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MASSIVE PARTICLE TUNNELS FROM THE TAUB-NUT BLACK HOLE

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In this paper, we investigate the Hawking radiation of the Taub-NUT black hole by Hamilton–Jacobi method. When the unfixed background space-time and self-gravitational interaction are considered, the tunnelling rate is related to the change of Bekenstein–Hawking entropy and the radiation spectrum deviates from the purely thermal one. This result is in accordance with Parikh and Wilczek's opinion and gives a correction to the Hawking radiation of the black hole.

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1. Introduction

Hawking radiation [1,2] is viewed as tunnelling process caused by vacuum fluctuations near the black hole horizon. It can be explained as a virtual particle pair spontaneously created inside the horizon of the black hole; the positive energy particle tunnels out the horizon and materializes as a true particle, while the negative energy particle is absorbed by the black hole. It can be also interpreted as the virtual particle pair created outside the horizon; the negative energy particle tunnels into the horizon and is absorbed by the black hole, while the positive energy particle is left outside the horizon and moves to infinite distance and forms the Hawking radiation.

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Based on the above tunnelling picture, Parikh and Wilczek studied the Hawking radiation of the static Schwarzschild and Reissner–Nordström black holes. After considering the self-gravitational interaction and unfixed background space-time, the derived radiation spectrum deviates from the purely thermal one and the tunnelling rate is related to the change of Bekenstein–Hawking entropy [3–5]. The essence of this methodology is dynamical treatment to the Hawking radiation. In their work, they pointed out the potential barrier is afforded by the outgoing particle self, thus the cause mechanism of the potential barrier is resolved. Meanwhile, there are two key points. Firstly, the unfixed background space-time and self-gravitational interaction, which were often overlooked, were considered. Secondly, to eliminate the coordinate singularity, the Painlevé coordinate was introduced, and this coordinate is quite appropriate to describe the Hawking radiation of slowly evaporating black holes. Following this work, people investigated the Hawking radiation of various space-times [6–14]. Hemming and Keski–Vakkuri studied the Hawking radiation of Anti-de Sitter background space-time [6], Medved researched that of de Sitter background space-time [7] and Zhao and Zhang *et al.* investigated the case of the stationary axisymmetric black holes [8–10]. However, all of these researches are limited to massless particles. In 2005, this work was extended to the case of massive and charged particles by Zhang and Zhao and a great deal of progress was made [15–18].

In the same year, Angheben *et al.* adopted another method to explore the action of radiation particles and discuss the Hawking radiation [19]. This method is different from Parikh and Wilczek's and shall be reffered to as Hamilton–Jacobi method [20]. In fact, this work is the extension of that of Srinivasan and Padmanabhan [21]. In this method, since the action of the radiation particle is derived by the Hamilton–Jacobi equation, one can avoid exploring the equation of motion of the radiation particle in Painlevé coordinate system and calculating the Hamilton equation. Meanwhile, although the equations of motion of massive and massless particles are different, one need not differentiate them. However, since the self-gravitational interaction and the unfixed background space-time were not considered, the derived radiation spectrum is only a leading term. To get the actual radiation spectrum, the self-gravitational interaction and unfixed background space-time should be considered.

Our work in this paper is to incorporate these and review the Hawking radiation of the Taub-NUT black hole by the Hamilton–Jacobi method. The result shows the tunnelling rate is related to the change of Bekenstein–Hawking entropy and the radiation spectrum deviates from the purely thermal one, which is fully in accordance with Parikh and Wilczek's opinion and gives another method to study the Hawking radiation. The Taub-NUT solution, where there is a NUT parameter [22], was first obtained by Newman *et al.* in 1963. In the subsequent researches, people found the parameter is related to gravitational monopole [23, 24]. The NUT space-time is asymptotically flat and its properties are very special. Due to the existence of the closed time-like geodesics, it violates the causality condition. There are half-closed time-like geodesics in Taub area that can be explored in NUT area, so the naked singularity exists. Meanwhile, for the Taub-NUT black hole, the angular velocity at the event horizon is equal to zero and there is not super-radiation. The axial-symmetry is not caused by the rotation of the black hole, while it is caused by the fact that the vector potential cannot be selected as spherical symmetry but should be axial-symmetric. In light of these properties, the research on the Taub-NUT black hole is necessary and meaningful.

The paper is organized as follows. In the next section, taking the unfixed background space-time and self-gravitational interaction into account, we review the Hawking radiation of the Taub-NUT black hole by the Hamilton–Jacobi method and get the actual radiation spectrum. Section 3 contains some discussion and conclusion.

2. Hawking radiation as tunnelling from the Taub-NUT black hole

The line element of the stationary axisymmetric Taub-NUT black hole is given by [25]

$$\begin{aligned} ds^2 &= \left(1-\frac{2\left(mr+l^2\right)}{r^2+l^2}\right)(dt-2l\sin\theta d\varphi)^2 \\ &\quad -\left(1-\frac{2\left(mr+l^2\right)}{r^2+l^2}\right)^{-1}dr^2-\left(r^2+l^2\right)\left(d\theta^2+\sin^2\theta d\varphi^2\right)\,, \end{aligned} \tag{1}$$

where t represents the time coordinate, m is the mass of the black hole, and l is the NUT parameter. For the convenience of the discussion, we define $\Delta = r^2 - 2mr - l^2$ and $\rho^2 = r^2 + l^2$, and then the line element can be written as

$$ds^2 = -\frac{\Delta}{\rho^2}dt^2+\frac{\rho^2}{\Delta}dr^2+\rho^2 d\theta^2+\frac{\left(\rho^4-4\Delta l^2\right)\sin^2\theta}{\rho^2}d\varphi^2-\frac{4\Delta l\sin\theta}{\rho^2}dtd\varphi\,. \tag{2}$$

The event horizon and entropy of the black hole are obtained as

$$r_{\mathrm{h}} = m+\sqrt{m^2+l^2}\,, \qquad S=\pi\left(r_{\mathrm{h}}^2+l^2\right)\,. \tag{3}$$

Obviously, the event horizon coincides with the outer infinite red-shift surface, which means the geometrical optics limit can be applied here. Using the WKB approximation [26], we can get the relationship between the tunnelling rate and the action of the radiation particle as $\Gamma \sim \exp\left(-2\mathrm{Im}I\right)$. Considering that the axial-symmetry is not caused by the rotation of the black hole, we need not perform the dragging coordinate transformation on the line element here. In the discussion of the Hawking radiation, the derivation of the action is crucial. There are two methods to derive it, namely the Hamilton–Jacobi method and radial geodesic method. The radial geodesic method was developed by Parikh and Wilczek *et al.* and the derivation of the action mainly relies on the exploration of the equation of motion in the Painlevé coordinate systems and the calculation of Hamilton equation. In the Hamilton–Jacobi method, the derivation of the action was dependent on the Hamilton–Jacobi equation. In this paper, we adopt the Hamilton–Jacobi equation to derive the action. Near the event horizon, the line element takes on the form

$$\begin{aligned} ds^2 &= -\frac{\Delta_{,r}\left(r_\mathrm{h}\right)\left(r-r_\mathrm{h}\right)}{\rho^2\left(r_\mathrm{h}\right)}dt^2+\frac{\rho^2\left(r_\mathrm{h}\right)}{\Delta_{,r}\left(r_\mathrm{h}\right)\left(r-r_\mathrm{h}\right)}dr^2+\rho^2\left(r_\mathrm{h}\right)d\theta^2 \\ &\quad +\rho^2\left(r_\mathrm{h}\right)\sin^2\theta d\varphi^2+\frac{4\Delta_{,r}\left(r_\mathrm{h}\right)\left(r-r_\mathrm{h}\right)l\sin\theta}{\rho^2\left(r_\mathrm{h}\right)}dtd\varphi\,, \end{aligned} \tag{4}$$

in which $\Delta_{,r}\left(r_\mathrm{h}\right)=\frac{\partial\Delta}{\partial r}\Big|_{r=r_\mathrm{h}}=2\left(r_\mathrm{h}-m\right)$ and $\rho^2\left(r_\mathrm{h}\right)=r_\mathrm{h}^2+l^2$. The action I of the outgoing particle satisfies relativistic Hamilton–Jacobi equation, namely

$$g^{\mu\nu}\left(\partial_\mu I\right)\left(\partial_\nu I\right)+u^2=0\,, \tag{5}$$

where u and $g^{\mu\nu}$ are the mass of the particle and the inverse metric tensors derived from the line element (4). The non-null inverse metric tensors are

$$\begin{aligned} g^{00} &= -\frac{\rho^2\left(r_\mathrm{h}\right)}{\Delta_{,r}\left(r_\mathrm{h}\right)\left(r-r_\mathrm{h}\right)}\,, \qquad g^{11}=\frac{\Delta_{,r}\left(r_\mathrm{h}\right)\left(r-r_\mathrm{h}\right)}{\rho^2\left(r_\mathrm{h}\right)}\,, \\ g^{22} &= \frac{1}{\rho^2\left(r_\mathrm{h}\right)}\,, \qquad g^{33}=\frac{1}{\rho^2\left(r_\mathrm{h}\right)\sin^2\theta}\,, \\ g^{03} &= g^{30}=\frac{2l}{\rho^2\left(r_\mathrm{h}\right)\sin\theta}\,. \end{aligned} \tag{6}$$

Substituting them into the Hamilton–Jacobi equation yields

$$\begin{aligned} &-\frac{\rho^2\left(r_\mathrm{h}\right)}{\Delta_{,r}\left(r_\mathrm{h}\right)\left(r-r_\mathrm{h}\right)}\left(\partial_t I\right)^2+\frac{\Delta_{,r}\left(r_\mathrm{h}\right)\left(r-r_\mathrm{h}\right)}{\rho^2\left(r_\mathrm{h}\right)}\left(\partial_r I\right)^2 \\ &+g^{22}\left(\partial_\theta I\right)^2+g^{33}\left(\partial_\varphi I\right)^2+2g^{03}\left(\partial_t I\right)\left(\partial_\varphi I\right)+u^2=0\,. \end{aligned} \tag{7}$$

Obviously, it is difficult to solve the action I for it is a function of t, r, θ and φ. Considering the properties of the black hole space-time, we carry out the separation of variables as

$$I = -\omega t + R(r) + H(\theta) + j\varphi\,, \tag{8}$$

where ω and j are the energy and angular momentum of the particle. At the event horizon, $\Omega_{\rm h} = \frac{d\varphi}{dt}\Big|_{r=r_{\rm h}} = 0$. Substituting Eq. (8) into (7) and solving $R(r)$ yields

$$\begin{aligned} R(r) &= \pm\frac{\rho^2(r_{\rm h})}{\varDelta_{,r}(r_{\rm h})}\int\frac{dr}{r-r_{\rm h}} \\ &\quad\times\sqrt{\omega^2-\frac{\varDelta_{,r}(r_{\rm h})(r-r_{\rm h})}{\rho^2(r_{\rm h})}\left[g^{22}\left(\partial_\theta H(\theta)\right)^2+g^{33}j^2+2g^{03}\omega j+u^2\right]} \\ &= \pm\frac{\pi i\rho^2(r_{\rm h})}{\varDelta_{,r}(r_{\rm h})}+\zeta\,, \end{aligned} \tag{9}$$

where $\pm$ sign comes from the square root and ζ is integral constant. Inserting Eq. (9) into Eq. (8), we can get two different actions which correspond to the outgoing and ingoing solution, respectively. Therefore the imaginary parts of the two actions are

$$\mathrm{Im}I_\pm = \pm\frac{\pi\rho^2(r_{\rm h})\,\omega}{\varDelta_{,r}(r_{\rm h})} + \mathrm{Re}(\zeta)\,. \tag{10}$$

According to Ref. [27], to ensure that the incoming probability is unity in the classical limit — when there is no reflection and everything is absorbed — instead of zero or infinity, one should select the appropriate value of ζ. We can let $\zeta = \frac{\pi i\rho^2(r_{\rm h})\omega}{\varDelta_{,r}(r_{\rm h})} + \mathrm{Re}(\zeta)$, and then $\mathrm{Im}I_- = 0$. Then I_+ with I give the imaginary part of the corresponding outgoing solution as

$$\mathrm{Im}I = \frac{2\pi\rho^2(r_{\rm h})\,\omega}{\varDelta_{,r}(r_{\rm h})} = \frac{\left(r_{\rm h}^2+l^2\right)\pi\omega}{r_{\rm h}-m}\,. \tag{11}$$

Using the WKB approximation, the tunnelling rate can be obtained. However, we find the radiation spectrum is only the leading term. The reason is that the unfixed background space-time and self-gravitational interaction were not taken into account. Now let us incorporate these and move on the discussion. Considering the unfixed background space-time, we fix the ADM mass of the total space-time and allow that of the black hole to fluctuate. When a particle with energy ω tunnels out, the mass of the black hole should change into $m-\omega$. At the event horizon, due to the angular velocity $\Omega_{\rm h}=0$, the angular momentum is equal to zero. Taking self-gravitational interaction into account, the imaginary part of the true action should be

$$\begin{aligned}
{\rm Im}I &= \pi\int\limits_0^{\omega}\frac{\left(r'^2_{\rm h}+l^2\right)d\omega'}{r'_{\rm h}-m} \\
&= -2\pi\int\limits_0^{\omega}\frac{(m-\omega')^2+l^2+(m-\omega')\sqrt{(m-\omega')^2+l^2}}{\sqrt{(m-\omega')^2+l^2}}d\omega' \\
&= -\pi\left[(m-\omega)^2-m^2+(m-\omega)\sqrt{(m-\omega)^2+l^2}-m\sqrt{m^2+l^2}\right]. \quad (12)
\end{aligned}$$

So the tunnelling rate is

$$\begin{aligned}
\Gamma &\sim \exp\left(-2{\rm Im}I\right)=\exp\left\{2\pi\left[(m-\omega)^2-m^2+(m-\omega)\sqrt{(m-\omega)^2+l^2}\right.\right. \\
&\left.\left.-m\sqrt{m^2+l^2}\right]\right\}=\exp\left[\pi\left(r_f^2-r_i^2\right)\right]=\exp\left(\Delta S_{\rm BH}\right), \quad (13)
\end{aligned}$$

where $r_f=(m-\omega)+\sqrt{(m-\omega)^2+l^2}$ and $r_i=m+\sqrt{m^2+l^2}$ are the locations of the event horizon before and after the particle emission, and $\Delta S_{\rm BH}=S_{\rm BH}\left(M-\omega\right)-S_{\rm BH}\left(M\right)$ is the change of Bekenstein–Hawking entropy. Clearly, the tunnelling rate is related to the change of Bekenstein–Hawking entropy and the radiation spectrum deviates from the purely thermal one. The result satisfies an underlying unitary theory and is fully in accordance with the well known result.

3. Discussion and conclusion

When $l=0$, the stationary axisymmetric Taub-NUT black hole is reduced to the Schwarzschild black hole, and the tunnelling rate can be accordingly obtained from Eq. (1) as $\Gamma\sim\exp\left(-2{\rm Im}I\right)=\exp\left[-8\pi\omega\left(m-\frac{\omega}{2}\right)\right]$, which is fully consistent with that obtained by Parikh and Wilczek.

From the discussion in Section 2, we find some virtues in the investigation of Hawking radiation by Hamilton–Jacobi method. Firstly, one does not introduce the Painlevé coordinate transformation. In fact, applying this method, we can get the same result in the Painlevé coordinate system and dragging coordinate system. But considering the simplicity, we would like to adopt the metric (4). Secondly, since the derivation of action depends on the

Hamilton–Jacobi equation, one can avoid differentiating massless and massive particles. In the former treatment, the massless and massive particles should be differentiated for their equations of motion are different, which correspond to light-like and time-like character, respectively. Thirdly, one does not need to solve the Hamilton canonical equations. Moreover, since one can obtain not only the actions of radiation particles from the stationary black holes but also these of the non-stationary black holes by Hamilton–Jacobi equations, this method can be easily extended to non-stationary black holes. Meanwhile, we should notice that the self-gravitational interaction and unfixed background space-time should be taken into account; otherwise the derived spectrum is only the leading term.

To summarize, we have considered the unfixed background space-time and self-gravitational interaction and revised the Hawking radiation of the Taub-NUT black hole. The Hawking radiation spectrum obtains a correction.

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