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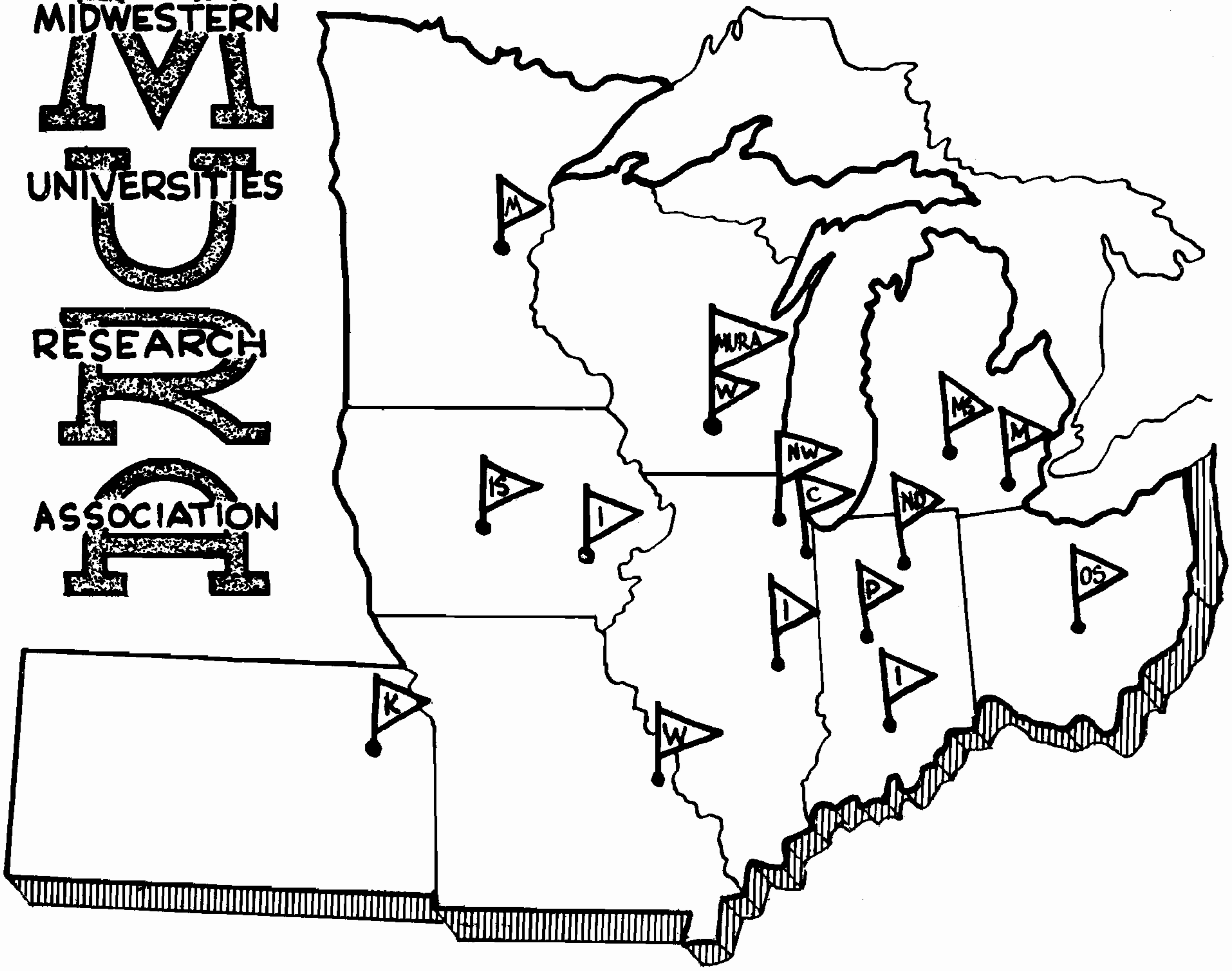

GENERAL FORMULAS FOR GAP-EXCITED LINEAR DUCTS

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ABSTRACT

Equations allowing determination of the electric and magnetic fields at each point within a gap-excited metallic duct are derived. With the application to accelerators in mind, formulas are established for the energy and lateral momentum kicks to which a particle is subjected upon crossing the gap region. The configuration where the gap is in a plane perpendicular to the duct's axis is given some special attention.

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I. INTRODUCTION

The purpose of the present investigation is to derive, in fairly general terms, formulas for the electromagnetic field in a metallic cylinder of arbitrary cross-section, cut in two by a gap of equally arbitrary contour. (See Figure 1) The field is excited by a voltage impressed across the lips of the gap, a voltage which varies continuously in phase and amplitude all along the gap contour. This variation, which depends on the external sources of excitation (cavities, two-wire lines, etc.--), is assumed to be given, and is the starting point of the present analysis. The problem is tackled with one well-defined application in mind: gap-excited accelerators, either linear, or so slightly curved that they can be considered as linear in the region surrounding the gap. In accordance with that point of view, the following restrictions to the generality of the approach will be made:

1. The operating frequency is less than the general cut-off frequency of the duct. As a result, fields are confined to the vicinity of the gap, and are not propagated. The tube will therefore be treated as an infinitely long wave-guide below cut-off.
2. The cross-section of the duct is simply-connected, and the walls are perfectly conducting.
3. The gap can be considered as infinitely narrow, its width being negligible with respect to wavelength and cross-sectional dimensions.
4. The gap contour, when developed, takes the aspect depicted in Figure 2, where c denotes a coordinate measured along the perimeter of the cross-section (See Figure 1). It is seen that the gap is allowed to run parallel

to the guide axis for a certain distance, but that no two of its points are otherwise permitted to lie on a parallel to that axis. In other words, the situation where the gap would "wind back" on itself is specifically excluded. Except for the segments parallel to the axis, therefore, z is a single-valued function of c.

II. FIELD CALCULATIONS

The fields within the duct are excited by the tangential component of the electric field in the gap. When the gap is very narrow, excitation takes place through the component of the field perpendicular to the gap.[1] If v and φ are respectively the magnitude and phase of the gap voltage, this component can be represented in δ function form by

$$\bar{e}_{tang}(c,t) = \delta(p)\, v(c) \cos\left[\omega t + \varphi(c)\right] \bar{u}_p \tag{1}$$

where $\bar{u}_p$ is the unit vector lying in the tangent plane, and perpendicular to the profile of the gap. It will be convenient to work with complex vectors, and to represent the tangential field in phasor form by

$$\bar{E}_{tang}(c) = \delta(p)\, V(c)\, \bar{u}_p \tag{2}$$

with $V(c) = v(c)\, e^{j\varphi(c)}$

The fields inside the wave-guide can be calculated by normal mode methods.[2] Adapted to the present situation, these consist of expanding $\bar{E}$ and $\bar{H}$ as*

$$\begin{aligned}\bar{E}(x,y,z) &= \sum_m a_m(z)\, \bar{\mathcal{E}}_{tm}(x,y) + \sum_n b_n(z)\, \bar{\mathcal{E}}_{tn}(x,y) + \sum_m c_m(z)\, \bar{\mathcal{E}}_{zm}(x,y)\\ \bar{H}(x,y,z) &= \sum_m \alpha_m(z)\, \bar{\mathcal{H}}_{tm}(x,y) + \sum_n \beta_n(z)\, \bar{\mathcal{H}}_{tn}(x,y) + \sum_n \gamma_n(z)\, \bar{\mathcal{H}}_{zn}(x,y)\end{aligned} \tag{3}$$

*See Footnote 1 on page 14

The unknowns are the complex expansion coefficients a_m, b_n, etc.; these satisfythe set of equations:

$$\left\{\begin{aligned} &\frac{d a_m}{d z} + j\omega\mu\,\alpha_m - c_m = -\frac{1}{N_m^2}\int_c (\bar{u}_n \times \bar{E}_t)\cdot\bar{\mathcal{H}}_{tm}\,dc = A_m(z) \\ &\frac{d\alpha_m}{dz} + j\omega\varepsilon\, a_m = 0 \\ &\alpha_m + \frac{j\omega\varepsilon}{k_m^2}\, c_m = 0 \end{aligned}\right.$$

$$\left\{\begin{aligned} &\frac{d\beta_n}{dz} - j\omega\varepsilon\, b_n - \gamma_n = 0 \\ &\frac{d b_n}{dz} - j\omega\mu\,\beta_n = \frac{1}{N_n^2}\int_c (\bar{u}_n \times \bar{E}_t)\cdot\bar{\mathcal{H}}_{tn}\,dc = B_n(z) \\ &b_n - \frac{j\omega\mu}{k_m^2}\,\gamma_n = \frac{1}{N_m^2}\int_c (\bar{u}_m \times \bar{E}_t)\cdot\bar{\mathcal{H}}_{zn}\,dc = C_m(z) \end{aligned}\right. \tag{4}$$

The integrals appearing in the right-hand members of the equations are known functions of z . Their actual value, which depends on the gap contour, gap voltage, etc., will ultimately determine the six unknown functions. The following general statements can be made about the behavior of A_m, B_n, and C_n. Let the developed curve consist of k arcs along which z varies monotonically with c (Figure 2). Then each integral consists of the sum of k contributions, one for each intersection of the gap with the cross-sectional contour.

a) If this intersection is an ordinary point such as $\underline{P}$, the contribution takes the following form:

$$A_m = \frac{1}{N_m^2}\left[V \cdot \frac{\partial \mathcal{E}_{zm}}{\partial n} \cdot \frac{1}{|tg\,\theta|}\right]_{at\,P} = \frac{1}{N_m^2}\left[V\ \frac{\partial \mathcal{E}_{zm}}{\partial n} \cdot \left|\frac{dc}{dz}\right|\right]_{at\,P}$$

$$B_n = -\frac{1}{N_m^2}\left[V \cdot \frac{\partial \mathcal{H}_{zn}}{\partial c} \cdot \frac{1}{|tg\,\theta|}\right]_{at\,P} = -\frac{1}{N_m^2}\left[V \cdot \frac{\partial \mathcal{H}_{zn}}{\partial c} \cdot \left|\frac{dc}{dz}\right|\right]_{at\,P}$$

$$C_m = -\frac{1}{N_m^2}\left[V\,\mathcal{H}_{zn}\right]_{at\,P} \quad \text{when } \theta \text{ is between } 0 \text{ and } \pi/2 \tag{5}$$

$$= +\frac{1}{N_m^2}\left[V\,\mathcal{H}_{zn}\right]_{at\,P} \quad \text{when } \theta \text{ is between } 0 \text{ and } -\pi/2$$

These formulas can also be applied for a point such as Q, where the gap runs parallel to the z axis, provided a value of $\pi/2$ (or $-\pi/2$) is substitued for θ.

b) At point B, where a change of slope from $tg\,\theta^-$ to $tg\,\theta^+$ is experienced, A_m and B_n are subjected to a jump in value equal to, respectively,

$$\frac{1}{N_m^2}\left[V\ \frac{\partial \mathcal{E}_{zm}}{\partial n}\right]_{at\,B} \cdot \left[\left|\frac{1}{tg\,\theta^+}\right| - \left|\frac{1}{tg\,\theta^-}\right|\right]$$

and

$$-\frac{1}{N_m^2}\left[V\ \frac{\partial \mathcal{H}_{zn}}{\partial c}\right]_{at\,B} \cdot \left[\left|\frac{1}{tg\,\theta^+}\right| - \left|\frac{1}{tg\,\theta^-}\right|\right]$$

C_n is continuous, but with a jump in its derivative (easily calculated from (5)).

c) At an axial distance z where the gap runs perpendicular to the axis for a while, such as between points E and F, the various integrals become:

$$A_m = \frac{1}{N_m^2}\,\delta(z - z_E)\int_E^F V(c)\,\frac{\partial \mathcal{E}_{zm}}{\partial n}\,dc \tag{6}$$

$$B_m = -\frac{1}{N_m^2}\,\delta(z - z_E)\int_E^F V(c)\ \frac{\partial \mathcal{H}_{zn}}{\partial c}\,dc$$

$$C_m = 0$$

d) At point C, where the developed curve has a horizontal tangent, and a radius of curvature R, A_m and B_n become infinite as

$$A_m = \frac{\sqrt{2R}}{N_m^2} \cdot \left[V \frac{\partial \mathcal{E}_{zm}}{\partial n} \right]_{\text{at} C} \cdot \frac{1}{\sqrt{|z - z_c|}} \qquad z < z_c \qquad (7)$$

$$B_m = - \frac{\sqrt{2R}}{N_m^2} \left[V \frac{\partial \mathcal{H}_{zm}}{\partial c} \right]_{\text{at} C} \cdot \frac{1}{\sqrt{|z - z_c|}}$$

formulas which can be derived from (5) by substituting $\sqrt{\frac{2|z - z_c|}{R}}$ for $\operatorname{tg} \theta$.

The infinity of A_m and B_n is weaker than a δ function discontinuity in that the "area under the curve" around C, i.e., $\lim_{\varepsilon \to 0} \int_{z_c - \varepsilon}^{z_c} A_m(z)\, dz$, , is equal to zero. Function C_m goes to zero at C, with a vertical tangent, and has the form

$$C_m = \frac{2\sqrt{R}}{N_m^2} \left[\frac{\partial}{\partial c} (V \mathcal{H}_{zm}) \right]_{\text{at} C} \sqrt{|z - z_c|} \qquad z < z_c$$

The behavior of A_m, B_n and C_n at point D, where the developed curve has an angular peak, can be deduced from the behavior at C by letting the radius of curvature R approach zero.

To summarize the preceding considerations in a graphical way, typical curves for A_m and C_n are shown in Figure 3. A curve for B_n would have the same general aspect as its A_m counterpart. At the discontinuity points, the unknown functions, a_m, b_n, etc., behave in a manner which can easily be deduced from a consideration of equations (4). For example:

---a_m, b_m, γ_m suffer a discontinuity in slope at B and D, a jump in value and slope at EF, and possess a vertical tangent at $z = z_c$.

--- α_m, β_m, c_m are smoother; they are subjected to a jump in slope at EF, but are continuous, with continuous first derivative, at all other singular points.

Finally, complete determination of the six unknown functions requires taking into account that they vanish at infinity (i. e., for $z = \pm \infty$), the guide being operated below cut-off.

III. ENERGY AND MOMENTUM KICKS

Calculation of a_m, b_m, etc., along the lines sketched in the preceding paragraph, allows determination of the electric and magnetic fields at each point of the vacuum chamber of the accelerator. In many situations, however, one is not so much interested in a detailed knowledge of these fields as in their integrated effect on a particle which flies through the gap region. Of particular physical significance, for instance, are the energy and lateral momentum kicks impressed on a particle which was originally travelling parallel to the accelerator axis. Remarkably simple formulas can be found for these kicks if one assumes that the gap voltage is not of such magnitude as to materially alter the particle's linear motion during the crossing of the gap region. Derivation of these formulas proceeds as follows:

a) Energy Kick $\Delta\mathcal{E}$.

Let v and q be the velocity and charge of the particle, φ_0 the phase of the voltage at A (See Figure 1) at the moment the particle crosses the $z=0$ plane. The longitudinal energy kick is equal to

$$\Delta\mathcal{E}(x,y) = q\int_{-\infty}^{\infty} e_z(x,y,z,t)\,dz = q \times \text{real part of} \int_{-\infty}^{\infty} \sum_m \mathcal{E}_{zm}(x,y).c_m(z).e^{j\left(\varphi_0 + \frac{2\pi}{\beta}\frac{z}{\lambda}\right)}.dz$$

There, as usual, β is equal to v/c, and λ is the free-space wavelength of the applied voltage. To evaluate an integral such as $\int_{-\infty}^{+\infty} c_m(z).e^{j\left(\varphi_0 + \frac{2\pi}{\beta}\frac{z}{\lambda}\right)}dz$, one first establishes a relation satisfied by c_m alone. This can be obtained from (4) as:

$$\frac{d^2 c_m}{dz^2} - (k_m^2 - k^2)\,c_m = k_m^2 A_m(z) \qquad \left(\text{with } k = \omega/c = 2\pi/\lambda\right)$$

The integral involving c_m is then obtained by multiplication of both members by $e^{j\left(\varphi_0+\frac{2\pi}{\beta}\frac{z}{\lambda}\right)}$, and integration between $z=-\infty$ and $z=+\infty$.

The integral involving $A_m(z)$ can be computed by applying the results derived in the preceding section. Detailed calculations will not be given here. They are fairly straightforward. The expression one ends up with is

$$\Delta\mathcal{E}(x,y) = -q\sum_m \frac{\mathcal{E}_{zm}(x,y)}{N_m^2}\,\frac{k_m^2}{k_m^2+k^2\left(\frac{1}{\beta^2}-1\right)}\int_c v(c)\,\frac{\partial\mathcal{E}_{zm}}{\partial n}\cos\left[\varphi_0+\Delta\varphi+\frac{kz}{\beta}\right]dc \qquad (8)$$

There, $\Delta\varphi$ is the phase difference between the voltage at a gap point c and the voltage at A. Expressed as a formula: $\Delta\varphi=\varphi(c)-\varphi(0)$. The angle appearing after the cosine has an interesting physical significance: it is the phase of the voltage in c, at the very moment the particle shoots through the cross-sectional plane to which c belongs.

b) <u>Lateral Momentum Kick $\Delta\bar{p}$.</u>

The lateral momentum kick due to the electric field is

$$\Delta\bar{p}_E = q\int_{-\infty}^{+\infty}\bar{e}_{\text{transverse}}(x,y,z,t)\,dt$$

which, upon substitution of the expansion for the electric field, yields:

$$\Delta\bar{p}_E=\frac{q}{v}\times\text{real part of}\left[\sum_m\bar{\mathcal{E}}_{tm}(x,y)\int_{-\infty}^{+\infty}a_m(z)\,e^{j\left(\varphi_0+\frac{2\pi}{\beta}\frac{z}{\lambda}\right)}dz+\sum_m\bar{\mathcal{E}}_{tm}(x,y)\int_{-\infty}^{+\infty}b_n(z)\,e^{j\left(\varphi_0+\frac{2\pi}{\beta}\frac{z}{\lambda}\right)}dz\right.$$

The lateral momentum kick due to the magnetic field is

$$\Delta\bar{p}_H=q\mu_0\int_{-\infty}^{+\infty}v\left[\bar{u}_z\times\bar{h}(x,y,z,t)\right]dt=q\mu_0\int_{-\infty}^{+\infty}(\bar{u}_z\times\bar{h})\,dz$$

an expression which can similarly be rewritten by insertion of the expansion for the magnetic field, as:

$$\Delta \bar{p}_H = q\mu_0 \times \text{real part of} \left[\sum_m \bar{\mathcal{E}}_{tm}(x,y) \int_{-\infty}^{+\infty} \alpha_m(z) \cdot e^{j(\varphi_0 + \frac{2\pi}{\beta}\frac{z}{\lambda})} dz + \sum_n \bar{\mathcal{E}}_{tn}(x,y) \int_{-\infty}^{+\infty} \beta_n(z) e^{j(\varphi_0 + \frac{2\pi}{\beta}\frac{z}{\lambda})} dz \right]$$

Upon computation of the integrals, the electric and magnetic kicks, consolidated in one expression, yield the fairly simple formula:

$$\Delta \bar{p}(x,y) = \frac{qk}{c}\left(1 - \frac{1}{\beta^2}\right) \sum_m \frac{\bar{\mathcal{E}}_{tm}(x,y)}{N_m^2} \cdot \frac{1}{k_{cm}^2 + k^2\left(\frac{1}{\beta^2} - 1\right)} \int_c v(c) \cdot \frac{\partial \mathcal{E}_{zm}}{\partial n} \cdot \sin\left(\varphi_0 + \Delta\varphi + \frac{kz}{\beta}\right) dc$$

$$+ \frac{q}{kc} \sum_n \frac{\bar{\mathcal{E}}_{tn}(x,y)}{N_n^2} \int_c v(c) \cdot \frac{\partial \mathcal{H}_{zn}}{\partial c} \cdot \sin\left(\varphi_0 + \Delta\varphi + \frac{kz}{\beta}\right) dc \qquad (9)$$

The value of the lateral momentum kick can now be obtained by merely substituting actual magnitude and phase of the applied voltage in the indicated integrals, and performing the required summations. The shape of the gap's profile enters the formula through the term $\frac{k\, z(c)}{\beta}$, the shape of the cross-section through the transverse vectors $\bar{\mathcal{E}}_{tm}$ and $\bar{\mathcal{E}}_{tn}$, and the eigenvalues k_m^2

IV. GAP PERPENDICULAR TO THE TUBE'S AXIS

A gap situated in a plane perpendicular to the z axis is often used in practical applications. The formulas relative to this important configuration are obtained by merely setting $z=0$ in the expressions derived in the preceding paragraphs. Two particular situations deserve some special attention.

a) The Particle Travelling at the Velocity of Light

The relevant expressions for voltage and momentum kick show that, if magnitude and phase of the applied voltage are maintained constant as frequency varies:

---The voltage kick is frequency independent at each point of the cross-section.

---The lateral momentum kick has a direction independent of frequency, and a magnitude inversely proportional to the latter.*

b) The Gap Voltage Constant in Phase and Amplitude all Along the Contour

Important simplifications in the formulas are associated with this voltage distribution, which is often desired in those applications where constant voltage throughout the cross-section is hoped for:

$$\Delta\mathcal{E} = qV\cos\varphi_0 \sum_m \mathcal{E}_{zm}(x,y) \frac{k_m^2}{k_m^2 + k^2\left(\frac{1}{\beta^2}-1\right)} \cdot \frac{\iint \mathcal{E}_{zm}\, dS}{\iint \mathcal{E}_{zm}^2\, dS}$$

$$\Delta\bar{p} = \frac{qV\sin\varphi_0}{c} k\left(\frac{1}{\beta^2}-1\right)\left[\sum_m \frac{\bar{\mathcal{E}}_{tm}(x,y)}{k_m^2 + k^2\left(\frac{1}{\beta^2}-1\right)} \cdot \frac{\iint \mathcal{E}_{zm}\, dS}{\iint \mathcal{E}_{zm}^2\, dS}\right]$$

*See Footnote 2 on page 14

It turns out that $\Delta \bar{p}$ is also equal to $-\frac{tg\varphi_0}{kc}\,\mathrm{grad}_{xy}(\Delta\mathcal{E})$.
In consequence, an accurate plot of the energy kick $\Delta\mathcal{E}$ throughout the cross-section allows one to dispense with the lengthy calculations associated with the lateral momentum kick. The latter is, for instance, perpendicular to the constant-energy curves. In particular, a particle travelling at the velocity of light experiences a constant voltage throughout the cross-section, and is not subjected to any net lateral momentum kick: magnetic and electric kicks balance each other.

The results obtained for a gap perpendicular to the ζ axis can be extrapolated to a gap of negligible longitudinal extent L_1 (See Figure 1). Such an extrapolation is valid as long as the term $\frac{k\zeta}{\beta}$ remains sufficiently small, i.e., as long as the particle flies through the gap region in a time negligible with respect to a quarter of a period.

FOOTNOTES

Footnote 1: Let $\mathcal{E}_{zm}(x,y)$ and $\mathcal{H}_{zn}(x,y)$ be the eigenfunctions of the following two-dimensional problems:

$$\nabla^2_{xy}\,\mathcal{E}_{zm} + k_m^2\,\mathcal{E}_{zm} = 0 \qquad \text{with} \quad \mathcal{E}_{zm} = 0 \quad \text{on the cross-section contour}$$

$$\nabla^2_{xy}\,\mathcal{H}_{zn} + k_n^2\,\mathcal{H}_{zn} = 0 \qquad \text{with} \quad \frac{\partial \mathcal{H}_{zn}}{\partial n} = 0 \quad \text{on the cross-section contour}$$

where a subscript such as m stands for a pair of indices. One defines:

$$\bar{\mathcal{E}}_{zm} = \mathcal{E}_{zm}\,\bar{u}_z \; ; \; \bar{\mathcal{H}}_{zn} = \mathcal{H}_{zn}\,\bar{u}_z \; ; \; \bar{\mathcal{E}}_{tm} = \text{grad}\,\mathcal{E}_{zm} \; ; \; \bar{\mathcal{E}}_{tn} = \bar{u}_z \times \text{grad}\,\mathcal{H}_{zn}$$

$$\bar{\mathcal{H}}_{tm} = \bar{u}_z \times \text{grad}\,\mathcal{E}_{zm} \; ; \; \bar{\mathcal{H}}_{tn} = \text{grad}\,\mathcal{H}_{zn}.$$

with the norms:

$$N_m^2 = \iint_S |\mathcal{E}_{tm}|^2\, dS \qquad \text{and} \qquad N_n^2 = \iint_S |\mathcal{E}_{tn}|^2\, dS$$

Footnote 2: As the frequency approaches zero, any phase difference along the gap will be reflected in a rapid increase in the wall currents and the magnetic fields associated with the latter. This, in turn, increases the lateral magnetic kick.

REFERENCES

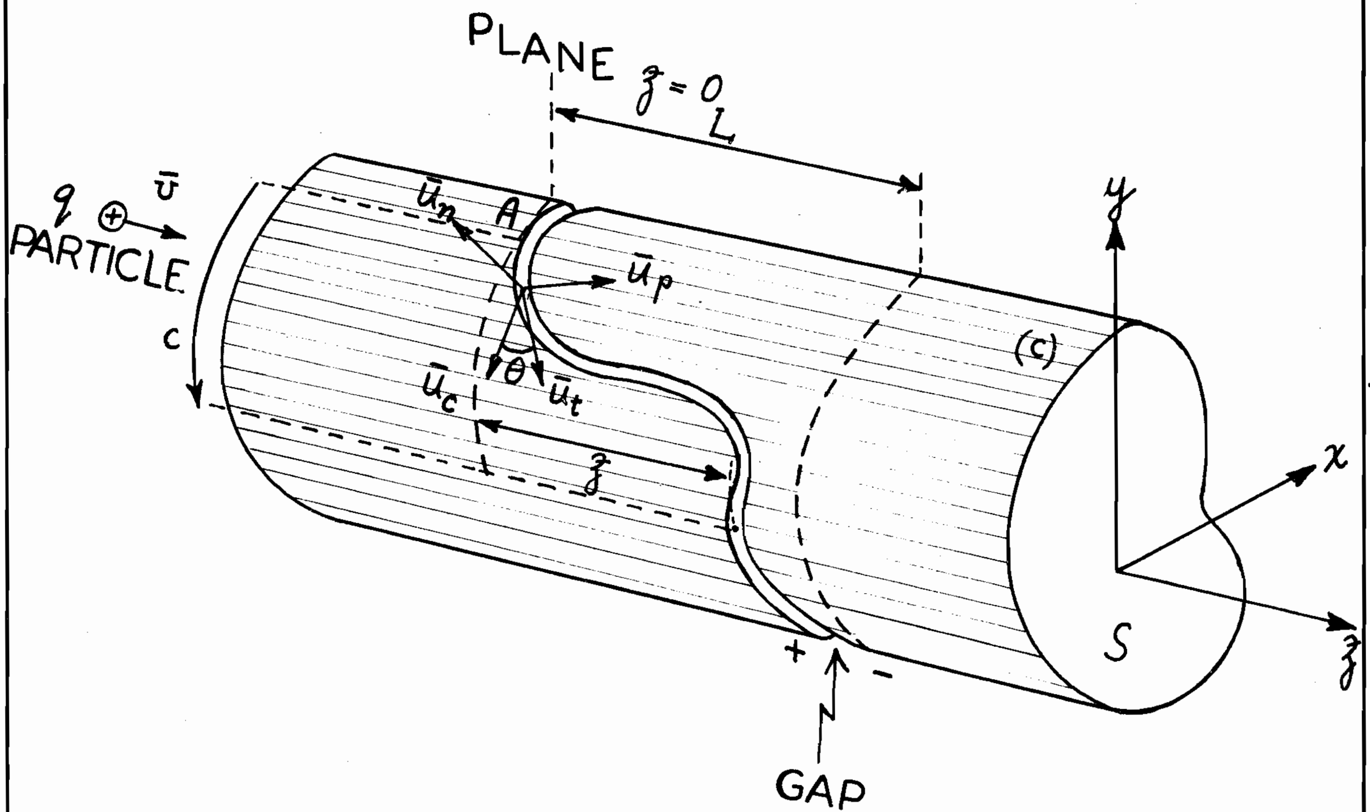
1. A. F. Stevenson, Journal of Applied Physics, 19, pp. 24-38, 1948.

2. J. Van Bladel, MURA Internal Report No. 301, June, 1957.

LIST OF CAPTIONS

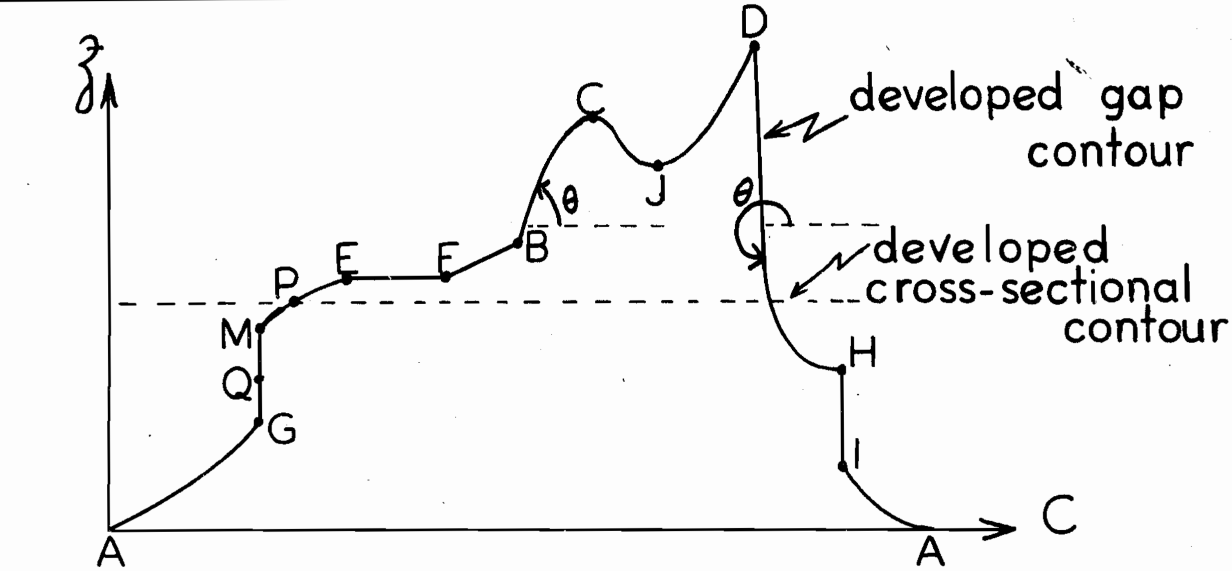
Figure 1: Gap-excited linear duct.

Figure 2: Developed gap-contour, and general behavior of the A_m and C_n integrals, which appear in the calculation of the fields.



PLANE $z=0$
L
q ⊕ $\bar{v}$
PARTICLE
$\bar{u}_n$
A
$\bar{u}_p$
c
θ
$\bar{u}_c$
$\bar{u}_t$
z
(c)
y
x
z
S
+
−
GAP

Fig. 1


ζ
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E
F
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θ
C
J
D
θ
developed gap contour
developed cross-sectional contour
H
I
A
C

Fig. 2

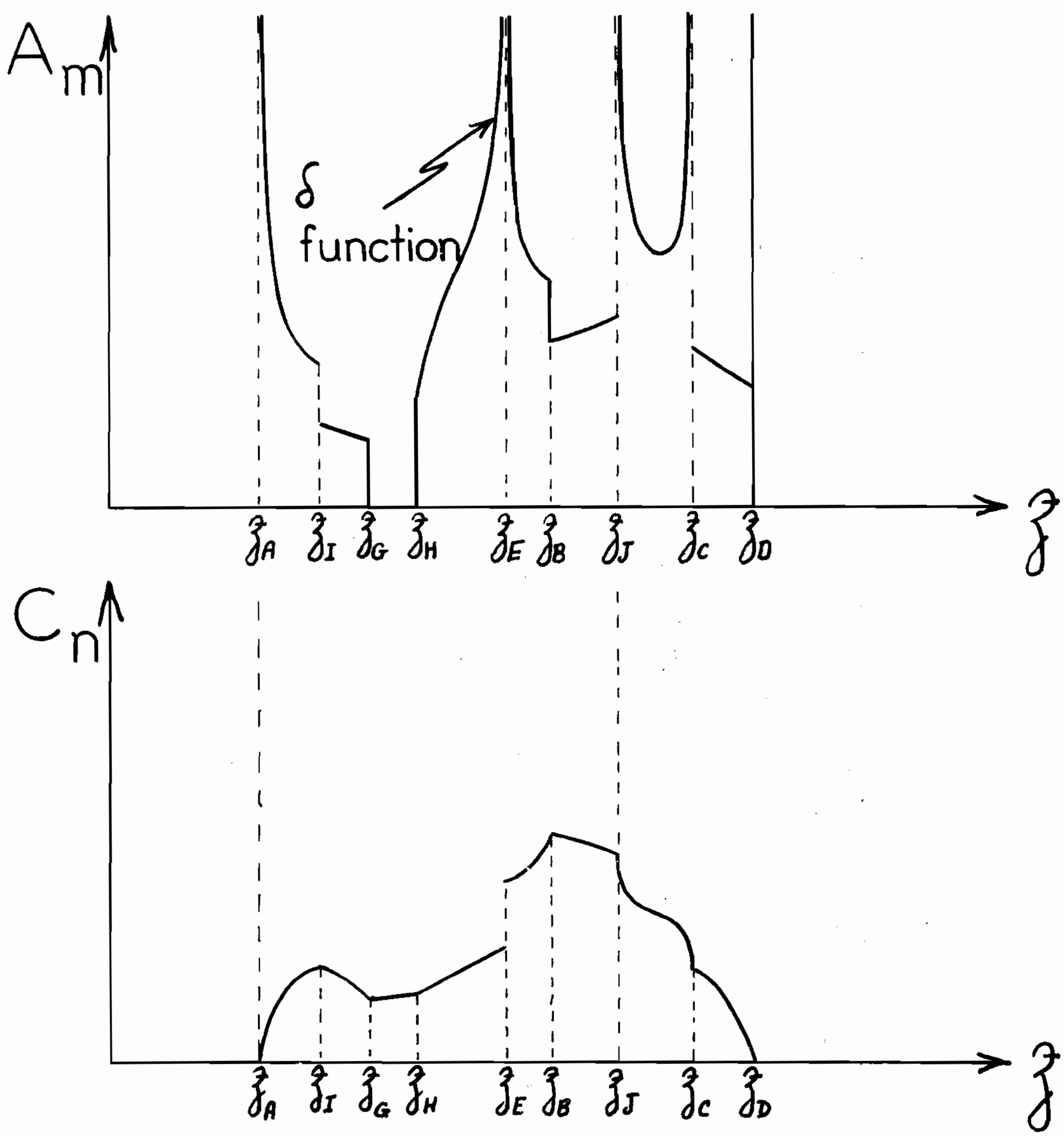
A_m
δ function
ζ_A ζ_I ζ_G ζ_H ζ_E ζ_B ζ_J ζ_C ζ_D
ζ
C_n
ζ_A ζ_I ζ_G ζ_H ζ_E ζ_B ζ_J ζ_C ζ_D
ζ

Fig. 3