Presented at 39th Rencontres De Moriond Workshop On Exploring The Universe:
Contents And Structures Of The Universe, 3/28/2004-4/4/2004, La Thuile, Italy

SLAC-PUB-10479
June 2, 2004

Quantizing Cosmology: A Simple Approach[a]

Marvin Weinstein
Stanford Linear Accelerator Center
Stanford University
Stanford, California 94309

I discuss the problem of inflation in the context of Friedmann-Robertson-Walker Cosmology and show how, after a simple change of variables, to quantize the problem in a way which parallels the classical discussion. The result is that two of the Einstein equations arise as exact equations of motion and one of the usual Einstein equations (suitably quantized) survives as a constraint equation to be imposed on the space of physical states. However, the Friedmann equation, which is also a constraint equation and which is the basis of the Wheeler-deWitt equation, acquires a welcome quantum correction that becomes significant for small scale factors. To clarify how things work in this formalism I briefly outline the way in which our formalism works for the exactly solvable case of de-Sitter space.

1 Introduction

The remarkable agreement of the WMAP[?,?,?] measurements of the anisotropy in the cosmic microwave background(CMB) radiation with the predictions of slow-roll inflation[?], strongly suggests that the paradigm for computing the fluctuations[?] in $\delta\rho/\rho$ is correct. These fluctuations are remarkable in that they represent an imprinting of the structure of the quantum state of the field theory, at the time inflation begins, onto the electromagnetic radiation that comes to us from the surface of last scattering. Unfortunately, derivations of this effect usually mix classical and quantum ideas and so, it is difficult to determine how they would change given a fully quantum mechanical treatment.

In this talk I will show how one can fill this gap by fully quantizing this problem in *cosmic time*. To be precise, I will show how to work in fixed, co-moving coordinates, and canonically quantize the theory of the Friedmann-Robertson-Walker(FRW) metric,

$$ds^2 = -dt^2 + a(t)^2 d\vec{x} \cdot d\vec{x}, \tag{1}$$

[a]This work was supported by the U. S. DOE, Contract No. DE-AC03-76SF00515.

and the spatially constant part of the inflaton field, $\Phi(t)$, in a straightforward manner. I will then show that the quantized system has states for which the expectation values of the scale factor and inflaton field satisfy the equations associated with the inflationary scenario.

I wish to emphasize that this approach assumes that getting quantum mechanics to describe the evolution of the system in *cosmic time* is paramount. Because of this, I will argue that imposing a strong form of the Wheeler-deWitt equation is not useful. Instead, in this formalism, geometry, defined by the condition that the Einstein equations be true, is an emergent phenomenon. It exists only for some quantum states and then, only when the scale factor becomes large. To clarify the subtle way in which this works I will exhibit the exact solution to the problem for the case of de Sitter space. As I will show, a bonus of this approach is that the quantum corrections to the Einstein equations, which become important when the scale factor is small, completely eliminate the problem of the big crunch.

Obviously, given the constraints of time, I cannot discuss all of the results presented in our earlier preprints[?]. What I will do is try to very briefly outline the main ideas and discuss what happens in the exactly solvable case of de Sitter space.

2 The Classical Problem

Before discussing this approach to the quantum treatment of FRW cosmology it is important to demonstrate that the classical version of this formalism does no violence to the usual Einstein theory. I will outline how this works.

To simplify the usual derivations of the Einstein equations for FRW cosmology I will assume, as is the case experimentally, that we are dealing with a spatially flat universe and choose co-moving coordinates, in which the metric takes the general form shown in Eq.**??**.

If we combine this form of the metric with the Lagrangian and ignore everything but the spatially constant part of the inflaton field, one obtains

$$\mathcal{S} = \mathbf{V} \int dt \sqrt{-g} \left[\frac{R(t)}{2\kappa^2} + \frac{1}{2} \frac{d\Phi(t)}{dt}^2 - V(\Phi(t)) \right]. \tag{2}$$

Note that $\mathbf{V}$ is the volume of the region in which the theory is being defined. Clearly $\mathbf{V}$ must be taken to be larger than the horizon volume at the time of inflation in order to avoid edge effects in $\delta\rho/\rho$.

If we write $\sqrt{-g}R(t)$ in terms of $a(t)$ and integrate by parts, to eliminate the term with $d^2a(t)/dt^2$, we obtain a form of the action which can be further simplified by making the change of variables $u(t)^2 = a(t)^3$. This results in the final, simple form for the action

$$\mathcal{S} = \mathbf{V} \left[-\frac{4}{3\kappa^2} \left(\frac{du(t)}{dt} \right)^2 + \frac{1}{2} u(t)^2 \left(\frac{d\Phi(t)}{dt} \right)^2 - u(t)^2 V(\Phi(t)) \right]. \tag{3}$$

Although this change of variables greatly simplifies the classical discussion, it has a greater significance for the quantized theory. This is because we can choose $-\infty \leq u \leq \infty$, whereas the only physically allowable range for a is $0 \leq a \leq \infty$. It is only for the space of square-integrable functions on the interval $-\infty \leq u \leq \infty$ that the Heisenberg equations of motion can be obtained by canonical manipulations.

It is obvious that there can only be two Euler-Lagrange equations for this system, one for $u(t)$ and one for $\Phi(t)$. However, there a four relevant Einstein equations. Thus, by quantizing in this fixed gauge, we fail to obtain the full set of Einstein equations. The missing equations are the Friedmann equation and its time derivative

$$\mathcal{H}(t)^2 = \frac{\kappa^2}{3} \left(\frac{1}{2} \left(\frac{d\Phi(t)}{dt} \right)^2 + V(\Phi(t)) \right) \quad \text{and} \quad \frac{d\mathcal{H}(t)}{dt} = -\frac{\kappa^2}{2} \left(\frac{d\Phi(t)}{dt} \right)^2. \tag{4}$$

where the Hubble parameter $\mathcal{H}$ is defined to be

$$\frac{1}{u(t)}\frac{da(t)}{dt} = \frac{2}{3u(t)}\frac{du(t)}{dt} \tag{5}$$

Since the Friedmann equation and its time derivative are not equations of motion they can, at best, be imposed as constraints on the space of physical solutions, in analogy to Coulomb's law in electrodynamics. Thus, we have to show that as a consequence of the equations we do have, if we impose the Friedmann equations at any one time, then they will continue to be true for all later times. This can be proven by first differentiating $\mathcal{H}$ with respect to t to obtain and identity which can be substituted into the equation of motion for $u(t)$ to obtain the equation

$$\frac{2u(t)}{\kappa^2}\left[\left(2\frac{d\mathcal{H}(t)}{dt} + \kappa^2\left(\frac{d\Phi(t)}{dt}\right)^2\right) + 3\left(\mathcal{H}(t)^2 - \frac{\kappa^2}{3}\left(\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 + V(\Phi(t))\right)\right)\right] = 0. \tag{6}$$

Identifying the first constraint equation as

$$\mathbf{G} = \mathcal{H}(t)^2 - \frac{\kappa^2}{3}\left(\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 + V(\Phi(t))\right), \tag{7}$$

we can use the Euler-Lagrange equation for $\Phi(t)$, to rewrite Eq.**??** as

$$\frac{2u(t)}{\kappa^2}\left(\frac{1}{\mathcal{H}(t)}\left(\frac{d\mathbf{G}}{dt}\right) + 3\mathbf{G}\right) = 0. \tag{8}$$

This first order differential equation for $\mathbf{G}(t)$ shows that if, at time $t = t_0$, $\mathbf{G} = 0$, then $d\mathbf{G}/dt$ will also vanish and so $\mathbf{G}(t) = 0$ exactly. In other words, we arrive at the desired result.

To show why it is possible, at the classical level, to confuse the Friedmann equation with the Hamiltonian, we follow the canonical procedure. The canonical momenta and Hamiltonian are written as

$$p_u = -\mathbf{V}\frac{8}{3\kappa^2}\frac{du(t)}{dt} \quad ; \quad p_\Phi = \mathbf{V}u^2\frac{d\Phi(t)}{dt}. \tag{9}$$

and

$$\mathbf{H} = p_u\frac{du(t)}{dt} + p_\Phi\frac{d\Phi(t)}{dt} - \mathcal{L} = -\frac{3\kappa^2}{16\mathbf{V}}p_u^2 + \frac{1}{2\mathbf{V}u^2}p_\phi^2 + \mathbf{V}u^2V(\Phi). \tag{10}$$

Next, rewriting the Hamiltonian in terms of the operators $\frac{du(t)}{dt}$ and $\frac{d\Phi(t)}{dt}$ and substituting the definition of $\mathcal{H}$ into the resulting equation we see that

$$\mathbf{H} == -\mathbf{V}u^2\left[\frac{3\mathcal{H}^2}{\kappa^2} - \left(\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 + V(\Phi)\right)\right] = -\mathbf{V}\frac{3u^2}{\kappa^2}\mathbf{G}. \tag{11}$$

In other words, the Hamiltonian, $\mathbf{H}$ is proportional to the constraint, $\mathbf{G}$. It follows that setting $\mathbf{G} = 0$ means $\mathbf{H} = 0$, which tells us that the Hamiltonian vanishes for physical solutions. The identification of the Hamiltonian with the constraint equation is the content of the Wheeler-DeWitt equation.

3 Canonical Quantization of the Theory

Now that we have seen that our formalism, including the change of variables from $a(t)$ to $u(t)$, does no violence to the classical theory, we will proceed to a discussion of the quantum mechanics.

The basic procedure is to use the same Hamiltonian and to define the operators u,Φ and their conjugate momenta to have the commutation relations

$$[p_u, u] = -i \quad ; \quad [p_\Phi, \Phi] = -i, \tag{12}$$

where all other commutators vanish. With this definition it is simple to derive the Heisenberg equations of motion, for any operator $\mathbf{O(t)}$, by commuting that operator with the Hamiltonian. (Note, for any operator, the Heisenberg operator is defined to be $O(t) = e^{i\mathbf{H}t}\mathbf{O}e^{-i\mathbf{H}t}$.)

Explicit computation shows that the operators u and Φ satisfy equations of motion which can be made to look identical to the classical equations of motion if we define the quantum version of the *Hubble* operator $\mathcal{H}$ to be

$$\mathcal{H} = -\frac{\kappa^2}{8\mathbf{V}}\left(p_u\frac{1}{u} + \frac{1}{u^3}p_u u^2\right). \tag{13}$$

Given this definition of the Hubble operator, we mimic the classical derivation; i.e., we compute the time derivative of the Hubble operator and use the identity

$$\frac{d^2u(t)}{dt^2} = \frac{3u}{2}\left(\frac{d\mathcal{H}}{dt} + \frac{3}{2}\mathcal{H}^2 - \frac{9\kappa^4}{128\mathbf{V}^2u^4}\right), \tag{14}$$

to rewrite the equation of motion for $u(t)$ as

$$\frac{3u}{4}\left(\frac{1}{\mathbf{A}}\frac{d\mathbf{G}}{dt} + 3\mathbf{G}\right) = 0. \tag{15}$$

where now $G(t)$ is defined to be

$$\mathbf{G} = \mathcal{H}^2 - \frac{\kappa^2}{3}\left(\frac{1}{2}\left(\frac{d\Phi(t)}{dt}\right)^2 + V(\Phi)\right) + \mathbf{Q}, \tag{16}$$

where the operator $\mathbf{Q}$ represents a quantum correction to the equation of motion

$$\mathbf{Q} = -\frac{3\kappa^4}{64\mathbf{V}^2u^4}. \tag{17}$$

(Note, the operator $\mathbf{A}$ can be solved for explicitly and is a non-vanishing expression and is just $\mathcal{H}$ plus explicit quantum corrections.) Since Eq.**??** is an exact operator equation of motion, we see that if we could define the space of states by the condition $\mathbf{G(t_0)}|\psi\rangle = 0$, then this condition would hold for all time. But, as in the classical theory, this definition of $\mathbf{G}$, implies that

$$\mathbf{G}(t) = -\frac{\kappa^2}{3\mathbf{V}u(t)^2}\mathbf{H}. \tag{18}$$

Thus, while we can define the space of physical states, to be which are annihilated by the Hamiltonian, obviously this immediately leads to a contradiction between the Schroedinger and Heisenberg picture. This is because $H|\psi\rangle = 0$ implies that the state does not evolve in the Schroedinger picture, whereas we have already shown that the operators $u(t)$ and $\Phi(t)$ do evolve in time.

Although it is possible to show that one can define variants of $G(t)$ which allow us to impose a state condition which does not manifestly ruin the connection between the Schroedinger and Heisenberg picture, unfortunately one can explicitly show that in exactly solvable theories the solutions to the equation $\mathbf{G}_\alpha|\psi\rangle = 0$ are not normalizable. Thus, attempting to impose such a strong condition for any α leads to problems interpreting the quantum mechanical theory. For this reason we propose a weak form of the condition, namely: a state is physical if

$$\lim_{t\to\pm\infty}\mathbf{G}(t)|\Psi\rangle = 0. \tag{19}$$

It should be clear from the fact that Q vanishes for large t that Eq.**??** guarantees that for these states geometry, in the sense that the familiar Einstein equations become arbitrarily accurate, emerges dynamically at large time.

In the next section, where we discuss the exact solution of de Sitter space, we show that this asymptotic condition is easily satisfied for a wide class of states. Furthermore, the exact solution demonstrates why imposing a stronger condition on physical states is neither necessary nor desirable.

4 de Sitter Space: An Exactly Solvable Problem

Since our assertion that it is unnecessary to adopt a strong version of the gauge condition flies in the face of conventional wisdom, it is important to show how things work in an exactly solvable example. For this reason I now turn to a discussion of de Sitter space.

Begin by considering the general action of the FRW problem, but with $V(\Phi)$ replaced by a cosmological constant Λ, so that the Hamiltonian takes the form

$$\mathbf{H} = -\frac{3\kappa^2}{16\mathbf{V}} p_u^2 + \frac{1}{2\mathbf{V}u^2} p_\Phi^2 + \mathbf{V}u^2\Lambda \tag{20}$$

Since the conjugate variable to p_Φ doesn't appear in the Hamiltonian, we are free to work in sectors of the Hilbert space in which p_Φ takes a definite value. The particular sector defined by the condition $p_\Phi|\psi\rangle = 0$ the Hamiltonian takes the simpler form

$$\mathbf{H} = -\frac{3\kappa^2}{16\mathbf{V}} p_u^2 + \mathbf{V}u^2\Lambda, \tag{21}$$

which we immediately recognize as a theory with a cosmological constant, whose solution at the classical level is just de Sitter space.

The Heisenberg equations of motion for $u(t)$ and $p_u(t)$ are:

$$\frac{du(t)}{dt} = -\frac{3\kappa^2}{8\mathbf{V}} p_u\,; \quad \frac{d^2u(t)}{dt^2} = \frac{3\kappa^2\Lambda}{4} u. \tag{22}$$

The exact solution to these equations, written in terms of the operators $u(t=0) = u$ and $p_u(t=0) = p_u$ are

$$\begin{aligned} u(t) &= \cosh(\omega t)u - \frac{3\kappa^2}{8\mathbf{V}\omega}\sinh(\omega t)p_u \\ p_u(t) &= \cosh(\omega t)p_u - \frac{8\mathbf{V}\omega}{3\kappa^2}\sinh(\omega t)u, \end{aligned} \tag{23}$$

where we have defined

$$\omega = \sqrt{\frac{3\kappa^2\Lambda}{4}}. \tag{24}$$

It is convenient to rewrite Eq.**??** in terms of exponentials; i.e.,

$$u(t) = \frac{e^{\omega t}}{2}\left(u - \frac{3\kappa^2}{8\mathbf{V}\omega}p_u\right) + \frac{3\kappa^2 e^{-\omega t}}{16\mathbf{V}\omega}\left(p_u + \frac{8\mathbf{V}\omega}{3\kappa^2}u\right) \tag{25}$$

and to introduce the canonically conjugate asymptotic operators

$$u_\infty = \frac{1}{\sqrt{2}}\left(u - \frac{3\kappa^2}{8\mathbf{V}\omega}p_u\right)\,;\; p_\infty = \frac{1}{\sqrt{2}}\left(p_u + \frac{8\mathbf{V}\omega}{3\kappa^2}u\right). \tag{26}$$

In terms of these operators the solution for the operator $u(t)$ and the Hamiltonian take the simple forms

$$u(t) = \frac{1}{\sqrt{2}}\,e^{\omega t}u_\infty + \frac{1}{\sqrt{2}}\frac{3\kappa^2}{8\mathbf{V}\omega}\,e^{-\omega t}p_\infty, \tag{27}$$

and

$$\mathbf{H} = \frac{\sqrt{3\Lambda}\kappa}{4} \left(u_\infty p_\infty + p_\infty u_\infty \right). \tag{28}$$

From this point on all of the technical work is finished, the only chore which remains is to extract the physical significance of these results.

Before discussing the physical states of the quantum theory, it is worth spending a few moments considering what the preceding results mean in the context of the classical theory. Obviously, Eqs.**??** and **??** are equally true for both the classical and quantum versions of the theory; the only difference between these cases being is that in the classical theory u_∞ and p_∞ are simply numbers, whereas in the quantum theory they are non-commuting operators. Thus, for the classical theory, imposing the condition that the energy vanishes is the same as requiring either u_∞ or p_∞ to vanish. This is, of course, just the usual result: i.e., for the case of a cosmological constant, the full, non-linear, set of Einstein equations, admit only an expanding, or contracting, solution for $a(t)$ or $u(t)$. This is why running the expanding solution back in time (or the contracting solution forward in time) always leads to a *big crunch*.

The situation is clearly different for the quantum theory since it is not possible to simply set an operator to zero. If one chooses the gauge-condition which corresponds to $\alpha = 1$, i.e. the Wheeler-deWitt equation, then one is looking for states annihilated by the Hamiltonian. Given that we can write $p_\infty = -i \frac{d}{du_\infty}$, for a function of the form $|\psi\rangle = e^{S(u_\infty)}$,this equation takes the simple form

$$2\, u_\infty \frac{dS(u_\infty)}{du_\infty} = -1, \tag{29}$$

which has the solution

$$S(u_\infty) = -\ln(\sqrt{u_\infty}). \tag{30}$$

This of course means that $|\psi\rangle$ is of the form

$$|\psi\rangle \approx \frac{1}{\sqrt{u_\infty}} \tag{31}$$

which is not normalizable.

Intuitively, given the exact solution for $u(t)$, we see that any state for which $\mathbf{H}|\Psi\rangle$ has a finite norm will, for sufficiently large $|t|$, satisfy Eq.**??** to arbitrary accuracy. This means that (modulo some technical details) essentially any Gaussian wave packet in u_∞ will be a physical state. It also means that for large times all the physics measured in such a state will be compatible with the full set of Einstein equations.

5 Defining Quantum Histories

Now that we have settled upon shifted Gaussian wavepackets as good candidates for physical states, we turn to a discussion of the only two physical observables in this theory; the expansion rate and the volume of the universe. Since we are working in the Heisenberg picture, where the choice of state determines the entire subsequent evolution of the system, we will henceforth refer to the choice of an allowed quantum state as a choice of *quantum history*. What we wish to ascertain is to what degree the value of each of the observables depends upon the specific choice of *quantum history*. The exact solution given in Eq.**??** shows that, at large times, the expansion rate is attached to the scale factor and is totally independent of the state. This is not true of the volume. Thus, I will now focus on the degree to which the measured properties of the volume operator differ from quantum history to quantum history.

Since we started off quantizing in a volume with coordinate size $\mathbf{V}$, the volume of the universe at any time is given by

$$V(t) \quad = \quad \mathbf{V} u(t)^2$$

$$= \frac{\mathbf{V}}{2}\left[e^{2\omega t}u_\infty^2 + \left(\frac{3\kappa^2}{8\mathbf{V}\omega}\right)^2 e^{-2\omega t}p_\infty^2 + \frac{3\kappa^2}{8\mathbf{V}\omega}\left(u_\infty p_\infty + p_\infty u_\infty\right)\right]. \tag{32}$$

A surprising feature of this formula is that for large times in the past and future the volume operator $V(t)$ behaves classically. By this I mean that, if one measures $V(t)$ at some late time, t_1, and obtain a definite value, then we will be able to predict the value we will obtain if we measure $V(t)$ at some later time t_2. To see that this is the case we note that Eq.**??** tells us that, for very large positive times, $V(t)$ is, to arbitrarily high accuracy, proportional to the single operator u_∞^2 (at large negative times it is proportional to p_∞^2). Thus we see that a measurement of $V(t_1)$, for sufficiently large t_1, corresponds to a measurement of u_∞^2, which means that we know $V(t)$ for all times $t_2 > t_1$.

From the fact that u_∞ and p_∞ are canonically conjugate variables we see that if we were to try and identify a quantum history with an eigenstate of p_∞, then the volume operator would be well-determined in the past, but completely undetermined in the future. Conversely, eigenstates of u_∞ correspond to states for which the volume operator is completely well determined in the future, but completely undetermined in the past. Fortunately, the condition that physical states must be normalizable states for which $\langle\psi|\mathbf{H}^2|\psi\rangle < \infty$ is true, tells us that we cannot identify such states with quantum histories. States which can be identified with allowed quantum histories are Gaussian packets of the form,

$$|\Psi\rangle = e^{-\frac{\gamma}{2}u_\infty^2} \tag{33}$$

and the coherent states, $|u_0, p_0, \gamma\rangle$, obtained from them. These coherent states are defined by

$$|u_0, p_0, \gamma\rangle = e^{ip_0 u_\infty}\, e^{-iu_0 p_\infty}|\Psi\rangle, \tag{34}$$

and the expectation values of u_∞ and p_∞ in these states are given by

$$\langle u_0, p_0, \gamma|u_\infty|u_0, p_0, \gamma\rangle = u_0,\ \langle u_0, p_0, \gamma|p_\infty|u_0, p_0, \gamma\rangle = p_0. \tag{35}$$

Moreover, the relevant products of these operators have the values

$$\begin{aligned}
\langle u_0, p_0, \gamma|u_\infty^2|u_0, p_0, \gamma\rangle &= u_0^2 + \frac{1}{2\gamma}, \\
\langle u_0, p_0, \gamma|p_\infty^2|u_0, p_0, \gamma\rangle &= p_0^2 + \frac{\gamma}{2}, \\
\langle u_0, p_0, \gamma|u_\infty p_\infty + p_\infty u_\infty|u_0, p_0, \gamma\rangle &= 2\Re(\langle u_\infty\, p_\infty\rangle) = 2\, u_0 p_0.
\end{aligned}$$

The nice thing about such coherent states is that they are the kind of states we would expect to obtain if, in the past, we make a measurement which determines $V(-t)$ to have a central value $\frac{\mathbf{V}}{2}\, e^{\omega|t|}p_0^2$, with a width parameterized by γ. For this same packet, measurements of $V(t)$ in the distant future will produce results centered about the value $\frac{\mathbf{V}}{2}\, e^{\omega|t|}u_0^2$, with a width parameterized by $1/\gamma$.

6 Equivalence Classes of Histories

From this point on we will restrict the term quantum history to mean a coherent state of the form defined above. To see that many of these histories are equivalent to one another consider the equation

$$\langle V(t)\rangle = \langle u_0, p_0, \gamma|V(t)|u_0, p_0, \gamma\rangle = \frac{\mathbf{V}}{2}\left[e^{2\omega t}\left\langle u_\infty^2\right\rangle + \left(\frac{3\kappa^2}{8\mathbf{V}\omega}\right)^2 e^{-2\omega t}\left\langle p_\infty^2\right\rangle + \frac{3\kappa^2}{8\mathbf{V}\omega}\left(2\Re(\langle u_\infty p_\infty\rangle)\right)\right]. \tag{36}$$

Clearly Eq.**??** shows that at large times the volume behaves as a single exponential, as expected from the solution of the classical Einstein equations. More interesting, however, is the fact that letting $t \to t + t_0$, where t_0 is defined by the condition

$$e^{2\omega t_0} = \frac{3\kappa^2}{8\mathbf{V}\omega}\sqrt{\frac{\langle p_\infty^2\rangle}{\langle u_\infty^2\rangle}}, \tag{37}$$

allows us to rewrite Eq.**??** as

$$\begin{aligned}\langle V(t)\rangle &= \frac{3\kappa^2\sqrt{\langle u_\infty^2\rangle\langle p_\infty^2\rangle}}{8\omega}\left[\cosh(\omega t) + \frac{\Re(\langle u_\infty p_\infty\rangle)}{\sqrt{\langle u_\infty^2\rangle\langle p_\infty^2\rangle}}\right] \\ &= \frac{\kappa^2\sqrt{\langle u_\infty^2\rangle\langle p_\infty^2\rangle}}{4\mathcal{H}}\left[\cosh(\omega t) + \frac{\Re(\langle u_\infty p_\infty\rangle)}{\sqrt{\langle u_\infty^2\rangle\langle p_\infty^2\rangle}}\right]\end{aligned} \tag{38}$$

Thus, we see $\langle V(t)\rangle$ corresponds to a system which is contracting at large times in the past and which then bounces and re-expands in the future. During most of this history the system satisfies the Friedmann equation to high accuracy and expands (or contracts) with a Hubble constant equal to

$$\mathcal{H} = \frac{2}{3}\omega = \sqrt{\frac{\kappa^2\Lambda}{3}}. \tag{39}$$

There is, however, a period in time where the quantum corrections to the Friedmann equation dominate the behavior; namely, at times $t \approx 1/\omega$. Assuming, for the sake of argument, that were to set $1/\kappa\mathcal{H} \approx 10^3$, as it is in many models of slow roll inflation, and assuming $\sqrt{\langle u_\infty^2\rangle\langle p_\infty^2\rangle}$ to be of order unity, then the minimum volume of the universe at the time of the bounce is on the order of 10^3 Planck volumes; i.e., on the order to 10 Planck-lengths in each dimension. This sets the order of magnitude of the scale at which the quantum corrections become important. It is gratifying that these quantum corrections keep the system from contracting forever and ending in a *big crunch*. Finally, Eq.**??** shows that any two quantum histories which give the same values for $\sqrt{\langle u_\infty^2\rangle\langle p_\infty^2\rangle}$ and $\Re(\langle u_\infty p_\infty\rangle)\sqrt{\langle u_\infty^2\rangle\langle p_\infty^2\rangle}$, see the same physics. They only differ by the time at which the bounce occurs.

7 Summary

In this talk I showed how, if one takes the point of view that getting a sensible evolution of a quantum system a a function of cosmic time takes precedence over forcing a purely geometrical interpretation, to fully quantize the theory of inflation and $\delta\rho/\rho$.

My focus throughout was on the formulation of the part of the problem that involved the spatially constant fields. As we demonstrated, in both the classical and quantum theory, working in a fixed gauge yields only two of the four relevant Einstein equations as equations of motion. I argued that in the classical theory the Friedmann equation and its time derivative must be treated as constraints whose constancy in time requires a proof. That proof followed from the two equations of motion we did have. Next, I showed that, in the quantum version of the theory, the same two Einstein equations appear as operator equations of motion, but the constraints appeared in a modified form. I then argued that the simplest of these constraint equations, that which corresponds to the Wheeler-de Witt equation, cannot be used to define the space of physical states, since it leads to a direct conflict between the Schroedinger and Heisenberg pictures. I then suggested that the Wheeler-de Witt equation should be replaced by a weaker asymptotic state condition. In order to clarify how things work in detail, I sketched the application of the general formalism to the case of de Sitter space. The most important result of this discussion is that, in the case of de Sitter space, the system deviates from the expected pure exponential expansion at a finite time in the past.

While, as it stands, the formalism I presented is by no means a candidate for a theory of everything, I feel that the interesting results obtained by proceeding along these lines suggests it is a good candidate for a theory of simething. Namely, a fully quantum theory of the measured fluctuations in the CMB radiation.

Acknowledgments

This work was supported by the U. S. DOE, Contract No. DE-AC03-76SF00515.

References

1. J. C. Mather *et al.*, Astrophys. J. **354**, L37 (1990).
2. H. V. Peiris *et al.*, Astrophys. J. Suppl. **148**, 213 (2003) [arXiv:astro-ph/0302225].
3. D. Polarski and A. A. Starobinsky, Class. Quant. Grav. **13**, 377 (1996) [arXiv:gr-qc/9504030].
4. E. Komatsu *et al.*, Astrophys. J. Suppl. **148**, 119 (2003) [arXiv:astro-ph/0302223].
5. M. R. Nolta *et al.*, arXiv:astro-ph/0305097.
6. For the original model of inflation which did not really work see A. H. Guth, Phys. Rev. **D 23**, 347356(1981). The current paradigm of slow-roll inflation which improved the situation was developed separately by Linde and Albrecht and Steinhardt, A. D. Linde, Phys. Lett. **B108** 389393 (1982) A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett.**48**, 12201223 (1982).
7. For references to the general theory of computing cosmological perturbations see V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rept. **215**, 203-333 (1992), and references cited therein.
8. M. Weinstein and R. Akhoury, arXiv:hep-th/0311189 and arXiv:hep-th/0312249. .
9. For references to the general theory of computing cosmological perturbations see V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rept. **215**, 203-333 (1992), and references cited therein.
10. For the original model of inflation which did not really work see A. H. Guth, Phys. Rev. **D 23**, 347356(1981). The current paradigm of slow-roll inflation which improved the situation was developed separately by Linde and Albrecht and Steinhardt, A. D. Linde, Phys. Lett. **B108** 389393 (1982) A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett.**48**, 12201223 (1982).