MEANS OF DAMPING RADIAL BETATRON OSCILLATIONS IN STRONG FOCUSING ELECTRON RINGS

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I. INTRODUCTION

In connection with a tentative proposal to build a pair of 100-m radius storage rings for the CERN Proton Synchrotron, we have looked at the possibility of also storing electrons in these rings, and adapting the C.P.S. so that it is capable of accelerating them. If one is content with an electron energy of some four or five Gev, the main problem is that of the betatron oscillation antidamping caused by radiation. This antidamping in strong-focusing accelerators has already been studied in several papers[1,2]; we consider here how it depends upon the choice of beam position across the available horizontal aperture of the vacuum chamber. A fuller account has been given in an internal report[3], which considers the numerical values for the CERN case in detail, including the stochastic quantum effects.

II. DAMPING RATES

If one of the modes of oscillation has its amplitude damped like e^{-kt}, we call k its damping rate. We shall use k_s for the synchrotron oscillations, k_1 and k_2 for betatron oscillations in general, k_h and k_v for horizontal and vertical betatron oscillations.

One may take from Robinson[1] the following formulae:—

$$k_s+k_1+k_2=2\ P/E \quad \text{always,} \tag{1}$$

$$k_v = \tfrac{1}{2}\ P/E \quad \text{if there is no coupling between horizontal and vertical motion,} \tag{2}$$

$$k_s = (2-\alpha/2)P/E \quad \text{if the principal orbit is isomagnetic.} \tag{3}$$

From these there follows:—

$$k_h = -(\tfrac{1}{2}-\alpha/2)\ P/E \tag{4}$$

if no coupling and principal orbit isomagnetic. Here P is the radiated power of one particle (energy radiated per unit time), E is the particle energy, α the momentum compaction factor. In a machine with field-free sections, no radiation and no damping occurs in them, so the formulae remain valid provided we take P to be the mean radiated power, and

$$\alpha = \frac{\Delta R}{R_0}\Big/\frac{\Delta p}{p_0}, \tag{5}$$

with R_0 the bending radius within the magnets.

From (1) and (3) we have

$$k_1+k_2=(\alpha/2)(P/E), \tag{6}$$

and, if one could introduce enough coupling between the horizontal and vertical motion to make both betatron modes polarised at exactly $\pm 45°$, one would have

$$k_1=k_2=\alpha/4\ P/E. \tag{7}$$

In practice, the initial splitting between horizontal and vertical Q-values makes it difficult to obtain polarisations sufficiently close to $\pm 45°$, and one mode is likely to be slightly more horizontal than vertical and to remain antidamped. It is therefore of interest to look for means of reducing k_s below the value (3), so that a more substantial amount of damping is available for the betatron modes.

It would be very interesting to be able to make:—

$$k_s=1.5\ P/E. \tag{8}$$

Then without coupling one would have:—

$$k_v=0.5\ P/E \qquad k_h=0. \tag{9}$$

This only just removes the horizontal antidamping, but it puts one into the position that any horizontal-vertical coupling will ensure that all modes are damped, and coupling that swamps the original h/v split by a modest factor like three or four will give one, very roughly,

$$k_1 \approx k_2 \approx 0.25\ P/E. \qquad (10)$$

The value (8) for k_s is a somewhat arbitrary aim, but it seems fair to regard any trick or device that is capable of reducing k_s to this value, or below, as a solution to the antidamping problem; while anything that does not reduce k_s so much as this merely eases the tuning tolerances involved in the use of the coupling method.

The most obvious way of achieving (8) is to put

$$\alpha = 1. \qquad (11)$$

This almost certainly means $Q_h < 1$, so the machine is a weak-focusing one at least in the horizontal plane, and we shall not discuss it further.

III. DISPLACED EQUILIBRIUM ORBITS

Another way of reducing the synchrotron oscillation damping rate is to use the wiggles of off-centre equilibrium orbits, which typically produce quite substantial changes in the orbit curvature. They can change the factor $(2-\alpha/2)$ is (3), which comes from a first-order theory, valid only for the case where the oscillations take place about the central (isomagnetic and circular, except in the field-free sectors) closed orbit, and where the wiggles can be and are neglected. A rough theory including the wiggles follows.

Ignoring the wiggles, a particle moves along, or makes betatron oscillations about, an orbit at $R_0+\Delta R$, with:—

$$\Delta R/R_0 = \alpha \Delta p/p_0; \qquad (12)$$

this is the definition of α. We suppose the wiggles have a radial amplitude of

$$a\ \alpha\ R_0\ \Delta p/p_0. \qquad (13)$$

These wiggles have M wavelengths per machine revolution, where M is the number of magnet periods. If we assume that they are sinusoidal:—

$$\Delta R/R_0 = \alpha(1 + a \sin M\ \theta)\Delta p/p_0. \qquad (14)$$

In a FOFDOD structure all closed orbits are straight and parallel in the field-free sections, so it is convenient to regard θ as a variable that changes by $2\pi/N$ (N is the number of magnet units, equal to $2M$) in each magnet unit, and is constant in the straight sections.

To first order in $\alpha\ \Delta p/p_0$ (which is typically of the order of 10^{-3}) the curvature C of such an orbit is $C_0+\Delta C$ with:—

$$\Delta C/C_0 = -\alpha\{1 - a(M^2-1) \sin M\theta\}\ \Delta p/p_0. \qquad (15)$$

For the C.P.S. $a(M^2-1)$ is about 350, so the wiggles introduce much larger curvature changes than does the mean orbit shift. Some small higher-order terms are neglected in (15).

The power P radiated by a particle of momentum p in a magnet field that produces a curvature C is proportional to

$$p^4\ C^2. \qquad (16)$$

What we are interested in, however, is the energy loss per machine radian (or turn); for this can legitimately be averaged over θ, and the rf system, which replaces the mean energy loss, produces a certain energy gain per turn, not per unit time. We may use U to represent the radiated energy per radian.

In the relativistic region, $\beta=1$, one has

$$\frac{d\theta}{dt} = c/R\ , \qquad (17)$$

so that U is proportional to RP, and so to

$$Rp^4\ C^2. \qquad (18)$$

Now we put $U=U_0+\Delta U$, and (18) gives:—

$$\Delta U/U_0 = \Delta R/R_0 + 4\Delta p/p_0 + 2\Delta C/C_0 + (\Delta C/C_0)^2. \qquad (19)$$

Although we have worked, so far, only to first order in Δ quantities, we include $(\Delta C\ C_0)^2$ in (19) because of the large value of $a(M^2-1)$. We shall retain terms like:—

$$(\alpha\Delta p/p_0)^2\ a^2\ (M^2-1)^2, \qquad (20)$$

while neglecting those like:—

$$(\Delta p/p_0)^2, \qquad (21)$$

and

$$\alpha(\Delta p/p_0)^2\ a\ (M^2-1). \qquad (22)$$

From (19), (15) and (14), and to this approximation, one finds

$$\Delta U/U_0 = \{4 - \alpha + a\ \alpha(2M^2-1) \sin M\ \theta\}\Delta p/p_0 + a^2\ (M^2-1)^2 \sin^2\ \theta\ (\alpha\Delta p/p_0)^2 \qquad (23)$$

and averaged over θ this gives:—

$$\left\langle\frac{\Delta U}{U_0}\right\rangle_\theta = \{4 - \alpha\}\ \Delta p/p_0 + \tfrac{1}{2}\alpha^2\ a^2\ (M^2 - 1)^2\ (\Delta\ p/p_0)^2. \qquad (24)$$

For synchrotron oscillations that are small, but are centred on some finite Δp, the quantity that determines their damping rate is the derivative of (24) with respect to $\Delta p/p_0$:—

$$4-\alpha+\alpha^2 a^2 (M^2-1)^2 \Delta p/p_0. \quad (25)$$

It can be shown, in a rather general way, that this is the ratio of the damping rate of the longitudinal phase-space to the radiation damping rate of the energy, so

$$k_s = \{2-\alpha/2+\tfrac{1}{2}\alpha^2 a^2 (M^2-1)^2 \Delta p/p_0\} \; P/E. \quad (26)$$

For practical purposes this is more convenient in the form:—

$$k_s = \{2-\alpha/2+\alpha a^2 (M^2-1)^2/2R_0 \cdot \overline{\Delta R}\} \; P/E, \quad (27)$$

where $\overline{\Delta R}$ represent the smoothed ΔR corresponding, (12), to Δp.

For $\overline{\Delta R}=0$, these expressions agree with (3), but the new term enables us to give k_s whatever value we like within limits set by the available width of the vacuum chamber. With the parameters of the C.P.S. one finds

$$\alpha a^2 (M^2-1)^2/2R_0 = 32 \text{ m}^{-1}. \quad (28)$$

The assumption that the wiggles are sinusoidal is only approximate, but one can evidently generalise (24) to

$$\left\langle \frac{\Delta U}{U_0} \right\rangle = (4-\alpha)\ \Delta p/p_0 + (b\ \Delta p/p_0)^2, \quad (29)$$

where b is the relative root mean square curvature change associated with the wiggles, per unit $\Delta p/p_0$. For a given amplitude and periodicity a sine curve has almost the lowest possible root-mean-square curvature, so one must expect (28) to be a low estimate. With a little numerical work we can obtain a better estimate from the usual expression for the closed orbits in terms of circular and hyperbolic functions in the F and D sectors (or alternatively use the known Fourier expansion of these orbits[4]). In this way we have calculated for the C.P.S.:

$$k_s = \{2-\alpha/2+39\overline{\Delta R}\} \; P/E, \quad (30)$$

where $\overline{\Delta R}$ is expressed in metres. We can, therefore, obtain the convenient value (8), of k_s by putting the beam at

$$\overline{\Delta R} = -12 \text{ mm}. \quad (31)$$

From (1), (2), and (30), one finds that the available aperture of a machine is divided into four distinct regions. With the C.P.S. parameters:—

(a) Almost all the outer half, $\overline{\Delta R} > 0.5$ mm, is "forbidden": in this region k_s is so big that no damping remains for the betatron oscillations.

(b) For $0.4 \text{ mm} > \overline{\Delta R} > -12$ mm there is damping available for the betatron modes, and they will both be damped if the coupling is adequate. "Adequate" coupling ranges from perfect coupling at 0.5 mm down to any coupling at all at -12 mm.

(c) For $-12 \text{ mm} > \overline{\Delta R} > -51$ mm all modes are damped, even without any coupling, but in the neighbourhood of -12 mm it is useful to have some coupling, for otherwise the horizontal betatron oscillations will be damped only very slowly.

(d) From -51 mm to the aperture limit at -62 mm is "forbidden" because the synchrotron oscillations are antidamped.

These results are illustrated by Fig. 1.

It should be mentioned that these regions and limits apply to the synchronous orbit

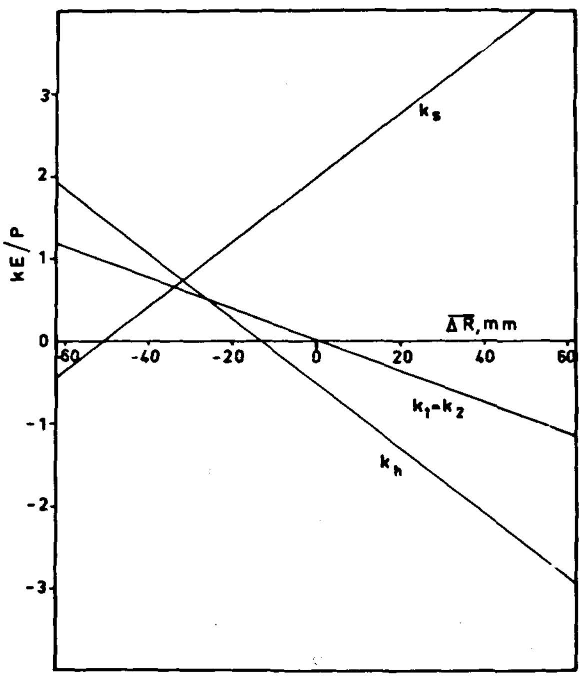

Fig. 1 Variation of damping rates with beam position across the aperture of the C.P.S. The line k_s refers to the synchrotron oscillations, k_h to the horizontal betatron oscillations without coupling, and $k_1=k_2$ applies to both betatron modes if they are fully coupled.

position, not to instantaneous positions of the particles. If a damping rate varies in course of the oscillations, there seems to be no objection to working with its time-average, provided only that the fastest instantaneous antidamping rate is slow compared with the slowest oscillation rate. To be absolutely sure of this statement a higher-order theory or numerical computations are needed; but, if it is true, then one can consider holding the synchronous orbit at say -15 ± 3 mm, and have 44 mm available for the sum of the betatron and synchrotron amplitudes.

It could almost be argued that the numerical value, 39 m^{-1}, appearing in (30) is just about ideal for accelerating electrons in the C.P.S. It makes the safe region (c) extend reasonably close to the center of the vacuum chamber, while still being wide enough to allow some space for beam manipulation. Up to say 3 Gev one can accelerate in the centre of the chamber, even with no coupling, without the radial antidamping being of any consequence. At 3 Gev even with the highest likely injector energy, the Liouvillian damping has reduced the betatron and synchrotron amplitudes by factors exceeding two, so one can comfortably move the synchronous orbit inwards to ~ -15 mm and accelerate on up to the limit set by available rf volts.

The same properties seem also to be satisfactory for the storage rings. The rather limited width of the safe region rules out stacking processes in which electrons are carried in buckets slowly across most of the aperture, but such processes are anyway not very interesting for particles that have substantial radiation loss and damping effects. However, it may be desirable to remove this restriction on the use of the horizontal aperture in order to have a more flexible machine. We consider means of doing so in the next section.

IV. THE ADDITION OF SPECIAL MAGNETS

One may add to a strong-focusing ring (for example in some of the field-free sectors) special magnets with alternations of sign, in the circumferential direction, sufficiently closely spaced that they have negligible effects on everything except the radiation rate. If they change the radial dependence of radiation rate in a suitable way, they will eliminate the horizontal betatron antidamping at the center of the chamber[1], given in their absence by (4). In the light of the previous section it is convenient to regard such magnets as devices for moving or widening the existing safe region (c).

The simplest type of special magnet is one that adds, to the radiation rate, a term linear in radial position. It is clear that this will add a constant term to k_s, and so slide the whole pattern of k values across the aperture without affecting the width of the safe region. To widen the safe region one would add another magnet or set of magnets which adds to the radiation rate a term proportional to $-\Delta R^2$, and so alters k_s by a term linear in ΔR.

As this second set of magnets is progressively energised, the whole pattern of k values expands horizontally about the centre of the aperture, so one gains in freedom of choice of beam position, but obviously loses progressively the possibility of altering the share-out of damping by moving the beam. They can be energised to the point where the safe region is infinitely wide and the k values are independent of ΔR, whereupon the share-out of damping can only be altered by changing the strength of the first set of magnets.

We have estimated the amount of straight-section length that would be required for two such sets of magnets in a structure like that of the C.P.S., and find that a total of the order of four metres would be sufficient.[3]

V. SUMMARY

In a strong-focusing structure like that of the CERN P.S., one could accelerate or store electrons without radiation antidamping by keeping the centre of the beam in a region a little inwards from the centre of the vacuum chamber. If it is desired to move or widen this safe region, this can be done by adding sets of special magnets in the straight-sections of the machine.

REFERENCES

1. K. W. Robinson, *Proc. International Conference, CERN 1959*, p. 293, and references therein.

2. V. F. Orlov, E. K. Tarasov, S. A. Khejfets, *Proc. International Conference, CERN 1959*, p. 306, and references therein.
3. H. G. Hereward, *Some Effects of Radiation Damping for Electrons in the C.P.S. or in Storage Rings of Similar Properties*, AR/Int. SR/61-15, CERN 1961.
4. A. Schoch, private communication.

DISCUSSION

E. D. Courant: I would like to add a comment about the formula you have written there: $k_s = (2 - \alpha/2)P/E$. I believe it is derived on the assumption that the radius of curvature is the same in the positive focusing and negative focusing magnets, and that both the scheme of moving the equilibrium orbit and the scheme of putting in special magnets will destroy that formula because the radii of curvature are different in the different focusing magnets.

H. G. Hereward: Yes, that is right. This formula is valid for the case where the orbit consists of parts of a circle; there can be straight sections but there must not be regions of different nonzero curvature. The consequence, in my case, is that the formula is valid in the centre of the vacuum chamber but not for a radially displaced beam. Perhaps I should mention that the Frascati group has looked at the possibility of removing this antidamping. One of the things that they have considered is to make the defocusing magnets a little stronger, a little greater in curvature, than the focusing magnets. Now, in my terminology, this corresponds to moving the beam over into the safe region and then moving the vacuum chamber as well. So, there is no real conflict of ideas here, but one should remember that the safe region does still remain of finite width unless something additional is done.

H. D. Bruck: Can you say a little more about the more fundamental fact that the sum of the three damping contants is something fixed?

H. G. Hereward: I think one could write rather a lot on this subject. About all I would like to say is that several persons have looked at this problem, and in different ways, and found that this formula, 2 P/E for the sum of the damping rates, seems to arise, whatever type of structure you have and whatever way you calculate its properties. It does seem to arise from something fundamental about the damping process, but I don't know of a really simple proof that it is inevitable.

H. D. Bruck: I don't know the answer but it is an evolution of the disorder or entropy of the beam. And I think there is a thermodynamic reason for its value, but I don't know exactly the explanation.

H. G. Hereward: I would be inclined to doubt that there are thermodynamic considerations involved here. This is entirely on a classical basis. We have said nothing about the quantum fluctuations and the associated stochastic antidamping. I think, rather, that this type of formula arises from some consideration of what happens to the multidimensional phase space involved if one takes the main motion and the three modes of oscillation together. As you know, if one has a complicated secular equation, it is easier to discover the sum of the roots than the roots individually and, in linear approximation, that is the way one calculates this formula. But I think it does go a little deeper than that—I think that one could probably obtain something of the sort by looking at the motion of the boundaries of an occupied region in the six-dimensional phase space.

K. R. Symon: I just wanted to comment, with respect to this formula, that it is not restricted to radiation damping but applies to any sort of damping process. If one attenuates the energy of the beam by letting it pass through a foil, one gets a similar formula and it can be obtained simply by writing down the corrections to Liouville's theorem that arise from the damping process.

H. G. Hereward: Yes, I think the calculation of what one has for deviation from Liouville's theorem is at the root of this and I imagine—I am just guessing now —that the only thing one needs to assume about the damping force, whether it is radiation or any other, is that it be directed in line with the orbit.

H. O. Wuster: I should like to ask Dr. Hereward if he has tried to see what the weakening of the damping of the synchronous oscillation means for the lifetime of the beam, for instance in a storage ring. There are quantum fluctuations in the synchronous oscillations which cause diffusion out of the phase stable region.

H. G. Hereward: Yes, I have looked at that. The answer is that, for this reason, one doesn't like to give away too much of the synchrotron damping. This is partly why I said that one would like to reduce k_s only to something like 1.5 P/E. Of course, you would have much more damping of the betatron oscillations if you were to cut this down still further. Another thing to be said in this connection is that one should choose the harmonic number of the machine not too high. One needs damping of the betatron oscillations to avoid radial loss at the chamber wall from the statistical fluctuations; but, in relation to the synchrotron oscillations, one must look at the available stable width rather than the vacuum-chamber width. By choice of a sufficiently low harmonic number one can obtain a reasonably wide bucket and then this sort of damping rate is not too bad. Perhaps I should mention that there is a CERN internal note (AR/INT SR/61-15) which deals with these matters, including some consideration of the quantum effects.