ON THE NON-ADIABATIC NEUTRINO OSCILLATIONS IN MATTER

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An exact expression (in terms of Weber–Hermite functions) for the two-neutrino transition (oscillation) probability in matter with linearly varying density in the case of finite initial and final densities is obtained and its properties are discussed. It takes a simple form when the resonance density differs considerably from the initial and final densities. The conditions under which the approximate expression for the probability derived in the latter case can be used for description of the neutrino transitions in the interior of the Sun are also briefly discussed.

In the present talk we shall discuss the problem of analytical description of the neutrino oscillations in matter [1-4] with non-adiabatically varying density. A possible solution of this problem for neutrinos passing through a resonance layer of matter, wherein the density changes linearly with the distance and only two neutrinos take part in the oscillations, has been suggested in Ref. [5]. Subsequently, it was generalized for transitions involving three neutrinos in Ref. [6]*). The result obtained in [5] is based on the Landau-Zener formula [11, 12] for the probability of the non-adiabatic transition between two states (say, $|\nu_e\rangle$ and $|\nu_\mu\rangle$ in the case of interest) of a system whose Hamiltonian coincides in form with the neutrino Hamiltonian for linearly changing matter density. There are several circumstances, however, which in our opinion make the derivation of this result unsatisfactory. First, the Landau-Zener probability amplitude describes the transition between two flavour neutrino states while in Ref. [5] it is used for the description of the transition between the neutrino matter eigenstates. Secondly, as can be shown, the transition probability amplitude of interest satisfies the Weber differential equation and the Landau-Zener result corresponds to a partial solution of the Weber equation satisfying certain initial conditions at time $t_0 = -\infty$; it describes the probability of transition at $t = +\infty$. In the problem of interest choosing $t_0 = -\infty$ and $t = +\infty$ corresponds to assuming infinite matter (electron number) densities in the regions of neutrino production and detection (we shall call them initial and final densities, respectively). In most of the physical examples of interest, these conditions are not realized (e.g. in the case of solar neutrinos, propagating from the centre to the surface of the Sun) and different solutions of the Weber equation have to be used to describe the transition between the flavour neutrinos. And, thirdly, the derivation of the neutrino transition probabilities used in Refs. [5] and [6] is based on rather qualitative arguments. As a consequence, it does not permit to formulate clear quantitative criteria for the conditions and the bounds of validity of the results obtained.

Exact analytical expressions for the probabilities of two-neutrino transitions in matter with linearly varying density in the case of finite initial and final densities have been obtained recently in Ref. [13] (see also [14]) using the approach of Zener [12]. When the initial and final densities are not close in value to the resonance density, one can deduce from them approximate expressions for the non-adiabatic transition probabilities with any given accuracy and clear quantitative conditions of their applicability. We give an example of such an expression derived in [13]. The relevance of the results obtained in Ref. [13] for the description of the propagation of the solar neutrinos in the Sun is also briefly discussed.

Consider the case of oscillations in matter involving two neutrinos only, say ν_e and ν_μ. As can be shown [1, 2, 15], the amplitude of the probability $A_\ell\,(t, t_0)$, $\ell = e, \mu$, to find neutrino ν_ℓ at time t satisfies the following system of evolution equations

$$i\frac{d}{dt}\begin{pmatrix} A_e(t,t_0) \\ A_\mu(t,t_0)\end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_1-\varepsilon_2\end{pmatrix}\begin{pmatrix} A_e(t,t_0) \\ A_\mu(t,t_0)\end{pmatrix} \tag{1}$$

Here

$$\varepsilon_{12} = \frac{\Delta m^2}{4p}\sin 2\theta \tag{2}$$

$$\varepsilon_1 - \varepsilon_2 = \frac{\Delta m^2}{2p}\cos 2\theta - \sqrt{2}\,G_F N_e(t) \tag{3}$$

where $\Delta m^2 = m_2^2 - m_1^2$, m_1 and m_2 being the masses of the neutrinos ν_1 and ν_2 with definite mass in vacuum ($m_2 > m_1$), p is the absolute value of the neutrino momentum, θ is the neutrino mixing angle in vacuum, and $N_e(t)$ is the electron number density at the point reached by the neutrino at distance $r = (t - t_0)$ from the production point. As usual, the neutrinos $\nu_{1,2}$ are assumed to be stable and relativistic:

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*) Three neutrino oscillations in matter were studied first in Refs. [7, 8]. They were investigated in greater detail recently in Refs. [9, 10, 6].

$$E_{1,2} = \sqrt{p^2 + m_{1,2}^2} \simeq p + \frac{m_{1,2}^2}{2p} .$$

We shall suppose that

$$\Delta m^2 \cos 2\theta > 0 \tag{4}$$

i.e. that a resonance amplification of the neutrino (but not of the antineutrino) transition probability in matter is possible. The resonance density N_e^{res} and the spatial width of the resonance layer ΔL^{res} are given respectively by [3]:

$$N_e^{res} = \frac{\Delta m^2 \cos 2\theta}{2p\sqrt{2}\, G_F} \tag{5}$$

$$\Delta L^{res} = 2N_e^{res} |\mathrm{tg}\, 2\theta| \Big/ \left(\left| \left(\frac{dN_e}{dt} \right)_{res} \right| \right) \tag{6}$$

where $(dN_e/dt)_{res} = (dN_e/dr)_{res}$ is the derivative of $N_e(t)$ at the resonance point. Following Landau and Zener, we shall consider the case

$$\varepsilon_1 - \varepsilon_2 = \alpha t \tag{7}$$

α being a real constant. For the problem we are trying to solve, this is equivalent to the assumption that the point of the neutrino path at which the density is equal to the resonance density is reached at $t = 0$, $N_e(0) = N_e^{res}$, and that

$$N_e(t) = N_e^{res} + \left(\frac{dN_e}{dt} \right)_{res} \cdot t \tag{8}$$

It follows then from (3), (7) and (8) that

$$\alpha = -\sqrt{2}\, G_F \left(\frac{dN_e}{dt} \right)_{res} \tag{9}$$

By making the change

$$A_\mu(t, t_0) = e^{-\frac{i}{2}\int_{t_0}^{t} (\varepsilon_1 - \varepsilon_2) dt'} A_\mu'(t, t_0) \tag{10}$$

and using the second equation of the system (1)

$$A_e(t, t_0) = \frac{1}{\varepsilon_{12}} e^{-i\int_{t_0}^{t} (\varepsilon_1 - \varepsilon_2) dt'} \left[i \frac{d}{dt} \left(e^{\frac{i}{2}\int_{t_0}^{t} (\varepsilon_1 - \varepsilon_2) dt'} A_\mu'(t, t_0) \right) \right] \tag{11}$$

to eliminate $A_e(t, t_0)$ from the first equation in (1), it is not difficult to convince oneself that the amplitude $A_\mu'(t, t_0)$ satisfies the Weber equation [16]:

$$\left\{ \frac{d^2}{dz^2} + \left[n + \frac{1}{2} - \frac{z^2}{4} \right] \right\} A_\mu'(t, t_0) = 0 . \tag{12}$$

Here

$$z = \sqrt{\alpha}\, t\, e^{i\frac{\pi}{4}} \tag{13}$$

and

$$n = -in_0 = -i\,\frac{\varepsilon_{12}^2}{\alpha} \tag{14}$$

The solutions of Eq. (12) are called functions of the parabolic cylinder, or Weber–Hermite functions. The Weber equation has two linearly independent solutions which can be chosen to be $D_n(z)$ and $D_{-n-1}(-iz)$ [17], where

$$D_n(z) = 2^{\frac{n}{2}+\frac{1}{4}}\, z^{-\frac{1}{2}}\, W_{\frac{n}{2}+\frac{1}{4},\,-\frac{1}{4}}\left(\frac{z^2}{2}\right) \tag{15}$$

$W_{a,b}(z)$ being the degenerate hypergeometric function.

Landau and Zener have calculated the probability $|A_\mu^{(\prime)}(t = +\infty, t_0 = -\infty)|^2$ under the initial conditions $A_\mu^{(\prime)}(t = -\infty, t_0 = -\infty) = 0$ and $|A_e(t = -\infty, t_0 = -\infty)| = 1$. They have found that

$$|A_e(t=+\infty, t_0=-\infty)|^2 = 1 - |A_\mu^{(\prime)}(t=+\infty, t_0=-\infty)|^2 = e^{-2\pi n_0} \tag{16}$$

which in the case of interest is the probability $P(\nu_e \to \nu_e;\, t = +\infty, t_0 = -\infty)$ to find neutrino ν_e at $t = +\infty$ provided that, at $t_0 = -\infty$, ν_e has been produced. In Eq. (16), as it follows from (14), (2), (5), (6), and (9):

$$2\pi n_0 = \frac{\pi}{4}\,\frac{N_e^{res}}{\left(-\left(\frac{dN_e}{dr}\right)_{res}\right)}\,\frac{\Delta m^2}{p}\,\frac{\sin^2 2\theta}{\cos 2\theta} = \frac{\pi^2}{2}\,\frac{\Delta L^{res}}{L_m^{res}} \tag{17}$$

where

$$L_m^{res} = \frac{4\pi p}{\Delta m^2 \sin 2\theta} \tag{18}$$

is the oscillation length at resonance [3].

We are seeking a solution of Eq. (12) which satisfies the initial conditions

$$A_\mu^{(\prime)}(t_0, t_0) = 0, \tag{19}$$

$$|A_e(t_0, t_0)| = 1 \tag{20}$$

for arbitrary t_0, i.e. for arbitrary initial density $N_e(t_0)$. It follows from the property of the functions $D_n(z)$ and $D_{-n-1}(-iz)$ [17]

$$D_n(z)\frac{d}{dz}D_{-n-1}(-iz) - D_{-n-1}(-iz)\frac{d}{dz}D_n(z) = -i\,e^{i\frac{\pi n}{2}} \tag{21}$$

that the solution we are interested in (up to a constant phase factor) has the form [13]:

$$A'_\mu(t, t_0) = \sqrt{n_0}\; e^{-\frac{\pi n_0}{2}}\left[D_n(z_0)D_{-n-1}(-iz) - D_{-n-1}(-iz_0)D_n(z)\right] \tag{22}$$

where

$$z_0 = \sqrt{\alpha}\, t_0\, e^{i\frac{\pi}{4}} \tag{23}$$

Note that the result obtained for $|A_\mu^{(\prime)}(t, t_0)|$ is symmetric with respect to the interchange of t $[N_e(t)]$ and t_0 $[N_e(t_0)]$:

$$|A_\mu^{(\prime)}(t,t_0)| = |A_\mu^{(\prime)}(t_0,t)|. \tag{24}$$

The neutrino transition probabilities are determined by:

$$P(\nu_e \to \nu_\mu; t, t_0) = |A_\mu^{(\prime)}(t,t_0)|^2 \tag{25}$$

$$P(\nu_e \to \nu_e; t, t_0) = 1 - |A_\mu^{(\prime)}(t,t_0)|^2 \tag{26}$$

The solution (22) found for $A_\mu^{(\prime)}(t, t_0)$ in [13] is exact. In principle, it can be used to calculate, e.g. $P(\nu_e \to \nu_e; t, t_0)$ for any set of values of the parameters characterizing the problem under discussion [Δm^2, θ, p, $N_e(t_0)$, $N_e(t)$ and $(dN_e/dt)_{res}$].

From the point of view of possible applications of the results (22), e.g. for a description of the transitions of the solar ν_e's in the interior of the Sun, it proves useful to find an approximate expression for the average probability $\bar{P}(\nu_e \to \nu_e; t, t_0)$ in the case

$$|z_{(0)}|^2 \equiv R_{(0)}^2 = \sqrt{2}\, G_F \frac{(N_e(t_{(0)}) - N_e^{res})^2}{|(dN_e/dt)_{res}|} \gg 1 \tag{27}$$

[we have used (8), (9) and (13) in (27)], i.e. when $N_e(t_0)$ and $N_e(t)$ differ considerably from N_e^{res}. We shall suppose that $N_e(t_0) \gg N_e^{res} \gg N_e(t)$ and that the density is decreasing along the neutrino path, so that $(dN_e/dt)_{res} < 0$. Then $\alpha > 0$ [see Eq. (9)]. Using the decomposition of $D_n(z_{(0)})$ and $D_{-n-1}(-iz_{(0)})$ for $\alpha > 0$ and $t > 0$ ($t_0 < 0$) in power series of $R_{(0)}^{-2k}$, $k = 0, 1, 2, \ldots$ [17], we get for the average probability to find ν_e at time t [13]:

$$\bar{P}(\nu_e \to \nu_e; t, t_0) = P_1(R_0^2, R^2) + e^{-2\pi n_0}\left(1 - P_2(R_0^2, R^2)\right) \tag{28}$$

where $P_{1,2}(R_0^2, R^2) = P_{1,2}(R^2, R_0^2)$ are polynomials in R_0^{-2} and R^{-2}. In order to give an idea about the structure of the terms appearing in the decomposition of interest, we present below the expressions for $P_{1,2}(R_0^2, R^2)$ calculated in Ref. [13] up to the third order in the parameters of decomposition R_0^{-2} and R^{-2}:

$$P_1(R_0^2, R^2) = P(R_0^2, R^2) + \frac{5}{8} n_0^3 (R_0^{-6} + R^{-6}) n_0^2 \tag{29}$$

$$P_2(R_0^2, R^2) = 2P(R_0^2, R^2) + \frac{5}{8} n_0^3 (R_0^{-6} + R^{-6}) n_0^2 \tag{30}$$

where

$$P(R_0^2, R^2) = n_0 (R_0^{-2} + R^{-2}) - 3n_0^2 (R_0^{-4} + R^{-4}) - 2n_0^2 R_0^{-2} R^{-2} + $$
$$+ 10 n_0^3 (R_0^{-6} + R^{-6}) + 6 n_0^3 R_0^{-2} R^{-2} (R_0^{-2} + R^{-2}) - 5 n_0 (R_0^{-6} + R^{-6}) \tag{31}$$

Let us discuss next the results (28)-(31). It follows from Eqs. (17), (5) and (27) that

$$\frac{n_0}{R_{(0)}^2} = \frac{1}{4} \frac{tg^2 2\theta}{(1 - N_e(t_{(0)})/N_e^{res})^2} = \frac{1}{4} tg^2 2\theta_m(t_{(0)}) \tag{32}$$

where $\theta_m(t_{(0)})$ is the mixing angle in matter at density $N_e(t_{(0)})$ [1-3]. In such a way, the decomposition under discussion is, in particular, a decomposition in power series of $\frac{1}{2}\, tg^2\, 2\theta_m(t_{(0)})$. Consequently, Eq. (28) will be valid provided

$$\frac{1}{2}\, tg^2 2\theta_m(t_{(0)}) \ll 1 . \tag{33}$$

Further, in the leading term of the type $n_0^2[{}^1\!/_4\ tg^2\ 2\theta_m(t_{(0)})]^k$ one has $k = 3$ [13]. Therefore, there is an additional condition [13] for the validity of Eq. (28):

$$n_0^2 \left(\frac{1}{4}\, tg^2 2\theta_m(t_{(0)})\right)^3 \ll 1 . \tag{34}$$

The decomposition we are discussing is valid for any value of $n_0 \geq 0$ satisfying (32) and (34).

Next, we shall compare briefly our result (28) for $\overline{P}(\nu_e \to \nu_e;\, t, t_0)$ with the result derived in [5]:

$$\overline{P}(\nu_e \to \nu_e;\, t, t_0) = \frac{1}{2} + \left(\frac{1}{2} - e^{-2\pi n_0}\right)\cos 2\theta_m(t)\cos 2\theta_m(t_0) \tag{35}$$

Using the relation [3]

$$\cos 2\theta_m(t_{(0)}) = \left(1 - \frac{N_e(t_{(0)})}{N_e^{res}}\right)\left(\left(1 - \frac{N_e(t_{(0)})}{N_e^{res}}\right)^2 + tg^2 2\theta\right)^{-\frac{1}{2}} = (\pm)\frac{1}{\sqrt{1+tg^2 2\theta_m(t_{(0)})}} \tag{36}$$

$[N_e(t_0) > N_e^{res},\ N_e(t) < N_e^{res}]$ and expressing $\cos 2\theta_m(t_{(0)})$ as power series in $tg^2\ 2\theta_m(t_{(0)})$, it is not difficult to show that if (33) is fulfilled, the expression (35) coincides with (28) up to second order in $tg^2\ 2\theta_m(t_{(0)})$ and $tg^2\ 2\theta_m(t)$. Terms of the type $n_0^2[{}^1\!/_4\ tg^2\ 2\theta_m(t_{(0)})]^3$ and $R_{(0)}^{\to}\ {}^1\!/_4\ tg^2\ 2\theta_m(t_{(0)})$ do not appear in the decomposition of (35). Consequently, expression (35) will describe the neutrino transition probability $\overline{P}(\nu_e \to \nu_e;\, t, t_0)$ with a rather good accuracy only if, in addition to (33), the conditions (27) and (34) do take place [13].

The calculation of $P(\nu_e \to \nu_e;\, t, t_0)$ suggested in Ref. [5] implies that the averaged probability to find ν_e will be given by Eq. (35) if one averages not only over the dimension of the region of neutrino production and the position of the detector, but also over the uncertainty in the position of the resonance (or in the value of N_e^{res}), and the oscillatory terms average out. The analysis performed in [13] shows that under the conditions discussed the probability to find the neutrino ν_e would be equal to (28) if the oscillatory terms would vanish as a result of the averaging only over the dimension of the region of neutrino production and the detector position.

As can be shown [13], for radially propagating neutrinos in the Sun, which pass through one resonance layer, the conditions (27) and (33) are fulfilled provided

$$10^5\ \frac{MeV}{eV^2} \lesssim \frac{p}{\Delta m^2 \cos 2\theta} \ll 2\times 10^8\ \frac{MeV}{eV^2} \tag{37}$$

For a given $tg^2\ 2\theta$ satisfying (33), the conditions (37) determine through Eq. (17) the interval of values of the quantity n_0 for which Eq. (28) is valid [13]:

$$0{,}24\ tg^2 2\theta \ll n_0 \lesssim 0{,}48\times 10^3\ tg^2 2\theta \tag{38}$$

Since $\pi(\Delta L^{res}/L_m^{res}) = 4n_0$, Eq. (38) indicates that expression (28) can describe both adiabatic $[\pi(\Delta L^{res}/L_m^{res}) \gg 1]$ and non-adiabatic $[\pi(\Delta L^{res}/L_m^{res}) \lesssim 1]$ transitions of the solar neutrinos depending on the values of $p/(\Delta m^2 \cos 2\theta)$ and $tg^2\ 2\theta$. For $tg^2\ 2\theta \lesssim 0.3$ the upper bound in Eq. (38) is more restrictive for n_0 than the bound (34)*).

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*) For further details, see Ref. [13]

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