Algebraic and Arithmetic Geometry in String Theory*

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In the first part we review how elements of algebraic geometry can be used to give an algebraic formula for the string partition function. In the second part we generalize these ideas to include arithmetic surfaces, i.e. surfaces defined over an algebraic number field K. We will calculate explicitly the volume of the lattice formed by K-rational tangent vectors at a K-rational point in moduli space, with respect to the Polyakov measure.

1. Introduction

Polyakov's formulation [1] for quantizing the closed bosonic string theory has various advantages over other quantization schemes due to its profound geometrical interpretation.

It is the purpose of this paper to discuss this in more detail. In section 1 we will give an outline of the proof of the Belavin Kniznik theorem [2] which states that after gauge fixing of the classical symmetries of the string, the so-called partition function for genus p surfaces becomes the square of the absolute value of a holomorphic function on the complex moduli space $\mathfrak{M}$ of stable curves. In section 2 we use elements of Faltings' work on Arakelov intersection theory to obtain an algebraic formula for the Polyakov measure i.e. the string partition function [14,16]. Subsequently, we consider the string partition function at K-rational points in moduli space. These are special points, corresponding to so-called arithmetic surfaces i.e. surfaces defined over an algebraic number field K.

As a new result we present a detailed calculation of the volume of the lattice spanned by K-integral vectors tangent to a K-rational point of $\mathfrak{M}$ with respect to the Polyakov measury using a Riemann Roch formula on Spec($\mathcal{O}_K$), $\mathcal{O}_K$ the ring of integers of K. At each infinite place of K the result reduces to the usual partition function. This will clarify some of the ideas presented in [3 (sect. 4.4), 4].

Part of this work, sect. 4.1 and 4.2 has been done in collaboration with B. Edixhoven. This part has also been reported in [22].

2. Determinants, isometries between holomorphic line bundles and the Belavin-Kniznik theorem.

We will first give a rough sketch of the content of the B-K theorem. The path integral for the bosonic string for genus p surfaces reads according to Polyakov:

$$Z_p = \int_{M_p \times \mathcal{E}} dg dx - S[x,g] \tag{2.1}$$

where M_p is the space of all metrics g that can be realized on the Euclidean (world) surface X and $\mathcal{E}$ is the space of all embeddings $x: X \rightarrow \mathbb{R}^d$ of the surface into d-dimensional Euclidean space-time. $S[x,g]$ is Polyakov's action:

*Talk presented at XVI-th Colloquium on Group Theoretical Methods in Physics at Varna, Bulgaria, June 1987

$$S[x,g] = \frac{1}{2}\int_X \sqrt{g}\, g^{ab}\partial_a x \partial_b x \tag{2.2}$$

The metric g^{ab} is defined on M and corresponds only at the stationary points of S with the induced metric from the embedding $x: X \to \mathbf{R}^d$.

In a number of papers [5] it is shown that after gauge fixing, (2.1) reduces to a finite dimensional integral over the complex moduli space $\mathfrak{M}$ of stable curves (corresponding to compact Riemann surfaces) of genus p (including those with a finite number of isolated nodes):

$$Z_p = \int_{\mathfrak{M}} \prod_{i=1}^{3p-3} \phi_i \wedge \overline{\phi_i} (\det(w_i, w_j))^{-d/2} G \tag{2.3}$$

$$G = \left[\frac{\det'\Delta_1}{\det(w_i,w_j)\int_X \sqrt{g}}\right]^{-d/2} \left[\frac{\det'\Delta_2}{\det(\phi_i,\phi_j)}\right]$$

where Δ_n, $n = 1,2,\cdots$ are Laplacians on holomorphic n-differentials on M. The prime denotes the ζ-function regularization of the determinants. That is, with

$$\zeta(s) = \sum_i \frac{1}{\lambda_i^s}, \quad \lambda_i \neq 0 \tag{2.4}$$

where λ_i are the eigenvalues of Δ_n we define

$$\det'\Delta_n \equiv \exp - \zeta'(0), \quad \zeta'(s) = \frac{d}{ds}\zeta(s) \tag{2.5}$$

The set $\{w_i\}_{i=1}^p$ forms a basis for $\Gamma(X, \Omega_X)$ the space of holomorphic 1-forms. The quadratic differentials $\{\phi_i\}_{i=1}^{3p-3}$ serve as holomorphic coordinates on $\mathfrak{M}$.

The B-K theorem says that for $d = 26$ the integrand in (2.3) is the squared modulus of a holomorphic function on $\mathfrak{M}$, so it represents a real valued volume form on $\mathfrak{M}$. (In physical terms this means that the left and right moving oscillator modes in the string fully decouple.) If $d = 26$ we refer to the integrand as the Polyakov integration measure. The proof of the theorem requires a detailed study of the ζ-function regularized determinants. For this we refer to [6,7]; for a more physical treatment see [3,8,9]. Here we recall some relevant facts.

The crucial point is that the determinants in (2.3) are really sections of a determinant line bundle $\mathfrak{L}_n$ over $\mathfrak{M}$, associated with the Cauchy-Riemann operator ∇_n. In our case this elliptic linear differential operator ∇_n acts on the bundle of holomorphic n-differentials, $\Omega_X^{\otimes n}$, on the Riemann surface X:

$$\nabla_n : \Omega_X^{\otimes n} \to \Omega_X^n \otimes \overline{\Omega}_X \tag{2.6}$$

Our first concern will be the precise definition of $\mathfrak{L}_n$ over $\mathfrak{M}$. The moduli space $\mathfrak{M}$ is the complex variety associated with the moduli space over $\mathbf{Z}$ (the integers) of stable curves of genus p over arbitrary ground fields. That is, $\mathfrak{M}$ is given by polynomial equations with coefficients in $\mathbf{Z}$. (The solution of such equation may lie in an arbitrary number field. For the moment we take $\mathbf{C}$, the complex numbers.) In what follows we also need the existence of a universal curve X over $\mathfrak{M}$. This is a family of smooth irreducible stable curves over $\mathfrak{M}$:

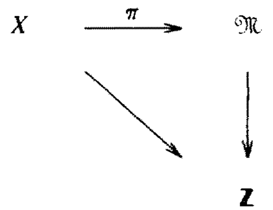

figure 1.

such that every curve occurs precisely once (up to isomorphism) in the family X and the curve over $y \in \mathcal{M}$ is precisely the curve y. In fact in order to avoid singularities, and at the same time to be able to define line bundles with sections over $\mathcal{M}$, one has to consider $\mathcal{M}$ as a slightly more general object viz. the moduli *stack* over $\mathbf{Z}$.

We now introduce on $\mathcal{M}$ the following sheaves: $\Omega_{\mathcal{M}/\mathbf{Z}}$ which is the sheaf of holomorphic 1-forms on $\mathcal{M}$; $R^i\pi_*\omega_{X/\mathcal{M}}^{\otimes}$, $i \geq 0$ which are the higher direct images of the holomorphic n-differentials relative to $\pi: X \to \mathcal{M}$. In addition we introduce $\mathcal{O}_{\mathcal{M}}(\Delta)$, the sheaf of meromorphic functions whose divisor is called the compactification divisor Δ defined on the boundary of $\mathcal{M}$

$$\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_{[g/2]} \tag{2.7}$$

where Δ_i, $i = 0, \ldots, [p/2]$ are the boundary components of $\mathcal{M}$.

The determinant bundle $\mathcal{L}_n$ introduced above can now be defined as

$$\mathcal{L}_n \equiv \det(R\pi_*\omega_{X/\mathcal{M}}^{\otimes n}) = (\wedge^{\max}\pi_*\omega_{X/\mathcal{M}}^{\otimes n})^* \otimes R^1_*\omega_{X/\mathcal{M}}^{\otimes n} \tag{2.8}$$

This definition can also be written as

$$\mathcal{L}_n = (\wedge^{\max}H^0(\pi^{-1}(y), \omega_{X/\mathcal{M}}^{\otimes n}))^* \otimes \wedge^{\max}H^1(\pi^{-1}(y), \omega_{X/\mathcal{M}}^{\otimes n})) \tag{2.9}$$

$y \in \mathcal{M}$. We call $\mathcal{L}_n$ the determinant line bundle associated with the Cauchy-Riemann operator ∇_n. For $p > 1$ and $n > 1$ is $H^1(X, \omega_{X/\mathcal{M}}^{\otimes n})$ is trivial hence we write $\mathcal{L}_n$ as $\det\pi_*\omega_{X/\mathcal{M}}^{\otimes n}$. At the basis of the B-K theorem are two isomorphisms which we will now describe. One gives a relation between $\mathcal{L}_2$ and the determinant line bundle of the Kähler differentials $\det\Omega_{\mathcal{M}/\mathbf{Z}}$, on $\mathcal{M}$.

We have the exact sequence

$$0 \to T_{X/\mathcal{M}} \to T_{X/\mathbf{Z}} \to \pi^* T_{\mathcal{M}/\mathbf{Z}} \to 0 \tag{2.10}$$

to which corresponds a homomorphism α, the socalled Kodaira-Spencer mapping

$$\alpha: T_{X/\mathbf{Z}} \to R^1\pi_*(T_{X/\mathcal{M}}) \tag{2.11}$$

where $T_{X/\mathcal{M}}$ is the dual of $\omega_{X/\mathcal{M}}$, and $T_{X/\mathbf{Z}}$, the dual of $\Omega_{X/\mathbf{Z}}$, etc. In fact we will consider the dual map. Taking the determinant (i.e. taking the maximum exterior power) one can prove that α is an isomorphism, called the Kodaira-Spencer isomorphism:

$$\det(\pi_*\omega_{X/\mathcal{M}}^{\otimes 2}) \simeq \det\Omega_{\mathcal{M}/\mathbf{Z}} \otimes \mathcal{O}_{\mathcal{M}}(\Delta) \tag{2.12}$$

The other isomorphism gives a relation among the $\mathcal{L}_n$, the determinant bundle of the relative differentials $\omega_{X/\mathcal{M}}^{\otimes n}$, for different values of n. To find it we use the Grothendieck-Riemann-Roch theorem. It characterizes relations between the line bundles by way of their Chern classes as elements in the Chow-ring $A(\mathcal{M})$, i.e. it determines $c_1(\mathcal{L}_n)$ up to *linear* equivalence.

The theorem states that

$$c_1(\pi_*\omega_{X/\mathcal{M}}^{\otimes n}) = \pi_*[Ch\,\omega_{X/\mathcal{M}}^{\otimes n} \cdot Td\,\Omega_{X/\mathcal{M}}^{-1}] \tag{2.13}$$

where Ch resp. Td denote the Chern character resp. Todd character.

Putting $c_1(\pi_*\omega_{X/\mathcal{M}}) = \lambda$, $c_n(\pi_*\omega_{X/\mathcal{M}}^{\otimes n}) = \lambda_n$ we obtain upon expanding the right hand side of

(2.13) Mumford's formula [10]

$$\lambda_n = \lambda + \binom{n}{2}(12\lambda - \delta) \tag{2.14}$$

where δ denotes the class of compactification divisor: $\delta = [\mathcal{O}_{\mathcal{M}}(\Delta)]$.

In particular we have the *holomorphic* isomorphism:

$$\det(\pi_* \omega_{X/\mathcal{M}}^{\otimes 2}) \simeq (\det\pi_* \omega_{X/\mathcal{M}})^{\otimes 13} \otimes \mathcal{O}_{\mathcal{M}}(-\Delta) \tag{2.15}$$

Combining (2.12-15), we than find

$$(\det\pi_* \omega_{X/\mathcal{M}})^{\otimes 13} \simeq (\det\Omega_{\mathcal{M}/\mathbf{Z}}) \otimes \mathcal{O}_{\mathcal{M}}(2\Delta) \tag{2.16}$$

This isomorphism is unique (up to an overall constant). The Polyakov measure arises by defining metrics on the determinant line bundles $\mathcal{L}_n$ which for $n=2$ can be transferred onto the canonical line bundle of $\mathcal{M}_{\mathbf{Z}}$, by means of (2.16). Up to a constant depending on the genus, the metric on $\det\Omega_{\mathcal{M}/\mathbf{Z}}$ obtained in this way is the Polyakov integration measure.

With the following theorem we put a suitable metric on $\mathcal{L}_n = \det\pi_* \omega_{X/\mathcal{M}}^{\otimes n}$:

THEOREM 1 [7]. Let the bundle $\omega_{X/\mathcal{M}}^{\otimes n}$ have a *smooth* Hermitian metric for each n, induced from the usual L_2-metric on each fibre. Furthermore let $\{\phi_i\}_{i=1}^m$, $m = (2n-1)(p-1)$ be a basis for $H^0(\pi^{-1}(y), \omega_{X/\mathcal{M}}^{\otimes n})$. Denote by s the section of $\mathcal{L}_n$ given by $s = (\phi_1 \wedge \cdots \wedge \phi_m)^{-1}$.

The Quillen norm $\|\cdot\|_Q$ defined as

$$\|s\|_Q^2 = \frac{\det'\Delta_n}{\det(\phi_i, \phi_i)} \tag{2.17}$$

is a smooth metric on $\mathcal{L}_n$. The curvature of this metric is given by

$$\mathrm{Curv}\,\mathcal{L}_n = \partial\bar{\partial}\log\|s\|_Q^2 \tag{2.18}$$

which represents a (1,1) form on $\mathcal{M}$.

The following remarks are in order.

The Laplacian Δ_n is computed using the Hermitian metric on each fibre $\pi^{-1}(y), y \in \mathcal{M}$. Note that the theorem is independent from the metric used to compute the Laplacian Δ_n. This fact will be used in the next section. Observe also that the Quillen norm differs from the usual L_2 norm: (cf. [7,11]): $\|\cdot\|_Q = \|\cdot\|_2 \det'\Delta_n$. (The L_2 norm does *not* very smoothly with the fibres.)

Using the definition and the theorem above, (2.3) can be rewritten as

$$Z_p = \int_{\mathcal{M}} \prod_{i=1}^{3p-3} \phi_i \wedge \bar{\phi}_i \, \det(w_i, w_i)^{-d/2} \|s_2\|_Q^2 \|s_1\|_Q^{-d} \tag{2.19}$$

We now apply a theorem of J. Bismut and D. Freed [12] which is a refinement of the G-R-R theorem to the level of differential forms. It states that the Chern class of $\mathcal{L}_n$ represented as a two form (using the Quillen metric), is given by:

$$c_1(\mathcal{L}_n, \|\cdot\|_Q) = -\int_X ch(\omega_{X/\mathcal{M}}^{\otimes n}) Td(\Omega_X^{-1}) \tag{2.20}$$

where the Chern character and the Todd character on the right hand side are computed with the Hermitian metric put on the bundles $\omega_{X/\mathcal{M}}^{\otimes n}$ and Ω_X^{-1}.

Using the isomorphism (2.16) we thus obtain a holomorphically flat metric on $(\mathcal{L}_1)^{-d/2} \otimes \mathcal{L}_2$ for $d=26$, which generates a second order pole at the boundary of $\mathcal{M}$. (The partition function therefore generates a fourth order pole, which physically signals the presence of a tachyon.)

3. An algebraic formulation of the Polyakov measure

One can be more explicit about the nature of the Polyakov partition function (i.e. the integrand in (2.19)), by giving an algebraic expression for the Quillen metric on $\mathcal{L}_n$. For this purpose we use some of the ideas of Faltings on Arakelov geometry [13,14].

We begin with constructing a special metric on an arbitrary line bundle L over a compact Riemann surface on X. As before let $w_1,...,w_p$ be an orthonormal basis for the space of holomorphic Kähler differentials $\Gamma(X,\Omega_X)$. Then one defines a Hermitian metric

$$<w_1,w_2> := \frac{i}{2}\int_X w_1 \wedge \overline{w_2} \tag{3.1}$$

Using the set $\{w_i\}_{i=1}^p$ one constructs a Kähler 1-1 form ω on X

$$\omega = \frac{i}{2p}\sum_{j=1}^{p} w_j \wedge \overline{w_j} \tag{3.2}$$

which is normalized by

$$\int_X \omega = 1 \tag{3.3}$$

Now by a theorem of Arakelov [13] that there exists for any line bundle L a metric $\|\cdot\|$ unique up to scalar mutliplication of which the first Chern class satisfies:

$$c_1(L) = 2\pi i \deg(L)\omega \tag{3.4}$$

where

$$c_1(L) = \partial\bar{\partial}\log\|s\|^2, s \in L \tag{3.5}$$

and

$$\deg L = \frac{1}{2\pi i}\int_X \partial\bar{\partial}\log\|s\|^2$$

Such a metric on L is referred as an *admissible* metric.

It's associated *Green's function* $g(P,Q) = \log G(P,Q)$, $P,Q \in X$, satisfies [14] $\partial\bar{\partial}\log G(P,Q) = 2\pi i\omega$. (It is not difficult to show, that $\log G(P,Q)$ is the inverse of the scalar Laplacian whence the terminology.) The function $G(P,Q)$ has a logarithmic singularity at $P = Q$; for $P \neq Q$ it is C^∞-function. Using the function $G(P,Q)$, one puts a metric on the bundle $\mathcal{O}_X(Q)$ of holomorphic functions at Q by setting the norm of the unit section **1** equal to

$$\| 1 \|_{\mathcal{O}_X(P)}(Q) = G(P,Q) \tag{3.6}$$

Taking tensor powers gives an admissible metric on $\mathcal{O}_X(D)$ called the Green's metric (D a divisor on X). The residue of a differential at P gives an isomorphism from the fibre at P of the line bundle

$$\Omega_X(P) = \Omega_X \otimes \mathcal{O}_X(P) \tag{3.7}$$

to $\mathbf{C}$ with its usual metric $|\cdot|$. There is unique metric on the relative differential $\omega_{X/\mathfrak{M}}$ (since Ω_X and $\omega_{X/\mathfrak{M}}$ are isomorphic as line bundles), for which the residue map is an isometry for all P.

It is possible to transfer this metric onto the associated determinant line bundle, by virtue of the following theorem:

Theorem 2 ([14]). There is a unique way of assigning to any line bundle L on X with an admissible metric a Hermitean metric on the space

$$\det R\Gamma(X,L) \equiv \wedge^{\max} H^0(X,L) \otimes (\wedge^{\max} H^1(X,L))^* \tag{3.8}$$

such that the following (functorial and compatibility) properties hold

1) An isometry $f: L \to L'$ induces an isometry from det $R\Gamma(X,L)$ to det $R\Gamma(X,L')$.
2) If the metric on L is changed by a factor $\alpha > 0$ then the metric on det $R\Gamma(X,L)$ is changed by $\alpha^{\chi(L)}$ where

$$\chi(L) = dimH^0(X,L) - dimH^1(X,L)$$
$$= degL - p + 1$$

3) The metrics on det $R\Gamma(X,L)$ are compatible with the Green's metrics on $\mathcal{O}(D)[P]$ in the following sense. Suppose D_1 and D are divisors on X such that $D = D_1 + P, P \in X$. Then the isomorphism

$$\det R\Gamma(X,(D)) \simeq \det R\Gamma(X,(D_1)) \otimes \mathcal{O}(D)[P], \tag{3.9}$$

which is induced by the exact sequence

$$0 \to \mathcal{O}(D_1) \to \mathcal{O}(D) \to \mathcal{O}(D)[P] \to 0 \tag{3.10}$$

is in fact an isometry.
4) The metric on $\det R\Gamma(X,\Omega) = \wedge^p H^0(X,\Omega_X)$ is the one determined by the canonical scalar product (3.1) on $H^0(X,\Omega)$.

We will give the line of reasoning in the proof of this theorem, because it has an interesting consequence in string theory. It is essentially enough to prove property 1, because the metric may be put on the $\det R\Gamma(X,L)$'s in a unique way so that properties 2,3,4 hold. Recall that one can construct always a divisor D such that $\mathcal{O}(D)$ and Ω are isomorphic as line bundles. Since we have the Green's metric on $\mathcal{O}(D)$, which is a scalar multiple of the fixed metric on Ω_X, conditions 2 and 4 determine the metric on $\det R\Gamma(X,\mathcal{O}(D))$. With property 3 it then follows that one may determine a metric on $\det R\Gamma(X,\mathcal{O}(D'))$, for any divisor D' by adding or subtracing points. $G(P,Q)$ is symmetric in P,Q so the order in which one adds or subtracts points is irrelevant for the metric on $\det R\Gamma(X,\mathcal{O}(D'))$.

It, therefore remains to prove that any isometry $\mathcal{O}(D) \simeq \mathcal{O}(D')$ induces an isometry

$$\det R\Gamma(X,\mathcal{O}(D)) \simeq \det R\Gamma(X,\mathcal{O}(D')).$$

For this purpose one adds or subtracts points such that the divisors D and D' are both of degree $p-1$. Then they can be written as

$$E - \sum_{i=1}^{r} P_i$$

for a fixed divisor E and some points $P_1,\ldots,P_r$ on M. For $\mathcal{P} = (P_1,\ldots,P_r) \in X^r$, ($X^r = X \otimes \cdots \otimes X, r$ factors). Define $L(\mathcal{P})$ to be $\mathcal{O}(E - \sum_{i=1}^{r} P^r)$. One then proceeds in proving property 1) by constructing a line bundle N on X^r whose fibre at $\mathcal{P}$ is naturally identified with $\det R\Gamma(X,L(\mathcal{P}))$. The isomorphism classes of this line bundle are easily obtained since they belong to the Picard group of line bundles of degree $p-1$, $Pic_{p-1}(X)$. We thus have a mapping

$$\psi: M^r \to Pic_{p-1}(X) \tag{3.11}$$

which sends $(P_1,\ldots,P_r)$ to $(E - \sum_{i=1}^{r} P_i)$.

Now, recall that in $Pic_{p-1}(X)$ there is the theta divisor, θ, of trivial bundles over X. One can prove that the bundle N is in fact given by the pull back of ψ [14]:

$$N = \psi^* \mathcal{O}(-\theta) \tag{3.12}$$

in which the unit section of $\mathcal{O}(-\theta)$ goes over into the meromorphic section s of N. Since the fibre of N at P has been identified with $\det R\Gamma(X,L(\mathcal{P}))$, the metrics that have been put on $\det R\Gamma(X,L)$ yield a metric $\|\cdot\|_N$ on N. Property 1) is thus proved if the curvature of this metric is equal to the curvature of the Hermitian metric $\|\cdot\|_\theta$ on $\mathcal{O}(\theta)$. This can be done by using the Green's metric on $\mathcal{O}(\theta)$. (We refer to [14] for this part of the proof.)

Important for us is, that apparently there exists a scalar Δ such that

$$\|\cdot\|_N = \Delta\|\cdot\|_\theta \tag{3.13}$$

When appropriately normalized, this factor can be expressed in terms of the new invariant $\delta(X)$ on a Riemann surface introduced in 14 (pp. 401-403).

In particular we have that for a given line bundle of degree $p-1$ without holomorphic sections, the norm on $\det R\Gamma(X,L)\simeq\mathbf{C}$ is independent of the metric on L, and by virtue of the theorem is proportional to the inverse of the norm of the unit section of $\mathcal{O}(\theta)$. The factor is most conveniently described by computing the metric on $\det R\Gamma(X,\Omega\otimes\mathcal{O}(Q\ \Sigma_{i=1}^{p}P_i))$. One then finds the equality [14]

$$\|\mathbf{1}_\theta\psi(P_i,\cdots,P_p,Q)\| = \exp(-\delta/8)\frac{\|\det w_i(P_j)\|}{\prod\limits_{i<j} G(P_i,P_j)}\prod_j G(P_j,Q) \tag{3.14}$$

The expression on the l.h.s. denotes the norm of the unit section of $\mathcal{O}(-\theta)$ in $(Pic_{p-1}-\theta)$ depending on the degree $p-1$ divisor $D=\Sigma_{i=1}^{p}P_i-Q$. (The representation of the proportionality factor is for convenience and will be clarified below).

One can lift the discussion above to a family of curves $\pi:X\rightarrow\mathfrak{M}$. In this case we have the analogue of the exact sequence (3.10) for L on X:

$$0\rightarrow L(-D)\rightarrow L\rightarrow\sigma^*L\rightarrow 0 \tag{3.15}$$

where σ^* is the pull-back of the section of π which assigns to each $y\in\mathfrak{M}$ a point on the fibre. The choice of D is such that $L(-D)$ is of degree $p-1$. Using (3.15) one has the isomorphism

$$\det R\pi_*L\simeq \det R\pi^*L(-D)\otimes\sigma^*L \tag{3.16}$$

Putting the Quillen metric on both sides and using the $G-R-R$ theorem in the form (2.20) one proves easily (see e.g. [3]) that (3.16) is indeed an isometry, for the Quillen metrics.

Applying this to $L=\det\pi_*\omega_{X/\mathfrak{M}}^{\otimes n}$ one finds

$$\|s\|_Q^2 = \|s'\|_Q^2\,\frac{\prod\limits_{i\neq j} G(P_i,P_i)}{\|\det\phi_i(P_j)\|} \tag{3.17}$$

where s is a section of $\mathfrak{L}_n$, and s' a section of the determinant line bundle associated to $\pi_*\omega_{X/\mathfrak{M}}^n(\mathfrak{M})$.

Because $\pi_*\omega_{X/\mathfrak{M}}^n(-D)$ is of degree $p-1$ by construction, the Quillen metric on the associated determinant bundle is isometric to the usual flat metric on $\mathbf{C}$, and only on the isometry class of $\pi_*\omega_{X/\mathfrak{M}}^n(-D)$. Hence, the Quillen metric on $\det\pi_*(\omega_{X/\mathfrak{M}}^n(-D)$ can be related to the metric on the unit section of the θ-divisor lifted to the universal curve [14] as follows. We consider the relative Picard variety $\mathrm{Pic}_\pi^{p-1}\rightarrow\mathfrak{M}$ whose fibre over $y\in\mathfrak{M}$ is $\mathrm{Pic}_{p-1}(X)$. We can associate with it a relative θ-divisor, which gives a line bundle on Pic_π^{p-1}, whose restriction to $\mathrm{Pic}_{p-1}(X)$ is the line bundle $\mathcal{O}(\theta_X)$. We denote by $\|\mathbf{1}_\theta\psi(P_1,\cdots,P_r)\|$ the norm of the unit section of $\mathcal{O}(-\theta)$ lifted to the universal curve. By virtue of the G-R-R theorem in the form (2.20) one has the isometric isomorphism:

$$\det\pi_*(\omega_{X/\mathfrak{M}}^n(-D))\simeq\pi_*\mathcal{O}(-\theta) \tag{3.18}$$

(Note that the definition of $\det R\Gamma(X,L)$ in (3.8) is dual to the definition of $\mathfrak{L}_n$. In the derivation of (3.18) and the formulae (3.21-23) one therefore has to incorporate an extra minus sign.) This implies that the Quillen metric $\|s'\|$ in (3.17) is proportional to the norm on $\mathbf{1}_\theta$, where the proportionality is given by the invariant δ. We will now make this explicit. Choosing an odd theta characteristic, or spin bundle, (which has degree $p-1$) on the (spin covering of the) universal curve X, one obtains using the G-R-R theorem, the isometry

$$(\mathfrak{L}_{\frac{1}{2}})^{-2}\simeq\mathfrak{L}_1 \tag{3.19}$$

where $\mathfrak{L}_{\frac{1}{2}}$ is the associated determinant line bundle of $\omega_{X/\mathfrak{M}}^{1/2}$. Subsequently, one may prove the following isometric isomorphism

$$(\mathfrak{L}_{\frac{1}{2}})^8 \simeq (\mathfrak{L}_2)^{-1} \otimes (\mathfrak{L}_1)^9. \tag{3.20}$$

As a result we can express the Quillen norm on $\mathfrak{L}_1$ entirely in terms of $\exp\delta$. First we conclude from (3.18-19) that the norm on s' in (3.17) is identified with

$$\|s'\|_Q^2 = \Lambda_p\Big[\frac{\det'\Delta_1}{\det(w_i,w_j)\int_{\pi^{-1}(y)}\sqrt{g}}\Big]^{-\frac{1}{2}} \|\mathbf{1}_\theta\psi(P_1,\cdots,P_p,Q)\|^2. \tag{3.21}$$

Where Λ_p is constant only depending on the genus. This together with the identity (2.14) generalized to the universal curve yields, for the Quillen norm on $\mathfrak{L}_1$:

$$\|s\|_Q^2 = \Big[\frac{\det'\Delta_1}{\det(w_i,w_j)\int_{\pi^{-1}(y)}\sqrt{g}}\Big] = \Lambda_p \exp(-\frac{\delta}{6}) \tag{3.22}$$

Computing a similar expression for the Quillen norm on the canonical line bundle, one finds that the partition function can be given as:

$$\Lambda_p|\phi_1\wedge\cdots\wedge\phi_{3p-3}|^2\exp(\frac{9\delta}{4})\frac{\prod_{i\neq j}G(P_i,P_j)}{\|\det\phi_i(P_j)\|^2}\|\mathbf{1}_\theta\psi(P_1,\cdots,P_m)\|^2. \tag{7.11}$$

This expression is independent of the points P_i and of the coordinates ϕ_i on $\mathfrak{M}$. It is thus possible to write the partition function in terms of Riemann theta functions [15,16].

4. Computation of the Polyakov volume for arithmetic surfaces

In this section we will describe a string partition function for arithmetic surfaces, which correspond to K-rational points in moduli space.

Let $\mathfrak{M}$ be the moduli stack over $\mathbf{Z}$ as before. Pick a K-rational point x in $\mathfrak{M}$ for some algebraic numberfield K and consider the tangent space $T_{\mathfrak{M}/\mathbf{Z}}(x)$ at x, which is the complexification of the vector space formed by the K-rational tangents. (The complexification arises by considering the product $K\otimes_K\mathbf{C}$.) Out of these vectors. We select the integral tangents to build a lattice and we will compute, using the techniques of [14,24], the volume of this lattice with respect to the Polyakov measure. The result suggests an alternative definition of the string partition function, as proposed in [3,4] which at each infinite place of K and upon taking the limit over the net of all K, reduces to the usual partition function. For genus 1 and 2 this can be checked explicitly [3,4].

Preliminaries on Arithmetic surfaces and metrized line bundles.

The definitions of the sheaves in section 2 commute with base changes; that is the following diagram defining the arithmetic surface A, is commutative

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\;P\;} & \mathfrak{M} \end{array}$$

figure 2.

Using the base change operator defined by P one can define the sheaves of section 2 on the algebraic stack $S = \mathrm{Spec}(\mathcal{O}_K)$ ($\mathcal{O}_K$ the ring of integers of K). In particular we have in this way the $\mathcal{O}_s$ module $P^* T_{\mathfrak{M}/\mathbf{Z}}$ on S which will be used frequently in the sequel.

The fibres of $p:A \to S$ are stable curves of genus $p > 1$. We denote by S_{inf} the set of infinite places of K associated to the infinite primes σ_i:

$$\sigma_i : K \hookrightarrow \mathbf{C},\ i = 1, \cdots n = [K:\mathbf{Q}] \tag{4.1.1}$$

$p: A \to S$ together with the fibres over each σ_i is referred as a compact arithmetic surface A of genus p (c.f. [14, 19]). To define line bundles on A, we introduce an Arakelov divisor D which is a finite formal linear combination

$$D = D_{\mathrm{fin}} + D_{\mathrm{inf}} \equiv \sum_i K_i C_i + \sum_{v \in S_{\mathrm{inf}}} \lambda_v F_v \tag{4.1.2}$$

where the K_i are integers, C_i is a sub-scheme of A (of codimension 1). The index i runs over the set of finite places of K. F_v denotes the fibre of A over the infinite place $v \in S_{\mathrm{inf}}$. For each infinite place v we introduce a Hermitian metric g_v on the surface $A_v \equiv A \otimes_v \mathbf{C}$. Denote by ω_v its volume element, and assume the normalization

$$\int_{A_v} \omega_v = 1 \tag{4.1.3}$$

One can now introduce a *metrized* line bundle on A as a line bundle L having a Hermitian metric $\|\cdot\|$ on L which is the line bundle over the infinite place included by L. The metric is induced by the inner product on each fibre. The bundle L is said to be *admissible* if each $\mathfrak{L}_v$ is admissible with respect to the metric on A_v, i.e.

$$c_1(L_v) = 2\pi \deg L_v\ \omega_v \tag{4.1.4}$$

To do explicit calculations, it is useful to express the relation between L and its Arakelov divisor D. For this purpose we define the bundle $\mathcal{O}_A(D)$ as the bundle of which $\mathcal{O}_A(D_{\mathrm{fin}})$ is the underlying bundle, which indudes a bundle over each infinite place. We can put a metric on the induced bundle by means of the Arakelov Green's function discussed earlier. Thus the admissible bundle $\mathcal{O}_A(D)$ has its Green's metric at each infinite place.

The divisor div of a section s of an admissible metrized line bundle is defined as [13]:

$$\mathrm{div}(s) = S_{\mathrm{fin}} + \sum_{v \in S_{\mathrm{inf}}} \gamma_v(s) F_v \tag{4.1.5}$$

$$\text{where } \gamma_v(s) = -\int_{A_v} \log\|s\|_v \omega_v$$

and S_{fin} is the finite part of the divisor of s. By construction, admissible metrics on a line bundle L are unique up to scalar multiples, so L is isometric to $\mathcal{O}_A(\mathrm{div}(s))$, where s is a section of L corresponding to the unit section $1_{\mathcal{O}_A}$ of $\mathcal{O}_A(div(s))$. Let $\omega_{A/S}$ be the dualizing sheaf for $p:A \to S$. This sheaf is isomorphic to Ω_A the sheaf of 1-forms A. At each infinite place we have the metric (3.1), and by the same construction as given in the previous section we obtain a Hermitian metric on $\omega_{A/S}$, at each infinite place.

We now use the commutativity of fig. 2, so that $P^* \omega_{X/\mathfrak{M}}$ becomes an admissible metrized line bundle over S. Such a line bundle corresponds to a projective rank-1 $\mathcal{O}_K$ module. One defines the Picard group Pic ($\mathcal{O}_K$) for these objects as the group of isomorphism classes of metrized $\mathcal{O}_K$ modules; i.e. modules with metrics at each infinite place of K. (One says that two modules L_1 and L_2 are isomorphic if there is a unit u in $\mathcal{O}_K$ such that it preserves the ν-norm for each $v \in S_{\mathrm{inf}}$. That is, if $\|\cdot\|_v^{(1)}$, $\|\cdot\|_v^{(2)}$ denote two Hermitian norms as L, then for $L_1 = L_2 = L$, we have

$$\|s\|_v^{(1)} = |u|_v\ \|s\|_v^{(2)} \quad \text{for all } s \in L). \tag{4.1.6}$$

The degree of an $\mathcal{O}_K$-module is a real valued function:

$$\deg : \mathrm{Pic}(\mathcal{O}_K) \to \mathbb{R} \tag{4.1.7}$$

defined as the degree of the associated divisor of L [17, 18]

$$\deg L = \log(\mathit{order}\ L/s\cdot\mathcal{O}_K) - \sum_{v\in S_{\inf}} \epsilon_v \log\|s\|_v \tag{4.1.8}$$

where ϵ_v is 1 (2) depending whether the embedding is real (complex). We will denote from now on the $\mathcal{O}_K$ module $P^*T_{\mathfrak{M}/\mathbf{Z}}$ by M. Observe that by means of the Polyakov measure, this is a metrized $\mathcal{O}_K$ module. The metrics on M at infinity define a Haar measure denoted by *vol* on the real vector space $M\otimes_{\mathbf{Z}}\mathbb{R}$. To describe it now we recall the following natural isomorphism [17]

$$\mathcal{O}_K\otimes_{\mathbf{Z}}\mathbf{C} \simeq \prod_{\sigma:K\hookrightarrow\mathbf{C}} \otimes_\sigma\mathbf{C} \tag{4.1.9}$$

where the product is over distinct embeddings of K into $\mathbf{C}$. By means of this isomorphism, together with the Euclidean Haar measure on each factor in the product on the right hand side of (4.1.9), the Haar measure on $\mathcal{O}_K\otimes_{\mathbf{Z}}\mathbf{C}$ is obtained.

So we conclude that M is a lattice in the vector space

$$M\otimes_{\mathbf{Z}}\mathbb{R} = M\otimes_{\mathcal{O}_K}K\otimes_{\mathbf{Q}}\mathbb{R} = \prod_{v\in S_{\inf}} M_v \tag{4.1.10}$$

The original problem, stated in the beginning of this section can thus be reformulated as, to compute the volume of

$$M\otimes_{\mathbf{Z}}\mathbb{R}/M \tag{4.1.11}$$

induced by the metric on M. We will show in the next sub-section how this can be done by applying a Riemann-Roch formula.

The Riemann-Roch theorem on Spec $(\mathcal{O}_K)$ and volume forms

Using the classical Riemann-Roch theorem [20] one can compute the Euler characteristic for a line bundle L on a Riemann surface Σ:

$$\chi(L) = \dim H^0(\Sigma,L) - \dim H^1(\Sigma,L) \tag{4.2.1}$$

In the previous sections we showed that volume forms on the formal difference

$$H^0(\Sigma,L) - H^1(\Sigma,L) \tag{4.2.2}$$

together with the isometric isomorphism (2.16) essentially determine the Polyakov measure. In this section we will develop a metrized Euler characteristic for $\mathcal{O}_K$ modules for which one has a Riemann-Roch formula. This formula is used to calculate the volume of the lattice (4.1.12). We will define the Euler characteristic χ for an $\mathcal{O}_K$ module in two steps. First we suppose that M is a rank 1 module over $\mathbf{Z}$. Denote as before by *vol* the Haar measure on $M\otimes_{\mathbf{Z}}\mathbb{R}$. Then one defines

$$\chi(M,\mathbf{Z}) \equiv -\log[\mathit{vol}(M\otimes_{\mathbf{Z}}\mathbb{R}/M)] \tag{4.2.3}$$

If M is a (finitely generated) $\mathcal{O}_K$ module one uses (4.2.3) to define

$$\chi(M,\mathcal{O}_K) \equiv \chi(M,\mathbf{Z}) - \mathrm{rank}\mathcal{O}_K(M)\chi(\mathcal{O}_K,\mathbf{Z}) \tag{4.2.4}$$

where the Euler characteristic for $\mathcal{O}_K$, $\chi(\mathcal{O}_K,\mathbf{Z})$ considered over $\mathbf{Z}$ is given by [19].

$$\chi(\mathcal{O}_K,\mathbf{Z}) = \log(2^{r_2}d_{K/\mathbf{Q}}^{-1/2}) \tag{4.2.5}$$

where r_2 denotes the number of complex embeddings of K into $\mathbf{C}$, and $d_{K/\mathbf{Q}}$ denotes the absolute value of the discriminant of $\mathcal{O}_K$ [17,21].

Before we proceed we will relate these definitions with the corresponding definitions for a line bundle on a complex Riemann surface. Recall the formula for the Euler characteristic for a line bundle L over a complex Riemann surface Σ [20]:

$$\chi(L,\mathbf{C}) = \deg L - p + 1 = \deg L - \chi(\Sigma,\mathbf{C}) \tag{4.2.6}$$

where $\chi(\Sigma,\mathbf{C})$ is the Euler characteristic of the surface Σ. In particular we have for the holomorphic one forms Ω_Σ:

$$\deg \Omega_\Sigma = -2\chi(\Sigma,\mathbf{C}) \tag{4.2.7}$$

We will check that such relation also holds in the arithmetic case using the definitions made above. That is, we will show that

$$\deg \Omega_{\mathcal{O}_{K/\mathbf{Z}}} = -2\chi(\mathcal{O}_K,\mathbf{Z}) \tag{4.2.8}$$

For this purpose we need the following more abstract definition of the dualizing sheaf for $p:A\to \mathrm{Spec}(\mathcal{O}_K)$ [19]:

$$\omega_{\mathcal{O}_K} = Hom_{\mathbf{Z}}(\mathcal{O}_K,\mathbf{Z}) \tag{4.2.9}$$

where $Hom_{\mathbf{Z}}$, denotes all homomorphisms of $\mathcal{O}_K$ linear in $\mathbf{Z}$.

In analogy with [20, p. 300 prop. 2.1] one has the following exact sequence

$$0\to\mathcal{O}_K\to\omega_{\mathcal{O}_K}\to\Omega_{\mathcal{O}_{K/\mathbf{Z}}}\to 0 \tag{4.2.10}$$

where $\Omega_{\mathcal{O}_{K/\mathbf{Z}}}$ are the relative Kähler differentials (i.e. 1 forms). The cananical section of $\omega_{\mathcal{O}_K}$ arises by taking the trace of the unit section in $\mathcal{O}_K$. (Note that this linear in $\mathbf{Z}$). At each infinite place v we put a metric on $\omega_{\mathcal{O}_K}$ such that

$$\|Tr\|_v = \|\mathbf{1}\|_v \tag{4.2.11}$$

The degree of $\omega_{\mathcal{O}_K}$ can be expressed as:

$$\begin{aligned}\deg\omega_{\mathcal{O}_K} &= \log(\text{order } \omega_{\mathcal{O}_K}/\mathcal{O}_K\cdot s) - \sum_{v\in S_{\inf}} \epsilon_v \log\|1\| \\ &= \log((\text{order } \Omega_{\mathcal{O}_{K/\mathbf{Z}}}),\ s\in\omega_{\mathcal{O}_K}.\end{aligned} \tag{4.2.12}$$

The module $\Omega_{\mathcal{O}_{K/\mathbf{Z}}}$ defines a lattice in the vector space

$$\Omega_{\mathcal{O}_{K/\mathbf{Z}}}\otimes_{\mathbf{Z}}\mathbf{R} = \prod_{v\in S_{\inf}} \Omega_v \tag{4.2.13}$$

Choosing a basis $\sigma_1,\cdots,\sigma_n$ of $\Omega_{\mathcal{O}K/\mathbf{Z}}$ at each infinite place $v_i, i=1,\cdots,n=[K:\mathbf{Q}]$, one can compute (4.2.12):

$$\begin{aligned}\log(order\Omega_{\mathcal{O}_{K/\mathbf{Z}}}) &= \log|\det(v_i(\sigma_j))|^2 \\ &= \log(2^{r_2/2} d_{K/\mathbf{Q}}^{1/4})\end{aligned} \tag{4.2.14}$$

where the second equality follows from the definition of the discriminant of $\mathcal{O}_K$ (see e.g. [21]).

Substituting (4.2.14) and (4.2.5) in (4.2.8) leads to the required equality. We will now proceed with the computation of the volume of $M\otimes_{\mathbf{Z}}\mathbf{R}/M$, $M=P^*\det T_{\mathfrak{M}/\mathbf{Z}}$, using the Riemann-Roch formula (4.2.4). We have

$$\log vol\,(M\otimes_{\mathbf{Z}}\mathbf{R}/M) = -\deg M - (3p-3)\log(2^{r_2} d_{K/\mathbf{Q}}^{-1/2}) \tag{4.2.15}$$

The first term on the right hand side can be evaluated, using the fact for rank 1 modules M we have:

$$\deg M = \deg\det M \tag{4.2.16}$$

(compare e.g. the situation with line bundles).
Using (2.16) we thus obtain

$$\begin{aligned} \deg \det M &= \deg P^*(\det \pi_* \omega_{X/\mathfrak{M}})^{\otimes 13} \mathcal{O}_{\mathfrak{M}}(-2\Delta) \\ &= 13 \deg \det p_* \omega_{A/S} - 2 \deg \mathcal{O}_S(\Delta) \end{aligned} \tag{4.2.17}$$

where we used the commutativity of fig. 2. Note that we have by our earlier discussion in the previous section a metric on $\det p_* \omega_{A/S}$. The metric on the sheaf $\mathcal{O}_s(\Delta)$ is trivial, so

$$\begin{aligned} \deg \mathcal{O}_s(\Delta) &= \log(\text{order } \mathcal{O}_s(\Delta)/\mathcal{O}_s(\Delta)\cdot 1) \\ &= \log |N_{K/\mathbf{Q}} \Delta_{A/S}| \end{aligned} \tag{4.2.18}$$

where $N_{K/\mathbf{Q}}$ is the absolute norm and $\Delta_{A/S}$ the discriminant of the curve associated to the surface $p: A \to S$ (see [18] for details).

At this point we recall the definition of Falting's modular height function [14] of a K-rational point x in the moduli space of stable surfaces $\pi: A \to S$ of genus p:

$$h(A) = \frac{1}{[K:\mathbf{Q}]} \deg \det (p_* \omega_{A/S}) \tag{4.2.19}$$

Then, (4.2.15) leads to

$$\begin{aligned} & [\text{vol } (M \otimes_{\mathbf{Z}} \mathbf{R}/M)]^{\frac{1}{[K:\mathbf{Q}]}} = \\ & \qquad \exp 13 h(A) \cdot (N_{K/\mathbf{Q}} \Delta_{A/S})^{\frac{-2}{[K:\mathbf{Q}]}} (2^{-r_2} d_{K/\mathbf{Q}}^{1/2})^{\frac{3p-3}{[K:\mathbf{Q}]}}, \quad \text{for } p > 1 \end{aligned} \tag{4.2.20}$$

for $p > 1$.

Using the formulae in [3,4,18] for genus 1 and 2 it follows that at each infinite place (4.2.20) reduces to the original partition function. (For genus 1, the factor 13 should be replaced by 14 in the exponent, and the factor $3p-3$ by 1).

4. Conclusions

In this paper, we have given an outline of the proof of the Belavin Kniznik theorem, which defines the partition function of Polyakov's string theory, as an invariant real valued measure on the moduli space of stable curves of given genus. Subsequently, we computed the volume of the lattice of K-rational tangents at a K-rational point with respect to this measure. At each infinite place of K one recovers for genus 1,2 the original partition function.

As was discussed already in [3,4], it thus seems natural to define the partition function as

$$Z = \lim_K \sum vol(M \otimes_{\mathbf{Z}} \mathbf{R}/M) \tag{5.1}$$

where the limit is taken over all finite field extensions of $\mathbf{Q}$, and the summation is over all K-rational points. Due to the higher dimensional Mordell conjecture, the convergence of (5.1) is presumably much better, thus showing that arithmetic geometry is more restrictive hence more powerful than the algebraic geometry of curves over the complex number field. It would be interesting to investigate, whether arithmetic geometry could provide a natural frame work to describe $d=2$ conformal quantum field theories in general.

Acknowledgements.

It is a pleasure to thank M. Hazewinkel and B. Edixhoven for useful comments and for reading the manuscript. This work was supported by "Stichting voor Fundamenteel Onderzoek der Materie".

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