Journal of Physics A: Mathematical and Theoretical

**TOPICAL REVIEW • OPEN ACCESS**

# Review of localization in geometry

To cite this article: Vasily Pestun 2017 *J. Phys. A: Math. Theor.* **50** 443002

OPEN ACCESS
IOP Publishing

Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **50** (2017) 443002 (31pp)

https://doi.org/10.1088/1751-8121/aa6161

Topical Review

# Review of localization in geometry*

**Vasily Pestun**

Institut des Hautes Études Scientifique, Bures-sur-Yvette, France

E-mail: pestun@ihes.fr

Received 31 October 2016, revised 8 February 2017
Accepted for publication 20 February 2017
Published 13 October 2017

**Abstract**

Review of localization in geometry: equivariant cohomology, characteristic classes, Atiyah–Bott–Berline–Vergne formula, Atiyah–Singer equivariant index formula, Mathai–Quillen formalism, and application to path integrals.

Keywords: localization, geometry, equivariant cohomology

The foundations of equivariant de Rham theory have been laid in two papers by Henri Cartan [2] [3]. The book by Guillemin and Sternberg [4] covers Cartan's papers and treats equivariant de Rham theory from the perspective of supersymmetry. See also the book by Berline–Getzler–Vergne [5], the lectures by Szabo [6] and by Cordes–Moore–Ramgoolam [7], and Vergne's review [8].

## 1. Equivariant cohomology

Let $G$ be a compact connected Lie group. Let $X$ be a $G$-manifold, which means that there is a defined action $G \times X \to X$ of the group $G$ on the manifold $X$.

If $G$ acts freely on $X$ (all stabilizers are trivial) then the space $X/G$ is an ordinary manifold on which the usual cohomology theory $H^\bullet(X/G)$ is defined. If the $G$ action on $X$ is free, the $G$-equivariant cohomology groups $H^\bullet_G(X)$ are defined to be the ordinary cohomology $H^\bullet(X/G)$.

If the $G$ action on $X$ is not free, the naive definition of the equivariant cohomology $H^\bullet_G(X)$ fails because $X/G$ is not an ordinary manifold. If non-trivial stabilizers exist, the corresponding points on $X/G$ are not ordinary points but fractional or stacky points.

* This is a contribution to the review issue 'Localization techniques in quantum field theories' (ed Pestun and Zabzine) which contains 17 chapters available at [1].

A proper topological definion of the *G-equivariant cohomology* $H_G(X)$ sets

$$H_G^\bullet(X) = H^\bullet(X \times_G EG) = H^\bullet((X \times EG)/G)) \tag{1.1}$$

where the space $EG$, called *universal bundle* [9, 10] is a (non-unique) topological space associated to $G$ with the following properties[1]

(1) The space $EG$ is contractible
(2) The group $G$ acts freely on $EG$.

Because of the property (1) the cohomology theory of $X$ is isomorphic to the cohomology theory of $X \times EG$, and because of the property (2) the group $G$ acts freely on $X \times EG$ and hence the quotient space $(X \times_G EG)$ has a well-defined ordinary cohomology theory.

## 2. Classifying space and characteristic classes

If $X$ is a point $pt$, the ordinary cohomology theory $H^\bullet(pt)$ is elementary

$$H^n(pt, \mathbb{R}) = \begin{cases} \mathbb{R}, & n = 0 \\ 0, & n > 0 \end{cases} \tag{2.1}$$

but the equivariant cohomology $H_G^\bullet(pt)$ is less trivial. Indeed,

$$H_G^\bullet(pt) = H^\bullet(EG/G) = H^\bullet(BG) \tag{2.2}$$

where the quotient space $BG = EG/G$ is called *classifying space*.

The terminology *universal bundle EG* and *classifying space BG* comes from the fact that any smooth principal $G$-bundle on a manifold $X$ can be induced by a pullback $f^*$ of the universal principal $G$-bundle $EG \to BG$ using a suitable smooth map $f : X \to BG$. Moreover, for two maps $f, g : X \to BG$, the $G$-bundles obtained by the pullbacks $f^*EG$ and $g^*EG$ are isomorphic if and only if $f$, $g$ are homotopic.

The cohomology groups of $BG$ are used to construct *characteristic classes* of principal $G$-bundles.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the real Lie algebra of a compact connected Lie group $G$. Let $\mathbb{R}[\mathfrak{g}]$ be the space of real valued polynomial functions on $\mathfrak{g}$, and let $\mathbb{R}[\mathfrak{g}]^{\mathrm{G}}$ be the subspace of $\mathrm{Ad}_G$ invariant polynomials on $\mathfrak{g}$.

For a principal $G$-bundle over a base manifold $X$ the Chern–Weil morphism

$$\begin{aligned} \mathbb{R}[\mathfrak{g}]^G &\to H^\bullet(X, \mathbb{R}) \\ p &\mapsto p(F_A) \end{aligned} \tag{2.3}$$

sends an adjoint invariant polynomial $p$ on the Lie algebra $\mathfrak{g}$ to a cohomology class $[p(F_A)]$ in $H^\bullet(X)$ where $F_A = \nabla_A^2$ is the curvature 2-form of any connection $\nabla_A$ on the $G$-bundle. The cohomology class $[p(F_A)]$ does not depend on the choice of the connection $A$ and is called the *characteristic class* of the $G$-bundle associated to the polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$.

The main theorem of Chern–Weil theory is that the ring of characteristic classes $\mathbb{R}[\mathfrak{g}]^G$ is isomorphic to the cohomology ring $H^\bullet(BG)$ of the classifying space $BG$: the Chern–Weil morphism (2.3) is an isomorphism

$$\mathbb{R}[\mathfrak{g}]^G \xrightarrow{\sim} H^\bullet(BG, \mathbb{R}) \tag{2.4}$$

[1] See dicussion at http://mathoverflow.net/questions/201209.

For the circle group $G = S^1 \simeq U(1)$ the universal bundle $ES^1$ and classifying space $BS^1$ can be modelled as the inductive limit of

$$ES^1 \simeq S^{2n+1}, \qquad BS^1 \simeq \mathbb{CP}^n \qquad \text{at} \qquad n \to \infty \tag{2.5}$$

Then the Chern–Weil isomorphism is explicitly

$$\mathbb{C}[\mathfrak{g}]^G \simeq H^\bullet(\mathbb{CP}^\infty, \mathbb{C}) \simeq \mathbb{C}[\epsilon] \tag{2.6}$$

where $\epsilon \in \mathfrak{g}^\vee$ is a linear function on $\mathfrak{g} = \mathrm{Lie}(S^1)$ and $\mathbb{C}[\epsilon]$ denotes the free polynomial ring on one generator $\epsilon$. This $\epsilon \in H^2(\mathbb{CP}^\infty, \mathbb{C})$ is negative of the first Chern class $c_1$ of the tautological line bundle $\gamma = \mathcal{O}_{\mathbb{CP}^\infty}(-1)$

$$-c_1(\gamma) = \epsilon = \frac{1}{2\pi\sqrt{-1}} \mathrm{tr}_{\mathbf{1}} F_A(\gamma) \tag{2.7}$$

where $\mathrm{tr}_{\mathbf{1}}$ denotes the trace of the curvature two-form $F_A = dA + A \wedge A$ in the fundamental complex 1-dimensional representation in which the Lie algebra of $\mathfrak{g} = \mathrm{Lie}(S^1)$ is represented by $\sqrt{-1}\,\mathbb{R}$. The cohomological degree of $\epsilon$ is

$$\deg \epsilon = \deg F_A(\gamma) = 2 \tag{2.8}$$

Generally, for a compact connected Lie group $G$ we reduce the Chern–Weil theory to the maximal torus $T \subset G$ and identify

$$\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{t}]^{W_G} \tag{2.9}$$

where $\mathfrak{t}$ is the Cartan Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$ and $W_G$ is the Weyl group of $G$.

For example, if $G = U(n)$ the Weyl group $W_{U(n)}$ is the permutation group of $n$ eigenvalues $\epsilon_1, \dots \epsilon_n$. Therefore

$$H^\bullet(BU(n), \mathbb{C}) = \mathbb{C}[\mathfrak{g}]^{U(n)} \simeq \mathbb{C}[\epsilon_1, \dots, \epsilon_n]^{W_{U(n)}} \simeq \mathbb{C}[c_1, \dots, c_n] \tag{2.10}$$

where $(c_1, \dots, c_n)$ are elementary symmetrical monomials called Chern classes

$$c_k = (-1)^k \sum_{i_1 \leqslant \dots \leqslant i_k} \epsilon_{i_1} \dots \epsilon_{i_k} \tag{2.11}$$

The classifying space for $G = U(n)$ is

$$BU(n) = \lim_{k \to \infty} \mathrm{Gr}_n(\mathbb{C}^{k+n}) \tag{2.12}$$

where $\mathrm{Gr}_n(V)$ denotes the space of $n$-planes in the vector space $V$.

To summarize, if $G$ is a connected compact Lie group with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, maximal torus $T$ and its Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$, and Weyl group $W_G$, then it holds that

$$\boxed{H^\bullet_G(pt, \mathbb{R}) \simeq H^\bullet(BG, \mathbb{R}) \simeq \mathbb{R}[\mathfrak{g}]^G \simeq \mathbb{R}[\mathfrak{t}]^{W_G}} \tag{2.13}$$

## 3. Weil algebra

The cohomology $H^\bullet(BG, \mathbb{R})$ of the classifying space $BG$ can also be realized in the Weil algebra

$$\mathcal{W}_{\mathfrak{g}} := \mathbb{R}[\mathfrak{g}[1] \oplus \mathfrak{g}[2]] = \Lambda \mathfrak{g}^\vee \otimes S\mathfrak{g}^\vee \tag{3.1}$$

Here $\mathfrak{g}[1]$ denotes a shift of degree so that elements of $\mathfrak{g}[1]$ are Grassmann. The space of polynomial functions $\mathbb{R}[\mathfrak{g}[1]]$ on $\mathfrak{g}[1]$ is the anti-symmetric algebra $\Lambda\mathfrak{g}^\vee$ of $\mathfrak{g}^\vee$, and the space of polynomial functions $\mathbb{R}[\mathfrak{g}[2]]$ on $\mathfrak{g}[2]$ is the symmetric algebra $S\mathfrak{g}^\vee$ of $\mathfrak{g}^\vee$.

The elements $c \in \mathfrak{g}[1]$ have degree 1 and represent the connection 1-form on the universal bundle. The elements $\phi \in \mathfrak{g}[2]$ have degree 2 and represent the curvature 2-form on the universal bundle. An odd differential on functions on $\mathfrak{g}[1] \oplus \mathfrak{g}[2]$ can be described as an odd vector field $\delta$ such that $\delta^2 = 0$. The odd vector field $\delta$ of degree 1 representing de Rham differential on the universal bundle is

$$\begin{aligned} \delta c &= \phi - \frac{1}{2}[c, c] \\ \delta\phi &= -[c, \phi] \end{aligned} \tag{3.2}$$

as follows from the standard relations between the connection $A$ and the curvature $F_A$

$$\begin{aligned} dA &= F_A - \frac{1}{2}[A, A] \\ dF_A &= -[A, F_A] \end{aligned} \tag{3.3}$$

This definition implies $\delta^2 = 0$. Indeed,

$$\begin{aligned} \delta^2 c &= \delta\phi - [\delta c, c] = -[c, \phi] - [\phi - \frac{1}{2}[c, c], c] = 0 \\ \delta^2\phi &= -[\delta c, \phi] + [c, \delta\phi] = -[\phi - \frac{1}{2}[c, c], \phi] - [c, [c, \phi]] = 0 \end{aligned} \tag{3.4}$$

Given a basis $T_\alpha$ of the Lie algebra $\mathfrak{g}$ with structure constants $[T_\beta, T_\gamma] = f^\alpha_{\beta\gamma} T_\alpha$ the differential $\delta$ has the form

$$\begin{aligned} \delta c^\alpha &= \phi^\alpha - \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \\ \delta\phi^\alpha &= -f^\alpha_{\beta\gamma} c^\beta \phi^\gamma \end{aligned} \tag{3.5}$$

The differential $\delta$ can be decomposed into the sum of two differentials

$$\delta = \delta_{\mathrm{K}} + \delta_{\mathrm{BRST}} \tag{3.6}$$

with

$$\begin{aligned} \delta_{\mathrm{K}} c &= \phi, \qquad & \delta_{\mathrm{BRST}} c &= -\frac{1}{2}[c, c] \\ \delta_{\mathrm{K}}\phi &= 0, \qquad & \delta_{\mathrm{BRST}}\phi &= -[c, \phi] \end{aligned} \tag{3.7}$$

The differential $\delta_{\mathrm{BRST}}$ is the BRST differential (Chevalley–Eilenberg differential for Lie algebra cohomology with coefficients in the Lie algebra module $S\mathfrak{g}^\vee$). The differential $\delta_{\mathrm{K}}$ is the Koszul differential (de Rham differential on $\Omega^\bullet(\Pi\mathfrak{g})$).

The field theory interpretation of the Weil algebra and the differential (3.6) was given in [11] and [12].

The Weil algebra $\mathcal{W}_\mathfrak{g} = \mathbb{R}[\mathfrak{g}[1] \oplus \mathfrak{g}[2]]$ is an extension of the Chevalley–Eilenberg algebra $CE_\mathfrak{g} = \mathbb{R}[\mathfrak{g}[1]] = \Lambda\mathfrak{g}^\vee$ by the algebra $\mathbb{R}[\mathfrak{g}[2]] = S\mathfrak{g}^\vee$ of symmetric polynomials on $\mathfrak{g}$

$$CE_\mathfrak{g} \leftarrow \mathcal{W}_\mathfrak{g} \leftarrow S\mathfrak{g}^\vee \tag{3.8}$$

which is quasi-isomorphic to the algebra of differential forms on the universal bundle

$$G \to EG \to BG \tag{3.9}$$

The duality between the Weil algebra $\mathcal{W}_{\mathfrak{g}}$ and the de Rham algebra $\Omega^\bullet(EG)$ of differential forms on $EG$ is provided by the Weil homomorphism

$$\mathcal{W}_{\mathfrak{g}} \to \Omega^\bullet(EG) \tag{3.10}$$

after a choice of a connection 1-form $A \in \Omega^1(EG) \otimes \mathfrak{g}$ and its field strength $F_A \in \Omega^2(EG) \otimes \mathfrak{g}$ on the universal bundle $EG \to BG$.

Indeed, the connection 1-form $A \in \Omega^1(EG) \otimes \mathfrak{g}$ and field strength $F \in \Omega^2(EG) \otimes \mathfrak{g}$ define maps $\mathfrak{g}^\vee \to \Omega^1(EG)$ and $\mathfrak{g}^\vee \to \Omega^2(EG)$

$$\begin{aligned} c^\alpha &\mapsto A^\alpha \\ \phi^\alpha &\mapsto F^\alpha \end{aligned} \tag{3.11}$$

The cohomology of the Weil algebra is trivial

$$H^n(\mathcal{W}_{\mathfrak{g}}, \delta, \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases} \tag{3.12}$$

corresponding to the trivial cohomology of $\Omega^\bullet(EG)$.

To define the $G$-equivariant cohomology we need to consider the action of $G$ on $EG$. To compute $H^\bullet_G(pt) = H^\bullet(BG)$, consider $\Omega^\bullet(BG) = \Omega^\bullet(EG/G)$.

For any principal $G$-bundle $\pi : P \to P/G$ the differential forms on $P$ in the image of the pullback $\pi^*$ of the space of differential forms on $P/G$ are called *basic*

$$\Omega^\bullet(P)_{\text{basic}} = \pi^*\Omega^\bullet(P/G) \tag{3.13}$$

Let $L_\alpha$ be the Lie derivative in the direction of a vector field $v_\alpha$ generated by a basis element $T_\alpha \in \mathfrak{g}$, and $i_\alpha$ be the contraction with $v_\alpha$.

An element $\omega \in \Omega^\bullet(P)_{\text{basic}}$ can be characterized by two conditions

(1) $\omega$ is invariant on $P$ with respect to the $G$-action: $L_\alpha\omega = 0$
(2) $\omega$ is horizontal on $P$ with respect to the $G$-action: $i_\alpha\omega = 0$.

In the Weil model the contraction operation $i_\alpha$ is realized as

$$\begin{aligned} i_\alpha c^\beta &= \delta^\beta_\alpha \\ i_\alpha \phi^\beta &= 0 \end{aligned} \tag{3.14}$$

and the Lie derivative $L_\alpha$ is defined by the usual relation

$$L_\alpha = \delta i_\alpha + i_\alpha \delta. \tag{3.15}$$

From the definition of $\Omega^\bullet(P)_{\text{basic}}$ for the case of $P = EG$ we obtain

$$H^\bullet_G(pt) = H^\bullet(BG, \mathbb{R}) = H^\bullet(\Omega^\bullet(EG)_{\text{basic}}, \mathbb{R}) = H^\bullet(\mathcal{W}_{\mathfrak{g}}, \delta, \mathbb{R})_{\text{basic}} = (S\mathfrak{g}^\vee)^G \tag{3.16}$$

## 4. Weil model and Cartan model of equivariant cohomology

The isomorphism

$$H(BG, \mathbb{R}) = H(EG, \mathbb{R})_{\text{basic}} = H(\mathcal{W}_{\mathfrak{g}}, \delta, \mathbb{R})_{\text{basic}} \tag{4.1}$$

suggests to replace the topological model for $G$-equivariant cohomologies

$$H_G(X, \mathbb{R}) = H((X \times EG)/G, \mathbb{R}) \tag{4.2}$$

of a real manifold $X$ by the *Cartan model*

$$H_G(X,\mathbb{R}) = H((\Omega^\bullet(X)\otimes S\mathfrak{g}^\vee)^G,\mathbb{R}) \tag{4.3}$$

or by the equivalent algebraic *Weil model*

$$H_G(X,\mathbb{R}) = H((\Omega^\bullet(X)\otimes \mathcal{W}_\mathfrak{g})_{\text{basic}},\mathbb{R}) \tag{4.4}$$

### *4.1. Cartan model*

Here $(\Omega^\bullet(X)\otimes S\mathfrak{g}^\vee)^G$ denotes the $G$-invariant subspace in $(\Omega^\bullet(X)\otimes S\mathfrak{g}^\vee)$ under the $G$-action induced from $G$-action on $X$ and adjoint $G$-action on $\mathfrak{g}$.

It is convenient to think about $(\Omega^\bullet(X)\otimes S\mathfrak{g}^\vee)$ as the space

$$\Omega^{\bullet,0}_{C^\infty,\text{poly}}(X\times\mathfrak{g}) \tag{4.5}$$

of smooth differential forms on $X\times\mathfrak{g}$ of degree 0 along $\mathfrak{g}$ and polynomial along $\mathfrak{g}$.

In $(T_a)$ basis on $\mathfrak{g}$, an element $\phi\in\mathfrak{g}$ is represented as $\phi=\phi^\alpha T_\alpha$. Then $(\phi^\alpha)$ is the dual basis of $\mathfrak{g}^\vee$. Equivalently $\phi^\alpha$ is a linear coordinate on $\mathfrak{g}$.

The commutative ring $\mathbb{R}[\mathfrak{g}]$ of polynomial functions on the vector space underlying $\mathfrak{g}$ is naturally represented in the coordinates as the ring of polynomials in generators $\{\phi^\alpha\}$

$$\mathbb{R}[\mathfrak{g}] = \mathbb{R}[\phi^1,\ldots,\phi^{\operatorname{rk}\mathfrak{g}}] \tag{4.6}$$

Hence, the space (4.5) can be equivalently presented as

$$\Omega^{\bullet,0}_{C^\infty,\text{poly}}(X\times\mathfrak{g}) = \Omega^\bullet(X)\otimes\mathbb{R}[\mathfrak{g}] \tag{4.7}$$

Given an action of the group $G$ on any manifold $M$

$$\rho_g : m\mapsto g\cdot m \tag{4.8}$$

the induced action on the space of differential forms $\Omega^\bullet(M)$ comes from the pullback by the map $\rho_{g^{-1}}$

$$\rho_g : \omega\mapsto \rho^*_{g^{-1}}\omega, \qquad \omega\in\Omega^\bullet(M) \tag{4.9}$$

In particular, if $M=\mathfrak{g}$ and $\omega\in\mathfrak{g}^\vee$ is a linear function on $\mathfrak{g}$, then (4.9) is the *co-adjoint action* on $\mathfrak{g}^\vee$.

The invariant subspace $(\Omega^\bullet(X)\otimes\mathbb{R}[\mathfrak{g}])^G$ forms a complex with respect to the *Cartan differential*

$$d_G = d\otimes 1 + i_\alpha\otimes\phi^\alpha \tag{4.10}$$

where $d:\Omega^\bullet(X)\to\Omega^{\bullet+1}(X)$ is the de Rham differential, and $i_\alpha:\Omega^\bullet(X)\to\Omega^{\bullet-1}(X)$ is the operation of contraction of the vector field on $X$ generated by $T_\alpha\in\mathfrak{g}$ with differential forms in $\Omega^\bullet(X)$.

The Cartan model of the $G$-equivariant cohomology $H_G(X)$ is

$$H_G(X) = H\left((\Omega^\bullet(X)\otimes\mathbb{R}[\mathfrak{g}])^G, d_G\right) \tag{4.11}$$

To check that $d_G^2=0$ on $(\Omega^\bullet(X)\otimes\mathbb{R}[\mathfrak{g}])^G$ we compute $d_G^2$ on $\Omega^\bullet(X)\otimes\mathbb{R}[\mathfrak{g}]$ and find

$$d_G^2 = L_\alpha\otimes\phi^\alpha \tag{4.12}$$

where $L_\alpha : \Omega^\bullet(X) \to \Omega^\bullet(X)$ is the Lie derivative on $X$

$$L_\alpha = di_\alpha + i_\alpha d \tag{4.13}$$

along vector field generated by $T_\alpha$.

The infinitesimal action by a Lie algebra generator $T_a$ on an element $\omega \in \Omega^\bullet(X) \otimes R[\mathfrak{g}]$ is

$$T_\alpha \omega = (L_\alpha \otimes 1 + 1 \otimes L_\alpha)\omega \tag{4.14}$$

where $L_\alpha \otimes 1$ is the geometrical Lie derivative by the vector field generated by $T_\alpha$ on $\Omega^\bullet(X)$ and $1 \otimes L_\alpha$ is the coadjoint action on $\mathbb{R}[\mathfrak{g}]$

$$L_\alpha = -f^\gamma_{\alpha\beta}\phi^\beta \frac{\partial}{\partial \phi^\gamma} \tag{4.15}$$

If $\omega$ is a $G$-invariant element, $\omega \in (\Omega^\bullet(X) \otimes R[\mathfrak{g}])^G$, then

$$(L_\alpha \otimes 1 + 1 \otimes L_\alpha)\omega = 0 \tag{4.16}$$

Therefore, if $\omega \in (\Omega^\bullet(X) \otimes R[\mathfrak{g}])^G$ it holds that

$$d_G^2 \omega = -(1 \otimes \phi^\alpha L_\alpha)\omega = \phi^\alpha f^\gamma_{\alpha\beta}\phi^\beta \frac{\partial \omega}{\partial \phi^\gamma} = 0 \tag{4.17}$$

by the antisymmetry of the structure constants $f^\gamma_{\alpha\beta} = -f^\gamma_{\beta\alpha}$. Therefore $d_G^2 = 0$ on $(\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}])^G$.

The grading on $\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}]$ is defined by the assignment

$$\deg d = 1 \quad \deg i_\alpha = -1 \quad \deg \phi^\alpha = 2 \tag{4.18}$$

which implies

$$\deg d_G = 1 \tag{4.19}$$

Let

$$\Omega^n_G(X) = \oplus_k (\Omega^{n-2k} \otimes \mathbb{R}[\mathfrak{g}]^k)^G \tag{4.20}$$

be the subspace in $(\Omega(X) \otimes \mathbb{R}[\mathfrak{g}])^G$ of degree $n$ according to the grading (4.18).

Then

$$\ldots \xrightarrow{d_G} \Omega^n_G(X) \xrightarrow{d_G} \Omega^{n+1}_G(X) \xrightarrow{d_G} \ldots \tag{4.21}$$

is a differential complex. The equivariant cohomology groups $H^\bullet_G(X)$ in the Cartan model are defined as the cohomology of the complex (4.21)

$$H^\bullet_G(X) \equiv \mathrm{Ker}\, d_G / \mathrm{Im}\, d_G \tag{4.22}$$

In particular, if $X = pt$ is a point then

$$H^\bullet_G(pt) = \mathbb{R}[\mathfrak{g}]^G \tag{4.23}$$

in agreement with (3.16).

If $x^\mu$ are coordinates on $X$, and $\psi^\mu = dx^\mu$ are Grassman coordinates on the fibers of $\Pi TX$, we can represent the Cartan differential (4.10) in the notations more common in quantum field theory traditions

$$\begin{aligned} \delta x^\mu &= \psi^\mu \\ \delta \psi^\mu &= \phi^\alpha v^\mu_\alpha \end{aligned} \qquad \delta\phi = 0 \tag{4.24}$$

where $v^\mu$ are components of the vector field on $X$ generated by a basis element $T_\alpha$ for the $G$-action on $X$. In quantum field theory, the coordinates $x^\mu$ are typically coordinates on the infinite-dimensional space of bosonic fields, and $\psi^\mu$ are typically coordinates on the infinite-dimensional space of fermionic fields.

### 4.2. Weil model

The differential in the Weil model can be presented in coordinate notations similar to (4.24) as follows

$$\begin{aligned} \delta x^\mu &= \psi^\mu + c^\alpha v^\mu_\alpha & \delta c^\alpha &= \phi^\alpha - \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \\ \delta \psi^\mu &= \phi^\alpha v^\mu_\alpha + \partial_\nu v^\mu_\alpha c^\alpha \psi^\nu & \delta \phi^\alpha &= -f^\alpha_{\beta\gamma} c^\beta \phi^\gamma \end{aligned} \tag{4.25}$$

In physical applications, typically $c$ is the BRST ghost field for gauge symmetry, and Weil differential is the sum of a supersymmetry transformation and BRST transformation, for example see [13].

## 5. Equivariant characteristic classes in the Cartan model

For a reference see [14] and [15].

Let $G$ and $T$ be compact connected Lie groups. We consider a $T$-equivariant $G$-principal bundle $\pi : P \to X$. This means that equivariant $T$-actions are defined on $P$ and on $X$ compatible with the $G$-bundle structure of $\pi : P \to X$. One can take that $G$ acts from the right and $T$ acts from the left.

The compatibility means that $T$-actions on the total space $P$ and base space $X$

- commute with the projection map $\pi : P \to X$
- commute with the $G$ action on the fibers of $\pi : P \to X$.

Let $D_A = d + A$ be a $T$-invariant connection on a $T$-equivariant $G$-bundle $P$. Here we think about connection $A$ is a $\mathfrak{g}$-valued $G$-equivariant 1-form on the total space of $P$ which evaluates to the Lie algebra generators on the fundamental vertical vector fields on $P$. Moreover, a $T$-invariant connection on $G$-bundle always can be constructed starting from any connection on $G$-bundle and using the averaging procedure with respect to the compact Lie group $T$.

Then we define the $T$-equivariant connection

$$D_{A,T} = D_A + \epsilon^\alpha \otimes i_{v_\alpha} \tag{5.1}$$

and the $T$-equivariant curvature

$$F_{A,T} = (D_{A,T})^2 - \epsilon^\alpha \otimes \mathcal{L}_{v_\alpha} \tag{5.2}$$

where $\epsilon^\alpha$ are coordinates on the Lie algebra $\mathfrak{t}$ (like the coordinates $\phi^\alpha$ on the Lie algebra $\mathfrak{g}$ used in the previous section to define the Cartan model of $G$-equivariant cohomology). The curvature is in fact an element of $\Omega^2_T(X) \otimes \mathfrak{g}$

$$F_{A,T} = F_A - \epsilon^\alpha \otimes \mathcal{L}_{v_\alpha} + [\epsilon^\alpha \otimes i_\alpha, 1 \otimes D_A] = F_A + \epsilon^\alpha i_\alpha A \tag{5.3}$$

Let $X^T$ be the $T$-fixed point set in $X$. If the equivariant curvature $F_{A,T}$ is evaluated on $X^T$, only the vertical component of $i_\alpha$ contributes to the formula (5.3) and $v_\alpha$ pairs with the vertical component of the connection $A$ on the $G$-fiber of $P$ over $X^T$. By definition of the connection 1-form $A$ on $G$-bundle $P$, the vertical component of $A$ is $g^{-1}dg$ for $g \in G$. The $T$-action on $G$-fibers over $X^T$ induces a homomorphism

$$\rho_x : \mathfrak{t} \to \mathfrak{g} \qquad \text{for } x \in X^T \tag{5.4}$$

Let $T_\alpha$ be elements of a basis of $\mathfrak{t}$. Then the equivariant curvature $F_{A,T}$ on $X^T$ evaluates to

$$F_A + \epsilon^\alpha \rho_x(T_\alpha) \tag{5.5}$$

An ordinary characteristic class for a principal $G$-bundle on $X$ is $[p(F_A)] \in H^{2d}(X)$ for a $G$-invariant degree $d$ polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$. Here $F_A$ is the curvature of any connection $A$ on the $G$-bundle.

In the same way, the $T$-equivariant characteristic class for a principal $G$-bundle associated to a $G$-invariant degree $d$ polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$ is $[p(F_{A,T})] \in H^{2d}_T(X)$. Here $F_{A,T}$ is the $T$-equivariant curvature of any $T$-equivariant connection $A$ on the $G$-bundle.

Restricted to $T$-fixed points $X^T$ the $T$-equivariant characteristic class associated to a polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$ is

$$p(F_A + \epsilon^\alpha \rho(T_\alpha)) \tag{5.6}$$

In particular, if $V$ is a representation of $G$ and $p = \mathrm{tr}_V \exp()$ is the Chern character of the vector bundle $V$ associated to principal $G$-bundle, then if $X^T$ is a point, the $T$-equivariant Chern character induced by $p$ is an ordinary $T$-character of $V$ induced by a homomorphism $T \to G$ and the $G$-module structure on $V$.

## 6. Standard characteristic classes

For a reference see the book by Bott and Tu [16].

### *6.1. Euler class*

Let $G = SO(2n)$ be the special orthogonal group which preserves a Riemannian metric $g \in S^2V^\vee$ on an oriented real vector space $V$ of $\dim_{\mathbb{R}} V = 2n$.

The Euler characterstic class is defined by the adjoint invariant polynomial

$$\mathrm{Pf} : \mathfrak{so}(2n, \mathbb{R}) \to \mathbb{R} \tag{6.1}$$

of degree $n$ on the Lie algebra $\mathfrak{so}(2n)$ called *Pfaffian* and defined as follows. For an element $x \in \mathfrak{so}(2n)$ let $x' \in V^\vee \otimes V$ denote the representation of $x$ on $V$ (fundamental representation), so that $x'$ is an antisymmetric $(2n) \times (2n)$ matrix in some orthonormal basis of $V$. Let $g \cdot x' \in \Lambda^2 V^\vee$ be the two-form associated by $g$ to $x'$, and let $v_g \in \Lambda^{2n} V^\vee$ be the standard volume form on $V$ associated to the metric $g$, and $v_g^* \in \Lambda^{2n} V$ be the dual of $v_g$. By definition

$$\mathrm{Pf}(x) = \frac{1}{n!} \langle v_g^*, (g \cdot x')^{\wedge n} \rangle \tag{6.2}$$

For example, for the $2 \times 2$-blocks diagonal matrix $x'$ in an orthonormal basis on $V$

$$\mathrm{Pf} \begin{pmatrix} 0 & \epsilon_1 & \cdots & \cdots & 0 & 0 \\ -\epsilon_1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & \epsilon_n \\ 0 & 0 & \cdots & \cdots & -\epsilon_n & 0 \end{pmatrix} = \epsilon_1 \ldots \epsilon_n \tag{6.3}$$

For an antisymmetric $(2n) \times (2n)$ matrix $x'$, the definition implies that $\mathrm{Pf}(x)$ is a degree $n$ polynomial of matrix elements of $x$ which satisfies

$$\mathrm{Pf}(x)^2 = \det x \tag{6.4}$$

Let $P$ be an $SO(2n)$ principal bundle $P \to X$.

In the standard normalization the Euler class $\mathrm{e}(P)$ is defined in such a way that it takes values in $H^{2n}(X, \mathbb{Z})$ and is given by

$$\mathrm{e}(P) = \frac{1}{(2\pi)^n}[\mathrm{Pf}(F)] \tag{6.5}$$

For example, the Euler characteristic of an oriented real manifold $X$ of real dimension $2n$ is an integer number given by

$$\mathrm{e}(X) = \int_X \mathrm{e}(T_X) = \frac{1}{(2\pi)^n}\int_X \mathrm{Pf}(R) \tag{6.6}$$

where $R$ denotes the curvature form of the tangent bundle $T_X$.

In quantum field theories the definition (6.2) of the Pfaffian is usually realized in terms of a Gaussian integral over Grassmann (anticommuting) variables $\theta$ that satisfy $\theta_i\theta_j = -\theta_j\theta_i$. The definition (6.2) is presented as

$$\mathrm{Pf}(x) = \int [\mathrm{d}\theta_{2n}\ldots \mathrm{d}\theta_1]\exp(\frac{1}{2}\theta_i x'_{ij}\theta_j) \tag{6.7}$$

By definition, the operation $\int[\mathrm{d}\theta_{2n}\ldots\mathrm{d}\theta_1]$ picks the coefficient of the monomial $\theta_1\ldots\theta_{2n}$.

### 6.2. Euler class of a vector bundle and Mathai–Quillen form

See Mathai–Quillen [17] and Atiyah–Jeffrey [18].

The Euler class of a vector bundle can be presented in a QFT formalism. Let $E$ be an oriented real vector bundle $E$ of rank $2n$ over a manifold $X$.

Let $x^\mu$ be local coordinates on the base $X$, and let their differentials be denoted $\psi^\mu = dx^\mu$.

Let $h^i$ be local coordinates on the fibers of $E$. Let $\Pi E$ denote the superspace obtained from the total space of the bundle $E$ by inverting the parity of the fibers, so that the coordinates in the fibers of $\Pi E$ are odd variables $\chi^i$. Let $g_{ij}$ be the matrix of a Riemannian metric on the bundle $E$. Let $A^i{}_\mu$ be the matrix valued 1-form on $X$ representing a connection on the bundle $E$.

Using the connection $A$ we can define an odd vector field $\delta$ on the superspace $\Pi T(\Pi E)$, or, equivalently, a de Rham differential on the space of differential forms $\Omega^\bullet(\Pi E)$. In local coordinates $(x^\mu, \psi^\mu)$ and $(\chi^i, h^i)$ the definition of $\delta$ is

$$\begin{aligned} \delta x^\mu &= \psi^\mu & \delta\chi^i &= h^i - A^i_{j\mu}\psi^\mu\chi^j \\ \delta\psi^\mu &= 0 & \delta h^i &= \delta(A^i_{j\mu}\psi^\mu\chi^j) \end{aligned} \tag{6.8}$$

Here $h^i = D\chi^i$ is the *covariant* de Rham differential of $\chi^i$, so that under the change of framing on $E$ given by $\chi^i = s^i_j\tilde{\chi}^j$ the $h^i$ transforms in the same way, that is $h^i = s^i_j\tilde{h}^j$.

The odd vector field $\delta$ is nilpotent

$$\delta^2 = 0 \tag{6.9}$$

and is called de Rham vector field on $\Pi T(\Pi E)$.

Consider an element $\alpha$ of $\Omega^\bullet(\Pi E)$ defined by the equation

$$\alpha = \frac{1}{(2\pi)^{2n}} \exp(-t\delta V) \tag{6.10}$$

where $t \in \mathbb{R}_{>0}$ and

$$V = \frac{1}{2}(g_{ij}\chi^i h^j) \tag{6.11}$$

Notice that since $h^i$ has been defined as $D\chi^i$ the definition (6.10) is coordinate independent.

To expand the definition of $\alpha$ (6.10) we compute

$$\delta(\chi, h) = (h - A\chi, h) - (\chi, dA\chi - A(h - A\chi)) = (h, h) - (\chi, F_A\chi) \tag{6.12}$$

where we suppresed the indices $i, j$, the $d$ denotes the de Rham differential on $X$ and $F_A$ the curvature 2-form on the connection $A$

$$F_A = dA + A \wedge A \tag{6.13}$$

The Gaussian integration of the form $\alpha$ along the vertical fibers of $\Pi E$ gives

$$\frac{1}{(2\pi)^{2n}} \int [\mathrm{d}h][\mathrm{d}\chi] \exp\left(-\frac{1}{2}\delta(\chi, h)\right) = \frac{1}{(2\pi)^n}\mathrm{Pf}(F_A) \tag{6.14}$$

which agrees with definition of the integer valued Euler class (6.5). The form (6.10) used to represent the Euler class is called the Gaussian *Mathai–Quillen representation* of the Thom class, defined later in section 8.1.

The Euler class of the vector bundle $E$ is an element of $H^{2n}(X, \mathbb{Z})$. If $\dim X = 2n$, the number obtained after integration of the fundamental cycle on $X$

$$e(E) = \int_{\Pi T(\Pi E)} \alpha \tag{6.15}$$

is an integer Euler characterstic of the vector bundle $E$.

If $E = TX$ the equation (6.15) provides the Euler characteristic of the manifold $X$ in the form

$$e(X) = \frac{1}{(2\pi)^{\dim X}} \int_{\Pi T(\Pi TX)} \exp(-t\delta V) \overset{t \to 0}{=} \frac{1}{(2\pi)^{\dim X}} \int_{\Pi T(\Pi TX)} 1 \tag{6.16}$$

Given a section $s$ of the vector bundle $E$, we can deform the form $\alpha$ in the same $\delta$-cohomology class by taking

$$V_s = \frac{1}{2t}\left(\chi, h + \sqrt{-1}s\right) \tag{6.17}$$

After integrating over $(h, \chi)$ the resulting differential form on $X$ has a factor

$$\exp\left(-\frac{1}{2t}s^2\right) \tag{6.18}$$

so it is concentrated in a neigborhood of the locus $s^{-1}(0) \subset X$ of zeroes of the section $s$.

In this way the Poincaré–Hopf theorem is proven: given an oriented vector bundle $E$ on an oriented manifold $X$, with rank $E = \dim X$, the Euler characteristic of $E$ is equal to the number of zeroes of a generic section $s$ of $E$ counted with orientation

$$e(E) = \sum_{x \in s^{-1}(0) \subset X} \mathrm{sign}\det \mathrm{d}s|_x \tag{6.19}$$

where $\mathrm{d}s|_x : T_x \to E_x$ is the differential of the section $s$ at a zero $x \in s^{-1}(0)$. The assumption that $s$ is a generic section implies that $\det \mathrm{d}s|_x$ is non-zero.

For a short reference on the Mathai–Quillen formalism see [19].

### 6.3. Chern character

Let $P$ be a principal $GL(n, \mathbb{C})$ bundle over a manifold $X$. The Chern character is an adjoint invariant function

$$\mathrm{ch} : \mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C} \tag{6.20}$$

defined as the trace in the fundamental representation of the exponential map

$$\mathrm{ch} : x \mapsto \mathrm{tr}\mathrm{e}^x \tag{6.21}$$

The exponential map is defined by formal series

$$\mathrm{tr}\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{tr} x^n \tag{6.22}$$

The eigenvalues of the $\mathfrak{gl}(n, \mathbb{C})$ matrix $x$ are called *Chern roots*. In terms of the Chern roots $x_i$ the Chern character is

$$\mathrm{ch}(x) = \sum_{i=1}^{n} \mathrm{e}^{x_i} \tag{6.23}$$

### 6.4. Chern class

Let $P$ be a principal $GL(n, \mathbb{C})$ bundle over a manifold $X$. The Chern class $c_k$ for $k \in \mathbb{Z}_{>0}$ of $x \in \mathfrak{gl}(n, \mathbb{C})$ is defined by expansion of the determinant

$$\det(1 + tx) = \sum_{k=0}^{n} t^n c_n \tag{6.24}$$

In particular

$$c_1(x) = \mathrm{tr} x, \qquad c_n(x) = \det x \tag{6.25}$$

In terms of Chern roots the Chern class $c_k$ is defined as the elementary symmetric monomial

$$c_k = \sum_{1 \leqslant i_1 < i_2 \cdots < i_k \leqslant n} x_{i_1} \ldots x_{i_n} \tag{6.26}$$

*Remark on integrality.* Our conventions for characteristic classes of $GL(n, \mathbb{C})$ bundles differ by a factor of $(-2\pi\sqrt{-1})^k$ from the frequently used conventions in which Chern classes $c_k$ take value in $H^{2k}(X, \mathbb{Z})$. In our conventions the characteristic class of degree $2k$ needs to be multiplied by $\frac{1}{(-2\pi\sqrt{-1})^k}$ to be integral.

### 6.5. Todd class

Let $P$ be a principal $GL(n, \mathbb{C})$ bundle over a manifold $X$. The Todd class of $x \in \mathfrak{gl}(n, \mathbb{C})$ is defined to be

$$\mathrm{td}(x) = \det \frac{x}{1 - \mathrm{e}^{-x}} = \prod_{i=1}^{n} \frac{x_i}{1 - \mathrm{e}^{-x_i}} \tag{6.27}$$

where det is evaluated in the fundamental representation. The ratio evaluates to a series expansion involving Bernoulli numbers $B_k$

$$\frac{x}{1-e^{-x}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} B_k x^k = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \dots \tag{6.28}$$

### 6.6. The $\hat{\boldsymbol{A}}$ class

Let $P$ be a principal $GL(n,\mathbb{C})$ bundle over a manifold $X$. The $\hat{A}$ class of $x \in GL(n,\mathbb{C})$ is defined as

$$\hat{A} = \det \frac{x}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} = \prod_{i=1}^{n} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}} \tag{6.29}$$

The $\hat{A}$ class is related to the Todd class by

$$\hat{A}(x) = \det e^{-\frac{x}{2}} \mathrm{td} x \tag{6.30}$$

## 7. Index formula

For a holomorphic vector bundle $E$ over a complex variety $X$ of $\dim_{\mathbb{C}} X = n$ the index $\mathrm{ind}(\bar{\partial}, E)$ is defined as p

$$\mathrm{ind}(\bar{\partial}, E) = \sum_{k=0}^{n} (-1)^k \dim H^k(X, E) \tag{7.1}$$

The localization theorem in $K$-theory gives the index formula of Grothendieck-Riemann–Roch–Hirzebruch–Atiyah–Singer relating the index to the Todd class

$$\boxed{\mathrm{ind}(\bar{\partial}, E) = \frac{1}{(-2\pi\sqrt{-1})^n} \int_X \mathrm{td}(T_X^{1,0}) \mathrm{ch}(E)} \tag{7.2}$$

Similarly, the index of the Dirac operator $\not{D} : S^+ \otimes E \to S^- \otimes E$ from positive chiral spinors $S^+$ to negative chiral spinors $S^-$, twisted by a vector bundle $E$, is defined as

$$\mathrm{ind}(\not{D}, E) = \dim \ker \not{D} - \dim \mathrm{coker}\, \not{D} \tag{7.3}$$

and is given by the Atiyah–Singer index formula

$$\boxed{\mathrm{ind}(\not{D}, E) = \frac{1}{(-2\pi\sqrt{-1})^n} \int_X \hat{A}(T_X^{1,0}) \mathrm{ch}(E)} \tag{7.4}$$

Notice that on a Kahler manifold the Dirac complex

$$\not{D} : S^+ \to S^- \tag{7.5}$$

is isomorphic to the Dolbeault complex

$$\cdots \to \Omega^{0,p}(X) \xrightarrow{\bar{\partial}} \Omega^{0,p+1}(X) \to \cdots \tag{7.6}$$

twisted by a square root of the canonical bundle $K = \Lambda^n (T_X^{1,0})^\vee$

$$\not{D} = \bar{\partial} \otimes K^{\frac{1}{2}} \tag{7.7}$$

consistently with the relation (6.30) and the Grothendieck–Riemann–Roch–Hirzebruch–Atiyah–Singer index formula.

*Remark on* $2\pi$ *and* $\sqrt{-1}$ *factors.* The vector bundle $E$ in the index formula (7.2) can be promoted to a complex

$$\to E^{\bullet} \to E^{\bullet+1} \to \tag{7.8}$$

In particular, the $\bar{\partial}$ index of the complex $E^{\bullet} = \Lambda^{\bullet}(T^{1,0})^{\vee}$ of $(\bullet, 0)$-forms on a Kahler variety $X$ equals the Euler characteristic of $X$

$$e(X) = \mathrm{ind}(\bar{\partial}, \Lambda^{\bullet}(T^{1,0})^{\vee}) = \sum_{q=0}^{n}\sum_{p=0}^{n}(-1)^{p+q}\mathrm{dim}H^{p,q}(X) \tag{7.9}$$

We find

$$\mathrm{ch}\Lambda^{\bullet}(T^{1,0})^{\vee} = \prod_{i=1}^{n}(1 - \mathrm{e}^{-x_i}) \tag{7.10}$$

where $x_i$ are Chern roots of the curvature of the $n$-dimensional complex bundle $T_X^{1,0}$. Hence, the Todd index formula (7.2) gives

$$\mathrm{e}(X) = \frac{1}{(2\pi\sqrt{-1})^n}\int c_n(T_X^{1,0}) \tag{7.11}$$

The above agrees with the Euler characteristic (6.6) provided it holds that

$$\det(\sqrt{-1}x_{\mathfrak{u}(n)}) = \mathrm{Pf}(x_{\mathfrak{so}(2n)}) \tag{7.12}$$

where $x_{\mathfrak{so}(2n)}$ represents the curvature of the $2n$-dimensional real tangent bundle $T_X$ as $2n \times 2n$ antisymmetric matrices, and $x_{\mathfrak{u}(n)}$ represents the curvature of the complex holomorphic $n$-dimensional tangent bundle $T_X^{(1,0)}$ as $n \times n$ anti-hermitian matrices. That (7.12) holds is clear from the $2 \times 2$ representation of $\sqrt{-1}$

$$\sqrt{-1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7.13}$$

## 8. Equivariant integration

See the papers by Atiyah–Bott [20] and Berline–Vergne [21]

### *8.1. Thom isomorphism and Atiyah–Bott–Berline–Vergne localization*

A map

$$f : F \to X \tag{8.1}$$

of manifolds induces a natural pushfoward map on the homology

$$f_* : H_{\bullet}(F) \to H_{\bullet}(X) \tag{8.2}$$

and pullback on the cohomology

$$f^* : H^{\bullet}(X) \to H^{\bullet}(F) \tag{8.3}$$

In situations where there is a Poincaré duality between homology and cohomology we can construct a pushforward operation on the cohomology

$$f_* : H^\bullet(F) \to H^\bullet(X) \tag{8.4}$$

We can display the pullback and pushforward maps on the diagram

$$H^\bullet(F) \underset{f^*}{\overset{f_*}{\rightleftarrows}} H^\bullet(X) \tag{8.5}$$

For example, if $F$ and $X$ are compact oriented manifolds and $f : F \hookrightarrow X$ is the embedding, then for the pushforward map $f_* : H^\bullet(F) \to H^\bullet(X)$ we find

$$f_* 1 = \Phi_F \tag{8.6}$$

where $\Phi_F$ is the cohomology class in $H^\bullet(X)$ that is Poincaré dual to the manifold $F \subset X$: for a form $\alpha$ on $X$ we have

$$\int_F f^* \alpha = \int_X \Phi_F \wedge \alpha \tag{8.7}$$

If $X$ is the total space of an oriented orthogonal vector bundle $\pi : X \to F$ over a manifold $F$ then $\Phi_F$ is called the Thom class of the vector bundle $X$ and $f_* : H^\bullet(F) \to H^\bullet(X)$ is the Thom isomorphism: to a form $\alpha$ on $F$ we associate a form $\Phi_F \wedge \pi^* \alpha$ on $X$. An important property of the Thom class $\Phi_F$ of a submanifold $F \hookrightarrow X$ is

$$f^* \Phi_F = \mathrm{e}(\nu_F) \tag{8.8}$$

where $\mathrm{e}(\nu_F)$ is the Euler class[2] of the normal bundle to $F$ in $X$. Combined with (8.6) the last equation gives

$$f^* f_* 1 = \mu_{\mathrm{e}(\nu_F)} \tag{8.9}$$

as a map $\mu_{\mathrm{e}(\nu_F)} : H^\bullet(F) \to H^\bullet(F)$ of multiplication by $\mathrm{e}(\nu_F)$

$$\alpha \mapsto e(\nu_F) \wedge \alpha \tag{8.10}$$

Now we consider $T$-equivariant cohomologies for a compact abelian Lie group $T$ acting on $X$. Let $F = X^T$ be the set of $T$ fixed points in $X$. Atiyah and Bott [20] have shown, that the rank of $H^\bullet_T(F)$ (viewed as a module over the coefficient ring $H^\bullet_T(pt)$ modulo torsion elements) coincides with the rank of $H^\bullet_T(X)$. The rank is equal to the number of points in $X^T$.

Moreover, the equivariant Euler class $\mathrm{e}_T(\nu_F)$ is invertible since at each component $x \in X^T$ it is given by the product of non-zero weights of the $T$-action on $T_x X$.

The facts that the spaces $H^\bullet_T(F)$ and $H^\bullet_T(X)$ have the same rank and that the map $\mu_{\mathrm{e}(\nu_F)} = f^* f_* : H^\bullet_T(F) \to H^\bullet_T(F)$ is invertible imply that the identity map $\mathrm{id}_{H^\bullet_T(X)}$ on $H^\bullet_T(X)$ can be represented in the form

$$\mathrm{id}_{H^\bullet_T(X)} = f_* \mu^{-1}_{\mathrm{e}_T(\nu_F)} f^* \tag{8.11}$$

[2] Notice that the definition of the Euler class (8.8) in terms of the Thom class does not require that rank of $\nu_F$ is even neither that $F$ is oriented, but only that $\nu_F$ is orientable vector bundle over $F$. In integral cohomology theory $H^\bullet(F, \mathbb{Z})$ it is possible that Euler class $e(\nu_F)$ is non-trivial for an oriented vector bundle $\nu_F$ of odd rank, but in this case the Euler class has to be a torsion element because $e(\nu_F) = -e(\nu_F)$ by the existence of orientation changing isomorphism of $\nu_F$ (See example at http://math.stackexchange.com/questions/1268751). The Pfaffian construction of the Euler class in sections 6.1 and 6.2 automatically produces zero for orientable vector bundles of odd rank because it is based on the de Rham cohomology theory with coefficients in $\mathbb{R}$ which does not detect torsion elements.

Let $\pi^X : X \to pt$ be the map from a manifold $X$ to a point $pt$. The pushforward operator $\pi^X_* : H^\bullet_T(X) \to H^\bullet_T(pt)$ corresponds to the integrating the cohomology class over $X$. The pushforward is functorial. For maps $F \xrightarrow{f} X \xrightarrow{\pi^X} pt$ we have the composition $\pi^X_* f_* = \pi^F_*$ for $F \xrightarrow{\pi^F} pt$. So we arrive at the Atiyah–Bott–Berline–Vergne integration formula

$$\pi^X_* = \pi^F_* \mu^{-1}_{\mathrm{e}_T(\nu_F)} f^* \tag{8.12}$$

or more explicitly

$$\boxed{\int_X \alpha = \int_F \frac{f^*\alpha}{\mathrm{e}_T(\nu_F)}} \tag{8.13}$$

### 8.2. Duistermaat–Heckman localization

A particular example where the Atiyah–Bott–Berline–Vergne localization formula can be applied is a symplectic space on which a Lie group $T$ acts in a Hamiltonian way. Namely, let $(X, \omega)$ be a real symplectic manifold of $\dim_{\mathbb{R}} X = 2n$ with symplectic form $\omega$ and let a compact connected Lie group $T$ act on $X$ in Hamiltonian way, which means that there exists a function, called *moment map* or Hamiltonian

$$\mu : X \to \mathfrak{t}^\vee \tag{8.14}$$

such that

$$\mathrm{d}\mu_a = -i_a \omega \tag{8.15}$$

in some basis $(T_a)$ of $\mathfrak{t}$ where $i_a$ is the contraction operation with the vector field generated by the $T_a$ action on $X$.

The degree 2 element $\omega_T \in \Omega^\bullet(X) \otimes S\mathfrak{t}^\vee$ defined by the equation

$$\omega_T = \omega + \epsilon^a \mu_a \tag{8.16}$$

is a $d_T$-closed equivariant differential form:

$$d_T \omega_T = (d + \epsilon^a i_a)(\omega + \epsilon^b \mu_b) = \epsilon^a \mathrm{d}\mu_a + \epsilon^a i_a \omega = 0 \tag{8.17}$$

This implies that the mixed-degree equivariant differential form

$$\alpha = \mathrm{e}^{\omega_T} \tag{8.18}$$

is also $d_T$-closed, and we can apply the Atiyah–Bott–Berline–Vergne localization formula to the integral

$$\int_X \exp(\omega_T) = \frac{1}{n!} \int_X \omega^n \exp(\epsilon^a \mu_a) \tag{8.19}$$

For $T = SO(2)$ so that $\mathfrak{so}(2) \simeq \mathbb{R}$ the integral (8.19) is the typical partition function of a classical Hamiltonian mechanical system in statistical physics with Hamiltonian function $\mu : X \to \mathbb{R}$ and inverse temperature parameter $-\epsilon$.

Suppose that $T = SO(2)$ and that the set of fixed points $X^T$ is discrete. Then the Atiyah–Bott–Berline–Vergne localization formula (8.13) implies

$$\boxed{\frac{1}{n!} \int_X \omega^n \exp(\epsilon^a \mu_a) = \sum_{x \in X^T} \frac{\exp(\epsilon^a \mu_a)}{\mathrm{e}_T(\nu_x)}} \tag{8.20}$$

where $\nu_x$ is the normal bundle to a fixed point $x \in X^T$ in $X$ and $\mathrm{e}_T(\nu_x)$ is the $T$-equivariant Euler class of the bundle $\nu_x$.

The rank of the normal bundle $\nu_x$ is $2n$ and its structure group is $SO(2n)$. In the notations of section 5 we evaluate the $T$-equivariant Euler class of the principal $G$-bundle for $T = SO(2)$ and $G = SO(2n)$ by equation (5.6) for the invariant polynomial on $\mathfrak{g} = \mathfrak{so}(2n)$ given by $p = \frac{1}{(2\pi)^n}\mathrm{Pf}$ according to definition (6.5).

### *8.3. Gaussian integral example*

To illustrate the localization formula (8.20) suppose that $X = \mathbb{R}^{2n}$ with symplectic form

$$\omega = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y_i \tag{8.21}$$

and $SO(2)$ action

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} \mapsto \begin{pmatrix} \cos w_i\theta & -\sin w_i\theta \\ \sin w_i\theta & \cos w_i\theta \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \tag{8.22}$$

where $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ parametrizes $SO(2)$ and $(w_1, \dots, w_n) \in \mathbb{Z}^n$ with all $w_i \neq 0$.

The point $0 \in X$ is the *fixed point* so that $X^T = \{0\}$, and the normal bundle $\nu_x = T_0X$ is an $SO(2)$-module of real dimension $2n$ and complex dimension $n$ that splits into a direct sum of $n$ irreducible $SO(2)$ modules with weights $(w_1, \dots, w_n)$.

We identify $\mathfrak{so}(2) \simeq \mathbb{R}$ with basis $\{1\}$ and coordinate function $\epsilon \in \mathfrak{so}(2)^\vee \simeq \mathbb{R}$. The $SO(2)$ action (8.22) is Hamiltonian with respect to the moment map

$$\mu = \mu_0 + \frac{1}{2}\sum_{i=1}^{n} w_i(x_i^2 + y_i^2) \tag{8.23}$$

Assuming that $\epsilon < 0$ and all $w_i > 0$ we find by direct Gaussian integration

$$\frac{1}{n!}\int_X \omega^n \exp(\epsilon\mu) = \frac{(2\pi)^n}{(-\epsilon)^n \prod_{i=1}^n w_i} \exp(\epsilon\mu_0) \tag{8.24}$$

and the same result by the localization formula (8.20) because

$$\mathrm{e}_T(\nu_x) = \frac{1}{(2\pi)^n}\mathrm{Pf}(\epsilon\rho(1)) \tag{8.25}$$

according to the definition of the $T$-equivariant class (5.6) and the Euler characteristic class (6.5), and where $\rho : \mathfrak{so}(2) \to \mathfrak{so}(2n)$ is the homomorphism in (5.4) with

$$\rho(1) = \begin{pmatrix} 0 & -w_1 & \cdots & \cdots & 0 & 0 \\ w_1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & -w_n \\ 0 & 0 & \cdots & \cdots & w_n & 0 \end{pmatrix} \tag{8.26}$$

according to (8.22).

### 8.4. Example of a two-sphere

Let $(X, \omega)$ be the two-sphere $S^2$ with coordinates $(\theta, \phi)$ and symplectic structure

$$\omega = \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi \tag{8.27}$$

Let the Hamiltonian function be

$$\mu = -\cos\theta \tag{8.28}$$

so that

$$\omega = \mathrm{d}\mu \wedge \mathrm{d}\phi \tag{8.29}$$

and the Hamiltonian vector field be $v = \partial_\phi$. The differential form

$$\omega_T = \omega + \epsilon\mu = \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi - \epsilon\cos\theta \tag{8.30}$$

is $d_T$-closed for

$$d_T = d + \epsilon i_\phi \tag{8.31}$$

Let

$$\alpha = \exp(t\omega_T) \tag{8.32}$$

Locally there is a degree 1 form $V$ such that $\omega_T = d_T V$, for example

$$V = -\cos\theta \mathrm{d}\phi \tag{8.33}$$

but globally $V$ does not exist. The $d_T$-cohomology class $[\alpha]$ of the form $\alpha$ is non-zero.

The localization formula (8.20) gives

$$\int_X \exp(\omega_T) = \frac{2\pi}{-\epsilon}\exp(-\epsilon) + \frac{2\pi}{\epsilon}\exp(\epsilon) \tag{8.34}$$

where the first term is the contribution of the $T$-fixed point $\theta = 0$ and the second term is the contribution of the $T$-fixed point $\theta = \pi$.

## 9. Equivariant index formula (Dolbeault and Dirac)

Let $T$ be a compact connected Lie group.

Suppose that $X$ is a complex variety of complex dimension $n$ and $E$ is a holomorphic $T$-equivariant vector bundle over $X$. Then the cohomology groups $H^\bullet(X, E)$ form representation of $T$. In this case the index of $E$ (7.1) can be refined to *an equivariant index* or *character*

$$\mathrm{ind}_T(\bar\partial, E) = \sum_{k=0}^{n}(-1)^k \mathrm{ch}_T H^k(X, E) \tag{9.1}$$

where $\mathrm{ch}_T H^i(X, E)$ is the character of the representation of $T$ on the vector space $H^i(X, E)$. More concretely, the equivariant index can be thought of as a gadget that attaches to the $T$-equivariant holomorphic bundle $E$ a complex valued adjoint invariant function

$$\mathrm{ind}_T(\bar\partial, E)(t) = \sum_{k=0}^{n}(-1)^k \mathrm{tr}_{H^k(X,E)} t \tag{9.2}$$

on elements $t \in T$. The sign alternating sum (9.2) is also known as *the supertrace*

$$\mathrm{ind}_T(\bar\partial, E)(t) = \mathrm{str}_{H^\bullet(X,E)} t \tag{9.3}$$

The index formula (7.2) is replaced by the equivariant index formula in which characteristic classes are promoted to $T$-equivariant characteristic classes in the Cartan model of $T$-equivariant cohomology with differential $d_T = d + \epsilon^\alpha i_\alpha$ as in (4.10)

$$\mathrm{ind}_T(\bar\partial, E)(\mathrm{e}^{\epsilon^\alpha T_\alpha}) = \frac{1}{(-2\pi\sqrt{-1})^n}\int_X \mathrm{td}_T(T_X)\mathrm{ch}_T(E) = \int_X e_T(T_X)\frac{\mathrm{ch}_T(E)}{\mathrm{ch}_T\Lambda^\bullet(T_X^\vee)} \tag{9.4}$$

Here $\{T_\alpha\}$ is a basis of the Lie algebra $\mathfrak{t}$ and $\{\epsilon^\alpha\}$ are the respective coordinates, so that $\epsilon^\alpha T_\alpha \in \mathfrak{t}$ and $\mathrm{e}^{\epsilon^\alpha T_\alpha} \in T$. In the right hand side $T_X$ denotes the holomorphic tangent bundle of the complex manifold $X$, and $T_X^\vee$ is its dual. The right hand side depends on $\epsilon^a$ by the definition (5.6) of $T$-equivariant characteristic classes in the Cartan model.

If the set $X^T$ of $T$-fixed points is discrete, then applying the localization formula (8.13) to the equivariant index (9.4) we find the equivariant Lefshetz formula

$$\boxed{\mathrm{ind}(\bar\partial, E)(t) = \sum_{x\in X^T}\frac{\mathrm{tr}_{E_x}(t)}{\det_{T_x^{1,0}X}(1-t^{-1})}} \tag{9.5}$$

The Euler character is cancelled against the numerator of the Todd character.

### *9.1. Example of* $\mathbb{CP}^1$

Let $X$ be $\mathbb{CP}^1$ and let $E = \mathcal{O}(n)$ be a complex line bundle of degree $n$ over $\mathbb{CP}^1$, and let $G = U(1)$ equivariantly act on $E$ as follows. Let $z$ be a local coordinate on $\mathbb{CP}^1$, and let an element $t \in U(1) \subset \mathbb{C}^\times$ send the point with coordinate $z$ to the point with coordinate $tz$ so that

$$\mathrm{ch}T_0^{1,0}X = t \qquad \mathrm{ch}T_\infty^{1,0}X = t^{-1} \tag{9.6}$$

where $T_0^{1,0}X$ denotes the fiber of the holomorphic tangent bundle at $z=0$ and similarly $T_\infty^{1,0}X$ the fiber at $z=\infty$. Let the action of $U(1)$ on the fiber of $E$ at $z=0$ be trivial. Then the action of $U(1)$ on the fiber of $E$ at $z=\infty$ is found from the gluing relation

$$s_\infty = z^{-n}s_0 \tag{9.7}$$

to be of weight $-n$, so that

$$\mathrm{ch}E|_{z=0} = 1, \qquad \mathrm{ch}E|_{z=\infty} = t^{-n}. \tag{9.8}$$

Then

$$\mathrm{ind}(\bar\partial, \mathcal{O}(n), \mathbb{CP}^1)(t) = \frac{1}{1-t^{-1}} + \frac{t^{-n}}{1-t} = \frac{1-t^{-n-1}}{1-t^{-1}} = \begin{cases}\sum_{k=0}^n t^{-k}, & n\geq 0\\ 0, & n=-1,\\ -t\sum_{k=0}^{-n-2}t^k, & n<-1\end{cases} \tag{9.9}$$

We can check against the direct computation. Assume $n \geqslant 0$. The kernel of $\bar\partial$ is spanned by $n+1$ holomorphic sections of $\mathcal{O}(n)$ of the form $z^k$ for $k = 0,\dots,n$, the cokernel is empty by Riemann–Roch. The section $z^k$ is acted upon by $t \in T$ with weight $t^{-k}$. Therefore

$$\mathrm{ind}_T(\bar\partial, \mathcal{O}(n), \mathbb{CP}^1) = \sum_{k=0}^n t^{-k}. \tag{9.10}$$

Even more explicitly, for illustration, choose a connection 1-form $A$ with constant curvature $F_A = -\frac{1}{2}in\omega$, denoted in the patch around $\theta = 0$ (or $z=0$) by $A^{(0)}$ and in the patch around $\theta = \pi$ (or $z = \infty$) by $A^{(\pi)}$

$$A^{(0)} = -\frac{1}{2}\mathrm{i}n(1-\cos\theta)\mathrm{d}\alpha \qquad A^{(\pi)} = -\frac{1}{2}\mathrm{i}n(-1-\cos\theta)\mathrm{d}\alpha \tag{9.11}$$

The gauge transformation between the two patches

$$A^{(0)} = A^{(\pi)} - \mathrm{i}n\,\mathrm{d}\alpha \tag{9.12}$$

is consistent with the defining $E$ bundle transformation rule for the sections $s^{(0)}, s^{(\pi)}$ in the patches around $\theta = 0$ and $\theta = \pi$

$$s^{(0)} = z^n s^{(\pi)} \qquad A^{(0)} = A^{(\pi)} + z^n d(z^{-n}). \tag{9.13}$$

The equivariant curvature $F_T$ of the connection $A$ in the bundle $E$ is given by

$$F_T = -\frac{1}{2}\mathrm{i}n(\omega + \epsilon(1-\cos\theta)) \tag{9.14}$$

as can be verified against the definition (5.3) $F_T = F + \epsilon i_v A$. Notice that to verify the expression for the equivariant curvature (9.14) in the patch near $\theta = \pi$ one needs to take into account contributions from the vertical component $g^{-1}dg$ of the connection $A$ on the total space of the principal $U(1)$ bundle and from the $T$-action on the fiber at $\theta = \pi$ with weight $-n$.

Then

$$\begin{aligned} \mathrm{ch}(E)|_{\theta=0} &= \exp(F_T)|_{\theta=0} = 1 \\ \mathrm{ch}(E)|_{\theta=\pi} &= \exp(F_T)|_{\theta=\pi} = \exp(-in\epsilon) = t^{-n} \end{aligned} \tag{9.15}$$

for $t = \exp(\mathrm{i}\epsilon)$ in agreement with (9.8).

A similar exercise gives the index for the Dirac operator on $S^2$ twisted by a magnetic field of flux $n$

$$\mathrm{ind}(\not{D}, \mathcal{O}(n), S^2) = \frac{t^{n/2} - t^{-n/2}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \tag{9.16}$$

where now we have chosen the lift of the $T$-action symmetrically to be of weight $n/2$ at $\theta = 0$ and of weight $-n/2$ at $\theta = \pi$, so that (9.14) is changed into

$$F_T = -\frac{1}{2}\mathrm{i}n(\omega - \cos\theta)) \tag{9.17}$$

Also notice that up to an overall multiplication by a power of $t$ related to the choice of lift of the $T$-action to the fibers of the bundle $E$, the relation (7.7) holds

$$\mathrm{ind}(\not{D}, \mathcal{O}(n), S^2) = \mathrm{ind}(\bar{\partial}, \mathcal{O}(n-1), \mathbb{CP}^1) \tag{9.18}$$

because on $\mathbb{CP}^1$ the canonical bundle is $K = \mathcal{O}(-2)$.

### 9.2. Example of $\mathbb{CP}^m$

Let $X = \mathbb{CP}^m$ be defined by the projective coordinates $(x_0 : x_1 : \cdots : x_m)$ and $L_n$ be the line bundle $L_n = \mathcal{O}(n)$. Let $T = U(1)^{(m+1)}$ act on $X$ by

$$(x_0 : x_1 : \ldots x_m) \mapsto (t_0^{-1}x_0 : t_1^{-1}x_1 : \cdots : t_m^{-1}x_m) \tag{9.19}$$

and by $t_k^n$ on the fiber of the bundle $L_n$ in the patch around the $k$th fixed point $x_k = 1, x_{i\neq k} = 0$. We find the index as a sum of contributions from $m+1$ fixed points

$$\mathrm{ind}_T(D) = \sum_{k=0}^{m} \frac{t_k^n}{\prod_{j\neq k}(1-(t_j/t_k))} \tag{9.20}$$

For $n \geqslant 0$ the index is a homogeneous polynomial in $\mathbb{C}[t_0, \ldots, t_m]$ of degree $n$ representing the character on the space of holomorphic sections of the $\mathcal{O}(n)$ bundle over $\mathbb{CP}^m$.

$$\mathrm{ind}_T(D) = \begin{cases} s_n(t_0, \ldots, t_m), & n \geqslant 0 \\ 0, & -m \leqslant n < 0 \\ (-1)^m t_0^{-1} t_1^{-1} \ldots t_m^{-1} s_{-n-m-1}(t_0^{-1}, \ldots, t_m^{-1}), & n \leqslant -m-1 \end{cases} \tag{9.21}$$

where $s_n(t_0, \ldots, t_m)$ are complete homogeneous symmetric polynomials. This result can be quickly obtained from the contour integral representation of the sum (9.20)

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{dz}{z} \frac{z^n}{\prod_{j=0}^m (1 - t_j/z)} = \sum_{k=0}^m \frac{t_k^n}{\prod_{j \neq k}(1 - (t_j/t_k))}, \tag{9.22}$$

If $n \geqslant -m$ we pick the contour of integration $\mathcal{C}$ to enclose all residues $z = t_j$. The residue at $z = 0$ is zero and the sum of residues is (9.20). On the other hand, the same contour integral is evaluated by the residue at $z = \infty$ which is computed by expanding all fractions in inverse powers of $z$, and is given by the complete homogeneous polynomial in $t_i$ of degree $n$.

If $n < -m$ we assume that the contour of integration is a small circle around the $z = 0$ and does not include any of the residues $z = t_j$. Summing the residues outside the contour, and noting that $z = \infty$ does not contribute, we get (9.20) with the $(-)$ sign . The residue at $z = 0$ contributes by (9.21).

Also notice that the last line of (9.21) relates[3] to the first line by the reflection $t_i \to t_i^{-1}$

$$\frac{t_k^n}{\prod_{j \neq k}(1 - t_j/t_k)} = \frac{(-1)^m (t_k^{-1})^{-n-m-1} (\prod_j t_j^{-1})}{\prod_{j \neq k}(1 - t_j^{-1}/t_k^{-1})} \tag{9.23}$$

which is the consequence of Serre duality on $\mathbb{CP}^m$.

## 10. Equivariant index and representation theory

The $\mathbb{CP}^1$ in example (9.16) can be thought of as a flag manifold $SU(2)/U(1)$, and (9.9) (9.16) as characters of $SU(2)$-modules. For index theory on general flag manifolds $G_{\mathbb{C}}/B_{\mathbb{C}}$, that is, for Borel-Weyl-Bott theorem[4], the shift of the form (9.18) is a shift by the Weyl vector $\rho = \sum_{\alpha > 0} \alpha$ where $\alpha$ are positive roots of $\mathfrak{g}$.

The index formula with localization to the fixed points on a flag manifold is equivalent to the Weyl character formula.

The generalization of formula (9.16) for a generic flag manifold appearing from a co-adjoint orbit in $\mathfrak{g}^*$ is called *Kirillov character formula* [22, 23], [24].

Let $G$ be a compact simple Lie group and $T \subset G$ its maximal torus. The Kirillov character formula equates the $T$-equivariant index $\mathrm{ind}_T(\not{D})$ of the Dirac operator on the $G$-coadjoint orbit of the element $\lambda + \rho \in \mathfrak{g}^\vee$ with the character $\chi_\lambda$ of the $G$ irreducible representation with highest weight $\lambda$.

The character $\chi_\lambda$ is a function $\mathfrak{g} \to \mathbb{C}$ determined by the representation of the Lie group $G$ with highest weight $\lambda$ as

$$\chi_\lambda : X \mapsto \mathrm{tr}_\lambda e^X, \qquad X \in \mathfrak{g} \tag{10.1}$$

[3] Thanks to Bruno Le Floch for the comment.

[4] For a short presentation see the exposition by J. Lurie at http://www.math.harvard.edu/~lurie/papers/bwb.pdf.

Let $X_\lambda$ be an orbit of the co-adjoint action by $G$ on $\mathfrak{g}^\vee$. Such an orbit is specified by an element $\lambda \in \mathfrak{t}^\vee/W$ where $\mathfrak{t}$ is the Lie algebra of the maximal torus $T \subset G$ and $W$ is the Weyl group. The co-adjoint orbit $X_\lambda$ is a homogeneous symplectic $G$-manifold with the canonical symplectic structure $\omega$ defined at point $x \in X \subset \mathfrak{g}^\vee$ on tangent vectors in $\mathfrak{g}$ by the formula

$$\omega_x(\bullet_1, \bullet_2) = \langle x, [\bullet_1, \bullet_2] \rangle \qquad \bullet_1, \bullet_2 \in \mathfrak{g} \tag{10.2}$$

The converse is also true: any homogeneous symplectic $G$-manifold is locally isomorphic to a coadjoint orbit of $G$ or a central extension of it.

The minimal possible stabilizer of $\lambda$ is the maximal abelian subgroup $T \subset G$, and the maximal co-adjoint orbit is $G/T$. Such an orbit is called *a full flag manifold*. The real dimension of the full flag manifold is $2n = \dim G - \mathrm{rk} G$, and is equal to the number of roots of $\mathfrak{g}$. If the stabilizer of $\lambda$ is a larger group $H$, such that $T \subset H \subset G$, the orbit $X_\lambda$ is called *a partial flag manifold* $G/H$. A partial flag manifold is a projection from the full flag manifold with fibers isomorphic to $H/T$.

Flag manifolds are equipped with natural complex and Kahler structures. There is an explicitly holomorphic realization of a flag manifold as a complex quotient $G_\mathbb{C}/P_\mathbb{C}$ where $G_\mathbb{C}$ is the complexification of the compact group $G$ and $P_\mathbb{C} \subset G_\mathbb{C}$ is a parabolic subgroup. Let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ be the standard decomposition of $\mathfrak{g}$ into the Cartan $\mathfrak{h}$ algebra and the upper triangular $\mathfrak{g}_+$ and lower triangular $\mathfrak{g}_-$ subspaces.

The minimal parabolic subgroup is known as the Borel subgroup $B_\mathbb{C}$, its Lie algebra is conjugate to $\mathfrak{h} \oplus \mathfrak{g}_+$. Lie algebra of a generic parabolic subgroup $P_\mathbb{C} \supset B_\mathbb{C}$ is conjugate to the direct sum of $\mathfrak{h} \oplus \mathfrak{g}_+$ and a proper subspace of $\mathfrak{g}_-$.

Full flag manifolds with integral symplectic structure are in bijection with irreducible $G$-representations $\pi_\lambda$ of highest weight $\lambda$

$$X_{\lambda+\rho} \leftrightarrow \pi_\lambda \tag{10.3}$$

This is known as the Kirillov correspondence in geometric representation theory.

Namely, if $\lambda \in \mathfrak{g}^\vee$ is a weight, the symplectic structure $\omega$ is integral and there exists a line bundle $L \to X_\lambda$ with a unitary connection of curvature $\omega$. The line bundle $L \to X_\lambda$ is acted upon by the maximal torus $T \subset G$ and we can study the $T$-equivariant geometric objects. The Kirillov–Berline–Getzler–Vergne character formula states that the equivariant index of the Dirac operator $\not{D}$ twisted by the line bundle $L \to X_{\lambda+\rho}$ on the co-adjoint orbit $X_{\lambda+\rho}$ is equal to the character $\chi_\lambda$ of the irreducible representation of $G$ with highest weight $\lambda$

$$\mathrm{ind}_T(\not{D})(X_{\lambda+\rho}) = \chi_\lambda \tag{10.4}$$

This formula can be easily proven using equivariant version of the Atiyah–Singer index formula (7.4)

$$\mathrm{ind}_T(\not{D})(X_{\lambda+\rho}) = \frac{1}{(-2\pi\sqrt{-1})^n} \int_{X_{\lambda+\rho}} \mathrm{ch}_T(L)\hat{A}_T(T_X) \tag{10.5}$$

and the Atiyah–Bott–Berline–Vergne formula to localize the integral over $X_{\lambda+\rho}$ to the set of fixed points $X^T_{\lambda+\rho}$.

The localization to $X^T_{\lambda+\rho}$ yields the Weyl formula for the character. The $T$-fixed points are in the intersection $X_{\lambda+\rho} \cap \mathfrak{t}$, and hence, the set of the $T$-fixed points is the Weyl orbit of $\lambda + \rho$

$$X^T_{\lambda+\rho} = \mathrm{Weyl}(\lambda+\rho) \tag{10.6}$$

At each fixed point $x = w(\lambda+\rho) \in X^T_{\lambda+\rho}$ the tangent space $T_{X_{\lambda+\rho}}|_p$ is generated by the root system of $\mathfrak{g}$. Indeed, the stabilizer of $\lambda+\rho$, where $\lambda$ is a dominant weight, is the Cartan torus $T \subset G$ thus the co-adjoint orbit $X_{\lambda+\rho}$ is a full flag manifold. The tangent space at $x$ is a complex $T$-module $\oplus_{\alpha>0}\mathbb{C}_\alpha$ with weights $\alpha$ given by the positive roots of $\mathfrak{g}$. Consequently, the denominator of $\hat{A}_T$ gives the Weyl denominator, the numerator of $\hat{A}_T$ cancels with the Euler class $e_T(T_X)$ in the localization formula, and the restriction to $x = w(\lambda+\rho)$ of $\mathrm{ch}_T(L) = \mathrm{e}^{F_T}$, where $F_T$ is $T$-equivariant curvature of a connection on line bundle $L$, is $\mathrm{e}^{w(\lambda+\rho)}$ (c.f. example (9.17))

$$\frac{1}{(-2\pi\sqrt{-1})^n}\int_{X_{\lambda+\rho}} \mathrm{ch}_T(L)\hat{A}(T_X) = \sum_{w\in W}\frac{\mathrm{e}^{\mathrm{i}w(\lambda+\rho)\epsilon}}{\prod_{\alpha>0}(\mathrm{e}^{\frac{1}{2}\mathrm{i}\alpha\epsilon}-\mathrm{e}^{-\frac{1}{2}\mathrm{i}\alpha\epsilon})} \tag{10.7}$$

We conclude that the localization of the equivariant index of the Dirac operator on $X_{\lambda+\rho}$ twisted by the line bundle $L$ to the set of fixed points $X^T_{\lambda+\rho}$ is precisely the Weyl formula for the character.

The Kirillov correspondence between the index of the Dirac operator of $L \to X_{\lambda+\rho}$ and the character is closedly related to the Borel–Weyl–Bott theorem.

Let $B_{\mathbb{C}}$ be a Borel subgroup of $G_{\mathbb{C}}$, $T_{\mathbb{C}}$ be the maximal torus, $\lambda$ an integral weight of $T_{\mathbb{C}}$. A weight $\lambda$ defines a one-dimensional representation of $B_{\mathbb{C}}$ by pulling back the representation of $T_{\mathbb{C}} = B_{\mathbb{C}}/U_{\mathbb{C}}$ where $U_{\mathbb{C}}$ is the unipotent radical of $B_{\mathbb{C}}$ (the unipotent radical $U_{\mathbb{C}}$ is conjugate to a subgroup generated by $\mathfrak{g}_+$). Let $L_\lambda \to G_{\mathbb{C}}/B_{\mathbb{C}}$ be the associated line bundle, and $\mathcal{O}(L_\lambda)$ be the sheaf of regular local sections of $L_\lambda$. For $w \in \mathrm{Weyl}_G$ define the action of $w$ on a weight $\lambda$ by $w * \lambda := w(\lambda+\rho) - \rho$.

The *Borel–Weyl–Bott* theorem is that for any weight $\lambda$ one has

$$H^\bullet(G_{\mathbb{C}}/B_{\mathbb{C}}, \mathcal{O}(L_\lambda)) = \begin{cases} 0 \text{ if none of the } w*\lambda \text{ are dominant} \\ R_\lambda \text{ at degree } l(w) \text{ if } w*\lambda \text{ is dominant} \end{cases} \tag{10.8}$$

where $R_\lambda$ is the irreducible $G$-module with highest weight $\lambda$, the $w$ is an element of Weyl group such that $w*\lambda$ is dominant weight, and $l(w)$ is the length of $w$. We remark that if there exists $w \in \mathrm{Weyl}_G$ such that $w*\lambda$ is dominant weight, then $w$ is unique. There is no $w \in \mathrm{Weyl}_G$ such that $w*\lambda$ is dominant if in the basis of the fundamental weights $\Lambda_i$ some of the coordinates of $\lambda+\rho$ vanish.

### *10.1. Example*

For $G = SU(2)$ one has $G_{\mathbb{C}}/B_{\mathbb{C}} = \mathbb{CP}^1$, an integral weight of $T_{\mathbb{C}}$ is an integer $n \in \mathbb{Z}$, and the line bundle $L_n$ is the $\mathcal{O}(n)$ bundle over $\mathbb{CP}^1$. The Weyl weight is $\rho = 1$.

The weight $n \geqslant 0$ is dominant and $H^0(\mathbb{CP}^1, \mathcal{O}(n))$ is the $SL(2,\mathbb{C})$ module of highest weight $n$ (in the basis of fundamental weights of $SL(2)$).

For weight $n = -1$ the $H^i(\mathbb{CP}^1, \mathcal{O}(-1))$ is zero for all $i$ as there is no Weyl transformation $w$ such that $w*n$ is dominant (equivalently, because $\rho + n = 0$).

For weight $n \leqslant -2$ the $\mathbb{Z}_2$ reflection $w$ makes $w*n = -(n+1) - 1 = -n-2$ dominant and $H^1(\mathbb{CP}^1, \mathcal{O}(n))$ is an irreducible $SL(2,\mathbb{C})$ module of highest weight $-n-2$.

The relation between the Borel–Weil–Bott theorem for $G_{\mathbb{C}}/B_{\mathbb{C}}$ and the Dirac complex on $G_{\mathbb{C}}/B_{\mathbb{C}}$ is that the Dirac operator is precisely the Dolbeault operator shifted by the square root of the canonical bundle

$$S^+(X) \ominus S^-(X) = K^{\frac{1}{2}} \sum (-1)^p \Omega^{0,p}(X) \tag{10.9}$$

and consequently

$$\operatorname{ind}(X_{\lambda+\rho}, \not{D} \otimes L_{\lambda+\rho}) = \operatorname{ind}(G_{\mathbb{C}}/B_{\mathbb{C}}, \bar{\partial} \otimes L_{\lambda}) \tag{10.10}$$

The Borel–Bott–Weyl theorem has a generalization for partial flag manifolds. Let $P_{\mathbb{C}}$ be a parabolic subgroup of $G_{\mathbb{C}}$ with $B_{\mathbb{C}} \subset P_{\mathbb{C}}$ and let $\pi : G_{\mathbb{C}}/B_{\mathbb{C}} \to G_{\mathbb{C}}/P_{\mathbb{C}}$ denote the canonical projection. Let $E \to G_{\mathbb{C}}/P_{\mathbb{C}}$ be a vector bundle associated to an irreducible finite dimensional $P_{\mathbb{C}}$ module, and let $\mathcal{O}(E)$ the the sheaf of local regular sections of $E$. Then $\mathcal{O}(E)$ is isomorphic to the direct image sheaf $\pi_* \mathcal{O}(L)$ for a one-dimensional $B_{\mathbb{C}}$-module $L$ and

$$H^{\bullet}(G_{\mathbb{C}}/P_{\mathbb{C}}, \mathcal{O}(E)) = H^{\bullet}(G_{\mathbb{C}}/B_{\mathbb{C}}, \mathcal{O}(L)) \tag{10.11}$$

For application of Kirillov theory to Kac-Moody and Virasoro algebra see [25].

## 11. Equivariant index for differential operators

See the book by Atiyah [26].

Let $E_k$ be vector bundles over a manifold $X$. Let $G$ be a compact Lie group acting on $X$ and the bundles $E_k$. The action of $G$ on a bundle $E$ induces canonically a linear action on the space of sections $\Gamma(E)$. For $g \in G$ and a section $\phi \in \Gamma(E)$ the action is

$$(g\phi)(x) = g\phi(g^{-1}x), \quad x \in X \tag{11.1}$$

Let $D_k$ be linear differential operators compatible with the $G$ action, and let $\mathcal{E}$ be the complex (that is $D_{k+1} \circ D_k = 0$)

$$\mathcal{E} : \Gamma(E_0) \xrightarrow{D_0} \Gamma(E_1) \xrightarrow{D_1} \Gamma(E_2) \to \dots \tag{11.2}$$

Assume that complex $\mathcal{E}$ is elliptic. Then the cohomology groups $H^k(\mathcal{E})$ are finite dimensional spaces. Since $D_k$ are $G$-equivariant operators, the $G$-action on $\Gamma(E_k)$ induces the $G$-action on the cohomology $H^k(\mathcal{E})$. The equivariant index of an elliptic complex $\mathcal{E}$ is the virtual character

$$\operatorname{ind}_G(D) : \mathfrak{g} \to \mathbb{C} \tag{11.3}$$

defined by

$$\operatorname{ind}_G(D)(g) = \sum_k (-1)^k \operatorname{tr}_{H^k(\mathcal{E})} g \tag{11.4}$$

### 11.1. Atiyah–Singer equivariant index formula for elliptic complexes

If the set $X^G$ of $G$-fixed points is discrete, the Atiyah–Singer equivariant index formula for elliptic complex is

$$\boxed{\operatorname{ind}_G(D) = \sum_{x \in X^G} \frac{\sum_k (-1)^k \operatorname{ch}_G(E_k)|_x}{\det_{T_x X}(1 - g^{-1})}} \tag{11.5}$$

For the Dolbeault complex $E_k = \Omega^{0,k}$ and $D_k = \bar{\partial} : \Omega^{0,k} \to \Omega^{0,k+1}$

$$\to \Omega^{0,\bullet} \xrightarrow{\bar{\partial}} \Omega^{0,\bullet+1} \to \tag{11.6}$$

the index (11.5) agrees with (9.5) because the numerator in (11.5) decomposes as $\operatorname{ch}_G E \operatorname{ch}_G \Lambda^{\bullet} T^*_{0,1}$ and the denominator as $\operatorname{ch}_G \Lambda^{\bullet} T^*_{0,1} \operatorname{ch}_G \Lambda^{\bullet} T^*_{1,0}$ and the factor $\operatorname{ch}_G \Lambda^{\bullet} T^*_{0,1}$ cancels out.

For example, the equivariant index of $\bar{\partial} : \Omega^{0,0}(X) \to \Omega^{0,1}(X)$ on $X = \mathbb{C}_{\langle x \rangle}$ under the $T = U(1)$ action $x \mapsto t^{-1}x$ where $t \in T$ is the fundamental character is contributed by the fixed point $x = 0$ as

$$\mathrm{ind}_T(\mathbb{C}, \bar{\partial}) = \frac{1 - \bar{t}}{(1-t)(1-\bar{t})} = \frac{1}{1-t} = \sum_{k=0}^{\infty} t^k \tag{11.7}$$

where the denominator is the determinant of the operator $1 - t$ over the two-dimensional normal bundle to $0 \in \mathbb{C}$ spanned by the vectors $\partial_x$ and $\partial_{\bar{x}}$ with eigenvalues $t$ and $\bar{t}$. In the numerator, 1 comes from the equivariant Chern character on the fiber of the trivial line bundle at $x = 0$ and $-\bar{t}$ comes from the equivariant Chern character on the fiber of the bundle of (0, 1) forms $\mathrm{d}\bar{x}$.

We can compare the expansion in power series in $t^k$ of the index with the direct computation. The terms $t^k$ for $k \in \mathbb{Z}_{\geqslant 0}$ come from the local $T$-equivariant holomorphic functions $x^k$ which span the kernel of $\bar{\partial}$ on $\mathbb{C}_{\langle x \rangle}$. The cokernel is empty by the Poincaré lemma. Compare with (9.10).

Similarly, for the $\bar{\partial}$ complex on $\mathbb{C}^r$ we obtain

$$\mathrm{ind}_T(\mathbb{C}^r, \bar{\partial}) = \left[ \prod_{k=1}^{r} \frac{1}{(1 - t_k)} \right]_+ \tag{11.8}$$

where $[]_+$ means expansion in positive powers of $t_k$.

For application to the localization computation on spheres of even dimension $S^{2r}$ we can compute the index of a certain transversally elliptic operator $D$ which naturally interpolates between the $\bar{\partial}$-complex in the neighborhood of one fixed point (north pole) of the $r$-torus $T^r$ action on $S^{2r}$ and the $\bar{\partial}$-complex in the neighborhood of another fixed point (south pole). The index is a sum of two fixed point contributions

$$\begin{aligned} \mathrm{ind}_T(S^{2r}, D) &= \left[ \prod_{k=1}^{r} \frac{1}{(1 - t_k)} \right]_+ + \left[ \prod_{k=1}^{r} \frac{1}{(1 - t_k)} \right]_- \\ &= \left[ \prod_{k=1}^{r} \frac{1}{(1 - t_k)} \right]_+ + \left[ \prod_{k=1}^{r} \frac{(-1)^r t_1^{-1} \ldots t_r^{-1}}{(1 - t_k^{-1})} \right]_- \end{aligned} \tag{11.9}$$

where $[]_+$ and $[]_-$ denotes the expansions in positive and negative powers of $t_k$.

### *11.2. Atiyah–Singer index formula for a free action G-manifold*

Suppose that a compact Lie group $G$ acts freely on a manifold $X$ and let $Y = X/G$ be the quotient, and let

$$\pi : X \to Y \tag{11.10}$$

be the associated $G$-principal bundle.

Suppose that $D$ is a $G \times T$ equivariant differential operator for a complex $(\mathcal{E}, D)$ of vector bundles $E_k$ over $X$ as in (11.2). The $G \times T$-equivariance means that the complex $\mathcal{E}$ and the operator $D$ are pullbacks by $\pi^*$ of a $T$-equivariant complex $\tilde{\mathcal{E}}$ and operator $\tilde{D}$ on the base $Y$

$$\mathcal{E} = \pi^* \tilde{\mathcal{E}}, \quad D = \pi^* \tilde{D} \tag{11.11}$$

We want to compute the $G \times T$-equivariant index $\mathrm{ind}_{G \times T}(D; X)$ for the complex $(\mathcal{E}, D)$ on the total space $X$ for a $G \times T$ transversally elliptic operator $D$ using $T$-equivariant index theory on the base $Y$. We can do that using Fourier–Peter–Weyl theory on $G$ (counting Kaluza–Klein modes in $G$-fibers).

Let $R_G$ be the set of all irreducible representations of $G$. For each irreducible representation $\alpha \in R_G$ we denote by $\chi_\alpha$ the character of this representation, and by $W_\alpha$ the vector bundle over $Y$ associated to the principal $G$-bundle (11.10). Then, for each irrep $\alpha \in R_G$ we consider a complex $\tilde{\mathcal{E}} \otimes W_\alpha$ on $Y$ obtained by tensoring $\tilde{\mathcal{E}}$ with the vector bundle $W_\alpha$ over $Y$. The Atiyah–Singer formula is

$$\operatorname{ind}_{G\times T}(D; X) = \sum_{\alpha\in R_G} \operatorname{ind}_T(\tilde{D} \otimes \mathrm{id}_{W_\alpha}; Y)\chi_\alpha. \tag{11.12}$$

*11.2.1. Example of $S^{2r-1}$.* We consider an example immediately relevant for localization on odd-dimensional spheres $S^{2r-1}$ which are subject to the equivariant action of the maximal torus $T^r$ of the isometry group $SO(2r)$. The sphere $\pi : S^{2r-1} \to \mathbb{CP}^{r-1}$ is the total space of the $S^1$ Hopf fibration over the complex projective space $\mathbb{CP}^{r-1}$.

We will apply the equation (11.12) for a transversally elliptic operator $D$ induced from the Dolbeault operator $\tilde{D} = \bar{\partial}$ on $\mathbb{CP}^{r-1}$ by the pullback $\pi^*$.

To compute the index of the operator $D = \pi^*\bar{\partial}$ on $\pi : S^{2r-1} \to \mathbb{CP}^{r-1}$ we apply (11.12) and use (9.21) and obtain

$$\operatorname{ind}(D, S^{2r-1}) = \sum_{n=-\infty}^{\infty} \operatorname{ind}_T(\bar{\partial}, \mathbb{CP}^{r-1}, \mathcal{O}(n)) = \left[\frac{1}{\prod_{k=1}^r (1-t_k)}\right]_+ + \left[\frac{(-1)^{r-1} t_1^{-1} \dots t_r^{-1}}{\prod_{k=1}^r (1-t_k^{-1})}\right]_- \tag{11.13}$$

where $[]_+$ and $[]_-$ denotes the expansion in positive and negative powers of $t_k$. See further review in contribution [27].

### 11.3. General Atiyah–Singer index formula

The Atiyah–Singer index formula for the Dolbeault and Dirac complexes and the equivariant index formula (11.5) can be generalized to the generic situation of an equivariant index of the transversally elliptic complex (11.2).

Let $X$ be a real manifold. Let $\pi : T^*X \to X$ be the cotangent bundle. Let $\{E^\bullet\}$ be an indexed set of vector bundles on $X$ and $\pi^*E^\bullet$ be the vector bundles over $T^*X$ defined by the pullback.

The symbol $\sigma(D)$ of a differential operator $D : \Gamma(E) \to \Gamma(F)$ (11.2) is a linear operator $\sigma(D) : \pi^*E \to \pi^*F$ which is defined by taking the highest degree part of the differential operator and replacing all derivatives $\frac{\partial}{\partial x^\mu}$ by the conjugate coordinates $p_\mu$ in the fibers of $T^*X$.

For example, for the Laplacian $\Delta : \Omega^0(X,\mathbb{R}) \to \Omega^0(X,\mathbb{R})$ with highest degree part in some coordinate system $\{x^\mu\}$ given by $\Delta = g^{\mu\nu}\partial_\mu\partial_\nu$ where $g^{\mu\nu}$ is the inverse Riemannian metric, the symbol of $\Delta$ is a $\mathrm{Hom}(\mathbb{R},\mathbb{R})$-valued (i.e. number valued) function on $T^*X$ given by

$$\sigma(\Delta) = g^{\mu\nu} p_\mu p_\nu \tag{11.14}$$

where $p_\mu$ are conjugate coordinates (momenta) on the fibers of $T^*X$.

A differential operator $D : \Gamma(E) \to \Gamma(F)$ is *elliptic* if its symbol $\sigma(D) : \pi^*E \to \pi^*F$ is an isomorphism of vector bundles $\pi^*E$ and $\pi^*F$ on $T^*X$ outside of the zero section $X \subset T^*X$.

If a compact connect Lie group $G$ acts on $X$ and $G$-action is lifted equivariantly on vector bundles $E$, $F$ and differential operator $D$, one can define a weaker property for $D$ to be *G-transversally elliptic*. For this purpose consider the family $T^*_G X$ consisting at each point $x \in X$ of covectors $p \in T^*_x X$ that are annihilated by all tangent vectors to the $G$-orbit at $x$.

A differential operator $D : \Gamma(E) \to \Gamma(F)$ is *G-transversally elliptic* if $\sigma(D) : \pi^*E \to \pi^*F$ is an isomorphism of vector bundles $\pi^*E$ and $\pi^*F$ on $T^*_G X$ outside of zero section.

The index of a differential operator $D$ depends only on the topological class of its symbol in the topological K-theory of vector bundles on $T^*X$. The Atiyah–Singer formula for the index of the complex (11.2) is

$$\boxed{\mathrm{ind}_G(D,X) = \frac{1}{(2\pi)^{\dim_{\mathbb{R}} X}} \int_{T^*X} \hat{A}_G(\pi^* T_X)\mathrm{ch}_G(\pi^* E^\bullet)} \tag{11.15}$$

Here $T^*X$ denotes the total space of the cotangent bundle of $X$ with canonical orientation such that $\mathrm{d}x^1 \wedge \mathrm{d}p_1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}p_2 \ldots$ is a positive element of $\Lambda^{\mathrm{top}}(T^*X)$.

Let $n = \dim_{\mathbb{R}} X$ and let $\pi^* T_X$ be the rank $n$ vector bundle over $T^*X$ defined by the pullback by $\pi : T^*X \to X$ of the tangent bundle $T_X$ over $X$. The $\hat{A}_G$-character of $\pi^* T_X$ is

$$\hat{A}_G(\pi^* T_X) = \det{}_{\pi^* T_X} \left( \frac{R_G}{\mathrm{e}^{R_G/2} - \mathrm{e}^{-R_G/2}} \right) \tag{11.16}$$

where $R_G$ denotes the $G$-equivariant curvature of the bundle $\pi^* T_X$. Notice that the argument of $\hat{A}$ is $n \times n$ matrix where $n = \dim_{\mathbb{R}} T_X$ (real dimension of $X$) while if general index formula is specialized to Dirac operator on Kahler manifold $X$ as in (7.4) the argument of the $\hat{A}$-character is an $n \times n$ matrix where $n = \dim_{\mathbb{C}} T_X^{1,0}$ (complex dimension of $X$).

Even though the integration domain $T^*X$ is non-compact the integral (11.16) is well-defined if $X$ is compact because of the ($G$-transversal) ellipticity of the complex $\pi^* E^\bullet$.

For illustration take the complex to be $E_0 \xrightarrow{D} E_1$. Since $\sigma(D) : \pi^* E_0 \to \pi^* E_1$ is an isomorphism outside of the zero section we can pick a smooth connection on $\pi^* E_0$ and $\pi^* E_1$ such that its curvature on $E_0$ is equal to the curvature on $E_1$ away from a compact tubular neighborhood $U_\epsilon X$ of $X \subset T^*X$. Then $\mathrm{ch}_G(\pi^* E^\bullet)$ is explicitly vanishing away from $U_\epsilon X$ and the integration over $T^*X$ reduces to integration over the compact domain $U_\epsilon X$.

It is clear that under localization to the fixed point set $X^G$ of the $G$-action on $X$ the general formula (11.16) reduces to the fixed point formula (11.5) for discrete $X^G$. If $X^G$ is discrete set then the set of $G$-fixed points for $G$-action on $T^*X$ coincides with $X^G$. The numerator in the $\hat{A}$-character $\det_{\pi^* T_X} R_G = \mathrm{Pf}_{T_{T^*X}}(R_G)$ is the Euler class of the tangent bundle $T_{T^*X}$ to $T^*X$ which cancels with the denominator in (8.13), while the restriction of the denominator of (11.16) to fixed points is equal to (11.5), because $\det \mathrm{e}^{R_G} = 1$, since $R_G$ is a curvature of the tangent bundle $T_X$ with orthogonal structure group. Finally, the restriction to $X^G$ of $\mathrm{ch}_G \pi^* E^\bullet$ gives the numerator in (11.5).

For more details see for example [28].

## 12. Equivariant cohomological field theories

Certain field theories have first been interpreted as cohomological and topological field theories by Witten, see [29, 30].

Often the path integral for supersymmetric field theories can be represented in the form

$$Z = \int_X \alpha \tag{12.1}$$

where $X$ is the superspace (usually of infinite dimension) of all fields of the theory. Moreover, the integrand measure $\alpha$ is closed with respect to an odd operator $\delta$ which is typically constructed as a sum of a supersymmetry algebra generator and a BRST charge

$$\delta\alpha = 0 \tag{12.2}$$

The integrand is typically a product of an exponentiated action functional $S$, perhaps with insertion of a non-exponentiated observable $\mathcal{O}$

$$\alpha = \mathrm{e}^{-S}\mathcal{O} \tag{12.3}$$

so that both $S$ and $\mathcal{O}$ are $\delta$-closed

$$\delta S = 0, \qquad \delta\mathcal{O} = 0. \tag{12.4}$$

If $X$ is a supermanifold, such as a total space $\Pi E$ of a vector bundle $E$ (over a base $Y$) with parity inversed fibers, the equivariant Euler characteristic class (Pfaffian) in the Atiyah–Bott–Berline–Vergne formula (8.13) is replaced by the graded (super) version of the Pfaffian. The weights associated to fermionic components contribute inversely compared to the weights associated to bosonic components.

Typically, in quantum field theories the base $Y$ of the bundle $E \to Y$ is the space of fields. Certain differential equations (like BPS equations) are represented by a section $s : Y \to E$. The zero set $s^{-1}(0) \subset Y$ of the section are the field configurations which solve the equations. For example, in topological self-dual Yang–Mills theory (Donaldson–Witten theory) the space $Y$ is the infinite-dimensional affine space of all connections on a principal $G$-bundle on a smooth four-manifold $M_4$. In a given framing, connections are represented by adjoint-valued 1-forms on $M_4$, so $Y \simeq \Omega^1(M_4) \otimes \mathrm{ad}\mathfrak{g}$. A fiber of the vector bundle $E$ at a given connection $A$ on the $G$-bundle on $M_4$ is the space of adjoint-valued two-forms $\Omega^{2+}(M_4) \otimes \mathrm{ad}\mathfrak{g}$. The section $s : \Omega^1(M_4) \otimes \mathrm{ad}\mathfrak{g} \to \Omega^{2+}(M_4) \otimes \mathrm{ad}\mathfrak{g}$ is represented by the self-dual part of the curvature form

$$A \mapsto F_A^+ \tag{12.5}$$

The zeroes $s^{-1}(0)$ of the section $s$ are connections $A$ that are solutions of the equation $F_A^+ = 0$. The integrand $\alpha$ is the Mathai–Quillen representative of the Thom class for the bundle $E \to Y$ like in (6.10) and (6.17). The integral over the space of all fields $X = \Pi E$ localizes to the integral over the zeroes $s^{-1}(0)$ of the section , which in the Donaldson–Witten example is the moduli space of self-dual connections, called *instanton moduli space*.

The functional integral version of the localization formula of Atiyah–Bott–Berline–Vergne has the same formal form as the finite-dimensional (8.13)

$$\boxed{\int_X \alpha = \int_F \frac{f^*\alpha}{\mathrm{e}(\nu_F)}} \tag{12.6}$$

except that in the quantum field theory version the space $X$ is an infinite-dimensional superspace of fields. Here $F$ denotes the localization locus in the space of fields. Let $\Phi_F \subset H^\bullet(X)$ be the Poincaré dual class to $F$, or Thom class of the inclusion $f : F \hookrightarrow X$, which is image $f_* 1$ under the pushforward morphism

$$f_* : H^\bullet(F) \to H^\bullet(X) \tag{12.7}$$

Let $\nu_F$ be the normal bundle to $F$ in $X$. In quantum field theory language the space $F$ is called the moduli space or localization locus, and $\nu_F$ is the space of linearized fluctuations of fields transversal to the localization locus. The pullback $f^*\Phi_F$ in $H^\bullet(F)$ is equal to the Euler class of the normal bundle $\nu_F$

$$f^*\Phi_F = e(\nu_F) \tag{12.8}$$

Let $\mu_{e(\nu_F)} : H^\bullet(F) \to H^\bullet(F)$ denote multiplication by $e(\nu_F)$, that is

$$\mu_{e(\nu_F)} : \alpha \mapsto e(\nu_F) \wedge \alpha, \qquad \alpha \in H^\bullet(F) \tag{12.9}$$

then definition of the Thom class and Euler class imply that

$$f^* f_* = \mu_{e(\nu_F)} \tag{12.10}$$

The localization (12.6) from $X$ to $F$ exists if the pushforward morphism $f_*$ (12.7) and the pullback morphism

$$f^* : H^\bullet(X) \to H^\bullet(F) \tag{12.11}$$

have inverses. In this case, the equation (12.10) implies equality

$$\mathrm{id}_{H^\bullet(X)} = f_* \mu^{-1}_{e(\nu_F)} f^* \tag{12.12}$$

of maps $H^\bullet(X) \to H^\bullet(X)$. After integration one get the general localization formula (12.6).

In finite-dimensional situation existence of the inverse maps to $f_*$ and $f^*$ is equivalent to the statement that the morphism $\mu_{e(\nu_F)}$ is invertible and that the ranks of $H^\bullet(X)$ and $H^\bullet(F)$ are equal.

Two examples of such localization have been considered above:

- (Section 6.2) if $X = \Pi E$ is the total space of a vector bundle $E \to Y$ with parity inversed fibers, then $F \subset Y \subset X$ can be taken to be the set of zeroes $F = s^{-1}(0)$ of a generic section $s : Y \to E$. In this case the (super) rank of $\nu_F$ is 0 since $\nu_F$ splits into $\mathrm{rank}\, E$ bosonic subbundle in the horizontal direction and $\mathrm{rank}\, E$ fermionic subbundle in the vertical direction. Therefore the (super) Euler class $e(\nu_F)$ at each point is concentrated in degree 0. It gives numerical $\pm 1$ factor entering into the localization formula (6.19).
- (Section 8.1) if $X$ is a $T$-manifold for a compact Lie group $T$, then $F$ can be taken to be $F = X^T$, the set of $T$-fixed points on $X$. As Atiyah–Bott [20] have shown, the localization (12.12) holds if Euler class $e_T(\nu_F)$ is replaced by the equivariant Euler class $e_T(\nu_F)$. Even though $e(\nu_F)$ is concentrated in the generically non-zero degree equal to the $\mathrm{rank}(\nu_F) = \mathrm{codim}(F, X)$, it is invertible if treated over the field of fractions of the polynomial coefficient ring $H^\bullet_T(pt) = \mathbb{R}[\mathfrak{t}]^T$ as dicussed in section 8.1: the equivariant Euler class $e_T(\nu_F)$ is contributed by 0-form in $\Omega^\bullet(F)$ multiplied by an element of $\mathbb{R}[\mathfrak{t}]^T$ equal to the non-zero product of weights of $T$-action on $\nu_F$.

The formula (12.6) is more general than these examples. In practice, in quantum field theory problems, the localization locus $F$ is found by deforming the form $\alpha$ to

$$\alpha_\tau = \alpha \exp(-\tau \delta V) \tag{12.13}$$

Here $\tau \in \mathbb{R}$ is a deformation parameter, and $V$ is a fermionic functional on the space of fields, such that $\delta V$ has a trivial cohomology class (the cohomology class $\delta V$ is automatically trivial on effectively compact spaces, but on a non-compact space of fields, which usually appears in quantum field theory path integrals, one has to take extra care of the contributions from the boundary at infinity to ensure that $\delta V$ has trivial cohomology class).

If the even part of the functional $\delta V$ is positive definite, then by sending the parameter $\tau \to \infty$ we can see that the integral

$$\int_X \alpha \exp(-\tau \delta V) \tag{12.14}$$

localizes to the locus $F \subset X$ where $\delta V$ vanishes. Such a locus $F$ has an invertible Euler class of its normal bundle in $X$ and the localization formula (12.6) holds.

In some quantum field theory problems, a compact Lie group $G$ acts on $X$ and $\delta$ is isomorphic to an equivariant de Rham differential in the Cartan model of $G$-equivariant cohomology of $X$, so that an element $\mathbf{a}$ of the Lie algebra of $G$ appears as a parameter of the partition function $Z$.

Then the partition function $Z(\mathbf{a})$ can be interpreted as an element of $H^{\bullet}_{\mathbf{G}}(pt)$, and the Atiyah–Bott–Berline–Vergne localization formula can be applied to compute $Z(\mathbf{a})$.

There are are two types of equivariant partition functions.

In the partition functions of the first type $Z(\mathbf{a})$, the variable $\mathbf{a}$ is a *parameter* of the quantum field theory such as a coupling constant, a background field, a choice of vacuum, an asymptotics of fields or a boundary condition. Such a partition function is typical for a quantum field theory on a non-compact space, such as the Nekrasov partition function of equivariant gauge theory on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ [31].

In the partition function of the second type, the variable $\mathbf{a}$ is actually a *dynamical field* of the quantum field theory, so that the complete partition function is defined by integration of the partial partition function $\tilde{Z}(\mathbf{a}) \in H^{\bullet}_{G}(pt)$

$$Z = \int_{\mathbf{a}\in\mathbf{g}} \mu(\mathbf{a})\tilde{Z}(\mathbf{a}) \tag{12.15}$$

where $\mu(\mathbf{a})$ is a certain adjoint invariant volume form on the Lie algebra $\mathfrak{g}$. The partition function $Z$ of second type is typical for quantum field theories on compact space-times reviewed in [1], such as the partition function of a supersymmetric gauge theory on $S^4$ [13] reviewed in contribution [32], or on spheres of other dimensions, see summary of results in contribution [27].

## Acknowledgments

The author is grateful to Bruno Le Floch and to a referee of *Journal of Physics A* for a very detailed proofreading of the manuscript and myriad of corrections and comments.

The research of VP on this project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (QUASIFT grant agreement 677368).

## References

[1] Pestun V and Zabzine M 2017 Localization techniques in quantum field theories *J. Phys. A: Math. Theor.* **50** 440301
[2] Cartan H 1951 Notions d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie *Colloque de Topologie (Espaces Fibrés) (Bruxelles, 1950)* pp 15–27
[3] Cartan H 1951 La transgression dans un groupe de Lie et dans un espace fibré principal *Colloque de Topologie (Espaces Fibrés) (Bruxelles, 1950)* pp 57–71
[4] Guillemin V W and Sternberg S 1999 *Supersymmetry and Equivariant de Rham Theory* (*Mathematics Past and Present*) (Berlin: Springer) (With an appendix containing two reprints by Henri Cartan [MR0042426 (13,107e); MR0042427 (13,107f)])
[5] Berline N, Getzler E and Vergne M 2004 *Heat Kernels and Dirac Operators* (*Grundlehren Text Editions*) (Berlin: Springer) (Corrected reprint of the 1992 original)
[6] Szabo R J 1996 Equivariant localization of path integrals (arXiv:hep-th/9608068 [hep-th])
[7] Cordes S, Moore G W and Ramgoolam S 1997 Large N 2D Yang–Mills theory and topological string theory *Commun. Math. Phys.* **185** 543–619
[8] Vergne M 2006 Applications of equivariant cohomology (arXiv:math/0607389 [math.DG])
[9] Milnor J 1956 Construction of universal bundles. I *Ann. Math.* **63** 272–84
[10] Milnor J 1956 Construction of universal bundles. II *Ann. Math.* **63** 430–6

[11] Kanno H 1989 Weyl algebra structure and geometrical meaning of BRST transformation in topological quantum field theory *Z. Phys.* C **43** 477
[12] Labastida J M F and Pernici M 1988 A gauge invariant action in topological quantum field theory *Phys. Lett.* B **212** 56
[13] Pestun V 2012 Localization of gauge theory on a four-sphere and supersymmetric Wilson loops *Commun. Math. Phys.* **313** 71–129
[14] Bott R and Tu L W 2001 *Equivariant Characteristic Classes in the Cartan Model* (River Edge, NJ: World Scientific)
[15] Berline N and Vergne M 1992 The equivariant Chern character and index of *G*-invariant operators *Lectures at CIME (Venise, 1992*)
[16] Bott R and Tu L W 1982 *Differential Forms in Algebraic Topology* (*Graduate Texts in Mathematics* vol 82) (New York: Springer)
[17] Mathai V and Quillen D 1986 Superconnections, Thom classes, and equivariant differential forms *Topology* **25** 85–110
[18] Atiyah M F and Jeffrey L 1990 Topological Lagrangians and cohomology *J. Geom. Phys.* **7** 119–36
[19] Wu S 2006 Mathai–Quillen formalism *Encyclopedia Math. Phys.* **2006** 390–9
[20] Atiyah M F and Bott R 1984 The moment map and equivariant cohomology *Topology* **23** 1–28
[21] Berline N and Vergne M 1982 Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante *C. R. Acad. Sci. Paris* I **295** 539–41
[22] Kirillov A A 2004 *Lectures on the Orbit Method* (*Graduate Studies in Mathematics* vol 64) (Providence, RI: American Mathematical Society)
[23] Kostant B 1970 Quantization and unitary representations (*Lectures in Modern Analysis and Applications III. Lect. Not. Math.* vol 170) ed C T Taam (Berlin, Heidelberg: Springer) pp 87-208
[24] Kirillov A A 1999 Merits and demerits of the orbit method *Bull. Am. Math. Soc.* **36** 433–88
[25] Alekseev A and Shatashvili S L 1989 Path integral quantization of the coadjoint orbits of the virasoro group and 2D gravity *Nucl. Phys.* B **323** 719
[26] Atiyah M F 1974 *Elliptic Operators and Compact Groups* (*Lecture Notes in Mathematics* vol 401) (Berlin: Springer)
[27] Pestun V and Zabzine M 2017 Introduction to localization in quantum field theory *J. Phys. A: Math. Theor.* **50** 443001
[28] Berline N and Vergne M 1996 The Chern character of a transversally elliptic symbol and the equivariant index *Invent. Math.* **124** 11–49
[29] Witten E 1991 Introduction to cohomological field theories *Int. J. Mod. Phys.* A **6** 2775–92
[30] Witten E 1988 Topological quantum field theory *Commun. Math. Phys.* **117** 353
[31] Nekrasov N A 2004 Seiberg–Witten prepotential from instanton counting *Adv. Theor. Math. Phys.* **7** 831–64 (To Arkady Vainshtein on his 60th anniversary)
[32] Hosomichi K 2017 $\mathcal{N}=2$ SUSY gauge theories on $S^4$ *J. Phys. A: Math. Theor.* **50** 443010