



Off-shell massive $\mathcal{N} = 1$ supermultiplets in three dimensions

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Abstract

This paper is mainly concerned with the construction of new off-shell higher spin $\mathcal{N} = 1$ supermultiplets in three spacetime dimensions. We elaborate on the gauge prepotentials and linearised super-Cotton tensors for higher spin $\mathcal{N} = 1$ superconformal geometry and propose compensating superfields required to formulate off-shell massless higher spin supermultiplets. The corresponding gauge-invariant actions are worked out explicitly using an auxiliary oscillator realisation. We construct, for the first time, off-shell massive higher spin supermultiplets. The gauge-invariant actions for these supermultiplets are obtained by adding Chern–Simons like mass terms (that is, higher spin extensions of the linearised action for $\mathcal{N} = 1$ conformal supergravity) to the actions for the massless supermultiplets. For each of the massive gravitino and supergravity multiplets, we propose two dually equivalent formulations.

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1. Introduction

In three spacetime dimensions (3D), the off-shell structure of $\mathcal{N} = 1$ supergravity was understood in the late 1970s [1–3] and further elaborated in [4]. Since then, there have appeared a number of important developments in minimal 3D supergravity, including the $\mathcal{N} = 1$ topologically massive supergravity with and without a cosmological term [5,6], various approaches to

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$\mathcal{N} = 1$ conformal supergravity [7–12], 3D compactifications of \mathcal{M} -theory with minimal local supersymmetry (see [13] and references therein), and higher-derivative models for massive $\mathcal{N} = 1$ supergravity [14–17]. The latter locally supersymmetric theories, which generalise the models for massive gravity proposed in [18,19], possess remarkable properties such as unitarity in the presence of curvature squared terms. Since these massive theories are nonlinear in the curvature tensor, their explicit construction would be extremely difficult to achieve without making use of the off-shell multiplet calculus for $\mathcal{N} = 1$ supergravity.

The general massive gravity models of [18,19] and their supersymmetric cousins, including those proposed in [14–17], may possess higher spin generalisations, see e.g. [20]. Surprisingly, to the best of our knowledge, off-shell massive higher spin $\mathcal{N} = 1$ supermultiplets in three dimensions have never been constructed. The on-shell massive higher spin 3D $\mathcal{N} = 1$ supermultiplets have been formulated recently, both for the Minkowski and anti-de Sitter (AdS) backgrounds [21,22], building on the elegant gauge-invariant construction of massive higher spin fields in AdS [23]. However, since the massive higher spin supermultiplets of [21,22] lack auxiliary fields, it could be difficult to use this approach to generate consistent cubic and possible higher-order couplings (as it often happens in supersymmetric field theory). The aim of this paper is to construct, for the first time, off-shell massive higher spin $\mathcal{N} = 1$ supermultiplets.

Our paper is a continuation of the recent work [24] in which the off-shell massive higher spin $\mathcal{N} = 2$ supermultiplets were constructed in three dimensions. The structure of these 3D $\mathcal{N} = 2$ massive supermultiplets is similar to that of the off-shell 4D $\mathcal{N} = 1$ massless supermultiplets [25,26] (see [27] for a review) in the sense that there are two dually equivalent series of off-shell formulations. As will be shown below, the 3D $\mathcal{N} = 1$ case is more similar to the non-supersymmetric Fronsda actions [28,29], for there is essentially a single off-shell formulation for each massive higher spin supermultiplet (modulo auxiliary superfields). A remarkable feature of our massive $\mathcal{N} = 1$ supermultiplets is that they are formulated in terms of *unconstrained* superfields, unlike their $\mathcal{N} = 2$ counterparts [24]. This makes the off-shell higher spin $\mathcal{N} = 1$ supersymmetric theories more tractable than the $\mathcal{N} = 2$ ones.

This paper is organised as follows. In section 2 we define on-shell massive superfields and present a manifestly supersymmetric expression for the superhelicity operator. In section 3 we elaborate on the higher spin superconformal gauge multiplets and the corresponding gauge invariant field strengths. Section 4 describes the massless higher spin gauge prepotentials. The off-shell realisations for massless low spin supermultiplets are given in section 5. In section 6 we present the off-shell massless higher spin supermultiplets, and the massive case is presented in section 7. Concluding comments and open problems are discussed in section 8. The main body of the paper is accompanied by two technical appendices. Our 3D notation and conventions correspond to those introduced in [10,30].

2. Massive (super)fields

In this section we discuss on-shell (super)fields which realise the massive representations of the 3D Poincaré and $\mathcal{N} = 1$ super-Poincaré groups. The material in subsection 2.1 is taken almost verbatim from [24].

2.1. Massive fields

Let P_a and $J_{ab} = -J_{ba}$ be the generators of the 3D Poincaré group. The Pauli–Lubanski scalar

$$W := \frac{1}{2} \varepsilon^{abc} P_a J_{bc} = -\frac{1}{2} P^{\alpha\beta} J_{\alpha\beta} \quad (2.1)$$

commutes with the generators P_a and J_{ab} . Irreducible unitary representations of the Poincaré group are labelled by two parameters, mass $m > 0$ and helicity λ , which are associated with the Casimir operators,

$$P^a P_a = -m^2 \mathbb{1}, \quad W = m\lambda \mathbb{1}. \quad (2.2)$$

One defines $|\lambda|$ to be the spin.

In the case of field representations, it holds that

$$W = \frac{1}{2} \partial^{\alpha\beta} M_{\alpha\beta}, \quad (2.3)$$

where the action of the Lorentz generator with spinor indices, $M_{\alpha\beta} = M_{\beta\alpha}$, on a field $\phi_{\gamma_1 \dots \gamma_n} = \phi_{(\gamma_1 \dots \gamma_n)}$ is defined by

$$M_{\alpha\beta} \phi_{\gamma_1 \dots \gamma_n} = \sum_{i=1}^n \varepsilon_{\gamma_i}(\alpha \phi_{\beta})_{\gamma_1 \dots \widehat{\gamma_i} \dots \gamma_n}, \quad (2.4)$$

where the hatted index of $\phi_{\beta\gamma_1 \dots \widehat{\gamma_i} \dots \gamma_n}$ is omitted.

For $n > 1$, a massive field, $\phi_{\alpha_1 \dots \alpha_n} = \bar{\phi}_{\alpha_1 \dots \alpha_n} = \phi_{(\alpha_1 \dots \alpha_n)}$, is a real symmetric rank- n spinor field which obeys the differential conditions [31] (see also [32])

$$\partial^{\beta\gamma} \phi_{\beta\gamma\alpha_1 \dots \alpha_{n-2}} = 0, \quad (2.5a)$$

$$\partial^{\beta}_{(\alpha_1} \phi_{\alpha_2 \dots \alpha_n)\beta} = m\sigma \phi_{\alpha_1 \dots \alpha_n}, \quad \sigma = \pm 1. \quad (2.5b)$$

In the spinor case, $n = 1$, eq. (2.5a) is absent, and the massive field is defined to obey the Dirac equation (2.5b). It is easy to see that (2.5a) and (2.5b) imply the mass-shell equation¹

$$(\square - m^2) \phi_{\alpha_1 \dots \alpha_n} = 0, \quad (2.6)$$

which is the first equation in (2.2). In the spinor case, $n = 1$, eq. (2.6) follows from the Dirac equation (2.5b). The second relation in (2.2) also holds, with

$$\lambda = \frac{n}{2} \sigma. \quad (2.7)$$

The spin of $\phi_{\alpha(n)}$ is $n/2$.

2.2. Massive $\mathcal{N} = 1$ superfields

Let P_a , $J_{ab} = -J_{ba}$, Q_α be the generators of the 3D $\mathcal{N} = 1$ super-Poincaré group. The supersymmetric extension of the Pauli–Lubanski scalar (2.1) is the following operator [34]

$$Z = W - \frac{i}{8} Q^2 = \frac{1}{2} \varepsilon^{abc} P_a J_{bc} - \frac{i}{8} Q^\alpha Q_\alpha, \quad (2.8)$$

which commutes with the supercharges,

$$[Z, Q_\alpha] = 0. \quad (2.9)$$

¹ The equations (2.5a) and (2.6) prove to be equivalent to the 3D Fierz–Pauli field equations [33].

The operator Z is analogous to the 4D $\mathcal{N} = 1$ superhelicity operator introduced in [27]. Irreducible unitary representations of the $\mathcal{N} = 1$ super-Poincaré group are labelled by two parameters, mass m and superhelicity κ , which are associated with the Casimir operators,

$$P^a P_a = -m^2 \mathbb{1}, \quad Z = m\kappa \mathbb{1}. \quad (2.10)$$

Our definition of the superhelicity agrees with [34]. The massive representation of superhelicity κ is a direct sum of two massive representations of the Poincaré group with helicity values $(\kappa - \frac{1}{4}, \kappa + \frac{1}{4})$. If $\kappa - \frac{1}{4}$ is not an integer, the supermultiplet describes anyons. The case $2\kappa \in \mathbb{Z}$ corresponds to the so-called semion supermultiplets, of which the $\kappa = \frac{1}{2}$ supermultiplet was first studied in [35].

When dealing with the supermultiplets containing particles of (half-)integer helicity, it appears more convenient, by analogy with the $\mathcal{N} = 1$ case in four dimensions [27], to define a shifted superhelicity operator, $\hat{\kappa} = \kappa - \frac{1}{4}$, which takes integer or half-integer values. However, here we will use the definition introduced in [34].

In the case of superfield representations of the $\mathcal{N} = 1$ super-Poincaré group, the infinitesimal super-Poincaré transformation of a tensor superfield is

$$\delta\Phi = i(-b^a P_a + \frac{1}{2}\Lambda^{ab} J_{ab} + \epsilon^\alpha Q_\alpha)\Phi = i\left(\frac{1}{2}b^{\alpha\beta} P_{\alpha\beta} + \frac{1}{2}\Lambda^{\alpha\beta} J_{\alpha\beta} + \epsilon^\alpha Q_\alpha\right)\Phi, \quad (2.11)$$

where the generators of spacetime translations ($P_{\alpha\beta}$), Lorentz transformations ($J_{\alpha\beta}$) and supersymmetry transformations (Q_α) are

$$P_{\alpha\beta} = -i\partial_{\alpha\beta}, \quad \partial_{\alpha\beta} = (\gamma^m)_{\alpha\beta}\partial_m, \quad (2.12a)$$

$$J_{\alpha\beta} = i\theta_{(\alpha}\partial_{\beta)} - iM_{\alpha\beta}, \quad (2.12b)$$

$$Q_\alpha = \partial_\alpha + i\theta^\beta\partial_{\alpha\beta}, \quad \partial_\alpha = \frac{\partial}{\partial\theta^\alpha}. \quad (2.12c)$$

Using the explicit expressions for the super-Poincaré generators, the superhelicity operator (2.8) can be written in a manifestly supersymmetric form

$$Z = \frac{1}{2}\partial^{\alpha\beta}M_{\alpha\beta} - \frac{i}{8}D^2. \quad (2.13)$$

For $n > 0$, a massive superfield $T_{\alpha(n)}$ is defined to be a real symmetric rank- n spinor, $T_{\alpha_1\dots\alpha_n} = \bar{T}_{\alpha_1\dots\alpha_n} = T_{(\alpha_1\dots\alpha_n)}$, which obeys the differential conditions [17]

$$D^\beta T_{\beta\alpha_1\dots\alpha_{n-1}} = 0 \implies \partial^{\beta\gamma} T_{\beta\gamma\alpha_1\dots\alpha_{n-2}} = 0, \quad (2.14a)$$

$$-\frac{i}{2}D^2 T_{\alpha_1\dots\alpha_n} = m\sigma T_{\alpha_1\dots\alpha_n}, \quad \sigma = \pm 1. \quad (2.14b)$$

It follows from (2.14a) that

$$-\frac{i}{2}D^2 T_{\alpha_1\dots\alpha_n} = \partial^\beta{}_{(\alpha_1} T_{\alpha_2\dots\alpha_n)\beta}, \quad (2.15)$$

and thus $T_{\alpha(n)}$ is an on-shell superfield,

$$\partial^\beta{}_{(\alpha_1} T_{\alpha_2\dots\alpha_n)\beta} = m\sigma T_{\alpha_1\dots\alpha_n}, \quad \sigma = \pm 1. \quad (2.16)$$

Making use of the identity (A.5d), we also deduce directly from (2.14b) that²

² The equations (2.14a) and (2.17) provide the $\mathcal{N} = 1$ supersymmetric extensions of the 3D Fierz–Pauli equations.

$$(\square - m^2)T_{\alpha(n)} = 0. \quad (2.17)$$

For the superhelicity of $T_{\alpha(n)}$ we obtain

$$\kappa = \frac{1}{2} \left(n + \frac{1}{2} \right) \sigma. \quad (2.18)$$

We define the superspin of $T_{\alpha(n)}$ to be $n/2$. The massive supermultiplet $T_{\alpha(n)}$ contains two ordinary massive fields of the type (2.5), which are

$$\phi_{\alpha_1 \dots \alpha_n} := T_{\alpha_1 \dots \alpha_n}|_{\theta=0}, \quad \phi_{\alpha_1 \dots \alpha_{n+1}} := i^{n+1} D_{(\alpha_1} T_{\alpha_2 \dots \alpha_{n+1})}|_{\theta=0}. \quad (2.19)$$

Their helicity values are $\frac{n}{2}\sigma$ and $\frac{n+1}{2}\sigma$, respectively.

As an example, let us consider the following model for a massive scalar multiplet

$$S_{\text{SM}}[X] = -\frac{i}{2} \int d^3|z| D^\alpha X D_\alpha X + m\sigma \int d^3|z| X^2, \quad \sigma = \pm 1. \quad (2.20)$$

Throughout this paper, the $\mathcal{N} = 1$ superspace integration measure³ is defined as follows:

$$\int d^3|z| L = \frac{i}{4} \int d^3x D^2 L \Big|_{\theta=0}. \quad (2.21)$$

The equation of motion for the action (2.20) is

$$-\frac{i}{2} D^2 X = m\sigma X, \quad (2.22)$$

which shows that the superhelicity of X is $\kappa = \frac{1}{4}\sigma$.

3. $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superspace reduction: superconformal gauge multiplets

In general, off-shell $\mathcal{N} = 1$ higher spins supermultiplets in three dimensions may be obtained by applying the $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superspace reduction to the $\mathcal{N} = 2$ supermultiplets constructed in [24]. We denote by \mathbb{D}_α and $\tilde{\mathbb{D}}_\alpha$ the spinor covariant derivatives of the $\mathcal{N} = 2$ Minkowski superspace $\mathbb{M}^{3|4}$. They obey the anti-commutation relations

$$\{\mathbb{D}_\alpha, \tilde{\mathbb{D}}_\beta\} = -2i\delta_{\alpha\beta}, \quad \{\mathbb{D}_\alpha, \mathbb{D}_\beta\} = \{\tilde{\mathbb{D}}_\alpha, \tilde{\mathbb{D}}_\beta\} = 0. \quad (3.1)$$

In order to carry out the $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superspace reduction, it is useful to introduce real Grassmann coordinates θ_I^α for $\mathbb{M}^{3|4}$, where $I = \underline{1}, \underline{2}$. We define these coordinates by choosing the corresponding spinor covariant derivatives D_α^I as in [30]:

$$\mathbb{D}_\alpha = \frac{1}{\sqrt{2}}(D_\alpha^{\underline{1}} - iD_\alpha^{\underline{2}}), \quad \tilde{\mathbb{D}}_\alpha = -\frac{1}{\sqrt{2}}(D_\alpha^{\underline{1}} + iD_\alpha^{\underline{2}}). \quad (3.2)$$

From (3.1) we deduce

$$\{D_\alpha^I, D_\beta^J\} = 2i\delta^{IJ}(\gamma^m)_{\alpha\beta}\partial_m, \quad I, J = \underline{1}, \underline{2}. \quad (3.3)$$

Given an $\mathcal{N} = 2$ superfield $U(x, \theta_I)$, we define its $\mathcal{N} = 1$ bar-projection

$$U| := U(x, \theta_I)|_{\theta_{\underline{2}}=0}. \quad (3.4)$$

³ This definition implies that $\int d^3|z| V = \int d^3x F$, for any scalar superfield $V(x, \theta) = \dots + i\theta^2 F(x)$.

It is clear that $U|$ is a superfield on $\mathcal{N} = 1$ Minkowski superspace $\mathbb{M}^{3|2}$ parametrised by real Cartesian coordinates $z^A = (x^a, \theta^\alpha)$, where $\theta^\alpha := \theta_1^\alpha$. The covariant derivative of $\mathcal{N} = 1$ Minkowski superspace $D_\alpha := D_{\underline{\alpha}}^1$ obeys the anti-commutation relation

$$\{D_\alpha, D_\beta\} = 2i(\gamma^m)_{\alpha\beta} \partial_m. \quad (3.5)$$

3.1. Higher spin superconformal gauge multiplets

In accordance with [24], the higher spin $\mathcal{N} = 2$ superconformal gauge multiplet is described in terms of a real unconstrained prepotential

$$\mathbb{H}_{\alpha(n)} := \mathbb{H}_{\alpha_1 \dots \alpha_n} = \mathbb{H}_{(\alpha_1 \dots \alpha_n)} = \bar{\mathbb{H}}_{\alpha(n)}, \quad (3.6)$$

which is defined modulo gauge transformations of the form

$$\delta \mathbb{H}_{\alpha(n)} = g_{\alpha(n)} + \bar{g}_{\alpha(n)}, \quad (3.7a)$$

where the complex gauge parameter $g_{\alpha(n)} = g_{\alpha_1 \dots \alpha_n} = g_{(\alpha_1 \dots \alpha_n)}$ is a longitudinal linear superfield constrained by

$$\bar{\mathbb{D}}_{(\alpha_1} g_{\alpha_2 \dots \alpha_{n+1})} = 0 \quad \implies \quad \bar{\mathbb{D}}^2 g_{\alpha(n)} = 0. \quad (3.7b)$$

This constraint can always be solved in terms of a complex unconstrained potential $L_{\alpha(n-1)}$ by the rule

$$g_{\alpha_1 \dots \alpha_n} = \bar{\mathbb{D}}_{(\alpha_1} L_{\alpha_2 \dots \alpha_n)}. \quad (3.8)$$

However we will not use this representation in the present paper.

Making use of the representation (3.2), the longitudinal linear constraint (3.7b) takes the form

$$D^2_{(\alpha_1} g_{\alpha_2 \dots \alpha_{n+1})} = i D^1_{(\alpha_1} g_{\alpha_2 \dots \alpha_{n+1})}. \quad (3.9)$$

This tells us that, upon reduction to $\mathcal{N} = 1$ superspace, $g_{\alpha(n)}$ is equivalent to two complex unconstrained $\mathcal{N} = 1$ superfields, which are obtained by Taylor-expanding the $\mathcal{N} = 2$ superfield $g_{\alpha(n)}(\theta_I) = g_{\alpha(n)}(\theta_1, \theta_2)$ in powers of θ_2^α and which may be chosen as

$$g_{\alpha_1 \dots \alpha_n}|, \quad D^{\underline{2}\beta} g_{\alpha_1 \dots \alpha_{n-1}\beta}|. \quad (3.10)$$

Upon reduction to $\mathcal{N} = 1$ superspace, the gauge prepotential $\mathbb{H}_{\alpha(n)}$ is equivalent to four unconstrained superfields

$$\mathbb{H}_{\alpha_1 \dots \alpha_n}|, \quad D^{\underline{2}}_{(\alpha_1} \mathbb{H}_{\alpha_2 \dots \alpha_{n+1})}|, \quad D^{\underline{2}\beta} \mathbb{H}_{\alpha_1 \dots \alpha_{n-1}\beta}|, \quad \frac{i}{4} (D^{\underline{2}})^2 \mathbb{H}_{\alpha_1 \dots \alpha_n}|. \quad (3.11)$$

Here the first and the fourth superfields are real, while the other superfields are real or imaginary depending on n .

Since the $\mathcal{N} = 1$ gauge parameters (3.10) are complex unconstrained, it is in our power to choose the $\mathcal{N} = 1$ supersymmetric gauge conditions

$$\mathbb{H}_{\alpha_1 \dots \alpha_n}| = 0, \quad D^{\underline{2}\beta} \mathbb{H}_{\alpha_1 \dots \alpha_{n-1}\beta}| = 0. \quad (3.12)$$

In this gauge we stay with the following real unconstrained $\mathcal{N} = 1$ superfields

$$H_{\alpha_1 \dots \alpha_{n+1}} := i^{n+1} D^2_{(\alpha_1} \mathbb{H}_{\alpha_2 \dots \alpha_{n+1})} |, \quad (3.13a)$$

$$H_{\alpha_1 \dots \alpha_n} := \frac{i}{4} (D^2)^2 \mathbb{H}_{\alpha_1 \dots \alpha_n} |. \quad (3.13b)$$

The residual gauge freedom, which preserves the gauge conditions (3.12), is described by real unconstrained $\mathcal{N} = 1$ superfield parameters $\zeta_{\alpha(n)}$ and $\zeta_{\alpha(n-1)}$ defined by

$$g_{\alpha_1 \dots \alpha_n} | = -\frac{i}{2} \zeta_{\alpha_1 \dots \alpha_n}, \quad \bar{\zeta}_{\alpha(n)} = \zeta_{\alpha(n)}, \quad (3.14a)$$

$$D^{2\beta} g_{\alpha_1 \dots \alpha_{n-1} \beta} | = -i^n \frac{n+1}{n} \zeta_{\alpha_1 \dots \alpha_{n-1}}, \quad \bar{\zeta}_{\alpha(n-1)} = \xi_{\alpha(n-1)}. \quad (3.14b)$$

This leads to

$$\begin{aligned} \delta H_{\alpha_1 \dots \alpha_{n+1}} &\propto D^2_{(\alpha_1} \delta \mathbb{H}_{\alpha_2 \dots \alpha_{n+1})} | = D^2_{(\alpha_1} g_{\alpha_2 \dots \alpha_{n+1})} | + D^2_{(\alpha_1} \bar{g}_{\alpha_2 \dots \alpha_{n+1})} | \\ &= i D^1_{(\alpha_1} g_{\alpha_2 \dots \alpha_{n+1})} | - i D^1_{(\alpha_1} \bar{g}_{\alpha_2 \dots \alpha_{n+1})} | = D_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_{n+1})}, \end{aligned} \quad (3.15)$$

where we have used the longitudinal linear constraint (3.9) and the explicit expression (3.14a) for the residual gauge transformation. The final result for the gauge transformation of (3.13a) is

$$\delta H_{\alpha_1 \dots \alpha_{n+1}} = i^{n+1} D_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_{n+1})}. \quad (3.16a)$$

In a similar way we determine the gauge transformation of (3.13b) to be

$$\delta H_{\alpha_1 \dots \alpha_n} = i^n D_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_n)}. \quad (3.16b)$$

This agrees with (3.16a) if we replace $n \rightarrow n+1$. The superconformal prepotential $H_{\alpha(n)}$ and its gauge transformation (3.16b) were introduced in [37].

In discussing $\mathcal{N} = 1$ superconformal multiplets, we follow the formalism described in [30, 36]. The $\mathcal{N} = 1$ superconformal transformations are generated by conformal Killing supervector field.

$$\xi = \xi^a \partial_a + \xi^\alpha D_\alpha. \quad (3.17)$$

By definition, the $\mathcal{N} = 1$ conformal Killing supervector field obeys the equation $[\xi, D_\alpha] \propto D_\beta$, or equivalently

$$[\xi, D_\alpha] = -K_\alpha{}^\beta D_\beta - \frac{1}{2} \sigma D_\alpha, \quad (3.18)$$

which implies

$$\xi^\alpha = \frac{i}{6} D_\beta \xi^{\beta\alpha}, \quad (3.19a)$$

$$D_{(\gamma} \xi_{\alpha\beta)} = 0, \quad (3.19b)$$

of which (3.19b) is the $\mathcal{N} = 1$ superconformal Killing equation. In (3.18) we have introduced the z -dependent parameters of Lorentz ($K_{\alpha\beta}$) and scale (σ) transformations

$$K_{\alpha\beta} := D_{(\alpha} \xi_{\beta)}, \quad \sigma := D_\alpha \xi^\alpha = \frac{1}{3} \partial_a \xi^a. \quad (3.20)$$

These parameters are related to each other by the relation

$$D_\alpha K_{\beta\gamma} = -\varepsilon_{\alpha(\beta} D_{\gamma)} \sigma, \quad (3.21)$$

which implies

$$D^2\sigma = 0. \quad (3.22)$$

A symmetric rank- n spinor superfield $\Phi_{\alpha(n)} = \Phi_{\alpha_1 \dots \alpha_n}$ is said to be primary of dimension d_Φ if its superconformal transformation is

$$\delta_\xi \Phi_{\alpha_1 \dots \alpha_n} = \xi \Phi_{\alpha_1 \dots \alpha_n} + n K^\beta_{(\alpha_1} \Phi_{\alpha_2 \dots \alpha_n)\beta} + d_\Phi \sigma \Phi_{\alpha_1 \dots \alpha_n}. \quad (3.23)$$

We now require both the gauge field $H_{\alpha(n)}$ and the gauge parameter $\zeta_{\alpha(n-1)}$ in (3.16b) to be primary superfields. This is consistent if and only if the dimension of $H_{\alpha(n)}$ is equal to $(1 - n/2)$, as stated in [37]. Thus the superconformal transformation law of $H_{\alpha(n)}$ is

$$\delta_\xi H_{\alpha_1 \dots \alpha_n} = \xi H_{\alpha_1 \dots \alpha_n} + n K^\beta_{(\alpha_1} H_{\alpha_2 \dots \alpha_n)\beta} + (1 - \frac{1}{2}n) \sigma H_{\alpha_1 \dots \alpha_n}. \quad (3.24)$$

3.2. Higher spin superconformal field strengths

To start with, it is worth recalling the $\mathcal{N} = 2$ superconformal gauge-invariant field strength, $\mathbb{W}_{\alpha(n)} = \bar{\mathbb{W}}_{\alpha(n)}$, introduced in [24]

$$\begin{aligned} \mathbb{W}_{\alpha_1 \dots \alpha_n} := & \frac{1}{2^n} \sum_{J=0}^{\lfloor n/2 \rfloor} \left\{ \binom{n}{2J} \Delta \square^J \partial_{(\alpha_1}^{\beta_1} \dots \partial_{\alpha_{n-2J}}^{\beta_{n-2J}} \mathbb{H}_{\alpha_{n-2J+1} \dots \alpha_n) \beta_1 \dots \beta_{n-2J}} \right. \\ & \left. + \binom{n}{2J+1} \Delta^2 \square^J \partial_{(\alpha_1}^{\beta_1} \dots \partial_{\alpha_{n-2J-1}}^{\beta_{n-2J-1}} \mathbb{H}_{\alpha_{n-2J} \dots \alpha_n) \beta_1 \dots \beta_{n-2J-1}} \right\}, \end{aligned} \quad (3.25)$$

where $\lfloor x \rfloor$ denotes the floor (or the integer part) of a number x , and the operator Δ is

$$\Delta = \frac{i}{2} \mathbb{D}^\alpha \bar{\mathbb{D}}_\alpha = \frac{i}{2} \bar{\mathbb{D}}^\alpha \mathbb{D}_\alpha. \quad (3.26)$$

There are three fundamental properties that $\mathbb{W}_{\alpha(n)}$ possesses. Firstly, it is invariant under the gauge transformations (3.7). Secondly, it obeys the Bianchi identity

$$\mathbb{D}^\beta \mathbb{W}_{\beta \alpha_1 \dots \alpha_{n-1}} = 0 \iff \bar{\mathbb{D}}^\beta \mathbb{W}_{\beta \alpha_1 \dots \alpha_{n-1}} = 0. \quad (3.27)$$

Thirdly, the real symmetric rank- n spinor $\mathbb{W}_{\alpha(n)}$ is a primary $\mathcal{N} = 2$ superfield of dimension $(1 + n/2)$. As explained in [24], the conditions that $\mathbb{W}_{\alpha(n)}$ is primary and obeys the constraints (3.27) are consistent if and only if the dimension of $\mathbb{W}_{\alpha(n)}$ is equal to $(1 + n/2)$. If the prepotential $\mathbb{H}_{\alpha(n)}$ is chosen to be primary of dimension $(-n/2)$, then its descendant (3.25) proves to be primary of dimensions $(1 + n/2)$. It is important to emphasise that the most general solution to the constraints (3.27) is given by (3.25), as discussed in [37].

In the $n = 2$ case, the field strength $W_{\alpha\beta}(H)$ coincides with the linearised version [38,24] of the $\mathcal{N} = 2$ super-Cotton tensor [39,12]. Thus the field strength (3.25) for $n > 2$ is the higher-spin extension of the super-Cotton tensor.

We now turn to reducing the field strength $\mathbb{W}_{\alpha(n)}$ to $\mathcal{N} = 1$ superspace. In the real basis for the $\mathcal{N} = 2$ spinor covariant derivatives, the Bianchi identities (3.27) read

$$D^{I\beta} \mathbb{W}_{\beta \alpha_1 \dots \alpha_{n-1}} = 0, \quad I = \underline{1}, \underline{2}. \quad (3.28)$$

These constraints imply that, upon reduction to $\mathcal{N} = 1$ superspace, $\mathbb{W}_{\alpha(n)}$ is equivalent to the following real $\mathcal{N} = 1$ superfields

$$W_{\alpha_1 \dots \alpha_n} := \mathbb{W}_{\alpha_1 \dots \alpha_n} |, \quad W_{\alpha_1 \dots \alpha_{n+1}} \propto i^{n+1} D^2_{(\alpha_1} \mathbb{W}_{\alpha_2 \dots \alpha_{n+1})} |, \quad (3.29)$$

each of which is divergenceless, in particular

$$D^\beta W_{\beta \alpha_1 \dots \alpha_{n-1}} = 0. \quad (3.30)$$

We now compute the bar-projection of (3.25) in the gauge (3.12) and make use of the identities

$$\Delta = -\frac{i}{4} \left\{ (D^\perp)^2 + (D^\perp)^2 \right\}, \quad \Delta^2 = \frac{1}{8} \left\{ 4\Box - (D^\perp)^2 (D^\perp)^2 \right\}. \quad (3.31)$$

Making use of these identities leads to the $\mathcal{N} = 1$ field strength⁴

$$W_{\alpha_1 \dots \alpha_n}(H) := \frac{1}{2^n} \sum_{J=0}^{\lfloor n/2 \rfloor} \left\{ \binom{n}{2J} \Box^J \partial_{(\alpha_1}^{\beta_1} \dots \partial_{\alpha_{n-2J}}^{\beta_{n-2J}} H_{\alpha_{n-2J+1} \dots \alpha_n) \beta_1 \dots \beta_{n-2J}} \right. \\ \left. - \frac{i}{2} \binom{n}{2J+1} D^2 \Box^J \partial_{(\alpha_1}^{\beta_1} \dots \partial_{\alpha_{n-2J-1}}^{\beta_{n-2J-1}} H_{\alpha_{n-2J} \dots \alpha_n) \beta_1 \dots \beta_{n-2J-1}} \right\}. \quad (3.32)$$

This real superfield, $W_{\alpha(n)} = \bar{W}_{\alpha(n)}$, is a descendant of the real unconstrained prepotential $H_{\alpha(n)}$ defined modulo the gauge transformations (3.16b). The field strength proves to be gauge invariant,

$$\delta H_{\alpha_1 \dots \alpha_n} = i^n D_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_n)} \implies \delta W_{\alpha(n)} = 0, \quad (3.33)$$

and obey the Bianchi identity (3.30). Using the superconformal transformation law of $H_{\alpha(n)}$, eq. (3.24), one may check that the superconformal transformation law of the field strength (3.32) is

$$\delta_\xi W_{\alpha_1 \dots \alpha_n} = \xi W_{\alpha_1 \dots \alpha_n} + n K^\beta_{(\alpha_1} W_{\alpha_2 \dots \alpha_n) \beta} + \left(1 + \frac{1}{2}n\right) \sigma W_{\alpha_1 \dots \alpha_n}, \quad (3.34)$$

and therefore $W_{\alpha(n)}$ is a primary superfield of dimension $(1 + n/2)$.

For $n = 1$ the field strength (3.32) is

$$2W_\alpha = -\partial_\alpha{}^\beta H_\beta + \frac{i}{2} D^2 H_\alpha = i D^\beta D_\alpha H_\beta, \quad (3.35)$$

as a consequence of the anti-commutation relation (A.4). The final expression for W_α in (3.35) coincides with the gauge-invariant field strength of a vector multiplet [4]. For $n = 2$ the field strength $W_{\alpha\beta}$ given by (3.32) can be seen to coincide with the gravitino field strength [4]. Finally, for $n = 3$ the field strength $W_{\alpha\beta\gamma}$ given by (3.32) is the linearised version [17] of the $\mathcal{N} = 1$ super-Cotton tensor [11,12]. At the component level, field strength (3.32) for $n = 2s$ contains (as the θ -independent component) the bosonic higher spin Cotton tensors proposed by Pope and Townsend [40], as shown in [37]. In the $n = 2s + 1$ case, the fermionic (θ -independent) component of $W_{\alpha(2s+1)}$ was given in [37]. The fermionic component of $W_{\alpha(3)}$, known as the Cottino tensor, was first introduced in [14].

It should be pointed out that (3.32) is the most general solution of the constraint (3.30), as was emphasised in [37]. The simplest way to prove this is the observation that the field strength (3.32) may be recast in the form⁵ [37]

⁴ It was given without derivation in [37].

⁵ The numerical coefficient in the right-hand side of (3.36) was not computed in [37].

$$W_{\alpha(n)} = \frac{(-i)^n}{2^n} D^{\beta_1} D_{\alpha_1} \dots D^{\beta_n} D_{\alpha_n} H_{\beta_1 \dots \beta_n} . \quad (3.36)$$

It is completely symmetric, $W_{\alpha_1 \dots \alpha_n} = W_{(\alpha_1 \dots \alpha_n)}$, as a consequence of (A.6). If the superconformal prepotential is constrained to be transverse, $D^\beta H_{\beta\alpha(n-1)} = 0$, the expression for the super-Cotton simplifies,

$$D^\beta H_{\beta\alpha_1 \dots \alpha_{n-1}} = 0 \implies W_{\alpha(n)} = \partial_{\alpha_1}^{\beta_1} \dots \partial_{\alpha_n}^{\beta_n} H_{\beta_1 \dots \beta_n} . \quad (3.37a)$$

This result can be fine-tuned as follows:

$$W_{\alpha(2s)} = \square^s H_{\alpha(2s)} , \quad (3.37b)$$

$$W_{\alpha(2s+1)} = \square^s \partial_{(\alpha_1}^{\beta} H_{\alpha_2 \dots \alpha_{2s+1})\beta} = \square^s \partial_{\alpha_1}^{\beta} H_{\alpha_2 \dots \alpha_{2s+1}\beta} . \quad (3.37c)$$

Associated with $W_{\alpha(n)}(H)$ is the gauge-invariant Chern–Simons action

$$S_{\text{CS}}[H] = i^n \int d^{3|2} z H^{\alpha(n)} W_{\alpha(n)}(H) , \quad (3.38)$$

which is also invariant under the superconformal transformations (3.24). The action (3.38) coincides for $n = 1$ with the topological mass term for the Abelian vector multiplet [3]. In the $n = 3$ case, (3.38) proves to be the linearised action for $\mathcal{N} = 1$ conformal supergravity, as may be shown using the results in [4,11].

4. $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superspace reduction: massless gauge multiplets

There are two series of the massless half-integer superspin $\mathcal{N} = 2$ multiplets [24], which are dual to each other. Here we describe their $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superspace reduction. Throughout this section, we fix an integer $s > 1$.

4.1. Longitudinal formulation

The longitudinal formulation is realised in terms of the following dynamical variables:

$$\mathcal{V}^{\parallel} = \{ \mathbb{H}_{\alpha(2s)}, \mathbb{G}_{\alpha(2s-2)}, \bar{\mathbb{G}}_{\alpha(2s-2)} \} , \quad (4.1)$$

where the real superfield $\mathbb{H}_{\alpha(2s)} = \mathbb{H}_{(\alpha_1 \dots \alpha_{2s})}$ is unconstrained, and the complex superfield $\mathbb{G}_{\alpha(2s-2)} = \mathbb{G}_{(\alpha_1 \dots \alpha_{2s-2})}$ is longitudinal linear,

$$\bar{\mathbb{D}}_{(\alpha_1} \mathbb{G}_{\alpha_2 \dots \alpha_{2n-1})} = 0 . \quad (4.2)$$

The dynamical superfields are defined modulo gauge transformations of the form

$$\delta \mathbb{H}_{\alpha_1 \dots \alpha_{2s}} = g_{\alpha_1 \dots \alpha_{2s}} + \bar{g}_{\alpha_1 \dots \alpha_{2s}} , \quad (4.3a)$$

$$\delta \mathbb{G}_{\alpha_1 \dots \alpha_{2s-2}} = \frac{s}{2s+1} \mathbb{D}^\beta \bar{\mathbb{D}}^\gamma g_{\beta\gamma\alpha_1 \dots \alpha_{2s-2}} + i s \partial^{\beta\gamma} g_{\beta\gamma\alpha_1 \dots \alpha_{2s-2}} , \quad (4.3b)$$

where the complex gauge parameter $g_{\alpha_1 \dots \alpha_{2s}} = g_{(\alpha_1 \dots \alpha_{2s})}$ is an arbitrary longitudinal linear superfield, eq. (3.7b). Clearly, $\mathbb{H}_{\alpha(2s)}$ is the higher spin superconformal gauge multiplet with $n = 2s$ introduced in section 3.1. The superfields $\mathbb{G}_{\alpha(2s-2)}$ and $\bar{\mathbb{G}}_{\alpha(2s-2)}$ should be viewed as compensators. The gauge-invariant action is

$$\begin{aligned}
S_{s+\frac{1}{2}}^{\parallel}[\mathbb{H}, \mathbb{G}, \bar{\mathbb{G}}] = & \left(-\frac{1}{2}\right)^s \int d^3|4_z \left\{ \frac{1}{8} \mathbb{H}^{\alpha(2s)} D^{\gamma} \bar{D}^2 D_{\gamma} \mathbb{H}_{\alpha(2s)} \right. \\
& - \frac{1}{16} ([D_{\beta_1}, \bar{D}_{\beta_2}] \mathbb{H}^{\beta_1 \beta_2 \alpha(2s-2)}) [D^{\gamma_1}, \bar{D}^{\gamma_2}] \mathbb{H}_{\gamma_1 \gamma_2 \alpha(2s-2)} \\
& + \frac{s}{2} (\partial_{\beta_1 \beta_2} \mathbb{H}^{\beta_1 \beta_2 \alpha(2s-2)}) \partial^{\gamma_1 \gamma_2} \mathbb{H}_{\gamma_1 \gamma_2 \alpha(2s-2)} \\
& + i \frac{2s-1}{2s} (\mathbb{G} - \bar{\mathbb{G}})^{\alpha(2s-2)} \partial_{\beta_1 \beta_2} \mathbb{H}_{\beta_1 \beta_2 \alpha(2s-2)} \\
& \left. + \frac{2s-1}{2s^2} \mathbb{G} \cdot \bar{\mathbb{G}} - \frac{2s+1}{4s^2} (\mathbb{G} \cdot \mathbb{G} + \bar{\mathbb{G}} \cdot \bar{\mathbb{G}}) \right\}. \quad (4.4)
\end{aligned}$$

The $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superspace reduction of the superconformal gauge multiplet $\mathbb{H}_{\alpha(2s)}$ was carried out in section 3.1. It remains to reduce the compensator $\mathbb{G}_{\alpha(2s-2)}$ to $\mathcal{N} = 1$ superspace. From the point of view of $\mathcal{N} = 1$ supersymmetry, $\mathbb{G}_{\alpha(2s-2)}$ is equivalent to two *complex unconstrained* superfields, which we define as follows:

$$G_{\alpha_1 \dots \alpha_{2s-2}} := \mathbb{G}_{\alpha_1 \dots \alpha_{2s-2}}|, \quad \Omega_{\alpha_1 \dots \alpha_{2s-3}} := i D^{\beta} \mathbb{G}_{\beta \alpha_1 \dots \alpha_{2s-3}}|. \quad (4.5)$$

Making use of the gauge transformation (4.3b) gives

$$\delta \mathbb{G}_{\alpha_1 \dots \alpha_{2s-2}} = \frac{2is^2}{2s+1} \partial^{\beta \gamma} g_{\beta \gamma \alpha_1 \dots \alpha_{2s-2}} - \frac{is}{2s+1} D^{\perp \beta} D^2 g_{\beta \gamma \alpha_1 \dots \alpha_{2s-2}}, \quad (4.6a)$$

$$D^2 \beta \delta \mathbb{G}_{\beta \alpha_1 \dots \alpha_{2s-3}} = \frac{2is^2}{2s+1} \partial^{\beta \gamma} D^{2\delta} g_{\beta \gamma \delta \alpha_1 \dots \alpha_{2s-3}} - \frac{s}{2s+1} D^{\perp \beta} \partial^{\gamma \delta} g_{\beta \gamma \delta \alpha_1 \dots \alpha_{2s-3}}. \quad (4.6b)$$

At this stage one should recall that upon imposing the $\mathcal{N} = 1$ supersymmetric gauge conditions (3.12) the residual gauge freedom is described by the gauge parameters (3.14a) and (3.14b). From (4.6) we read off the gauge transformations of the $\mathcal{N} = 1$ complex superfields (4.5)

$$\delta G_{\alpha(2s-2)} = \frac{s^2}{2s+1} \partial^{\beta \gamma} \zeta_{\beta \gamma \alpha(2s-2)} - (-1)^s \frac{i}{2} D^{\beta} \zeta_{\beta \alpha(2s-2)}, \quad (4.7a)$$

$$\delta \Omega_{\alpha(2s-3)} = -\frac{s}{2(2s+1)} D^{\beta} \partial^{\gamma \delta} \zeta_{\beta \gamma \delta \alpha(2s-3)} + (-1)^s s \partial^{\beta \gamma} \zeta_{\beta \gamma \alpha(2s-3)}. \quad (4.7b)$$

In the $\mathcal{N} = 1$ supersymmetric gauge (3.12), $\mathbb{H}_{\alpha(2s)}$ is described by the two real unconstrained superfields $H_{\alpha(2s+1)}$ and $H_{\alpha(2s)}$ defined according to (3.13), and their gauge transformation laws are given by eqs. (3.16a) and (3.16b), respectively. Now it is useful to split each of $G_{\alpha(2s-2)}$ and $\Omega_{\alpha(2s-3)}$ into their real and imaginary parts,

$$G_{\alpha(2s-2)} = X_{\alpha(2s-2)} + iY_{\alpha(2s-2)}, \quad \Omega_{\alpha(2s-3)} = \Phi_{\alpha(2s-3)} + i\Psi_{\alpha(2s-3)}. \quad (4.8)$$

Then it follows from the gauge transformations (3.16a), (3.16b) and (4.7) that in fact we are dealing with two different gauge theories. One of them is formulated in terms of the real unconstrained gauge superfields

$$\mathcal{V}_{s+\frac{1}{2}}^{\parallel} = \{H_{\alpha(2s+1)}, X_{\alpha(2s-2)}, \Psi_{\alpha(2s-3)}\}, \quad (4.9)$$

which are defined modulo gauge transformations of the form

$$\delta H_{\alpha_1 \dots \alpha_{2s+1}} = (-1)^s i D_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_{2s+1})} , \quad (4.10a)$$

$$\delta X_{\alpha_1 \dots \alpha_{2s-2}} = \frac{s^2}{2s+1} \partial^{\beta\gamma} \zeta_{\beta\gamma\alpha_1 \dots \alpha_{2s-2}} , \quad (4.10b)$$

$$\delta \Psi_{\alpha_1 \dots \alpha_{2s-3}} = \frac{is}{2(2s+1)} D^\beta \partial^{\gamma\delta} \zeta_{\beta\gamma\delta\alpha_1 \dots \alpha_{2s-3}} , \quad (4.10c)$$

where the gauge parameter $\zeta_{\alpha(2s)}$ is real unconstrained. The other theory is described by the gauge superfields

$$\mathcal{V}_s^\parallel = \{ H_{\alpha(2s)}, Y_{\alpha(2s-2)}, \Phi_{\alpha(2s-3)} \} , \quad (4.11)$$

with the following gauge freedom

$$\delta H_{\alpha_1 \dots \alpha_{2s}} = (-1)^s D_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_{2s})} , \quad (4.12a)$$

$$\delta Y_{\alpha_1 \dots \alpha_{2s-2}} = -\frac{1}{2} (-1)^s D^\beta \zeta_{\beta\alpha_1 \dots \alpha_{2s-2}} , \quad (4.12b)$$

$$\delta \Phi_{\alpha_1 \dots \alpha_{2s-3}} = (-1)^s s \partial^{\beta\gamma} \zeta_{\beta\gamma\alpha_1 \dots \alpha_{2s-3}} , \quad (4.12c)$$

with the gauge parameter $\zeta_{\alpha(2s-1)}$ being real unconstrained.

4.2. Transverse formulation

The transverse formulation is realised in terms of the following dynamical variables:

$$\mathcal{V}^\perp = \{ \mathbb{H}_{\alpha(2s)}, \mathbf{\Gamma}_{\alpha(2s-2)}, \bar{\mathbf{\Gamma}}_{\alpha(2s-2)} \} , \quad (4.13)$$

where the real superfield $\mathbb{H}_{\alpha(2s)} = \mathbb{H}_{(\alpha_1 \dots \alpha_{2s})}$ is unconstrained, and the complex superfield $\mathbf{\Gamma}_{\alpha(2s-2)} = \mathbf{\Gamma}_{(\alpha_1 \dots \alpha_{2s-2})}$ is transverse linear,

$$\bar{\mathbb{D}}^\beta \mathbf{\Gamma}_{\beta\alpha_1 \dots \alpha_{2s-3}} = 0 \quad \implies \quad \bar{\mathbb{D}}^2 \mathbf{\Gamma}_{\alpha(2s-2)} = 0 . \quad (4.14)$$

The dynamical superfields are defined modulo gauge transformations of the form

$$\delta \mathbb{H}_{\alpha_1 \dots \alpha_{2s}} = g_{\alpha_1 \dots \alpha_{2s}} + \bar{g}_{\alpha_1 \dots \alpha_{2s}} , \quad (4.15a)$$

$$\delta \mathbf{\Gamma}_{\alpha(2s-2)} = \frac{s}{2s+1} \bar{D}^\beta D^\gamma \bar{g}_{\alpha(2s-2)\beta\gamma} . \quad (4.15b)$$

The gauge-invariant action is

$$\begin{aligned} S_{s+\frac{1}{2}}^\perp[\mathbb{H}, \mathbf{\Gamma}, \bar{\mathbf{\Gamma}}] = & \left(-\frac{1}{2}\right)^s \int d^3|4_z \left\{ \frac{1}{8} \mathbb{H}^{\alpha(2s)} D^\beta \bar{D}^2 D_\beta \mathbb{H}_{\alpha(2s)} \right. \\ & + \mathbb{H}^{\alpha(2s)} (D_{\alpha_1} \bar{D}_{\alpha_2} \mathbf{\Gamma}_{\alpha_3 \dots \alpha_{2s}} - \bar{D}_{\alpha_1} D_{\alpha_2} \bar{\mathbf{\Gamma}}_{\alpha_3 \dots \alpha_{2s}}) \\ & \left. + \frac{2s-1}{s} \bar{\mathbf{\Gamma}} \cdot \mathbf{\Gamma} + \frac{2s+1}{2s} (\mathbf{\Gamma} \cdot \mathbf{\Gamma} + \bar{\mathbf{\Gamma}} \cdot \bar{\mathbf{\Gamma}}) \right\} . \end{aligned} \quad (4.16)$$

From the point of view of $\mathcal{N} = 1$ supersymmetry, $\mathbf{\Gamma}_{\alpha(2s-2)}$ is equivalent to two *complex unconstrained* superfields, which we define as follows:

$$\Gamma_{\alpha_1 \dots \alpha_{2s-2}} := \mathbf{\Gamma}_{\alpha_1 \dots \alpha_{2s-2}}| , \quad \Upsilon_{\alpha_1 \dots \alpha_{2s-1}} := i D^2_{(\alpha_1} \mathbf{\Gamma}_{\alpha_2 \dots \alpha_{2s-1})}| . \quad (4.17)$$

Making use of the gauge transformation (4.15b) gives

$$\delta \Gamma_{\alpha(2s-2)} = -\frac{is}{2s+1} \partial^{\beta\gamma} \bar{g}_{\alpha_1 \dots \alpha_{2s-2} \beta\gamma} + \frac{is}{2s+1} D^\perp{}^\beta D^{\perp\gamma} \bar{g}_{\alpha_1 \dots \alpha_{2s-2} \beta\gamma} , \quad (4.18a)$$

$$\begin{aligned} iD^2_{(\alpha_1} \delta \Gamma_{\alpha_2 \dots \alpha_{2s-1})} &= \frac{s}{2s+1} \left\{ i\partial^\gamma{}_\beta D^\perp{}^\beta \bar{g}_{\alpha_1 \dots \alpha_{2s-1} \gamma} + iD^\perp{}^\beta \partial^\gamma{}_{(\alpha_1} \bar{g}_{\alpha_2 \dots \alpha_{2s-1}) \beta\gamma} \right. \\ &\quad \left. + \partial^\beta{}_{(\alpha_1} D^{\perp\gamma} \bar{g}_{\alpha_2 \dots \alpha_{2s-1}) \beta\gamma} - \frac{i}{2} (D^\perp)^2 D^{\perp\gamma} \bar{g}_{\alpha_1 \dots \alpha_{2s-1} \gamma} \right\} . \end{aligned} \quad (4.18b)$$

From here we read off the gauge transformations of the $\mathcal{N} = 1$ superfields (4.17)

$$\delta \Gamma_{\alpha(2s-2)} = \frac{s}{2(2s+1)} \partial^{\beta\gamma} \zeta_{\alpha_1 \dots \alpha_{2s-2} \beta\gamma} - (-1)^s \frac{i}{2} D^\beta \zeta_{\alpha_1 \dots \alpha_{2s-2} \beta} , \quad (4.19a)$$

$$\begin{aligned} \delta \Upsilon_{\alpha(2s-1)} &= -\frac{2}{2(2s+1)} \left\{ \partial^\gamma{}_\beta D^\beta \zeta_{\alpha_1 \dots \alpha_{2s-1} \gamma} + D^\beta \partial^\gamma{}_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_{2s-1}) \beta\gamma} \right\} \\ &\quad - \frac{1}{2} (-1)^s \left\{ \partial^\beta{}_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_{2s-1}) \beta} - \frac{i}{2} D^2 \zeta_{\alpha_1 \dots \alpha_{2s-1}} \right\} . \end{aligned} \quad (4.19b)$$

Now it is useful to split each of $\Gamma_{\alpha(2s-2)}$ and $\Upsilon_{\alpha(2s-1)}$ into their real and imaginary parts,

$$\Gamma_{\alpha(2s-2)} = X_{\alpha(2s-2)} + iY_{\alpha(2s-2)} , \quad \Upsilon_{\alpha(2s-1)} = \Phi_{\alpha(2s-1)} + i\Psi_{\alpha(2s-1)} . \quad (4.20)$$

Then it follows from the gauge transformations (3.16a), (3.16b) and (4.19) that in fact we are dealing with two different gauge theories. One of them is formulated in terms of the real unconstrained gauge superfields

$$\mathcal{V}_{s+\frac{1}{2}}^\perp = \{ H_{\alpha(2s+1)}, X_{\alpha(2s-2)}, \Psi_{\alpha(2s-1)} \} , \quad (4.21)$$

which are defined modulo gauge transformations of the form

$$\delta H_{\alpha_1 \dots \alpha_{2s+1}} = (-1)^s iD_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_{2s+1})} , \quad (4.22a)$$

$$\delta X_{\alpha_1 \dots \alpha_{2s-2}} = \frac{s}{2(2s+1)} \partial^{\beta\gamma} \zeta_{\beta\gamma\alpha_1 \dots \alpha_{2s-2}} , \quad (4.22b)$$

$$\delta \Psi_{\alpha_1 \dots \alpha_{2s-1}} = \frac{is}{2(2s+1)} \left\{ \partial^\gamma{}_\beta D^\beta \zeta_{\alpha_1 \dots \alpha_{2s-1} \gamma} + D^\beta \partial^\gamma{}_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_{2s-1}) \beta\gamma} \right\} , \quad (4.22c)$$

where the gauge parameter $\zeta_{\alpha(2s)}$ is real unconstrained. The other theory is described by the gauge superfields

$$\mathcal{V}_s^\perp = \{ H_{\alpha(2s)}, Y_{\alpha(2s-2)}, \Phi_{\alpha(2s-1)} \} , \quad (4.23)$$

with the following gauge freedom

$$\delta H_{\alpha_1 \dots \alpha_{2s}} = (-1)^s D_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_{2s})} , \quad (4.24a)$$

$$\delta Y_{\alpha_1 \dots \alpha_{2s-2}} = -\frac{1}{2} (-1)^s D^\beta \zeta_{\beta\alpha_1 \dots \alpha_{2s-2}} , \quad (4.24b)$$

$$\delta \Phi_{\alpha_1 \dots \alpha_{2s-1}} = -\frac{1}{2} (-1)^s \left\{ \partial^\beta{}_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_{2s-1}) \beta} - \frac{i}{2} D^2 \zeta_{\alpha_1 \dots \alpha_{2s-1}} \right\} , \quad (4.24c)$$

with the gauge parameter $\zeta_{\alpha(2s-1)}$ being real unconstrained.

4.3. Off-shell formulations for linearised $\mathcal{N} = 2$ supergravity

The limiting case $s = 1$ for the longitudinal and transverse formulations corresponds to linearised $\mathcal{N} = 2$ supergravity. As discussed in [24], the longitudinal $s = 1$ model is equivalent to the linearised action for type I minimal $\mathcal{N} = 2$ supergravity [41]. The transverse $s = 1$ model is equivalent to the linearised action for $w = -1$ non-minimal $\mathcal{N} = 2$ supergravity [41].

4.3.1. Longitudinal formulation: type I minimal $\mathcal{N} = 2$ supergravity

In the $s = 1$ case, the constraint (4.2) means that the $\mathcal{N} = 2$ superfield \mathbb{G} is chiral. The specific feature of $s = 1$ is that the second $\mathcal{N} = 1$ superfield in (4.5) does not exist in this case. According to (4.7), the scalar $G = \mathbb{G}|$ transforms as

$$\delta G = \frac{1}{3} \partial^{\alpha\beta} \zeta_{\alpha\beta} + \frac{i}{2} D^\alpha \zeta_\alpha. \quad (4.25)$$

We introduce the real and imaginary parts of G , $G = X + iY$. From the point of view of $\mathcal{N} = 1$ supersymmetry, the original $\mathcal{N} = 2$ theory is equivalent to a sum of two models. One of them realises an off-shell $\mathcal{N} = 1$ supergravity multiplet. It is described by the real gauge fields

$$\mathcal{V}_{3/2}^{\parallel} = \{H_{\alpha\beta\gamma}, X\}, \quad (4.26)$$

with the following gauge transformation law:

$$\delta H_{\alpha\beta\gamma} = -iD_{(\alpha} \zeta_{\beta\gamma)}, \quad \delta X = \frac{1}{3} \partial^{\alpha\beta} \zeta_{\alpha\beta}. \quad (4.27)$$

The second model realises an off-shell $\mathcal{N} = 1$ gravitino multiplet. It is described by the real gauge fields

$$\mathcal{V}_1^{\parallel} = \{H_{\alpha\beta}, Y\}, \quad (4.28)$$

with the following gauge transformation laws:

$$\delta H_{\alpha\beta} = -D_{(\alpha} \zeta_{\beta)}, \quad \delta Y = \frac{1}{2} D^\alpha \zeta_\alpha. \quad (4.29)$$

4.3.2. Transverse formulation: non-minimal $\mathcal{N} = 2$ supergravity

For $s = 1$ the transverse linear constraint (4.14) is not defined. However, its corollary $\mathbb{D}^2 \Gamma_{\alpha(2s-2)} = 0$ can be used; for $s = 1$ it defines a complex linear superfield. Then the gauge-invariant action (4.16) corresponds to the linearised action for $w = -1$ non-minimal $\mathcal{N} = 2$ supergravity [41]. Upon reduction to $\mathcal{N} = 1$ superspace, this dynamical system describes two off-shell $\mathcal{N} = 1$ supermultiplets, a supergravity multiplet and a gravitino multiplet. The supergravity multiplet is described by the real gauge superfields

$$\mathcal{V}_{3/2}^{\perp} = \{H_{\alpha\beta\gamma}, X, \Psi_\alpha\}. \quad (4.30)$$

The gravitino multiplet is described by the real gauge superfields

$$\mathcal{V}_1^{\perp} = \{H_{\alpha\beta}, Y, \Phi_\alpha\}. \quad (4.31)$$

4.3.3. Type II minimal $\mathcal{N} = 2$ supergravity

The linearised action for type II minimal $\mathcal{N} = 2$ supergravity [41] is

$$S^{(II)}[\mathbb{H}, \mathbb{S}] = \int d^3|4z \left\{ -\frac{1}{16} \mathbb{H}^{\alpha\beta} D^\gamma \bar{D}^2 D_\gamma \mathbb{H}_{\alpha\beta} - \frac{1}{4} (\partial_{\alpha\beta} \mathbb{H}^{\alpha\beta})^2 + \frac{1}{16} ([D_\alpha, \bar{D}_\beta] \mathbb{H}^{\alpha\beta})^2 \right. \\ \left. + \frac{1}{4} \mathbb{S} [D_\alpha, \bar{D}_\beta] \mathbb{H}^{\alpha\beta} + \frac{1}{2} \mathbb{S}^2 \right\}, \quad (4.32)$$

where the real compensator $\bar{\mathbb{S}} = \mathbb{S}$ is a linear superfield,

$$\bar{\mathbb{D}}^2 \mathbb{S} = \mathbb{D}^2 \mathbb{S} = 0. \quad (4.33)$$

Such a superfield describes the field strength of an Abelian $\mathcal{N} = 2$ vector multiplet. The action (4.32) is invariant under the gauge transformations

$$\delta \mathbb{H}_{\alpha\beta} = g_{\alpha\beta} + \bar{g}_{\alpha\beta}, \quad \delta \mathbb{S} = -\frac{1}{3} (\mathbb{D}^\alpha \bar{\mathbb{D}}^\beta g_{\alpha\beta} - \bar{\mathbb{D}}^\alpha \mathbb{D}^\beta \bar{g}_{\alpha\beta}). \quad (4.34)$$

The linear constraint (4.33) is equivalent to two constraints in the real basis for the covariant derivatives. The constraints are

$$\left\{ (D^2)^2 - (D^\perp)^2 \right\} \mathbb{S} = 0, \quad (4.35a)$$

$$D^{1\alpha} D^2_{\alpha} \mathbb{S} = 0. \quad (4.35b)$$

Thus \mathbb{S} is equivalent to the following real $\mathcal{N} = 1$ superfields

$$X := |\mathbb{S}|, \quad W_\alpha := -i D^2_{\alpha} \mathbb{S}, \quad (4.36)$$

of which the former is unconstrained and the latter is the field strength of an Abelian $\mathcal{N} = 1$ vector multiplet (see, e.g., [4]),

$$D^\alpha W_\alpha = 0. \quad (4.37)$$

To derive the gauge transformations of X and W_α , we should rewrite the gauge transformation of \mathbb{S} , eq. (4.34), as well as its corollary $D^2_{\alpha} \delta \mathbb{S}$, in the real basis for the covariant derivatives. We obtain

$$\delta \mathbb{S} = \frac{i}{3} \partial^{\alpha\beta} (g_{\alpha\beta} - \bar{g}_{\alpha\beta}) + \frac{i}{3} D^{1\alpha} (D^{2\beta} g_{\alpha\beta} + D^{2\beta} \bar{g}_{\alpha\beta}), \quad (4.38a)$$

$$-i D^2_{\alpha} \delta \mathbb{S} = -\frac{i}{3} \left(D^\beta \partial_{\alpha}{}^{\gamma} g_{\beta\gamma} + \partial^{\beta}{}_{\gamma} D^{\gamma} g_{\alpha\beta} + i \partial_{\alpha}{}^{\beta} D^{2\gamma} g_{\beta\gamma} \right. \\ \left. - \frac{1}{2} (D^\perp)^2 D^{2\beta} g_{\alpha\beta} \right) + \text{c.c.} \quad (4.38b)$$

From the point of view of $\mathcal{N} = 1$ supersymmetry, the dynamical system under consideration splits into two $\mathcal{N} = 1$ supersymmetric theories. One of them describes the off-shell $\mathcal{N} = 1$ supergravity multiplet realised in terms of the gauge superfields

$$\mathcal{V}_{3/2}^{(II)} = \{H_{\alpha\beta}, X\}, \quad (4.39)$$

with the gauge transformation of X being identical to that of X in (4.27). The other provides an off-shell realisation for $\mathcal{N} = 1$ gravitino multiplet realised in terms of the gauge superfields

$$\mathcal{V}_1^{(II)} = \{H_{\alpha\beta}, W_\alpha\}. \quad (4.40)$$

Their gauge transformation laws are:

$$\delta H_{\alpha\beta} = -D_{(\alpha} \zeta_{\beta)}, \quad \delta W_\alpha = i D^\beta D_\alpha \zeta_\beta. \quad (4.41)$$

4.3.4. Type III minimal $\mathcal{N} = 2$ supergravity

Type III supergravity [41] is described by the action

$$S^{(III)}[\mathbb{H}, \mathbb{T}] = \int d^3|4z \left\{ -\frac{1}{16} \mathbb{H}^{\alpha\beta} D^\gamma \bar{D}^2 D_\gamma \mathbb{H}_{\alpha\beta} - \frac{1}{8} (\partial_{\alpha\beta} \mathbb{H}^{\alpha\beta})^2 + \frac{1}{32} ([D_\alpha, \bar{D}_\beta] \mathbb{H}^{\alpha\beta})^2 \right. \\ \left. + \frac{1}{4} \mathbb{T} \partial_{\alpha\beta} \mathbb{H}^{\alpha\beta} + \frac{1}{8} \mathbb{T}^2 \right\}, \quad (4.42)$$

where the real compensator $\bar{\mathbb{T}} = \mathbb{T}$ is a linear superfield,

$$\bar{\mathbb{D}}^2 \mathbb{T} = \mathbb{D}^2 \mathbb{T} = 0. \quad (4.43)$$

The action (4.42) is invariant under the gauge transformations

$$\delta \mathbb{H}_{\alpha\beta} = g_{\alpha\beta} + \bar{g}_{\alpha\beta}, \quad \delta \mathbb{T} = -\frac{i}{3} (\mathbb{D}^\alpha \bar{\mathbb{D}}^\beta g_{\alpha\beta} + \bar{\mathbb{D}}^\alpha \mathbb{D}^\beta \bar{g}_{\alpha\beta}) \quad (4.44)$$

compare with (4.34).

The compensator \mathbb{T} is equivalent to the following real $\mathcal{N} = 1$ superfields

$$T := \mathbb{T}|, \quad Z_\alpha := -i D^2{}_\alpha \mathbb{T}|, \quad (4.45)$$

of which the former is unconstrained and the latter is the field strength of an Abelian $\mathcal{N} = 1$ vector multiplet,

$$D^\alpha Z_\alpha = 0. \quad (4.46)$$

Upon reduction to $\mathcal{N} = 1$ superspace, the theory describes two off-shell $\mathcal{N} = 1$ supermultiplets, a supergravity multiplet and a gravitino multiplet. The supergravity multiplet is realised in terms of the gauge superfields

$$\mathcal{V}_{3/2}^{(III)} = \{H_{\alpha\beta\gamma}, Z_\alpha\}. \quad (4.47)$$

Their gauge transformations are:

$$\delta H_{\alpha\beta\gamma} = -i D_{(\alpha} \zeta_{\beta\gamma)}, \quad \delta Z_\alpha = -\frac{1}{3} D^\beta D_\alpha D^\gamma \zeta_{\beta\gamma}. \quad (4.48)$$

The gravitino multiplet is realised in terms of the gauge superfields

$$\mathcal{V}_1^{(III)} = \{H_{\alpha\beta}, T\}, \quad (4.49)$$

with the gauge freedom

$$\delta H_{\alpha\beta} = -D_{(\alpha} \zeta_{\beta)}, \quad \delta T = D^\alpha \zeta_\alpha. \quad (4.50)$$

5. Off-shell formulations for massless low spin supermultiplets

Consider an arbitrary $\mathcal{N} = 2$ supersymmetric theory with action

$$S = \int d^3|4z L_{(\mathcal{N}=2)}, \quad (5.1)$$

where the Lagrangian $L_{(\mathcal{N}=2)}$ is a real scalar $\mathcal{N} = 2$ superfield. The action can be reduced to component fields by the rule

$$S = \int d^3x L_{(\mathcal{N}=0)} , \quad L_{(\mathcal{N}=0)} := \frac{1}{16} \mathbb{D}^2 \bar{\mathbb{D}}^2 L_{(\mathcal{N}=2)} \Big|_{\theta_I=0} , \quad (5.2)$$

or to $\mathcal{N} = 1$ superspace

$$S = \int d^3|z L_{(\mathcal{N}=1)} , \quad L_{(\mathcal{N}=1)} := -\frac{i}{4} (D^2)^2 L_{(\mathcal{N}=2)} \Big| . \quad (5.3)$$

Here the Lagrangian $L_{(\mathcal{N}=1)}$ is a real scalar $\mathcal{N} = 1$ superfield.

5.1. Scalar and vector multiplets

As an example, we consider the low-energy model for an Abelian $\mathcal{N} = 2$ vector multiplet with Lagrangian $L_{(\mathcal{N}=2)} = \mathcal{F}(\mathbb{S})$, where the vector multiplet field strength \mathbb{S} is a real linear superfield constrained as in (4.33). Upon reduction to $\mathcal{N} = 1$ superspace, the action becomes

$$\begin{aligned} S &= \frac{i}{4} \int d^3|z \left\{ \mathcal{F}''(X) W^\alpha W_\alpha - \mathcal{F}'(X) D^2 X \right\} \\ &= \frac{i}{4} \int d^3|z \mathcal{F}''(X) \left\{ D^\alpha X D_\alpha X + W^\alpha W_\alpha \right\} , \end{aligned} \quad (5.4)$$

where the $\mathcal{N} = 1$ components of \mathbb{S} , X and W_α , are defined as in (4.36). We recall that the $\mathcal{N} = 1$ field strength W_α obeys the Bianchi identity (4.37), which is solved according to (3.35). For a free $\mathcal{N} = 2$ vector multiplet, $\mathcal{F}(\mathbb{S}) = -\mathbb{S}^2$.

The model for a massless $\mathcal{N} = 1$ scalar multiplet is

$$S_{\text{SM}} = -\frac{i}{2} \int d^3|z D^\alpha X D_\alpha X . \quad (5.5)$$

The model for a massless $\mathcal{N} = 1$ vector multiplet is

$$S_{\text{VM}} = -\frac{i}{2} \int d^3|z W^\alpha W_\alpha = \frac{1}{4} \int d^3|z \left\{ -i H^\alpha \square H_\alpha + \frac{1}{2} H^\alpha \partial_{\alpha\beta} D^2 H^\beta \right\} . \quad (5.6)$$

Here we have made use of the gauge prepotential H_α for the vector multiplet, eq. (3.35). The models S_{SM} and S_{VM} are dual to each other [44]. It is worth reviewing this duality, for we will meet other examples of dual $\mathcal{N} = 1$ supersymmetric field theories. Consider a model for the vector multiplet W_α coupled to a background superfield Λ with action

$$S = -\frac{i}{2} \int d^3|z \Lambda W^\alpha W_\alpha , \quad D^\alpha W_\alpha = 0 . \quad (5.7)$$

This model is equivalent to a first-order model with action

$$S_{\text{first-order}} = -\frac{i}{2} \int d^3|z \Lambda \mathcal{W}^\alpha \mathcal{W}_\alpha + \int d^3|z \mathcal{W}^\alpha D_\alpha X , \quad (5.8)$$

in which the dynamical variables are an unconstrained real spinor superfield \mathcal{W}_α and a real scalar superfield X . Varying X gives $D^\alpha \mathcal{W}_\alpha = 0$, and hence $\mathcal{W}_\alpha = W_\alpha$. As a result, the action (5.8) reduces to (5.7). On the other hand, we can integrate out the auxiliary superfield \mathcal{W}_α from (5.8) to result with the dual action

$$S_{(\text{dual})} = -\frac{i}{2} \int d^3|z \Lambda^{-1} D^\alpha X D_\alpha X . \quad (5.9)$$

The inverse duality transformation is obtained by replacing (5.9) with an equivalent first-order action

$$\tilde{\mathcal{S}}_{\text{first-order}} = -\frac{i}{2} \int d^3|z| \Lambda^{-1} \mathcal{Y}^\alpha \mathcal{Y}_\alpha + \int d^3|z| W^\alpha \mathcal{Y}_\alpha, \quad (5.10)$$

in which \mathcal{Y}_α is an unconstrained *imaginary* spinor superfield, and W_α the field strength of a vector multiplet.

5.2. Gravitino multiplet

An off-shell formulation for the massless $\mathcal{N} = 1$ gravitino multiplet can be realised in terms of two real unconstrained gauge superfields, a three-vector $H_{\alpha\beta} = H_{\beta\alpha}$ and a scalar X . The superfield Lagrangian has the form

$$L_{\text{GM}} = -\frac{i}{2} \left\{ H^{\alpha\beta} D^2 H_{\alpha\beta} + D_\alpha H^{\alpha\beta} D_\gamma H^\gamma{}_\beta - 2D_\alpha H^{\alpha\beta} D_\beta X - D^\beta X D_\beta X \right\}, \quad (5.11)$$

and proves to be equivalent to the Lagrangian introduced by Siegel [3]. The action associated with (5.11) is invariant under the gauge transformations

$$\delta H_{\alpha\beta} = D_\alpha \zeta_\beta + D_\beta \zeta_\alpha = 2D_{(\alpha} \zeta_{\beta)}, \quad \delta X = D^\alpha \zeta_\alpha \quad (5.12)$$

where the gauge parameter ζ_α is real unconstrained. The superfield X is a compensator, for its kinetic term in (5.11) has a wrong sign as compared with the scalar multiplet (5.5).

The gravitino multiplet possesses a dual formulation obtained by dualising the scalar compensator in (5.11) into a vector multiplet. The dual Lagrangian is

$$L_{\text{GM}}^{(\text{dual})} = -\frac{i}{2} \left\{ H^{\alpha\beta} D^2 H_{\alpha\beta} + 2D_\alpha H^{\alpha\beta} D_\gamma H^\gamma{}_\beta + 2iW^\alpha D^\beta H_{\alpha\beta} - W^\alpha W_\alpha \right\}. \quad (5.13)$$

It is invariant under the gauge transformations

$$\delta H_{\alpha\beta} = D_\alpha \zeta_\beta + D_\beta \zeta_\alpha, \quad \delta W_\alpha = 2iD^\beta D_{\alpha\beta} \zeta_\beta. \quad (5.14)$$

5.3. Supergravity multiplet

An off-shell formulation for the massless $\mathcal{N} = 1$ supergravity multiplet can be realised in terms of two real unconstrained gauge superfields, a symmetric rank-3 spinor $H_{\alpha\beta\gamma} = H_{(\alpha\beta\gamma)}$ and a scalar X . The superfield Lagrangian is

$$L_{\text{SGM}} = \frac{i}{4} H^{\alpha\beta\gamma} \square H_{\alpha\beta\gamma} - \frac{1}{8} H^{\alpha\beta\gamma} \partial_{\gamma\rho} D^2 H_{\alpha\beta}{}^\rho - \frac{i}{4} \partial_{\alpha\beta} H^{\alpha\beta\gamma} \partial^{\rho\sigma} H_{\rho\sigma\gamma} \\ + \frac{1}{2} \partial_{\alpha\beta} H^{\alpha\beta\gamma} D_\gamma X + \frac{i}{2} D^\gamma X D_\gamma X. \quad (5.15)$$

The Lagrangian (5.15) is invariant under the gauge transformations [4]

$$\delta H_{\alpha\beta\gamma} = i(D_\alpha \zeta_{\beta\gamma} + D_\beta \zeta_{\alpha\gamma} + D_\gamma \zeta_{\alpha\beta}) = 3iD_{(\alpha} \zeta_{\beta\gamma)}, \quad \delta X = -\partial^{\alpha\beta} \zeta_{\alpha\beta}. \quad (5.16)$$

The supergravity multiplet possesses a dual formulation obtained by dualising the scalar compensator in (5.15) into a vector multiplet. The dual Lagrangian is

$$L_{\text{SGM}}^{(\text{dual})} = \frac{i}{4} H^{\alpha\beta\gamma} \square H_{\alpha\beta\gamma} - \frac{1}{8} H^{\alpha\beta\gamma} \partial_{\gamma\rho} D^2 H_{\alpha\beta}{}^\rho - \frac{i}{8} \partial_{\alpha\beta} H^{\alpha\beta\gamma} \partial^{\rho\sigma} H_{\rho\sigma\gamma} \\ + \frac{i}{2} \partial_{\alpha\beta} H^{\alpha\beta\gamma} W_\gamma + \frac{i}{2} W^\alpha W_\alpha. \quad (5.17)$$

It is invariant under gauge transformations

$$\delta H_{\alpha\beta\gamma} = i(D_\alpha \zeta_{\beta\gamma} + D_\beta \zeta_{\alpha\gamma} + D_\gamma \zeta_{\alpha\beta}) , \quad \delta W_\alpha = \frac{1}{2} D^\beta D_\alpha D^\gamma \zeta_{\beta\gamma} . \quad (5.18)$$

At this stage it is worth pausing in order to discuss some of the results obtained. According to the classification of linearised off-shell actions for 4D $\mathcal{N} = 1$ supergravity [42], there are three minimal models with $12 + 12$ off-shell degrees of freedom. The three-dimensional $\mathcal{N} = 2$ analogues of these models (with $8 + 8$ off-shell degrees of freedom) were constructed in [41] and called the type I, type II and type III minimal supergravity theories. We discussed these models in sections 4.3.1, 4.3.3 and 4.3.4, respectively. The difference between the 3D $\mathcal{N} = 2$ minimal supergravity models becomes quite transparent upon their reduction to $\mathcal{N} = 1$ superspace. Every $\mathcal{N} = 2$ action becomes a sum of two $\mathcal{N} = 1$ actions, one of which describes the gravitino multiplet and the other corresponds to the supergravity multiplet. Each of the $\mathcal{N} = 1$ actions is realised in terms of two $\mathcal{N} = 1$ superfields, of which one is universally the superconformal gauge field ($H_{\alpha\beta}$ for the gravitino multiplet, $H_{\alpha\beta\gamma}$ for the supergravity multiplet), while the other is a compensator. The difference between the three minimal $\mathcal{N} = 2$ supergravity models is encoded in different types of $\mathcal{N} = 1$ compensators. In the case of type I supergravity, both the $\mathcal{N} = 1$ supergravity and gravitino multiplets are characterised by scalar compensators, devoted X and Y , respectively, in section 4.3.1. The type II and type III formulations are obtained by dualising one of the scalar X and Y into an $\mathcal{N} = 1$ vector multiplet. In principle, it is possible to dualise both X and Y into vector multiplets. This would lead to a new linearised action for $\mathcal{N} = 2$ supergravity involving a double vector multiplet [43] as the corresponding $\mathcal{N} = 2$ compensator. However, such an action proves to possess $\mathcal{N} = 2$ supersymmetry with an intrinsic central charge, which is less interesting than the standard $\mathcal{N} = 2$ Poincaré supersymmetry.

5.4. Transverse formulation

The models (5.11) and (5.15) correspond to the longitudinal formulation discussed in section 4.3.1. It is of interest to compare them with the models originating within the transverse formulation sketched in section 4.3.2.

In addition to the dynamical variables $H_{\alpha\beta}$ and X , the gravitino multiplet now contains an auxiliary spinor superfield Φ_α . The Lagrangian has the form

$$\begin{aligned} L_{\text{GM}}^\perp = & -\frac{i}{2} H^{\alpha\beta} D^2 H_{\alpha\beta} - \frac{i}{2} D_\alpha H^{\alpha\beta} D_\beta X - \frac{1}{2} \Phi^\alpha D^\beta H_{\alpha\beta} \\ & + \frac{i}{4} D^\alpha X D_\alpha X - \frac{i}{2} \Phi^\alpha \Phi_\alpha - \frac{1}{2} \Phi^\alpha D_\alpha X . \end{aligned} \quad (5.19)$$

The Lagrangian (5.19) is invariant under the gauge transformations

$$\delta H_{\alpha\beta} = D_\alpha \zeta_\beta + D_\beta \zeta_\alpha , \quad \delta X = D^\alpha \zeta_\alpha , \quad \delta \Phi_\alpha = \partial_{\alpha\beta} \zeta^\beta + \frac{i}{2} D^2 \zeta_\alpha . \quad (5.20)$$

The superfield Φ_α can be integrated out using its equation of motion

$$\Phi^\alpha = -\frac{i}{2} D_\beta H^{\alpha\beta} + \frac{i}{2} D^\alpha X . \quad (5.21)$$

Then the Lagrangian (5.19) reduces to (5.11).

Within the transverse formulation, the supergravity multiplet contains not only the dynamical variables $H_{\alpha\beta\gamma}$ and X , but also an auxiliary spinor superfield Ψ_α . The corresponding Lagrangian is

$$\begin{aligned}
L = & \frac{i}{4} H^{\alpha\beta\gamma} \square H_{\alpha\beta\gamma} - \frac{1}{8} H^{\alpha\beta\gamma} D^2 \partial_{\gamma\rho} H_{\alpha\beta}{}^\rho - \frac{1}{2} \partial_{\alpha\beta} H^{\alpha\beta\gamma} (i\Psi_\gamma + D_\gamma X) \\
& + \frac{i}{4} \Psi^\alpha \Psi_\alpha + \Psi^\alpha D_\alpha X - \frac{i}{2} D^\alpha X D_\alpha X .
\end{aligned} \tag{5.22}$$

The action associate with this Lagrangian is invariant under the gauge transformations

$$\delta H_{\alpha\beta\gamma} = i(D_\alpha \zeta_{\beta\gamma} + D_\beta \zeta_{\alpha\gamma} + D_\gamma \zeta_{\alpha\beta}) , \tag{5.23a}$$

$$\delta \Psi_\alpha = i(\partial^{\beta\gamma} D_\beta \zeta_{\alpha\gamma} - D_\beta \partial_{\alpha\gamma} \zeta^{\beta\gamma}) , \tag{5.23b}$$

$$\delta X = -i\partial^{\alpha\beta} \zeta_{\alpha\beta} . \tag{5.23c}$$

The auxiliary superfield Ψ_α can be integrated out using its equation of motion

$$\Psi^\alpha = \partial_{\beta\gamma} H^{\alpha\beta\gamma} + 2iD^\alpha X . \tag{5.24}$$

Then the Lagrangian (5.22) turns into (5.15).

6. Massless higher spin supermultiplets

To derive off-shell formulations for the massless higher spin $\mathcal{N} = 1$ supermultiplets, one can apply the $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ superspace reduction to the actions (4.4) and (4.16). Here we will follow a different approach. We make use of the two pieces of input information: (i) the four sets of dynamical variables $\mathcal{V}_{s+\frac{1}{2}}^\parallel$, \mathcal{V}_s^\parallel , $\mathcal{V}_{s+\frac{1}{2}}^\perp$ and \mathcal{V}_s^\perp defined by eqs. (4.9) (4.11), (4.21) and (4.23), respectively; and (ii) the corresponding gauge transformation laws given by eqs. (4.10), (4.12), (4.22) and (4.24), respectively. To construct gauge-invariant actions, we will make use of the oscillator realisation for higher spin fields, see [45] for a review.⁶ The oscillator construction is expected to be useful for deriving interaction vertices for higher spin supermultiplets.

Before we proceed, a comment on the terminology used below is in order. In three dimensions, the notion of superspin is defined only in the massive case, see section 2.2. When speaking of a massless higher superspin theory in three dimensions, we will refer to the kinematic structure of the field variables, their gauge transformation laws and the gauge-invariant action. Given an integer $s > 1$, the massless supersymmetric gauge theories described by the dynamical variables \mathcal{V}_s^\parallel or \mathcal{V}_s^\perp will be referred to as massless integer superspin multiplets, for the gauge superfield $H_{\alpha(2s)}$ carries an even number of spinor indices. When speaking of massless half-integer superspin multiplets, we mean the massless supersymmetric gauge theories described by the dynamical variables $\mathcal{V}_{s+\frac{1}{2}}^\parallel$ or $\mathcal{V}_{s+\frac{1}{2}}^\perp$, for the gauge superfield $H_{\alpha(2s+1)}$ carries an odd number of spinor indices.

6.1. Auxiliary oscillators

In order to simplify computations for higher spin superfields, let us introduce auxiliary oscillators defined by the commutation relations

$$[a^\alpha, a^{\beta+}] = \varepsilon^{\alpha\beta} . \tag{6.1}$$

An “ n -particle” ket-state $|\Phi_n\rangle$ in this auxiliary Fock space is defined as

⁶ The oscillator formulation for the off-shell massless higher spin $\mathcal{N} = 1$ supermultiplets in four dimensions [25,26] was presented in [46].

$$|\Phi_n\rangle = \frac{1}{n!} \Phi_{\alpha_1 \alpha_2 \dots \alpha_n}(z) a^{\alpha_1+} a^{\alpha_2+} \dots a^{\alpha_n+} |0\rangle, \quad (6.2)$$

with the Fock vacuum defined by $a^\alpha |0\rangle = 0$. Here $\Phi_{\alpha(n)}(z)$ is a symmetric rank- n spinor superfield. The bra-state $\langle \Phi_n|$ is defined similarly,

$$\langle \Phi_n| = \frac{1}{n!} \langle 0| a^{\alpha_1} a^{\alpha_2} \dots a^{\alpha_n} \Phi_{\alpha_1 \alpha_2 \dots \alpha_n}(z). \quad (6.3)$$

We introduce the following operators

$$\gamma = a^\alpha D_\alpha, \quad \gamma^+ = a^{\alpha+} D_\alpha, \quad (6.4a)$$

$$P = a^\alpha \partial_{\alpha\beta} D^\beta, \quad P^+ = a^{\alpha+} \partial_{\alpha\beta} D^\beta, \quad (6.4b)$$

$$K_l = a^{\alpha_1+} \dots a^{\alpha_l+} \partial_{\alpha_1\beta_1} \dots \partial_{\alpha_l\beta_l} a^{\beta_1} \dots a^{\beta_l}. \quad (6.4c)$$

Some properties of these operators are listed in [Appendix B](#).

The action of the operators (6.4a)–(6.4c) on a state of the form (6.2) can be translated as follows

$$\gamma |\Phi_n\rangle \rightarrow D_\beta \Phi_{\alpha_1 \dots \alpha_{n-1}}^\beta, \quad \gamma^+ |\Phi_n\rangle \rightarrow (n+1) D_{(\alpha_1} \Phi_{\alpha_2 \dots \alpha_{n+1})}, \quad (6.5a)$$

$$P |\Phi_n\rangle \rightarrow D^\beta \partial_{\beta\gamma} \Phi_{\alpha_1 \dots \alpha_{n-1}}^\gamma, \quad P^+ |\Phi_n\rangle \rightarrow (n+1) D^\beta \partial_{\beta(\alpha_1} \Phi_{\alpha_2 \dots \alpha_{n+1})}, \quad (6.5b)$$

$$K_l |\Phi_n\rangle \rightarrow (-1)^l \frac{n!}{(n-l)!} \partial^{\beta_1}_{(\alpha_1} \dots \partial^{\beta_l}_{\alpha_l} \Phi_{\alpha_{l+1} \dots \alpha_n) \beta_{\alpha} \dots \beta_l}. \quad (6.5c)$$

We also introduce the “number operator” $N = a^{\alpha+} a_\alpha$ which acts on $|\Phi_n\rangle$ as

$$N |\Phi_n\rangle = n |\Phi_n\rangle. \quad (6.6)$$

6.2. Integer superspin multiplets

In this and the next subsections, we present massless gauge theories realised in terms of the dynamical variables \mathcal{V}_s^\parallel and $\mathcal{V}_{s+\frac{1}{2}}^\parallel$ defined by eqs. (4.11) and (4.9), respectively.

A Lagrangian formulation for a massless multiplet of integer superspin s , with $s > 1$, contains a gauge superfield $|H_{2s}\rangle$, a compensator $|Y_{2s-2}\rangle$ and an auxiliary superfield $|\Phi_{2s-3}\rangle$. The superfield Lagrangian, \mathcal{L}_s^\parallel , is

$$\begin{aligned} \frac{(-1)^s}{(2s-1)!} \mathcal{L}_s^\parallel &= \frac{i}{2} \langle H_{2s} | N D^2 | H_{2s} \rangle - \frac{i}{2} \langle H_{2s} | \gamma^+ \gamma | H_{2s} \rangle \\ &\quad - \frac{i}{2} \langle Y_{2s-2} | \gamma^2 | H_{2s} \rangle + \frac{i}{2} \langle H_{2s} | \gamma^{+2} | Y_{2s-2} \rangle + \frac{i}{2} \langle Y_{2s-2} | D^2 | Y_{2s-2} \rangle \\ &\quad - \langle \Phi_{2s-3} | \gamma | Y_{2s-2} \rangle - \langle Y_{2s-2} | \gamma^+ | \Phi_{2s-3} \rangle + 2i \langle \Phi_{2s-3} | \Phi_{2s-3} \rangle. \end{aligned} \quad (6.7)$$

The corresponding action proves to be invariant under gauge transformations

$$\delta |H_{2s}\rangle = \gamma^+ |\zeta_{2s-1}\rangle, \quad (6.8a)$$

$$\delta |Y_{2s-2}\rangle = \gamma |\zeta_{2s-1}\rangle, \quad (6.8b)$$

$$\delta |\Phi_{2s-3}\rangle = -\frac{i}{2} \gamma^2 |\zeta_{2s-1}\rangle. \quad (6.8c)$$

The equation of motion for $|\Phi_{2s-3}\rangle$ expresses this field in terms of $|Y_{2s-2}\rangle$,

$$|\Phi_{2s-3}\rangle = -\frac{i}{2}\gamma|Y_{2s-2}\rangle. \quad (6.9)$$

Plugging this expression back into the Lagrangian (6.7) gives

$$\begin{aligned} \frac{(-1)^s}{(2s-1)!}\mathcal{L}_s &= \frac{i}{2}\langle H_{2s}|ND^2|H_{2s}\rangle - \frac{i}{2}\langle H_{2s}|\gamma^+\gamma|H_{2s}\rangle - \frac{i}{2}\langle Y_{2s-2}|\gamma^2|H_{2s}\rangle \\ &\quad + \frac{i}{2}\langle H_{2s}|\gamma^{+2}|Y_{2s-2}\rangle + \frac{i}{2}\langle Y_{2s-2}|D^2|Y_{2s-2}\rangle \\ &\quad + \frac{i}{2}\langle Y_{2s-2}|\gamma^+\gamma|Y_{2s-2}\rangle. \end{aligned} \quad (6.10)$$

The above results can be readily recast in terms of ordinary superfields. We introduce the gauge superfields $H_{\alpha(2s)}$, $Y_{\alpha(2s-2)}$ and $\Phi_{\alpha(2s-3)}$ as follows:

$$|H_{2s}\rangle = \frac{1}{(2s)!}H_{\alpha_1\dots\alpha_{2s}}a^{\alpha_1+}\dots a^{\alpha_{2s}+}|0\rangle, \quad (6.11a)$$

$$|Y_{2s-2}\rangle = \frac{1}{(2s-2)!}Y_{\alpha_1\dots\alpha_{2s-2}}a^{\alpha_1+}\dots a^{\alpha_{2s-2}+}|0\rangle, \quad (6.11b)$$

$$|\Phi_{2s-3}\rangle = \frac{1}{(2s-3)!}\Phi_{\alpha_1\dots\alpha_{2s-3}}a^{\alpha_1+}\dots a^{\alpha_{2s-3}+}|0\rangle. \quad (6.11c)$$

The gauge parameters $\zeta_{\alpha(s2-1)}$ are introduced similarly,

$$|\zeta_{2s-1}\rangle = \frac{1}{(2s-1)!}\zeta_{\alpha_1\dots\alpha_{2s-1}}a^{\alpha_1+}\dots a^{\alpha_{2s-1}+}|0\rangle. \quad (6.12)$$

The Lagrangian (6.7) is equivalently written as

$$\begin{aligned} (-1)^s\mathcal{L}_s^{\parallel} &= \frac{i}{2}H_{\alpha_1\dots\alpha_{2s}}D^2H^{\alpha_1\dots\alpha_{2s}} + \frac{i}{2}D^\beta H_{\beta\alpha_1\dots\alpha_{2s-1}}D_\gamma H^{\gamma\alpha_1\dots\alpha_{2s-1}} \\ &\quad + (2s-1)Y_{\alpha_1\dots\alpha_{2s-2}}\partial_{\beta\gamma}H^{\beta\gamma\alpha_1\dots\alpha_{2s-2}} + \frac{i}{2}(2s-1)Y_{\alpha_1\dots\alpha_{2s-2}}D^2Y^{\alpha_1\dots\alpha_{2s-2}} \\ &\quad + 2i(2s-1)(2s-2)\Phi_{\alpha_1\dots\alpha_{2s-3}}\Phi^{\alpha_1\dots\alpha_{2s-3}} \\ &\quad - 2(2s-1)(2s-2)\Phi_{\alpha_1\dots\alpha_{2s-3}}D_\beta Y^{\beta\alpha_1\dots\alpha_{2s-3}}, \end{aligned} \quad (6.13)$$

while the Lagrangian (6.10) coincides with

$$\begin{aligned} (-1)^s\mathcal{L}_s &= \frac{i}{2}H_{\alpha_1\dots\alpha_{2s}}D^2H^{\alpha_1\dots\alpha_{2s}} + \frac{i}{2}D^\beta H_{\beta\alpha_1\dots\alpha_{2s-1}}D_\gamma H^{\gamma\alpha_1\dots\alpha_{2s-1}} \\ &\quad + (2s-1)Y_{\alpha_1\dots\alpha_{2s-2}}\partial_{\beta\gamma}H^{\beta\gamma\alpha_1\dots\alpha_{2s-2}} + \frac{i}{2}(2s-1)Y_{\alpha_1\dots\alpha_{2s-2}}D^2Y^{\alpha_1\dots\alpha_{2s-2}} \\ &\quad - i(s-1)(2s-1)D^\beta Y_{\beta\alpha_1\dots\alpha_{2s-3}}D_\gamma Y^{\gamma\alpha_1\dots\alpha_{2s-3}}. \end{aligned} \quad (6.14)$$

The gauge transformation laws (6.8) turn into

$$\delta H_{\alpha_1\dots\alpha_{2s}} = 2sD_{(\alpha_1}\zeta_{\alpha_2\dots\alpha_{2s})}, \quad (6.15a)$$

$$\delta Y_{\alpha_1\dots\alpha_{2s-2}} = -D^\beta\zeta_{\beta\alpha_1\dots\alpha_{2s-2}}, \quad (6.15b)$$

$$\delta\Phi_{\alpha_1\dots\alpha_{2s-3}} = \frac{1}{2}\partial^{\beta\gamma}\zeta_{\beta\gamma\alpha_1\dots\alpha_{2s-3}}. \quad (6.15c)$$

Upon inspecting (6.14) one may see that the Lagrangian is also well defined for the cases $s=0$ and $s=1$ which have been excluded from the above consideration. For $s=0$ only the first

term in the right-hand side of (6.14) remains, and the resulting Lagrangian corresponds to the massless scalar multiplet described by the action (5.5). In the $s = 1$ case, (6.14) coincides with the gravitino multiplet Lagrangian (5.11).

Since the Lagrangian (6.14) defines an off-shell massless supermultiplet for every integer $s = 0, 1, 2, \dots$, we may introduce a generating formulation for the massless multiplets of arbitrary superspin. It is described by the Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{i}{2} \langle H | (-1)^{N/2} (D^2 - N^{-1} \gamma^+ \gamma) | H \rangle \\ &\quad + \frac{i}{2} \langle Y | (-1)^{N/2} \gamma^2 N^{-1} | H \rangle - \frac{i}{2} \langle H | N^{-1} \gamma^{+2} (-1)^{N/2} | Y \rangle \\ &\quad - \frac{i}{2} \langle Y | (-1)^{N/2} (D^2 + \gamma^+ \gamma) (N + 2)^{-1} | Y \rangle \\ &= \sum_{s=0}^{\infty} \frac{1}{(2s)!} \mathcal{L}_s, \end{aligned} \quad (6.16)$$

in which the dynamical variables are given by

$$|H\rangle = \sum_{s=0}^{\infty} \frac{1}{(2s)!} H_{\alpha_1 \dots \alpha_{2s}} a^{\alpha_1+} \dots a^{\alpha_{2s}+} |0\rangle, \quad (6.17a)$$

$$|Y\rangle = \sum_{s=0}^{\infty} \frac{1}{(2s)!} \mathcal{G}_{\alpha_1 \dots \alpha_{2s}} a^{\alpha_1+} \dots a^{\alpha_{2s}+} |0\rangle. \quad (6.17b)$$

The action associated with (6.16) is invariant under gauge transformations of the form

$$\delta|H\rangle = \gamma^+ |\zeta\rangle, \quad \delta|Y\rangle = \gamma |\zeta\rangle, \quad (6.18)$$

where the gauge parameter is

$$|\zeta\rangle = \sum_{s=0}^{\infty} \frac{1}{(2s+1)!} \zeta_{\alpha_1 \dots \alpha_{2s+1}} a^{\alpha_1+} \dots a^{\alpha_{2s+1}+} |0\rangle. \quad (6.19)$$

6.3. Half-integer superspin multiplets

A Lagrangian formulation for a massless multiplet of half-integer superspin $(s + \frac{1}{2})$, with $s > 1$, contains a gauge superfield $|H_{2s+1}\rangle$, a compensator $|X_{2s-2}\rangle$ and an auxiliary superfield $|\Psi_{2s-3}\rangle$. The superfield Lagrangian, $\mathcal{L}_{s+\frac{1}{2}}^{\parallel}$, is

$$\begin{aligned} \frac{(-1)^s}{(2s)!} \mathcal{L}_{s+\frac{1}{2}}^{\parallel} &= \frac{i}{4} \langle H_{2s+1} | 2N \square + iK_1 D^2 - \gamma^{+2} \gamma^2 | H_{2s+1} \rangle \\ &\quad + \frac{i}{2} \langle H_{2s+1} | \gamma^{+3} | X_{2s-2} \rangle + \frac{i}{2} \langle X_{2s-2} | \gamma^3 | H_{2s+1} \rangle \\ &\quad + i \langle X_{2s-2} | (N + 2) D^2 | X_{2s-2} \rangle + \langle \Psi_{2s-3} | \gamma | X_{2s-2} \rangle \\ &\quad + \langle X_{2s-2} | \gamma^+ | \Psi_{2s-3} \rangle - i \langle \Psi_{2s-3} | | \Psi_{2s-3} \rangle. \end{aligned} \quad (6.20)$$

The corresponding action proves to be invariant under the gauge transformations

$$\delta|H_{2s+1}\rangle = i\gamma^+|\zeta_{2s}\rangle, \quad (6.21a)$$

$$\delta|X_{2s-2}\rangle = \frac{i}{2}\gamma^2|\zeta_{2s}\rangle, \quad (6.21b)$$

$$\delta|\Psi_{2s-3}\rangle = \frac{1}{2}\gamma^3|\zeta_{2s}\rangle. \quad (6.21c)$$

The field $|\Psi_{2s-3}\rangle$ can be integrated out using its equation of motion

$$|\Psi_{2s-3}\rangle = -i\gamma|X_{2s-2}\rangle. \quad (6.22)$$

Then the Lagrangian (6.20) turns into

$$\begin{aligned} \frac{(-1)^s}{(2s)!}\mathcal{L}_{s+\frac{1}{2}} &= \frac{i}{4}\langle H_{2s+1}|2N\Box + iK_1D^2 - \gamma^{+2}\gamma^2|H_{2s+1}\rangle + \frac{i}{2}\langle H_{2s+1}|\gamma^{+3}|X_{2s-2}\rangle \\ &\quad + \frac{i}{2}\langle X_{2s-2}|\gamma^3|H_{2s+1}\rangle - i\langle X_{2s-2}|\gamma^+\gamma - (N+2)D^2|X_{2s-2}\rangle. \end{aligned} \quad (6.23)$$

Introducing the expansions

$$|H_{2s+1}\rangle = \frac{1}{(2s+1)!}H_{\alpha_1\dots\alpha_{2s+1}}a^{\alpha_1+}\dots a^{\alpha_{2s+1}+}|0\rangle, \quad (6.24a)$$

$$|X_{2s-2}\rangle = \frac{1}{(2s-2)!}X_{\alpha_1\dots\alpha_{2s-2}}a^{\alpha_1+}\dots a^{\alpha_{2s-2}+}|0\rangle, \quad (6.24b)$$

$$|\Psi_{2s-3}\rangle = \frac{1}{(2s-3)!}\Psi_{\alpha_1\dots\alpha_{2s-3}}a^{\alpha_1+}\dots a^{\alpha_{2s-3}+}|0\rangle \quad (6.24c)$$

for the fields, and

$$|\zeta_{2s}\rangle = \frac{1}{(2s)!}\zeta_{\alpha_1\dots\alpha_{2s}}a^{\alpha_1+}\dots a^{\alpha_{2s}+}|0\rangle \quad (6.25)$$

for the gauge parameters, one gets from (6.20) the Lagrangian in terms of the three fields

$$\begin{aligned} (-1)^s\mathcal{L}_{s+\frac{1}{2}}^{\parallel} &= \frac{i}{2}H_{\alpha_1\dots\alpha_{2s+1}}\Box H^{\alpha_1\dots\alpha_{2s+1}} - \frac{1}{4}H_{\alpha_1\dots\alpha_{2s}}\beta^\beta\gamma D^2 H^{\gamma\alpha_1\dots\alpha_{2s}} \\ &\quad - \frac{i}{2}s\partial^{\beta\gamma}H_{\beta\gamma\alpha_1\dots\alpha_{2s-1}}\partial_{\delta\lambda}H^{\delta\lambda\alpha_1\dots\alpha_{2s-1}} \\ &\quad - 2s(2s-1)X_{\alpha_1\dots\alpha_{2s-2}}\partial_{\beta\gamma}D_\delta H^{\beta\gamma\delta\alpha_1\dots\alpha_{2s-2}} \\ &\quad + i(2s)^2(2s-1)X_{\alpha_1\dots\alpha_{2s-2}}D^2 X^{\alpha_1\dots\alpha_{2s-2}} \\ &\quad - i2s(2s-1)(2s-2)\Psi_{\alpha_1\dots\alpha_{2s-3}}\Psi^{\alpha_1\dots\alpha_{2s-3}} \\ &\quad + 4s(2s-1)(2s-2)\Psi_{\alpha_1\dots\alpha_{2s-3}}D_\beta X^{\beta\alpha_1\dots\alpha_{2s-3}} \end{aligned} \quad (6.26)$$

from (6.23) the Lagrangian in terms of two fields

$$\begin{aligned} (-1)^s\mathcal{L}_{s+\frac{1}{2}} &= \frac{i}{2}H_{\alpha_1\dots\alpha_{2s+1}}\Box H^{\alpha_1\dots\alpha_{2s+1}} - \frac{1}{4}H_{\alpha_1\dots\alpha_{2s}}\beta^\beta\gamma D^2 H^{\gamma\alpha_1\dots\alpha_{2s}} \\ &\quad - \frac{i}{2}s\partial^{\beta\gamma}H_{\beta\gamma\alpha_1\dots\alpha_{2s-1}}\partial_{\delta\lambda}H^{\delta\lambda\alpha_1\dots\alpha_{2s-1}} \\ &\quad - 2s(2s-1)X_{\alpha_1\dots\alpha_{2s-2}}\partial_{\beta\gamma}D_\delta H^{\beta\gamma\delta\alpha_1\dots\alpha_{2s-2}} \\ &\quad + i(2s)^2(2s-1)X_{\alpha_1\dots\alpha_{2s-2}}D^2 X^{\alpha_1\dots\alpha_{2s-2}} \\ &\quad + 4is(2s-1)(2s-2)D^\beta X_{\beta\alpha_1\dots\alpha_{2s-3}}D_\gamma X^{\gamma\alpha_1\dots\alpha_{2s-3}}. \end{aligned} \quad (6.27)$$

From (6.21) we read off the gauge transformations

$$\delta H_{\alpha_1 \dots \alpha_{2s+1}} = i(2s+1) D_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_{2s+1})} , \quad (6.28a)$$

$$\delta X_{\alpha_1 \dots \alpha_{2s-2}} = -\frac{1}{2} \partial^{\beta\gamma} \zeta_{\beta\gamma \alpha_1 \dots \alpha_{2s-2}} , \quad (6.28b)$$

$$\delta \Psi_{\alpha_1 \dots \alpha_{2s-3}} = \frac{i}{2} \partial^{\beta\gamma} D^\delta \zeta_{\beta\gamma \delta \alpha_1 \dots \alpha_{2s-3}} . \quad (6.28c)$$

It can be seen that the Lagrangian (6.27) is well defined for the cases $s = 0$ and $s = 1$ excluded from the above consideration. For $s = 0$ it coincides (modulo an overall factor of) with the Lagrangian for the vector multiplet, eq. (5.6). For $s = 1$ it coincides (modulo an overall factor) with the Lagrangian for the supergravity multiplet, eq. (5.15).

Generating formulation

$$\begin{aligned} \mathcal{L} &= \frac{i}{4} \langle H | (-1)^{(N-1)/2} (2N\Box + iK_1 D^2 - \gamma^{+2} \gamma^2) N^{-1} | H \rangle \\ &\quad + \frac{i}{2} \langle H | (-1)^{(N-1)/2} N^{-1} \gamma^{+3} | X \rangle + \frac{i}{2} \langle X | \gamma^3 (-1)^{(N-1)/2} N^{-1} | H \rangle \\ &\quad + i \langle X | (-1)^{N/2} (\gamma^+ \gamma - (N+2) D^2) (N+3)^{-1} | X \rangle \\ &= \sum_{s=0}^{\infty} \frac{1}{(2s+1)!} \mathcal{L}_{s+\frac{1}{2}} , \end{aligned} \quad (6.29)$$

where

$$|H\rangle = \sum_{s=0}^{\infty} \frac{1}{(2s+1)!} H_{\alpha_1 \dots \alpha_{2s+1}} a^{\alpha_1+} \dots a^{\alpha_{2s+1}+} |0\rangle , \quad (6.30a)$$

$$|X\rangle = \sum_{s=0}^{\infty} \frac{1}{(2s)!} g_{\alpha_1 \dots \alpha_{2s}} a^{\alpha_1+} \dots a^{\alpha_{2s}+} |0\rangle . \quad (6.30b)$$

Gauge transformation

$$\delta |H\rangle = i\gamma^+ |\zeta\rangle , \quad \delta |X\rangle = \frac{i}{2} \gamma^2 |\zeta\rangle , \quad (6.31)$$

where the gauge parameter is

$$|\zeta\rangle = \sum_{s=0}^{\infty} \frac{1}{(2s)!} \zeta_{\alpha_1 \dots \alpha_{2s}} a^{\alpha_1+} \dots a^{\alpha_{2s}+} |0\rangle . \quad (6.32)$$

6.4. Transverse formulation

In this subsection, we briefly describe massless gauge theories realised in terms of the dynamical variables \mathcal{V}_s^\perp and $\mathcal{V}_{s+\frac{1}{2}}^\perp$ defined by eqs. (4.23) and (4.21), respectively.

6.4.1. Integer superspins

A Lagrangian formulation for a massless multiplet of integer superspin s , with $s > 1$, contains a gauge superfield $|H_{2s}\rangle$, a compensator $|Y_{2s-2}\rangle$ and an auxiliary superfield $|\Phi_{2s-1}\rangle$. The Lagrangian for this supermultiplet, \mathcal{L}_s^\perp , is

$$\begin{aligned}
\frac{(-1)^s}{(2s-1)!} \mathcal{L}_s^\perp &= \frac{i}{2} \langle H_{2s} | N D^2 | H_{2s} \rangle - i \langle Y_{2s-2} | \gamma^2 | H_{2s} \rangle + i \langle H_{2s} | \gamma^{+2} | Y_{2s-2} \rangle \\
&+ \langle H_{2s} | \gamma^+ | \Phi_{2s-1} \rangle + \langle \Phi_{2s-1} | \gamma | H_{2s} \rangle + \frac{i}{2} \langle Y_{2s-2} | (N+2) D^2 | Y_{2s-2} \rangle \\
&- 2i \langle \Phi_{2s-1} | \Phi_{2s-1} \rangle - \langle Y_{2s-2} | \gamma | \Phi_{2s-1} \rangle + \langle \Phi_{2s-1} | \gamma^+ | Y_{2s-2} \rangle . \quad (6.33)
\end{aligned}$$

The corresponding action is invariant under gauge transformations

$$\delta |H_{2s}\rangle = \gamma^+ |\zeta_{2s-1}\rangle , \quad (6.34a)$$

$$\delta |Y_{2s-2}\rangle = \gamma |\zeta_{2s-1}\rangle , \quad (6.34b)$$

$$\delta |\Phi_{2s-1}\rangle = (K_1 + \frac{i}{2} D^2) |\zeta_{2s-1}\rangle . \quad (6.34c)$$

The auxiliary superfield $|\Phi_{2s-1}\rangle$ can be integrated out using its equation of motion

$$|\Phi_{2s-1}\rangle = -\frac{i}{2} (\gamma |H_{2s}\rangle + \gamma^+ |Y_{2s-2}\rangle) . \quad (6.35)$$

Then the Lagrangian (6.33) reduces to (6.10).

The fields and the gauge parameters can be expanded in terms of the oscillators, in complete analogy with our analysis in subsection 6.2. Then the gauge transformation laws (6.34) turn into

$$\delta H_{\alpha_1 \dots \alpha_{2s}} = 2s D_{(\alpha_1} \zeta_{\alpha_2 \dots \alpha_{2s})} , \quad (6.36a)$$

$$\delta Y_{\alpha_1 \dots \alpha_{2s-2}} = -D^\beta \zeta_{\beta \alpha_1 \dots \alpha_{2s-2}} , \quad (6.36b)$$

$$\delta \Phi_{\alpha_1 \dots \alpha_{2s-1}} = -2s \partial^\beta (\alpha_1 \zeta_{\alpha_2 \dots \alpha_{2s-1}) \beta} + \frac{i}{2} D^2 \zeta_{\alpha_1 \dots \alpha_{2s-1}} , \quad (6.36c)$$

and the Lagrangian (6.33) becomes

$$\begin{aligned}
(-1)^s \mathcal{L}_s^\perp &= \frac{i}{2} H_{\alpha_1 \dots \alpha_{2s}} D^2 H^{\alpha_1 \dots \alpha_{2s}} + 2(2s-1) Y_{\alpha_1 \dots \alpha_{2s-2}} \partial_\beta \gamma H^{\beta \gamma \alpha_1 \dots \alpha_{2s-2}} \\
&+ 2\Phi_{\alpha_1 \dots \alpha_{2s-1}} D_\beta H^{\beta \alpha_1 \dots \alpha_{2s-1}} + i s (2s-1) Y_{\alpha_1 \dots \alpha_{2s-2}} D^2 Y^{\alpha_1 \dots \alpha_{2s-2}} \\
&- 2i \Phi_{\alpha_1 \dots \alpha_{2s-1}} \Phi^{\alpha_1 \dots \alpha_{2s-1}} - 2(2s-1) Y_{\alpha_1 \dots \alpha_{2s-2}} D_\beta \Phi^{\beta \alpha_1 \dots \alpha_{2s-2}} . \quad (6.37)
\end{aligned}$$

6.4.2. Half-integer superspins

A Lagrangian formulation for a massless multiplet of half-integer superspin $(s + \frac{1}{2})$, with $s > 1$, contains a gauge superfield $|H_{2s+1}\rangle$, a compensator $|X_{2s-2}\rangle$ and an auxiliary superfield $|\Psi_{2s-1}\rangle$. The Lagrangian, $\mathcal{L}_{s+\frac{1}{2}}^\perp$, is

$$\begin{aligned}
\frac{(-1)^s}{(2s)!} \mathcal{L}_{s+\frac{1}{2}}^\perp &= \frac{i}{4} \langle H_{2s+1} | 2N \square + i K_1 D^2 | H_{2s+1} \rangle + \langle H_{2s+1} | \gamma^{+2} | \Psi_{2s-1} \rangle \\
&- \langle \Psi_{2s-1} | \gamma^2 | H_{2s+1} \rangle + \langle H_{2s+1} | \gamma^{+3} | X_{2s-2} \rangle + \langle X_{2s-2} | \gamma^3 | H_{2s+1} \rangle \\
&+ i \langle X_{2s-2} | D^2 | X_{2s-2} \rangle + 4i \langle \Psi_{2s-1} | \Psi_{2s-1} \rangle - 2 \langle \Psi_{2s-1} | \gamma^+ | X_{2s-2} \rangle \\
&+ 2 \langle X_{2s-2} | \gamma | \Psi_{2s-1} \rangle . \quad (6.38)
\end{aligned}$$

The corresponding action is invariant under gauge transformations

$$\delta|H_{2s+1}\rangle = i\gamma^+|\zeta_{2s}\rangle, \quad (6.39a)$$

$$\delta|X_{2s-2}\rangle = \frac{i}{2}\gamma^2|\zeta_{2s}\rangle, \quad (6.39b)$$

$$\delta|\Psi_{2s-1}\rangle = \left(\frac{1}{2}\gamma^+\gamma^2 + \frac{i}{2}P\right)|\zeta_{2s}\rangle. \quad (6.39c)$$

The auxiliary superfield $|\Psi_{2s-1}\rangle$ can be integrated out using its equation of motion

$$|\Psi_{2s-1}\rangle = -\frac{i}{4}\gamma^2|H_{2s}\rangle - \frac{i}{2}\gamma^+|X_{2s-2}\rangle, \quad (6.40)$$

which turns the Lagrangian (6.38) into (6.23).

The above results can be rewritten in terms of ordinary superfields (compare with subsection 6.3). The gauge transformation laws (6.39) are equivalent to

$$\delta H_{\alpha_1\dots\alpha_{2s+1}} = (2s+1)iD_{(\alpha_1}\zeta_{\alpha_2\dots\alpha_{2s+1})}, \quad (6.41a)$$

$$\delta X_{\alpha_1\dots\alpha_{2s-2}} = -\frac{1}{2}\partial^{\beta\gamma}\zeta_{\beta\gamma\alpha_1\dots\alpha_{2s-2}}, \quad (6.41b)$$

$$\delta\Psi_{\alpha_1\dots\alpha_{2s-1}} = \frac{i}{2}(2s-1)\partial^{\beta\gamma}D_{(\alpha_1}\zeta_{\alpha_2\dots\alpha_{2s-1})\beta\gamma} + \frac{i}{2}\partial^{\beta\gamma}D_\beta\zeta_{\gamma\alpha_1\dots\alpha_{2s-1}}. \quad (6.41c)$$

The Lagrangian (6.38) is equivalent to

$$\begin{aligned} (-1)^s\mathcal{L}_{s+\frac{1}{2}} = & \frac{i}{2}H_{\alpha_1\dots\alpha_{2s+1}}\square H^{\alpha_1\dots\alpha_{2s+1}} - \frac{1}{4}H_{\alpha_1\dots\alpha_{2s}\beta}\partial^\beta{}_\gamma D^2 H^{\gamma\alpha_1\dots\alpha_{2s}} \\ & - 4is\Psi_{\alpha_1\dots\alpha_{2s-1}}\partial_{\beta\gamma}H^{\beta\gamma\alpha_1\dots\alpha_{2s-1}} \\ & - 4s(2s-1)X_{\alpha_1\dots\alpha_{2s-2}}\partial_{\beta\gamma}D_\delta H^{\beta\gamma\delta\alpha_1\dots\alpha_{2s-2}} \\ & + 2is(2s-1)X_{\alpha_1\dots\alpha_{2s-2}}D^2 X^{\alpha_1\dots\alpha_{2s-2}} + 8is\Psi_{\alpha_1\dots\alpha_{2s-1}}\Psi^{\alpha_1\dots\alpha_{2s-1}} \\ & + 8s(2s-1)X_{\alpha_1\dots\alpha_{2s-2}}D_\beta\Psi^{\beta\alpha_1\dots\alpha_{2s-2}}. \end{aligned} \quad (6.42)$$

7. Massive higher spin supermultiplets

Before presenting the Lagrangians for massive supermultiplets and analysing the corresponding equations of motion, it is useful to reformulate some of the results given in subsection 3.2 in terms of the auxiliary oscillators used in the previous section.

7.1. Higher spin super-Cotton tensor

One can check that the higher spin super-Cotton tensor (3.32) can be written in the form

$$|W_n\rangle = (-1)^n \left(\sum_{p=0}^{[n/2]} a_p \square^p K_{n-2p} + i \sum_{p=0}^{[n/2]} b_p \square^p D^2 K_{n-2p-1} \right) |H_n\rangle. \quad (7.1)$$

The expression (7.1) is invariant under gauge transformations

$$\delta|H_n\rangle = \gamma^+|\Lambda_{n-1}\rangle \quad (7.2)$$

provided the constant coefficients a_p and b_p satisfy the equations

$$a_p(n-2p) - 2b_p = 0, \quad a_{p+1} - 2b_p(n-2p-1) = 0. \quad (7.3)$$

These recurrence relations are solved by

$$a_p = \binom{n}{2p} (2p)! a_0, \quad b_p = \frac{1}{2} \binom{n}{2p+1} (2p+1)! a_0. \quad (7.4)$$

In order to match the overall coefficient in (3.32), a_0 has to be

$$a_0 = \frac{1}{n! 2^n}. \quad (7.5)$$

One may check that the gauge-invariant field strength (7.1) satisfies the Bianchi identity

$$\gamma |W_n\rangle = 0. \quad (7.6)$$

7.2. Superfield Lagrangian

Now let us turn to describing our off-shell massive higher spin $\mathcal{N} = 1$ supermultiplets. The Lagrangian for a massive superspin- $\frac{n}{2}$ multiplet is defined by

$$\mathcal{L}_{\text{massive}}^{(n/2)} = \frac{1}{(n-1)!} \mathcal{L}_{n/2} + \frac{i^n}{2} n \lambda (\langle H_n | |W_n\rangle + \langle W_n | |H_n\rangle), \quad (7.7)$$

where the massless Lagrangian $\mathcal{L}_{n/2}$ is given either by the equation (6.10) for $n = 2s$, or by (6.23) for $n = 2s + 1$. (We recall that the oscillator realisation for the super-Cotton tensor $|W_n\rangle$ is described in subsection 7.1.) Thus the massive Lagrangian is obtained from the massless one by adding the Chern–Simons like term.

The action generated by (7.7) is gauge invariant since both terms in the action are separately gauge invariant. Indeed the term proportional to λ is invariant under the transformations (7.2) due to the gauge invariance of $|W_n\rangle$ and the Bianchi identities (7.6). The gauge invariance of the first term was established in section 6. Note also that the mass term contains only the physical gauge superfield $|H_n\rangle$ and does not depend on the compensator.

In the integer superspin case, $n = 2s$, the gauge-invariant equations of motion derived from (7.7) are

$$|E_{2s}\rangle + 2s\lambda |W_{2s}\rangle = 0, \quad (7.8a)$$

$$|Q_{2s-2}\rangle = 0, \quad (7.8b)$$

where we have introduced the following gauge-invariant field strengths:

$$|E_{2s}\rangle = \frac{i}{2} (ND^2 - \gamma^+ \gamma) |H_{2s}\rangle + \frac{i}{2} \gamma^{+2} |Y_{2s-2}\rangle, \quad (7.9a)$$

$$|Q_{2s-2}\rangle = -\frac{i}{2} \gamma^2 |H_{2s}\rangle + \frac{i}{2} (D^2 + \gamma^+ \gamma) |Y_{2s-2}\rangle. \quad (7.9b)$$

These field strengths are gauge invariant, since they are proportional to the equations of motion for the massless model \mathcal{L}_s . The field strengths $|E_{2s}\rangle$ and $|Q_{2s-2}\rangle$ obey the Bianchi identity

$$\gamma |E_{2s}\rangle = \gamma^+ |Q_{2s-2}\rangle, \quad (7.10)$$

which expresses the gauge invariance of the massless action. Therefore the field strength $|E_{2s}\rangle$ is transverse provided the equation of motion (7.8b) holds, that is

$$|Q_{2s-2}\rangle = 0 \implies \gamma |E_{2s}\rangle = 0. \quad (7.11)$$

We point out that (7.8b) is the equation of motion for the compensator $|Y_{2s-2}\rangle$.

One can consider the half-integer superspin case in a similar way. The corresponding equations of motion are

$$|E_{2s+1}\rangle + (-1)^s(2s+1)\lambda|W_{2s+1}\rangle = 0, \quad (7.12a)$$

$$|R_{2s-2}\rangle = 0, \quad (7.12b)$$

where we have introduced the gauge-invariant field strengths:

$$|E_{2s+1}\rangle = \frac{1}{4}(2N\Box + iK_1D^2 - \gamma^{+2}\gamma^2)|H_{2s+1}\rangle + \frac{1}{2}\gamma^{+3}|X_{2s-2}\rangle, \quad (7.13a)$$

$$|R_{2s-2}\rangle = -\frac{i}{2}\gamma^3|H_{2s+1}\rangle - i((N+2)D^2 - \gamma^+\gamma)|X_{2s-2}\rangle. \quad (7.13b)$$

The Bianchi identity relating the field strengths (7.13a) and (7.13b) reads

$$\gamma|E_{2s+1}\rangle = -\frac{i}{2}\gamma^{+2}|R_{2s-2}\rangle. \quad (7.14)$$

Therefore the field strength $|E_{2s+1}\rangle$ satisfies the equation (7.11) provided $|R_{2s-2}\rangle = 0$.

Now we are in a position to analyse equations of motions for massive supermultiplets. We start with gravitino and supergravity multiplets and then generalise the analysis for an arbitrary superspin.

7.3. Massive gravitino multiplet

For massive gravitino multiplet one has the following expressions for the fields strength $|E_2\rangle$ and $|Q_0\rangle$:

$$|E_2\rangle = \frac{i}{2}(D^2 - iK_1)|H_2\rangle + \frac{i}{2}\gamma^{+2}|Y_0\rangle, \quad (7.15a)$$

$$|Q_0\rangle = -\frac{i}{2}\gamma^2|H_2\rangle + \frac{i}{2}D^2|Y_0\rangle. \quad (7.15b)$$

Taking the linear combination

$$|W_2\rangle := \frac{i}{4}(D^2|E_2\rangle - \gamma^{+2}|Q_0\rangle) \quad (7.16)$$

and using the expressions (B.13) and (B.14) one obtains

$$|W_2\rangle = \frac{1}{8}(K_2 + 2\Box + iK_1D^2)|H_2\rangle, \quad (7.17)$$

which is the same as the superconformal field strength (7.1) for $n = 2$.

Let us analyse the equations of motion. As mentioned above, the equation of motion (7.8b) for the compensator is $|Q_0\rangle = 0$. Furthermore, the equation of motion for the field strength $|E_2\rangle$ is

$$|E_2\rangle + \frac{i}{2}\lambda D^2|E_2\rangle = 0 \quad (7.18)$$

and, due to the Bianchi identity (7.10), the field strength satisfies $\gamma|E_2\rangle = 0$. The equation (7.18) in turn implies

$$(\square - m^2)|E_2\rangle = 0, \quad m^2 = \frac{1}{\lambda^2}. \quad (7.19)$$

It is important to notice that since the Chern–Simons like mass term in (7.7) does not contain compensator superfields one can immediately recover two dually invariant formulations for the gravitino multiplet, alike those in the subsection 5.2. Explicitly the field strength has the form

$$W_{\alpha_1\alpha_2} = -\frac{1}{4}D^{\beta_1}D_{\alpha_1}D^{\beta_2}D_{\alpha_2}H_{\beta_1\beta_2} \quad (7.20)$$

and, therefore, the Lagrangian for the massive gravitino multiplet has the form

$$L_{\text{GM}} = -\frac{i}{2}\left\{H^{\alpha\beta}D^2H_{\alpha\beta} + D_\alpha H^{\alpha\beta}D_\gamma H^\gamma{}_\beta - 2D_\alpha H^{\alpha\beta}D_\beta X - D^\beta X D_\beta X\right\} \\ - \lambda H^{\alpha\beta}W_{\alpha\beta}, \quad (7.21)$$

while the dual Lagrangian is

$$L_{\text{GM}}^{(\text{dual})} = -\frac{i}{2}\left\{H^{\alpha\beta}D^2H_{\alpha\beta} + 2D_\alpha H^{\alpha\beta}D_\gamma H^\gamma{}_\beta + 2iW^\alpha D^\beta H_{\alpha\beta} - W^\alpha W_\alpha\right\} \\ - \lambda H^{\alpha\beta}W_{\alpha\beta}. \quad (7.22)$$

7.4. Massive supergravity multiplet

One can perform a similar procedure for the supergravity multiplet. The corresponding field strengths are

$$|E_3\rangle = \frac{1}{4}(6\square + iK_1D^2 - \gamma^{+2}\gamma^2)|H_3\rangle + \frac{1}{2}\gamma^{+3}|X_0\rangle, \quad (7.23a)$$

$$|R_0\rangle = -\frac{i}{2}\gamma^3|H_3\rangle - 2iD^2|X_0\rangle. \quad (7.23b)$$

Taking a linear combination

$$|W_3\rangle := \frac{1}{3!}(-iD^2|E_3\rangle + \gamma^{+3}|R_0\rangle), \quad (7.24)$$

one obtains

$$|W_3\rangle = -\frac{1}{3!\cdot 8}(K_3 + 6K_1\square + \frac{3i}{2}K_2D^2 + 3i\square D^2)|H_3\rangle, \quad (7.25)$$

which is the field strength (7.1) for $n = 3$. The equation of motion for the compensator is $|R_0\rangle = 0$, whereas the equation of motion with respect to $|H_3\rangle$ is

$$|E_3\rangle + \frac{i}{2}\lambda D^2|E_3\rangle = 0 \quad (7.26)$$

which in turn implies

$$(\square - m^2)|E_3\rangle = 0, \quad m^2 = \frac{1}{\lambda^2}. \quad (7.27)$$

Similar to the case of the gravitino multiplet, one can present two dual formulations for the massive supergravity multiplet. Since the linearised super-Cotton tensor is independent of the compensator,

$$W_{\alpha_1\alpha_2\alpha_3} = \frac{i}{8} D^{\beta_1} D_{\alpha_1} D^{\beta_2} D_{\alpha_2} D^{\beta_3} D_{\alpha_3} H_{\beta_1\beta_2\beta_3} , \quad (7.28)$$

one can write the Lagrangian for the massive supergravity multiplet

$$\begin{aligned} L_{\text{SGM}} = & \frac{i}{4} H^{\alpha\beta\gamma} \square H_{\alpha\beta\gamma} - \frac{1}{8} H^{\alpha\beta\gamma} \partial_{\gamma\rho} D^2 H_{\alpha\beta}{}^{\rho} - \frac{i}{4} \partial_{\alpha\beta} H^{\alpha\beta\gamma} \partial^{\rho\sigma} H_{\rho\sigma\gamma} \\ & + \frac{1}{2} \partial_{\alpha\beta} H^{\alpha\beta\gamma} D_{\gamma} X + \frac{i}{2} D^{\gamma} X D_{\gamma} X + i\lambda H^{\alpha\beta\gamma} W_{\alpha\beta\gamma} , \end{aligned} \quad (7.29)$$

as well as its dual form

$$\begin{aligned} L_{\text{SGM}}^{(\text{dual})} = & \frac{i}{4} H^{\alpha\beta\gamma} \square H_{\alpha\beta\gamma} - \frac{1}{8} H^{\alpha\beta\gamma} \partial_{\gamma\rho} D^2 H_{\alpha\beta}{}^{\rho} - \frac{i}{8} \partial_{\alpha\beta} H^{\alpha\beta\gamma} \partial^{\rho\sigma} H_{\rho\sigma\gamma} \\ & + \frac{i}{2} \partial_{\alpha\beta} H^{\alpha\beta\gamma} W_{\gamma} + \frac{i}{2} W^{\alpha} W_{\alpha} + i\lambda H^{\alpha\beta\gamma} W_{\alpha\beta\gamma} . \end{aligned} \quad (7.30)$$

7.5. Arbitrary superspin

In order to analyse the case of an arbitrary integer superspin, let us consider the gauge-invariant action for $n = 2s$

$$S_{\text{massive}}^{(s)} = \int d^3z \left\{ \mathcal{L}_s(H_{\alpha(2s)}, Y_{\alpha(2s-2)}) + (-1)^s \lambda H^{\alpha(2s)} W_{\alpha(2s)}(H) \right\} , \quad (7.31)$$

where the Lagrangian \mathcal{L}_s is given by (6.14). Let us analyse the equations of motion. The gauge invariance (6.15) allows us to choose a gauge $Y_{\alpha(2s-2)} = 0$ in which the equation of motion for $Y_{\alpha(2s-2)}$ amounts to $\partial^{\beta\gamma} H_{\beta\gamma\alpha(2s-2)} = 0$, and the residual gauge freedom is constrained by $D^{\beta} \zeta_{\beta\alpha(s2-2)} = 0$. On the mass shell, the residual gauge freedom can be used to impose a stronger condition on the gauge prepotential,

$$D^{\beta} H_{\beta\alpha(2s-1)} = 0 . \quad (7.32)$$

In this gauge the super-Cotton tensor becomes

$$W_{\alpha(2s)} = \square^s H_{\alpha(2s)} , \quad (7.33)$$

in accordance with (3.37). Under the same gauge condition, the equation of motion for $H_{\alpha(2s)}$ reduces to

$$\frac{i}{2} D^2 H_{\alpha(2s)} + \lambda W_{\alpha(2s)} = 0 . \quad (7.34)$$

Combining the equations (7.33) and (7.34) gives

$$\square \left(\square^{2s-1} - \lambda^{-2} \right) H_{\alpha(2s)} = 0 . \quad (7.35)$$

If we choose the solution $\square H_{\alpha(2s)} = 0$, the super-Cotton tensor vanishes, $W_{\alpha(2s)} = 0$, in accordance with (7.33). Then the equations of motion reduce to the massless ones, which means the gauge field can be completely gauged away. Thus the nontrivial solutions obey the equations

$$\left(\square^{2s-1} - \lambda^{-2} \right) H_{\alpha(2s)} = 0 , \quad (7.36)$$

which implies⁷

⁷ Compare with a similar analysis in the $\mathcal{N} = 2$ case [24].

$$\left(\square - m^2\right)H_{\alpha(2s)} = 0, \quad m = \frac{1}{|\lambda|^{1/(2s-1)}}. \quad (7.37)$$

The equations (7.33) and (7.34) also imply that $W_{\alpha(2s)}$ is an on-shell massive superfield in the sense of (2.14),

$$\left(\frac{i}{2}D^2 + m\sigma\right)W_{\alpha(2s)} = 0, \quad \sigma = (-1)^s \frac{\lambda}{|\lambda|}, \quad (7.38)$$

and hence the superhelicity of $W_{\alpha(2s)}$ is $\kappa = \left(s + \frac{1}{4}\right)\sigma$.

It is instructive to repeat the above analysis by making use of the oscillator realisation. The relation between the field strength (7.1) and $|E_{2s}\rangle$ is

$$|W_{2s}\rangle = \left(\sum_{p=0}^{s-1} c_p \square^p K_{2s-2p-1}\right) |E_{2s}\rangle, \quad (7.39)$$

where the coefficients c_p are related to the coefficients b_p given in (7.4) as $c_p = \frac{1}{s}b_p$. After gauging away the compensator $|Y_{2s-2}\rangle$, imposing the corresponding equation of motion $|Q_{2s-2}\rangle = 0$ and the condition (7.32) on the gauge potential

$$\gamma |H_{2s}\rangle = 0, \quad (7.40)$$

one obtains

$$|W_{2s}\rangle = \square^s |H_{2s}\rangle \quad (7.41)$$

and

$$\left(\square - \frac{2}{|\lambda|^{1/(N-1)}}\right) |H_{2s}\rangle = 0, \quad (7.42)$$

with the latter being the same equation as (7.36). The equation (7.42) implies that the field strength $|E_{2s}\rangle$ satisfies the same Klein–Gordon equation

$$\left(\square - \frac{2}{|\lambda|^{1/(N-1)}}\right) |E_{2s}\rangle = 0. \quad (7.43)$$

The analysis of the equations for the half-integer superspin case simplifies due to an observation that there is a transformation that connects the systems of field equations for integer and half-integer superspins. Again we are considering a partially gauge fixed system when the compensators $|X_{2s-2}\rangle$ and $|Y_{2s-2}\rangle$ are gauged away. Then one can check that the transformation

$$|H_{2s+1}\rangle = \xi P^+ |H_{2s}\rangle, \quad (7.44)$$

where ξ is some Grassmann even constant parameter, transforms the solutions of the equations (7.8) into solutions of the system (7.12) in the limit of zero mass (i.e., when $\lambda \rightarrow 0$). Moreover, defining the operators E_{2s+1} and E_{2s} as

$$E_{2s+1} |H_{2s+1}\rangle = |E_{2s+1}\rangle, \quad E_{2s} |H_{2s}\rangle = |E_{2s}\rangle, \quad (7.45)$$

one has the following chain of equations:

$$E_{2s+1} |H_{2s+1}\rangle = \xi E_{2s+1} P^+ |H_{2s}\rangle = -\frac{1}{4} \xi P^+ \gamma^{+2} \gamma^2 |H_{2s}\rangle = -\frac{i}{2} \xi P^+ \gamma^{+2} |Q_{2s-2}\rangle. \quad (7.46)$$

Using the Bianchi identity (7.10) one finally gets

$$E_{2s+1}|H_{2s+1}\rangle = |E_{2s+1}\rangle = -\frac{i}{2}\xi P^+\gamma^+\gamma|E_{2s}\rangle. \quad (7.47)$$

After establishing this connection between field strengths for integer and half-integer superspins one can multiply the equation (7.43) with the operator $P^+\gamma^+\gamma$, to obtain the result that the field strength $|E_{2s+1}\rangle$ satisfies the Klein–Gordon equation

$$\left(\square - \frac{2}{|\lambda|^{1/(N-2)}}\right)|E_{2s+1}\rangle = 0, \quad (7.48)$$

as was the case of integer superspins. From the equations (7.43) and (7.48), one can see that the fields with integer and half-integer superspins have the same mass, as a result of the original $\mathcal{N} = 2$ supersymmetry. Let us note however, that as soon as one considers $\mathcal{N} = 1$ supersymmetry the parameter λ does not have to be the same for integer and half-integer superspins. Moreover for the case of free fields, which is our concern in the present paper, the parameter λ can be different from each separate value of a superspin, either it is integer or half-integer.

8. Discussion

In this paper we constructed the off-shell higher spin $\mathcal{N} = 1$ supermultiplets in three dimensions, both in the massless and massive cases. Our massive actions are actually defined for arbitrary non-zero superspin. They are labelled by a positive integer, $n = 1, 2, \dots$, and have the form

$$S_{\text{massive}}^{(n/2)} = \int d^3z \left\{ \mathcal{L}_{n/2}(H_{\alpha(n)}, \mathcal{X}_{\alpha(2\lfloor n/2 \rfloor - 2)}) + i^n \lambda H^{\alpha_1 \dots \alpha_n} W_{\alpha_1 \dots \alpha_n}(H) \right\}. \quad (8.1)$$

Here the compensator $\mathcal{X}_{\alpha(2\lfloor n/2 \rfloor - 2)}$ is not present in the case $n = 1$, which corresponds to the topologically massive vector multiplet. In section 6, the compensator $\mathcal{X}_{\alpha(2\lfloor n/2 \rfloor - 2)}$ was denoted $Y_{\alpha(2s-2)}$ for even $n = 2s$, and $X_{\alpha(2s+1)}$ for odd $n = 2s + 1$. The cases $n = 2$ and $n = 3$ correspond to the topologically massive gravitino and supergravity multiplets, respectively. The action (8.1) is gauge invariant. It may be shown that the massless actions

$$S_{\text{massless}}^{(n/2)} = \int d^3z \mathcal{L}_{n/2}(H_{\alpha(n)}, \mathcal{X}_{\alpha(2\lfloor n/2 \rfloor - 2)}) \quad (8.2)$$

do not describe any propagating degrees of freedom for $n > 1$. Nontrivial dynamics in the massive case is due to the presence of the Chern–Simons like term (8.1).

In section 5, we constructed two dual formulations for the massless gravitino multiplet and for the linearised supergravity multiplet. Deforming the dual massless actions by Chern–Simons like mass terms according to (8.1), we end up with two dual formulations for the corresponding massive multiplets. At the nonlinear level, only one off-shell formulation for $\mathcal{N} = 1$ supergravity has been constructed so far, and its conformal compensator is a scalar superfield, see [10] for a review. The fact that we now have two different off-shell actions for linearised supergravity is intriguing, and it may imply the existence of a new off-shell supergravity formulation.

Our massive supermultiplets can be coupled to external sources $\mathcal{J}_{\alpha(n)}$ using an action functional of the form

$$S_{\text{massive}}^{(n)}[H_{\alpha(n)}, \mathcal{X}_{\alpha(2\lfloor n/2 \rfloor - 2)}] + i^n \int d^3z H^{\alpha_1 \dots \alpha_n} \mathcal{J}_{\alpha_1 \dots \alpha_n}. \quad (8.3)$$

In order for such an action to be invariant under the gauge transformations (6.15a) and (6.15b) for $n = 2s$ or under the gauge transformations (6.28a) and (6.28b) for $n = 2s + 1$, the source must be conserved, that is

$$D^\beta \mathcal{J}_{\beta\alpha_1\ldots\alpha_{n-1}} = 0. \quad (8.4)$$

Such a superfield contains two ordinary conserved currents [47], which can be chosen as

$$j_{\alpha_1\ldots\alpha_n}(x) = \mathcal{J}_{\alpha_1\ldots\alpha_n} \Big|_{\theta=0}, \quad j_{\alpha_1\ldots\alpha_{n+1}}(x) = i^{n+1} D_{(\alpha_1} \mathcal{J}_{\alpha_2\ldots\alpha_{n+1})} \Big|_{\theta=0}. \quad (8.5)$$

It follows from (8.4) that

$$\partial^{\beta\gamma} j_{\beta\gamma\alpha_1\ldots\alpha_{n-2}} = 0, \quad \partial^{\beta\gamma} j_{\beta\gamma\alpha_1\ldots\alpha_{n-1}} = 0. \quad (8.6)$$

In 3D $\mathcal{N} = 1$ superconformal field theory, $\mathcal{J}_{\alpha\beta\gamma}$ describes the supercurrent multiplet, \mathcal{J}_α is present if the theory possesses a flavour symmetry, and $\mathcal{J}_{\alpha\beta}$ emerges if the theory possesses an extended supersymmetry, see [36] for more details.

In the $\mathcal{N} = 2$ supersymmetric case, the massive higher spin supermultiplets constructed in [24] are gauge theories with linearly dependent generators, following the terminology of the Batalin–Vilkovisky quantisation [48] (see [49] for a pedagogical review). The Lagrangian quantisation of such gauge theories is nontrivial.⁸ The remarkable feature of our 3D $\mathcal{N} = 1$ massive higher spin supermultiplets is that they are irreducible gauge theories that can be quantised using the standard Faddeev–Popov procedure.

Our construction of the massive higher spin supermultiplets may be viewed as a generalisation of the topologically massive vector multiplet model [3]

$$S_{\text{TMVM}} = -\frac{i}{2} \int d^3|2z W^\alpha W_\alpha - \frac{i}{2} m\sigma \int d^3|2z H^\alpha W_\alpha, \quad (8.7)$$

where $\sigma = \pm 1$. The equation of motion in this theory is

$$-\frac{i}{2} D^2 W_\alpha = m\sigma W_\alpha. \quad (8.8)$$

In conjunction with the Bianchi identity $D^\alpha W_\alpha = 0$, this amounts to (2.14) with $n = 1$. Similar to (8.7), our higher spin gauge theories describe irreducible massive supermultiplets that propagate a single superhelicity state. For low spin fields, however, there is a more traditional way to generate off-shell massive supermultiplets that are parity-invariant and, therefore, do not describe a single superhelicity. They are extensions of the massive vector multiplet model

$$S_{\text{MVM}} = -\frac{i}{2} \int d^3|2z (W^\alpha W_\alpha - m^2 H^\alpha H_\alpha), \quad (8.9)$$

in which the mass term involves the naked prepotential H_α squared such that the action is not gauge invariant. The equation of motion for this action is

$$0 = -\frac{i}{2} D^\beta D_\alpha W_\alpha + m^2 H_\alpha = \partial_\alpha{}^\beta W_\beta + m^2 H_\alpha, \quad (8.10)$$

which implies

⁸ This is similar to the off-shell 4D $\mathcal{N} = 1$ massless higher spin supermultiplets [25,26], which are also reducible gauge theories. For the off-shell $\mathcal{N} = 1$ massless higher spin supermultiplets in AdS₄ [50], the Lagrangian quantisation was carried out in [51].

$$D^\alpha H_\alpha = 0, \quad (\square - m^2)H_\alpha = 0. \quad (8.11)$$

Due to the identity

$$(\square - m^2) = \left(\frac{i}{2}D^2 + m\right)\left(\frac{i}{2}D^2 - m\right), \quad (8.12)$$

it follows that the theory propagates two irreducible on-shell multiplets with superhelicity values $\kappa = \pm 3/4$, compare with (2.18). In the case of 4D $\mathcal{N} = 1$ Poincaré supersymmetry, there have appeared various off-shell realisations for the massive gravitino and supergravity multiplets [52–57] that, conceptually, are similar to (8.9). Analogous massive models without gauge invariance may be constructed in the 3D $\mathcal{N} = 1$ case as well. An interesting point is that our *off-shell* massless 3D $\mathcal{N} = 1$ higher spin supermultiplets appear to be suitable to lift the component *on-shell* massive gauge-invariant construction of [21] to superspace.

A topic of our particular interest is an application of our results to the systems of interacting higher spin fields on AdS backgrounds first developed in [58–61], which have received much interest in the last years. In relation to higher spin gauge theories our results can be a step towards a few further developments which we hope do address in future work. We conclude by listing possible future lines of work:

- The massive higher spin supermultiplets constructed in this paper can be extended to 3D $\mathcal{N} = 1$ AdS superspace⁹ $\text{AdS}^{3|2}$ (defined, e.g., in [62]). It would be interesting to quantise the gauge-invariant massive higher spin theories in $\text{AdS}^{3|2}$ and to compute the corresponding partition functions.
- It would be interesting to construct a BRST formulation for these systems both on flat and AdS_3 backgrounds. We would like to mention that in this respect, in terms of their structure, the field equations for the integer and half-integer 3D $\mathcal{N} = 1$ higher spin supermultiplets are very similar to so-called triplet formulations for massless [63–69] and massive [70–72] reducible higher spin fields, usually formulated in terms of the BRST formalism (see also [76–78] for a recent work on BRST-FV approach for massive and massless higher spin fields). Indeed, in both cases the Lagrangian system of equations contains a physical field and two auxiliary fields. One of these fields is eliminated via its own equation of motion, whereas the other one can be gauged away in a complete analogy with the higher super-spin systems constructed in the present paper. On the other hand, since in the present paper we deal with irreducible higher spin supermultiplets, it would be also interesting to find a connection with the BRST formulation [73–75] for irreducible higher spin models as well.
- It would be of particular interest to consider cubic and possibly higher order Lagrangians on AdS_3 in the “metric-like” approach following the lines of [79] (see also [80–88] for cubic interactions of higher spin fields on AdS background). One can investigate also a possibility of constructing cubic and higher order Lagrangians for analogous systems with 3D $\mathcal{N} = 2$ supersymmetry which in principle can be more restrictive on the level of interactions comparing to $\mathcal{N} = 1$ supersymmetry considered here.
- Vasiliev’s higher spin gauge theory [58,59] was extended to superspace [89,90], although no analysis appeared as to whether this approach reproduces the off-shell higher spin supermultiplets in AdS_4 [50] at the linearised level. Studying such issues in the 3D case seems to be less involved than in four dimensions.

⁹ To appear soon.

- There has been much interest to higher spin (super)conformal field theories in three dimensions [24,37,40,91–95]. We hope our results will be useful for formulating interacting higher spin superconformal theories.
- Higher spin gauge fields possess interesting patterns of duality, both in the bosonic (see [96,97] and references therein) and supersymmetric cases [37]. It would be interesting to continue studying the duality aspects of higher spin gauge fields.

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Appendix A. Some useful identities

Our 3D notation and conventions correspond to those introduced in [30,10]. In particular, the spinor indices are raised and lowered using the $SL(2, \mathbb{R})$ invariant tensors

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon^{\alpha\gamma} \varepsilon_{\gamma\beta} = \delta_{\beta}^{\alpha} \quad (\text{A.1})$$

using the standard rule:

$$\theta^{\alpha} = \varepsilon^{\alpha\beta} \theta_{\beta}, \quad \theta_{\alpha} = \varepsilon_{\alpha\beta} \theta^{\beta}. \quad (\text{A.2})$$

The spinor covariant derivative of $\mathcal{N} = 1$ Minkowski superspace is

$$D_{\alpha} = \frac{\partial}{\partial \theta^{\alpha}} + i(\gamma^m)_{\alpha\beta} \theta^{\beta} \partial_m = \frac{\partial}{\partial \theta^{\alpha}} + i\theta^{\beta} \partial_{\beta\alpha}, \quad (\text{A.3})$$

and obeys the anti-commutation relation

$$\{D_{\alpha}, D_{\beta}\} = 2i\partial_{\alpha\beta}. \quad (\text{A.4})$$

As a result of (A.4) we have the identities

$$D_{\alpha} D_{\beta} = i\partial_{\alpha\beta} + \frac{1}{2} \varepsilon_{\alpha\beta} D^2, \quad (\text{A.5a})$$

$$D^{\beta} D_{\alpha} D_{\beta} = 0, \quad (\text{A.5b})$$

$$D^2 D_{\alpha} = -D_{\alpha} D^2 = 2i\partial_{\alpha\beta} D^{\beta}, \quad (\text{A.5c})$$

$$D^2 D^2 = -4\Box, \quad (\text{A.5d})$$

where $D^2 = D^{\alpha} D_{\alpha}$ and $\Box = \partial^a \partial_a = -\frac{1}{2} \partial^{\alpha\beta} \partial_{\alpha\beta}$. An important corollary of (A.5b) is

$$[D_{\alpha} D_{\beta}, D_{\gamma} D_{\delta}] = 0. \quad (\text{A.6})$$

As compared with the supersymmetry in four dimensions, the spinor covariant derivative possesses unusual conjugation properties. Specifically, given an arbitrary superfield F and $\bar{F} := (F)^*$ its complex conjugate, the following relation holds

$$(D_\alpha F)^* = -(-1)^{\epsilon(F)} D_\alpha \bar{F} , \quad (\text{A.7})$$

where $\epsilon(F)$ denotes the Grassmann parity of F .

The supersymmetry generator is

$$Q_\alpha = i \frac{\partial}{\partial \theta^\alpha} + (\gamma^m)_{\alpha\beta} \theta^\beta \partial_m = i \frac{\partial}{\partial \theta^\alpha} + \theta^\beta \partial_{\beta\alpha} . \quad (\text{A.8})$$

It anti-commutes with the spinor covariant derivative

$$\{Q_\alpha, D_\beta\} = 0 . \quad (\text{A.9})$$

Appendix B. Some useful identities for the operators

The operators introduced in the subsection 6.1 obey some useful relations

$$\{\gamma, \gamma^+\} = 2iK_1 - D^2 , \quad \gamma^+ \gamma = iK_1 + \frac{1}{2}ND^2 , \quad (\text{B.1})$$

$$P^+ K_1 = -i\gamma^{+2}P + \gamma^+ N \square , \quad K_1 P = -iP^+ \gamma^2 - N\gamma \square , \quad (\text{B.2})$$

$$K_1 \gamma = -i\gamma^+ \gamma^2 - NP , \quad \gamma^+ K_1 = -i\gamma^{+2} \gamma + P^+ N , \quad (\text{B.3})$$

$$[K_1, P] = \square \gamma , \quad [K_1, P^+] = \gamma^+ \square , \quad (\text{B.4})$$

$$PD^2 = -2i\square \gamma , \quad D^2 P = 2i\square \gamma , \quad (\text{B.5})$$

$$P^+ D^2 = -2i\gamma^+ \square , \quad D^2 P^+ = 2i\gamma^+ \square , \quad (\text{B.6})$$

$$[\gamma, \gamma^{+2}] = -2iP^+ , \quad [\gamma^2, \gamma^+] = 2iP , \quad (\text{B.7})$$

$$\gamma D^2 = -2iP , \quad D^2 \gamma = 2iP , \quad (\text{B.8})$$

$$\gamma^+ D^2 = -2iP^+ , \quad D^2 \gamma^+ = 2iP^+ , \quad (\text{B.9})$$

$$[N, \gamma^+] = \gamma^+ , \quad [N, P^+] = P^+ , \quad (\text{B.10})$$

$$[N, \gamma] = -\gamma , \quad [N, P] = -P , \quad (\text{B.11})$$

$$[N, \square] = [N, D^2] = 0 , \quad [N, K_l] = 0 . \quad (\text{B.12})$$

One has also the identity

$$\gamma^{+2} \gamma^2 = -K_2 + N(N-1)\square , \quad (\text{B.13})$$

as well as “reduction” rules for the operators K_1 , K_2 and K_l

$$K_1 K_l = K_{l+1} + lK_{l-1}(N-(l-1))\square , \quad (\text{B.14})$$

$$K_2 K_l = K_{l+2} + 2lK_l(N-l)\square + l(l-1)K_{l-2}(N-(l-1))(N-(l-2))\square^2 ,$$

where $K_0 = 1$.

References

- [1] P.S. Howe, R.W. Tucker, Local supersymmetry in $(2+1)$ dimensions. 1. Supergravity and differential forms, J. Math. Phys. 19 (1978) 869;
P.S. Howe, R.W. Tucker, Local supersymmetry in $(2+1)$ dimensions. 2. An action for a spinning membrane, J. Math. Phys. 19 (1978) 981.
- [2] M. Brown, S.J. Gates Jr., Superspace Bianchi identities and the supercovariant derivative, Ann. Phys. 122 (1979) 443.

- [3] W. Siegel, Unextended superfields in extended supersymmetry, Nucl. Phys. B 156 (1979) 135.
- [4] S.J. Gates Jr., M.T. Grisaru, M. Roček, W. Siegel, Superspace, or One Thousand and One Lessons in Supersymmetry, Benjamin/Cummings, Reading, MA, 1983, arXiv:hep-th/0108200.
- [5] S. Deser, J.H. Kay, Topologically massive supergravity, Phys. Lett. B 120 (1983) 97.
- [6] S. Deser, Cosmological topological supergravity, in: S.M. Christensen (Ed.), Quantum Theory of Gravity, Adam Hilger, Bristol, 1984, pp. 374–381.
- [7] P. van Nieuwenhuizen, $D = 3$ conformal supergravity and Chern–Simons terms, Phys. Rev. D 32 (1985) 872.
- [8] T. Uematsu, Structure of $N = 1$ conformal and Poincaré supergravity in $(1 + 1)$ -dimensions and $(2 + 1)$ -dimensions, Z. Phys. C 29 (1985) 143;
T. Uematsu, Constraints and actions in two-dimensional and three-dimensional $N = 1$ conformal supergravity, Z. Phys. C 32 (1986) 33.
- [9] P.S. Howe, J.M. Izquierdo, G. Papadopoulos, P.K. Townsend, New supergravities with central charges and Killing spinors in $2 + 1$ dimensions, Nucl. Phys. B 467 (1996) 183, arXiv:hep-th/9505032.
- [10] S.M. Kuzenko, U. Lindström, G. Tartaglino-Mazzucchelli, Off-shell supergravity-matter couplings in three dimensions, J. High Energy Phys. 1103 (2011) 120, arXiv:1101.4013 [hep-th].
- [11] S.M. Kuzenko, G. Tartaglino-Mazzucchelli, Conformal supergravities as Chern–Simons theories revisited, J. High Energy Phys. 1303 (2013) 113, arXiv:1212.6852 [hep-th].
- [12] D. Butter, S.M. Kuzenko, J. Novak, G. Tartaglino-Mazzucchelli, Conformal supergravity in three dimensions: new off-shell formulation, J. High Energy Phys. 1309 (2013) 072, arXiv:1305.3132 [hep-th];
D. Butter, S.M. Kuzenko, J. Novak, G. Tartaglino-Mazzucchelli, Conformal supergravity in three dimensions: off-shell actions, J. High Energy Phys. 1310 (2013) 073, arXiv:1306.1205 [hep-th].
- [13] M. Becker, D. Constantin, S.J. Gates Jr., W.D. Linch III, W. Merrell, J. Phillips, M-theory on Spin(7) manifolds, fluxes and 3D, $N = 1$ supergravity, Nucl. Phys. B 683 (2004) 67, arXiv:hep-th/0312040.
- [14] R. Andringa, E.A. Bergshoeff, M. de Roo, O. Hohm, E. Sezgin, P.K. Townsend, Massive 3D supergravity, Class. Quantum Gravity 27 (2010) 025010, arXiv:0907.4658 [hep-th].
- [15] E.A. Bergshoeff, O. Hohm, J. Rosseel, E. Sezgin, P.K. Townsend, More on massive 3D supergravity, Class. Quantum Gravity 28 (2011) 015002, arXiv:1005.3952 [hep-th].
- [16] E. Bergshoeff, M. Ozkan, 3D Born–Infeld gravity and supersymmetry, J. High Energy Phys. 1408 (2014) 149, arXiv:1405.6212 [hep-th].
- [17] S.M. Kuzenko, J. Novak, G. Tartaglino-Mazzucchelli, Higher derivative couplings and massive supergravity in three dimensions, J. High Energy Phys. 1509 (2015) 081, arXiv:1506.09063 [hep-th].
- [18] E.A. Bergshoeff, O. Hohm, P.K. Townsend, Massive gravity in three dimensions, Phys. Rev. Lett. 102 (2009) 201301, arXiv:0901.1766 [hep-th].
- [19] E.A. Bergshoeff, O. Hohm, P.K. Townsend, More on massive 3D gravity, Phys. Rev. D 79 (2009) 124042, arXiv:0905.1259 [hep-th].
- [20] E.A. Bergshoeff, O. Hohm, P.K. Townsend, On higher derivatives in 3D gravity and higher spin gauge theories, Ann. Phys. 325 (2010) 1118, arXiv:0911.3061 [hep-th].
- [21] I.L. Buchbinder, T.V. Snegirev, Y.M. Zinoviev, Lagrangian formulation of the massive higher spin supermultiplets in three dimensional space–time, J. High Energy Phys. 1510 (2015) 148, arXiv:1508.02829 [hep-th].
- [22] I.L. Buchbinder, T.V. Snegirev, Y.M. Zinoviev, Unfolded equations for massive higher spin supermultiplets in AdS_3 , J. High Energy Phys. 1608 (2016) 075, arXiv:1606.02475 [hep-th].
- [23] Y.M. Zinoviev, On massive high spin particles in AdS, arXiv:hep-th/0108192.
- [24] S.M. Kuzenko, D.X. Ogburn, Off-shell higher spin supermultiplets in three dimensions, arXiv:1603.04668 [hep-th].
- [25] S.M. Kuzenko, V.V. Postnikov, A.G. Sibiryakov, Massless gauge superfields of higher half-integer superspins, JETP Lett. 57 (1993) 534, Pisma Zh. Eksp. Teor. Fiz. 57 (1993) 521.
- [26] S.M. Kuzenko, A.G. Sibiryakov, Massless gauge superfields of higher integer superspins, JETP Lett. 57 (1993) 539, Pisma Zh. Eksp. Teor. Fiz. 57 (1993) 526.
- [27] I.L. Buchbinder, S.M. Kuzenko, Ideas and Methods of Supersymmetry and Supergravity: Or a Walk Through Superspace, IOP, Bristol, 1995 (revised edition: 1998), 656p.
- [28] C. Fronsdal, Massless fields with integer spin, Phys. Rev. D 18 (1978) 3624.
- [29] J. Fang, C. Fronsdal, Massless fields with half-integral spin, Phys. Rev. D 18 (1978) 3630.
- [30] S.M. Kuzenko, J.-H. Park, G. Tartaglino-Mazzucchelli, R. Unge, Off-shell superconformal nonlinear sigma-models in three dimensions, J. High Energy Phys. 1101 (2011) 146, arXiv:1011.5727 [hep-th].
- [31] I.V. Tyutin, M.A. Vasiliev, Lagrangian formulation of irreducible massive fields of arbitrary spin in $(2 + 1)$ dimensions, Teor. Mat. Fiz. 113 (1) (1997) 45, Theor. Math. Phys. 113 (1997) 1244, arXiv:hep-th/9704132.
- [32] E.A. Bergshoeff, O. Hohm, P.K. Townsend, On higher derivatives in 3D gravity and higher spin gauge theories, Ann. Phys. 325 (2010) 1118, arXiv:0911.3061 [hep-th].

- [33] M. Fierz, W. Pauli, On relativistic wave equations for particles of arbitrary spin in an electromagnetic field, *Proc. R. Soc. Lond. A* 173 (1939) 211.
- [34] L. Mezincescu, P.K. Townsend, Quantum 3D superstrings, *Phys. Rev. D* 84 (2011) 106006, arXiv:1106.1374 [hep-th].
- [35] D.P. Sorokin, D.V. Volkov, (Anti)commuting spinors and supersymmetric dynamics of semions, *Nucl. Phys. B* 409 (1993) 547.
- [36] E.I. Buchbinder, S.M. Kuzenko, I.B. Samsonov, Superconformal field theory in three dimensions: correlation functions of conserved currents, *J. High Energy Phys.* 1506 (2015) 138, arXiv:1503.04961 [hep-th].
- [37] S.M. Kuzenko, Higher spin super-Cotton tensors and generalisations of the linear-chiral duality in three dimensions, *Phys. Lett. B* 763 (2016) 308, arXiv:1606.08624 [hep-th].
- [38] C. Closset, T.T. Dumitrescu, G. Festuccia, Z. Komargodski, N. Seiberg, Comments on Chern–Simons contact terms in three dimensions, *J. High Energy Phys.* 1209 (2012) 091, arXiv:1206.5218 [hep-th].
- [39] S.M. Kuzenko, Prepotentials for $N = 2$ conformal supergravity in three dimensions, *J. High Energy Phys.* 1212 (2012) 021, arXiv:1209.3894 [hep-th].
- [40] C.N. Pope, P.K. Townsend, Conformal higher spin in $(2 + 1)$ dimensions, *Phys. Lett. B* 225 (1989) 245.
- [41] S.M. Kuzenko, G. Tartaglino-Mazzucchelli, Three-dimensional $N = 2$ (AdS) supergravity and associated supercurrents, *J. High Energy Phys.* 1112 (2011) 052, arXiv:1109.0496 [hep-th].
- [42] S.J. Gates Jr., S.M. Kuzenko, J. Phillips, The off-shell $(3/2, 2)$ supermultiplets revisited, *Phys. Lett. B* 576 (2003) 97, arXiv:hep-th/0306288.
- [43] E. Ivanov, S. Krivonos, O. Lechtenfeld, Double vector multiplet and partially broken $N = 4$, $D = 3$ supersymmetry, *Phys. Lett. B* 487 (2000) 192, arXiv:hep-th/0006017.
- [44] N.J. Hitchin, A. Karlhede, U. Lindström, M. Roček, Hyperkähler metrics and supersymmetry, *Commun. Math. Phys.* 108 (1987) 535.
- [45] A. Fotopoulos, M. Tsulaia, Gauge invariant Lagrangians for free and interacting higher spin fields. A review of the BRST formulation, *Int. J. Mod. Phys. A* 24 (2009) 1, arXiv:0805.1346 [hep-th].
- [46] S.J. Gates Jr., S.M. Kuzenko, 4D, $N = 1$ higher spin gauge superfields and quantized twistors, *J. High Energy Phys.* 0510 (2005) 008, arXiv:hep-th/0506255.
- [47] A.A. Nizami, T. Sharma, V. Umesh, Superspace formulation and correlation functions of 3d superconformal field theories, *J. High Energy Phys.* 1407 (2014) 022, arXiv:1308.4778 [hep-th].
- [48] I.A. Batalin, G.A. Vilkovisky, Quantization of gauge theories with linearly dependent generators, *Phys. Rev. D* 28 (1983) 2567.
- [49] M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems*, Univ. Pr., Princeton, USA, 1992, p. 520.
- [50] S.M. Kuzenko, A.G. Sibiryakov, Free massless higher-superspin superfields on the anti-de Sitter superspace, *Phys. At. Nucl.* 57 (1994) 1257, *Yad. Fiz.* 57 (1994) 1326, arXiv:1112.4612 [hep-th].
- [51] I.L. Buchbinder, S.M. Kuzenko, A.G. Sibiryakov, Quantization of higher spin superfields in the anti-de Sitter superspace, *Phys. Lett. B* 352 (1995) 29, arXiv:hep-th/9502148.
- [52] V.I. Ogievetsky, E. Sokatchev, Superfield equations of motion, *J. Phys. A* 10 (1977) 2021.
- [53] I.L. Buchbinder, S.J. Gates Jr., W.D. Linch, J. Phillips, New 4D, $N = 1$ superfield theory: model of free massive superspin-3/2 multiplet, *Phys. Lett. B* 535 (2002) 280;
I.L. Buchbinder, S.J. Gates Jr., W.D. Linch, J. Phillips, Dynamical superfield theory of free massive superspin-1 multiplet, *Phys. Lett. B* 549 (2002) 229.
- [54] T. Gregoire, M.D. Schwartz, Y. Shadmi, Massive supergravity and deconstruction, *J. High Energy Phys.* 0407 (2004) 029.
- [55] I.L. Buchbinder, S.J. Gates Jr., S.M. Kuzenko, J. Phillips, Massive 4D, $N = 1$ superspin 1 and 3/2 multiplets and dualities, *J. High Energy Phys.* 0502 (2005) 056.
- [56] S.J. Gates Jr., S.M. Kuzenko, G. Tartaglino-Mazzucchelli, New massive supergravity multiplets, *J. High Energy Phys.* 0702 (2007) 052, arXiv:hep-th/0610333.
- [57] S.J. Gates Jr., K. Koutrolikos, A dynamical theory for linearized massive superspin 3/2, *J. High Energy Phys.* 1403 (2014) 030, arXiv:1310.7387 [hep-th].
- [58] M.A. Vasiliev, Nonlinear equations for symmetric massless higher spin fields in (A)dS(d), *Phys. Lett. B* 567 (2003) 139, arXiv:hep-th/0304049.
- [59] M.A. Vasiliev, Consistent equation for interacting gauge fields of all spins in $(3 + 1)$ -dimensions, *Phys. Lett. B* 243 (1990) 378.
- [60] E.S. Fradkin, M.A. Vasiliev, Cubic interaction in extended theories of massless higher spin fields, *Nucl. Phys. B* 291 (1987) 141.
- [61] S.F. Prokushkin, M.A. Vasiliev, Higher spin gauge interactions for massive matter fields in 3-D AdS space-time, *Nucl. Phys. B* 545 (1999) 385, arXiv:hep-th/9806236.

- [62] S.M. Kuzenko, U. Lindström, G. Tartaglino-Mazzucchelli, Three-dimensional (p, q) AdS superspaces and matter couplings, *J. High Energy Phys.* 1208 (2012) 024, arXiv:1205.4622 [hep-th].
- [63] S. Ouvry, J. Stern, Gauge fields of any spin and symmetry, *Phys. Lett. B* 177 (1986) 335.
- [64] D. Francia, A. Sagnotti, On the geometry of higher spin gauge fields, *Class. Quantum Gravity* 20 (2003) S473, *Comment. Phys. Math. Soc. Sci. Fenn.* 166 (2004) 165, PoS JHW 2003 (2003) 005, arXiv:hep-th/0212185.
- [65] A. Sagnotti, M. Tsulaia, On higher spins and the tensionless limit of string theory, *Nucl. Phys. B* 682 (2004) 83, arXiv:hep-th/0311257.
- [66] G. Barnich, N. Bouatta, M. Grigoriev, Surface charges and dynamical Killing tensors for higher spin gauge fields in constant curvature spaces, *J. High Energy Phys.* 0510 (2005) 010, arXiv:hep-th/0507138.
- [67] D.P. Sorokin, M.A. Vasiliev, Reducible higher-spin multiplets in flat and AdS spaces and their geometric frame-like formulation, *Nucl. Phys. B* 809 (2009) 110, arXiv:0807.0206 [hep-th].
- [68] A. Campoleoni, D. Francia, Maxwell-like Lagrangians for higher spins, *J. High Energy Phys.* 1303 (2013) 168, arXiv:1206.5877 [hep-th].
- [69] A. Agugliaro, F. Azzurli, D. Sorokin, Fermionic higher-spin triplets in AdS, *Nucl. Phys. B* 907 (2016) 633, arXiv:1603.02251 [hep-th].
- [70] F. Hussain, G. Thompson, P.D. Jarvis, Massive and massless gauge fields of any spin and symmetry, *Phys. Lett. B* 216 (1989) 139.
- [71] A. Pashnev, M.M. Tsulaia, Dimensional reduction and BRST approach to the description of a Regge trajectory, *Mod. Phys. Lett. A* 12 (1997) 861, arXiv:hep-th/9703010.
- [72] X. Bekaert, I.L. Buchbinder, A. Pashnev, M. Tsulaia, On higher spin theory: strings, BRST, dimensional reductions, *Class. Quantum Gravity* 21 (2004) S1457, arXiv:hep-th/0312252.
- [73] I.L. Buchbinder, A. Pashnev, M. Tsulaia, Lagrangian formulation of the massless higher integer spin fields in the AdS background, *Phys. Lett. B* 523 (2001) 338, arXiv:hep-th/0109067.
- [74] I.L. Buchbinder, A.V. Galajinsky, V.A. Krykhtin, Quartet unconstrained formulation for massless higher spin fields, *Nucl. Phys. B* 779 (2007) 155, arXiv:hep-th/0702161.
- [75] I.L. Buchbinder, V.A. Krykhtin, A.A. Reshetnyak, BRST approach to Lagrangian construction for fermionic higher spin fields in (A)dS space, *Nucl. Phys. B* 787 (2007) 211, arXiv:hep-th/0703049.
- [76] R.R. Metsaev, BRST invariant effective action of shadow fields, conformal fields, and AdS/CFT, *Theor. Math. Phys.* 181 (3) (2014) 1548, arXiv:1407.2601 [hep-th].
- [77] R.R. Metsaev, The BRST-BV approach to conformal fields, *J. Phys. A* 49 (17) (2016) 175401, arXiv:1511.01836 [hep-th].
- [78] R.R. Metsaev, BRST-BV approach to massless fields adapted to AdS/CFT correspondence, *Theor. Math. Phys.* 187 (2) (2016) 730, *Teor. Mat. Fiz.* 187 (2) (2016) 323, arXiv:1508.07928 [hep-th].
- [79] I.L. Buchbinder, A. Fotopoulos, A.C. Petkou, M. Tsulaia, Constructing the cubic interaction vertex of higher spin gauge fields, *Phys. Rev. D* 74 (2006) 105018, arXiv:hep-th/0609082.
- [80] N. Boulanger, E.D. Skvortsov, Y.M. Zinoviev, Gravitational cubic interactions for a simple mixed-symmetry gauge field in AdS and flat backgrounds, *J. Phys. A* 44 (2011) 415403, arXiv:1107.1872 [hep-th].
- [81] M.A. Vasiliev, Cubic vertices for symmetric higher-spin gauge fields in (A)dS_d, *Nucl. Phys. B* 862 (2012) 341, arXiv:1108.5921 [hep-th].
- [82] E. Joung, M. Taronna, Cubic interactions of massless higher spins in (A)dS: metric-like approach, *Nucl. Phys. B* 861 (2012) 145, arXiv:1110.5918 [hep-th].
- [83] E. Joung, L. Lopez, M. Taronna, On the cubic interactions of massive and partially-massless higher spins in (A)dS, *J. High Energy Phys.* 1207 (2012) 041, arXiv:1203.6578 [hep-th].
- [84] E. Joung, M. Taronna, A. Waldron, A calculus for higher spin interactions, *J. High Energy Phys.* 1307 (2013) 186, arXiv:1305.5809 [hep-th].
- [85] D. Polyakov, A string model for AdS gravity and higher spins, *Phys. Rev. D* 84 (2011) 126004, arXiv:1106.1558 [hep-th].
- [86] R. Manvelyan, R. Mkrtchyan, W. Ruehl, Radial reduction and cubic interaction for higher spins in (A)dS space, *Nucl. Phys. B* 872 (2013) 265, arXiv:1210.7227 [hep-th].
- [87] N. Boulanger, D. Ponomarev, E.D. Skvortsov, Non-abelian cubic vertices for higher-spin fields in anti-de Sitter space, *J. High Energy Phys.* 1305 (2013) 008, arXiv:1211.6979 [hep-th].
- [88] C. Sleight, M. Taronna, Higher spin interactions from conformal field theory: the complete cubic couplings, *Phys. Rev. Lett.* 116 (18) (2016) 181602, arXiv:1603.00022 [hep-th].
- [89] J. Engquist, E. Sezgin, P. Sundell, On $N = 1$, $N = 2$, $N = 4$ higher spin gauge theories in four-dimensions, *Class. Quantum Gravity* 19 (2002) 6175, arXiv:hep-th/0207101.
- [90] J. Engquist, E. Sezgin, P. Sundell, Superspace formulation of 4-D higher spin gauge theory, *Nucl. Phys. B* 664 (2003) 439, arXiv:hep-th/0211113.

- [91] E.S. Fradkin, V.Y. Linetsky, A superconformal theory of massless higher spin fields in $D = (2 + 1)$, *Ann. Phys.* 198 (1990) 293.
- [92] B.E.W. Nilsson, Towards an exact frame formulation of conformal higher spins in three dimensions, *J. High Energy Phys.* 1509 (2015) 078, [arXiv:1312.5883 \[hep-th\]](#).
- [93] B.E.W. Nilsson, On the conformal higher spin unfolded equation for a three-dimensional self-interacting scalar field, *J. High Energy Phys.* 1608 (2016) 142, [arXiv:1506.03328 \[hep-th\]](#).
- [94] M. Henneaux, S. Hörtner, A. Leonard, Higher spin conformal geometry in three dimensions and prepotentials for higher spin gauge fields, *J. High Energy Phys.* 1601 (2016) 073, [arXiv:1511.07389 \[hep-th\]](#).
- [95] H. Linander, B.E.W. Nilsson, The non-linear coupled spin 2-spin 3 Cotton equation in three dimensions, *J. High Energy Phys.* 1607 (2016) 024, [arXiv:1602.01682 \[hep-th\]](#).
- [96] K. Hinterbichler, A. Joyce, Manifest duality for partially massless higher spins, [arXiv:1608.04385 \[hep-th\]](#).
- [97] M. Henneaux, S. Hörtner, A. Leonard, Twisted self-duality for higher spin gauge fields and prepotentials, [arXiv:1609.04461 \[hep-th\]](#).