu} c_3^2 e^{-2\beta H(t)}$$
$$\times U\Big(c_3 e^{-\beta H(t)} x - \beta c_2 c_3 \int e^{-F(t)-2\beta H(t)} dt + c_4, \frac{\gamma}{\mu} c_3 \int e^{-3\beta H(t)} dt + c_5\Big), \tag{2.3}$$

under the following condition

$$\alpha\gamma c_3^3 e^F(\beta G + c_1) + \beta\mu e^{3\beta H} - \mu a e^{F+3\beta H}(\beta G + c_1) = 0, \tag{2.4}$$

where $F = F(t)$, $G = G(t)$ and $H = H(t)$ are given by (2.2), c_1 and $c_3 \neq 0$ are arbitrary constants. In other words, under the condition (2.4), Eq. (1.1) can be transformed into the constant-coefficient equation (2.1) through the transformation (1.2) with the functions A, B, X and T given by (2.2).

Summering the above discussion, we have

Theorem 2.2. *If $U = U(X, T)$ is a solution to Eq. (2.1), then $u = A + BU(X, T)$ is a solution to Eq. (1.1) under the condition (2.4), where A, B, X and T are given by (2.2).* □

Furthermore, in view of (2.2), it is easy to see that the parameters λ and μ are nonzero constants, while the other parameter α is arbitrary. In particular, if $\alpha = 0$, then Eq. (1.1) can be transformed into the following classical KdV equation

$$U_T + \lambda U U_X + \mu U_{XXX} = 0, \tag{2.5}$$

through the equivalent transformation (1.2) with the coefficients given by (2.2). In this case, the compatibility condition (2.4) becomes

$$a e^F(\beta G + c_1) = \beta, \tag{2.6}$$

where $F = F(t)$ and $G = G(t)$ are given by (2.2), c_1 is an arbitrary constant. That is, under the condition (2.6), Eq. (1.1) can be transformed into the constant-coefficient KdV equation (2.5) through the transformation (1.2).

In view of the forms of Eqs. (2.1) and (2.5) are the same as Eqs. (1.3) and (1.4), respectively, we have the conclusion: *If the exact solutions to cc-NLPDEs (1.3) and (1.4) are obtained, then the exact solutions to vc-NLPDE (1.1) are presented immediately through the equivalent transformation (1.2), under the conditions (2.4) and (2.6).* In what follows, we only consider the exact solutions to cc-NLPDE (1.3).

Now, we make some further discussion on the compatibility condition (2.6). Substituting $F = \int a(t)dt$ and $G = \int e^{-F(t)}dt$ into (2.6) and differentiating this equation, we get

$$a' + 2a^2 = 0. \tag{2.7}$$

Solving Eq. (2.7), we get the general solution to this equation as follows

$$a(t) = \frac{1}{2t + k}, \tag{2.8}$$

where k is an arbitrary constant. In view of (2.8), we can see that Eq. (1.1) becomes the classical cylindrical KdV (cKdV) equation under the condition. Summarizing, we obtain the following results:

Theorem 2.3. *For the classical cKdV equation of the form*

$$u_t + \frac{1}{2t + k}u + \lambda u u_x + \mu u_{xxx} = 0, \tag{2.9}$$

it can be transformed into the classical KdV equation (2.5) through the equivalent transformation (1.2) with the coefficients given by (2.2), where k, λ and μ are constants, and $\lambda\mu \neq 0$. □

Corollary 2.4. *In general, if $a(t) \neq \frac{1}{2t+k}$, then the generalized cKdV type of equation (1.1) can be transformed into cc-NLPDE (2.1) rather than the classical KdV equation (2.5) by the equivalent transformation (1.2).* □

In other words, if $a(t) \neq \frac{1}{2t+k}$, then the equivalent transformation (1.2) cannot transform Eq. (1.1) into classical KdV equation (2.5) directly.

Remark 2.1. Generally, it is not easy to get exact solutions to the classical cKdV equation (2.9). However, through the CK-type equivalent transformation (1.2), this equation can be transformed into the classical KdV equation (2.5), i.e. Eq. (1.4), so the exact solutions to the cKdV equation are presented based on the solutions to Eq. (1.4). Considering there are lots of results on the solutions to Eq. (1.4), such as traveling wave solutions, solition solutions, and so on, thus the exact solutions to Eq. (2.9) are given in terms of the solutions to classical KdV equation, the details are omitted here.

3. Exact solutions to vc-NLPDEs

3.1. Exact solutions to cKdV equation (2.9)

First of all, considering the cKdV equation can be transformed into classical cc-KdV equation, so the exact solutions to the cKdV equation (2.9) are obtained based on the solutions to the cc-KdV equation (1.4). In other words, if the exact solutions to classical KdV equation (1.4) are given, then the exact solutions to classical cKdV equation (2.9) are provided through the equivalent transformation given in Sect. 2. Summarizing, we have

Corollary 3.1. *If the exact solutions to classical KdV equation (1.4) are given, then the exact solutions to cKdV equation (2.9) are provided through the ET (2.3), where $a = a(t)$ is given by (2.8), $F = F(t)$, $G = G(t)$ and $H = H(t)$ are defined in (2.2).* □

To our best knowledge, this is the first time to get exact solutions of the cKdV equation based on the exact solutions of the KdV equation by the ET method.

3.2. Exact solutions to generalized cKdV equation (1.1)

In this subsection, we only consider the exact solutions to the generalized cylindrical equation (1.1). By the Lie group analysis method, we give all of the point symmetries of Eq. (1.3) as follows

$$V_1 = \partial_x, \quad V_2 = \partial_t, \quad V_3 = \lambda e^{-\alpha t}\partial_x - \alpha e^{-\alpha t}\partial_u, \tag{3.1}$$

where α and λ are nonzero constants.

Now, we consider the symmetry reductions and exact solutions to the generalized cylindrical equation (1.1) based on the symmetries (3.1) and equivalent transformation between Eqs. (1.1) and (2.1).

(i) For V_1, we have

$$u = f(\xi), \tag{3.2}$$

where $\xi = t$. Substituting (3.2) into Eq. (1.3), we reduce the equation to the following ordinary differential equation (ODE):

$$f' + \alpha f = 0, \tag{3.3}$$

where $f' = df/d\xi$. Solving this equation, we have $f = ce^{-\alpha\xi}$. So the exact solution to Eq. (1.3) is $u(x, t) = ce^{-\alpha t}$. Thus, we obtain the exact solution to Eq. (1.1) as follows

$$\begin{aligned} u(x,t) = {} & \frac{x}{e^{F(t)}(\beta G(t) + c_1)} + c_2 e^{-F(t)-\beta H(t)} + \frac{\gamma\lambda}{\beta\mu} c c_3^2 e^{-2\beta H(t)} \\ & \times \exp\left\{ -\frac{\alpha\gamma}{\mu} c_3 \int e^{-3\beta H(t)} dt - \alpha c_5 \right\}, \end{aligned} \tag{3.4}$$

where $F = F(t)$, $G = G(t)$ and $H = H(t)$ are given by (2.2), c and c_i $(i = 1, 2, 3, 5)$ are arbitrary constants.

(ii) For V_3, we have

$$u = f(\xi) - \frac{\alpha}{\lambda} x, \tag{3.5}$$

where $\xi = t$. Substituting (3.5) into Eq. (1.3), we reduce the equation to the following ODE:

$$f' = 0, \tag{3.6}$$

where $f' = df/d\xi$. Solving this equation, we have $f = c$. So the exact solution to Eq. (1.3) is $u(x, t) = c - \frac{\alpha}{\lambda} x$. Thus, we obtain the exact solution to Eq. (1.1) as follows

$$\begin{aligned} u(x,t) = {} & \frac{x}{e^{F(t)}(\beta G(t) + c_1)} + c_2 e^{-F(t)-\beta H(t)} \\ & + \frac{\gamma\lambda}{\beta\mu} c_3^2 e^{-2\beta H(t)} \Big(c - \frac{\alpha}{\lambda} c_3 e^{-\beta H(t)} x - \frac{\alpha\beta}{\lambda} c_2 c_3 \int e^{-F(t)-2\beta H(t)} dt - \frac{\alpha}{\lambda} c_4 \Big), \end{aligned} \tag{3.7}$$

where $F = F(t)$, $G = G(t)$ and $H = H(t)$ are given by (2.2), c and c_i $(i = 1, 2, 3, 4)$ are arbitrary constants.

(iii) For $V = vV_1 + V_2$, we have

$$u = f(\xi), \tag{3.8}$$

where $\xi = x - vt$. Substituting (3.8) into Eq. (1.3), we reduce the equation to the following ODE:

$$\mu f''' + \lambda f f' - v f' + \alpha f = 0, \tag{3.9}$$

where $f' = df/d\xi$, v is a constant denotes the wave speed in wave motion.

(iv) For $V = V_2 + vV_3$ (v is a constant), we have

$$u = f(\xi) - \frac{\alpha}{\lambda} x, \tag{3.10}$$

where $\xi = x + \frac{v\lambda}{\alpha} e^{-\alpha t}$. Substituting (3.10) into Eq. (1.3), we reduce the equation to the following ODE:

$$\alpha\mu f''' + \alpha\lambda f f' - \xi f' = 0, \tag{3.11}$$

where $f' = df/d\xi$.

Thus we reduce NLPDE (1.3) to ODEs (3.9) and (3.11), respectively. If the exact solutions to the two ODEs are obtained, then the solutions to Eq. (1.3), and so the solutions to vc-PDE (1.1) are given accordingly through the equivalent transformation. However, Eqs. (3.9) and (3.11) are nonlinear ODEs, they cannot be solved by the integration generally. Now we give the exact power series solution to Eq. (3.11).

Suppose that Eq. (3.11) has a solution in the power series form

$$f(\xi)=\sum_{n=0}^{\infty}c_n\xi^n=c_0+\sum_{n=1}^{\infty}c_n\xi^n, \tag{3.12}$$

where the coefficients c_n $(n=0,1,2,...)$ are constants to be determined.

Clearly, $f(0)=c_0$ is a constant solution to Eq. (3.11). Substituting (3.12) into Eq. (3.11), we have

$$c_{n+3}=\frac{1}{\alpha\mu(n+1)(n+2)(n+3)}\Big(nc_n-\alpha\lambda\sum_{k=0}^{n}c_kc_{n+1-k}\Big),\quad n=0,1,2,.... \tag{3.13}$$

Thus, for arbitrarily chosen constants c_0, c_1 and c_2, in terms of the recursion formula (3.13), we get all of the coefficients of (3.12). For example, $c_3=-\frac{\lambda}{6\mu}c_0c_1$, $c_4=\frac{1}{24\alpha\mu}(c_1-2\alpha\lambda c_0c_2-\alpha\lambda c_1^2)$, $c_5=\frac{1}{60\alpha\mu}(2c_2-3\alpha\lambda c_0c_3-3\alpha\lambda c_1c_2)$, and so on.

Hence, the general solution to Eq. (3.11) in power series form is

$$f(\xi)=c_0+c_1\xi+c_2\xi^2+\sum_{n=0}^{\infty}c_{n+3}\xi^{n+3}, \tag{3.14}$$

where c_0, c_1 and c_2 are arbitrary constants, the other coefficients c_{n+3} $(n=0,1,2,...)$ are given by (3.13). Accordingly, the power series solution to Eq. (1.3) is

$$u(x,t)=c_0+c_1(x+\frac{v\lambda}{\alpha}e^{-\alpha t})+c_2(x+\frac{v\lambda}{\alpha}e^{-\alpha t})^2+\sum_{n=0}^{\infty}c_{n+3}(x+\frac{v\lambda}{\alpha}e^{-\alpha t})^{n+3}-\frac{\alpha}{\lambda}x. \tag{3.15}$$

Thus, through the ET (2.3), we obtain the exact power series solution to vc-PDE (1.1) as follows

$$u(x,t)=c_0+c_1(X+\frac{v\lambda}{\alpha}e^{-\alpha T})+c_2(X+\frac{v\lambda}{\alpha}e^{-\alpha T})^2+\sum_{n=0}^{\infty}c_{n+3}(X+\frac{v\lambda}{\alpha}e^{-\alpha T})^{n+3}-\frac{\alpha}{\lambda}X, \tag{3.16}$$

where $A=A(x,t)$, $B=B(t)$, $X=X(x,t)$ and $T=T(t)$ are given by (2.2), the coefficients c_{n+3} $(n=0,1,2,...)$ are given by (3.13), c_0, c_1 and c_2 are arbitrary constants.

Similarly, the exact traveling wave solution in power series form of Eq. (3.9) can be given, so the exact solution to vc-PDE (1.1) in power series form are obtained, the details are omitted.

4. Conclusions and remarks

In the present paper, by constructing equivalent transformation, the generalized cKdV type of equation is transformed into a nonlinear cc-PDE. Correspondingly, the classical cKdV equation be transformed into classical KdV equation, so the exact solutions to classical cKdV equation can be obtained by the solutions to the latter. In this case, we can say that the solutions to the classical cKdV equation are given completely for the first time. As far as we know, this is the first systematic work to get exact solutions of the cKdV equation from the solutions of the KdV equation through ET. In addition, based on the symmetry reductions and exact solutions to the cc-NLPDE, the exact solutions to the generalized cKdV equation are presented. From the above discussion, we can see that the equivalent transformation is a systematic and effective method for dealing with exact solutions to vc-PDEs based on the cc-PDEs. However, there are a lot of problems to study further, for example, are there other types of equivalent transformations, and they can transform a vc-PDE into the other types of cc-PDEs? We hope to investigate it in the future.

Declaration of competing interest

The authors of the paper declare that they have no conflict of interest.

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