Parallel Session 13

Recent Developments in 2-Dimensional Gravity

Organisers

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The Liouville mode and String Fields in Matrix Models

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ABSTRACT

Using the collective field method we show that the $d = 1$ matrix model may be rewritten as a field theory of a single massless scalar field in *two* dimensions. The extra dimension, identified as the liouville mode, comes from the space of eigenvalues of the matrix. The extent of this extra dimension is proportional to the logarithm of the cosmological constant, which explains the logarithmic scaling violations of the model. The field theory has interactions only at the boundary.

In critical string theory one considers conformally invariant matter coupled to two dimensional gravity. Since the matter is critical, the couplings of the theory are constrained to have their fixed point (under renormalization group) values and strict conformal invariance further restricts the central charge, and hence the dimension of the embedding space, to its critical value. These constraints are significant in critical strings. For a string moving in arbitrary backgrounds, the couplings of the two dimensional theory are essentially the background fields and the fixed point conditions turn out to be the classical equations of motion of the background space-time fields. In this way one uncovers "string field theory".

In non-critical strings, one is interested in coupling any matter to 2d gravity. The couplings and the central charge are not constrained by anything and it appears that the background fields are not determined. In that case how does one recover "string field theory" in non-critical strings ?

The answer lies in the realization that the Liouville mode acts as an extra dimension[1] (see also[2]). Matter couplings get dressed by gravity and thus acquire a dependence on the Liouville mode. The point is that this renormalization of the couplings is determined purely by the reparametrization invariance[3][4]. Consider for example a string moving in d dimensions in the presence of some Tachyon background $T(X)$. The background gets dressed by gravity into $T(\varphi, X)$ and the above mentioned condition of reparametrization invariance then implies that[1][3]

$$(-\partial_t^2 + \partial_\varphi^2 + (2 - \frac{Q^2}{4}))T + \lambda e^{-\frac{Q}{2}\phi} T^2 + O(T^3) = 0. \tag{1}$$

where $3Q^2 = 25 - d$. This is the equation of motion of a scalar field in $d+1$ dimensions with φ as the extra coordinate. The mass of the field $T(\phi, X)$ is $(2 - \frac{Q^2}{4})$. For $d = 1$ the scalar particle is massless, while it becomes tachyonic for $d > 1$. For $d \neq 25$ there is no translation invariance in the ϕ direction. At $d = 25$ there is complete symmetry and one has the standard 26-dimensional bosonic string[5]. Furthermore for $d > 25$ the coordinate ϕ has a time-like signature, while for $d < 25$ it has a space-like signature.

Let us now turn to the matrix model representation of non-critical strings. Consider a $N \times N$ hermitian matrix field $M_{ij}(x)$ living in d dimensions and having the action:

$$S = \int d^d x [Tr\ M(x) e^{-\nabla^2} M(y) + \frac{N}{g} V(M(x))] \tag{2}$$

where $V(M)$ is some invariant potential. Then the feynman diagrams of this matrix field theory may be regarded as discretizations of a two-dimensional surface with cosmological constant g.

Matrix models may be used to provide a non-perturbative definition of the continuum string. To do this one has to perform a *double scaling*

limit[6] in which the cosmological constant g goes to its critical value g_c where the partition function is singular, and $N \to \infty$ such that the combination $(g_{0c} - g_0)^\alpha \; N = \frac{1}{g_{st}}$ is held fixed. The value of the quantity α depends on the particular model.

In the single matrix problem, the free energy is completely finite in the double scaling limit, the contribution of each genus being equally important[6]. In contrast, the $d = 1$ model has finite contributions for genus 2 and higher, but has logarithmically divergent contributions for the genus 0 and genus 1 contributions[7]. This is usually referred to as the logarithmic scaling violations of the $d = 1$ model and had created a lot of confusion regarding the physical interpretation of the model.

In order to provide a physical interpretation of the matrix model one has to identify the liouville mode. In the rest of the talk I will indicate how to do that[8]. We have shown that the $d = 1$ matrix model may be rewritten as a field theory of a single massless scalar field in two dimensions, the extra dimension (the liouville mode) coming from the space of eigenvalues of the matrix M_{ij}. This "string field theory" provides a natural explanation of the scaling violations mentioned above. The scaling violations have also been explained from the point of view of the continuum theory equations (1) in[9].

$d = 1$ Matrix Model

In one dimension the absence of ultraviolet problems allows one to replace the gaussian propagator of the action in (2) by a standard feynman propagator. Considering the single X as time, the quantum.theory is defined by the Hamiltonian:

$$\hat{H} = \mathrm{Tr}\left(-\frac{1}{2}\frac{\partial^2}{\partial \mathrm{M}^2} + \beta \mathrm{v(M)}\right) \qquad (3)$$

where $\beta = \frac{N}{g}$. We shall restrict ourselves to the sector of the theory which is singlet under the symmetry group of $U(N)$ rotations. In terms of the eigenvalues λ_i of the matrix M, the hamiltonian becomes:

$$\hat{H} = \sum_i [\frac{1}{2}\frac{\partial^2}{\partial \lambda_i^2} + \beta v(\lambda_i)] \qquad (4)$$

acting on fermionic wave functions which are antisymmetric under interchanges of the λ-s. The ground state energy is

$$E_{g_{st}} = \int_0^{\mu_F} de \; e \; \rho(e) \qquad (5)$$

where $\rho(e)$ is the density of states and μ_F is the fermi energy determined by the filling condition $\int_0^{\mu_F} de \; \rho(e) = N$. Let the potential have a maximum at the point λ_c. Then criticality is obtained when the fermi energy μ_F approaches $V(\lambda_c) = \mu_{Fc}$, which happens when the cosmological constant g approaches a critical value g_c. A crucial role is played by the quantity

$$\mu = \mu_{Fc} - \mu_F \qquad (6)$$

The double scaling limit is defined by

$$\beta \to \infty, \qquad \mu \to 0, \qquad (\beta\mu) = \text{fixed} \qquad (7)$$

In this limit the expression for the ground state energy has the expansion[7]

$$E_{g_{st}} = -\frac{1}{4\pi}(\beta\mu)^2 \ln\mu + \frac{1}{12\pi}\ln\mu + \sum_{m=1}^{\infty} C_m (\beta\mu)^{-m} \qquad (8)$$

We also write down the expression for the finite temperature free energy F for the singlet sector at temperature T[10]

$$\frac{F}{T} = -\frac{1}{4\pi T}(\beta\mu)^2 \ln\mu + \frac{1}{12}\ln\mu(\pi T + \frac{1}{\pi T}) + \ldots. \qquad (9)$$

The higher terms are all finite in the double scaling limit and respect the duality symmetry $\pi T \to \frac{1}{\pi T}$. The expressions (8) and (9) display the scaling violations mentioned above.

The Collective Field Theory

The main idea of our work is to rewrite the problem in terms of invariant variables using the method of collective fields developed in[11]. Correlation functions of any number singlet operators may be written in terms of the collective fields,

$$\phi(x) = \int \frac{dk}{2\pi} e^{-ikx}\phi_k = \int \frac{dk}{2\pi} e^{-ikx}\,\mathrm{Tr}(e^{ikM}). \qquad (10)$$

$\phi(x)$ is simply the density of eigenvalues λ_i.

We now make a change of variables from the fields M_{ij} to the collective fields $\phi(x)$. The field $\phi(x)$ is, of course, constrained to satisfy:

$$\int dx\ \phi(x) = N \tag{11}$$

The change of variables is made using the procedure of[11]. We also perform the rescalings $x \to \sqrt{\beta}x, \phi \to \sqrt{\beta}\ \phi$. One finally gets a Hamiltonian[8]

$$H_\phi = \int dx(\frac{1}{2}\partial_x\pi(x)\phi(x)\partial_x\pi(x)\ + \beta^2 V_0 + \Delta V) \tag{12}$$

where $\pi(x)$ is the momentum conjugate to the field $\phi(x)$, and

$$V_0 = \frac{\pi^2}{6}\phi^3 + (v(x) - \mu_F)\phi(x) \tag{13}$$

$$\Delta V = \frac{1}{2}\int\limits_{y=x} dx\phi(x)\partial_x\partial_y ln|x-y| \tag{14}$$

In this hamiltonian, the constraint (11) which has a g instead of N on the right hand side, has been implemented by a lagrange multiplier μ_F. The lagrangian which follows from the above hamiltonian may be written as

$$\mathcal{L} = \int dx\left\{\frac{1}{2}\partial_x^{-1}\dot{\phi}\frac{1}{\phi}\partial_x^{-1}\dot{\phi} - \beta^2 V_0(\phi(x,t)) - \Delta V\right\} \tag{15}$$

Leading Order

It is clear that β is the bare string coupling constant. In the leading order of the WKB expansion the free energy is dominated by the saddle point solution for $\phi(x)$ which is given by

$$\phi_0(x) = \frac{1}{\pi}\sqrt{2(\mu_F - v(x))} \tag{16}$$

while its integral must be equal to g. In the critical limit of small μ one may easily show that the energy evaluated at the saddle point coincides with the leading order ground state energy in (8).

The Liouville mode and the Spectrum

The lagrangian density (15) is clearly the lagrangian of a two dimensional field theory with x as the extra dimension. However x is not quite the liouville mode we are looking for. To find the liouville mode we have to study the fluctuations around the saddle point solution[12]:

$$\phi(x) = \phi_0(x) + \frac{\partial\psi}{\partial x} \tag{17}$$

It is convenient to introduce the variable q defined by:

$$q = \frac{1}{\pi}\int^x \frac{dx}{\phi_0(x)}, \qquad dq = \frac{dx}{\phi_0} \tag{18}$$

Note that q is simply the time of flight of a classical particle moving in the potential $v(x)$. In the following we will use the specific form of the potential

$$v(x) = x^2 - 2x^4 \tag{19}$$

The results are of course independent of the detailed form of the potential provided the maximum is generic, i.e. the second derivative $v''(x_0)$ is non-vanishing.

The range of q is given by $-L < q < L$ where $4L$ is the time period of classical motion. In the critical limit $L \to \infty$ as $L = -\frac{1}{4}\ln\mu$ (as follows from the saddle point solution above). The action quadratic in the fluctuations

$$S_2 = \pi^3 \int dt \int\limits_{-L}^{L} dq\left[\frac{1}{2}(\partial_t\psi)^2 - \frac{1}{2}(\partial_q\psi)^2\right] \tag{20}$$

which is the lagrangian of a massless scalar field in two dimensions. This is exactly what we expected from the continuum equation (1)for $d = 1$! In fact what we have found that q is precisely the liouville mode hidden in the matrix model.

The boundary conditions on the fluctuation $\psi(q,t)$ are determined from the time independence of the constraint, i.e. $\frac{d}{dt}(\int dx\phi(x)) = 0$, which leads to Dirichlet boundary conditions on $\psi : \psi(-L,t) = \psi(L,t) = 0$. The eigenfunctions are therefore

$$\psi_n(q) = \begin{cases} \frac{1}{\sqrt{L}}\sin(\frac{n\pi q}{L}) & n = 0,\ 1,\ 2,\ 3... \\ \frac{1}{\sqrt{L}}\cos(n+\frac{1}{2})\frac{\pi q}{L} \end{cases} \tag{21}$$

with the frequencies

$$\omega_j = \frac{j\pi}{2L} = j\omega_c; \qquad j = 0,1,2.... \tag{22}$$

The propagator is then

$$D(t-t';q,q') = \int \frac{dE}{\pi} e^{iE(t-t')} \sum_j \frac{\psi_j(q)\psi_j(q')}{E^2-\omega_j^2+i\epsilon}. \tag{23}$$

In the scaling limit we have $L \to \infty$ and we define continuum momenta $p = \frac{\pi x}{2L}$. The propagator now becomes the standard massless scalar propagator in two dimensions. The dispersion relation becomes $E^2 - p^2 = 0$.

Using the above basic two point function it is straightforward to evaluate the class of two-point functions in the matrix model

$$< Tr\ M^p(t) Tr\ M^q(0) >_{conn} = \int dx \int dx' x^p\ x^q\ \ \partial_x \partial_{x'} < \psi(x,t)\psi(x',t) > \tag{24}$$

Using (21) and (23) it is easy to check that the result is identical to that obtained from a direct calculation of the above correlator in the matrix model[13].

One loop free energies

To obtain the one-loop (torus) free energy at zero temperature we need to calculate the expression

$$\Delta E_0 = \pi^3 \int dq [\partial_q \partial_{q'} G(q,q')]_{q=q'} \tag{25}$$

where $G(q,q')$ isthe standard propagator following from the action (20)

$$G(q,q') = -\frac{1}{2\pi^3} \ln|q-q'| \tag{26}$$

One must also add the term ΔV in the original collective field lagrangian (15). To treat the various singular pieces, we make a change of variables in (25) to the original variable x. Using the definition of $q(x)$ one finds that the singularity as $x \to y$ cancels between ΔV and (25), leaving with the finite answer

$$\Delta E_1 = \frac{1}{24\pi} \int \frac{v''(x)dx}{[2(\mu_F - v(x))]^{\frac{1}{2}}} \tag{27}$$

Using the form of the potential (19) one can find the singular (as $\mu \to 0$) piece in the one loop free energy density to be

$$\Delta E_1 = \frac{1}{12\pi} \ln\mu \tag{28}$$

which agrees with the corresponding term in (8). The expression (27) also agrees with the leading WKB correction obtained in[14], and the collective field calculation in[15]

To obtained the free energy at finite temperature one has to add to ΔE_1 the contribution of the thermal free energy of a massless scalar field in (1 + 1) dimensions having the dispersion relation (22). At temperature T the free energy is easily seen to be

$$F/T = \frac{\pi LT}{3} = -\frac{\pi T}{12} \ln\mu \tag{29}$$

The total thermal free energy is obatined by adding (28) and (29)

$$\frac{F}{T} = \frac{1}{12} \ln\mu (\pi T + \frac{1}{\pi T})$$

which displays duality and agrees with (9).

Thus the genus zero and one free energies are proportional to $\ln\mu$ simply because the effective field theory lives in a box of length $L \sim \ln\mu$.

Interactions

To obtain the leading form of the interactions, one has to compute the pieces of the collective field lagrangian which are cubic in ψ. There are two contributions, one coming from the ϕ^3 piece and one coming from the expansion of $\frac{1}{\phi}$ in the kinetic energy. The full result is

$$\int dq\ dt\ \frac{\pi^3}{(2\beta\phi_0^2)} [(\partial_q\psi)(\partial_t\psi)^2 + \frac{1}{3}(\partial_q\psi)^2] \tag{30}$$

For our potential $\phi_0(q)$ may be obtained in terms of standard elliptic functions, and in the critical limit, one can show that

$$\frac{1}{\phi_0^2} \sim \cosh^4(2x_+ q) \tag{31}$$

The coupling has the exponential dependence on the liouville direction as has been found in the continuum theory in equation (1). The coupling grows near the boundaries. To define a sensible continuum theory it is natural to require that the coupling at the boundary to be held fixed, which means

$$g_{st} = \frac{1}{\beta} \exp(8x_+ L) = \text{fixed.} \tag{32}$$

For our potential $x_+ = \frac{1}{2}$ and $L = -\frac{1}{4}\ln\mu$ so that

$$g_{st} = \frac{1}{\beta\mu} = \text{fixed.} \tag{33}$$

We have found the double scaling law.

In the double scaling limit the interactions of the theory are nonzero only at the boundary, as is evident from (31). This immediately provides a qualitative explanation of why there are no $\ln\mu$ factors in front of the higher genus contributions to the free energy. The higher genus contributions are purely due to interactions which are only at the boundary; therefore no overall volume factor is present.

Conclusions

What happened to the other modes of the string ? The answer is that for our "$d = 1$" theory the gauge invariant degrees of freedom are : (i) a single massless scalar field and (ii) a set of discrete states with energies given by $E = \frac{n}{\sqrt{2}}$ for integers n. This follows from a BRST analysis[16] The latter are not particle like states in two dimensions. Nevertheless, there does not seem to be any trace of them in the matrix model ! This is an unresolved question at the moment. Perhaps the answer lies in a more careful treatment at the boundaries. Furthermore, it may be useful to obtain a gauge-invariant form of the theory with all the redundant variables present. Presumably the pure gauge degrees of freedom have automatically disappeared in our treatment because we have already chosen a direction of time by adopting the hamiltonian approach.

While we have found that the qualitative form of the interaction agrees with that obtained from the continuum theory the interaction terms are not really identical, as is clear by comparing (30) and (1). It is not clear at this moment why this is so. This may be related to the non universality of the higher terms of the beta function from which (1) has been derived.

Finally we note that recently the string field theory underlying the $d = 1$ matrix model has also been obtained starting from the second quantized fermion description[17]. The results are in agreement with ours.

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DISCUSSION

Q. **E. Brezin***(ENS, Paris)*: (1) The master field is a pure $U(N)$-singlet. Therefore the finite temperature calculation is not really a compactified $c = 1$ string.
(2) The master field Hamiltonian includes $1/N^2$ corrections to the usual large-N limit. In order to recover the full double scaling limit, does one need to include further terms or is this all?

A. (1) Dr. Das does agree with the comment.
(2) Dr Das told us that in momentum space there are no further terms needed to recover the continuum theory (this is my understanding of what has been said!).

MULTI-BAND STRUCTURE AND ORTHOGONAL POLYNOMIALS

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ABSTRACT

We discuss the multi-band phase structure that arises in hermitian one-matrix models with bounded potentials having several local minima by relating orthogonal polynomial recursion coefficients R_n and S_n to the large-N limit of the generating function $F(z) \equiv \frac{1}{N}\langle tr\frac{1}{z-\Phi}\rangle$. We show how periodicity structure in these coefficients naturally leads to multi-band structure, and in particular, we provide an explicit example of a three-band phase. We also briefly comment on the stability of the $k = 2$ pure gravity solution.

INTRODUCTION

Matrix models have proved to be a powerful tool for analyzing sums over random surfaces.[1,2] Some of the more interesting features of non-critical strings have been derived from the large-N hermitian matrix models using polynomial potentials which are unbounded from below.[3] Attempts have been made in regularizing these models by considering bounded potentials of higher orders[4,5] by making potentials periodic thus leading to unitary matrix models.[6,7] In both cases, one must now face the question of tunneling, and one is led to the phenomenon of multi-band structure, which was first observed and studied in the early eighties for unitary matrix models.[8]

In this talk, we shall concentrate on discussing this phenomenon for hermitian one-matrix models by relating orthogonal polynomial recursion coefficients R_n and S_n to the large-N limit of the generating function $F(z) \equiv \frac{1}{N}\langle tr\frac{1}{z-\Phi}\rangle$. We show how periodicity structure in these coefficients naturally leads to multi-band structure and we provide an explicit example of a three-band phase. We also briefly comment on the $k = 2$ pure gravity solution.

We are interested in the phase structure of the one-matrix model defined by the partition function $Z \equiv \mathrm{e}^{-N^2\Gamma} = \int d\Phi\ \mathrm{e}^{-N\,\mathrm{tr}\,U(\Phi)}$ where Φ is an $N \times N$ Hermitian matrix and $U = U_L(\phi) = \sum_{j=1}^{L} \frac{1}{j}\lambda_j\phi^j$, λ_j real. Unless otherwise stated, we consider $\lambda_L > 0$ and L even so that U is bounded from below and the partition function Z is well-defined.

The $k = 2$ pure gravity solution also exists in the model U_3 and in the "inverted Mexican-hat" U_4, both unbounded from below. Thus U_6 is the simplest well-defined model containing a $k = 2$ solution. Although much of our analysis is general, we have concentrated on the U_6 model in Ref. 5. Certain aspects of the tree level phase structure of this model has recently been studied also by others.[4] Other work on related questions includes discussions of the impossibility of flowing from $k = 3$ to $k = 2$, and regulating Z by allowing complex couplings.[9]

PHASE DIAGRAM

At the usual large-N limit (the tree level), the model is completely described in terms of the density of eigenvalues $\rho(z) \equiv \frac{1}{N}\sum_{i=1}^{N}\delta(z - x_i)$ where $(x_1, x_2, \ldots, x_N)$ is the large-N saddle point configuration of eigenvalues of Φ that dominates the integral after the $SU(N)$ degrees of freedom are integrated out. The phase diagram indicates the region of coupling constant space where different types of solutions for ρ characterized by the number of bands on which ρ has support, are present. The solutions for ρ satisfy a saddle point equation, which is obtained by minimizing the free energy $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, where $\Gamma_1 \equiv \int dz\rho(z)U(z)$ is the "potential term", $\Gamma_2 \equiv -\int\int dzdz'\rho(z)\ln|z - z'|\rho(z')$ is the "repulsion term", and $\Gamma_3 \equiv \gamma\Big(\int dz\rho(z) - 1\Big)$ is an additional term involving the "chemical potential" γ to enforce the constraint $\int dz\rho(z) = 1$.

If all couplings λ_j are positive, U has a unique minimum at $\phi = 0$. The saddle point solution for ρ then has a single finite band centered at the origin on which ρ has support. If, on the other hand, some λ_j, $1 \leq j \leq L-1$, are allowed to take on negative values, U has multiple local minima, and then $\rho(z)$ can have support in more than one bands.

Broadly speaking, when U has one minimum, one has the one-band phase, and in most of the regions where it has two or three minima, one has two- or three-band phase respectively. (The correspondence is not exact because of the repulsion term.) For instance, for the symmetric quartic potential $U_4(\phi) = \frac{\mu}{2}\phi^2 + \frac{g}{4}\phi^4$, with $g > 0$, one is always in a one-band phase for $\mu > 0$. As μ turns sufficiently negative, one moves into a two-band phase.

The phase diagram can also be determined using the Schwinger-Dyson equation satisfied by $F(z)$: $F(z)^2 - U'(z)F(z) + b(z) = 0$, where $b(z)$ is a polynomial. Its solution must satisfy the following constraints: (i) $F(z)$ must have only real singularities in the complex z-plane (the eigenvalues of Φ are real), (ii) it must be consistent with the large z expansion $F(z) = \frac{1}{z}(1 + \frac{w_1}{z} + \frac{w_2}{z^2} \ldots)$, and, (iii) it must lead to a density function that is positive semidefinite for real z. In terms of $F(z)$, the spectral density $\rho(z)$ for real z is given by

$$\rho(z) = -\frac{1}{\pi}\mathrm{Im}F(z) = -\frac{1}{2\pi}\mathrm{Im}\sqrt{\Delta(z)},$$

where $\Delta(z) \equiv [U'(z)]^2 - 4b(z)$ is a polynomial of order $2L-2$. Every odd degree zero of $\Delta(z)$ is a branch point of $F(z)$, hence must be real. With $F(z)$ real analytic, it follows that the branch cuts of F on the real axis are the bands on which ρ has support. On these cuts, $\mathrm{Re}F = \frac{1}{2}U'(z)$. A detailed analysis for the tree-level phase structure from the viewpoint of Schwinger-Dyson equation for U_4 and U_6 can be found in Ref. 5.

ORTHQGONAL POLYNOMIALS

We next discuss how $F(z)$ can be obtained in the large-N spherical limit using the method of orthogonal polynomials. Recall that for each normalizable measure $d\mu(x) = \exp(-NU(x))dx$, $-\infty < x < \infty$, a set of orthogonal polynomials, $P_n(x)$, can be defined by $\int d\mu(x)P_n(x)P_m(x) = h_n\delta_{n,m}$. $P_n(x)$ one. They satisfy a recursion relation: $xP_n(x) = P_{n+1}(x) + S_nP_n(x) + R_nP_{n-1}(x)$, with $R_0 \equiv 0$. For even potentials, $P_n(-x) = (-1)^nP_n(x)$, and $S_n = 0$.

One next introduces an operator formalism by associating each normalized orthogonal polynomial, $P_n(x)/\sqrt{h_n}$, an orthonormal basis vector, $|n\rangle$, $n = 0, 1, 2, \ldots$. For instance, the recursion relation for $P_n(x)$ is equivalent to the existence of an operator $\hat{\phi}$, with matrix elements

$$\langle n|\hat{\phi}|m\rangle = \delta_{n,m+1}\sqrt{R_n} + \delta_{n,m}S_n + \delta_{n,m-1}\sqrt{R_m}\,.$$

$F(z)$ can now be expressed in terms of $\hat{\phi}$,

$$F(z) = \frac{1}{N}\sum_{n=0}^{N-1}\langle n|(z-\hat{\phi})^{-1})|n\rangle\,. \qquad (1)$$

Given a potential $U_L(\phi)$, R_n and S_n obey recursion relations which can be obtained by evaluating the $(n-1,n)$ and (n,n) matrix elements of the differential operator $\frac{\partial}{\partial\phi}$. Upon integrating by parts, these relations, in an operator form, read

$$\begin{aligned} n &= N\sqrt{R_n}\langle n-1|U_L'(\hat{\phi})|n\rangle\,, \\ 0 &= \langle n|U_L'(\hat{\phi})|n\rangle\,. \end{aligned} \qquad (2)$$

where the right-hand sides involve R_j and S_k. Eq. (2) can be considered as a set of recursion relations, which allows one to find R_n and S_n iteratively. Alternatively, for the large-N limit, one can directly analyze Eq. (2) once an ansatz on the continuum structure of R_n and S_n as a function of $x = n/N$ is made. We have carried out both analyses and have demonstrated that the continuum limit is characterized by certain periodicity structure in n.[5] This periodicity is crucial for understanding the multi-band structure of a matrix model when the potential possesses multiple local minima.

A) Scalar Ansatz and Single-Band Structure

The simplest continuum ansatz one can make is one in which R_n and S_n approach continuous functions $R(x)$ and $S(x)$. This shall be referred to as a scalar ansatz (or period-1).

It is convenient to approach the large-N limit by introducing a pair of conjugate operators $\hat{\ell}$ and $\hat{\theta}$, where $\hat{\ell}|n\rangle = x_n|n\rangle$, $x_n \equiv n/N$, and $[\hat{\theta}, \hat{\ell}] = i/N$.

Under our scalar ansatz, the operator $\hat{\phi}$ can be expressed as $\hat{\phi} = \sqrt{R(\hat{\ell})}e^{i\hat{\theta}} + S(\hat{\ell}) + e^{-i\hat{\theta}}\sqrt{R(\hat{\ell})}$. In the θ-basis, an operator-valued function of $\hat{\phi}$ becomes a local differential operator with $\hat{\ell} \to -\frac{i}{N}\frac{d}{d\theta}$, and wave functions are periodic in θ with a period 2π. Since $[\hat{\theta}, \hat{\ell}] \to 0$ as $N \to \infty$, one finds, for the diagonal matrix elements of the resolvent operator, $\langle n|(z - \hat{\phi})^{-1}|n\rangle \to \int_0^{2\pi} \frac{d\theta}{2\pi}(z - \phi(x,\theta))^{-1}$, where $\phi(x,\theta) = S(x) + (e^{i\theta} + e^{-i\theta})\sqrt{R(x)}$. It follows that, in the large-N limit, after performing the θ-integration, the generating function approaches

$$F^{(1)}(z) = \int_0^1 dx \frac{1}{\sqrt{[z - S(x)]^2 - 4R(x)}} , \quad (3)$$

where the superscript signifies that (3) is derived under a scalar ansatz. In what follows, we shall simplify the discussion by restricting to the case of even potentials where $S(x) = 0$. The general case where $S(x) \neq 0$ will be reported elsewhere.

In the large-N limit, Eq. (2) for even potentials, under our scalar ansatz becomes $x = W_L(R(x))$, which can be solved for $R(x)$ analytically. For symmetric potentials whose coupling constants are all positive, a unique real solution $R(x)$ exists which is monotonically increasing over the positive x-axis with $R(0) = 0$. We shall refer to this class of symmetric potentials as "positive" and shall concentrate on this special class first.

Since $R(x)$ is bounded for $0 \leq x \leq 1$, Eq. (3) defines an analytic function for $|z|$ sufficiently large. Since $R(0) = 0$ and $R(x)$ is monotonic, it follows that $R(z)$ is real-analytic with a pair of symmetric square-root branch points located at $\pm z_1$, $z_1 \equiv 2\sqrt{R(1)}$. That is, Eq. (3) represents a single-band structure. Furthermore, since $F^{(1)}(z) \to 1/z$ as $|z| \to \infty$, it automatically leads to a properly normalized spectral density, $\rho(z)$, on the cut.

Changing the integration variable from x to $r = R(x)$, Eq. (3) becomes

$$F^{(1)}(z) = \int_0^{R(1)} dr \frac{W'(r)}{\sqrt{z^2 - 4r}} . \quad (4)$$

Since $W'(r)$ is real, for a fixed value of z in the range $|z| < z_1 = 2\sqrt{R(1)}$, only the region $0 \leq r \leq z^2/4$ contributes to $ReF(z)$. One easily verifies that $ReF^{(1)}(z) = \frac{1}{2}U'(z)$, as required. Similarly for the imaginary part one gets

$$\rho(z) = \frac{1}{2\pi} \int_0^{(z_1^2 - z^2)/4} \frac{dr}{\sqrt{r}} W'(r + \frac{z^2}{4}) , \quad (5)$$

and since $W'(r) > 0$ for positive potentials, $\rho(z)$ automatically satisfies the positivity requirement on the cut. Eq. (4) therefore represents a *bona fide* solution to the Schwinger-Dyson equation. Eq. (5) was first obtained in Ref. 2, and it follows more readily from our general expression, Eq. (4).

In generally, for a positive symmetric polynomial potential of order L, Eq. (5) leads to a spectral density $\rho(z) = \mathcal{P}_L(z^2)\sqrt{z_1^2 - z^2}$, where $\mathcal{P}_L(z^2)$ is an $(\frac{L}{2} - 1)th$ order polynomial in z^2, positive over the region $|z| \leq z_1$.

What will happen if we are dealing with potentials involving negative couplings? One can explicitly demonstrate that Eq. (3) no longer represents a solution of the Schwinger-Dyson equation. A possible approach is to always start with a *positive* potential and then analytically continue $F^{(1)}(z)$ into regions involving negative couplings. Indeed, (4) provides such a continuation. In terms of this representation, $F^{(1)}(z)$ always satisfies the Schwinger-Dyson equation, provided that the upper limit of the integral, $R(1)$, is real and well-defined. It also automatically leads to a single-band spectral density, given by (5). However, three obstacles can stop this continuation process: (i) $R(1)$ starts to turn complex, or takes a discontinuous jump, (ii) $\mathcal{P}_L(z^2)$ turns negative over the branch cut while $R(1)$ remains well-defined, (iii) this single-band solution becomes "subdominant".[5]

B) Two-component Ansatz

In order to deal directly with potentials involving negative couplings, one must go beyond the scalar ansatz. We first consider the period-2 ansatz, namely, that in the large-N limit $R_n \to A(x)$ for n even and $R_n \to B(x)$ for n odd, where $x = n/N$.

Since $\hat{\phi}$ connects only "nearest-neighbor" even-odd and odd-even basis states, it is convenient to think of the Hilbert space as a direct sum of an "even-space" and an "odd-space", *i.e.*, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. One can then arrange $\hat{\phi}$ into a 2×2 matrix form where $\hat{\phi}_{\sigma,\sigma'}$ maps $\mathcal{H}_{\sigma'} \to \mathcal{H}_\sigma$. Under

our two-component ansatz, one finds that $\hat{\phi}_{0,0} = \hat{\phi}_{1,1} = 0$, $\hat{\phi}_{1,0} = e^{-i\hat{\theta}}\sqrt{B(\hat{\ell})} + \sqrt{A(\hat{\ell})}e^{i\hat{\theta}}$, and $\hat{\phi}_{0,1} = \hat{\phi}^{\dagger}_{1,0}$. Similarly, any operator-valued function of $\hat{\phi}$ can also be arranged in a 2×2 matrix form, defined by a power series expansion in $\hat{\phi}$. In particular, the resolvent operator $\hat{G} = (z - \hat{\phi})^{-1}$ now has four components, $\hat{G}_{\sigma,\sigma'}$. In the large-N limit where $[\hat{\ell}, \hat{\theta}] \to 0$, one finds that the diagonal matrix element $\langle n|(z-\hat{\phi})^{-1}|n\rangle$, $n = 2p+\sigma$, approaches $\int_0^{2\pi} \frac{d\theta}{2\pi} G_{\sigma,\sigma}(x,\theta)$ where $G(x,\theta) = (z - \phi(x,\theta))^{-1}$, and $\phi(x,\theta)$ is an ordinary 2×2 hermitian matrix, *e.g.*, $\phi_{0,1}(x,\theta) = \sqrt{A(x)}e^{-i\theta} + \sqrt{B(x)}e^{i\theta}$.

It immediately follows from Eq. (1) that in the large-N limit a general representation for the generating function under a two-component ansatz is

$$F^{(2)}(z) = z \int_0^1 dx \frac{1}{\sqrt{[z^2 - \eta_2(x)]^2 - 4\xi_2(x)}}, \quad (6)$$

where we have again added a superscript. Instead of using A and B, we have also expressed Eq. (6) in terms of symmetric combinations $\eta_2 \equiv A + B$ and $\xi_2 \equiv AB$. As a consistency check, we note that Eq. (6) reduces to Eq. (3) if $A(x) = B(x)$.

Under what conditions is the period-2 ansatz valid? Detailed analysis for both U_4 and U_6 can be found in Ref. 5. The answer verifies our expectation from the numerical calculation and from the fact that the scalar ansatz applies for positive potentials that the period-2 ansatz is valid when the potential has two mimina. In particular, we show that Eq. (6) provides a unified representation for the transition region between the one-band and the two-band phase.

C) Multi-Component Ansatz

When a potential has many local minima, it is possible to have a multi-band structure where the number of bands increases with the number of local minima. Although this correspondence is not exact, it is a qualitative feature which can be understood by using a Dyson gas picture. Let us next assume that R_n has a period-q structure in the large-N limit so that $R_n \to A_\sigma(x), x = n/N$ where $\sigma = n \pmod q$, $q \geq 3$. This corresponds to a q-component ansatz.

We can again consider our Hilbert space as a direct sum of q subspaces $\mathcal{H}_\sigma, \mathcal{H} = \oplus_\sigma \mathcal{H}_\sigma$, where each $\mathcal{H}_\sigma$ is spanned by $\{|n\rangle, n = qp + \sigma, p = 0, 1, 2, \ldots\}$. Therefore $\hat{G}$ can be written as a $q \times q$ matrix where each component $\hat{G}_{\sigma,\sigma'}$ maps $\mathcal{H}_{\sigma'}$ to $\mathcal{H}_\sigma$. In the large-N limit, the generating function becomes

$$F^{(q)} = \frac{1}{q} \int_0^1 dx \sum_\sigma \int_0^{2\pi} \frac{d\theta}{2\pi} G_{\sigma,\sigma}(x,\theta), \quad (7)$$

where $G(x,\theta) = (z - \phi(x,\theta))^{-1}$ and $\phi(x,\theta)$ is a $q \times q$ hermitian matrix whose matrix elements are

$$\phi_{\sigma,\sigma'} = \sqrt{A_\sigma(x)}\delta_{\sigma,\sigma'+1}e^{i\theta} + \sqrt{A_{\sigma'}(x)}\delta_{\sigma,\sigma'-1}e^{-i\theta},$$

$\sigma = 1, 2, \ldots, q \pmod q$. Instead of $A_\sigma(x)$, it is sometimes more convenient to use symmetric combinations $\eta_q \equiv \sum_{\sigma=1}^q A_\sigma$ and $\xi_q \equiv \prod_{\sigma=1}^q A_\sigma$. For $q = 3$, (7) can be greatly simplified and one obtains

$$F^{(3)}(z) = \int_0^1 dx \frac{[3z^2 - \eta_3(x)/3]}{\sqrt{z^2[z^2 - \eta_3(x)]^2 - 4\xi_3(x)}}. \quad (8)$$

It is *a priori* unclear for which potentials a particular q-ansatz would be appropriate in the large-N limit. It is plausible to assume that a large-q ansatz would involve situations where a large number of local minima exist. For our current applications, we are primarily interested in situations where a three-band structure can occur. We have seen earlier that, generically, when a potential has a pair of degenerate local minima, a two-component ansatz becomes operative. The simplest generalization is one where a potential has three degenerate minima, *e.g.*, for $U_6(\phi) \equiv \frac{\mu}{2}\phi^2 + \frac{g}{4}\phi^4 + \frac{\lambda}{6}\phi^6 = (\frac{\lambda}{6})\phi^2(\phi^2 - \phi_0^2)^2$, $\lambda > 0$. The minima occur at $\phi = 0$, and $\phi = \pm\phi_0$ where $\phi_0^2 = -3g/4\lambda = -4\mu/g$.

Under a three-component ansatz, Eq. (2) leads to three coupled equations in the spherical limit, which allow one to find $\eta_3(x)$ and $\xi_3(x)$. Denote $\bar{x} \equiv \lambda r_0^3$ where $r_0 \equiv \sqrt{\frac{\mu}{3\lambda}}$. For $\bar{x} > 1$, with $g = -\sqrt{16\mu\lambda/3}$, Eq. (8) can be integrated easily. One obtains $F^{(3)}(z) = \frac{1}{2}U_6'(z) - i\pi\rho(z)$, where

$$\rho(z) = \frac{1}{2\pi}\lambda(z^2 - r_0)\sqrt{(z^2 - r_1)(z^2 - r_2)(r_3 - z^2)},$$

with $0 < r_1 < r_0 < r_2 < r_3$, $r_1 r_2 r_3 = 2/3\lambda$, $r_1 + r_2 + r_3 = 3r_0$, and $r_1 r_2 + r_2 r_3 + r_3 r_1 = 9r_0^2$. The positive normalized spectral density represents a three-band structure and solves the Schwinger-Dyson equation when the potential has three degenerate minima.

For $\bar{x} < 1$, $F^{(3)}(z)$ can be shown to agree with $F^{(1)}(z)$ obtained by analytic continuation, thus describing a single-band structure. The transition between the three-band and the single-band phases occurs at $\bar{x} = 1$, where $r_0 = r_1 = r_2$.

Surprisingly, a three-component ansatz as formulated above is incompatible for U_6 away from the situation of three degenerate minima. Although Eq. (8) would produce a real analytic function $F(z)$ having the structure of a three-band phase, it turns out that in general $ReF(z) \neq \frac{1}{2}U'(z)$ on the cuts. It is apparent that, to produce an acceptable three-band solution, one needs to work with an ansatz involving q being a multiple of three. This will be discussed in another publication.

REMARKS

In Ref. 5, we have provided numerical evidence for the periodicity structure in the orthogonal polynomial recursion coefficients. We have also identified different double scaling limits from various multi-band phases. In particular, a $k = 2$ type multi-critical solution from the two-band phase in the ϕ^8 potential and a $k = 1$ type solution from the three-band phase in the ϕ^6 potential are found. Both solutions are described by differential equations related to the modified KdV hierarchy, (type-II). A similar analysis for the multi-band structure of unitary one-matrix model has also been carried out in Ref. 7.

We have stressed in Ref. 5 that all the $k = 2$ Painleve-I solution found in hermitian one-matrix models so far arise either when the potential is unbounded from below or when the solution that gives rise to $k = 2$ behavior is sub-dominant. We have also shown in Ref. 7 that a similar situation occurs when we adopt a periodic regularization. By carrying out an analysis for a prototypic unitary model, we find that the $k = 2$ Painleve-I equation occurs at boundaries of phase regions which are again sub-dominant. However, unlike the case of a bounded ϕ^6 hermitian matrix model, a $k = 2$ type-II scaling equation is also obtained. This solution could serve as a new candidate for the pure 2d gravity. Significance of this new scaling solution deserves further examination.

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Critical Exponents in Matrix Models

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Abstract

Critical behaviour in D-dimensional matrix models is considered within the field theory approach to critical phenomena and ε-expansion. For real (Hermitian) $N \times N$ matrix order parameter $\hat{\Phi}$ the Landau-Ginzburg type Lagrangians are constructed and all the critical points are found. The relevant critical point is a saddle point existing when the potential is essentially $V \sim (Tr\hat{\Phi}^2)^2$. At the critical point the model is equivalent to the n-vector one with an appropriate number of parameters. In the latter case, the critical exponents can be calculated with high accuracy. Empirical relation for the correlation length exponent ν for arbitrary n and D is presented. Large N limit admits an exact solution coinciding with that of the spherical model.

1 Introduction

There are two main motivations to study matrix models. The first one is the existence of real physical objects such as liquid crystals [1] or liquid Helium-3 [2], which can be described within the models with a 3×3 matrix order parameter. In both cases, different phases could exist and a phase transition occurs. Though its nature is not yet clear, several characteristics have a singular behaviour which presumably can be described by the standard renormalization group approach.

The second motivation is connected with lattice formulation of 2-dim. quantum gravity. In the discrete approach based on dynamically triangulated random surfaces the theory can be regarded as $N \times N$ matrix field theory, where the sum over various genuses is simply the large N expansion [3]. Continuum limit actually corresponds to the critical point of the underlined field theory. For a number of models corresponding to conformal field theories with central charge $c \leq 1$ the critical exponents have been calculated from the discrete approach [3], while for the string theory $c > 1$, and in this regime calculations have so far failed.

In the present paper we write down matrix models and treat them according to the standard field theory approach and renormalization group method. Critical behaviour in the infrared region is studied and all the critical points are found. We show that the critical point, if it exists, corresponds to an n-vector model with an appropriate number of parameters, where the critical exponents have been calculated already with great accuracy.

2 The Model. Relation to Critical Phenomena

We consider the model with a single order parameter (field) which is an $N \times N$ real (Hermitian) matrix $\hat{\Phi}$. Symmetry properties are dictated by the form of this matrix and could be different in different phases. We concentrate below on three particular cases being irreducible representations of $SO(N)$ and $SU(N)$. Namely, we consider $\hat{\Phi}$ to be real traceless symmetric, antisymmetric and hermitian traceless matrices.

In the field theoretical approach to critical phenomena a crucial role is played by a Lagrangian rather than by free energy. To construct a Lagrangian, which is invariant under an appropriate symmetry group, we consider all possible invariants restricted by the renormalizability requirement. Having in mind ε-expansion, where dimension is $D = 4 - 2\varepsilon$, we are left with quadratic and quartic terms. Thus, we come to the Landau-Ginzburg type Lagrangian for a traceless field $\hat{\Phi}$:

$$\mathcal{L} = \frac{1}{2} Tr(\partial\hat{\Phi})^2 - \frac{m^2}{2} Tr\hat{\Phi}^2 - \frac{h_1}{4!} Tr\hat{\Phi}^4 - \frac{h_2}{4!} (Tr\hat{\Phi}^2)^2 \quad (1)$$

Three different choices of the matrix $\hat{\Phi}$ are distinguished by the form of the propagator. We have

respectively :

$$\textit{symmetric}\ \widehat{\hat{\Phi}^{ab}\hat{\Phi}^{cd}} \sim \frac{1}{2}(\delta^{ad}\delta^{bc}+\delta^{ac}\delta^{bd}-\frac{2}{N}\delta^{ab}\delta^{cd}),$$

$$\textit{antisymm}\ \widehat{\hat{\Phi}^{ab}\hat{\Phi}^{cd}} \sim \frac{1}{2}(\delta^{ad}\delta^{bc}-\delta^{ac}\delta^{bd}),$$

$$\textit{Hermitian}\ \widehat{\hat{\Phi}^{ab}\hat{\Phi}^{cd}} \sim (\delta^{ad}\delta^{bc}-\frac{1}{N}\delta^{ab}\delta^{cd}),$$

where $a,b,c,d = 1,2,\ldots,N$.

Critical phenomena are associated with the infrared properties of the model. Scaling behaviour in the vicinity of the critical point caused by the appearance of a long-range order can be described in terms of Euclidean quantum field theory possessing an infra-red stable fixed point [4].

A systematic approach to the description of infra-red asymptotics is based on the renormalization group. In the presence of infra-red stable fixed points defined by the vanishing of RG β-functions the dimensionless Green functions obey the scaling laws for small p^2

$$\Gamma_R \overset{p^2\to 0}{\sim} (p^2)^{-\gamma_\Gamma(h^*)}$$

with the powers γ_Γ equal to the anomalous dimensions at $h = h^*$. There exist direct relations between the anomalous dimensions and critical exponents, which characterize the scaling behaviour of various quantities in the neighbourhood of a second order phase transition. For example, the critical exponents η (correlation function) and ν (correlation length) can be expressed through the anomalous dimension of the field and mass, respectively [5,6]

$$\eta = 2\gamma_2(h^*), \quad \nu = \frac{1}{2(1-\gamma_m(h^*))}. \qquad (2)$$

All other critical exponents are not independent and can be evaluated via the scaling laws

$$\gamma = (2-\eta)\nu,\ \alpha = 2-\nu D,\ \beta = \frac{\gamma}{\delta-1},\ \delta = \frac{D+2-\eta}{D-2+\eta}.$$

Critical point h^* is the infra-red stable fixed point of the renormalization group equation. Within the ε-expansion method it is a power series of ε calculated in perturbation theory.

3 Renormalization Group Equations. Fixed Points

In this section we consider RG equations for the effective couplings of the model at hand in $4-2\varepsilon$ dimensions. Remind that in the MS-scheme the β-functions in $4-2\varepsilon$ dimensions are connected with those in 4 dimensions by the equation [7]

$$\beta_{4-2\varepsilon}(h) = -\varepsilon h + \beta_4(h).$$

Fixed points correspond to the r.h.s equal to zero. They are all the power series of ε :

$$h_i^* = \varepsilon u_i^1 + \varepsilon^2 u_i^2 + \cdots,$$

where the coefficients u_i^k are determined in k-th order of perturbation theory.

Having this in mind we get the following RG equations written to one-loop order (we consider below the Hermitian case for definiteness).

$$\dot{h_1} = -\varepsilon h_1 + \frac{N^2-9}{3N}h_1^2 + 2h_1h_2, \qquad (3)$$

$$\dot{h_2} = -\varepsilon h_2 + \frac{N^2+3}{2N^2}h_1^2 + \frac{2N^2-3}{3N}h_1h_2 + \frac{N^2+7}{6}h_2^2.$$

According to the general analysis [4, Sect XI] there are four types of fixed points of eq.(3):

1. $u_1 = u_2 = 0$;
2. $u_1 = 0, u_2 = \frac{6}{N^2+7}$;
3. $u_1 = \begin{cases} 6/121 & N=2 \\ -1/4 & N=3 \\ \textit{absent} & N>3 \end{cases}$
   $u_2 = \begin{cases} 63/121 & N=2 \\ 1/2 & N=3 \\ \textit{absent} & N>3 \end{cases}$
4. $u_1 = \begin{cases} -6/11 & N=2 \\ -1 & N=3 \\ \textit{absent} & N>3 \end{cases}$
   $u_2 = \begin{cases} 3/11 & N=2 \\ 1/2 & N=3 \\ \textit{absent} & N>3 \end{cases}$

Before analysing the stability properties of these fixed points it is useful to note that Hermitian traceless $N\times N$ matrices for $N = 2$ and 3 obey the following equation:

$$Tr\ \hat{\Phi}^4 = \frac{1}{2}\,(\,Tr\ \hat{\Phi}^2)^2. \qquad (4)$$

Hence for $N = 2,3$ there exists only one independent coupling in eq.(1) equal to $(h_1+2h_2)(Tr\ \hat{\Phi}^2)^2$. Looking for the value of the coupling h_1+2h_2 at the critical points (1 - 4) we find out that it is the same

for the points 1 and 4 as well as for 2 and 3, respectively. Thus, the presence of four different points for $N = 2, 3$ is just an artefact, and there are only two relevant fixed points for any value of N.

Stability properties of the fixed points can be investigated in a standard way. The fixed point 1 is absolutely infra-red unstable and the fixed point 2 is a saddle point. The phase portrait of the trajectories is shown in Fig.1. One can see that the fixed point 2 can be reached only when the coupling $h_1 = 0$. In this case, the fixed point is infra-red stable and according to a general belief corresponds to a second order phase transition one. Otherwise, there is no fixed point solution of eq.(3).

So far, we have considered the leading approximation. However, the obtained results are stable with respect to higher order corrections. In any loop order the infra-red fixed point will lie on the h_2 axis being the power series of ε

$$h_1^* = 0, \quad h_2^* = u_2^1\varepsilon + u_2^2\varepsilon^2 + \cdots.$$

It is a saddle point in the coupling constant space.

It should be stressed that the conclusion is valid for any value of ε , i.e. for any value of D. The problem arises when one tryes to sum the series. Here we come to the problem of validity of the ε -expansion.

The situation is qualitatively the same for the other cases mentioned above. In case of complex matrices the potential is more complicated, however even here nothing is changed. The only fixed point is a saddle point with only Tr^2 interaction surviving.

4 Critical Exponents

To find the critical exponents one has to calculate the anomalous dimensions at the infra-red fixed point. The results will be expressed via the power series of ε. However, there is no necessity to perform any new calculation. Indeed, if one looks at the Lagrangian, eq.(1), at the fixed point, one finds out that the only coupling which survives is $(Tr\ \hat{\Phi}^2)^2$. Then, expanding the matrix field $\hat{\Phi}$ over the irreducible set of matrices in an appropriate representation $\hat{\Phi} = \sum T^i\phi^i$, we get

$$(Tr\ \hat{\Phi}^2)^2 \sim (\phi^i\phi^i)^2,$$

where we have taken into account that $Tr\ T^iT^j \sim \delta^{ij}$.

Thus, what we finally get is the n-vector model with the number of components equal to that of the original matrix. For the three cases of interest we have respectively

$$n = \begin{cases} \frac{(N-1)(N+2)}{2} & symmetric\ SO(N) \\ \frac{(N-1)N}{2} & antisymmetric\ SO(N) \\ N^2-1 & Hermitian\ SU(N) \end{cases}.$$

Critical exponents in the n-vector model have been calculated with high accuracy. Recent most accurate estimates have been achieved in the framework of ε-expansion, where the calculations are done up two five loop order [9,8].

The coefficients of ε-expansion grow very fast which is a manifestation of asymptotical character of this expansion. This means that to get a numerical result for $D = 2$ or 3 (i.e. $\varepsilon = 1$ or $1/2$) one needs a special summation procedure. The latter was proposed in a number of papers [5,6,8]. The results obtained for small values of n are in very good agreement with experiment as well as with other approaches. The procedure can be repeated for any value of n.

In a recent paper [10] we have proposed an empirical expression for the correlation length critical exponent ν. It is valid for arbitrary n and D and fits all known exact and numerical values. Even if it is not an exact solution, the advocated result can serve as a very accurate approximation to the true value. It has the following form:

$$\nu = \frac{(D-2)(3+2x) - x}{2(D-2)(3+2x) - 2x - (4-D)(x+2)}, \quad (5)$$

where the parameter x is connected with n by the equation

$$x = \begin{cases} \frac{n-6}{4} & for\ even\ n \\ \frac{n-[\frac{n}{4}]-5}{3} & for\ odd\ n \end{cases} \quad (6)$$

For $D = 3$ eq.(5) gives a smooth curve

$$\nu = \frac{3+x}{4+x}. \quad (7)$$

Thus, to get the values of the critical exponents for the matrix model one has to substitute an appropriate value of $n = n(N)$ into ε- expansion or directly into eqs.(5),(6) and (7).

Special attention is paid to the $N = \infty$ case. For the matrix model, eq.(1), it corresponds to taking into account of planar diagrams in all orders of perturbation theory. However, at the critical point due

to the absence of the $Tr\ \hat{\Phi}^4$ term the situation is drastically simplified. For the $(Tr\ \hat{\Phi}^2)^2$ interaction (or equivalently for the n-vector model) in the large N limit only one-loop diagrams survive. As can be seen e.g. from ref.[6] the results for $n = \infty$ are

$$\eta = 0, \quad \nu = \frac{1}{D-2}. \tag{8}$$

This corresponds to the so called spherical model which admits an exact solution [11]. Strictly speaking, eq.(8) is valid only for $D = 3$ or 4 as far as for $D \leq 2$ the phase transition for large $n \geq 2$ disappears. For $D = 3$ we get $\eta = 0$, $\nu = 1$ in accordance with eq.(7).

5 Conclusion

We have shown above that the critical behaviour in matrix models of eq.(1) is determined by the presence of nontrivial infra-red fixed point. Our conclusion is true in all orders of perturbation theory for any value of ε and N. This fixed point is believed to be associated with the second order phase transition with the matrix order parameter. As we have seen, it exists only when the interaction is essentially $(Tr\ \hat{\Phi}^2)^2$. Addition of arbitrary small amount of $Tr\ \hat{\Phi}^4$ destroys a phase transition.

At this point, our conclusion contradicts that of ref.[12] where in $D = 0$ matrix model in $N \to \infty$ limit different phases were obtained with critical exponents having negative values. From our point of view the negative values of exponents mean that the formulas are out of the range of applicability. This really happens in the vector model, where for some values of parameters a phase transition disappears, while formally the critical exponent may take a negative value (see Fig. 2). A way out of this descrepancy is probably in different meaning attached to a phase transition and connection between the field theory model and statistical system. Strictly speaking it is not obvious, what kind of a phase transition occurs in $D = 0$ case and what is an order parameter. It would be interesting to find a relation between the two approaches and to understand better the nature of different phases.

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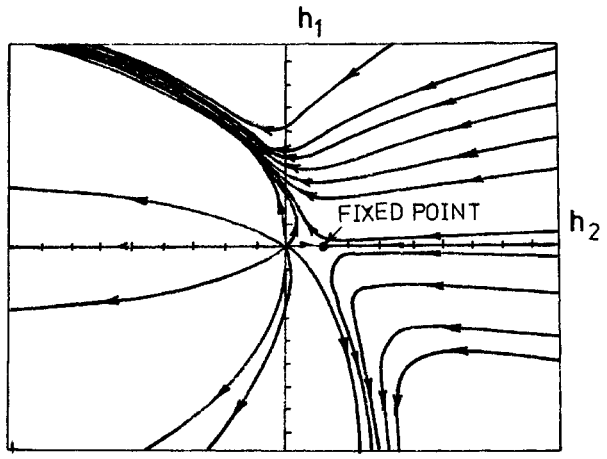

Figure 1: Phase portrait of solutions for $N > 3$. The arrows show the direction of decreasing argument t corresponding to the infra-red limit.

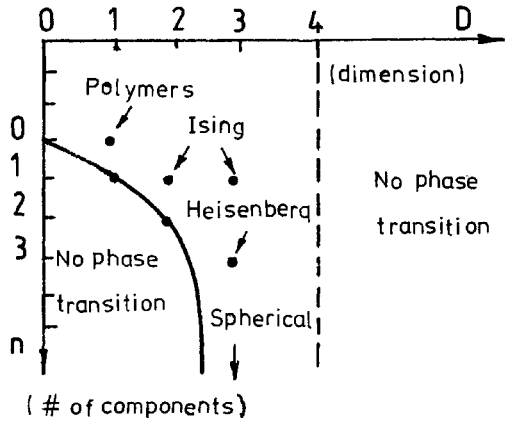

Figure 2: Structure of vector model in the space of parameters.

INTEGRABILITY IN MATRIX MODELS

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1. Some aspects of the recently discovered non-perturbative solutions to non-critical strings [1] can be better understood and clarified directly in terms of the integrability properties of the random matrix model used to define the coupling of (p,q)-conformal matter to $2D$-gravity. Soon after the appearance of the original papers on the subject , Douglas [2] showed that the continuum limit of these models can be described in terms of Heisenberg algebras and the KdV hierarchy [3]

The non-pertubative definition of strings [1] requires two different types of equations. Denoting the string susceptibility by $u(x)$ as a function of x, the cosmological constant, the first equation is

$$-x = \sum_{n=1}^{\infty}(2n+1)t_n\mathcal{R}_n(u) \qquad (1)$$

involving the Gel'fand-Dickii potentials $\mathcal{R}_n$ and providing a non-perturbative characterization of the different critical points of a generic hermitean matrix model. The second type of equations describe the renormalization group flows of the models defined by (1) . These equations coincide with the flows of the KdV hierarchy [3] The geometrical meaning of the string equations has been clarified [4] using the Schwinger-Dyson equations for loop functionals [5] (see also [6]).

In collaboration with C. Gomez and J. Lacki [7] we consider a different derivation of (1) using the Volterra hierarchy for even potentials [8]. The two basic steps in this approach are: i) to show the connection between discrete integrable models and the matrix models defined in terms of orthogonal polynomials, and ii) understand the extension of the double scaling limit to non-linear lattices. The outputs of this construction are (1) and the flow equations where (1) appears as an initial condition of the non-linear lattice.

An interesting fact about the Volterra equation which gives rise to some interesting speculation is its relation to the Liouville model on a regular lattice discussed in [9]. The existence of an infinite number of conserved charges in involution for the Volterra equation defines an infinite set of hamiltonian equations and a collection of compatible hamiltonian structures which in the continuum limit reproduce the well-know Gel'fand-Dickii bihamiltonian properties of the KdV equation [10].

2 For simplicity we consider only the case of one matrix models with even potentials. The generic even potential is

$$V(M) = \sum_i g_i \ M^{2i} \qquad (2)$$

if one defines monic orthogonal polynomials with respect to the measure

$$d\mu(\lambda) = d\lambda e^{-\beta V(\lambda)} \tag{3}$$

the partition function of the one matrix model can be written as [11]

$$Z(g) = h_0 h_1 \dots h_{N-1} \tag{4}$$

the orthogonal polynomials satisfy the two step recursion relation

$$\lambda P_n = P_{n+1} + R_n P_{n-1} \tag{5}$$

hence

$$Z(g) = \prod_i R_i^{N-i} \tag{6}$$

changing basis from the P_n's to an orthonormal basis , the operation of multiplication by λ becomes

$$\lambda \mathcal{P}_n = \sum L_{nm} \mathcal{P}_m \tag{7}$$

where L is the Jacobi matrix

$$L = \sum \sqrt{R_k} E^+_{k,k+1} \tag{8}$$

and the E_{ij} are matrix units. As we change the couplings g_i the polynomials change as well as the matrix elements of L. Using simple homogeneity arguments and a theorem of Moser [12] it is not difficult to show that the matrix L changes with the couplings according to

$$\frac{\partial L}{\partial g_i} = [L^{2i}_+, L] \tag{9}$$

in (9) L^{2i}_+ is defined as the antisymmetric matrix such that the strictly upper triangular parts of L^{2i}_+ and L^{2i} coincide. In the theory of integrable systems [3] an equation of the form (9) is called a Lax equation and its complete integrability is a consequence of the existence of an infinite number of conservation laws in involution (with vanishing Poisson brackets). In this context the flows are generated only by the even powers of L^{2i}_+. The form of the matrix (8) implies that the only non-vanishing entries of the matrix L^n appear in positions $(i, i+n), (i, i+n-2), \dots$ and their transpose, and furthermore in $[L^{2i}_+, L]$ the only non-vanishing entries appear at positions $(i, i+1)$ and $(i+1, i)$ whereas for $[L^{2n+1}_+, L]$ only the diagonal elements are different from zero. Consequently L^{2n}_+ generates flows and L^{2n+1}_+ provides initial conditions. This cannot be seen so clearly in the continuum limit where the form of the operators generated by $[L^{2n}_+, L]$ and $[L^{2n+1}, L]$ are essentially the same.

If the Volterra flows are associated to a potential with a finite number of non-zero couplings, the operator $d/d\lambda$ is also represented by a Jacobi matrix P [13] such that $P_{ij} \neq 0$ for $|i-j| \leq 2k-1$ if the potential has degree $2k$. The matrix P can be chosen to be antisymmetric and it obviously satisfies $[P, L] = 1$. Since P is a Jacobi matrix and $[P, L]$ is diagonal, we can write P as a linear combination of L^{2n+1}_+'s

$$P = \sum_{j=1}^{k} 2j c_j L^{2j-1}_+ \qquad [P, L] = 1 \tag{10}$$

Conversely, it is possible to prove using Favard's theorem [14] that beginning with a Volterra hierarchy together with the initial condition (10) for the flows, the set of polynomials (5) defined by the operator L are an orthogonal set with respect to the measure (3) .

3. It is very instructive to to study the hierarchy (9) from a hamiltonian point of view. This approach exhibits the discrete version of the Virasoro conditions found in [4]. The hierarchy (9) admits a collection of compatible Poisson structures. The most natural one is [8]

$$\{R_n, R_m\}_1 = R_n R_m(\delta_{n,m+1} - \delta_{n,m-1}) \qquad (11)$$

and the flows (9) are generated by the hamiltonians

$$H_k = \frac{1}{2k} tr L^{2k} \qquad (12)$$

It is more convenient for our purposes to introduce a new variable $u_m = \log R_m$ which will become one of the scaling variables in the continuum limit, and

$$\frac{\partial u_m}{\partial g_n} = \{H_n, u_m\}_{=}(\frac{\partial}{\partial u_{m+1}} - \frac{\partial}{\partial u_{m-1}})H_n \qquad (13)$$

and one can check that the flows (13) commute.

The second hamiltonian structure which can be defined for the Volterra hierarchy [9] is

$$\{R_n, R_m\}_2 = R_n R_m(R_n + R_m)(\delta_{n,m+1} - \delta_{n,m-1})$$

$$+R_n R_m(\delta_{n,m+2}R_{m+1} - \delta_{n,m-2}R_{m-1}) \qquad (14)$$

the origin of the Virasoro conditions in [4] lies in the discrete analogue of the Gel'fand-Dickii relation [10] between the two hamiltonian structures of the KdV hierarchy. In the case of the Volterra hierarchy we find

$$\{H_n, R_m\}_2 = \{H_{n+1}, R_m\}_1 \qquad (15)$$

the proof can be found in [7]. The origin of the Virasoro conditions on the partition function in the continuum limit [4] lies in the discrete Gel'fand-Dickii relation and in the fact that the second hamiltonian structure of the Volterra hierarchy gives in the continuum limit the Virasoro algebra. This discrete version of the Virasoro algebra appeared for the first time in [9] and it is also shown in this reference that the classical equations of motion for the discretized Liouville theory are equivalent to the classical Volterra equations. One wonders whether there is a direct connection between the discrete Liouville system described in [9] and the matrix model definition of $2D$-gravity [1].

In terms of hamiltonian flows the string equations become

$$\sum_j j g_j(\frac{\partial H_j}{\partial u_{m+1}} - \frac{\partial H_j}{\partial u_{m-1}}) = \beta^{-1} \qquad (16)$$

and in terms of the partition function (6) we obtain

$$\sum_j j g_j \frac{\partial log Z}{\partial g_j} = \frac{N(N+1)}{\beta} \qquad (17)$$

using (16) and the Gel'fand-Dickii property to relate higher and lower flows one obtains an infinite number of conditions on the partition function. In the continuum limit they become the Virasoro conditions of [4]. To understand these statements more clearly, we briefly analyze the double scaling limit of the Volterra hierarchy.

We will follow the notation of Gross and Migdal in ref.[1] The string equation in Douglas' form [2] has the discrete form

$$\sum_j 2j g_j [L_+^{2j-1}, L] = 2\beta^{-1} \qquad (18)$$

since only the (l, l) components of this equation are different from zero, summing the first m diagonal entries in (18) we obtain

$$\sum_j 2 g_j R_m^{1/2} L_{m,m+1}^{2j-1} = m\beta^{-1} \qquad (19)$$

going to the continuum limit one introduces the continuous variable $x = n/\beta$ and the k-th critical point is obtained by setting $x = 1 - \beta^{-\frac{2k}{2k+1}}t$ and $R = 1 - \beta^{\frac{-2}{2k+1}}f(t)$. By tuning the couplings in (19) one obtains the k-th critical point. Let $\bar{g}_j^{(k)}$ be the corresponding values of the couplings, then we know from [1] that

$$\sum_l \mu_l \alpha_l^{(k)}(\beta) \sum_{j=1}^{k} 2j\bar{g}_j^{(k)} R_m^{1/2} L_{m,m+1}^{2j-1} = m\beta^{-1} \tag{20}$$

has a well defined scaling limit describing the perturbations by scaling operators of the kth critical point, with the constants $\alpha_l^{(k)}$ chosen appropriately [1]. Since the hamiltonians (12) are given as functionals of R_n which is not a scaling variable in the continuum limit, we have to find the linear combinations of hamiltonians with well defined scaling properties. In terms of u_m we obtain

$$\frac{\partial u_m}{\partial g_n} = D\frac{\partial H_n}{\partial u_m}$$

with

$$\frac{\partial H_n}{\partial u_m} = R_m^{1/2} L_{m,m+1}^{2n-1} \tag{21}$$

D is a discrete derivative defined by $Df_m = f_{m+1} - f_{m-1}$ with good scaling properties in the continuum limit. Hence

$$\sum_l \mu_l \alpha_l^{(k)} \sum_j 2j\bar{g}_j^{(k)} \frac{\partial H_j}{\partial u_m} = m\beta^{-1} \tag{22}$$

and therefore

$$\mathcal{H}_l = \sum_j^l 2l\bar{g}_j^{(l)} H_j \tag{23}$$

is a combination of hamiltonians with good scaling properties. Therefore in the continuum limit we expect the scaling hamiltonians (23) to turn into the conserved charges of the KdV hierarchy. To show that this is indeed the case all we have to do is to show that the continuum limit of (15) is the standard Gel'fand-Dickii relation [10], or equivalently that the second Poisson structure (14) becomes the Virasoro algebra. Although the conserved quantities for the Volterra hierarchy were written explicitly in (12) , we can read (15) as a recursion relation for the conserved quantities of the hierarchy. From H_n we can compute H_{n+1} with a unique result if we require H_n to be homogeneous of degree n in R. Therefore, knowing H_1 we can derive all other conserved quantities H_n and they are guaranteed to be commuting due to the compatibility of the Poisson structures. The continuum limit of the Poisson structures is easier to derive in terms of the variables u_m . For the first Poisson structure (11) we have

$$\{u_n, u_m\} = \delta_{n,m+1} - \delta_{n,m-1} \tag{24}$$

in the continuum limit $R_n \mapsto R(x) = 1 - \beta^{\frac{-2}{2k+1}}f(t)$ with $x = 1 - \beta^{\frac{-2k}{2k+1}}t$ and $u_m = \log R_m$, defining $\lambda = \beta^{\frac{-1}{2k+1}}$ we have $u_m = -\lambda^2 f_m \mapsto -\lambda^2 f(t)$. Thus the renormalized first Poisson bracket becomes in the continuum limit

$$\{f(t), f(t')\}_1 = lim_{\lambda \mapsto 0} \lambda^2 \{f_n, f_m\} \tag{25}$$

Next to define the continuum limit of the second Poisson bracket with good scaling properties we have to subtract the first Poisson structure with a coefficient depending appropriately on the scaling parameter λ. This is legitimate because the two hamiltonian structures are compatible for any value of the cut-off. Up to some trivial numerical factors we can write the result as

$$\{f(t), f(t')\} = (\frac{1}{2}D^3 + Df + Df)\ \delta(t-t') \quad (26)$$

which is the classical form of the Virasoro algebra. As a consequence of the compatibility of the lattice Poisson structures these two brackets are compatible and as before they can be used to generate all the conserved quantities for the continuum system. Once we have the string equations and the Gelfand-Dickii relations one only needs to repeat the arguments on [4] to see that the infinite number of conditions on the partition function become the Virasoro constraints.

4. We have shown that many of the properties of the recently found non-critical strings follow directly from the complete integrability of the Volterra hierarchy equivalent to the original matrix model formulation. If one considers general potentials instead of even ones the three step recursion relation satisfied by the orthogonal polynomials leads to the Toda hierarchy and the arguments presented go through virtually unchanged. To summarize we have found that the integrability of the Volterra hierarchy together with a very particular initial condition gives the discrete string equations together with an infinite number of conditions on the partition function as a consequence of the discrete version of the Gel'fand-Dickii relations. continuum limit as the Virasoro constraints.

It is known that many of the known conformal field theories can be obtained as special limits of integrable systems. In the examples described in this work we have found that the effective action of some (non-unitary) conformal models coupled to $2-D$ gravity are described by simple integrable systems subject to particular initial conditions. To what extent a similar procedure can be carried out for other known intergrable systems is an interesting question currently under investigation.

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String Field Theory in One Dimension and Matrix Models

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ABSTRACT

We discuss the singlet sector of the $d = 1$ matrix model in terms of a Dirac fermion formalism. The leading order two- and three- point functions of the density fluctuations are obtained by this method. This allows us to construct the effective action to that order and hence provide the equation of motion. This equation is compared with the one obtained from the continuum approach.

In recent works[1][2][3] we have studied the problem of two-dimensional quantum field theories coupled to gravity. Our original motivation to do this was to arrive at a natural setting for the theory space formulation[4][5] of string theory, where, (1) there is no restriction on the central charge of the matter sector, and, (2) the theory has, within it, the ingredients to describe trajectories which join special points in the theory space, namely the classical vacua which correspond to conformally invariant theories. One of our main results has been that the matter + gravity system can be regarded as a field theory of the Liouville mode and matter fields in the background of the fiducial metric. Generic couplings or backgrounds now depend both on the Liouville mode and on the matter degrees of freedom and satisfy equations of motion in $d+1$ variables (d = matter, 1 = Liouville). This is because reparametrization invariance of the theory implies that all objects, in which the conformal mode has been integrated, should be Weyl-invariant in its dependence upon the fiducial metric. This condition, stated as the vanishing of the 'B function', gives rise to the equations of motion[3]. Other related works are due to J. Polchinski[6], and, T. Banks and J. Lykken[7].

These ideas were illustrated in various situations:

(a) For d-scalar fields interacting with 2-dim. gravity, we proved that this system quantized in the light cone gauge is exactly mapped into the conformally invariant field theory of d + 1-scalar fields, with background charge. At $d = 25$ we obtained the exact tree level S-matrix and spectrum of the "$d = 26$ critical string".

(b) In the case of $d < 1$, we considered the $(m, m+1)$ minimal models coupled to gravity, and could effectively describe the interpolation between two minimal models, for m very large, by means of a 'string field' that depends only on the Liouville mode, a function $\kappa(\eta)$ which satisfies the field equation[3]

$$\left(\partial_\eta{}^2 + Q\partial_\eta + h\right)\kappa(\eta) = b\kappa^2(\eta) + o(\kappa^3) \quad (1)$$

(c) In the case of d-scalar fields coupled to gravity, perturbed by a 'tachyon' background, the tachyon coupling T,which depends on d coordinates ϕ_i and the Liouville mode η satisfies the $d + 1$ dimensional field equation[1][3]

$$\left(\partial_\eta{}^2 + Q\partial_\eta + \partial_i\partial_i + 2\right)T(\phi,\eta) + T^2(\phi,\eta) + \cdots = 0 \quad (2)$$

where

$$Q = \sqrt{\frac{25-d}{3}} \quad (3)$$

The coupling of other backgrounds like the metric, antisymmetric tensor and dilaton can be discussed similarly. To see the spectrum from equation (2), one eliminates the

linear derivative piece by defining $\tilde{T} = e^{-\frac{1}{2}Q_\eta}T$

$$\Big(\partial_\eta{}^2 + \partial_i\partial_i + \frac{1}{4}(8-Q^2)\Big)\tilde{T}(\phi,\eta) + e^{-\frac{1}{2}Q_\eta}\tilde{T}^2(\phi,\eta) + \cdots = 0 \tag{4}$$

This equation tells us that the spectrum at $d = 1$ (i.e., $Q^2 = 8$) is that of a massless particle. For $d > 1$, there is a tachyon in the spectrum and hence for much the same reasons as in 26-dimensional critical string theories, where it ruins the perturbation expansion, these theories may not exist. It is likely that the tachyon perturbation drives $d > 1$ theories to a stable point which is $d = 1$. It would also be interesting to understand how one can reach models with $d < 1$ by appropriate perturbations of the $d = 1$ model.

Our main purpose here is to discuss the cut-off string field theory at $d = 1$[9][9][10], formulated as the quantum mechanics of the matrix hamiltonian which was originally discussed by Brézin, Itzykson, Parisi and Zuber[11]

$$H = -\frac{1}{2N}\nabla_M^2 + N\ tr\ V(M) \tag{5}$$

where ∇_H^2 is the laplacian in the space of hermetian matrices and $V(M)$ is a polynomial. We can expect the results of the continuum theory and that from the matrix model approach to agree in the low momentum region only.

Since this hamiltonian is invariant under $U(N)$ transformations, $M \to UMU^\dagger$, there would be wavefunctions transforming according to various different representations of $U(N)$. (To be more precise, these consist of the trivial representation and the representations that can be generated by taking products of the adjoint.) It is not yet clear whether states which transform nontrivially under $U(N)$ are related to the string degrees of freedom. Presently we will analyse the singlet sector of model. We use the fermionic representation of this sector as explained below. This representation has two major advantages.

(*a*) The model is well defined even for finite N and for noncritical values of the coupling. Hence the nature of the various regularizations are most clearly recognized in this picture.

(*b*) It is easier to see various approximate and exact symmetries of the system from this point of view.

As in well known, ∇_M^2 acting on the singlet sector wave function $\psi(\Lambda)$ has the form

$$\nabla_M^2\psi(\Lambda) = \frac{1}{\Delta(\Lambda)}\Big(\sum_i \frac{\partial^2}{\partial\lambda_i^2}\Big)\Delta(\Lambda)\psi(\Lambda) \tag{6}$$

where $\Lambda = (\lambda_1, \cdots, \lambda_N)$, λ_i being the eigenvalues of M. $\Delta(\Lambda)$ is the Vandermonde determinant

$$\Delta(\Lambda) = \prod_{i<j}(\lambda_i - \lambda_j) \tag{7}$$

If we use $\chi(\Lambda) = \Delta(\Lambda)\psi(\Lambda)$ instead of $\psi(\Lambda)$ as the wave function, the effective hamiltonian becomes

$$H_f = -\frac{1}{2N}\sum_i \frac{\partial^2}{\partial\lambda_i^2} + N\sum_i V(\lambda_i) \tag{8}$$

This is the hamiltonian for non-interacting particles. However, since $\psi(\Lambda)$ is symmetric, $\chi(\Lambda)$ is antisymmetric. Hence the problem reduces to that of noninteracting fermions moving in an external potential.

The ground state is obtained by filling the N states which are lowermost in energy. The corresponding total energy becomes singular when the Fermi energy of the system approaches the value of the potential at a stationary point. This is related to the fact that the time period of the classical orbits corresponding to the states near the fermi surface starts diverging when they can approach the stationary point. In the semiclassical analysis one can obtain the nature of the singularities. Also one can take the double-scaling limit by keeping fixed the energy difference between the value of the potential at the stationary point and that of Fermi energy as N goes to infinity. Inverse of this energy difference can be identified as the string coupling constant, g_{str}[9].

We have developed a Dirac fermion formalism for the states near the Fermi surface which works well for the leading contributions and lends valuable insight into many results.

To obtain the equations of motion we write this theory as a second quantized theory with the

action

$$S = \int dt\, d\lambda\, \chi^\dagger(\lambda,t)(i\partial_t + \frac{1}{2N}\partial_\lambda^2 - NV(\lambda))\chi(\lambda,t) + \int dt\, d\lambda\, Na_0(\lambda,t)\chi^\dagger(\lambda,t)\chi(\lambda,t) \quad (9)$$

where $\chi(\lambda,t)$ is a second-quantized fermion field in two dimensions and $a_0(\lambda,t)$ is the source function conjugate to density. It corresponds a coupling of the form $\sum_n tr\ M^n(t)a_n(t)$ in the original matrix model, where $a_n(t) = \int d\lambda\ \lambda^n\ a_0^n(\lambda,t)$. The corresponding vacuum to vacuum amplitude $Z[a_0]$ contains the information of all correlation function of density. Let

$$F[a_0] = \ln Z[a_0] \quad (10)$$

By taking the Legendre transformation of $F[a_0]$ we obtain the effective action $\Gamma[\rho]$ where

$$\rho(\lambda,t) = \frac{\delta F[a_0]}{\delta a_0(\lambda,t)} \quad (11)$$

and

$$\Gamma[\rho] = \int d\lambda\ dt\ \rho(\lambda,t)a_0(\lambda,t) - F[a_0] \quad (12)$$

When $a_0 = 0$ we have

$$\frac{\delta\Gamma[\rho]}{\delta\rho(\lambda,t)} = 0 \quad (13)$$

which is the quantum equation of motion.

Since we are looking at an effective bosonic theory we define the field variable that is used for bosonizing relativistic fermion theories,

$$\phi(\lambda,t) = \int^\lambda (\rho(\lambda,t) - <\rho(\lambda,t)>) \quad (14)$$

We will find, perturbatively, the leading order terms in the equation of motion in terms of this variable.

Dirac fermion representation

In the following we briefly indicate how this result was obtained using a Dirac fermion representation. This is essentially by a change of variables from $\lambda \to \tau = \int^\lambda \rho_o(\lambda')d\lambda'$, and scaling the hamiltonian by the fermi level wave function,

$$\hat{H} = \left(e^{\mp iN\Theta_f}\rho_f^{-1/2}\right)H\left(\rho_f^{1/2}e^{\pm iN\Theta_f}\right) = \mp i\frac{d}{d\tau} - \frac{1}{2N}\frac{d}{d\tau}\rho_o^2\frac{d}{d\tau}. \quad (15)$$

where $H(\rho_f^{1/2}e^{\pm iN\Theta_f}) = E_f(\rho_f^{1/2}e^{\pm iN\Theta_f})$.

To get the scales right, let us make some estimates. The leading order large N solution of the equation

$$H\phi = E_f\phi \quad (16)$$

is

$$\phi = \text{constant} \times \frac{1}{\sqrt[4]{2(\epsilon_f - V(\lambda))}} \times e^{\pm iN\int^\lambda d\lambda'\sqrt{2(\epsilon_f - V(\lambda'))}} \quad (17)$$

where

$$\epsilon_f = \frac{E_f}{N} \quad (18)$$

If we choose the constant to be 1, we have

$$\rho_f(\lambda) = \frac{1}{\sqrt{2(\epsilon_f - V(\lambda))}}$$
$$\Theta_f(\lambda) = \int^\lambda d\lambda'\sqrt{2(\epsilon_0 - V(\lambda))} \quad (19)$$

and

$$\rho_f\Theta_f' = 1 \quad (20)$$

Let the potential have a maximum at λ_0 with $V''(\lambda_0) \neq 0$. If we take a solution for ϵ_0 very near $V(\lambda_0)$ then most of the probability is concentrated near that tip. Classically this is manifested by the particle spending a lot of time near the turning point, which is very close to the rather flat region around the potential maximum.

In this region $V(\lambda) \approx V(\lambda_0) - \frac{1}{2}|V''(\lambda_0)|(\lambda - \lambda_0)^2$

$$\tau - a \sim \int_{\lambda_1}^{\lambda} d\lambda' \frac{1}{\sqrt{|V''(\lambda_0)|(\lambda' - \lambda_0)^2 - 2(V(\lambda_0) - \epsilon_0)}} \sim \frac{1}{(|V''(\lambda_0)|)^{1/2}} \int_{\lambda_0+\sqrt{2\mu}}^{\lambda} d\lambda' \frac{1}{\left((\lambda' - \lambda_0)^2 - 2\mu\right)^{1/2}} \quad (21)$$

By convention we make $V''(\lambda_0) = -1$ and define μ to be $V(\lambda_0) - \epsilon_0$.

Upon integration we get

$$\tau - a = \cosh^{-1}\left(\frac{\lambda - \lambda_0}{\sqrt{2\mu}}\right) \tag{22}$$

or

$$\lambda = \lambda_0 + \sqrt{2\mu}\cosh(\tau - a) \tag{23}$$

where a is the value of τ at the turning point.

Now

$$\rho_f^2 \sim \frac{1}{4\mu \sinh^2(\tau - a)} \tag{24}$$

$$\hat{H} \sim \mp i\frac{\partial}{\partial\tau} - \frac{1}{8N\mu}\frac{\partial}{\partial\tau}\frac{1}{\sinh^2(\tau - a)}\frac{\partial}{\partial\tau} \tag{25}$$

This estimate can be trusted, when τ is not too near a.

To recover an approximate relativistic fermion picture from a nonrelativistic one, the most natural thing to do is to take the reference energy level E_0 to be the Fermi level E_f. If we now want the expression of $\hat{H}$ in terms of τ to be a scaled expression, that is, if we want to keep $\tau - a$ as a scaled variable, we have to have $N\mu$ = fixed. (This is true irrespective of the semiclassical approximation that we made to reach this expression of $\hat{H}$.)

Strictly speaking, for this problem, the wave functions are not exactly like $\rho^{1/2}e^{+iN\Theta}$ and $\rho^{1/2}e^{-iN\Theta}$, but a specific linear combination which depends upon the energy and the the boundary conditions. In terms of τ variables, $\rho^{1/2}e^{\pm iN\Theta}$, after the relevant transformation looks like a plane wave in the leading order. The extent of classically allowed $\lambda - \lambda_0$ is roughly from 0 to say 1. Corresponding range of $\tau - a$ is from 0 to $\ln\frac{1}{\sqrt{\mu}}$. The level spacing goes as inverse of this range. Hence the boundary condition can give rise to mixing of left moving and right moving plane waves which can change the energy atmost by $\frac{1}{\ln\frac{1}{\sqrt{\mu}}}$. This vanishes in the scaling limit.

Thus we are allowed, in the scaling limit to deal with chiral states which are almost exact eigenstates. The hamiltonian which makes the right moving states near the Fermi surface look like plane waves is

$$\hat{H}_R = -i\partial_\tau - \frac{1}{2N}\partial_\tau\rho_f^2\partial_\tau \tag{26}$$

The hamiltonian which does the same for the left moving states is

$$\hat{H}_L = i\partial_\tau - \frac{1}{2N}\partial_\tau\rho_f^2\partial_\tau \tag{27}$$

Both the hamiltonians have the information about all the states. However, for the left moving states the second term in $\hat{H}_R$ cannot be considered as a small perturbation. Similar problem arise for right moving states and $\hat{H}_L$.

Thus, for the calculations where only states near the Fermi surface matter, one can describe the left moving states by $\hat{H}_L$ and the right moving by $\hat{H}_R$. This gives a Dirac hamiltonian. In the second-quantized notation the hamiltonian is

$$H = \int d\tau\left(\psi_+^\dagger\hat{H}_L\psi_+ + \psi_-^\dagger\hat{H}_R\psi_-\right) \tag{28}$$

To be honest one should discard half the solutions of each of the hamiltonians, not to over count the states. This would be some ultraviolet cut off in the theory. This cutoff would refer to the value of the momenta where the second term starts dominating over the first. For calculations involving processes near the Fermi surface, this cutoff is not important.

In many of the leading order calculations, this problem does not show up. This effective ultraviolet cutoff parameter, in a certain region of τ, is finite in the scaled picture (as opposed to the semiclassical case). Hence one has to be careful about it.

We now briefly indicate the results of the calculation of the 2 and 3 point functions of the density which corroborate our guess of the effective action to leading order.

The two-point function of density fluctuations

$$G^{(2)}(1,2) = <0|T\rho(\lambda_1,t_1)\rho(\lambda_2,t_2)|0>_c \tag{29}$$

$\rho(\lambda,t)$ is the eigenvalue/fermion density. If we look only at the connected part, we would see the correlation of density fluctuation, $\rho(\lambda,t) - <\rho(\lambda,t)>$, only. This density fluctuation can be represented also by $\chi^\dagger\chi$ normal ordered with respect to the Fermi sea.

If we change over to τ variables we have

$$\begin{aligned}&\tilde{G}^{(2)}(1,2)\\&=< 0|T\tilde{\rho}(\tau_1,t_1)\tilde{\rho}(\tau_2,t_2)|0>_c\\&=\frac{d\lambda_1}{d\tau_1}\frac{d\lambda_2}{d\tau_2}< 0|T\rho(\lambda_1,t_1)\rho(\lambda_2,t_2)|0>\end{aligned} \tag{30}$$

We call $\tilde{\chi}_L = \psi_+$ and $\tilde{\chi}_R = \psi_-$

$$:\chi^\dagger\chi:\rightarrow:\psi_+^\dagger\psi_+:+:\psi_-^\dagger\psi_-: \tag{31}$$

$$\begin{aligned}&\tilde{G}^{(2)}(1,2)\\&=< 0|T:\psi_+^\dagger(1)\psi_+(1)::\psi_+^\dagger(2)\psi_+(2):|0>\\&+(+\rightarrow -)\end{aligned} \tag{32}$$

Take $t_1 > t_2$ and consider

$$\begin{aligned}&< 0|:\psi_-^\dagger(1)\psi_-(1)::\psi_-^\dagger(2)\psi_-(2)::|0>\\&=S_p^{(-)}(1,2)S_h^{(-)}(1,2)\end{aligned} \tag{33}$$

In the leading order the particle and hole propagators are identical and charge conjugation symmetry is explicit,

$$S_p^{(-)} = S_h^{(-)} = \frac{\omega}{2\pi}\sum_{n=0}^{\infty}e^{-i(n+\frac{1}{2})\omega t_{12}^-} \tag{34}$$

Using these formulae the 2-point function is calculated to be

$$\begin{aligned}&\tilde{G}^{(2)}(1,2) =\\&\frac{\partial}{\partial\tau_1}\frac{\partial}{\partial\tau_2}\Big[\sum_{j=-\infty}^{\infty}\frac{i\omega}{4\pi^3}\int_{-\infty}^{\infty}dE\frac{e^{-i(Et_{12}-j\omega\tau_{12})}}{E^2-(j\omega)^2+i\epsilon}\Big]\end{aligned} \tag{35}$$

The expression inside the square bracket is the correlator of a free bose field. This is not surprising since what we have done is to bosonize the noninteracting fermions in a finite volume. We identify the free Bose field through the well known relation

$$:\psi^\dagger\psi:=\partial_\tau\phi \tag{36}$$

One can then see equation (37) coming out immediately from the Bose field correlator.

The three-point function of density fluctuations

For fermions satisfying the Dirac equation, the three point function of density is zero. This is a consequence of the charge conjugation symmetry of the Dirac hamiltonian. In other words, it is a consequence of the symmetry of the problem under reflection about the Fermi level. However, we know that this symmetry is broken in the nonrelativistic model and this is caused by the second term in the hamiltonian. This term, treated as a perturbation, should provide systematic order by order contributions to the three point function.

The lowest order contribution to the three-point function

$$\begin{aligned}\tilde{G}^{(3)}(1,2,3) =&< 0|:\psi_+^\dagger(1)\psi_+(1)::\psi_+^\dagger(2)\psi_+(2):\\&:\psi_+^\dagger(3)\psi_+(3):|0>+(+\rightarrow -)\end{aligned} \tag{37}$$

$(t_1 > t_2 > t_3)$ turns out to be the following lengthy expression after a long calculation.

$$\begin{aligned}\tilde{G}^{(3)}(1,2,3) =& \sum_{j_1,j_2=1}^{\infty}\frac{\omega^4}{8\pi^3}e^{-ij_1\omega t_1^- - i(j_2-j_1)t_2^- + ij_2t_3^-}\\&\Big[\omega^{-1}\frac{\partial\omega}{\partial\epsilon}\Big\{j_2(j_1-j_2)\Theta(j_1-j_2)(1-ij_1\omega t_1^-)\\&+j_2j_1(1-i(j_2-j_1)\omega t_2^-)\\&+j_1(j_2-j_1)\Theta(j_2-j_1)(1+ij_2\omega t_3^-)\Big\}\\&-j_2(j_1-j_2)\Theta(j_1-j_2)\Big(\frac{1}{2(\epsilon-V(1))}+\\&ij_1\omega\int^{\tau_1}\frac{d\tau}{2(\epsilon-V)}\Big)\\&-j_2j_1\Big(\frac{1}{2(\epsilon-V(2))}\\&+i(j_2-j_1)\omega\int^{\tau_2}\frac{d\tau}{2(\epsilon-V)}\Big)\\&-j_1(j_2-j_1)\Theta(j_2-j_1)\Big(\frac{1}{2(\epsilon-V(3))}-\\&ij_2\omega\int^{\tau_3}\frac{d\tau}{2(\epsilon-V)}\Big)\Big]\\&+\Big(t^-\rightarrow t^+\text{ and }\int^{\tau}\frac{d\tau'}{2(\epsilon-V)}\rightarrow -\\&\int^{\tau}\frac{d\tau'}{2(\epsilon-V)}\Big)\end{aligned} \tag{38}$$

$\Theta(x)$ being the Heaviside function.

The structure of the effective action

We want to keep only terms upto the order

of $\frac{1}{N}$ in the equation of motion. It can be easily seen from N counting that if we normalize the two-point connected Green's function to be order 1, then the order of the n-point connected function is N^{2-n}. Hence we need to consider only the two-point and the three-point function. The leading contribution to the three-point function is of the order $1/N$. The $1/N$ contribution to the two-point function cancells off. This is because the two-point function of the density is $S_h(1,2)S_p(1,2)$ which is $S(1,2)^2$ in the lowest order. The next order is $\delta(S_h(1,2)S_p(1,2))$. Since, in the lowest order $\Delta S_h = -\Delta S_p$, the first correction to the two-point funcion is zero. The correction is therefore $\sim O(\frac{1}{N^2})$.

Hence, from what we have done till now, we can reconstruct in the lowest order quadratic and cubic pieces of the effective action. The quadratic piece is going to be that of a free boson field which is $2\pi \int dtd\tau \partial_+\phi\partial_-\phi$. We need to choose a three-vertex which gives the correct three point function. This three point function has two pieces. One is proportional to $\omega^{-1}\frac{\partial\omega}{\partial\epsilon}$, the other is not. This first term is the dominant one for fixed λ_i, if we calculate $< \prod_i \rho(\lambda_i, t_i) >$. However if we change over to scaled variable like $\tau - a$ then since

$$\omega \sim |\ln \Delta\epsilon|^{-1}, \quad \Delta\epsilon = V(\lambda_0) - \epsilon$$
$$\frac{1}{N\omega}\frac{\partial\omega}{\partial\epsilon} \sim \frac{1}{|\ln\Delta\epsilon|}\frac{1}{N\Delta\epsilon} \to 0 \tag{39}$$

if $N\Delta\epsilon$ is held fixed when $N \to \infty$. On the other hand quantities like

$$\frac{1}{N}\frac{1}{2(\epsilon - V)} \sim \frac{1}{N\Delta\epsilon}\frac{1}{\sinh^2(\tau - a)} \tag{40}$$

remain finite. Hence we pay less attention to the piece proportional to $\frac{\partial\omega}{\partial\epsilon}$. The other piece is a sum of two chiral contributions. This indicates that the vertex is made of $\partial_+\phi$ and $\partial_-\phi$. $\partial_\pm = \frac{\partial}{\partial t^\pm}$. In fact one can show that the required interaction piece of the effective action is of the form

$$\Gamma_{\rm int} = \frac{-2\pi^2}{3N}\int dtd\tau\rho_f^2(\tau)\{(\partial_+\phi)^3 - (\partial_-\phi)^3\}. \tag{41}$$

It is remarkable that some very similar action can be obtained if one tries to bosonize the fermion theory naively by using Mandelstam formulae[12],

$$\psi_\pm^\dagger(\tau_1)\psi_\pm(\tau_2) = \mp\frac{i}{2\pi(\tau_1 - \tau_2)}$$
$$\exp\Big(-\pi i\int_{\tau_1}^{\tau_2} d\tau(\dot\phi \pm \phi') + O(\tau_1 - \tau_2)^2\Big) \tag{42}$$

(Note that our normalization of ϕ is different from Mandelstam's.) Now, one can separately differentiate in τ_1 and τ_2 and then take the limit $\tau_1 \to \tau_2$ and use the result in equation ( ) to obtain the bosonic expression for the perturbation.

We know that Mandelstam formulae depend crucially on the short distance properties of the Green's function, which can be modified if the perturbation is singular. This is precisely the case here. Yet this procedure gives the same leading order effective action,except for a $\frac{1}{N}\int dtd\tau\rho_f^2\partial_\tau^3\phi$ term (which, if genuinely present, should shifts the background ϕ from zero to a value $\sim 0(\frac{1}{N})$ and in that process give $0(\frac{1}{N^2})$ correction to the two point function which no longer remain
translation-invariant). It is possible that there is a generalization of the Mandelstam formulea in our case, where terms more singular than $\frac{1}{\tau_1 - \tau_2}$ appear, but they are always multiplied by higher powers of $1/N$ (or $g_{\rm str}$).

The equation of motion in the lowest order looks like

$$\partial_+\partial_-\phi = \frac{\pi}{2N}\Big[\partial_+\{\rho_f^2(\partial_+\phi)^2\} - \partial_-\{\rho_f^2(\partial_-\phi)^2\}\Big] \tag{43}$$

since

$$\rho_f^2(\tau) \sim \frac{1}{4\mu\sinh^2(\tau - a)} \tag{44}$$

for large $\tau - a$, i.e. for points far away from the turning point,

$$\rho_f^2(\tau) \sim \frac{e^{-2(\tau-a)}}{\mu} \tag{45}$$

Then

$$\partial_+\partial_-\phi = \frac{\pi}{N\mu}e^{-2(\tau-a)}\Big[-(\partial_+\phi)^2 + (\partial_-\phi)^2 + \partial_+\phi\partial_+^2\phi - \partial_-\phi\partial_-^2\phi\Big] \tag{46}$$

This is very similar to the tachyon equation. Note, however, that the interaction terms consist solely

of derivatives of ϕ and not ϕ itself. Also it can be written entirely in terms of the currents $j_{\pm} = \partial_{\pm}\phi +$ higher order terms.

Note added

While this work was in progress we became aware of similar works by S.R. Das and A. Jevicki[13] and[14] J. Polchinski.

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