A COMPLEX GIBBS-HEAVISIDE VECTOR ALGEBRA FOR SPACETIME

By G. Sobczyk

Institute for Theoretical Physics, University of Wrocław*

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The Gibbs-Heaviside vector algebra is widely used in problems pertaining to three dimensional Euclidean space. In this paper we introduce a remarkably similar complex three dimensional vector algebra for use in four dimensional spacetime. A complex vector has the geometric interpretation of a bivector in spacetime.

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Introduction

It is widely believed that the Gibbs-Heaviside vector algebra is stricktly limited to three dimensions because only in three dimensions there is a unique vector (up to an orientation) which is perpendicular to the plane defined by two vectors. This belief is usually accompanied with lamentation, because it is generally recognized that greater conceptual clarity is possible when geometric ideas are expressed in a vector formalism than when expressed in the tensor/matrix/spinor formalisms that are usually resorted to when studying properties of spaces of dimension greater than three. In this paper we show that the above mentioned belief is unfounded by constructing a complex vector algebra for use in the Minkowskian spacetime. Whereas the present work only concerns spacetime, the appropriate generalization to n dimensions, based on Clifford algebra, has been developed in [1].

In Section 1, we set down the axioms for the complex three dimensional vector algebra. A complex vector will be interchangeably referred to as a spacetime bivector to emphasize its geometric interpretation as a bivector in the abstrat Dirac-Clifford algebra of spacetime

* Address: Instytut Fizyki Teoretycznej, Uniwersytet Wrocławski, Cybulskiego 36, 50-205 Wrocław, Poland.

[2; p. 20]. The linear space structure of the algebra is best described as a formal sum of the space of complex numbers, and a three dimensional complex vector space. The complex vector algebra is algebraically isomorphic to both the algebra of Pauli matrices, and the complex quaternions [3; p. 186].

In Section 2, we derive basic identities which give the complex vector algebra the appearance of being a complexified version of the Gibbs-Heaviside vector algebra of Euclidean space. However, the *geometric product* of complex vectors, which unites the symmetric complex-valued inner product and the anti-symmetric vector-valued cross product into a single sum, has no parallel in the ordinary Gibbs-Heaviside vector algebra, and it is this feature which gives the complex vector algebra its manifestly rich structure. Canonical forms for complex vectors are derived and it is shown that, with respect to the complex symmetric inner product, a complex vector is either null, or can be uniquely factored into a positive magnitude, a complex phase, and a unit direction in spacetime.

In Section 3, the product of complex vectors is expressed in exponential form, and a generalized Euler formula involving a complex angle between unit directions in spacetime is derived. Complex angles have a geometric interpretation which, for real angles, reduces down to the well-known representation on the unit circle; this representation is illustrated in a figure.

In Section 4, we introduce the operation of complex conjugation with respect to a given inertial frame. The *space vectors* of this frame will appear in the inertial frame of a stationary observer, to be *real* vectors (with respect to the conjugation). On the other hand, if the observer has a relative velocity, then the space vectors of the given inertial frame will appear in the frame of the observer to be complex vectors with both real and imaginary vector parts. Imaginary vectors of an inertial frame have the geometric interpretation of *space bivectors*. Lorentz rotations between different inertial frames are defined and characterized in terms of the exponential mapping of a spacetime bivector.

The complex vector algebra set down in this paper has already demonstrated its merit by greatly simplifying the classification of the Riemann curvature tensor and the so-called Petrov classification, and by revealing the hidden structure of a linear operator, found in its characteristic equation when eigenvalues are given a geometric interpretation [4], [5]. Here-to-fore, spinor methods were considered to be the most powerful [6, p. 1165]. It is anticipated that the complex vector algebra can be fruitfully applied to any problem which is today expressed in the spinor formalism. Future papers demonstrating the power of this new formalism are planned along two lines. In a paper entitled "Geometric Extension of Linear Operators", it will be shown that complex eigenvalues can always be given a direct geometric interpretation when the original operator is extended in the proper way. A second paper entitled: "Geometry of Null Bivectors", will make clear the relationship of the complex vector algebra to the transformation groups SL(2, C), SU(3), and SU(2, 2), groups of recognized importance in physics [7]. Finally, we note that a powerful complex vector analysis of spacetime can be developed by suitably combining the pertinent ideas and methods from both the Gibbs-Heaviside vector analysis, and the rich complex analysis of the complex number plane [8], [9].

1. *Axioms of the algebra*

Let $\mathscr{C}$ be the complex number field. We adopt the somewhat unusual convention of denoting the imaginary unit by a capital I, because in addition to its usual algebraic property $I^2 = -1$, I is to have the geometric interpretation of being the *unit pseudoscalar* of the associated abstract Dirac-Clifford algebra, as will be explained in Section 4. Thus, a complex number $z = x+yI \in \mathscr{C}$, consists of a *scalar part* x, and a *pseudoscalar part* yI, where x and y are real numbers. We will also need a complex three dimensional vector space $\mathscr{B}$ over $\mathscr{C}$. Complex vectors $B \in \mathscr{B}$ will also be referred to as *spacetime* (*s.t.*) *bivectors* to emphasize their geometric interpretation as bivectors in the associated abstract Dirac--Clifford algebra. If $\{E_k | k = 1, 2, 3\}$ is a linearly independent set of complex vectors in $\mathscr{B}$, then an arbitrary complex vector $B \in \mathscr{B}$ can be uniquely expressed in the form $B = \beta^k E_k$, where summation over the indices $k = 1, 2, 3$ is assumed and where the complex scalars $\beta^k \in \mathscr{C}$.

Now let $\mathscr{P} = \mathscr{C} \oplus \mathscr{B}$ be the formal sum of the elements of $\mathscr{C}$ and $\mathscr{B}$. An arbitrary element $P \in \mathscr{P}$ has the form $P = z+B$. For $P_1 = z_1+B_1$, $P_2 = z_2+B_2 \in \mathscr{P}$ and $\alpha, \beta \in \mathscr{C}$, we assume the following properties

Axiom 1a. $\alpha P = \alpha z+\alpha B = P\alpha$, and $(\alpha\beta)P = \alpha(\beta P)$.

Axiom 1b. $P_1+P_2 = (z_1+z_2)+(B_1+B_2)$.

Axiom 1c. $(\alpha+\beta)P = \alpha P+\beta P$.

Axiom 1d. $\alpha(P_1+P_2) = \alpha P_1+\alpha P_2$.

We give $\mathscr{P}$ the structure of an associative algebra by defining the geometric product: $\mathscr{B} \otimes \mathscr{B} \to \mathscr{P}$. Let $A, B, C \in \mathscr{B}$, then the geometric product $AB \in \mathscr{P}$ is characterized by the following properties

Axiom 2a. $AB \in \mathscr{P}$ and $AB \equiv \frac{1}{2}(AB+BA)+\frac{1}{2}(AB-BA)$, where (*i*) $A \bigcirc B = \frac{1}{2}(AB+BA) \in \mathscr{C}$, and (*ii*) $A \times B = \frac{1}{2}(AB-BA) \in \mathscr{B}$.

Axiom 2b. $A(B+C) = AB+AC$ and $(B+C)A = BA+CA$.

Axiom 2c. $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

Axiom 2d. $A(BC) = (AB)C$.

The geometric product just defined on $\mathscr{B} \otimes \mathscr{B}$ can be extended in a natural way to $\mathscr{P} \otimes \mathscr{P}$ by assuming that

Axiom 3. $P_1 P_2 = z_1 z_2+z_2 B_1+z_1 B_2+B_1 B_2$.

Finally, we assume an axiom which guarantees non-degeneracy:

Axiom 4. There exist s.t. bivectors $A, B, C \in \mathscr{B}$ such that $ABC = I$. We sometimes refer to $\mathscr{P}$ as the abstract Pauli algebra because it is algebraically isomorphic to the algebra of Pauli matrices; $\mathscr{P}$ is also isomorphic to the algebra of complex quaternions, see [3, p. 186]. The remaining sections of this paper are devoted to revealing the geometric structure of $\mathscr{P}$ as a consequence of the above axioms.

2. *Algebraic identities*

Recalling axiom 2a, the geometric product of s.t. bivectors A, B is the sum of a symmetric, complex-valued inner product $A \bigcirc B$, and an anti-symmetric, bivector-valued cross product $A \times B$. In symbols,

$$AB = A \bigcirc B + A \times B, \tag{1}$$

where

$$A \bigcirc B = B \bigcirc A \quad \text{and} \quad A \times B = -B \times A. \tag{2}$$

An immediate consequence of the linearity and distributivity of the geometric product AB, as given in axioms 1d, 2abc is the linearity and distributivity of the inner and cross products $A \bigcirc B$ and $A \times B$. We state this important result in the following

Theorem 1. (*i*) $\alpha(A \bigcirc B) = (\alpha A) \bigcirc B = A \bigcirc (\alpha B)$. (*ii*) $\alpha(A \times B) = (\alpha A) \times B = A \times (\alpha B)$. (*iii*) $A \bigcirc (B+C) = A \bigcirc B + A \bigcirc C$. (*iv*) $A \times (B+C) = A \times B + A \times C$.

Since, by axiom 2a, $A \bigcirc B \in \mathscr{C}$, we can make the following

DEFINITION 1. $A \bigcirc B = A \cdot B + A \wedge B$, where $A \cdot B \equiv \langle A \bigcirc B \rangle_{\text{scalar part}}$, and $A \wedge B \equiv \langle A \bigcirc B \rangle_{\text{pseudoscalar part}}$.

In definition 1, we have introduced unorthodox symbolism for the real and imaginary parts of the complex number $A \bigcirc B$ because this symbolism suggests the geometric interpretation of the inner and outer products of bivectors. We will have more to say about this geometric interpretation in Section 4. We could have also defined $A \cdot B \equiv \frac{1}{2}(A \bigcirc B + \overline{A \bigcirc B})$ and $A \wedge B \equiv \frac{1}{2}(A \bigcirc B - \overline{A \bigcirc B})$, where $\overline{A \bigcirc B}$ is the complex conjugate of $A \bigcirc B$.

We will now prove a theorem expressing the *duality* of the scalar and pseudoscalar parts of $A \bigcirc B$.

Corollary 1. (*i*) $(IA) \cdot B = I(A \wedge B)$. (*ii*) $(IA) \wedge B = I(A \cdot B)$.

Proof: First note the identities:

$$I(A \bigcirc B) = I(A \cdot B) + I(A \wedge B) \quad \text{and} \quad (IA) \bigcirc B = (IA) \cdot B + (IA) \wedge B,$$

which are obvious consequences of definition 1 and the distributivity of complex multiplication. By using part (*i*) of theorem 1, we see that the left hand sides of these identities are equal, and therefore we can equate the scalar and pseudoscalar parts on the right hand sides, which gives (*i*) and (*ii*).

Corollary 2. $A \bigcirc B = A \cdot B + I(-IA) \cdot B$.

Proof: By definition 1, $A \bigcirc B \equiv A \cdot B + A \wedge B$. Multiplying part (*i*) of corollary 1 by $-I$ gives $A \wedge B = -I(IA) \cdot B$, and so the corollary quickly follows.

Next, we wish to establish a number of identities which bear striking resemblance to their counterparts in the Gibbs-Heaviside vector algebra. First we establish the

Lemma 1. $A \bigcirc (B \times C) = \frac{1}{2}(CAB - BAC) = \frac{1}{2}(BCA - ACB)$.

Proof: To show the first equality, we write the steps

$$\begin{aligned} \tfrac{1}{2}(CAB - BAC) &= \tfrac{1}{2}C(A \bigcirc B + A \times B) - \tfrac{1}{2}(B \bigcirc A + B \times A)C \\ &= \tfrac{1}{2}(CA \times B + A \times BC) = C \bigcirc (A \times B) = (C \times A) \bigcirc B \end{aligned} \tag{3}$$

which can be justified by using axiom 2d, the identity (1), axioms 3, 1a, and 2a, respectively. In a similar way it can be established that

$$\tfrac{1}{2}(BCA-ACB) = (B\times C)\bigcirc A = B\bigcirc(C\times A). \tag{4}$$

Because of (2), the right hand sides of (3) and (4) are equal, and therefore, so are the left hand sides, and the proof is complete.

Theorem 2. (*i*) $A\bigcirc(B\times C) = (A\times B)\bigcirc C = B\bigcirc(C\times A)$. (*ii*) $A\times(B\times C) = (A\bigcirc B)C - (A\bigcirc C)B$. (*iii*) $A\times(B\times C) = (A\times B)\times C + B\times(A\times C)$.

Proof: (*i*) The proof of this part is contained in the proof of lemma 1. (*ii*) The proof of this part is contained in the steps

$$A(B\times C) = \tfrac{1}{2}(ABC-ACB) = \tfrac{1}{2}(AB+BA)C-\tfrac{1}{2}(AC+CA)B+\tfrac{1}{2}(CAB-BAC)$$

$$= A\bigcirc BC - A\bigcirc CB + A\bigcirc B\times C,$$

which can be justified by using axioms 2ab and lemma 1. The proof is completed by noting from (1) that

$$A(B\times C) = A\bigcirc(B\times C)+A\times(B\times C),$$

so that the term $A\bigcirc(B\times C)$ can be subtracted from both the left and right sides of the first and last equalities. (*iii*) This part is an expression of the Jacobi identity, and can be established directly by using the definition of $A\times B$.

We have refrained from introducing an orthonormal basis in $\mathscr{B}$ to emphasize the independence and simplicity of the proof of theorem 2, without any reference to a basis. In fact, the notion of linear independence, itself, can be characterized by the symmetric and cross products:

DEFINITION 2. (*i*) Two complex vectors A, B are said to be *linearly independent* iff $A\times B \neq 0$. (*ii*) Three complex vectors A, B, C are said to be *linearly independent* iff $A\bigcirc B\times C \neq 0$.

We will now show how the existence of an orthonormal frame of s.t. bivectors is a consequence of axiom 4.

Theorem 3. There exists an orthonormal basis $\{E_k|\, k = 1, 2, 3\}$ satisfying the properties: (*i*) $E_i\bigcirc E_j = \delta_{ij}$. (*ii*) $E_1\times E_2 = E_1E_2 = IE_3$, $E_2\times E_3 = E_2E_3 = IE_1$, $E_3\times E_1 = E_3E_1 = IE_2$. (*iii*) $E_1\bigcirc E_2\times E_3 = E_1E_2E_3 = I$.

Proof: By axiom 4, there exist complex vectors A, B, C such that $ABC = I$. Multiplying both sides of this relationship by A gives $A^2BC = IA$. If $A^2 = 0$, then $IA = 0$ so that $A = 0$ also. But this is impossible, for then we would have that $0 = ABC = I$. Similarly, it follows that $C^2 \neq 0$. To see that $B^2 \neq 0$, assume the contrary, and note that this implies

$$0 = A^2B^2C^2 = ICBA \Leftrightarrow CBA = 0.$$

But $CBA = 0$ implies that $C^2BA^2 = 0$ which implies that $B = 0$ contradicting that $ABC = I$. Hence we are free to define $E_1 = A(A^2)^{-1}$, $E_2 = B(B^2)^{-1}$, and $E_3 = C(C^2)^{-1}$. The proof is easily completed by checking that the E_k's satisfy the properties (*i*), (*ii*) and (*iii*).

In terms of the orthonormal basis $\{E_k\}$ defined in theorem 3, any complex vector $A \in \mathscr{B}$ can be expressed in the form

$$A = \alpha^k E_k = \alpha_k E^k \equiv \sum_{k=1}^{3} A \bigcirc E^k E_k,$$

where $\alpha^k = \alpha_k \equiv A \bigcirc E_k \in \mathscr{C}$, and $E_k = E^k$. Suppose now that complex vectors $B = \beta^k E_k$, and $C = \gamma^k E_k$ are also given. We have the following corollary to theorem 3:

Corollary. (*i*) $A \bigcirc B = \alpha^k \beta_k = \alpha_k \beta^k$.

(*ii*)
$$A \times B = \begin{vmatrix} E_1 & E_2 & E_3 \\ \alpha^1 & \alpha^2 & \alpha^3 \\ \beta^1 & \beta^2 & \beta^3 \end{vmatrix} = \begin{vmatrix} \alpha^2 & \alpha^3 \\ \beta^2 & \beta^3 \end{vmatrix} IE_1 + \begin{vmatrix} \alpha^3 & \alpha^1 \\ \beta^3 & \beta^1 \end{vmatrix} IE_2 + \begin{vmatrix} \alpha^1 & \alpha^2 \\ \beta^1 & \beta^2 \end{vmatrix} IE_3.$$

(*iii*)
$$A \times B \bigcirc C = \begin{vmatrix} \alpha^1 & \alpha^2 & \alpha^3 \\ \beta^1 & \beta^2 & \beta^3 \\ \gamma^2 & \gamma^2 & \gamma^3 \end{vmatrix} I.$$

Parts (*ii*) and (*iii*) of this corollary show the equivalence of definition 2 to the usual notions of linear independence. This corollary reveals the Gibbs–Heaviside like structure of the spacetime bivector algebra, but the greater generality of the latter should be apparent.

3. *Geometry of spacetime bivectors*

We begin this section by proving a theorem regarding the structure of a bivector in spacetime. First, we make the following

DEFINITION 3. A s.t. bivector $N \in \mathscr{B}$ is said to be a *null bivector* if $N^2 = 0$. Otherwise, a bivector $B \in \mathscr{B}$, with $B^2 \neq 0$, is said to be *non-null.*

Theorem 4. (*i*) A non-null s.t. bivector B can always be uniquely expressed in the canonical form $B = e^{\phi + I\theta}\hat{B}$ where $-\infty < \phi < \infty$, $0 \leqslant \theta < \pi$, and where $\hat{B}$ is a unit s.t. bivector with $\hat{B}^2 = 1$. (*ii*) A non-trivial null bivector N can be put into the canonical form $N = \alpha(1 + A_1)A_2$ where $\alpha \in \mathscr{C}$, and A_1 and A_2 are spacetime bivectors satisfying the conditions $A_1^2 = 1 = A_2^2$, and $A_1 \bigcirc A_2 = 0$.

Proof: (*i*) Since $B^2 \neq 0$ is a complex scalar, it can be uniquely expressed in the form $B^2 = e^{2(\phi + I\theta)}$ for real numbers ϕ and θ where $-\infty < \phi < \infty$ and $0 \leqslant \theta < \pi$. It then follows that $B = e^{\phi + I\theta}\hat{B}$, where $\hat{B} \equiv \pm B/(B^2)$ with the $\pm$ sign chosen appropriately.

(*ii*) Since $N \in \mathscr{B}$, it can be expressed in terms of the orthonormal basis $\{E_k\}$ in the form $N = \eta^k E_k = \eta_k E^k$ where $N^2 = \eta_1^2 + \eta_2^2 + \eta_3^2 = 0$. Without loss of generality, we can assume that $\eta_1 \neq 0$. Now define $E_1' = E_1$, $E_2' = (-I/\eta_1)(\eta_2 E_2 + \eta_3 E_3)$. The it is not hard to check that $N = \eta_1(E_1' + IE_2')$, $(E_1')^2 = (E_2')^2 = 1$, and $E_1' \bigcirc E_2' = 0$. By defining $E_3' = IE_2'E_1'$, N can be written in the form

$$N = \eta_1(E_1' + E_3'E_1') = \eta_1(1 + E_3')E_1'.$$

To complete the proof let $\alpha = \eta_1$, $A_1 = E_3'$, and $A_2 = E_1'$.

Now let $B = e^{\phi+I\theta}\hat{B}$ be the representation of a non-null complex vector, and $N = e^{\phi+I\theta}(1+A_1)A_2$ be the representation of a null complex vector as given in parts (*i*) and (*ii*) of theorem 4. It is natural to make the following

DEFINITION 4. (*i*) For the complex vector $B = e^{\phi}e^{I\theta}\hat{B}$, e^{ϕ} is said to be the *magnitude*, $e^{I\theta}$ is said to be the *phase*, and $\hat{B}$ is said to be the *direction* of the complex vector B. (*ii*) For the null complex vector $N = e^{\phi}e^{I\theta}(1+A_1)A_2$, e^{ϕ} is said to be the *magnitude*, $e^{I\theta}$ is said to be the *phase*, and $(1+A_1)A_2$ the *nullity* of N *with respect to* the orthonormal basis $A_1, A_2, A_3 = -IA_1A_2$.

Part (*i*) of definition 4 is the natural generalization of the concept of the magnitude and direction of a real vector. In the case of the null complex vector N, such concepts can only be defined relative to a given basis.

Consider now the identities:

$$(A+B)^2 = A^2+2A\bigcirc B+B^2 \tag{5}$$

and

$$A^2B^2 = (AB)(BA) = (A\bigcirc B+A\times B)(A\bigcirc B-A\times B) = (A\bigcirc B)^2-(A\times B)^2, \tag{6}$$

which are easily verified. We refer to (5) and (6) as the generalized law of cosines and sines, respectively, because of their obvious similarity to the laws by these names. For unit bivectors $\hat{A}$ and $\hat{B}$ we make the following

DEFINITION 5. (*i*) $\cos(\varphi) \equiv \hat{A}\bigcirc\hat{B}$. (*ii*) $I\sin(\varphi) \equiv \hat{C}\bigcirc(\hat{A}\times\hat{B})$ if $(\hat{A}\times\hat{B})^2 \neq 0$, where $\hat{C}$ is the unit direction of $\hat{A}\times\hat{B}$ as defined in part (*i*) of definition 4. If $(\hat{A}\times\hat{B})^2 = 0$, then $I\sin(\varphi) \equiv 0$.

By using part (*i*) of theorem 4, definition 5 can be extended to apply to arbitrary non-null complex vectors. For $A = e^{\alpha}\hat{A}$, and $B = e^{\beta}\hat{B}$, we have

Theorem 5. (*i*) $A\bigcirc B = e^{\alpha+\beta}\hat{A}\bigcirc\hat{B} = e^{\alpha+\beta}\cos(\varphi)$, and if $(A\times B)^2 \neq 0$, then $A\times B = e^{\alpha+\beta}\hat{A}\times\hat{B} = e^{\alpha+\beta}IC\sin(\varphi)$. (*ii*) If $(A\times B)^2 \neq 0$, then $AB = e^{\alpha+\beta}e^{I\hat{C}\varphi}$. (*iii*) If $(A\times B)^2 = 0$, then $AB = e^{\alpha+\beta}(1+N) = e^{\alpha+\beta}e^{N}$, where $N = \hat{A}\times\hat{B}$.

Proof: The proof of (*i*) is an immediate consequence of definition 5. The proof of (*ii*) follows from the steps

$$AB = A\bigcirc B+A\times B = e^{\alpha+\beta}[\cos(\varphi)+I\hat{C}\sin(\varphi)] = e^{\alpha+\beta}e^{I\hat{C}\varphi}.$$

The first step uses identity (1), the second step uses part (*i*) above, and the third step is just a statement of the Euler formula for the complex plane of the bivector $I\hat{C}$ since $(I\hat{C})^2 = -1$.

The proof of part (*iii*) follows from the steps

$$AB = e^{\alpha+\beta}[\hat{A}\bigcirc\hat{B}+\hat{A}\times\hat{B}] = e^{\alpha+\beta}[1+N] = e^{\alpha+\beta}e^{N},$$

where $N \equiv \hat{A}\times\hat{B}$. The first step uses the basic identity (1), the second step uses the generalized law of sines as expressed in (6), and the third step is a simple consequence of the fact that

$$e^{N} \equiv \sum_{k=0}^{\infty}(1/k!)N^k = 1+N,$$

since the powers of N vanish for $k \geqslant 2$.

Let us now consider a simple example: Let $A = E_1$, and $B = E_1 + N$ where $N = (1 + E_1)E_2$. It easily follows that $A^2 = 1 = B^2 = A \bigcirc B$, and $A \times B = N$. Therefore, from definition 5, the complex angle φ between the unit complex vectors A and B is zero, but, unlike the case for real vectors, $A \neq B$. This circumstance is possible for complex vectors because of the existence of null bivectors in the Lorentzian geometry of spacetime.

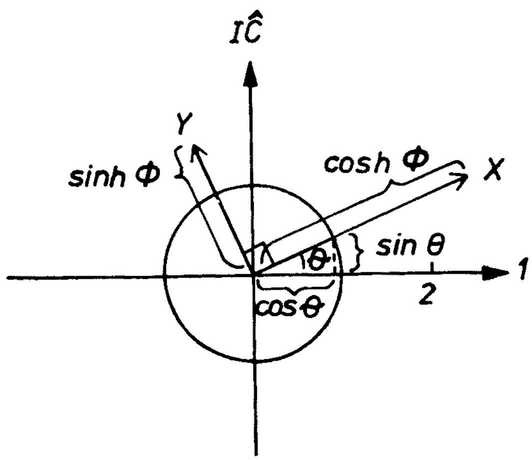


Fig. 1

We close this section by giving a geometric interpretation to the complex angle $\varphi = \theta + I\phi$ between two unit complex vectors $\hat{A}$ and $\hat{B}$, assuming that $(\hat{A} \times \hat{B})^2 \neq 0$. For this case, we know from part (*ii*) of theorem 5 that $\hat{A}\hat{B} = \cos\varphi + I\hat{C}\sin\varphi = e^{I\hat{C}\varphi}$. We decompose $e^{I\hat{C}\varphi}$ in the following steps:

$$e^{I\hat{C}\varphi} = e^{I\hat{C}(\theta + I\phi)} = e^{\theta I\hat{C}}e^{-\phi\hat{C}} = e^{\theta I\hat{C}}\cosh\phi + I\hat{C}e^{\theta I\hat{C}}I\sinh\phi$$

$$= e^{\theta I\hat{C}}\cosh\phi + e^{(\theta + \pi/2)I\hat{C}}I\sinh\phi = X + IY$$

where $X \equiv e^{\theta I\hat{C}}\cosh\phi$ and $Y \equiv e^{(\theta + \pi/2)I\hat{C}}\sinh\phi$. Thus, we have decomposed $e^{\varphi I\hat{C}}$ into the sum of two orthogonal complex vectors X and Y in the complex plane of $I\hat{C}$. Note also that $|X| = \cosh\phi$ and $|Y| = \sinh\phi$, so that $|X|^2 - |Y|^2 = \cosh^2\phi - \sinh^2\phi = 1$. Figure 1 below gives the pictorial description of these facts.

4. *Conjugations and Lorentz rotations in spacetime*

Let $\{E_k\}$ be an orthonormal frame of s.t. bivectors, and let $A = \alpha^k E_k$.

DEFINITION 6. The spacetime bivector $\bar{A} = \bar{\alpha}^k E_k$ is said to be the *complex conjugate of A w.r.t* $\{E_k\}$.

A conjugation in a complex linear space has the following properties:

Theorem 6. (*i*) $\overline{A+B} = \bar{A}+\bar{B}$. (*ii*) $\overline{\alpha A} = \bar{\alpha}\bar{A}$. (*iii*) $\bar{\bar{A}} = A$.

Proof: The proof is an immediate consequence of the definition.

Because $\mathscr{B}$ has the structure of an algebra, we have the following additional properties:

Theorem 7. (*i*) $\overline{AB} = \bar{B}\bar{A}$. (*ii*) $A\bigcirc\bar{A} \geqslant 0$ and $A\bigcirc\bar{A} = 0$ iff $A = 0$.

Proof: The proof of (*i*) follows from the steps

$$\overline{AB} = \overline{A\bigcirc B+A\times B} = \overline{A\bigcirc B}+\overline{A\times B} = \bar{A}\bigcirc\bar{B}-\bar{A}\times\bar{B}$$

$$= \bar{B}\bigcirc\bar{A}+\bar{B}\times\bar{A} = \bar{B}\bar{A}.$$

The first and last steps are applications of the basic identity (1), the second step assumes the natural extension of the conjugation to elements of $\mathscr{P}$, and the third step uses parts (*i*) and (*ii*) of the corollary to theorem 3. The proof of part (*ii*) is an immediate consequence of the positive definiteness of $A\bigcirc\bar{A} \equiv \alpha^k\bar{\alpha}_k$.

Corollary. The inner product defined by $(A, B) \equiv A\bigcirc\bar{B}$ gives $\mathscr{B}$ the structure of the unitary space U(3). See Ref. [10, p. 310]. Thus, to each orthonormal basis $\{E_k\}$ there is associated a unique unitary space with the inner product (A, B).

Now let $F\in\mathscr{B}$. The identity

$$F = \tfrac{1}{2}(F+\bar{F})+\tfrac{1}{2}(F-\bar{F}) = \tfrac{1}{2}(F+\bar{F})-\tfrac{1}{2}I(IF+\overline{IF}) = E+IB, \tag{7}$$

where $E = \langle F\rangle_{\text{real}} \equiv \frac{1}{2}(F+\bar{F})$ and $B = \langle F\rangle_{\text{imag.}} \equiv -\frac{1}{2}(IF+\overline{IF})$, decomposes an arbitrary complex vector into real and imaginary vector parts w.r.t the frame $\{E_k\}$. For a discussion of the bivector representation of an electromagnetic field, see [2, p. 29].

DEFINITION 7. (*i*) If $A = \bar{A}$, then A is said to be a *space vector of the orthonormal frame* $\{E_k\}$. (*ii*) If $A = -\bar{A}$, then A is said to be a *space bivector of the orthonormal frame* $\{E_k\}$.

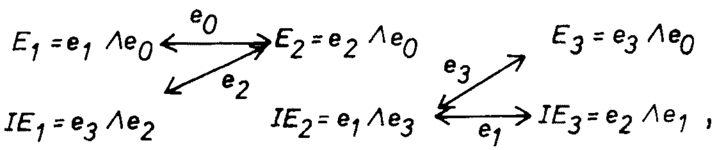
Thus, the *real vectors* are space vectors of the frame $\{E_k\}$, and *imaginary vectors* are space bivectors of the frame $\{E_k\}$. Note also the *duality* of the space vectors and bivectors, that is, if $\bar{A} = A$, then $\overline{IA} = -IA$, and conversely.

We can now identify the *Gibbs-Heaviside vectors* of an *inertial frame* as the real vectors of the orthonormal frame $\{E_k\}$. Let $A = \bar{A}$ and $B = \bar{B}$ be real space vectors of the frame $\{E_k\}$. We wish now to define the Gibbs-Heaviside "dot" and "cross" products of the space vectors A, B:

DEFINITION 8. (*i*) $A \overset{g}{\cdot} B \equiv \langle A\bigcirc B\rangle_{\text{real}} = A\bigcirc B$. (*ii*) $A \overset{g}{\times} B \equiv \langle A\times B\rangle_{\text{im}} = -IA\times B$. With this definition, all the usual properties of the Gibbs-Heaviside dot and cross products of space vectors follow as an immediate consequence of their complex counterparts given in the corollary to theorem 3. Thus, our complex vector algebra can be regarded as the appropriate "complexification" of the Gibbs-Heaviside vector algebra to apply in spacetime, where each inertial system is distinguished by its particular choice of an orthonormal frame of real vectors.

Alternatively, we can regard the elements of $\mathscr{B}$ to be the spacetime bivectors of an encompassing Clifford-Dirac algebra, [2, p. 24]. Then, to each orthonormal frame of s.t. bivectors $\{E_k\}$, there will correspond a unique orthonormal frame of Dirac vectors $\{e_u\}$

satisfying



(8)

where $I = E_1E_2E_3 = e_0e_1e_2e_3$. Thus, the $\{e_u\}$ for $u = 0, 1, 2, 3$ can be considered to be uniquely determined by the intersections of the spacetime planes of appropriately chosen pairs of s.t. bivectors, as indicated by the arrows in (8) above. The $\{e_u\}$ form an orthonormal basis of a Dirac algebra $\mathcal{D}$ with signature $e_0^2 = 1 = -e_1^2 = -e_2^2 = -e_3^2$.

We wish now to derive several consequences of the decomposition given in (7).

Theorem 8. (*i*) $F^2 = E^2 - B^2 + 2E \cdot BI$. (*ii*) If $F^2 \neq 0$ then F can be uniquely expressed in the form $F = e^{\omega}e^{\alpha A_1}A_2$, where ω is given by part (*i*) of theorem 4, α is a real scalar, and A_1 and A_2 are orthonormal space vectors w.r.t the inertial frame $\{E_k\}$. (*iii*) If $F^2 = 0$, then $F = e^{\phi}(1 + A_1)A_2$ for a unique real scalar ϕ, and A_1 and A_2 are unique orthonormal space vectors of the inertial frame $\{E_k\}$.

Proof: (*i*) By (7), $F = E + IB$. Squaring this equality gives $F^2 = E^2 + 2IE\bigcirc B - B^2$. Using the definition of E and B, we can further calculate

$$EB = -(I/4)(F+\bar{F})(F-\bar{F}) = -(I/4)(F^2 + 2\bar{F} \times F - \bar{F}^2).$$

Equating the scalar and bivector parts of this equation gives

$$E\bigcirc B = -(I/4)(F^2 - \bar{F}^2) = E \cdot B$$

and

$$E \times B = \tfrac{1}{2} IF \times \bar{F},$$

so the proof of this part is complete.

(*ii*) Using theorem 4(*i*), we can write $F = e^{\omega}\hat{F}$. Applying the decomposition (7) to $\hat{F}$, we find that

$$\hat{F} = \alpha_2 A_2 + \alpha_3 I A_3 = (\alpha_2 A_2 + \alpha_3 I A_3) A_2 A_2$$

$$= (\alpha_2 + \alpha_3 A_1) A_2 = e^{\alpha A_1} A_2,$$

where $\alpha_2 > 0$ and α, α_3 are real non-negative scalars such that

$$F^2 = \alpha_2^2 - \alpha_3^2 = 1,$$

and $A_1 \equiv IA_3A_2$, A_2, A_3 are real orthonormal vectors w.r.t $\{E_k\}$. The fact that A_2, A_3 are real and orthogonal follows from part (*i*). The proof of part (*ii*) is now easily completed.

(*iii*) The proof of this part is similar to part (*ii*) and is omitted.

Corollary. (*i*) $E\bigcirc B = -(I/4)(F^2 - \bar{F}^2) = E \cdot B$ and $E \times B = \frac{1}{2} IF \times \bar{F}$. (*ii*) $F^2 = F \cdot F = E^2 - B^2 \Leftrightarrow E \cdot B = 0$.

We close this section with a discussion of Lorentz rotations.

DEFINITION 9. A transformation $L:\mathscr{B}\to\mathscr{B}$ is said to be a Lorentz rotation iff $L(A)\bigcirc L(B) = A\bigcirc B$ for all $A, B\in\mathscr{B}$.

Equivalently, L is a Lorentz rotation if for each orthonormal frame $\{E_k\}$, the frame $\{E'_k\}$, defined by $E'_k \equiv L(E_k)$, is an orthonormal frame.

Theorem 9. $A' = L(A)$ is a Lorentz rotation if there exists a s.t. bivector $D\in\mathscr{B}$- such that $A' = L(A) = e^D A e^{-D}$.

Proof: First, suppose that $D\in\mathscr{B}$. Then

$$L(A)\bigcirc L(B) \equiv \langle e^D A e^{-D} e^D B e^{-D}\rangle_{\text{complex scalar part}} = \langle e^D A B e^{-D}\rangle_{\text{com. sc. pt.}}$$

Similarly,

$$L(B)\bigcirc L(A) = \langle e^D B A e^{-D}\rangle_{\text{complex scalar part}}.$$

Hence

$$L(A)\bigcirc L(B) = \tfrac{1}{2}\,[L(A)\bigcirc L(B)+L(B)\bigcirc L(A)] = \tfrac{1}{2}\,e^D(AB+BA)e^{-D} = A\bigcirc B.$$

To prove the "only if" part of the theorem, assume that a Lorentz rotation L is given. We shall show that a s.t. bivector C exists such that

$$A' = L(A) = e^C A e^{-C}$$

by directly solving this equation for the bivector C. First note the identity

$$e^C \equiv \sum_{n=0}^{\infty} (1/n!)C^n = \cosh(C)+\sinh(C),$$

where $\cosh(C)\in\mathscr{C}$, and $\sinh(C)\in\mathscr{B}$. We can now directly proceed with the proof by writing the steps

$$\partial L = \partial_A L(A) = \partial_A e^C A e^{-C} = [\cosh(C)\partial_A A + \partial_A \sinh(C) A]e^{-C}$$

$$= [3\cosh(C)-\sinh(C)]e^{-C} = 4\cosh(C)e^{-C}-1,$$

where we are incorporating the *normalized bivector derivative* $\partial = \partial_A$, and simple formulas for differentiation w.r.t the bivector variable A, first introduced and developed in [4] and [5]. Equating the complex scalar and bivector parts of the first and last expressions in the above steps gives

$$\partial\bigcirc L = 4\cosh^2(C)-1 \quad\text{and}\quad \partial\times L = 4\cosh(C)\sinh(C),$$

from which it follows that

$$e^C = \frac{1}{4\cosh(C)}\,[4\cosh^2(C)+4\cosh(C)\sinh(C)] = \frac{\pm 1}{2(1+\partial\bigcirc L)^{1/2}}\,(1+\partial L),$$

provided that $\cosh(C)\neq 0$. For the cases when $\cosh(C) = 0$,

$$\partial\bigcirc L = -1 \quad\text{and}\quad \partial\times L = 0,$$

so that we can appeal to the Petrov classification as worked out in [4]. We can conclude that L is type I, and since $L = L^{-1}$, the eigenvalues of L must be $+1, -1, -1$, which in turn implies that L can be put into the canonical form

$$L(A) = DAD,$$

where $D = A_1 + \alpha N$ and $D^2 = A_1^2 = 1$, $N^2 = 0$. Defining $C = (\pi/2)I(A_1 + \alpha N)$, it can be readily checked that

$$L(A) = e^C A e^{-C} = \dot{D}AD$$

as required, and the proof is complete.

The proof of this theorem corrects and generalizes to spacetime a proof which can be found in [11].

Corollary. L is a Lorentz rotation iff $L(A) \bigcirc B = A \bigcirc L^{-1}(B)$ for all $A, B \in \mathscr{B}$.

DEFINITION 10. By the *matrix* of L w.r.t the orthonormal inertial frame $\{E_k\}$, we mean the matrix whose elements are given by $L_{ij} = L(E_i) \bigcirc E_j$.

In terms of the orthonormal frame $\{E_k\}$, the *characteristic complex scalar* $\mathfrak{d} \bigcirc L$, and the *characteristic bivector* $\mathfrak{d} \times L$ of the transformation L can be evaluated by the formulas

$$\mathfrak{d} \bigcirc L \equiv E^k \bigcirc L(E_k) = \sum_{k=1}^{3} L_{kk}, \tag{9}$$

and

$$\mathfrak{d} \times L \equiv E^k \times L(E_k) = (L_{23} - L_{32})IE_1 + (L_{31} - L_{13})IE_2 + (L_{12} - L_{21})IE_3, \tag{10}$$

as can be verified with the help of the corollary to theorem 3, and properties of the bivector derivative given in [4].

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