BEAM EXPANSION AND INCREASE OF

THE AMMAN - RITSON LUMINOSITY LIMIT

BY COUPLING AND UNDAMPING OF

BETATRON OSCILLATIONS

BY

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Abstract:

In the following we describe a method for expanding the beam
cross section in a storage ring by coupling the betatron oscillations
and damping* the synchrotron oscillations.

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*Translator's note: Undamping?

It is shown that in the case of resonance

$$Q_z - Q_x \approx \text{integer}$$

the vertical beam expansion is increased and the horizontal expansion
decreased if coupling of the betatron oscillations is introduced.

In connection with coupling a further enlargement of the beam
cross section in the x-direction, as well as in the z-direction, occurs
by damping of the synchrotron oscillations.

Rotation of the beam at the crossing points generated by
coupling, which could give rise to additional coupling, does not
appear if

1) $Q_z - O_x$ is an odd number;

2)   the rotated quadrupole which produces the coupling is

     precisely in the middle between the interaction sections.

The expansion of the beam cross section leads to

1)   balancing of the strong vertical and the relatively

       weak horizontal space-charge forces and

2)   weakening of the space-charge forces.

In this way the limit of luminosity due to the space-charge effect is

raised.

Equations to determine the beam cross section and the Q-shift

induced by the beam-beam interaction are derived and evaluated numerically

for the case of the DESY storage ring.

Contents:

I. Introduction and Statement of the Problem

In the present work we describe a method for considerably

expanding the beam in the interaction regions with a view to raising

the luminosity limit given by the Amman-Ritson effect. This is achieved

by coupling of the betatron oscillations and a shift in the damping

constants for betatron and synchrotron oscillations.

First we consider the coupling alone. We see how the beam cross

section can be altered, with the aid of a rotated quadrupole which causes

a coupling of the betatron oscillations, in such a way that the beam

expands in the vertical direction at the expense of its comparatively

large width.

Such deformation of the beam cross section leads to a balancing

of the strong vertical space-charge forces and the relatively weak

horizontal ones in the interaction regions.

A measure of the strength of the space-charge forces is the

Q-shift caused by the Amman-Ritson effect. Experience has shown that

this shift must be smaller than 0.025 in the x- and z-directions if the

beam is to remain stable. The beam expansion in the z-direction caused

by coupling works in such a way that, for a constant beam current I,

adjustments in the Q-shifts ΔQ_I and ΔQ_{II} occur for both forms of oscillation, such that the smaller shift grows and the larger decreases. In consequence, the luminosity limit due to the Amman-Ritson effect is found to rise. The most favorable distribution of the space-charge forces, at which both forms of oscillation are influenced by the beam-beam interaction in the proper manner relative to the focussing power, is attained precisely when ΔQ_I and ΔQ_{II} assume equal values.

Our task, therefore, is first to determine the Q-shift that occurs upon switching on of the coupling and the related change in the luminosity limit. We shall also give the general conditions for coupling strength and the machine parameters that must be fulfilled if the luminosity limit is to be appreciably raised.

Following the study of beam expansion by coupling, we shall show, by means of the general equations for the beam cross section, that a further magnification of the beam size is possible. This is done by simultaneously switching on the coupling in the x- and z-directions, whereby the betatron oscillations are undamped, by the installation of a FODO channel, and the synchrotron oscillations are simulta-

neously damped[1]. This additional expansion of the beam by means of

a shift of the damping constants is only limited by the size of the

vacuum chamber. The effect is to further weaken the space-charge forces

in the z-direction as well as in x-direction, so that a further rise in

the luminosity limit of the space-charge effect is produced.

The beam expansion is of importance for the storage ring

planned at DESY, particularly in the energy region

$$E \lesssim 2 \text{ GeV},$$

where the lifetime of the beam is essentially determined by the

Amman-Ritson effect.

II. The General Equations for the Beam Cross Section and the Amman-Ritson Effect in a Coupled Machine·

To calculate the beam expansion and the shift of the luminosity

limit due to the space-charge effect in the presence of coupling, we begin

with the general equations for the beam cross section and for the Amman-

Ritson effect in a "coupled" machine[2, 3, 4].

We designate the eigenvectors of the circulation matrix

$\mathfrak{M}$ (s + C, s) (C is the orbit circumference) by:

$$w_1 \equiv w_I ,$$
$$w_2 = w_I^* ,$$
$$w_3 \equiv w_{II} ,$$
$$w_4 = w_{II}^* , \qquad \qquad (2.1)$$

If the normalization conditions

$$\frac{1}{2i} \, w_I^+ \, \mathcal{T} \, w_I = 1 \quad ;$$

$$\frac{1}{2i} \, w_{II}^+ \, \mathcal{T} \, w_{II} = 1 \quad ; \qquad \qquad (2.2)$$

$$\mathcal{T} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

are fulfilled, the following equations should hold:

Damping constant for synchrotron oscillations:

$$\alpha_s = \frac{\bar{W}}{E \cdot \langle k^2 \rangle} \left\{ \langle \chi^2 \rangle + \frac{1}{2} \cdot \langle k^2 (C_x D_x + C_z D_z) \rangle \right\} \qquad (2.3)$$

where we have

$$C_x = K_x (1 - 2n_x) ;$$

$$C_z = K_z (1 - 2n_z) ; \qquad \qquad (2.4)$$

$$K^2 = K_x^2 + K_z^2 ; \qquad \qquad (2.5)$$

$$\bar{W} = \frac{2}{3} r_e c \gamma^3 E \cdot \langle K^2 \rangle ; \qquad \qquad (2.6)$$

η_i = field index; (i = x, z)

K_i = curvature

D_i = dispersion

E = particle energy

$\overline{W}$ = average energy loss for a particle

Damping constants for betatron oscillation:

$$\alpha_n = \frac{\overline{W}}{2E \cdot \langle K^2 \rangle} \cdot \Big\langle K^2 \Big(1 + \eta_n \{ (C_x v_{n1}^* + C_z v_{n3}') \cdot$$
$$\cdot (D_x v_{n2} - D_x' v_{n1} + \tag{2.7}$$
$$+ D_z v_{n4} - D_z' v_{n3}) \} \Big\rangle \; ;$$

$$(n = I, II) ;$$

Emittances:

$$\varepsilon_I = \frac{\gamma_I}{\alpha_I} \quad ; \quad \varepsilon_{II} = \frac{\gamma_{II}}{\alpha_{II}} \tag{2.8}$$

with (n = I, II)

$$\gamma_n = \frac{\overline{Q}_s}{4 E^2 \cdot \langle |K^3| \rangle} \cdot \Big\langle |K|^3 \; | D_x v_{n2} - D_x' v_{n1} + D_z v_{n4} - D_z' v_{n3} |^2 \Big\rangle ; \tag{2.9}$$

$$\overline{Q}_s = \frac{55}{24 \sqrt{3}} r_e c^2 \hbar \gamma^6 E \cdot \langle |K^3| \rangle . \tag{2.10}$$

Average beam size in the x and z directions:

$$\sigma_x^2 = \varepsilon_I \cdot | v_{I1}(s) |^2 + \varepsilon_{II} \cdot | v_{II1}(s) |^2 \; ;$$
$$\sigma_z^2 = \varepsilon_I \cdot | v_{I3}(s) |^2 + \varepsilon_{II} \cdot | v_{II3}(s) |^2 . \tag{2.11}$$

Semi-axes of the elliptical beam cross section:

$$\sigma_{1,2}^2 = \frac{1}{2} \left\{ [\sigma_x^2 + \sigma_z^2] \pm [\sigma_x^2 - \sigma_z^2] \cdot \sqrt{1 + \frac{4\sigma_{xz}^2}{[\sigma_x^2 - \sigma_z^2]^2}} \right\} \quad (2.12)$$

with

$$\sigma_{xz} = \varepsilon_I \cdot Re\{v_{Ix} v_{Iz}^*\} + \varepsilon_{II} \cdot Re\{v_{IIx} v_{IIz}^*\} ; \quad (2.13)$$

Rotation angle of the beam:

$$tg\, 2\theta = \frac{2\sigma_{xz}}{\sigma_x^2 - \sigma_z^2} ; \quad (2.14)$$

$$\sin 2\theta = \frac{2\sigma_{xz}}{\sqrt{[\sigma_x^2 - \sigma_z^2]^2 + 4\sigma_{xz}^2}} ;$$

Q-shift by space-charge forces in the interaction zone:

$$\Delta Q_n = \frac{1}{4\pi} \left\{ [h_1 \cos^2\theta + h_2 \sin^2\theta] \cdot |v_{n1}|^2 \right.$$

$$+ [h_1 \sin^2\theta + h_2 \cos^2\theta] \cdot |v_{n3}|^2$$

$$\left. + Re\{v_{n1} v_{n3}^*\} \cdot (h_1 - h_2) \sin 2\theta \right\} ; \quad (2.15)$$

$$(n = I, II)$$

with

$$h_1 \cdot \sigma_1 = h_2 \cdot \sigma_2 = \frac{2 r_e N}{q \, \beta \, (\sigma_1 + \sigma_2)} , \quad (2.16)$$

N = number of particles

q = number of bunches

(the crossing angle is neglected in Equation 2.16.)

We assume that the following relations hold for the DESY

storage ring:

$$K_z \cdot D_x = K_x \cdot D_z = 0 \; ;$$

$$K_z \cdot D'_x = K_x \cdot D'_z = 0 \; ;$$

$$D_x \cdot D_z = 0 \; ;$$

$$D_x \cdot D'_z = D_z \cdot D'_x = 0 \; ;$$ <div align="right">(2.17)</div>

$$D'_x \cdot D'_z = 0 \; ;$$

$$K_x \cdot K_z = 0$$

Then Equations (2.3), (2.7), and (2.9) simplify further to:

$$\alpha_s = \frac{\bar{W}}{E \cdot \langle \varkappa^2 \rangle} \cdot \left\{ \langle \varkappa^2 \rangle + \frac{1}{2} \tilde{A}_x + \frac{1}{2} \tilde{A}_z \right\} \qquad (2.3a)$$

where

$$\tilde{A}_x = \langle \varkappa_x^3 (1 - 2 n_x) D_x \rangle \; ;$$

$$\qquad\qquad\qquad\qquad (2.3b)$$

$$\tilde{A}_z = \langle \varkappa_z^3 (1 - 2 n_z) D_z \rangle \; ;$$

$$\alpha_n = \frac{\bar{W}}{2 E \cdot \langle \varkappa^2 \rangle} \cdot \left\{ \langle \varkappa^2 \rangle + \langle \varkappa_x^3 (1 - 2 n_x) D_x \cdot \mathcal{R} \{ v_{n1}^* v_{n2} \} \rangle \right.$$

$$\qquad\qquad\qquad\qquad\qquad\qquad (2.7a)$$

$$\left. + \langle \varkappa_z^3 (1 - 2 n_z) D_z \cdot \mathcal{R} \{ v_{n3}^* v_{n4} \} \rangle \right\} \; ;$$

$$\gamma_n \;=\; \frac{\bar{Q}_3}{4\,E^2\,\langle|x^3|\rangle} \;\times$$

(2.9a)

$$\times\;\Big\{\big\langle\,|x_x|^3\cdot\big(\,D_x^2\cdot v_{n2}\,v_{n2}^* - 2\,D_x\,D_x'\cdot\mathcal{R}_e\,\{v_{n2}\,v_{n1}^*\} + D_x'^2\cdot v_{n1}\,v_{n1}^*\big)\big\rangle$$

$$+\;\big\langle\,|x_z|^3\cdot\big(\,D_z^2\cdot v_{n4}\,v_{n4}^* - 2\,D_z\,D_z'\cdot\mathcal{R}_e\,\{v_{n4}\,v_{n3}^*\} + D_z'^2\cdot v_{n3}\,v_{n3}^*\big)\big\rangle\Big\}\,,$$

$$(\,n\;=\;I,\,II\,)\,.$$

To evaluate Equations (2.1-2.16), we still require the

eigenvectors of the circulation matrix $\mathfrak{M}$ (s + C, s) for the coupling

of the betatron oscillations by a rotated quadrupole. They are calculated

in the following section.

III. The Eigenvectors of the Circulation Matrix for Coupling of the
     Betatron Oscillations by a Rotated Quadrupole

To determine the eigenvectors, imagine that the rotated

quadrupole is replaced by a thin lens with the transfer matrix

$$\mathfrak{M}_L \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \delta & 0 \\ 0 & 0 & 1 & 0 \\ \delta & 0 & 0 & 1 \end{pmatrix}$$

(3.1)

and denote the lens center by $s = s_L$. Then the complete matrix will have the form:

$$\mathcal{M}(s_L + C, s_L) = \mathcal{M}_L \cdot \mathcal{M}_0(s_L + C, s_L), \, ,$$

(3.2)

where $\mathcal{M}_0(s_L + s, s_L)$ stands for the transfer matrix of an ideal machine (no coupling). For the eigenvectors

$$w_k(s) = \mathcal{M}_0(s, s_L) \, w_k(s_L)$$

(3.3)

we obtain the equation[2]

$$\{ \mathcal{M}_L \, \mathcal{M}_0(s_L + C, s_L) - e^{-i \cdot 2\pi Q_k} \cdot \mathbf{1} \} \, w_k(s_0) = 0 .$$

(3.4)

If we then write

$$\mathcal{M}_0(s, s_L) = (w_1(s), w_2(s), w_3(s), w_4(s))$$

(3.5)

where we define

$$w_1 = \begin{pmatrix} \dfrac{\sqrt{\beta_x(s)}}{\sqrt{\beta_x(s_L)}} \cdot \left\{ \cos[\varphi_x(s) - \varphi_x(s_L)] + \alpha_x(s_L) \cdot \sin[\varphi_x(s) - \varphi_x(s_L)] \right\} \\[4mm] \dfrac{1}{\sqrt{\beta_x(s_L)} \, \sqrt{\beta_x(s)}} \left\{ [\alpha_x(s_L) - \alpha_x(s)] \cdot \cos[\varphi_x(s) - \varphi_x(s_L)] - \right. \\[2mm] \left. - [1 + \alpha_x(s_L) \alpha_x(s)] \cdot \sin[\varphi_x(s) - \varphi_x(s_L)] \right\} \\[4mm] 0 \\[2mm] 0 \end{pmatrix} ;$$

$$\psi_2 = \begin{pmatrix} \sqrt{\beta_x(s)}\,\sqrt{\beta_x(s_L)}\,\sin[\varphi_x(s) - \varphi_x(s_L)] \\[2mm] \dfrac{\sqrt{\beta_x(s_L)}}{\sqrt{\beta_x(s)}}\left\{\cos[\varphi_x(s) - \varphi_x(s_L)] - \alpha_x(s)\sin[\varphi_x(s) - \varphi_x(s_L)]\right\} \\[2mm] 0 \\[2mm] 0 \end{pmatrix};$$

$$\psi_3 = \begin{pmatrix} 0 \\[2mm] 0 \\[2mm] \dfrac{\sqrt{\beta_z(s)}}{\sqrt{\beta_z(s_L)}}\left\{\cos[\varphi_z(s) - \varphi_z(s_L)] + \alpha_z(s_L)\cdot\sin[\varphi_z(s) - \varphi_z(s_L)]\right\} \\[2mm] \dfrac{1}{\sqrt{\beta_z(s_L)}\,\sqrt{\beta_z(s)}}\left\{[\alpha_z(s_L) - \alpha_z(s)]\cdot\cos[\varphi_z(s) - \varphi_z(s_L)] - [1 + \alpha_z(s_L)\,\alpha_z(s)]\cdot\sin[\varphi_z(s) - \varphi_z(s_L)]\right\} \end{pmatrix};$$

$$\psi_4 = \begin{pmatrix} 0 \\[2mm] 0 \\[2mm] \sqrt{\beta_z(s_L)}\,\sqrt{\beta_z(s)}\cdot\sin[\varphi_z(s) - \varphi_z(s_L)] \\[2mm] \dfrac{\sqrt{\beta_z(s_L)}}{\sqrt{\beta_z(s)}}\left\{\cos[\varphi_z(s) - \varphi_z(s_L)] - \alpha_z(s)\sin[\varphi_z(s) - \varphi_z(s_L)]\right\} \end{pmatrix},$$

the following equation follows from Equations (3.1-3.5), if we also

use the normalization conditions (Equation 2.2) (n = I, II)[5]:

$$w_n = N_n \cdot \begin{pmatrix} \sqrt{\beta_x} \cdot G_{nx} \\[1ex] \frac{1}{\sqrt{\beta_x}} \cdot H_{nx} \\[1ex] \sqrt{\beta_z} \cdot G_{nz} \cdot \frac{\Delta \cdot \sin \mu_z}{2 \left(\cos 2\pi Q_n - \cos \mu_z \right)} \\[2ex] \frac{1}{\sqrt{\beta_z}} \cdot H_{nz} \cdot \frac{\Delta \cdot \sin \mu_z}{2 \left(\cos 2\pi Q_n - \cos \mu_z \right)} \end{pmatrix} \qquad (3.6)$$

where (using $k = x$ or z) we define:

$$G_{nk} = \cos \left[\varphi_k - \varphi_k(s_L) \right] + \sin \left[\varphi_k - \varphi_k(s_L) \right] \cdot \frac{e^{-i \cdot 2\pi Q_n} - \cos \mu_k}{\sin \mu_k} \; ; \qquad (3.7a)$$

$$H_{nk} = - \left(\sin \left[\varphi_k - \varphi_k(s_L) \right] + \alpha_k \cdot \cos \left[\varphi_k - \varphi_k(s_L) \right] \right) + $$

$$+ \left(\cos \left[\varphi_k - \varphi_k(s_L) \right] - \alpha_k \sin \left[\varphi_k - \varphi_k(s_L) \right] \right) \cdot \frac{e^{-i \cdot 2\pi Q_n} - \cos \mu_k}{\sin \mu_k} \; ; \qquad (3.7b)$$

$$N_I = \sqrt{ \frac{\sin \mu_x}{\sin 2\pi Q_I} \cdot \frac{x+1}{2x} } \; ;$$

$$\qquad (3.8)$$

$$N_{II} = \sqrt{ \frac{\sin \mu_x}{\sin 2\pi Q_{II}} \cdot \frac{x-1}{2x} } \; ;$$

$$\cos 2\pi Q_I = \frac{1}{2} \left(\cos \mu_x + \cos \mu_z \right) + \frac{1}{2} \left(\cos \mu_x - \cos \mu_z \right) x \; ;$$

$$\qquad (3.9)$$

$$\cos 2\pi Q_{II} = \frac{1}{2} \left(\cos \mu_x + \cos \mu_z \right) - \frac{1}{2} \left(\cos \mu_x - \cos \mu_z \right) \cdot x \; ;$$

$$\Delta = \delta \cdot \sqrt{\beta_x(s_L)} \sqrt{\beta_z(s_L)} \quad ; \tag{3.10}$$

$$\varkappa = \sqrt{1 + \frac{\Delta^2 \sin \mu_x \sin \mu_z}{(\cos \mu_x - \cos \mu_z)^2}} \quad . \tag{3.11}$$

Here the sign of Q_I and Q_{II} is to be determined by Equation (2.2),

so that the following relations are satisfied:

$$\frac{\sin 2\pi Q_I}{\sin \mu_x} > 0 \, ,$$

$$\tag{3.12}$$

$$\frac{\sin 2\pi Q_{II}}{\sin \mu_x} \cdot \frac{\varkappa - 1}{2\varkappa} > 0 \quad \Longleftrightarrow \quad \frac{\sin 2\pi Q_{II}}{\sin \mu_z} > 0$$

IV. Beam Expansion by Coupling

A) General Case (Q_x, Q_z are arbitrary).

With the help of Equations (3.6-3.11) we are now able to

calculate the beam expansion and the corresponding shift of the luminosity

limit of the Ritson-Amman effect for the case in which the coupling of

the betatron oscillations is brought about by a rotated quadrupole.

First we insert the expressions Equations (3.6-3.8) in Equations

(2.7a, 2.9a, 2.8, and 2.11), and obtain the following for the damping

<u>constants</u>[5]:

$$\alpha_I = \frac{\bar{W}}{2E \cdot \langle \mathcal{K}^2 \rangle} \cdot \left\{ \langle \mathcal{K}^2 \rangle - \frac{x+1}{2x} A_x - \frac{x-1}{2x} A_z \right\} ; \qquad (4.1a)$$

$$\alpha_{II} = \frac{\bar{W}}{2E \cdot \langle \mathcal{K}^2 \rangle} \cdot \left\{ \langle \mathcal{K}^2 \rangle - \frac{x-1}{2x} A_x - \frac{x+1}{2x} A_z \right\} ; $$

$$\qquad (4.1b)$$

$$A_x = \langle \mathcal{K}_x^3 (1 - 2n_x) D_x \rangle ;$$

$$A_z = \langle \mathcal{K}_z^3 (1 - 2n_z) D_z \rangle ;$$

and for the quantities γ_I and γ_{II} :

$$\gamma_I = \frac{\bar{Q}_s}{4E^2 \cdot \langle |\mathcal{K}^3| \rangle} \times$$

$$\times \left\{ \frac{\sin \mu_x}{\sin \ln \tilde{\Omega}_I} \cdot \frac{x+1}{2x} \left[\left\langle |\mathcal{K}_x|^3 \cdot \left(D_x^2 \cdot \frac{1+\alpha_x^2}{\beta_x} + 2 D_x D_x' \cdot \alpha_x + D_x'^2 \cdot \beta_x \right) \right\rangle \right. \right.$$

$$+ (x-1) \cdot \frac{(\omega \mu_x - \omega \mu_z)}{\sin \mu_x} \cdot \left\langle |\mathcal{K}_x|^3 \cdot \left(D_x^2 \cdot \frac{R_x}{\beta_x} + 2 D_x D_x' \cdot S_x + D_x'^2 \cdot \beta_x T_x \right) \right\rangle \right]$$

$$\qquad (4.2a)$$

$$+ \frac{\sin \mu_z}{\sin \ln \tilde{\Omega}_I} \cdot \frac{x-1}{2x} \left[\left\langle |\mathcal{K}_z|^3 \cdot \left(D_z^2 \cdot \frac{1+\alpha_z^2}{\beta_z} + 2 D_z D_z' \cdot \alpha_z + D_z'^2 \cdot \beta_z \right) \right\rangle \right.$$

$$+ (x+1) \cdot \frac{(\omega \mu_x - \omega \mu_z)}{\sin \mu_z} \cdot \left\langle |\mathcal{K}_z|^3 \cdot \left(D_z^2 \cdot \frac{R_z}{\beta_z} + 2 D_z D_z' \cdot S_z + D_z'^2 \cdot \beta_z T_z \right) \right\rangle \right] \right\} ;$$

$$\gamma_{II} = \frac{\bar{Q}_s}{4E^2 \cdot \langle |\kappa|^3 \rangle} \times$$

$$\times \left\{ \frac{\partial m\, \mu_x}{\partial m\, 2\pi\, U_{II}} \cdot \frac{x-1}{2x} \left[\left\langle |\kappa_x|^3 \cdot \left(D_x^2 \cdot \frac{1+\alpha_x^2}{\beta_x} + 2 D_x D_x' \cdot \alpha_x + D_x'^2 \cdot \beta_x \right) \right\rangle \right.\right.$$

$$\left. - (x+1) \cdot \frac{(\omega\mu_x - \omega\mu_z)}{\partial m\, \mu_x} \cdot \left\langle |\kappa_x|^3 \cdot \left(D_x^2 \cdot \frac{R_x}{\beta_x} + 2 D_x D_x' \cdot S_x + D_x'^2 \cdot \beta_x T_x \right) \right\rangle \right] \qquad (4.2b)$$

$$+ \frac{\partial m\, \mu_z}{\partial m\, 2\pi\, U_{II}} \cdot \frac{x+1}{2x} \left[\left\langle |\kappa_z|^3 \cdot \left(D_z^2 \cdot \frac{1+\alpha_z^2}{\beta_z} + 2 D_z D_z' \cdot \alpha_z + D_z'^2 \cdot \beta_z \right) \right\rangle \right.$$

$$\left.\left. - (x-1) \cdot \frac{(\omega\mu_x - \omega\mu_z)}{\partial m\, \mu_z} \cdot \left\langle |\kappa_z|^3 \cdot \left(D_z^2 \cdot \frac{R_z}{\beta_z} + 2 D_z D_z' \cdot S_z + D_z'^2 \cdot \beta_z T_z \right) \right\rangle \right] \right\}$$

where (k = x or z)

$$R_k = - \left(\cos[\varphi_k - \varphi_k(s_L)] - \alpha_k \sin[\varphi_k - \varphi_k(s_L)] \right) \times$$

$$\times \left\{ \left(\sin[\varphi_k - \varphi_k(s)] + \alpha_k \cos[\varphi_k - \varphi_k(s_L)] \right) + \right. \qquad (4.3)$$

$$\left. + \frac{\omega\mu_k}{\partial m\, \mu_k} \cdot \left(\cos[\varphi_k - \varphi_k(s_L)] - \alpha_k \sin[\varphi_k - \varphi_k(s_L)] \right) \right\};$$

$$S_k = \frac{1}{2} \sin[\varphi_k - \varphi_k(s_L)] \cdot \left\{ \left(\sin[\varphi_k - \varphi_k(s_L)] + \alpha_k \cos[\varphi_k - \varphi_k(s_L)] \right) + \right.$$

$$\left. + 2 \frac{\omega\mu_k}{\partial m\, \mu_k} \cdot \left(\cos[\varphi_k - \varphi_k(s_L)] - \alpha_k \sin[\varphi_k - \varphi_k(s_L)] \right) \right\}; \qquad (4.4)$$

$$T_k = \sin[\varphi_k - \varphi_k(s_L)] \cdot \left(\cos[\varphi_k - \varphi_k(s_L)] - \frac{\omega\mu_k}{\partial m\, \mu_k} \sin[\varphi_k - \varphi_k(s_L)] \right); \qquad (4.5)$$

For the emittances, we have:

$$\epsilon_I = \frac{\sin\mu_x}{\sin 2n\theta_I}\cdot\frac{x+1}{2x}\cdot\left\{\epsilon_x + \frac{(\cos\mu_x - \cos\mu_z)}{\sin\mu_x}(x-1)\,\tilde\epsilon_x\right\}\times$$

$$\times\ \frac{\langle\mathcal{K}^2\rangle - A_x}{\langle\mathcal{K}^2\rangle - \dfrac{x+1}{2x}A_x - \dfrac{x-1}{2x}A_z}$$

(4.6a)

$$+\ \frac{\sin\mu_z}{\sin 2n\theta_I}\cdot\frac{x-1}{2x}\cdot\left\{\epsilon_z + \frac{(\cos\mu_x - \cos\mu_z)}{\sin\mu_z}(x+1)\cdot\tilde\epsilon_z\right\}\times$$

$$\times\ \frac{\langle\mathcal{K}^2\rangle - A_z}{\langle\mathcal{K}^2\rangle - \dfrac{x+1}{2x}A_x - \dfrac{x-1}{2x}A_z}\ ;$$

$$\epsilon_{II} = \frac{\sin\mu_x}{\sin 2n\theta_{II}}\cdot\frac{x-1}{2x}\cdot\left\{\epsilon_x - \frac{(\cos\mu_x - \cos\mu_z)}{\sin\mu_x}(x+1)\cdot\tilde\epsilon_x\right\}\times$$

$$\times\ \frac{\langle\mathcal{K}^2\rangle - A_x}{\langle\mathcal{K}^2\rangle - \dfrac{x-1}{2x}A_x - \dfrac{x+1}{2x}A_z}$$

(4.6b)

$$+\ \frac{\sin\mu_z}{\sin 2n\theta_{II}}\cdot\frac{x+1}{2x}\cdot\left\{\epsilon_z - \frac{(\cos\mu_x - \cos\mu_z)}{\sin\mu_z}(x-1)\cdot\tilde\epsilon_z\right\}\times$$

$$\times\ \frac{\langle\mathcal{K}^2\rangle - A_z}{\langle\mathcal{K}^2\rangle - \dfrac{x-1}{2x}A_x - \dfrac{x+1}{2x}A_z}$$

where

$$\epsilon_x = \frac{55\,\hbar c\gamma^3}{32\sqrt{3}\,E}\cdot\frac{\left\langle |\mathcal{K}_x|^3\cdot\left(D_x^2\cdot\dfrac{1+\alpha_x^2}{\beta_x} + 2D_x D_x' \cdot\alpha_x + D_x'^2\cdot\beta_x\right)\right\rangle}{\langle\mathcal{K}^2\rangle - A_x}\ ;$$

$$\tilde\epsilon_x = \frac{55\,\hbar c\gamma^3}{32\sqrt{3}\,E}\cdot\frac{\left\langle |\mathcal{K}_x|^3\cdot\left(D_x^2\cdot\dfrac{R_x}{\beta_x} + 2D_x D_x'\cdot S_x + D_x'^2\cdot\beta_x T_x\right)\right\rangle}{\langle\mathcal{K}^2\rangle - A_x}\ ;$$

(4.7)

$$\epsilon_z = \frac{55\,\hbar c\gamma^3}{32\sqrt{3}\,E}\cdot\frac{\left\langle |\mathcal{K}_z|^3\cdot\left(D_z^2\cdot\dfrac{1+\alpha_z^2}{\beta_z} + 2D_z D_z'\cdot\alpha_z + D_z'^2\cdot\beta_z\right)\right\rangle}{\langle\mathcal{K}^2\rangle - A_z}\ ;$$

$$\tilde\epsilon_z = \frac{55\,\hbar c\gamma^3}{32\sqrt{3}\,E}\cdot\frac{\left\langle |\mathcal{K}_z|^3\cdot\left(D_z^2\cdot\dfrac{R_z}{\beta_z} + 2D_z D_z'\cdot S_z + D_z'^2\cdot\beta_z T_z\right)\right\rangle}{\langle\mathcal{K}^2\rangle - A_z}$$

Finally, for the <u>average beam size</u>

$$\sigma_x^2 = \beta_x \cdot \left\{ \epsilon_I \cdot \frac{\sin \mu_x}{\sin 2\pi Q_I} \frac{x+1}{2x} \left[1 + \frac{(\cos \mu_x - \cos \mu_z)}{\sin \mu_x} (x-1) \cdot T_x \right] \right.$$

(4.8a)

$$\left. + \epsilon_{II} \cdot \frac{\sin \mu_x}{\sin 2\pi Q_{II}} \frac{x-1}{2x} \left[1 - \frac{(\cos \mu_x - \cos \mu_z)}{\sin \mu_x} (x+1) \cdot T_x \right] \right\} ;$$

$$\sigma_z^2 = \beta_z \cdot \left\{ \epsilon_I \cdot \frac{\sin \mu_z}{\sin 2\pi Q_I} \frac{x-1}{2x} \left[1 + \frac{(\cos \mu_x - \cos \mu_z)}{\sin \mu_z} (x+1) \cdot T_z \right] \right.$$

(4.8b)

$$\left. + \epsilon_{II} \cdot \frac{\sin \mu_z}{\sin 2\pi Q_{II}} \frac{x+1}{2x} \left[1 - \frac{(\cos \mu_x - \cos \mu_z)}{\sin \mu_z} (x-1) \cdot T_z \right] \right\} ;$$

$$\sigma_{xz} = \sigma_{xz}^{(1)} + \sigma_{xz}^{(2)} ;$$

$$\sigma_{xz}^{(1)} = \frac{1}{2} \cos \left[\left(\varphi_x(s) - \varphi_x(s_L) \right) - \left(\varphi_z(s) - \varphi_z(s_L) \right) \right] \cdot \sqrt{\beta_x \beta_z} \times$$

$$\times \frac{\Delta \cdot \sin \mu_x \sin \mu_z}{(\cos \mu_x - \cos \mu_z) \cdot \sqrt{1 + \frac{\Delta^2 \sin \mu_x \sin \mu_z}{(\cos \mu_x - \cos \mu_z)^2}}} \times$$

(4.9)

$$\times \left\{ \frac{\epsilon_I}{\sin 2\pi Q_I} + \frac{\epsilon_{II}}{\sin 2\pi Q_{II}} \right\} ;$$

$$\sigma_{xz}^{(2)} = \frac{1}{2} \left[\cos \mu_x - \cos \mu_z \right] \cdot \sqrt{\beta_x \beta_z} \times$$

$$\times \frac{\Delta \cdot \sin \mu_x \sin \mu_z}{(\cos \mu_x - \cos \mu_z) \cdot \sqrt{1 + \frac{\Delta^2 \sin \mu_x \sin \mu_z}{(\cos \mu_x - \cos \mu_z)^2}}} \times$$

(4.9a)

$$\times \left\{ \frac{\epsilon_I}{\sin 2\pi Q_I} \left[\frac{x+1}{\sin \mu_z} \cdot U_{xz} + \frac{x-1}{\sin \mu_x} \cdot V_{xz} \right] \right.$$

$$\left. - \frac{\epsilon_{II}}{\sin 2\pi Q_{II}} \left[\frac{x-1}{\sin \mu_z} \cdot U_{xz} + \frac{x+1}{\sin \mu_x} \cdot V_{xz} \right] \right\}$$

where

$$U_{xz} = \cos[\varphi_x(s) - \varphi_x(s_L)] \cdot \cos[\varphi_z(s) - \varphi_z(s_L)] -$$

$$- \frac{\cos\mu_x}{\sin\mu_x} \sin[\varphi_x(s) - \varphi_x(s_L)] \cdot \sin[\varphi_z(s) - \varphi_z(s_L)];$$

(4.9b)

$$V_{xz} = \sin[\varphi_x(s) - \varphi_x(s_L)] \cdot \cos[\varphi_z(s) - \varphi_z(s_L)] -$$

$$- \frac{\cos\mu_z}{\sin\mu_z} \sin[\varphi_x(s) - \varphi_x(s_L)] \cdot \sin[\varphi_z(s) - \varphi_z(s_L)].$$

(The quantity σ_{xz} determines the rotation angle of the beam cross section, by Equation (2.14)).

B)   General Requirements for Beam Expansion

Equations (4.8) and (4.9) determine the beam cross section at every position s of the ring, particularly in the interaction regions, if the position s = s_L of the lens that causes the coupling, and the lens power δ are given. For the case δ = 0 (or Δ = 0) these equations simplify to the well-known equations for an uncoupled machine:

$$\sigma_x = \epsilon_x \cdot \sqrt{\beta_x} \,,$$

$$\sigma_z = \epsilon_z \cdot \sqrt{\beta_z} \,,$$

$$\sigma_{xz} = 0$$

(4.10)

According to the program set out in the introduction, we must now determine whether the beam cross section can be expanded in the vertical direction at the intersection points, in order to weaken the space-charge forces in this direction and thereby stabilize the betatron oscillations.

The following conditions are generally set for such beam expansion:

(I) The Q-shift $(Q_x, Q_z) \rightarrow (Q_I, Q_{II})$ caused by coupling must remain sufficiently small so that the operating point (Q_I, Q_{II}) does not reach a stop band.

In other words, the conditions of stability of the beam steering must be fulfilled[4]:

$$Q_I, Q_{II} \neq n, \frac{n}{2}$$

(4.11)

where n is an integer.

(II)   The beam expansion must occur for the smallest possible lens power δ.

(III)   When the coupling is introduced, the dispersion orbits $D_x(s)$, $D_z(s)$ may not be disturbed. The lens must be positioned at a point $s = s_L$ such that D_x and D_z vanish:

$$D_x(s_L) = D_z(s_L) = 0 \quad . \tag{4.12}$$

(IV)   Since space-charge forces arising from the interpenetration of two rotated beams act, in a linear approximation, as a quadrupole lens rotated through the same angle, the beam's rotation in the interaction region must be so small that the additional coupling caused by the beam-beam interaction does not noticeably affect the beam cross section. Such additional coupling must be negligible compared to that arising from the quadrupole lens.

C)   The Resonance Condition

In order to fulfill the requirements (I-IV), we first consider that, in accordance with Equations (3.10) and (4.8), the average beam dimensions in the $x-$ and $z-$directions change appreciably only if

the following relationships hold[*]:

$$\sin\mu_x \cdot \sin\mu_z > 0 \qquad (4.13)$$

$$\Delta^2 \cdot \sin\mu_x \cdot \sin\mu_z > (\cos\mu_x - \cos\mu_z)^2 \qquad (4.14)$$

On the other hand, it follows from Equation (3.9) that the quantity $\varkappa$ (and correspondingly the coupling strength of the lens Δ) may not be arbitrarily large. Otherwise the quantity

$$\omega \, 2\pi Q_n, \quad (n = I, II)$$

from Equation (3.9) would exceed unity, Q_n would be complex, and the beam would be unstable[4]:

$$|\cos 2\pi Q_n| < 1 , \quad (n = I, II);$$

(condition of stability). $\qquad (4.15)$

All three conditions (4.13, 4.14, 4.15) are simultaneously satisfied only if we pass to the resonant case:

$$Q_z - Q_x = n + \delta Q;$$

$$n = \text{integer}; \qquad (4.16)$$

$$|\delta Q| \ll 1$$

---

[*] The case $\sin\mu_x \sin\mu_z < 0$ has an unfavorable effect on the Amman-Ritson effect, and we do not consider it. Besides, requirement (IV) cannot be fulfilled in the case $\sin\mu_x \sin\mu_z < 0$ (see Equation 4.18).

Actually, the requirement (4.13) is automatically fulfilled with:

$$\sin\mu_x \equiv \sin 2\pi Q_x$$
$$= \sin(2\pi Q_x + \pi + \delta Q)$$
$$\approx \sin 2\pi Q_x \equiv \sin\mu_x$$

Further, we have from Equation (4.14):

$$\Delta^2 \gg (2\pi \cdot \delta Q)^2 . \qquad (4.17)$$

For $|\delta Q| \ll 1$, this condition can be satisfied with relatively small

coupling strength Δ , without violating the conditions of stability

(4.15). These may now be written in the form:

$$|\cos 2\pi Q_I| < 1 , \qquad |\cos 2\pi Q_{II}| < 1 \qquad (4.18a)$$

where

$$\begin{cases} \cos 2\pi Q_I = \cos\mu_x + (x-1) \cdot \pi \cdot \delta Q \cdot \sin\mu_x \\ \cos 2\pi Q_{II} = \cos\mu_x - (x+1) \cdot \pi \cdot \delta Q \cdot \sin\mu_x \end{cases} \qquad (4.18b)$$

It is clear from the above that with the resonance condition (4.16)

alone the requirements (I) and (II) can be satisfied.

In order to show that the requirements (III) and (IV) can also

be fulfilled, we choose for the planned DESY storage ring the lens

position

$$s_L = \frac{C}{4}$$

(C = orbit circumference),

so that the rotated quadrupole is exactly in the middle between the two interaction regions. The following relationship then holds for the DESY storage ring, in agreement with requirement III:

$$D_x \left(\frac{C}{4}\right) = D_z \left(\frac{C}{4}\right) = 0$$

We require further that the difference between the Q-values $(Q_x - Q_z)$ is nearly an odd number:

$$Q_z - Q_x = (2m + 1) + \delta Q ;$$

$$m = \text{integer}.$$

Then, considering the phase difference (which is determining for the quantity δ_{xz}):

$$\Delta \varphi (s) = [\varphi_x (s) - \varphi_x (s_L)] - [\varphi_z (s) - \varphi_z (s_L)]$$

at the intersection points $s = 0, \frac{C}{2}$ where

$$\varphi_x (0) - \varphi_x (s_L) = -\frac{1}{4} \cdot 2\pi Q_x = -\frac{\pi}{2} \cdot Q_x ;$$

$$\varphi_z (0) - \varphi_z (s_L) = -\frac{1}{4} \cdot 2\pi Q_z = -\frac{\pi}{2} \cdot Q_z ;$$

$$\varphi_x \left(\frac{C}{2}\right) - \varphi_x (s_L) = \frac{1}{2} \cdot 2\pi Q_x - \frac{1}{4} \cdot 2\pi Q_x = \frac{\pi}{2} Q_x ;$$

$$\varphi_z \left(\frac{C}{2}\right) - \varphi_z (s_L) = \frac{1}{2} \cdot 2\pi Q_z - \frac{1}{4} \cdot 2\pi Q_z = \frac{\pi}{2} Q_z$$

We shall have

$$\Delta \varphi (0) = \frac{\pi}{2} (Q_z - Q_x) = \frac{\pi}{2} \cdot (2m + 1) + \frac{\pi}{2} \cdot \delta Q ;$$

$$\Delta \varphi \left(\frac{C}{2}\right) = -\Delta \varphi (0) ,$$

For the factor $\cos(\Delta\phi)$ in Equation (4.9a), we have

$$\cos(\Delta\phi) = -(-1)^m \cdot \frac{\pi}{2} \cdot \delta Q .$$

As a consequence, we can make the quantity σ_{xz} arbitrarily small

as required by condition (IV), by adjusting to a narrow resonance

$(|\delta Q| \ll 1)$. This quantity, according to Equation (2.14), determines

the rotation of the beam and therefore the rotation angle Θ in the

interaction regions.

In the following section we determine the beam cross section

$(\sigma_x, \sigma_z, \text{ and } \sigma_{xz})$ that is produced at the intersection points at

resonance.

D)   Beam Cross Section at Resonance

The quantities σ_x, σ_z and σ_{xz} , which characterize the beam

size, may now be calculated for the case of resonance from the general

formulae, Equations (4.8) and (4.9), taking into account the following

relation, obtained from the resonance condition (4.16):

$$\cos\mu_x - \cos\mu_z = \cos 2\pi Q_z - \cos 2\pi [Q_x + n + \delta Q] \approx 2\pi \cdot \delta Q \cdot \sin\mu_x . \quad (4.19)$$

In principle, we are now in a position to determine the beam cross

section for an arbitrary coupling strength Δ, if the beam cross section

"detuning" δQ is specified.

However, Equations (4.8) and (4.9) can still be appreciably

simplified if the following assumptions are made. These assumptions

apply largely to the case of the DESY storage ring, or are realizable in

this case (by adjusting to sharp resonance $|\delta Q| \ll 1$ and relatively

weak coupling):

$$\varepsilon_z \ll \varepsilon_x \quad ; \quad |\tilde{\varepsilon}_z| \ll \varepsilon_x \quad ; \tag{4.20a}$$

$$2\pi \cdot |\delta Q| \cdot x \cdot |\tilde{\varepsilon}_x| \ll \varepsilon_x \quad ; \tag{4.20b}$$

$$2\pi \cdot |\delta Q| \cdot x \cdot |T_{x,z}| \ll 1 \quad \text{for} \quad z = 0, \frac{C}{2} \quad ; \tag{4.20c}$$

$$2\pi \cdot |\delta Q| \cdot x \cdot \left| \frac{\omega \mu_z}{2 m \mu_x} \right| \ll 1 \quad . \tag{4.20d}$$

With these assumptions we have

$$\frac{2m^2 \mu_x}{2m^2 2\pi Q_I} \approx \frac{2m^2 \mu_z}{2m^2 2\pi Q_{II}} \approx 1$$

and all the terms in Equations (4.6) and (4.8) which contain ε_z, $\tilde{\varepsilon}_x$, $\tilde{\varepsilon}_z$

as a factor may be ignored in these equations. Therefore, we obtain from

Equation (4.8) the following expressions for the beam cross section at

resonance, with the aid of (4.6):

$$\sigma_x^2 = \beta_x \cdot F_x(x) \quad ; \tag{4.21a}$$

$$\sigma_z^2 = \beta_z \cdot F_z(x) \quad ; \tag{4.21b}$$

$$\sigma_{xz} = \sqrt{\beta_x} \sqrt{\beta_z} \cdot F_{xz}(x) \tag{4.21c}$$

where

$$F_x(x) = \varepsilon_x \cdot \left\{ \frac{(x+1)^2}{4x^2} \cdot \frac{\langle k^2 \rangle - A_x}{\langle k^2 \rangle - \frac{x+1}{2x} A_x - \frac{x-1}{2x} A_z} \right. +$$

$$\left. + \frac{(x-1)^2}{4x^2} \cdot \frac{\langle k^2 \rangle - A_x}{\langle k^2 \rangle - \frac{x-1}{2x} A_x - \frac{x+1}{2x} A_z} \right\} \quad ; \tag{4.22a}$$

$$F_z(x) = \varepsilon_x \cdot \frac{x^2 - 1}{4x^2} \cdot \left\{ \frac{\langle k^2 \rangle - A_x}{\langle k^2 \rangle - \frac{x+1}{2x} A_x - \frac{x-1}{2x} A_z} \right. +$$

$$\left. + \frac{\langle k^2 \rangle - A_x}{\langle k^2 \rangle - \frac{x-1}{2x} A_x - \frac{x+1}{2x} A_z} \right\} \quad ; \tag{4.22b}$$

$$F_{x_2}(x) = F_{x_2}^{(1)}(x) + F_{x_2}^{(2)}(x) \; ;$$

$$F_{x_2}^{(1)}(x) = -(-1)^n \cdot \frac{\pi}{4} \, \delta Q \; \times$$

$$\times \; \frac{\Delta \cdot \sin \mu_x}{(\cos \mu_x - \cos \mu_z) \cdot \sqrt{1 + \frac{\Delta^2 \cdot \sin \mu_x \sin \mu_z}{(\cos \mu_x - \cos \mu_z)^2}}} \; \times$$

$$\times \; \varepsilon_x \cdot \left\{ \frac{x+1}{2x} \cdot \frac{\langle \varkappa^2 \rangle - A_x}{\langle \varkappa^2 \rangle - \frac{x+1}{2x} A_x - \frac{x-1}{2x} A_z} + \right.$$

$$\left. + \; \frac{x-1}{2x} \cdot \frac{\langle \varkappa^2 \rangle - A_x}{\langle \varkappa^2 \rangle - \frac{x-1}{2x} A_x - \frac{x+1}{2x} A_z} \right\} \; ;$$

$$F_{x_2}^{(2)}(x) = \pi \cdot \delta Q \; \times \qquad\qquad (4.22c)$$

$$\times \; \frac{\Delta \cdot \sin \mu_x}{(\cos \mu_x - \cos \mu_z) \cdot \sqrt{1 + \frac{\Delta^2 \cdot \sin \mu_x \sin \mu_z}{(\cos \mu_x - \cos \mu_z)^2}}} \; \times$$

$$\times \; \varepsilon_x \cdot \left\{ \frac{(x+1)^2}{2x} \cdot \frac{\langle \varkappa^2 \rangle - A_x}{\langle \varkappa^2 \rangle - \frac{x+1}{2x} A_x - \frac{x-1}{2x} A_z} - \right.$$

$$\left. - \; \frac{(x-1)^2}{2x} \cdot \frac{\langle \varkappa^2 \rangle - A_x}{\langle \varkappa^2 \rangle - \frac{x-1}{2x} A_x - \frac{x+1}{2x} A_z} \right\} \times$$

$$\times \; \left\{ \cos \left[\varphi_x(s) - \varphi_x(s_L) \right] \cdot \sin \left[\varphi_z(s) - \varphi_z(s_L) \right] - \right.$$

$$\left. - \; \frac{\cos \mu_x}{\sin \mu_x} \cdot \sin \left[\varphi_x(s) - \varphi_x(s_L) \right] \cdot \sin \left[\varphi_z(s) - \varphi_z(s_L) \right] \right\} .$$

In order to put conditions (4.20) in a form more convenient for

evaluation, we put from the Eqs. (4.3, 4, 5, 7) for $\tilde{\varepsilon}_x$ and $\left(T_{x,z}\right)_{s=0,\frac{c}{2}}$

$$\tilde{\varepsilon}_x = \tilde{\varepsilon}_x^{(1)} + \frac{\omega \mu_x}{\sin \mu_x} \cdot \tilde{\varepsilon}_x^{(2)} ; \qquad (4.23)$$

$$T_{x,z} = T_{x,z}^{(1)} + \frac{\omega \mu_{x,z}}{\sin \mu_{x,z}} \cdot T_{x,z}^{(2)} \quad (\text{mit } s = 0, \frac{c}{2}) \quad (4.24)$$

where we have

$$\tilde{\varepsilon}_x^{(v)} = \frac{55 \hbar c \gamma^3}{32 \sqrt{3} E} \cdot \frac{\left\langle |\kappa_x|^3 \left(D_x^2 \cdot \frac{R_x^{(v)}}{\beta_x} + 2 D_x D_x' \cdot S_x^{(v)} + D_x'^2 \cdot \beta_x T_x^{(v)} \right) \right\rangle}{\left\langle \kappa^2 \right\rangle - \bar{\eta}_x}$$

$$(v = 1, 2);$$

$$R_x^{(1)} = -\left\{ \cos[\varphi_x - \varphi_x(s_L)] - d_x \cdot \sin[\varphi_x - \varphi_x(s_L)] \right\} \times$$

$$\times \left\{ \sin[\varphi_x - \varphi_x(s_L)] + d_x \cdot \cos[\varphi_x - \varphi_x(s_L)] \right\};$$

$$R_x^{(2)} = -\left\{ \cos[\varphi_x - \varphi_x(s_L)] - d_x \sin[\varphi_x - \varphi_x(s_L)] \right\}^2;$$

$$\qquad\qquad (4.25)$$

$$S_x^{(1)} = \frac{1}{2} \sin[\varphi_x - \varphi_x(s_L)] \cdot \left\{ \sin[\varphi_x - \varphi_x(s_L)] + d_x \cos[\varphi_x - \varphi_x(s_L)] \right\};$$

$$S_x^{(2)} = \sin[\varphi_x - \varphi_x(s_L)] \cdot \left\{ \cos[\varphi_x - \varphi_x(s_L)] - d_x \sin[\varphi_x - \varphi_x(s_L)] \right\};$$

$$T_x^{(1)} = \sin[\varphi_x - \varphi_x(s_L)] \cdot \cos[\varphi_x - \varphi_x(s_L)];$$

$$T_x^{(2)} = -\sin^2[\varphi_x - \varphi_x(s_L)],$$

$$T_z^{(1)} = \sin[\varphi_z - \varphi_z(s_L)] \cdot \cos[\varphi_z - \varphi_z(s_L)],$$

$$T_z^{(2)} = -\sin^2[\varphi_z - \varphi_z(s_L)]$$

as well as

$$\left(T_x^{(1)}\right)_{s=0} = -\left(T_x^{(1)}\right)_{s=\frac{c}{2}} = -\frac{1}{2}\sin\left(\mathcal{Q}_x \cdot n\right) ;$$

$$\left(T_x^{(2)}\right)_{s=0} = +\left(T_x^{(2)}\right)_{s=\frac{c}{2}} = -\sin^2\left(\mathcal{Q}_x \cdot \frac{n}{2}\right) ;$$

$$\left(T_z^{(1)}\right)_{s=0} = -\left(T_x^{(1)}\right)_{s=\frac{c}{2}} = -\frac{1}{2}\sin\left(\mathcal{Q}_z \cdot n\right) ;$$

$$\left(T_z^{(2)}\right)_{s=0} = +\left(T_z^{(2)}\right)_{s=\frac{c}{2}} = -\sin^2\left(\mathcal{Q}_z \cdot \frac{n}{2}\right) .$$

$$(4.26)$$

Then from Equation (4.20b) we have the conditions

$$2n \cdot \delta\mathcal{Q} \cdot \varkappa \cdot \left|\tilde{\varepsilon}_x^{(1)}\right| \ll \varepsilon_x ; \qquad (4.27a)$$

$$2n \cdot \delta\mathcal{Q} \cdot \varkappa \cdot \left|\tilde{\varepsilon}_x^{(2)}\right| \cdot \left|\frac{\cos\mu_x}{\sin\mu_x}\right| \ll \varepsilon_x \qquad (4.27b)$$

and from Equation (4.20c)

$$n \cdot |\delta\mathcal{Q}| \cdot \varkappa \cdot \left|\sin\left(\mathcal{Q}_x \cdot n\right)\right| \ll 1 ; \qquad (4.28a)$$

$$2n \cdot |\delta\mathcal{Q}| \cdot \varkappa \cdot \left|\frac{\cos\mu_x}{\sin\mu_x}\right| \cdot \sin^2\left(\mathcal{Q}_x \cdot \frac{n}{2}\right) \ll 1 ; \qquad (4.28b)$$

$$2n \cdot |\delta\mathcal{Q}| \cdot \varkappa \cdot \left|\frac{\cos\mu_x}{\sin\mu_x}\right| \cdot \sin^2\left(\mathcal{Q}_z \cdot \frac{n}{2}\right) \ll 1 , \qquad (4.28c)$$

of which Equations (4.28b, c) along with Equation (4.20d) are

automatically fulfilled.

If we also have the following inequalities

$$| \tilde{\epsilon}_x^{(1)} | \; << \; \epsilon_x \; ;$$

$$| \tilde{\epsilon}_x^{(2)} | \; \lesssim \; \epsilon_x \; , \tag{4.29}$$

which hold for the DESY storage ring, we may ignore Equation (4.27a),

and Equation (4.27b) becomes identical to Equation (4.20d). Altogether,

we thus arrive at the following requirements which are equivalent

to Equations (4.20):

$$1) \quad \epsilon_z \; << \; \epsilon_x \; ; \quad | \tilde{\epsilon}_z | \; << \; \epsilon_x \; ;$$

$$2) \quad n \cdot \delta Q \cdot \varkappa \cdot | \sin (Q_x \cdot \pi) | \; << \; 1 \; ; \tag{4.20'}$$

$$3) \quad 2\pi \cdot | \delta Q | \cdot \varkappa \cdot \left| \frac{\cos \mu_x}{\sin \mu_x} \right| \; << \; 1 \; ,$$

These are to be satisfied, together with Equation (4.29), for

the equations (4.21) and (4.22) for the beam cross section to be

valid. A suitable shift of the operating point may be required.

For the DESY storage ring (without FODO channel), we may

subsequently use the following to determine the beam cross section:

$$\frac{\langle \varkappa^2 \rangle - \beta_y}{\langle \varkappa^2 \rangle - \frac{\varkappa + 1}{2\varkappa} \beta_x - \frac{\varkappa - 1}{2\varkappa} \beta_z} \; \approx \; 1 \; , \tag{4.30a}$$

$$\frac{\langle \varkappa^2 \rangle - \mathcal{A}_x}{\langle \varkappa^2 \rangle - \frac{\varkappa - 1}{2\varkappa} \mathcal{A}_x - \frac{\varkappa + 1}{2\varkappa} \mathcal{A}_z} \approx 1 \tag{4.30b}$$

This is because we have

$$|\mathcal{A}_x|, \quad |\mathcal{A}_z| \ll \langle \varkappa^2 \rangle \tag{4.31}$$

so that Equation (4.22) goes over into

$$F_x(\varkappa) = \varepsilon_x \cdot \frac{\varkappa^2 + 1}{2\varkappa^2} ; \tag{4.32a}$$

$$F_z(\varkappa) = \varepsilon_z \cdot \frac{\varkappa^2 - 1}{2\varkappa^2} ; \tag{4.32b}$$

$$F_{xz}(\varkappa) = \varepsilon_x \cdot n \cdot \delta Q \cdot$$

$$\cdot \frac{\Delta \cdot \sin \mu_x}{(\cos \mu_x - \cos \mu_z)\sqrt{1 + \frac{\Delta^2 \cdot \sin \mu_x \sin \mu_z}{(\cos \mu_x - \cos \mu_z)^2}}} \cdot \tag{4.32c}$$

$$\cdot \left\{ -(-1)^m \cdot \frac{1}{4} + \right.$$

$$+ 2 \cdot \left[\cos\left(\varphi_x(s) - \varphi_x(s_L)\right) \sin\left(\varphi_z(s) - \varphi_z(s_L)\right) - \right.$$

$$\left. - \frac{\cos \mu_x}{\sin \mu_x} \cdot \sin\left(\varphi_x(s) - \varphi_x(s_L)\right) \sin\left(\varphi_z(s) - \varphi_z(s_L)\right) \right] \right\} .$$

If $\varkappa^2 \gg 1$, we have a nearly circular beam cross section, particularly

in the interaction regions where we have $\beta_x = \beta_z$ (for the DESY storage

ring).

The radius is given by

$$r = \frac{1}{\sqrt{2}} \sqrt{\varepsilon_x} \cdot \sqrt{\beta_x} \ .$$

In the case of decoupling ($\varkappa = 1$), we have from Equations (4.22) and

(4.32):

$$F_x(1) = \varepsilon_x \ ;$$

$$F_z(1) = 0 \ ;$$

$$F_{xz}(1) = 0 \ .$$

That is, the beam height vanishes. In practice, however, there is a

small vertical expansion due to scattering on gas:

$$\sigma_{zo}^2 = \beta_z \cdot (\varepsilon_z)_{Gas} \ ,$$

We have ignored this expansion in calculating the beam cross section

(Equations 4.8 and 4.9). In order that Equations (4.22) and (4.32)

remain valid, we must assume a coupling sufficiently strong for the

following to hold:

$$F_z(\varkappa) \gg (\varepsilon_z)_{Gas} \tag{4.33}$$

In Section VB, we shall estimate more accurately the remaining

coupling due to the space-charge forces together with a small beam

rotation $\Theta(0)$ or $\Theta(\frac{C}{2})$. This coupling is determined by the quantity

σ_{xz} or by F_{xz} . Before that, we shall determine the Q-shift caused

by the Amman-Ritson effect.

V.   Amman-Ritson Effect with Coupling

A)   The Q-Shift

The Amman-Ritson effect is in general described by Equations

(2.15) and (2.16) which give the shift of the operating point due to

the beam-beam interaction for an arbitrary rotation angle Θ . As

we have seen in Section IV, one can set $\Theta = 0$ as an approximation in

the case of resonance (if $Q_x = Q_z$ 2m + 1 is valid). If the

relations given by (3.6), (3.7) and (3.8) are taken into consideration,

we have, from Equations (2.15) and (2.16):

$$\Delta Q_I = \frac{r_e \cdot N}{2 \pi q \cdot (\sigma_x + \sigma_z)} \cdot \left\{ \frac{|v_{I1}|^2}{\sigma_x} + \frac{|v_{I3}|^2}{\sigma_z} \right\} ;$$

$$\Delta Q_{II} = \frac{r_e \cdot N}{2 \pi q \cdot (\sigma_x + \sigma_z)} \cdot \left\{ \frac{|v_{II1}|^2}{\sigma_x} + \frac{|v_{II3}|^2}{\sigma_z} \right\} ;$$

(5.1a)

r_e = classical electron radius

q = number of bunches

N = number of particles

where

$$|v_{I1}|^2 = \beta_x \cdot \frac{\sin \mu_x}{\sin 2\pi Q_I} \cdot \frac{\kappa + 1}{2\kappa} \cdot \left\{ 1 + 2\pi \cdot \delta Q \cdot (\kappa - 1) \cdot T_x \right\};$$

$$|v_{I3}|^2 = \beta_z \cdot \frac{\sin \mu_z}{\sin 2\pi Q_I} \cdot \frac{\kappa - 1}{2\kappa} \cdot \left\{ 1 + 2\pi \cdot \delta Q \cdot (\kappa + 1) \cdot T_z \right\};$$

$$(5.1b)$$

$$|v_{II1}|^2 = \beta_x \cdot \frac{\sin \mu_x}{\sin 2\pi Q_{II}} \cdot \frac{\kappa - 1}{2\kappa} \cdot \left\{ 1 - 2\pi \cdot \delta Q \cdot (\kappa + 1) \cdot T_x \right\};$$

$$|v_{II3}|^2 = \beta_z \cdot \frac{\sin \mu_z}{\sin 2\pi Q_{II}} \cdot \frac{\kappa + 1}{2\kappa} \cdot \left\{ 1 - 2\pi \cdot \delta Q \cdot (\kappa - 1) \cdot T_z \right\},$$

where the functions $T_x(s)$ and $T_z(s)$ are to be taken from Equation (4.5).

Equations (5.1a, b) are valid for arbitrary lens coupling strengths (arbitrary κ), assuming that the σ_x and σ_z are given. In particular, for $\kappa = 1$ Equations (5.1a, b) yield the following well known relations for a decoupled machine $(\sigma_{zo} \ll \sigma_{xo})$:

$$\Delta Q_x = \frac{r_e \cdot N \cdot \beta_x}{2\pi \cdot \gamma \cdot \sigma_{xo} (\sigma_{xo} + \sigma_{zo})} \approx \frac{r_e \cdot N}{2\pi \cdot \gamma \cdot \varepsilon_x} ;$$

$$(5.2)$$

$$\Delta Q_z = \frac{r_e \cdot N \cdot \beta_z}{2\pi \cdot \gamma \cdot \sigma_{zo} (\sigma_{xo} + \sigma_{zo})} \approx \frac{r_e \cdot N \cdot \beta_z}{2\pi \cdot \gamma \cdot \sigma_{zo} \sigma_{xo}} ;$$

σ_{xo} = beam half width ⎫
⎬ decoupled
σ_{zo} = beam half height ⎭

Fur further analysis of these equations we again assume that the conditions given in Equation (4.20) are fulfilled.

By inserting Equation (4.21) in Equation (5.1), we obtain:

$$\Delta Q_I = \frac{r_e \cdot N}{\ln q \cdot (c_x + c_2)} \cdot \left\{ \frac{x+1}{2k} \cdot \frac{\beta_x}{c_x} + \frac{x-1}{2k} \cdot \frac{\beta_2}{c_2} \right\}$$

$$= \frac{r_e \cdot N}{\ln q \cdot \left(\sqrt{\beta_x F_x} + \sqrt{\beta_2 F_2} \right)} \cdot \left\{ \frac{x+1}{2x} \cdot \sqrt{\frac{\beta_x}{F_x}} + \frac{x-1}{2x} \cdot \sqrt{\frac{\beta_2}{F_2}} \right\};$$

(5.3a)

$$\Delta Q_{II} = \frac{r_e \cdot N}{\ln q \cdot (c_x + c_2)} \cdot \left\{ \frac{x-1}{2x} \cdot \frac{\beta_x}{c_x} + \frac{x+1}{2k} \cdot \frac{\beta_2}{c_2} \right\}$$

$$= \frac{r_e \cdot N}{\ln q \cdot \left(\sqrt{\beta_x F_x} + \sqrt{\beta_2 F_2} \right)} \cdot \left\{ \frac{x-1}{2x} \cdot \sqrt{\frac{\beta_x}{F_x}} + \frac{x+1}{2x} \cdot \sqrt{\frac{\beta_2}{F_2}} \right\},$$

(5.3b)

with the functions $F_1(\kappa)$ and $F_2(\kappa)$ given in Equations (4.22) and (4.32).

For the special case

$$|\beta_x|, \quad |\beta_2| \quad << \quad \langle \kappa^2 \rangle$$

we have from Equation (4.32)

$$\Delta Q_I = \Delta Q_x \cdot \frac{x}{\sqrt{\beta_x} \cdot \sqrt{x^2+1} + \sqrt{\beta_2} \cdot \sqrt{x^2-1}} \times$$

$$\times \left\{ \sqrt{\beta_x} \cdot \frac{x+1}{\sqrt{x^2+1}} + \sqrt{\beta_2} \cdot \frac{x-1}{\sqrt{x^2-1}} \right\};$$

(5.4a)

$$\Delta Q_z = \Delta Q_x \cdot \frac{x}{\sqrt{\beta_x} \cdot \sqrt{x^2+1} + \sqrt{\beta_z} \cdot \sqrt{x^2-1}} \times \tag{5.4b}$$

$$\times \left\{ \sqrt{\beta_x} \cdot \frac{x-1}{\sqrt{x^2+1}} + \sqrt{\beta_z} \cdot \frac{x+1}{\sqrt{x^2-1}} \right\} .$$

Here the condition given by (4.33) is to be observed. This condition

restricts the validity of Equations (5.3) and (5.4) to values of κ that

satisfy the relation:

$$\sigma_z \gg \sigma_{zo} \times \sqrt{(\epsilon_z)_{Gas}} \cdot \sqrt{\beta_z} .$$

It follows from these equations that when the coupling is

introduced, an equalization of the space-charge forces takes place,

as stated in the introduction. This is because the quantities ΔQ_I and

ΔQ_{II} approach each other with increasing coupling strength, and become

equal for $\kappa^2 \gg 1$.

The factor by which the Amman-Ritson luminosity limit is

raised with this new and more favorable distribution of the space-charge

forces is given by the ratio:

$$\frac{\Delta Q_z}{\Delta Q_{II}} .$$

For the case

$$|\eta_x|, \ |\eta_z| \ll \langle \kappa^2 \rangle ,$$

Equations (5.2) and (5.4) give the following expression for this ratio:

$$\frac{\Delta Q_1}{\Delta Q_{II}} = \frac{\sigma_{x_0}}{\sigma_{30}} \cdot \frac{\beta_2}{\beta_x} \cdot \frac{\sqrt{\beta_x} \sqrt{x^2 + 1} + \sqrt{\beta_2} \sqrt{x^2 - 1}}{x} \cdot x$$

$$x \cdot \frac{\sqrt{x^4 - 1}}{\sqrt{\beta_x} \cdot (x - 1) \sqrt{x^2 - 1} + \sqrt{\beta_2} \cdot (x + 1) \sqrt{x^2 + 1}} \cdot$$

(5.5)

Numerical Example:

$$L = 0,60 \ m \qquad \text{(Lens distance)}$$

$$k = 0,015 \ m^{-2} \qquad \text{(Lens power)}$$

$$\beta_x(s_L) = 11,24 \ m$$

$$\beta_z(s_L) = 32,35 \ m$$

$$\Rightarrow \quad \delta = k \cdot L = 0,009 \ m^{-1}$$

$$\sqrt{\beta_x(s_L)\,\beta_z(s_L)} = 19,07 \ m$$

$$\Delta = \delta \cdot \sqrt{\beta_x(s_L)\,\beta_z(s_L)} = 0,172$$

$$2n \cdot \delta Q = 0,1 \quad \Rightarrow \quad x \approx 2$$

$$\sigma_x^2 = \varepsilon_x \beta_x \cdot \frac{5}{8} \quad ; \quad \sigma_x = 0,79 \cdot \sqrt{\varepsilon_x \beta_x}$$

$$\sigma_z^2 = \varepsilon_x \beta_z \cdot \frac{3}{8} \quad ; \quad \sigma_z = 0,61 \cdot \sqrt{\varepsilon_x \beta_x}$$

$$\Delta Q_I = \Delta Q_x \cdot 0,97 \quad ;$$

$$\Delta Q_{II} = \Delta Q_x \cdot 1,10 \quad ;$$

The difference $(\Delta Q_{II} - \Delta Q_I)$ is now only

$$\Delta Q_{II} - \Delta Q_I = 0,13 \quad \Delta Q_x \quad :$$

The luminosity limit is raised by the factor:

$$\frac{\Delta Q_z}{\Delta Q_{II}} = 0,9 \cdot \frac{\sigma_{x0}}{\sigma_{z0}}$$

In this discussion, we have still not taken into account the

requirement (IV) of Section IVB for coupling caused by the space-charge

forces. This requirement, together with

$$| \Delta Q_{I,II} | \leq 0,025$$

gives rise to an additional restriction on the luminosity limit. We

shall consider this in the following section.

B)   Estimate of the Coupling Caused by   Space-Charge Forces

It was mentioned in Section IVB that the space-charge forces

of a rotated beam act, in linear approximation, as a thin lens which

is rotated by the same angle as the beam. These forces thereby

provide a coupling that is superimposed upon that from the built-in lens.

The effect of the lens action of the space-charge forces can be

mathematically described by the transfer matrix:

$$
\mathcal{M}_W = \begin{pmatrix} 1 & 0 & 0 & \sigma \\ (k_1 \cos^2\theta + k_2 \sin^2\theta) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & (k_1 \sin^2\theta + k_2 \cos^2\theta) & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \delta_W & 0 \\ 0 & 0 & 1 & 0 \\ \delta_W & 0 & 0 & 1 \end{pmatrix}}_{} \quad (5.6)
$$

Transfer matrix of a rotated quadrupole

with the coupling term

$$\delta_W = (h_1 - h_2) \sin 2\theta \qquad (5.7)$$

and coupling strength

$$\Delta_W = \delta_W \cdot \sqrt{\beta_x(s_W) \beta_z(s_W)} \; ; \qquad (5.8)$$

$$\left(s_W = 0, \; \frac{C}{2} \; ; \; \text{Intersection points} \right),$$

where the quantities h_1 and h_2 as well as the rotation angle θ can

be obtained from Equations (2.12)-(2.16).

The following must hold if the beam is not to be appreciably

disturbed by the coupling term:

$$\Delta_W^2 \sin \mu_z \sin \mu_x \ll (\cos \mu_x - \cos \mu_z)^2 \; , \qquad (5.9)$$

or, at resonance

$$\Delta_W^2 \ll |2 n \cdot \delta Q|^2 \; . \qquad (5.10)$$

This is because the quantity,

$$\varkappa_W = \sqrt{1 + \frac{\Delta_W^2 \cdot \sin \mu_x \sin \mu_z}{(\cos \mu_x - \cos \mu_z)^2}} \; , \qquad (5.11)$$

which according to Equations (3.11) and (4.8), is a measure of the effect

of coupling on the beam, does not significantly deviate from unity, the

value assumed by $\varkappa$ for vanishing coupling.

By inserting Equations (2.14) and (2.16) in Equation (5.7),

and taking Equation (2.12) into account, we obtain

$$| \Delta_W | = \sqrt{\beta_x(s_W) \beta_z(s_W)} \cdot \frac{1}{\sigma_1 \sigma_2} \cdot \frac{2 \cdot |\sigma_{x2}|}{(\sigma_1 + \sigma_2)^2} \cdot \frac{2 r_e N}{4 \gamma} \quad . \quad (5.12)$$

Under the assumptions given by Equation (4.20), together with the

following:

$$\sigma_1 \approx \sigma_x \quad , \quad \sigma_2 \approx \sigma_z \quad ,$$

and making use of the relation, valid for the DESY storage ring

$$\beta_x(s_W) = \beta_z(s_W) \quad , \quad (5.13)$$

we have, according to Equation (5.3b):

$$\Delta Q_{II} \approx \frac{r_e \cdot N}{2 \pi \gamma \cdot (\sigma_1 + \sigma_2)} \cdot \beta_x \cdot \left(\frac{x-1}{2x} \frac{1}{\sigma_1} + \frac{x+1}{2x} \frac{1}{\sigma_2} \right)$$

$$= \frac{r_e \cdot N}{2 \pi \gamma \cdot (\sigma_1 + \sigma_2)} \cdot \beta_x \cdot \frac{x(\sigma_1 + \sigma_2) + (\sigma_1 - \sigma_2)}{2x \cdot \sigma_1 \sigma_2} \quad (5.14)$$

$$\approx \frac{r_e \cdot N}{4 \pi \gamma} \cdot \beta_x \cdot \frac{1}{\sigma_1 \sigma_2} \quad ,$$

Therefore, we may write in place of Equation (5.13):

$$| \Delta_W | \lesssim 8 \pi \cdot \Delta Q_{II} \cdot \frac{2 \cdot |\sigma_{x2}|}{(\sigma_x + \sigma_2)^2} = 8 \pi \cdot \Delta Q_{I} \cdot \frac{2 \cdot |F_{x2}|}{(\sqrt{F_x} + \sqrt{F_z})^2} \quad . \quad (5.15)$$

Using the beam cross section given by Equations (4.22) and

(4.32) (we restrict our discussion to the case $|\tilde{\eta}_x|, |\tilde{\eta}_z| \ll \langle K^2 \rangle$),

the preceding yields:

$$|\Delta_W| \lesssim 8\pi \cdot \Delta Q_{\mathbb{I}} \cdot \pi \cdot |\delta Q| \cdot |(M_{xz}^{(1)} + M_{xz}^{(2)})| \, ;$$

$$M_{xz}^{(1)} = -(-1)^m \cdot \frac{1}{4} -$$

$$\quad - 2 \, \frac{\cos \mu_x}{\sin \mu_x} \, \sin[\varphi_x(s) - \varphi_x(s_L)] \cdot \sin[\varphi_z(s) - \varphi_z(s_L)] \, ;$$

$$M_{xz}^{(2)} = 2 \cos[\varphi_x(s) - \varphi_x(s_L)] \, \sin[\varphi_z(s) - \varphi_z(s_L)] \, .$$

With this, the following is valid:

$$\left(M_{xz}^{(1)} \right)_{s=0} = \left(M_{xz}^{(1)} \right)_{s=\frac{C}{2}} \, ;$$

$$\left(M_{xz}^{(2)} \right)_{s=0} = -\left(M_{xz}^{(2)} \right)_{s=\frac{C}{2}} \, .$$

As we wish to obtain a minimal degree of perturbation, we assume

that the operating point (Q_x, Q_z) is so adjusted that the following

quantity

$$\left(M_{xz}^{(1)} + M_{xz}^{(2)} \right) \quad \text{or} \quad \Delta_W$$

vanishes at the first crossing point (at $s = 0$ or $s = \frac{C}{2}$), and as

a consequence the coupling due to the space-charge forces appears only in

the second interaction region. For the DESY storage ring we have:

$$Q_x \approx 9,23 \; ;$$

$$Q_z \approx 4,23 \; ;$$

$$\left| \left(M_{xz}^{(1)} + M_{xz}^{(21)} \right) \right|_{s=0} \approx 0,5 \; ,$$

and for Δ_w from Equation (5.15):

$$\left| \Delta_w \right|_{s=0} \approx 8\pi \cdot \left| \Delta Q_{II} \right| \cdot 0,5 \cdot n \cdot \left| \delta Q \right| \; ;$$

$$\left| \Delta_w \right|_{s=\frac{C}{2}} \approx 0 \; .$$

If we further consider that ΔQ_{II} can assume at the most a value of

0.025, we finally obtain:

$$\left| \Delta_w \right|_{s=0} \approx 0,3 \cdot n \cdot \left| \delta Q \right| \; .$$

Therefore the condition (5.10) is satisfied, and the coupling of the

betatron oscillations caused by the space-charge forces can be neglected,

even at the Amman-Ritson luminosity limit.

VI.   Beam Expansion by Shifting the Damping Constants

A)   The Damping Constants for Synchrotron and Betatron Oscillations

It was shown in Sections IV and V that through coupling alone

only the beam height is increased (if $\sigma_x > \sigma_z$ holds for no coupling)

while the beam contracts horizontally. The vertical beam expansion

through coupling therefore results at the expense of the beam width.

If the coupling is strong enough $(\kappa^2 \gg 1)$, the horizontal beam dimension

decreases according to Equations (4.21) and (4.32) by the factor $\dfrac{1}{\sqrt{2}}$,

and the beam cross section itself becomes almost circular in the

interaction region (if $\beta_x = \beta_z$).

It was mentioned in the introduction that as a result of coupling

of the betatron oscillations a further beam expansion in the x- and z-

directions (with decoupling, however, only in the x- direction), can be

simultaneously obtained by damping the synchrotron oscillations and

undamping the betatron oscillations. In particular, one can restore the

beam width to its original value (the decoupled value) with this method.

In the case where the betatron oscillations are coupled by means

of a rotated quadrupole, the damping constants α_s, α_I, and α_{II} are

given by Equations (2.3) and (4.1). By means of these equations, one

can easily confirm the relationship proven in general by K. W. Robinson

$$\alpha_s \; + \; \alpha_I \; + \; \alpha_{II} \; = \; 2 \cdot \frac{\overline{W}}{E} \; , \tag{6.1}$$

This equation shows that the damping of synchrotron oscillations and

undamping of betatron oscillations (or damping of the betatron and

undamping of the synchrotron oscillations) are interdependent.

The following is valid for the DESY storage ring:

$$| \mathcal{A}_z | \; << \; | \mathcal{A}_x | \tag{6.2}$$

so that we can write instead of (2.3) and (4.1):

$$\alpha_s \; = \; \frac{\overline{W}}{E \cdot \langle \mathcal{R}^2 \rangle} \cdot \left\{ \langle \mathcal{R}^2 \rangle + \frac{1}{2} \mathcal{A}_x \right\} ; \tag{6.3a}$$

$$\alpha_I \; = \; \frac{\overline{W}}{2 E \cdot \langle \mathcal{R}^2 \rangle} \cdot \left\{ \langle \mathcal{R}^2 \rangle - \frac{x + 1}{2 x} \mathcal{A}_x \right\} ; \tag{6.3b}$$

$$\alpha_{II} \; = \; \frac{\overline{W}}{2 E \cdot \langle \mathcal{R}^2 \rangle} \cdot \left\{ \langle \mathcal{R}^2 \rangle - \frac{x - 1}{2 x} \mathcal{A}_x \right\} . \tag{6.3c}$$

A shift of the damping constants in the direction corresponding to

an expansion of the beam therefore means that the machine-dependent

constant

$$\mathcal{A}_x \; = \; \langle \mathcal{R}_x^3 (1 - 2 n_x) D_x \rangle$$

is increased. Such a change in A_x can be achieved with a FODO channel[1].

B) Calculation of the Beam Cross Section

To calculate the new beam cross section that appears

when the betatron oscillations are undamped we have only to examine

the way in which the quantities σ_x, σ_z, and σ_{xz}, which determine the

beam expansion, depend on the constant A_x.

For this purpose, we consider that the product

$$\varepsilon_{x0} = : \varepsilon_x \frac{\langle k^2 \rangle - A_x}{\langle k^2 \rangle}$$

$$= \frac{\left\langle |k_x|^3 \cdot \left(D_x^2 \cdot \frac{1 + \alpha_x^2}{\beta_x} + 2 D_x D_x' \cdot \alpha_x + D_x'^2 \cdot \beta_x \right) \right\rangle}{\langle k^2 \rangle}$$

(6.4)

is independent of A_x since, according to Equation (4.7), the factor

$$\left(\langle k^2 \rangle - A_x \right)$$

cancels out with the denominator of ε_x. As a result, the quantity ε_{x0}

defined by Equation (6.4) remains unchanged following a shift of

the damping constants α_s, α_I, and α_{II}.

Therefore, according to Equations (4.21) and (4.22) and taking

Equation (6.2) into account, we may write:

$$\sigma_x^2 = \beta_x \, \varepsilon_{x_0} \cdot \left\{ \frac{(x+1)^2}{4x^2} \cdot \frac{\langle k^2 \rangle}{\langle k^2 \rangle - \frac{x+1}{2x} \Pi_x} + \right.$$

$$\left. + \frac{(x-1)^2}{4x^2} \cdot \frac{\langle k^2 \rangle}{\langle k^2 \rangle - \frac{x-1}{2x} \Pi_x} \right\} ; \tag{6.4a}$$

$$\sigma_z^2 = \beta_z \, \varepsilon_{x_0} \cdot \frac{x^2-1}{4x^2} \left\{ \frac{\langle k^2 \rangle}{\langle k^2 \rangle - \frac{x+1}{2x} \Pi_x} + \frac{\langle k^2 \rangle}{\langle k^2 \rangle - \frac{x-1}{2x} \Pi_x} \right\} ; \tag{6.4b}$$

$$\sigma_{x2} = \sqrt{\beta_x \beta_2} \cdot \varepsilon_{x0} \cdot \pi \cdot \delta Q \; \times$$

$$\times \; \frac{\Delta \cdot \sin \mu_x}{(\cos \mu_x - \cos \mu_2)\sqrt{1 + \dfrac{\Delta^2 \sin \mu_x \sin \mu_2}{(\cos \mu_x - \cos \mu_2)^2}}} \, ,$$

$$\times \; \left(M_{x2}^{(1)} + M_{x2}^{(2)} \right) ;$$

$$M_{x2}^{(1)} = -(-1)^m \cdot \frac{1}{4} \; \times$$

$$\times \; \left\{ \frac{\varkappa+1}{2\varkappa} \cdot \frac{\langle \varkappa^2 \rangle}{\langle \varkappa^2 \rangle - \dfrac{\varkappa+1}{2\varkappa} \Pi_x} \right.$$

$$\left. + \frac{\varkappa-1}{2\varkappa} \cdot \frac{\langle \varkappa^2 \rangle}{\langle \varkappa^2 \rangle - \dfrac{\varkappa-1}{2\varkappa} \Pi_x} \right\}$$

$$- \left\{ \frac{(\varkappa+1)^2}{2\varkappa} \cdot \frac{\langle \varkappa^2 \rangle}{\langle \varkappa^2 \rangle - \dfrac{\varkappa+1}{2\varkappa} \Pi_x} \right.$$

$$\left. - \frac{(\varkappa-1)^2}{2\varkappa} \cdot \frac{\langle \varkappa^2 \rangle}{\langle \varkappa^2 \rangle - \dfrac{\varkappa-1}{2\varkappa} \Pi_x} \right\} \; \times \qquad (6.4c)$$

$$\times \; \frac{\cos \mu_x}{\sin \mu_x} \cdot \sin \left[\varphi_x(s) - \varphi_x(s_L) \right] \cdot \sin \left[\varphi_2(s) - \varphi_2(s_L) \right],$$

$$M_{x2}^{(2)} = \left\{ \frac{(\varkappa+1)^2}{2\varkappa} \cdot \frac{\langle \varkappa^2 \rangle}{\langle \varkappa^2 \rangle - \dfrac{\varkappa+1}{2\varkappa} \Pi_x} \right.$$

$$\left. - \frac{(\varkappa-1)^2}{2\varkappa} \cdot \frac{\langle \varkappa^2 \rangle}{\langle \varkappa^2 \rangle - \dfrac{\varkappa-1}{2\varkappa} \Pi_x} \right\} \; \times$$

$$\times \; \cos \left[\varphi_x(s) - \varphi_x(s_L) \right] \cdot \sin \left[\varphi_2(s) - \varphi_2(s_L) \right]$$

where

$$\left(M_{x2}^{(1)} \right)_{s=0} = \left(M_{/2}^{(1)} \right)_{s=\frac{L}{2}} ;$$

$$\left(M_{x2}^{(2)} \right)_{s=0} = - \left(M_{x2}^{(1)} \right)_{s=L} .$$

The dependence of the beam cross section on the parameter A_x , which also determines the size of the damping constants, is therefore characterized by the factors

$$\frac{\langle \varkappa^2 \rangle}{\langle \varkappa^2 \rangle - \frac{\varkappa+1}{2\varkappa} A_x} \quad \text{and} \quad \frac{\langle \varkappa^2 \rangle}{\langle \varkappa^2 \rangle - \frac{\varkappa-1}{2\varkappa} A_x} \; .$$

It follows from Equation (6.4) that the beam width in the interaction region $(\beta_x \cdot \beta_z \; ! \;)$ is always larger than the beam height:

$$\sigma_x > \sigma_z \; .$$

However, for $\varkappa \gg 1$, we have an almost circular beam cross section of radius

$$r = \frac{1}{\sqrt{2}} \; \beta_x \; \varepsilon_{x_0} \cdot \sqrt{\frac{\langle \varkappa^2 \rangle}{\langle \varkappa^2 \rangle - \frac{1}{2} A_z}} \; , \qquad (6.5)$$

as long as the assumption of Equations (4.20) hold.

If the beam half-width σ_x alone shrinks after switching on coupling of a given strength Δ , and the original width $\sqrt{\beta_x} \; \sqrt{\varepsilon_{x_0}}$ is to be restored, the following relation must hold:

$$\frac{(\varkappa+1)^2}{4\varkappa^2} \cdot \frac{\langle \varkappa^2 \rangle}{\langle \varkappa^2 \rangle - \frac{\varkappa+1}{2\varkappa} A_x} + \frac{(\varkappa-1)^2}{4\varkappa^2} \cdot \frac{\langle \varkappa^2 \rangle}{\langle \varkappa^2 \rangle - \frac{\varkappa-1}{2\varkappa} A_z} = 1 \, .$$

This relation follows from Equation (6.4a). From it we may calculate

A_x as follows:

$$A_x = \langle x^2 \rangle \cdot \left\{ \frac{1}{2} \cdot \frac{3x^2+1}{x^2-1} - \frac{1}{2} \sqrt{\left(\frac{3x^2+1}{x^2-1}\right)^2 - 8} \right\} \quad (6.6)$$

C) Q-Shift

If the assumptions of Equation (4.20) are correct, we may

use Equation (5.3) to calculate the Q-shift that appears due to

space-charge forces following the additional beam expansion from the

undamping of the betatron oscillations. The quantities σ_x and σ_z

appearing in Equation (5.3) should then be taken from Equation (6.4).

Since the beam size is increased when the damping constants are shifted,

the luminosity limit is again raised.

Together with Equation (5.3) there remains the relation given by

(5.15), which sets an upper bound for the permissible coupling strength

arising from the beam-beam interaction.

In order to keep the coupling caused by rotation of the beam

as small as possible, it is useful to choose Q_x and Q_z such that

the following holds:

$$\left(M_{xz}^{(1)} + M_{xz}^{(2)} \right)_{s = \frac{C}{2}} = 0 \ .$$

Under these conditions Δ_w vanishes at the crossing point $s = \frac{C}{2}$.

If we put

$$\Delta Q_{II} = \alpha \cdot 0,025 \ ,$$

Equation (5.3) provides an estimate of the factor α as well as of

the maximum possible Q-shift for which the beam remains stable:

$$\Delta Q = \alpha \cdot 0,025 \ .$$

Consequently, the maximum luminosity limit connected with the Amman-

Ritson effect with $\Delta Q_{II} = 0.025$ can only be achieved if Equation

(5.3) can be satisfied with $\alpha = 1$.

Numerical Example:

We put $\kappa = 2$.

In order to achieve the original value for the beam width

$$\sigma_x = \sqrt{\varepsilon_{x_0}} \cdot \sqrt{\beta_x} \quad ,$$

we must choose $\beta_x = 0,5 \cdot \langle \kappa^2 \rangle$, according to Equation (6.6).

From this we have the corresponding damping constants according

to Equation (6.3):

$$\alpha_3 = \frac{5}{4} \cdot \frac{\overline{W}}{E} \quad ;$$

$$\alpha_I = \frac{5}{16} \cdot \frac{\overline{W}}{E} \quad ;$$

$$\alpha_{II} = \frac{7}{16} \cdot \frac{\overline{W}}{E} \quad .$$

We obtain the beam half-height from Equation (6.4b)

$$\sigma_z = 0,42 \cdot \sigma_x \quad ,$$

and Δ_w vanishes at the crossing point $s = \frac{C}{2}$ if the following

values are chosen:

$$\varrho_x \approx 9,23$$
$$\varrho_z \approx 4,23$$

The factor appearing in Equation (5.15)

$$\frac{2 \cdot |\sigma_{xz}|}{(\sigma_x + \sigma_z)^2}$$

assumes the following value at $s = \frac{C}{2}$

$$\frac{2 \cdot |\sigma_{xz}|}{(\sigma_x + \sigma_z)^2} \approx 0,5 \cdot \pi \cdot |\delta a|$$

so that the condition given by (5.15) may be considered to be satisfied.

Finally, we have from Equation (5.3) that the luminosity limit is raised by a total factor of

$$\frac{\Delta Q_{II}}{\Delta Q_{1}} = \frac{\sigma_{xo}}{\sigma_{zo}} \cdot 1,25 \quad .$$

Therefore, after the undamping of the betatron oscillations the luminosity rises 40% more than in the case of coupling alone (see Section VA).

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