

EXTENSIVE AND NON-EXTENSIVE THERMODYNAMICS

ASMAA G. SHALABY

Physics Department, Faculty of Science, Benha University
13815 Benha, Egypt
`asmaa.shalaby@fsc.bu.edu.eg`

(Received February 11, 2016; revised version received March 10, 2016)

This research addresses extensive and non-extensive thermodynamics. A comparison between the entropy for both different statistics are presented. The non-extensive parameter, entropic index q , is discussed. We attempt to explore the limit of the non-extensive parameter by comparing the theoretical results with lattice and the available experimental results. The two thermal parameters T , μ_B are calculated with the freeze-out condition $S/T^3 = 7$ for different q . The motivation of this research comes from recent non-extensive statistics studies which showed that this standard thermodynamics failed to reproduce the freeze-out parameters. As an application, the black-hole entropy is calculated in the quantum Generalized Uncertainty Principle (GUP) modification form. Black-hole entropy may reveal information about the thermodynamics it belongs. This discrimination is essential to quantify the entropy in the hadron production evolution stage and in the black-hole thermodynamics. It is concluded that lattice QCD reproduces the extensive thermodynamics very well. Also, the black hole appears as an extensive system.

DOI:10.5506/APhysPolB.47.1301

1. Introduction

The nature of entropy cannot be interpreted and known exactly. In a common, entropy is a measure of disorder. The information of the system is represented by entropy. In this special case, entropy represents the uncertainty in the system. As the information about the system increases, entropy increases [1]. A century ago, a wonderful summary of thermodynamics was presented by Clausius [2]:

- The energy of the Universe is constant.
- The entropy of the Universe strives toward a maximum.

The number of particles states in a system can be represented statistically in coordinate-momentum space as $6N$ -dimensional (phase space). The entropy is expressed as a logarithm of the number of states W times the Boltzmann constant

$$S = k_B \ln W, \quad (1)$$

where $W = \Omega/(\Delta\Omega)^N$, Ω is the phase-space volume, and $\Delta\Omega$ is the elementary cell volume. Shannon [3], Renyi [4], and Tsallis [5] have their statistical framework to describe the entropy. Shannon entropy reads

$$S = -k_B \sum_i P_i \ln P_i. \quad (2)$$

In 1988, Tsallis [5] postulated a generalization of Boltzmann-Gibbs (BG) statistics. Tsallis entropy reads

$$S_q \equiv k \frac{1 - \sum_i^W P_i^q}{q - 1}, \quad (q \in \mathbb{R}), \quad (3)$$

where k is a constant, q is the new parameter known as the entropic index and it should be greater than 1 [5], W is the total number of possible configurations and P_i are the probabilities associated with the configurations.

The deformed form of the exponential function, q -exponential, and the q -logarithmic have been applied in Tsallis statistics [6] recently to get the non-extensive partition function [7, 8].

The BG entropy is an additive quantity for any system and subsystems, for example, if one has a system A composed of two subsystems A_1, A_2 , so that the total entropy $S(A) = S(A_1) + S(A_2)$. The latter is true in BG statistics. On the other hand, Tsallis statistics leads to non-extensive property in which ($q \neq 1$) [9]

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B). \quad (4)$$

More properties of the q -sum as in equation (4), q -logarithmic and q -exponential can be found in [9].

This work is organized as follows. Section 2 describes briefly the thermal model and the calculation of entropy from the partition function (extensive thermodynamics). Section 3 discusses Tsallis (non-extensive) thermodynamics and the non-extensive parameter q (entropic index). Results and discussion are represented in Section 4. Finally, a conclusion is presented Section 5.

2. The thermal model

The thermal model is one of the most powerful tools to describe the system thermodynamics and the particle number measured in heavy-ion collision experiments. After the heavy-ion collision, the hot matter created begins to expand forming a hadron gas. This leads to the freezing out of the hadronic matter. Then, the particle produced moves freely to the detector. The thermal model or Hadron Resonance Gas (HRG) describes the hadron gas stage at which the hadrons are a free gas of resonances. The chemical and thermal freeze-out stages and freeze-out conditions can be extracted and described by the thermal model. Generally, the freeze-out process can be categorized into two stages:

- Chemical freeze-out stage at which the inelastic collisions between hadrons cease.
- Thermal freeze-out stage at which the elastic collisions cease and the momenta of the final state particles (hadrons) are fixed. In other words, it is the moment at which the hadrons cease to interact and begin to escape freely to the detector.

Furthermore, it satisfactorily describes different experimental measurements such as AGS, SPS, and RHIC [10]. In addition, the thermal model agrees with lattice QCD in the hadronic phase very well.

Different thermodynamic quantities (*e.g.* pressure, energy density, entropy, *etc.*) can be measured and studied by the thermal model. The calculation based on the grand canonical partition function in the Boltzmann–Gibbs statistics, in which all thermodynamics can be derived

$$\ln Z = \text{Tr} \left(\exp^{\beta(\mu \hat{\mathbf{b}} - \hat{\mathbf{H}})} \right), \quad (5)$$

where $\beta = 1/T$, and $\hat{\mathbf{b}}, \hat{\mathbf{H}}$ are the baryon number and the Hamiltonian of the system, respectively. The partition function is additive, which suggests that it is an extensive quantity. In the HRG model, thermodynamic quantities can be directly derived from the logarithm of the partition function and summed over all resonances i

$$\ln Z = \sum_i \frac{V g_i}{(2\pi)^3} \int \pm d^3p \ln [1 \pm \exp(-(\varepsilon_i - \mu_i)/T)], \quad (6)$$

where $\pm$ refers to fermions and bosons, respectively. The chemical potential is given by $\mu_i = \mu_s S + \mu_q Q + \mu_b B$, with μ_s, μ_q, μ_b the strange, quark and baryon chemical potentials, respectively, multiplied by the corresponding quantum numbers S, Q, B , $\varepsilon_i = \sqrt{p^2 + m_i^2}$, g_i is the degeneracy of the particles where masses m_i are taken up to 10 GeV in this work.

For canonical system, the entropy can be defined by the derivative of the partition function with respect to the temperature T . Then, the entropy reads

$$s_i = \frac{1}{V} \left(\frac{\partial (T \ln Z_i)}{\partial T} \right)_{V,\mu} = \pm \frac{g_i}{(2\pi)^3} \int d^3p \left[\ln \left(1 \pm \exp \left(-\frac{(\varepsilon_i - \mu_i)}{T} \right) \right) \pm \frac{(\varepsilon_i - \mu_i)}{T(\exp((\varepsilon_i - \mu_i)/T) \pm 1)} \right]. \quad (7)$$

3. Non-extensive thermodynamics

The non-extensive thermodynamics emerged from experimental results of spectra that showed deviations from the Boltzmann exponential behavior. This was the motivation of Tsallis [5] who suggested a new statistics to describe non-equilibrium systems with a new parameter known as the entropic index q . A good fit for the transverse momentum distribution can be obtained using the parameter q [11]. The entropic index q is claimed to be limited in the range of $q \geq 1$ [5], while others [12, 13] have assumed that q should be $q \sim 1.12$ – 1.14 . The values of the Tsallis entropic index q , Fig. 1, are found experimentally to lie between 1.11 and 1.15, they increase with the center-of-mass energy.

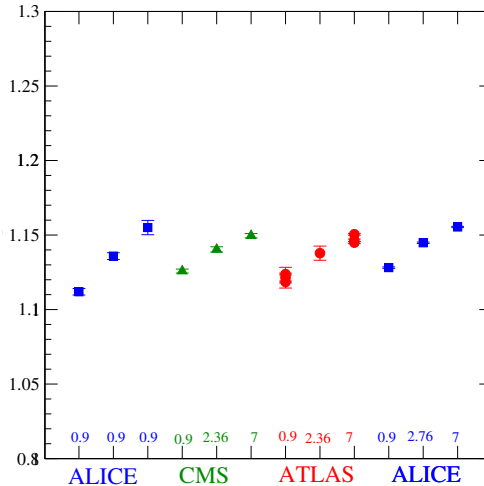


Fig. 1. The Tsallis parameter q obtained from fits to the p_T spectrum. Results from the ALICE Collaboration are indicated with square points, CMS by triangles and ATLAS by circles. The beam energy is given on the x axis. The figure is taken from Ref. [14].

In the present work, the entropic index q is increased up to $q = 1.2$. However, the Tsallis freeze-out temperature lies in the thermal model at smaller q . We discuss this result in Section 4. For an ideal quantum gas, it is argued that the grand-canonical partition function represented by [7]

$$\log \Xi_q(V, T, \mu) = -\xi V \int \frac{d^3p}{(2\pi)^3} \sum_{r=\pm} \Theta(rx) \log_q^{(-r)} \left(\frac{e_q^{(r)}(x) - \xi}{e_q^{(r)}(x)} \right) \quad (8)$$

and

$$\log \Xi_q(V, T, \{\mu\}) = \sum_i \log \Xi_q(V, T, \mu_i), \quad (9)$$

where $x = \beta(\varepsilon - \mu)$, the dispersion relation represents the particle energy as $\varepsilon = \sqrt{p^2 + m^2}$, with the hadron mass m , and the chemical potential μ . Finally, Θ is taken as the step function [7], while $\xi = \pm 1$ for bosons and fermions, respectively. It is also essential that the partition function is a sum over all resonances. The q -exponential $e_q^{(r)}$ and the q -logarithm $\log_q^{(r)}$ are defined as

$$\begin{aligned} e_q^{(+)}(x) &= [1 + (q-1)x]^{1/(q-1)}, & x \geq 0, \\ e_q^{(-)}(x) &= [1 + (1-q)x]^{1/(1-q)}, & x < 0, \\ \log_q^{(+)}(x) &= \frac{x^{q-1} - 1}{q-1}, \\ \log_q^{(-)}(x) &= \frac{x^{1-q} - 1}{1-q}. \end{aligned} \quad (10)$$

In Fig. 1, the values of the entropic index (Tsallis parameter) q are shown. It can be seen that they increase with the beam energy [14]. However, the highest value is approximately $q = 1.15$.

4. Results and discussion

4.1. Extensive and non-extensive entropy

In the present work, we have compared the entropy calculated extensively and non-extensively at vanishing chemical potential ($\mu_B = 0$), and at non-zero chemical potential ($\mu_B = 0.3, 0.8$ GeV). Figure 2 shows the entropy calculated by HRG and non-extensive thermodynamics. The latter is calculated at different values of $q = 1.0001, 1.05, 1.14, 1.2$.

It is argued that at $q = 1$, Boltzmann statistics is recovered [7]. From Fig. 2, where q is roughly very close to $q = 1$, the entropies from HRG and non-extensive are comparable. Also, it appears, by increasing q up to

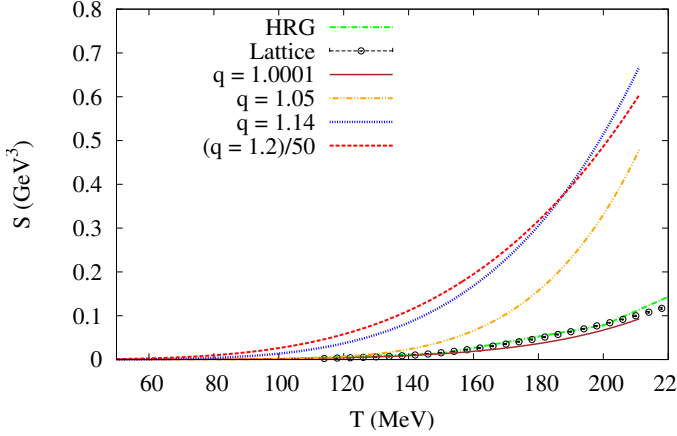


Fig. 2. The online colored line and symbols are the calculated entropy by HRG (extensive), non-extensive entropy for different entropic index, $q = 1.0001, 1.05, 1.14, 1.2$, and vanishing baryon chemical potential compared with lattice results [15].

1.2 that the non-extensive entropy needs to be scaled. In order to compare with the lattice results [15], the non-extensive entropy at $q = 1.2$ is scaled (divided) by 50. The different lines match all together till $T \simeq 100$ MeV and diverge after that point.

It seems that, at the non-vanishing chemical potential ($\mu_B \neq 0$), the extensive and non-extensive calculated entropy are comparable without scaling. Figure 3 shows the entropy calculated by HRG and non-extensive thermodynamics at $\mu_B = 0.3$ GeV and at $q = 1.14$. It is noticed that the different lines begin to grow apart at $T \approx 100$ MeV.

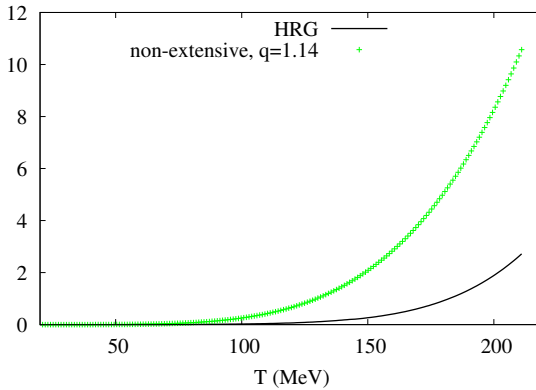


Fig. 3. The solid colored line and symbols are the calculated entropy by HRG compared with non-extensive entropy calculated for $q = 1.14$ and at $\mu_B = 0.3$ GeV.

4.2. Black-hole thermodynamics (BH)

The black-hole entropy is merely an application to the black-hole thermodynamics. The early study of Benkenstein and Hawking [16, 17] showed that the black-hole entropy is given by

$$S_{\text{BH}} = \frac{A}{4\hbar}, \quad (11)$$

where A is the area of the black-hole horizon. It is argued that the horizon increases with the entropy. This was the kick-off of what has become black-hole thermodynamics [18]. The modified black-hole thermodynamics due to modified gravity has been studied in [19, 20].

In Ref. [19], the specific heat of the black hole is studied. It is shown that a black hole is thermodynamically unstable. For this reason, we have compared the black-hole entropy within the different statistics mentioned above. One point of particular interest is the calculation of the modified black-hole entropy due to a generalized uncertainty principle and quantum correction (QGUP), see details [20, 21]

$$\frac{S}{k_{\text{B}}} = 1 + \frac{\beta^2 E_{\text{P}}^2}{16\pi} - \ln \left(\sqrt{\frac{3}{2\pi}} \beta E_{\text{P}} \right) - \frac{15m_{\text{P}}\alpha^2}{\beta} + \mathcal{O}(\alpha^4), \quad (12)$$

where the Planck energy $E_{\text{P}} = \sqrt{\frac{\hbar c^5}{G}}$, $\beta = 1/T$ and Planck mass $m_{\text{P}} = \sqrt{\frac{\hbar c}{G}}$ and α is a free parameter.

It is known that black hole is a logarithm of a number of independent states [22, 23]. In Ref. [24], 't Hooft stated that the quantum states in a finite region must have finite dimensions. It is supposed that those states are associated with the two-dimensional boundaries of that region. Other arguments of black-hole entropy have been suggested, such as the thermal radiation with Unruh effect in which the black-hole entropy arises from the quantum field outside the event horizon [25]. Entanglement entropy can also be considered as a part of the black-hole entropy [26] in which

$$S_{\text{ent}} = \text{Tr} \rho_{\text{ext}} \ln \rho_{\text{ext}}, \quad (13)$$

where ρ_{ext} is the reduced density matrix. For different reasons beside the listed above, we elaborated a comparison between extensive and non-extensive thermodynamics for a real thermodynamical system with temperature T and entropy S (black-hole). To the authors' best knowledge this is the first time to classify the black-hole entropy between the different statistics (extensive and non-extensive thermodynamics).

Figure 4 shows the calculated black-hole entropy (dashed line) compared with the non-extensive entropy calculated at $q = 1.0001, 1.05, 1.14$, respectively. Again, the entropy at $q = 1.2$ has to be scaled to be compared with the corresponding entropies at different q , lattice results and QGUP. The modified black-hole entropy is calculated from Eq. (12) at $\alpha = 0.01$. The QGUP entropy are elaborated in units where ($\hbar = C = G = k_B = 1$). QGUP entropy is a good matching with the lattice and the non-extensive entropy at $q = 1.0001$.

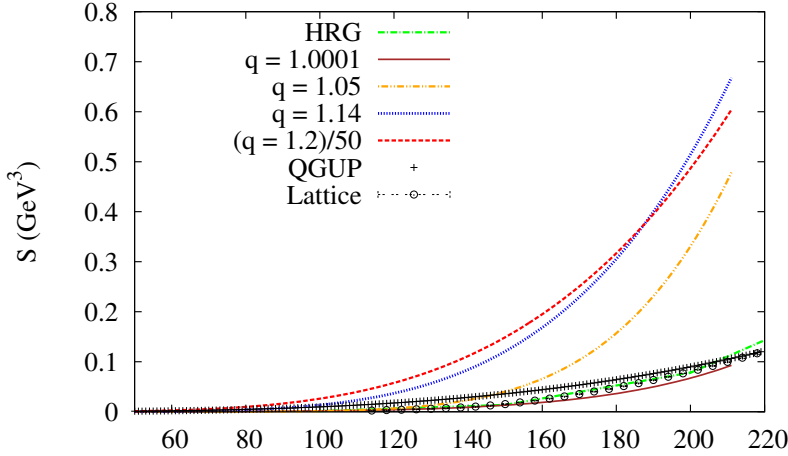


Fig. 4. The online colored lines and symbols are the calculated black-hole entropy compared with extensive and non-extensive entropy for different $q = 1.0001, 1.05, 1.14, 1.2$, and lattice results [15].

4.3. Freeze-out condition (S/T^3)

The temperature *versus* the baryon chemical potential, ($T-\mu_B$), has been studied at different freeze-out conditions such as $\frac{\langle E \rangle}{\langle N \rangle} \approx 1$ [27, 28], and for ($S/T^3 = 7$) [29, 30]. In the present work, $T-\mu_B$ for the freeze-out condition from non-extensive entropy ($S/T^3 = 7$) is studied at different values of $q = 1.05, 1.14, 1.2$. Figure 5 represents the temperature T *versus* the baryon chemical potential μ_B at the freeze-out condition $S/T^3 = 7$. These calculations are estimated for non-extensive entropy for different q at the mentioned values. It shows very slight change in the temperature within the chemical potential up to $\mu_B = 0.8$ GeV.

It is clear that at $q = 1.05, 1.14, 1.2$, Tsallis temperature must be scaled by 2 to be comparable with the experimental data [31, 32, 34] and [33]. The scaled temperature at $q = 1.05$ is well comparable with RHIC [34] and SPS [31].

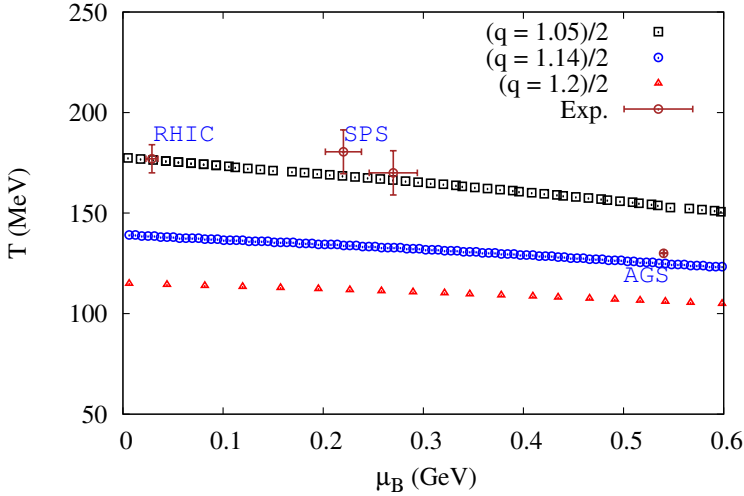


Fig. 5. The symbols (colored on-line) are temperature *versus* the baryon chemical potential, at different $q = 1.05, 1.14, 1.2$ calculated for freeze-out condition $S/T^3 = 7$ compared with different experiments.

A parameterization of the temperature depending on the chemical potential in Fig. 5 at $q = 1.05, 1.14, 1.2$ is done. This parameterization can be written as a function of μ_B as follows:

$$T_s = (-11.1 \mu_B^2 - 38.1 \mu_B + 177.4), \quad q = 1.05, \quad (14)$$

$$T_s = (-8.1 \mu_B^2 - 21.6 \mu_B + 139.1), \quad q = 1.14, \quad (15)$$

$$T_s = (-5.6 \mu_B^2 - 13.5 \mu_B + 115.1), \quad q = 1.2. \quad (16)$$

By substituting $\mu_B = 0$ in Eqs. (14)–(16), the freeze-out temperatures are obtained as 177.4, 139.1, and 115.1 MeV, respectively. At $q = 1.14$, the scaled freeze-out temperature lies in the range of the thermal model freeze-out temperature. In addition, it is remarkable that at $q = 1.05$, there is better agreement at RHIC and SPS. From the parameterized relations (14), (15) and (16) at $\mu_B = 0$ (say T_s).

The freeze-out temperature T_s *versus* the entropic index q is extracted from Fig. 5. These extracted values are plotted in Fig. 6. From this plot, the parameterization of the line can be written as

$$T_s \sim C \exp^{-2.88 q}, \quad (17)$$

where C is a parameterization constant ($C = 3.66 \times 10^3$ MeV). There is no limit apparent for μ_B at $T = 0$.

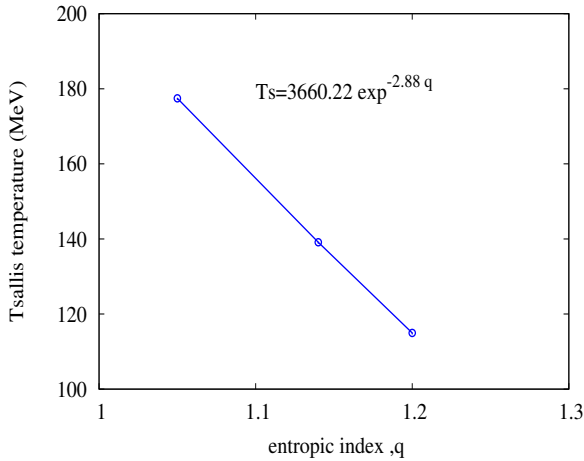


Fig. 6. The symbols are the Tsallis temperature *versus* q (entropic index) estimated from Fig. 5.

5. Conclusions

We have employed the hadron resonance gas model and non-extensive statistics to describe heavy-ion collision thermodynamics. From this study, it is concluded that the non-extensive thermodynamics should be scaled to be comparable with the extensive thermodynamics at certain values of q . The freeze-out temperature tends to be compatible with the corresponding one in the extensive thermal model at entropic index slightly larger than 1.

We also conclude that the black hole appeared as an extensive system, this is very clear from its closely coincidence to the extensive one and the lattice results as shown in Fig. 4. Finally, we conclude that if the entropic index is very close to 1, the non-extensive and extensive thermodynamics are comparable as expected.

REFERENCES

- [1] P. Jizba, J. Korbela, *Physica A* **444**, 808 (2016) [arXiv:1501.07386v2 [math-ph]]
- [2] R. Clausius, *The Mechanical Theory of Heat, with Its Applications to the Steam Engine and to Physical Properties of Bodies*, 1865. London: John van Voorst, 1 Paternoster Row. MDCCCLXVII.
- [3] C.E. Shannon, *Bell Syst. Tech. J.* **27**, 379 (1948).
- [4] A. Renyi, *Some Fundamental Questions of Information Theory, MTA III Oszt. Közl.* **10**, 251 (1960).
- [5] C. Tsallis, *J. Stat. Phys.* **52**, 479 (1988).

- [6] C. Tsallis, *Quimica Nova* **17**, 468 (1994).
- [7] E. Megias, D.P. Menezes, A. Deppman, *Physica A* **421**, 15 (2015).
- [8] E. Megias, D.P. Menezes, A. Deppman, *EPJ Web Conf.* **80**, 00040 (2014) [arXiv:1407.8044v1 [hep-ph]].
- [9] S. Umarov, C. Tsallis, S. Steinberg, *Milan J. Math.* **76**, 307 (2008).
- [10] M. Michalec, Ph.D. Thesis, 2001, arXiv:nuc1-th/0112044.
- [11] J. Cleymans, D. Worku, *Eur. Phys. J. A* **48**, 160 (2012).
- [12] M.D. Azmi, J. Cleymans, *Eur. Phys. J. C* **75**, 430 (2015).
- [13] J. Cleymans *et al.*, *Phys. Lett. B* **723**, 351 (2013) [arXiv:1302.1970 [hep-ph]].
- [14] M.D. Azmi, J. Cleymans, *J. Phys. G: Nucl. Part. Phys.* **41**, 065001 (2014) [arXiv:1401.4835 [hep-ph]].
- [15] G.S. Bali *et al.*, *J. High Energy Phys.* **1408**, 177 (2014) [arXiv:1406.0269v2 [hep-lat]].
- [16] J.D. Bekenstein, *Phys. Rev. D* **7**, 2333 (1973).
- [17] S.W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).
- [18] A. Ghosh, P. Mitra, *Class. Quantum Grav.* **32**, 165006 (2015) [arXiv:1506.08068 [gr-qc]].
- [19] K. Bamba, C.-q. Geng, S. Tsujikawa, *Int. J. Mod. Phys. D* **20**, 1339 (2011).
- [20] P. Bargueñoa, E.C. Vagenas, *Phys. Lett. B* **742**, 15 (2015).
- [21] B. Majumder, *Gen. Relativ. Gravitation* **45**, 2403 (2013).
- [22] G. Gourm, *Phys. Rev. D* **66**, 104022 (2002) [arXiv:gr-qc/0210024].
- [23] T. Jacobson, *AIP Conf. Proc.* **493**, 85 (1999) [arXiv:gr-qc/9908031].
- [24] G. 't Hooft, *On the Quantization of Space and Time, Quantum Gravity Proceedings*, Editors M.A. Markov, V.A. Berezin, V.P. Frolov, World Scientific, 1988.
- [25] G. 't Hooft, *Nucl. Phys. B* **256**, 727 (1985).
- [26] L. Bombelli, R.K. Koul, J. Lee, R.D. Sorkin, *Phys. Rev. D* **34**, 373 (1986).
- [27] J. Cleymans, K. Redlich, *Phys. Rev. Lett.* **81**, 5284 (1998).
- [28] J. Cleymans, K. Redlich, *Phys. Rev. C* **60**, 054908 (1999).
- [29] J. Cleymans, H. Oeschler, K. Redlich, S. Wheaton, *J. Phys. G: Nucl. Part. Phys.* **32**, S165 (2006).
- [30] A. Tawfik, *Nucl. Phys. A* **764**, 387 (2006).
- [31] F. Becattini, M. Gazdzicki, J. Sollfrank, *Eur. Phys. J. C* **5**, 143 (1998).
- [32] P. Braun-Munzinger, J. Stachel, J.P. Wessels, N. Xu, *Phys. Lett. B* **344**, 43 (1995).
- [33] J. Cleymans, D. Elliott, A. Keränen, E. Suhonen, *Phys. Rev. C* **57**, 3319 (1998).
- [34] P. Braun-Munzinger, D. Magestro, K. Redlich, J. Stachel, *Phys. Lett. B* **518**, 41 (2001); N. Xu, M. Kaneta, *Nucl. Phys. A* **698**, 306 (2002) [arXiv:nuc1-ex/0104021].