Asymptotic iteration method for spheroidal harmonics of higher-dimensional Kerr-(A)dS black holes

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Abstract

In this work we calculate the angular eigenvalues of the $(n+4)$-dimensional *simply* rotating Kerr-(A)dS angular equation using the Asymptotic Iteration Method (AIM). We then compare this method with the Continued Fraction Method (CFM) thereby checking our results.

1 Introduction

Recently a new method for obtaining solutions of second order ordinary differential equations has been developed called the asymptotic iteration method (AIM) [1]. The AIM provides a simple approach to obtaining eigenvalues of bound state problems, even for spheroidal harmonics with c a general complex number, large or small [1], and even to quasinormal mode (QNM) calculations [2]. It has also been shown that the AIM is closely related to the continued fractions method (CFM)[4] derived from an exact solution to the Schrödinger equation via a WKB ansatz [5], where a related CFM is often employed in numerical calculations of spheroidal eigenvalues and QNM of black hole equations [6].

In this letter we will demonstrate that the AIM can also be applied to the generalized scalar hyper-spheroidal equation, $S_{kjm}(\theta)$, derived from an $(n+4)$-dimensional *simply* rotating Kerr-(A)dS angular separation equation [3, 7, 8]:

$$\frac{\partial_\theta\Big((1+\alpha\cos^2\theta)\sin\theta\cos^n\theta\partial_\theta S\Big)}{\sin\theta\cos^n\theta}+\left(A_{kjm}-\frac{m^2(1+\alpha)}{\sin^2\theta}-\frac{c^2\sin^2\theta}{1+\alpha\cos^2\theta}-\frac{j(j+n-1)}{\cos^2\theta}\right)S=0 \quad (1)$$

where we have defined $\alpha = a^2\Lambda$ with a the angular rotation parameter, and that the frequency ω is contained in the dimensionless parameter $c = a\omega$.

Higher dimensional spheroids have already been discussed by Berti et al. [9], who use a 3-term CFM to solve the angular eigenvalues, however, the generalized scalar hyper-spheroidal equation under investigation here contains four regular singular points[5] which leads to a 4-term recurrence relation [10]. The simplest brute force approach to deal with an n-term recurrence relation is to use n Gaussian eliminations to reduce the problem to a tri-diagonal matrix form [11] but this can often be very tedious.

Even in four-dimensions the Kerr-(A)dS case does not allow for a simple 3-term continued fraction relation nevertheless an elegant method has been developed to deal with situations of this type [8], where such techniques can only be applied if there are exactly four regular singular points. In contrast to this the appeal of the AIM is that it can be applied somewhat independently of the singularity structure of the ordinary differential equation and thus to a larger class of equations.

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[5]Unlike the asymptotically flat limit ($\Lambda = 0$) which only has three.

2 The Asymptotic Iteration Method

To write the angular equation in a form suitable for the AIM we substitute $x = \cos\theta$ and obtain:

$$(1-x^2)(1+\alpha x^2)S''(x) + \left(\frac{n(1-x^2)-x^2}{x} + \alpha x(n+2-(n+3)x^2) - x(1+\alpha x^2)\right)S'(x) + \left(A_{kjm} - \frac{c^2(1-x^2)}{1+\alpha x^2} - \frac{m^2(1+\alpha)}{1-x^2} - \frac{j(j+n-1)}{x^2}\right)S(x) = 0 \ . \quad (2)$$

Note that the separation constant A_{kjm} above corresponds to a simple eigenvalue shift in the asymptotically flat cases studied thus far [9], as can be verified by setting $\alpha = 0$.

The AIM can be implemented by multiplying S_{kjm} by the characteristic exponents; however, we have found that the most suitable form (fastest converging) is obtained by multiplying the angular mode function by [1]:

$$S_{kjm}(x) = (1-x^2)^{\frac{|m|}{2}} y_{kjm}(x) \ , \quad (3)$$

which leads to a differential equation in the AIM form:

$$y'' = \lambda_0 y' + s_0 y \ , \quad (4)$$

where (for Kerr-(A)dS) λ_0 and s_0 are given in Ref. [3], and where the primes of y denote derivatives with respect to x. Differentiating equation (4) p times with respect to x, leads to:

$$y^{(p+2)} = \lambda_p y' + s_p y \ , \quad (5)$$

where the superscript p indicates the p-th derivative with respect to x and

$$\lambda_p = \lambda'_{p-1} + s_{p-1} + \lambda_0\lambda_{p-1} \qquad \text{with} \qquad s_p = s'_{p-1} + s_0\lambda_{p-1} \ . \quad (6)$$

For sufficiently large p the asymptotic aspect of the "method" is introduced, that is:

$$\frac{s_p(x)}{\lambda_p(x)} = \frac{s_{p-1}(x)}{\lambda_{p-1}(x)} \equiv \beta(x) \ , \quad (7)$$

which leads to the general eigenfunction solution given in Ref. [1]. Within the framework of the AIM, a sufficient condition for imposing termination of the iterations is when $\delta_p(x) = 0$, for a given choice of x, where $\delta_p(x) = s_p(x)\lambda_{p-1}(x) - s_{p-1}(x)\lambda_p(x)$ [1]. For each value of m and k (or j), in a given $(n+4)$-dimensions, the roots of δ_p lead to a tower of eigenvalues $(m, \ell_1, \ell_2, \dots)$, where larger iterations give more roots and better convergence for higher ℓ modes in the tower.

It has also been noticed that the AIM converges fastest at the maximum of the potential [1], which in four dimensions occurs at $x = 0$ (even with $\alpha \neq 0$ and for general spin-s). However, in the higher dimensional case we could not determine the relevant Schrödinger like form and thus the maximum of the potential could not be analytically obtained. Nevertheless, as can be seen from the plots in Fig. 1 we found that the point $x = \frac{1}{2} = \cos\frac{\pi}{3}$, in general, gave the fastest convergence.

3 Heun's method for de-Sitter case

As we mentioned earlier we could also work with a 4-term recurrence relation directly and use Gaussian elimination to obtain a 3-term recurrence, which then allows for the eigenvalues to be solved using the CFM. However, if we write the angular equation (1) in terms of the variable $x = \cos(2\theta)$ [8]:

$$(1-x^2)(2+\alpha(1+x))S''(x) + \left(n-1-(n+3)x + \frac{\alpha}{2}(1+x)(n+1-(n+5)x)\right)S'(x) + \left(\frac{A_{klm}}{2} + \frac{c^2(x-1)}{2(2+\alpha(1+x))} + \frac{m^2(1+\alpha)}{x-1} - \frac{j(j+n-1)}{x+1}\right)S(x) = 0 \quad (8)$$

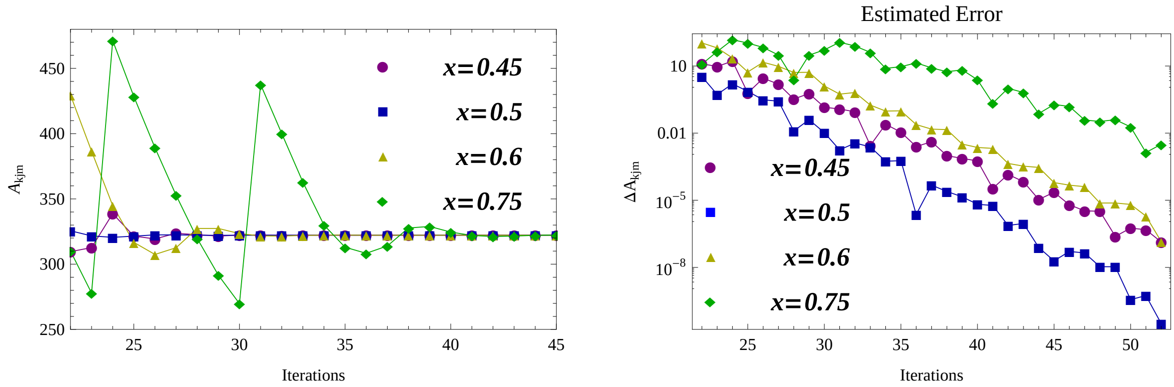

Figure 1: *Plot of the convergence of a typical eigenvalue A_{711} ($n = 1$, $c = 1$ and $\alpha = 1$) under p iterations of the AIM for various choices of $x = \{0.45, 0.5, 0.6, 0.75\}$. Shown on the left is the eigenvalue verses p, while on the right is a log plot of the estimated error, $|A_{kjm}(p) - A_{kjm}(\infty)|$.*

Table 1: *Comparison of selected eigenvalues, A_{kjm}, the Kerr-AdS case with $c = 1$, $\alpha = -0.05$, $n = 1$ (extra dimensions) and $m = 0$. Numbers in brackets represent the number of iterations required to reach convergence at the quoted precision, where subscript A and C are shorthand for AIM and CFM respectively.*

k	$j = 0$	$j = 1$
0	0.4978643318 $(14)_A$ $(3)_C$	3.317784170 $(14)_A$ $(3)_C$
1	8.304871188 $(15)_A$ $(4)_C$	15.12466814 $(15)_A$ $(4)_C$
2	23.89847347 $(16)_A$ $(5)_C$	34.63440913 $(17)_A$ $(5)_C$
3	47.29227791 $(17)_A$ $(6)_C$	61.93248179 $(19)_A$ $(6)_C$
4	78.48442957 $(20)_A$ $(8)_C$	97.02600365 $(21)_A$ $(8)_C$
5	117.4747381 $(23)_A$ $(9)_C$	139.9165956 $(23)_A$ $(9)_C$
6	164.2631560 $(25)_A$ $(10)_C$	190.6047952 $(24)_A$ $(10)_C$
7	218.8496664 $(27)_A$ $(11)_C$	249.0908236 $(27)_A$ $(11)_C$

and define $x = 2z - 1$, with the mode functions scaled by the characteristic exponents:

$$Q(x) = 2^{\frac{|m|}{2}} (z-1)^{\frac{|m|}{2}} (2z)^{\frac{j}{2}} \left(z + \frac{1}{\alpha}\right)^{\pm \frac{ic}{2\sqrt{\alpha}}} y(z) \ , \tag{9}$$

then the angular mode equation can now be written in the Heun form [8]:

$$\left[\frac{d^2}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z+\frac{1}{\alpha}}\right)\frac{d}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z+\frac{1}{\alpha})}\right] y(z) = 0 \ , \tag{10}$$

where α, β, γ, δ, ϵ and q are given in Ref. [3], and where these results are identical to the Kerr-AdS case considered in Ref. [8] by choosing $\alpha = -a^2/R^2$.

To compare with the AIM method we shall use the fact that a three-term recurrence relation is guaranteed for any solution to Heun's differential equation [8]:

$$\alpha_0 c_1 + \beta_0 c_0 = 0 \ , \tag{11}$$

$$\alpha_p c_{p+1} + \beta_p c_p + \gamma_p c_{p-1} = 0 \ , \qquad (p = 1, 2, \dots) \ , \tag{12}$$

where for the Kerr-(A)dS case α_p, β_p and γ_p are given in Ref. [8]. Once a 3-term recurrence relation is obtained the eigenvalue A_{kjm} can be found (for a given ω) by solving a continued fraction of the form [6, 9]:

$$\beta_0 - \frac{\alpha_0\gamma_1}{\beta_1 -}\frac{\alpha_1\gamma_2}{\beta_2 -}\frac{\alpha_2\gamma_3}{\beta_3 -}\dots = 0 \ . \tag{13}$$

Table 2: *Comparison of selected eigenvalues between the AIM and the CFM for different numbers of dimensions of the Kerr-dS case with $c = 1$, $\alpha = 1$, $m = j = k = 0$.*

n	2	3	4	5
A_{CFM}	0.284	0.2254	0.1861	0.1581
A_{AIM}	0.284049	0.225367	0.18606	0.158068

4 Analysis & Discussion

We have calculated the eigenvalues shown in Table 1 of the $(n+4)$-dimensional *simply* rotating Kerr-(A)dS angular separation equation using the AIM and the CFM. Although we only considered a real parameter $c = a\omega$, we could also have used a purely imaginary or complex value of c, see Ref. [2]. For brevity we presented results for $n = 1$ extra dimensions only, but we have also checked the dependence on dimension, as can be seen in Table 2 for the fundamental $k = 0$ mode.

We found that the CFM eigenvalue solutions converged very quickly with accurate results even after a continued fraction depth of only $p = 15$. One point worth mentioning is that the $\alpha \to 0$ limit cannot be taken via Heun's method, because the recurrence relation (and hence the continued fraction) diverges for this case. In contrast the AIM has no such problem. The AIM also gives an alternative approach to obtaining the eigenfunctions in terms of simple integrals, which may be useful for symbolic computations.

In conclusion, we have highlighted how the AIM can be applied to higher-dimensional scalar or tensor gravitational (for $n \geq 3$) spheroidal harmonics, which arise in the separation of metrics. We have seen that the AIM requires very little manipulation in order to obtain a fast route to the angular eigenvalues, which may be useful for cases where Heun's method may not apply. However, the AIM does have some shortfalls. Note that while we did not attempt to optimise either algorithm, our implementation of the AIM was found to be much slower than that of the CFM. Considering that the CFM essentially involves expanding out p nested fractions, whereas the AIM involves taking p^{th} order derivatives, this behaviour is not surprising. However, for most of the cases we considered only a few seconds were required to reach the desired level of accuracy and thus the time was not a large concern.

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