Proceedings of the secoud workshop on

"Quantum field Theory under the Influence of External Conditions"

held at University of Leipzig,
September 14 to September 20, 1992

Naturwissenschaftlich-Theoretisches Zentrum
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Preface

The second workshop on

Quantum Field Theory under the Influence of External Conditions

was held at the University of Leipzig, September 14 to September 20, 1992. This workshop was sponsored by the Deutsche Forschungsgemeinschaft (DFG), the Naturwissenschaftlich-Theoretisches Zentrum (NTZ) of the University of Leipzig and supported by the Interdisciplinary Seminar (INTSEM) of the University.

The aim of this workshop was to discuss current research problems as well as to offer young scientists an introduction into this field and to encourage them to join the Graduiertenkolleg on

Quantum Field Theory and its Application
in Elementary Particle and Solid State Physics

to be founded at the University of Leipzig.

Main emphasis has been placed on Quantum Electrodynamics in the presence of conducting surfaces, of external fields or a gravitational background. Investigations of the vacuum state have been presented from various perspectives, and applications of different methods (in particular, the zeta function method) played a major role. Besides QED also other important field theories, mainly Quantum Chromodynamics and the Standard Model in general, in special background fields were discussed.

These proceedings contain the main part of the contributions of the workshop. For some contributions which covered issues under current investigation or which are published elsewhere, an abstract stands only. Therefore, we included the complete program into this collection, as well as the addresses of the speakers, so that the interested reader could easily ask the authors for further information.

I would like to thank all those who contributed to make the workshop as succesful as it was. This concerns all participants, the session leaders and the speakers. It was a good idea of M. Bordag to collect the available talks for proceedings. Especially, I thank A. Uhlmann and B. Geyer for their support and my coworkers M. Bordag and K. Scharnhorst who did a lot of administrative and technical work related to the meeting.

D. Robaschik

October 1992

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NEW ASPECTS OF THE CASIMIR EFFECT:
FLUCTUATIONS AND RADIATIVE REACTION

(based on a review to appear under the same title in "Cavity Quantum Electrodynamics", ed. P. R. Berman Supplement: Advances in Atomic, Molecular, and Optical Physics; Academic Press)

G. Barton
Physics and Astronomy Division
University of Sussex
Brighton BN1 9QH, England

ABSTRACT

The much-discussed Casimir stresses, exerted on conducting ("mirror") surfaces by the zero-point electromagnetic fields in adjacent regions of space, are not constants of the motion, but merely mean values subject to quantum-statistical fluctuations, whose so-called correlation function W is reported on a single flat mirror. The stress is observable only when averaged over finite times T and finite surface regions of diameter a. From W we derive integral representations of the mean-square deviations of these averaged stresses, and their very different asymptotic values when a<<cT and a>>cT respectively.

Finally, as pointed out by Jaekel and Reynaud and by Braginsky and Khalili, from W the fluctuation-dissipation theorem yields the damping force on a moving mirror.

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Photon Pair Creation By Moving Dielectrics

Gabriel Barton and Claudia Eberlein

University of Sussex

School of Mathematical and Physical Sciences

Brighton BN1 9QH

UK

August, 1992

Abstract

For a scalar field in 1+1 dimensions a Hamiltonian formalism is set up to describe a moving dielectric of constant, but arbitrary refractive index. The velocity-dependent Hamiltonian is found and dealt with in first order of perturbation theory with respect to the velocity. The transition amplitude from the vacuum to the two-photon state is calculated in the Heisenberg picture via a Bogoliubov transformation of the photon annihilation and creation operators, and in the Schrödinger picture by expanding the wavefunction into the instantaneous set of eigenfunctions of the parameter-dependent Hamiltonian. The application of the stress-momentum-flow conservation law allows one to write down an effective Hamiltonian for an arbitrary arrangement of moving dielectrics. Transition amplitudes for photon pair creation are given for dielectric half-spaces and slabs. The total radiated energy and the frictional force corresponding to this energy loss are determined. The dissipative terms agree with established results for the photon creation by perfect mirrors.

In a consistency check the fluctuation-dissipation theorem is employed to provide the link between the correlation-function of the fluctuations of the stress on the surface and the motional force as obtained from the radiated energy. Furthermore, the resonance pattern of the photon radiation rate by an uniformly oscillating dielectric slab is looked at. Cavities of variable length are treated in order to provide comparison with previously known results.

Fluctuations of the Casimir Pressure in Lowest Order Quantum Electrodynamics

D. Robaschik
Fachbereich Physik der Universität Leipzig
E.Wieczorek
DESY-IfH Zeuthen

Abstract

Using standard techniques of quantum field ~~theory we determine~~ the fluctions of the Casimir pressure for different models of the plates. For ideally conducting plates the stress tensor fluctuations on both sides of the plates are uncorrelated and add up simply to the resulting fluctuation of the Casimir pressure. In the case of two plates we find a resonance structure of the correlation function. For penetrable plates the fluctuations on both sides of the plates are correlated and which leads to a reduction of the fluctuations.

1 Introduction

In the following we continue the investigation of the vacuum state in Quantum Field Theory. In Quantum Mechanics the ground state is equally well investigated compared with other states. The wave functions yields the necessary basic information. In Quantum Field Theory the situation is quite different. The vacuum state is usually represented by the formal Fock space vector $|0>$, only the Green functions of the field operators contain further information. Already in free field theory simple expectation values of the stress tensor or energy densities lead to divergent quantities. because this appears to be unphysical in most applications these infinitties are subtracted by the normal ordering procedure.

But this is not the right way. At least in part these infinities are direct consequences of the quantization procedure. For example the infinities of the ground state energy of QED could be understood as the added up zero point energies of the harmonic oszillators describing all the field modes. So possible infinities should be handeld carefully and one should look for physically interesting finite parts of it.

A wellknown nontrivial example is the Casimir pressure. Another interesting quantity is the fluctuation of the electromagnetic field strength (considered in Quantum Optics).

In the last time G.Barton [1] has raised the question of the fluctuations of the Casmir pressure. He pointed out that the infinities appearing by a direct calculation of the mean squared deviation makes sense physically. In a second step he started the treatment of these phenomena using correlation functions [1] [2] [3]. This allows a discussion of the expected fluctuations of observables in dependence of the measuring procedure. G. Barton investigated the fluctuation of the Casmir pressure first. He evaluated the matrix elements of products of operators by introducing complete sets of intermediat states.

Here we prefer another method. We write down explicitely the complete expression for the product of the two operators considered and apply then standard Quantum Field Theoretic methods. These methods allow a simultaneous treatment of different interesting cases for which the Green functions are explicitely known.

As physical situation we consider QED with one or two parallel conducting pla- tes [4]. We start with the treatment of ideal conductors characterized by vanishing the tangential component of the electric field strenght E_t and the normal component of the magnetic field strenght B_n on the plates. Comparing the pressure fluctuations of the free field in the free space or the space with one conducting plate (no Casimir pressure) the resulting fluctuations are enhanced by a factor two, for the Casmir pressure results a factor four. In the case of two mirrors we observe a resonance structure of the correlation functions . Such resonances appear if the distances between the considered points correspond to a classical light signal n-times reflected at the plates.

As next we study what happens when we consider more realistic plates. Real plates are simulated in different ways. The most simple possiblities are a freqency cut-offs or exponential damping factors which make the plate transparent for high frequencies. In this manner the boundary conditions for the electromagnetic field are changed. Another possiblitiy is to apply boundary conditions (known from delta-functions) which contain the ideal plates as limiting case [5], [6]. As already expected by G.Barton for non-ideal plates the fluctuations of both sides of the plates are correlated and lead to a reduction of the fluctuations.

2 Field theoretic description of fluctuations

In general the fluctuation of an observable T in the vacuum state is defined by

$$(\Delta T)^2 =< 0|(T - \bar{T})^2|0 >=< 0|T^2|0 > - < 0|\bar{T}|0 >^2 \qquad (2.1)$$

$$\bar{T} =< 0|T|0 >$$

where T is usually

$$T = \int f(x)T(x)\,dx. \qquad (2.2)$$

Here T(x) is the local field theoretic observable and $f(x)$ describes the measuring procedure. Therefore the essential information for the fluctuation is contained in the expectation values

$$\begin{aligned} W(x,x') &= \langle 0|T(x)T(x')|0\rangle - \langle 0|T(x)|0\rangle\langle 0|T(x')|0\rangle \\ &= \langle |T(x)T(x')| \rangle' \end{aligned} \tag{2.3}$$

In our case we have to consider the 33-component of the energy momentum tensor T_{33} in Quantum Electrodynamics. From this quantity the Casimir pressure on a plate located at $x_3 = a$ can be obtained as the difference of T_{33} across the plates

$$p(x) = T_{33}(x_3 = a + \epsilon) - T_{33}(x_3 = a - \epsilon) \tag{2.4}$$

For the energy momentum tensor we use the structure

$$T_{\mu\nu} = F^{\rho}_{\mu} F_{\rho\nu} - 1/4 g_{\mu\nu} F_{\sigma\tau} F^{\sigma\tau} \tag{2.5}$$

with the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

which leads in point splitting technique to

$$T_{33} = \lim_{y\to x} \frac{1}{2} \{ \partial^x_{\underline{\sigma}} A_{\underline{\tau}}(x) \partial^y_{\underline{\sigma}} A_{\underline{\tau}}(y) - \partial^x_{\underline{\sigma}} A_{\underline{\tau}}(x) \partial^y_{\underline{\tau}} A_{\underline{\sigma}}(y) \}. \tag{2.6}$$

Here and in the following we use the notation

$$\begin{aligned} a_{\underline{\sigma}} b_{\underline{\sigma}} &= a_0 b_0 - a_1 b_1 - a_2 b_2 + a_3 b_3 \\ \tilde{a}\tilde{b} &= a_0 b_0 - a_1 b_1 - a_2 b_2 \\ \partial^{\bar{x}y} &= \partial^x_{\underline{\sigma}} \partial^y_{\underline{\sigma}} = \tilde{\partial}^x \tilde{\partial}^y + \partial^x_3 \partial^y_3 \end{aligned} \tag{2.7}$$

Taking into account (2.3) and (2.4) the product $T_{33}(x,y)T_{33}(x',y')$ appears as a product of four field operators

$$\begin{aligned} \langle 0|T_{33}(x,y)T_{33}(x',y')|0\rangle' = \frac{1}{4} \{ & \partial^{\bar{x}y} \partial^{\bar{x}'y'} g_{\sigma\tau} g_{\sigma'\tau'} A_\tau(x) A_\sigma(y) A_{\tau'}(x') A_{\sigma'}(y') \\ & + \partial^x_{\underline{\sigma}} \partial^y_{\underline{\tau}} \partial^{x'}_{\underline{\sigma}'} \partial^{y'}_{\underline{\tau}'} A_{\underline{\tau}}(x) A_{\underline{\sigma}}(y) A_{\underline{\tau}'}(x') A_{\underline{\sigma}'}(y') \\ & - \{ \partial^{\bar{x}y} \partial^{x'}_{\sigma'} \partial^{y'}_{\tau'} g_{\sigma\tau} A_\tau(x) A_\sigma(y) A_{\tau'}(x') A_{\sigma'}(y') \\ & - \partial^{\bar{x}'y'} \partial^x_{\underline{\sigma}} \partial^y_{\underline{\tau}} g_{\sigma'\tau'} A_{\underline{\tau}}(x) A_{\underline{\sigma}}(y) A_{\tau'}(x') A_{\sigma'}(y') \}. \end{aligned} \tag{2.8}$$

Because we restrict the consideration to free field theory the Wick theorem can be applied immediately

$$\begin{aligned} \langle 0|A(x)A(y)A(x')A(y')|0\rangle = & \\ & + \langle 0|A(x)A(y)|0\rangle\langle 0|A(x')A(y')|0\rangle \\ & + \langle 0|A(x)A(x')|0\rangle\langle 0|A(y)A(y')|0\rangle \\ & + \langle 0|A(x)A(y')|0\rangle\langle 0|A(y)A(x')|0\rangle . \end{aligned} \tag{2.9}$$

Therefore the correlation function is reduced to a sum of products of elementary Wightman functions. Due to the subtracted structure of the correlation function $< 0|T_{33}(x,y)T_{33}(x',y')|0 >'$ the first term of the r.h.s. of eq.(2.9) drops out and the point splitting can be removed in principle. In view of an later application to QED including ideally conducting plates perpendicular to the x_3-axis we write down the gegneral structure of the corresponding Wightman functions

$$
\begin{aligned}
< 0|A_\mu(x)A_\nu(y)|0 > &= \frac{1}{i} g_{\mu\nu} D^-(x-y) \\
&+ \frac{1}{i}(\bar{g}_{\mu\nu} - \frac{\tilde{\partial}^x_\mu \tilde{\partial}^y_\nu}{\tilde{\partial}^x \tilde{\partial}^y}) \bar{D}^-(x,y). \qquad (2.10)
\end{aligned}
$$

D^- denotes the free space function and $\bar{D}^-$ an additional function which is necessary to satisfy the boundary conditions [4]. According to the symmetry of the situation we have translation invariance in the $\tilde{x}$-subspace only which allows the representations

$$
\begin{aligned}
D^-(x-y) &= \int \frac{d\tilde{p}}{(2\pi)^3} e^{i\widetilde{p(x-y)}} d(x_3, y_3, \tilde{p}) \qquad (2.11) \\
\bar{D}^-(x,y) &= \int \frac{d\tilde{p}}{(2\pi)^3} e^{i\widetilde{p(x-y)}} \bar{d}(x_3, y_3, \tilde{p}). \qquad (2.12)
\end{aligned}
$$

Inserting equs.(2.9),(2.10),(2.11) and (2.12) into the correlation functions for the stress tensor we obtain

$$
\begin{aligned}
&< 0|T_{33}(x)T_{33}(x')|0 >'= \\
&\quad -2\int \frac{d\tilde{p}}{(2\pi)^3} \int \frac{d\tilde{p}}{(2\pi)^3} e^{i\widetilde{(p+p')}\widetilde{(x-x')}} \\
&\quad \{+(\frac{1}{2})[(-\tilde{p}\tilde{p}' + \partial_3^x\partial_3^y)(-\tilde{p}\tilde{p}' + \partial_3^{x'}\partial_3^{y'})] \\
&\quad [d(x,x'\tilde{p})d(y,y',\tilde{p}') + \bar{d}(x,x'\tilde{p})d(y,y',\tilde{p}') + d(x,x'\tilde{p})\bar{d}(y,y',\tilde{p}')] \\
&\quad +(\frac{1}{4})[(-\tilde{p}\tilde{p}' + \partial_3^x\partial_3^y)(-\tilde{p}\tilde{p}' + \partial_3^{x'}\partial_3^{y'})(1 + \frac{(\widetilde{pp'})^2}{\Gamma^2\Gamma'^2}) \\
&\quad +(\tilde{p}^2 - \frac{(\tilde{pp'})^2}{\Gamma'^2})((\tilde{p}')^2 - \frac{(\tilde{pp'})^2}{\Gamma^2}) \\
&\quad -((-\tilde{p}\tilde{p}' + \partial_3^x\partial_3^y) + (-\tilde{p}\tilde{p}' + \partial_3^{x'}\partial_3^{y'}))(\tilde{pp'})(1 - \frac{(\tilde{pp'})^2}{\Gamma^2\Gamma'^2})]\bar{d}(x,x'\tilde{p})\bar{d}(y,y',\tilde{p}') \\
&\quad +(\frac{1}{4})[(\tilde{p}'^2 - \partial_3^y\partial_3^{y'})(\tilde{p}^2 - \frac{(\tilde{pp'}^2}{\Gamma'^2})]d(x,x'\tilde{p})\bar{d}(y,y',\tilde{p}') \\
&\quad +(\frac{1}{4})[(\tilde{p}^2 - \partial_3^x\partial_3^{x'})(\tilde{p}'^2 - \frac{(\tilde{pp'})^2}{\Gamma^2})]\bar{d}(x,x'\tilde{p})d(y,y',\tilde{p}')\}|_{x=y,x'=y'}.
\end{aligned}
$$

The fluctuation of the Casimir pressure on a plate located at $x_3 = a$ can be reduced to the correlation function (2.13) due to the relation (2.4). One obtains

$$< 0|p(y)p(x')|0 >'| \,|_{x_3=y_3=a} =< 0|T_{33}(x)T_{33}(x')|0 >' |_{x_3=x_3'=a+\epsilon} + \\ < 0|T_{33}(x)T_{33}(x')|0 >' |_{x_3=x_3'=a+\epsilon} \tag{2.13}$$

for ideally conducting plates. The reason for the absence of crossing terms originates from the fact, that physical modes cannot propagate across the plates for ideal conductors.

3 Wightman functions in the presence of plates

In principle, the Wightman functions could be obtained directly from the representations of the fields in terms of eigenmodes obeying the boundary conditions. It is however more appropriate to start from the expressions for the propagators D^c (or T-Products) which are well-known in the case of one or two plates. This procedure exploits the relation

$$D^c(x,y) - D^{c*}(x,y) = D^-(x,y) - D^+(x,y). \tag{3.1}$$

The Wightman function D^- is that part in $D^c - D^{c*}$ which allows an analytic continuation $z_0 = x_0 - y_0 \to z_0 - i\eta, (\eta > 0)$ and correspondingly carries a $\Theta(-p_0)$ in its Fourier representaion. For completeness let us begin with the free photon propagator. The corresponding Wightman function D_0^- is well-known

$$\begin{aligned} D_0^-(z) &= i\int \frac{dp}{(2\pi)^3} e^{ipz}\Theta(-p_0)\delta(p^2) \\ &= \frac{-i}{4\pi^2[(z_0 - i\epsilon)^2 - \vec{z}^2]} \\ &= \int \frac{d\tilde{p}}{(2\pi)^3} e^{i\tilde{p}\tilde{z}}\Theta(-p_0)\Theta(\tilde{p}^2)\frac{i}{2\Gamma}(e^{i\Gamma z_3} + e^{-i\Gamma z_3}) \end{aligned} \tag{3.2}$$

with $\Gamma = \sqrt{\tilde{p}^2}$. For the QED with mirrors the photon propagator has the structure

$$\begin{aligned} {}^s D^c_{\mu\nu}(x,y) &= (g_{\mu\nu} - (1-1/\alpha)D^c(x-y) \\ &\quad + (\tilde{g}_{\mu\nu} - \frac{\tilde{\partial}^x_\mu \tilde{\partial}^y_\nu}{\tilde{\partial}^x \tilde{\partial}^y})\bar{D}(x,y). \end{aligned} \tag{3.3}$$

In the case of one plate at $x_3 = 0$ the boundary dependent term $\bar{D}_1$ reads

$$\bar{D}_1(x,y) = \int \frac{d\tilde{p}}{(2\pi)^3} e^{i\tilde{p}\tilde{z}}(\frac{-i}{2\Gamma})(e^{i\Gamma(|x_3|+|y_3|)} \tag{3.4}$$

with $\Gamma = \sqrt{\tilde{p}^2 + i\epsilon}$. Following the procedure (3.1) we obtain

$$\bar{D}_1^-(x,y) = \int \frac{d\tilde{p}}{(2\pi)^3} e^{i\tilde{p}\tilde{z}}\Theta(-p_0)\Theta(\tilde{p}^2)(\frac{i}{-2\Gamma})(e^{i\Gamma(|x_3|+|y_3|)} + e^{-i\Gamma(|x_3|+|y_3|)}). \tag{3.5}$$

In the case of two plates located at $x_3 = 0$ and $x_3 = d$ the second part of the propagator has the form [4]

$$
\begin{aligned}
\bar{D}_2(x,y) &= \int \frac{d\tilde{p}}{(2\pi)^3} e^{i\tilde{p}\tilde{z}} \left(\frac{-i}{2\tilde{\Gamma}}\right) \frac{1}{2i\sin(\Gamma d)} \\
&\quad \{ e^{i\Gamma(|x_3|+|y_3-d|)} + e^{i\Gamma(|x_3-d|+|y_3|)} \\
&\quad - e^{i\Gamma(|x_3|+|y_3|-d)} - e^{i\Gamma(|x_3-d|+|y_3-d|-d)} \}.
\end{aligned} \tag{3.6}
$$

This propagator is valid for all x. Here it is appropriate to consider the corresponding Wightman function separately in the exterior regions $x_3 < 0$ or $x_3 > 0$ and in the inner region $0 < x_3 < d$. For $x_3 < 0$ one immediately obtains an expression which coincides with $\bar{D}_1^-$ (eq.3.5). For the inner region $0 < x_3 < d$ the propagator can equivalently be written

$$
\begin{aligned}
\bar{D}_2(x,y) &= \int \frac{d\tilde{p}}{(2\pi)^3} e^{i\tilde{p}\tilde{z}} \frac{-i}{2\Gamma} \{ \cos\Gamma(x_3 - y_3) + \\
&\quad \frac{1}{2i\sin(\Gamma d)} [(2\cos\Gamma(x_3-y_3)\cos\Gamma d - 2\cos\Gamma(x_3+y_3-d)] \}.
\end{aligned} \tag{3.7}
$$

Notice that this expression contains poles at $\Gamma d = n\pi$, which have to be correctly taken into account when applying the construction (3.1). Note that the first term in eq.(3.7) leads to D_0^- up to a sign:

$$
\begin{aligned}
\bar{D}_2^-(x,y) &= -\bar{D}_0^-(x,y) + \int \frac{d\tilde{p}}{(2\pi)^3} e^{i\tilde{p}\tilde{z}} \frac{-i}{2\Gamma} \Theta(-p_0)\Theta(\tilde{p}^2) \\
&\quad \left\{ \frac{1}{2i\sin(\Gamma d)} - \frac{1}{2i\sin(\Gamma^* d)} \right\} \\
&\quad [(2\cos\Gamma(x_3-y_3)\cos\Gamma d - 2\cos\Gamma(x_3+y_3-d)].
\end{aligned} \tag{3.8}
$$

Evaluation of (3.8) leads to two equaivalent representations

$$
\bar{D}_2^-(x,y) = -\bar{D}_0^-(x,y) + \frac{2i}{d} \sum_{n=1}^{\infty} \int \frac{dp_\perp}{(2\pi)^2} e^{-i\tilde{p}\tilde{z}} \sin\frac{\pi n x_3}{d} \sin\frac{\pi n y_3}{d} \tag{3.9}
$$

with $p_0 = \sqrt{p_\perp{}^2 + \frac{\pi n}{d}^2}$ and

$$
\begin{aligned}
\bar{D}_2^-(x,y) &= -\bar{D}_0^-(x,y) - \frac{1}{8\pi d\zeta} \left\{ \frac{1}{e^{\frac{i\pi}{d}(\zeta - x_3 - y_3)} - 1} + \frac{1}{e^{\frac{i\pi}{d}(\zeta + x_3 + y_3)} - 1} \right. \\
&\quad \left. - \frac{1}{e^{\frac{i\pi}{d}(\zeta - x_3 + y_3)} - 1} - \frac{1}{e^{(\frac{i\pi}{d}(\zeta + x_3 - y_3)} - 1} \right\}
\end{aligned} \tag{3.10}
$$

with $\zeta = \sqrt{(z_0 - i\eta)^2 - z_\perp{}^2}$. Obviously the sum $\bar{D}_2^-(x,y) + D_0^-(x,y)$ in (3.9) coincides with the naively constructed Wightman function for a massless scalar field obeying the Dirichlet boundary condition at $x_3 = 0$ and $x_3 = d$

4 Correlation functions

As a first example we consider the correlation function of the stress tensor for the free electromagnetic field restricted to $x_3 = x_3'$. The evaluation of (2.13) with $\bar{d} = 0$ leads to

$$
\begin{aligned}
&< 0|T_{33}(x)T_{33}(x')|0 >' |_{x_3=x_3'} = \\
&\quad 2\int \frac{d\tilde{p}}{(2\pi)^3}\int \frac{d\tilde{p}'}{(2\pi)^3} e^{i\widetilde{(p+p')}\widetilde{(x-x')}}[\Theta(-p_0)\Theta(\tilde{p}^2)\Theta(-p_0')\Theta((\tilde{p}')^2)] \\
&\quad \frac{\tilde{p}^2\tilde{p}'^2 + (\tilde{p}\tilde{p}')^2}{\Gamma\Gamma'} \\
&= \frac{1}{15(2\pi)^2}\int \frac{d\tilde{q}}{(2\pi)^3} e^{-i\tilde{q}\tilde{z}}\Theta(q_0)(\tilde{q}^2)_+^{5/2} &(4.1)\\
&= \frac{3}{\pi^4}\frac{1}{[(z_0 - i\eta)^2 - z_\perp^2]^4}. &(4.2)
\end{aligned}
$$

Without the restriction $x_3 = x_3'$ we would have obtained

$$
\begin{aligned}
&< 0|T_{33}(x)T_{33}(x')|0 >' = \\
&\quad [4(\zeta^2 + (x_3 - x_3')^2)^2 - (\zeta^2 - (x_3 - x_3')^2)^2]\frac{1}{\pi^4}\frac{1}{[\zeta^2 - (x_3 - x_3')^2]^6} &(4.3)
\end{aligned}
$$

as one of the characteristic functions for the fluctuations of a free electromagnetic field.

Let us treat now the case of one plate located at $x_3 = 0$. Here we have to take into account the Wightman function (3.5) in the representation (2.11) for the evaluation of (2.13). The result is

$$
\begin{aligned}
< 0|T_{33}(x)T_{33}(x')|0 >' |_{x_3=x_3'=0} &= \\
&= \frac{2}{15(2\pi)^2}\int \frac{d\tilde{q}}{(2\pi)^3} e^{-i\tilde{q}\tilde{z}}\Theta(q_0)(\tilde{q}^2)_+^{5/2} \\
&= \frac{6}{\pi^4}\frac{1}{[(z_0 - i\eta)^2 - z_\perp^2]^4} \qquad (4.4)
\end{aligned}
$$

which coincides with the result of G.Barton [1]. Comparing (4.3) with (4.1) we notice that the fluctuations near the plate are twice that of a free field. The unrestricted correlation function would be

$$
\begin{aligned}
&< 0|T_{33}(x)T_{33}(x')|0 >' = \\
&\quad \frac{1}{\pi^4}\{[4(\zeta^2 + (x_3 - x_3')^2)^2 - (\zeta^2 - (x_3 - x_3')^2)^2]\frac{1}{[\zeta^2 - (x_3 - x_3')^2]^6} \\
&\quad +[4(\zeta^2 + (x_3 + x_3')^2)^2 - (\zeta^2 - (x_3 + x_3')^2)^2]\frac{1}{[\zeta^2 - (x_3 + x_3')^2]^6}\}. &(4.5)
\end{aligned}
$$

In the case of two plates the correlation functions for the inner and exterior regions can be treated separately. This is no restriction because in lowest order of perturbation theory both regions are not correlated. Because the Wightman functions $\bar{D}_1^-$ and $\bar{D}_2^-$ for $x_3 < 0$ are identical, the corresponding correlation functions coincide, too.

The investigation of the inner region is more complicated. Following the standard procedure, i.e. combining (3.8),(2.11),(2.12) and (2.13) we obtain after a straightforward calculation

$$\begin{aligned}
&< 0|T_{33}(x)T_{33}(x')|0 >' |_{x_3=x_3'=0_+} = \\
&2\int \frac{d\tilde{p}}{(2\pi)^3}\int \frac{d\tilde{p}'}{(2\pi)^3} e^{i(\widetilde{p+p'})(\widetilde{x-x'})}[\Theta(-p_0)\Theta(\tilde{p}^2)\Theta(-p_0')\Theta((\tilde{p}')^2)] \\
&(\tilde{p}^2(\tilde{p}')^2 + (\tilde{p}\tilde{p}')^2)\frac{\cos\Gamma d\cos\Gamma' d}{\Gamma\Gamma'} \\
&\{\frac{1}{2i\sin(\Gamma d)} - \frac{1}{2i\sin(\Gamma^* d)}\}\{\frac{1}{2i\sin(\Gamma' d)} - \frac{1}{2i\sin(\Gamma'^* d)}\} \qquad (4.6)
\end{aligned}$$

(compare (4.1)). With

$$\frac{\cos\Gamma d}{\Gamma}\{\frac{1}{2i\sin(\Gamma d)} - \frac{1}{2i\sin(\Gamma^* d)}\} = -\frac{\pi}{d}\frac{1}{|p_0|}\delta(p_0 - \sqrt{(p_\perp)^2 + (\frac{\pi n}{d})^2})$$

we get

$$\begin{aligned}
&< 0|T_{33}(x)T_{33}(x')|0 >' |_{x_3=x_3'=0_+} = \\
&\frac{1}{2d^2}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\int \frac{d^2p_\perp}{(2\pi)^2}\int \frac{d^2p'_\perp}{(2\pi)^2} e^{i(p_0+p_0')(x_0-x_0')-i(p+p')_\perp(x-x')_\perp} \\
&[(\frac{\pi n}{d})^2(\frac{\pi n'}{d})^2 + (\widetilde{pp'})^2]\frac{1}{p_0 p'_0}. \qquad (4.7)
\end{aligned}$$

Further evaluation will be much simplified if we exploit the Lorentz invariance in the (x_0, x_1, x_2)-subspace. This allows to put $z_\perp = 0$ and to exploit in addition rotation invariance in the (p_1, p_2)-plane. This leads to

$$\begin{aligned}
&< 0|T_{33}(x)T_{33}(x')|0 >' |_{x_3=x_3'=0_+, z_\perp=0} = \\
&\frac{1}{(2\pi)^2 2d^2}\{A_1^2(z_0, d) + A_2^2(z_0, d) + 2A_3^2(z_0, d)\} \qquad (4.8)
\end{aligned}$$

with

$$\begin{aligned}
A_1 &= \sum_{n=1}^{\infty}\int \frac{d^2p_\perp}{(2\pi)} e^{-ip_0(x_0-x_0')}\frac{1}{p_0}(\frac{n\pi}{d})^2 \\
A_2 &= \sum_{n=1}^{\infty}\int \frac{d^2p_\perp}{(2\pi)} e^{-ip_0(x_0-x_0')}p_0 \\
A_3 &= \sum_{n=1}^{\infty}\int \frac{d^2p_\perp}{(2\pi)} e^{-ip_0(x_0-x_0')}\frac{1}{p_0}\frac{p_\perp^2}{2} \qquad (4.9)
\end{aligned}$$

$$\begin{aligned}
\lim_{d\to\infty} W_2(\zeta,d) &= W_1(\zeta) \\
\lim_{\zeta\to 0} W_2(\zeta,d) &= W_1(\zeta) \\
W_2(\zeta,d)_{\frac{|\zeta|}{d}<1} &= \frac{1}{d^8}(\frac{d}{\zeta})^2 f(\frac{\zeta}{d}).
\end{aligned}$$

The function $f(\frac{\zeta}{d})$ is an analytic and integrable function with poles at $\frac{\zeta}{d} = 2n$.

Besides the already given physical interpretations we see that the correlations at $\zeta^2 \approx 0$ approximately coincide with those of the one mirror problem. It can be understood in a simple physical picture: there is not enough time to receive the reflected signals.

As a simple consequence of the Wightman structure of the correlation functions (the poles are located in the upper x_0-plane) we conclude

$$\int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dx_0' W_i(\zeta,d) = 0$$

i.e. the fluctuations of observables measured over an infinite time interval tend to zero. According to eq.(2.14) the fluctuations of the Casimir pressure are the sum of the fluctuations on both sides of the plates.

5 Penetrable plates

In this section we restrict the considerations to the case of one plate only. Up to now we have characterized the plates by the standard boundary conditions of Electrodynamics $E_t = B_n = 0$ This correponds to

$$D_0^- + \tilde{D}_1^- = 0 \tag{5.1}$$

at the boundary for the Greens functions of the physical modes. To make the plates penetrable one has to modify the boundary conditions. In a phenomenological approach we introduce a special function $g(p_0)$ into the Fourier representation (3.4) of the boundary dependent part of the Green function. this obviously does not violate the field equation but disturbs the boundary condition (5.1) at higher frequencies. Special choices to be studied are

$$g(\tilde{p}) = \begin{cases} \Theta(\Lambda - |p_0|) \\ e^{-\mu|p_0|} \\ \frac{a}{a+i\Gamma},\ a<0. \end{cases}$$

The last function originates from the study of δ -potentials in field equations The Wightman functions are modified according to the rule

$$\begin{aligned}
\tilde{D}_1^-(x,y) = \int \frac{d\tilde{p}}{(2\pi)^3} e^{i\tilde{p}\tilde{z}} \Theta(-p_0)\Theta(\tilde{p}^2 \frac{i}{-2\Gamma} \\
(g(\tilde{p})e^{i\Gamma(|x_3|+|y_3|)} + g^*(\tilde{p})e^{-i\Gamma(|x_3|+|y_3|)}).
\end{aligned} \tag{5.2}$$

and $p_0 = \sqrt{p_\perp^2 + (\frac{\pi n}{d})^2}$.

Taking into acount the analytic properties of the Wightman functions $x_0 \to x_0 - i\eta$, $(\eta > 0)$ the integrations and summations can be carried out without problems. The final result can be written in terms of the variable $\zeta = \sqrt{(x_0 - i\eta)^2 - (x_\perp^2)}$

$$
\begin{aligned}
A_1(\zeta, d) &= \frac{1}{i\zeta}(\frac{\pi}{d})^2 \frac{e^{i\frac{2\pi\zeta}{d}} + e^{i\frac{\pi\zeta}{d}}}{(e^{i\frac{\pi\zeta}{d}} - 1)^3} \\
A_2(\zeta, d) &= \frac{2}{(i\zeta)^3} \frac{1}{(e^{i\frac{\pi\zeta}{d}} - 1)} \\
&+ \frac{2}{(i\zeta)^2}(\frac{\pi}{d}) \frac{e^{i\frac{\pi\zeta}{d}}}{(e^{i\frac{\pi\zeta}{d}} - 1)^2} \\
&+ \frac{1}{i\zeta}(\frac{\pi}{d})^2 \frac{e^{i\frac{2\pi\zeta}{d}} + e^{i\frac{\pi\zeta}{d}}}{(e^{i\frac{\pi\zeta}{d}} - 1)^3} \\
A_3(\zeta, d) &= \frac{1}{(i\zeta)^3} \frac{1}{(e^{i\frac{\pi\zeta}{d}} - 1)} + \frac{1}{(i\zeta)^2}(\frac{\pi}{d}) \frac{e^{i\frac{\pi\zeta}{d}}}{(e^{i\frac{\pi\zeta}{d}} - 1)^2}
\end{aligned} \tag{4.10}
$$

so that

$$
\begin{aligned}
&W_2(\zeta, d) \equiv < 0|T_{33}(x)T_{33}(x')|0 >' |_{x_3 = x_3' = 0_+} = \\
&\frac{1}{(2\pi)^2 2d^2}\{A_1^2(\zeta, d) + A_2^2(\zeta, d) + 2A_3^2(\zeta, d)\}.
\end{aligned} \tag{4.11}
$$

This result generalizes the correponding investigation of G.Barton. It is very interesting that the correlation function $W_2(\zeta, d)$ contains infinitely many multiple poles at

$$(\zeta)^2 = (x_0 - x_0')^2 - (x_\perp - x_\perp')^2 = 4n^2 d^2.$$

In a physical interpretation these values of ζ^2 correpond to pairs of events $(x_0, x_\perp, x_3 = 0)$ and $(x_0', x'_\perp, x_3' = 0)$ connected by n-times reflected light-signals. This implies a resonance behaviour of the fluctuations for such distances ζ^2 For the limiting case $d \to \infty$ the results for one mirror can be recovered.

$$
\begin{aligned}
A_1(\zeta, d)|_{d \to \infty} &= \frac{2d}{\pi\zeta^4} \\
A_2(\zeta, d)|_{d \to \infty} &= \frac{6d}{\pi\zeta^4} \\
A_3(\zeta, d)|_{d \to \infty} &= \frac{2d}{\pi\zeta^4}
\end{aligned} \tag{4.12}
$$

The correlation function W_2 has the following scaling and limiting poperties (W_1 denotes the correlation function corresponding to one plate)

$$W_2(\lambda\zeta, \lambda d) = \frac{1}{\lambda^8} W_2(\zeta, d)$$

In the present case of a penetrable plate it is interesting to concentrate on the fluctuation of the Casimir pressure. Now the field modes on both sides of the plates are correlated, at least for high frequencies. Therefore a lowering of the fluctuations of the Casimir pressure can be expected The final results are

$$< 0|p(x)p(x')|0 >' |_{x_3=x_3'=0} = \int \frac{d\tilde{q}}{(2\pi)^3} e^{-i\tilde{q}x} \Theta(q_0) f(\tilde{q}) \tag{5.3}$$

with

$$f(\tilde{q}) = \begin{cases} \frac{4}{15\pi^2}(\tilde{q}^2)_+^{5/2} & \text{for} \quad |q_0| < \Lambda \\ \frac{4}{15\pi^2}(\tilde{q}^2)_+^{5/2}\frac{\Lambda}{|q_0|} & \text{for} \quad |q_0| \gg \Lambda \end{cases}$$

(cut off case)

$$f(\tilde{q}) \sim (\tilde{q}^2)_+^{5/2}[\mu q_0]^{-4} \quad \text{for} \quad \mu|q_0| \gg 1$$

(exponential damping) and finally

$$f(\tilde{q}) = \frac{4}{15\pi^2}(\tilde{q}^2)_+^{5/2}\frac{a^2}{a^2 + \tilde{q}^2} \quad \text{for} \quad (\tilde{q})^2 \to \infty.$$

In all three cases the asymptotic behaviour of $f(\tilde{q})$ is damped in comparison with the ideal plate behaviour $f(\tilde{q}^2) \sim (\tilde{q}^2)_+^{5/2}$.

For interesting discussions D.R. is indepted to his colleagues K.Scharnhorst, A.Uhlmann, W.Weller and M.Bordag.

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Solubel Models at Finite Temperature
or
Q(F)T under the Infuence of External Conditions

A. Wipf

Institut für Theoretische Physik
Eidgenössische Technische Hochschule
Hönggerberg, Zürich CH-8093, Switzerland

1. Introduction

The response of physical systems, classical and quantum ones, to the change of external conditions is a vast and interesting area in theoretical and experimental physics. From the point of view of a theoretician these kind of problems can be (subjectively) divided into 3 classes:

a) Quantum systems in (possibly finite) spaces $\mathcal{M}$ with boundary $\partial\mathcal{M}$
Here one is interested in the dependence of the ground state $|0\rangle$, excited states or thermal state when $\mathcal{M}$ is changed or an external field is applied. The historical example being the measured Casimir effect where the ground state energy

$$E_0(\mathcal{M}) = \langle 0|H|0\rangle \tag{1}$$

depends on the shape of $\mathcal{M}$, since $|0\rangle$ does. This leads then to the Casimir force acting on the walls enclosing $\mathcal{M}$. Whether this force attracts or repels the walls depends on the form of $\partial\mathcal{M}$, the particle species enclosed in $\mathcal{M}$ and the imposed boundary conditions. For example, the electromagnetic vacuum-energy between 2 plates decreases if the plates come closer together [1] or the vacuum-energy inside a sphere increases if the radius of the sphere decreases [2]. Also, one can find fermionic boundary conditions such that the Casimir force due to fermions has the opposite sign to the electromagnetic Casimir force. This cancellation between bosonic and fermionic Casimir forces typically happens for supersymmetric systems (with supersymmetric boundary conditions) [3]. More generally, ground state expectation values of other observables besides the energy may also be sensitive to the geometry of $\mathcal{M}$, to applied external fields or the imposed boundary conditions. For example, for systems without mass-gap (e.g. Ising model, Higgsmodels or QCD in the chiral limit) we are interested in whether a symmetry is spontaneously broken or not, or whether

$$\langle 0|\text{order parameter}|0\rangle \tag{2}$$

in zero or not. The answer to this (also phenomenologically) important question depends very much on $\mathcal{M}$ and the imposed boundary conditions (see below).

Other interesting questions to ask in this context are (some of them are addressed at this workshop):

i. How moving boundaries influence a quantum system and how the system reacts back. The simplest example of this kind are accelerated mirrors leading to particle production [4].

ii. How a system responds to the application of external gauge- or gravitational fields, for example what is the number and distribution of particles produces by such fields. Also, a quantum system may lose some of its classical symmetries in the presence of external fields, that is, may be afflicted with anomalies.

iii. In curved spacetime there is the intriguing problem of how to define a vacuum state and the related one of defining particles. Recently a gauge invariant normal ordering has been discussed which addresses these problems [5].

iv Also, Minkowski type prescriptions to extract a finite Casimir energy do not apply to curved spacetimes. A method to find a finite Casimir energy in the presence of gravity has recently been derived in [6].

v. How do physical walls and the associated boundary conditions influence first quantized matter, e.g. a hydrogen atom. The atom vacuum interaction changes the transition probabilities of the atomic states quite considerably [7].

b) Quantum systems in finite euclidean space-times $\mathcal{M}$

In a functional approach to quantum field theory it is often convenient to work in euclidean space time. At finite temperature one is even forced to do that [8]. The relevant object containing all information of the quantum system is the effective action (quantum action, free energy) Γ. One maybe interested how Γ changes if spacetime $\mathcal{M}$ is changed. The problem of these socalled finite size effects are quite important in quantum field theory [9] and recently there has been some progress, e.g.:

i. For massless particles the trace anomaly determines to a great extend the finite size effects [10].

ii. The effective actions on the spacetime region $\mathcal{M}$ and $\lambda\mathcal{M}$ (the stretched region) are the same, if the coupling constants and field are scaled in certain way [11]

$$\Gamma[\lambda\mathcal{M},\phi,g,m]=\Gamma[\mathcal{M},\sqrt{Z}\phi,g(\lambda),m(\lambda)]. \tag{3}$$

In the semiclasical approximation the coupling constants and field run ecactly according to the wellknown 1-loop renormalization flow. This way one gets for example a Casimir type interpretation for the β-function and anomalous dimension γ in gauge theories.

iii. For $\mathcal{M}=[0,\beta]$ the effective action

$$\log\int\limits_{\substack{B(\beta)=B(0)\\F(\beta)=F(0)}}\mathcal{D}(\text{fields})\,e^{-S_E(\text{fields})}=F(\beta) \tag{4}$$

is just the free energy at temperature $T=1/\beta$. Here the temperature dependence only enters thru the geometry of the euclidean spacetime and the boundary conditions: the bosonic (fermionic) fields, denoted by B (F), obey periodic (antiperiodic) boundary conditions in the imaginary time direction. We see that in the functional approach to finite temperature field theories the temperature dependence is just a finite size effect.

c) Topological effects

For compact spacetimes $\mathcal{M}$ the configuration space of gauge theories

$$\mathcal{C}=\text{gauge potentials, matter fieds/ gauge transformations}$$

is generically topological non-trivial. As a consequence of this fact we find instantons or other topological objects with winding numbers [12] (for example, for QCD on the torus). Recently arguments have been put forward that fields with winding numbers are not suppressed in QCD if $Vm_q\gg 1$, where V is the volume of spacetime and m_q the mass of the lightest quark [13]. Due to these topologically non-trivial configuration the vacuum structure of gauge theories is quite non-trivial. We may imagine that there are infinitely many vacua $|\Psi_n\rangle$, labelled by an integer n. None of them is a 'good' vacuum (they do not cluster) but the good ones, the socalled θ-vacua, are superpositions of them

$$|\theta\rangle=\sum_n e^{in\theta}|\Psi_n\rangle. \tag{5}$$

In standard perturbation theory one only calculates

$$\langle\Psi_0|\text{operator}|\Psi_0\rangle, \tag{6}$$

hence all operators which transform $|\Psi_n\rangle$ into $|\Psi_m\rangle$ with $n \neq m$ have vanishing vacuum expectation value in perturbation theory. The possibly simplest and most important such operator is $\bar{\psi}\psi$, where ψ denotes a quark field. In the true QCD vacuum

$$\langle\theta|\bar{\psi}\psi|\theta\rangle = F_\pi G_\pi \neq 0, \tag{7}$$

where F_π is the pion decay constant. As a consequence the axial $SU(N)$ in QCD with N flavours (and in the chiral limit) is spontaneously broken. The non-zero expectation value (7) is at the heart of the effective low energy theory of strong interaction. It implies the existence of pions and the correct low energy strong interaction (via the $PCAC$-hypothesis).

Unfortunately, although several mechanisms leading to the spontaneous symmetry breakdown of $SU(N)_A$ in QCD have been proposed [14], a full understanding of this important problem is still lacking.

Many of the listed problems (in particular the last ones) are very intricate and we should not expect to find satisfactory solutions in the near future. So we decided to study simpler models, complicated enough to ask the interesting questions, yet simple enough to obtain some answers. Before discussing these models let us summarize some of the questions we have in mind which should be addressed when studying such models:

- Role of non-perturbative *vacuum structure*, e.g. what mechanism leads to chiral condensates $\langle\bar{\psi}\psi\rangle \neq 0$.
- *Finite size effects*, temperature dependence, e.g. size and temperature dependence of chiral condensates.
- Dependence of expectation values on imposed *boundary conditions*.
- Dependence of expectation values on *external fields*. For example, the thermodynamics of black holes relates the (surface) gravity to temperature. So it has been speculated [15] that in spacetime regions with large curvature QCD may deconfine and chiral symmetry maybe restored, as it happens for high temperatures.
- Does the *Hawking radiation* stay thermal for interacting theories [16].

2. The extended gauged Thirring model in curved space

We have studied a theory containing scalars, pseudoscalars, Dirac fermions and 'photons' in a classical gravitational field in two spacetime dimensions [17]. The theory is described by the action

$$\begin{aligned} S &= \frac{1}{2}\int_{\mathcal{M}} \frac{1}{\sqrt{g}} E^2 + S_F + S_B \\ S_F &= -\int \sqrt{g}\bar{\psi}\, i\gamma^\mu \left[\nabla_\mu - ig_1\partial_\mu\lambda + ig_2\eta_\mu^{\ \nu}\partial_\nu\phi\right]\psi \\ S_B &= \int \sqrt{g}g^{\mu\nu}\left[\partial_\mu\phi\partial_\nu\phi + \partial_\mu\lambda\partial_\nu\lambda\right] + g_3\int\sqrt{g}R\lambda, \end{aligned} \tag{8a}$$

where $E = \partial_0 A_1 - \partial_1 A_0$ is the field strength, the γ's are the 'curved' gamma matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{8b}$$

and the covariant derivative contains the gauge potential and the spin connection

$$\nabla_\mu = \partial_\mu - ieA_\mu + i\omega_\mu. \tag{8c}$$

The gravitation field $g_{\mu\nu}$ (or rather the 2-bein e^a_μ, since the theory contains fermions) is treated as a classical background field, whereas the scalars λ, pseudoscalars ϕ, 'electrons' ψ and 'photons' A_μ are fully quantized. This theory allows one to address many of the problems raised above.

For certain values of the coupling constants g_1, g_2 and g_3 it reduces to wellknown soluble models:

- For $g_3 = 0$ and $g_1 = g_2 = g$ it reduces to the gauged *Thirring model*, coupled to gravity

$$S = \frac{1}{2}\int d^2x \frac{1}{\sqrt{g}}E^2 - \int \sqrt{g}\left[\bar{\psi} i\gamma^\mu\nabla_\mu\psi + \frac{g^2}{4}j^\mu j_\mu\right], \qquad j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{9}$$

  On the classical level this can be seen at once when eliminating ϕ and λ by using their equations of motion. Quantum mechanically one integrates over these fields (the functional integrals over ϕ and λ are gaussian and can be performed) and the remaining effective action for the gauge bosons and fermions is just (9). If we switch off the gravitational and gauge fields, then (9) reduces further to the well-studied and exactly soluble Thirring model [18]. However, when switching off the gauge fields the configuration space becomes topologically trivial and one looses the θ-vacuum structure. As a consequence the chiral condensate $\langle\bar{\psi}\psi\rangle$ vanishes in the ungauged model.

- For $g_1 = g_2 = g_3 = 0$ one finds (up to trivial free bosons which decouple from the system) the *Schwinger model* coupled to gravity

$$S = \frac{1}{2}\int d^2x \frac{1}{\sqrt{g}} E^2 - \int \sqrt{g}\bar{\psi}\, i\gamma^\mu \nabla_\mu \psi. \tag{10}$$

  If one further assumes 2-dimensional space-time to be euclidean, $g_{\mu\nu} = \delta_{\mu\nu}$, then one recovers the well-known Schwinger model, or QED_2[19].
- If $g_1 = g_2 = 0$ then the theory reduces to QED_2 coupled to gravity and and free bosons coupled to a socalled *background charge*, a model playing a prominent role in conformal field theory, The two sectors decouple in this limiting case.

The physical role of the various coupling constants is the following:
Increasing the coupling constant g_2 decreases the effective electromagnetic interaction beween fermions. For example, the chiral condensate depends on g_2 as $\sim (1 + g_2/4\pi)^{-\frac{1}{2}}$ and vanishes for large g_2 as in the model without gauge fields.
The constant g_3 amplifies the Hawking radiation. The Hawking radiation of the model is $(3 + 24\pi g_3^2)$ times as strong as that of a free massless scalar field.
The constant g_1 seems to have no direct physical interpretation. However, it is needed to obtain a local effective theory for the fermions and gauge bosons after integrating out the (pseudo) scalars.
The general model (8) has recently been solved on the torus (that is the finite temperature model) and on the sphere and several of the above raised questions have been investigated [17]. For reasons of time and simplicity I shall present here only the results for vanishing coupling constants and on the euclidean torus, i.e. for the finite temperature Schwinger model [20]. In the last part I comment on some new results about the multi-flavour Schwinger model subject to bag boundary conditions.

2. The finite temperature Schwinger model

We assume spacetime to be $\mathcal{M} = [0, \beta] \times [0, L]$, where β is the inverse temperature and the finite length L of space is introduces as an infrared cutoff. According to (4) we need to impose the following boundary conditions in the euclidean time direction

$$A_\mu(x^0 + \beta, x^1) = A_\mu(x^0, x^1) \quad , \quad \psi(x^0 + \beta, x^1) = -\psi(x^0, x^1). \tag{11a}$$

A priori it is not clear what are the 'correct' boundary conditions in the spacial direction x^1. Actually there are only three kind of consistent boundary conditions we can assume for fermions:

i) (quasi) periodic boundary conditions (for which $\mathcal{M}$=torus)

ii) bag boundary conditions [21]

iii) non-local Atiyah-Patodi-Singer boundary conditions [22]

In what follows we shall assume the first kind of boundary conditions and comment on the results with bag boundary conditions at the end of this contribution.
Actually we can demand periodicity in the spacial direction only up to a non-trivial gauge transformation. To see that we rewrite the flux as

$$\int E = \oint A = -\int dx^1 \left[A_1(0,x^1) - A_1(\beta,x^1)\right] - \int dx^0 \left[A_0(x^0,L) - A_0(x^0,0)\right] \tag{12}$$

and observe that the x^1-integral vanishes due to the finite temperature boundary conditions (11a). On the other hand, the flux need not be zero, it is only quantized in integer multiple of 2π:

$$\Phi = e\int E = 2\pi \cdot k, \qquad k = \text{integer}. \tag{13}$$

This follows immediately from the index theorem which in the present situation states that k is the index of the Dirac operator, and hence must be an integer. From (12) we conclude that A_μ cannot be periodic in the spacial direction. It can be periodic only up to a (non-trivial) gauge transformation

$$A_\mu(x^0, x^1 + L) = A_\mu(x^0,x^1) + \partial_\mu\alpha \quad , \quad \psi(x^0, x^1 + L) = e^{ie\alpha}\,\psi(x^0,x^1), \tag{11b}$$

with $\alpha(0,x^1) - \alpha(\beta,x^1) = \Phi/e$. Actually one can alway choose

$$\alpha(x^0,x^1) = -\frac{\Phi}{e\beta}x^0, \tag{11c}$$

and this choice corresponds to a certain trivialization of the $U(1)$-bundle over the torus.

Quantizing the finite temperature model is equivalent to evaluating the generating functional

$$\begin{aligned} \mathcal{Z}[J,\eta,\bar\eta] &= \frac{1}{\mathcal{N}}\int \mathcal{D}A\; e^{-\frac{1}{2}\int E^2 + \int A_\mu J^\mu} \cdot \mathcal{Z}_F[A,\eta,\bar\eta] \\ \mathcal{Z}_F[A,\eta,\bar\eta] &= \int \mathcal{D}\psi\mathcal{D}\bar\psi\, e^{\int \bar\psi i \not{D}\psi + \int \bar\eta\psi + \int \bar\psi\eta}. \end{aligned} \tag{14}$$

All Green functions can be gotten by functional differentiation with respect to the external current J and the grassmannian sources η and $\bar{\eta}$. We shall determine $\mathcal{Z}$ in two steps: first we shall quantize the fermions with A_μ treated as external field. This is equivalent of evaluating the fermionic generating functional $\mathcal{Z}_F$. After having solved this external field problem we evaluate the remaining functional integral over the 'photons' A_μ to find the complete generating functional $\mathcal{Z}$. Note that as an intermediate step one quantizes part of the system only. This is how one typically encounters the external field problem in the euclidean framework.

Using both the index theorem and the supersymmetry of $i\not{D}$ on can show that $i\not{D}$ has exactly $k = \Phi/2\pi$ zero modes. This complicates the evaluation of the grassmannian functional integral leading to $\mathcal{Z}_F$. In any case, a careful treatment [20] leads to

$$\mathcal{Z}_F = \prod_1^k (\bar{\eta}, \psi_p)(\bar{\psi}_p, \eta)\, e^{-\int \bar{\eta} G' \eta} \det{}'(i\not{D}), \tag{15a}$$

where the ψ_p are the orthonormal zeromodes of $i\not{D}$,

$$G'(A, x, y) = \sum_{\lambda_q \neq 0} \frac{\psi_q(x)\psi_q^\dagger(y)}{\lambda_q} \tag{15b}$$

is the 'excited' Greenfunction in the given background potential A_μ, and the primed determinant

$$\det{}' i\not{D} = \prod_{\lambda_q \neq 0} \lambda_q \tag{15c}$$

is the product of the non-zero eigenvalues of $i\not{D}$. This divergent determinant is calculated with the help of the ζ-function regularization.

Before doing any explicit calculations we can draw the following conclusions from (14) and (15a):

i) The fermionic partition $\mathcal{Z}_F[A, 0, 0]$ is only non-zero if the Dirac operator has no zero modes or equivalently, if $\Phi = 0$

ii) The 2-point functions are non-zero only if $i\not{D}$ has no or one zero mode or equivalently if $\Phi = 0$ or $\Phi = \pm 2\pi$. In particular, $\langle \bar{\psi}\psi \rangle \neq 0$ only if $\Phi = \pm 2\pi$.

iii) For the higher order correlators one gets similar selection rules.

Everything said so far holds in arbitrary dimensions with the obvious replacements, e.g $E \to$ Chern density in (13) etc. Let us now assume that spacetime is 2-dimensional and sketch how one can perform the functional integrals. In 2-dimensions one decomposes the gauge potential as

$$\begin{aligned} eA_0 &= -\frac{\Phi}{V}x^1 + \frac{2\pi}{\beta}h_0 - \partial_1\phi + \partial_0\lambda \\ eA_1 &= \qquad\quad \frac{2\pi}{L}h_1 + \partial_0\phi + \partial_1\lambda, \end{aligned} \tag{16}$$

that is into an instanton potential ($\sim \Phi$), a harmonic ($\sim h$), a coexact ($\sim \phi$) and an exact ($\sim \lambda$) piece. In (16) V denotes the volume of spacetime, $V = \beta L$. It is then not difficult to see that the gauge field measure becomes

$$\mathcal{D}A = (2\pi)^2 \det{}'(-\triangle) \sum_{k=-\infty}^{\infty} d^2h_\mu \mathcal{D}\phi\mathcal{D}\lambda\ , \qquad \int\phi = \int\lambda = 0\ , \qquad h_\mu \in [0,1]\ , \tag{17}$$

where the Jacobian in front of the sum is independent of the fields and hence chancels in expectation values against the normalization. The field λ is the pure gauge term and also chancels in expectation values of gauge invariant operators. The ϕ dependence of the zero-modes and primed determinant in (15) can be gotten from the chiral transformation and the chiral anomaly. Their $\Phi(= 2\pi k)$ and h_μ dependence can be calculated by explicit mode analysis and is given by certain Jacobi theta functions and Dedekinds eta function. The result of all this manipulations yields for the chiral condensate [20]

$$\langle\bar\psi P_\pm \psi\rangle = -\frac{|\eta|^2}{\beta}\exp\left(\frac{2\pi^2}{e^2V}\right)\frac{\int_0^1 d^2h|\theta_\pm| \int \mathcal{D}\phi e^{-\Gamma[\phi]\mp 2e\phi(x)}}{\int_0^1 d^2h|\theta|^2 \int \mathcal{D}\phi e^{-\Gamma[\phi]}}\ , \tag{18a}$$

where $P_\pm = \frac{1}{2}(1\pm\gamma_5)$ are the chiral projectors and

$$\Gamma[\phi] = \frac{1}{2}\int \phi[\triangle^2 - m_\gamma^2\triangle]\phi \tag{18b}$$

is the induced action with the 'photon' mass $m_\gamma = e/\sqrt{\pi}$. The theta and eta functions in (18a) are

$$\theta_\pm = \theta\begin{bmatrix} \frac{x^1}{L} \mp (\frac{1}{2}+h_0) \\ \frac{x^0}{\beta} \pm h_1 \end{bmatrix}(i\tau), \qquad \theta = \theta\begin{bmatrix} h_0 \\ \frac{1}{2} - h_1 \end{bmatrix}(i\tau), \qquad \eta = \eta(i\tau), \tag{18c}$$

where $\tau = L/\beta$ is proportional to the temperature. The ϕ-integration is gaussian and can be performed. The remaining integration over the harmonics is the most difficult one. Using certain properties of the elliptic functions in the integrand one finally ends up with the following exact result for the chiral condensate

$$\langle\bar{\psi}P_\pm\psi\rangle = -\frac{1}{\beta}e^{-\frac{\pi}{\beta m_\gamma}\coth(Lm_\gamma/2)}e^{F(\beta m_\gamma)}e^{-2H(\beta m_\gamma,\tau)}, \tag{19a}$$

where

$$\begin{aligned} F(x) &= \sum_{n>0}\Big(\frac{1}{n} - \frac{1}{\sqrt{n^2+(x/2\pi)^2}}\Big) \\ H(x,\tau) &= \sum_{n>0}\frac{1}{\sqrt{n^2+(x/2\pi)^2}}\frac{1}{e^{\tau\sqrt{(2\pi n)^2+x^2}}-1}. \end{aligned} \tag{19b}$$

In the limit $L \to \infty$ the temperature dependence simplifies as

$$\langle\bar{\psi}\psi\rangle \longrightarrow -\frac{2}{\beta}e^{-\pi/\beta m_\gamma}\,e^{F(\beta m_\gamma)}, \tag{20}$$

and the low and high temperature limits are found to be

$$\langle\bar{\psi}\psi\rangle \longrightarrow \begin{cases} -m_\gamma e^\gamma/2\pi, & T \to 0 \\ -2Te^{-\pi T/m_\gamma}, & T \to \infty, \end{cases} \tag{21}$$

where γ is the Euler constant. Note that for low temperature the chiral condensate approaches the constant value of the zero-temperature model which has been solved some time ago [23]. For high temperature the condensate vanishes exponentially with T. At $T \sim m_\gamma$ the condensate drops sharply from the constant zero-temperature value to almost zero. The T dependence of the condensate is almost the same as for a second order phase transition, but not exactly so, since it vanishes only exponentially for high temperature.

It seems that the above exact result for the temperature dependence of the chiral condensate in QED_2 has not been properly derived before, at least not in the functional approach. This approach is well suited for quantizing gauge theories with topologically non-trivial configurations like QCD. Actually it is believed that 1-flavour QCD behaves very similar to QED_2 with regard to the chiral condensate. Thus the above calculation should shed some light on realistic theories. For example, it can be seen from the above considerations that the chiral condensate is due to the presence of fermionic zero modes which are supported by instanton configurations only. Thus the non-zero condensate is due to the infrared properties of the theory. Of course, the numerical value of the condensate depends on the ultraviolett sector of the theory.

Let us now put n heavy (that is static) external charges, described by the charge density

$$j^0_{ext}(x) = \sum_1^n e_i\delta(x - u_i) \tag{22}$$

into the system. This is eqivalent to changing the action in the functional integral as

$$S \to S - \int j^0_{ext} \cdot A_0 \Longrightarrow e^{-S} \to e^{-S}\mathcal{P}(u_1)\cdots\mathcal{P}(u_n), \tag{23a}$$

where we have introduce the Wilson line operators

$$\mathcal{P}(u) = \exp\left(ie\int_0^\beta A_0(x^0, u)dx^0\right). \tag{23b}$$

The expectation values

$$p(u_1,\ldots,u_n) = \frac{\langle\mathcal{P}(u_1)\cdots\mathcal{P}(u_n)\rangle}{\langle\mathcal{P}(0)\rangle^n} \tag{24}$$

are then to be interpreted as the finite temperature partition function of the quantum system containing n heavy external charges at positions u_i, divided by the same expectation value for non-interacting external charges. Hence

$$F(\underline{u}) = -T\log p(\underline{u}) \tag{25}$$

is the zero-energy subtracted free energy of n static external charges at positions u_i. The result of the functional integration for this free energy is

$$F(\underline{u}) = \sum_{i<j} F(u_i, u_j) = \sum F(u_i - u_j), \tag{26a}$$

where for large L the pair potential takes the simple Yukawa form

$$F(u) \to \frac{\pi m_\gamma}{2}\, e^{-m_\gamma|u|}. \tag{26b}$$

We see that the interaction between external charges interacting with the thermal gas of QED_2 particles falls off exponentially, contrary to external charges in the vacuum for which the Coulomb force is long range. This socalled charge-shielding is actually independent of the temperature and is present in the low- and high temperature phases.

Recently most of the above formulae have been rederived in the bosonised version of the Schwinger model [24]. Actually the bosonization techniques leads more quickly to these results. Unfortunately this method does not work in 4 dimensions (although some progress has been made in 3 dimensions), whereas some of the direct methods described above should be applicable in one or the other form.

3. Multi-flavour Schwinger model with bag boundary conditions

If we repeat the above calculation for N flavours $u, d, \cdots$ we find that $\langle \bar{u}u \rangle = 0$ for all L and all temperatures, as required by the Mermin-Wagner theorem and the fact that the boundary conditions do not break the flavour $SU(N)$ explicitly. The technical reason for that is very simple and the same as in the naive instanton calculation in QCD. For example, for 2 flavours $\mathcal{Z}_F$ in (15a) becomes

$$\mathcal{Z}_F = \prod_1^k (\bar{\eta}, \psi_p)(\bar{\psi}_p, \eta) \prod_1^k (\bar{\theta}, \psi_p)(\bar{\psi}_p, \theta)\, e^{-\int \bar{\eta} G_c \eta - \int \bar{\theta} G_c \theta} \det{}'^2 (i\not{D}), \tag{27}$$

where η (θ) is the source for the u (d) 'quarks'. Since $\langle \bar{u}u \rangle$ is gotten by taking the derivative with respect to $\bar{\eta}$ and η and then setting all sources to zero, it follows at once that the u-condensate vanishes (recall that G_c is off-diagonal in a representation where γ_5 is diagonal).

To get a better understanding of how the multi-flavour system escapes the chiral symmetry breaking we must break the symmetry explicitly and study how the symmetry is restored when the breaking is removed. The $SU(N)_A$ symmetry can be broken explicitly by introducing small 'quark' masses or by assuming symmetry breaking boundary conditions. We prefer the second alternative since it allows for an analytic treatment of the problem.

Then we need to impose certain boundary conditions at $\partial\mathcal{M}$. If we assume that the fermionic boundary conditions are local and that $i\not{D}$ is selfadjoint (such that the partition function is real) then only the bag-boundary conditions remain [21]

$$\psi = i\gamma_5 e^{\theta\gamma_5} \gamma_n \psi \qquad \text{on} \quad \partial\mathcal{M}; \qquad\qquad \gamma_n = (n, \gamma) \tag{28}$$

where n is the outward oriented normal vector on $\partial\mathcal{M}$. Note that there is a 1-parametric family boundary conditions and it is no coincidence that we named the corresponding parameter θ. Actually the boundary conditions (28) break the chiral symmetry as required: if ψ obeys the boundary condition (28) then the transformed field $U\psi$, $U \in SU_A(N)$, does not obey them anymore.

If $\mathcal{M}$ has no holes the configuration space C is trivial and any gauge potential can be decomposed as

$$eA_0 = -\partial_1\phi + \partial_0\lambda \qquad eA_1 = \partial_0\phi + \partial_1\lambda \tag{29}$$

and (17) is replaced by

$$\mathcal{D}A = (2\pi)^2\det(-\triangle)\mathcal{D}\phi\mathcal{D}\lambda, \qquad \phi|_{\partial\mathcal{M}} = 0. \tag{30}$$

Furthermore, it can be shown that $i\not{D}$ has no zero modes obeying the bag boundary conditions (28) so that for $\mathcal{Z}_F$ we obtain (15a) but without zero-mode part and without primes. So the calculation is actua lly simpler as on the torus. The only problem is to find the fermionic determinant and Green function for the bag boundary condition. Using the chiral anomaly together with a generalization of the Feynman-Hellman theorem one can show that [25]

$$Z[\eta,\bar\eta] = \frac{\int \mathcal{D}\phi e^{-\Gamma[\phi]-\theta W[\phi]}\; e^{-\sum\int \bar\eta_i G_\theta \eta_i}}{\int \mathcal{D}\phi e^{-\Gamma[\phi]-\theta W[\phi]}}, \tag{31a}$$

where

$$\Gamma[\phi] = \frac{1}{2}\int \phi[\triangle^2 - Nm_\gamma^2\triangle]\phi \quad \text{and} \quad W[\phi] = \frac{e}{2\pi}\int E. \tag{31b}$$

Note that θ couples to the flux and thus plays (almost) the same role as the θ-parameter in QCD. Now we choose for $\mathcal{M}$ a ball with radius R. Then we can find the explicit form of the Green function G_θ (by using conformal techniques) and finally obtain for the chiral condensate

$$\langle \bar u P_\pm u\rangle = \mp\frac{1}{2\pi}\left[1-\frac{r^2}{R^2}\right]^{\frac{1}{N}-1} R^{\frac{1}{N}-1}\left[\frac{1}{2}\sqrt{N}m_\gamma\right]^{\frac{1}{N}} e^{\gamma/N}. \tag{32}$$

Note that for one flavour, $N=1$, this result is consistent with the zero temperature result (21), since for $R\to\infty$

$$\langle\bar\psi P_\pm\psi\rangle \longrightarrow \mp\frac{m_\gamma}{4\pi}e^\gamma. \tag{33}$$

More interestingly, for several flavours the condensate vanishes as

$$\langle\bar\psi P_\pm\psi\rangle \longrightarrow \text{const.}\, R^{\frac{1}{N}-1} \tag{34}$$

for large radii R of the euclidean spacetime. We see that for $R\to\infty$ the condensate vanishes, but rather reluctantly. This is reminiscent of spontaneous symmetry breaking characterizing similar theories in higher dimensions.

Actually, if one introduces small fermion masses a similar statement holds [24]. If μ is the mass of the lightest particle in the theory, then the condensate vanishes as $\mu^{1-1/N}$. We see that the chirality breaking radius R plays indeed the same role as small quark masses and that the size of spacetime should be identified with the Compton wavelength of the lightes particle in the massive multi-flavour model. Finally note that the asymptotics for the chiral condensate for large R or small μ is essentially nonperturbative in nature.

To learn more about the dependence on the imposed boundary conditions we reconsidered finite temperature QED_2, but now subject to bag boundary conditions [25]. That is we assumed $\mathcal{M} = [0, \beta] \times [0, L]$, imposed the boundary conditions (11a) in the euclidean time direction and the bag boundary condition (28) in the spacial direction. In this case the result for the chiral condensate depends on the order of limits L $\to \infty$ and $T \to 0$:

$$\lim_{L\to\infty}\lim_{\beta\to\infty}\langle\bar{\psi}P_\pm\psi\rangle = \begin{cases} \mp m_\gamma e^\gamma/4\pi & \text{N=1} \\ 0 & N > 1 \end{cases} \tag{35a}$$

but

$$\lim_{\beta\to\infty}\lim_{L\to\infty}\langle\bar{\psi}P_\pm\psi\rangle = 0 \qquad \text{all N.} \tag{35b}$$

Clearly such a behaviour can only occur for systems with long range order (see [26] for similar properties in spin-models). For example, for scalar theory with global continuous symmetry one has [27]

$$\lim_{j\to 0}\lim_{V\to\infty} \neq \lim_{V\to\infty}\lim_{j\to 0} \tag{36}$$

in the broken phase.

What have we learned from the above discussions (and the ones for the more general model (8) which I could not present here):

- Boundary effect are very important for gauge theories
- We better choose boundary conditions such that clustering holds. For example, the finite temperature bag boundary conditions are not of this type.
- Bag boundary conditions can probably play the same role as small quark masses.
- 4 dimensional realistic theories are not solvable by any known method. Thus one has to retreat to lattice simulations and/or try to find the dominant configuations. Starting from the Schwinger model some progress has been made which maybe relevant for QCD [28].

Simple models may still provide some insights which could help us to understand the structure of more realistic theories, like the properties of the *QCD* vacuum. The main problem is to find the right questions to ask

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VORTICES IN A CHERN-SIMONS THEORY WITH ANOMALOUS MAGNETIC MOMENT

Manuel Torres

Instituto de Física, Universidad Nacional Autónoma de México

Apartado Postal 20-364, 01000 México D.F., MEXICO

Abstract

Topological as well as nontopological vortices are obtained in an Abelian Chern-Simons (CS) model which includes both the CS term and an anomalous magnetic contribution. We show that nontopological solitons satisfy a set of Bogomol'nyi-type equations for a $\frac{m^2}{2}|\phi|^2$ potential, when m and the topological mass are equal.

I. INTRODUCTION

Vortex solutions in $2+1$ dimensions have recently received a lot of attention, one of the reasons for this is that charged vortices are in fact anyons.

It has been known for a long time that the Ginzburg-Landau model of superconductivity admits topological solitons of the vortex type [1]. This is also true for its relativistic generalization, *i.e.* the abelian Higgs model [2]. Characteristically, these vortices carry magnetic flux but are electrically neutral. Furthermore, when the parameters are chosen to make the vector and scalar masses equal, minimum energy vortex configurations arise that satisfy first order differential equations [3-4]. In this limit, known as the Bogomol'nyi limit, the vortices become non-interacting [5].

Vortex solutions also exist in 2+1 dimensions when a Chern-Simon term is added to the lagrangian of an abelian [6] or non-abelian [7] Higgs model. These CS vortices are different from the Nielsen-Olesen vortices in that they carry electric charge as well as magnetic flux. More recently topological and non-topological solitons have been studied in the Chern-Simons Higgs theory whitout the Maxwell term and the corresponding Bogomol'nyi limit were derived for an specific sixth-order Higgs potential [8-9].

In this work we consider a generalization of the abelian CS Higgs model, in which we add an extra non minimal contribution to the covariant derivative. The extra term couples the scalar field directly to the electromagnetic field tensor. The added term can be interpretated as an anomalous magnetic moment. Vortex solutions are obtained for this model in a particular limit in which the gauge field equations reduce to a set of first order differential equations. As we sall see, Bogomol'nyi equations are obtained for nontopological solitons, if the potential is chosen as $V(|\phi|) = \frac{m^2}{2}|\phi|^2$ with $m = \kappa$ (κ is the topological mass). These solitons carry finite energy, magnetic flux and angular momentum.

II. THE MODEL

We first define our theory by writing the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{4}\epsilon^{\mu\nu\alpha}A_\mu F_{\nu\alpha} + \frac{1}{2}|D_\mu\phi|^2 - V(|\phi|), \qquad (1)$$

where the most general renormalizable potential in $(2+1)$ dimensions is of the form

$$V(|\phi|) = a_6|\phi|^6 + a_4|\phi|^4 + a_2|\phi|^2 . \qquad (2)$$

Our notation is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \mu = (0,1,2), g_{\mu\nu} = diag(+1,-1,-1), \hbar = c = 1$. The generalized covariant derivative D_μ is given by

$$D_\mu\phi = (\partial_\mu - ieA_\mu - i\frac{g}{4}\epsilon_{\mu\nu\alpha}F^{\nu\alpha})\phi\,. \tag{3}$$

The interpretation of g as anomalous magnetic moment follows from the fact that in $(2+1)$ dimensions the Pauli coupling for a fermion field (ψ) can be written as [10] $\overline{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha}\overline{\psi}\gamma_\mu\psi F_{\nu\alpha}$. Therefore, it is an specific feature of $(2+1)$ dimensions that the Pauli coupling can be incorporated into a generalized covariant derivative, that can be used without any reference to a spin degree of freedom. In particular, it can be introduced for scalar fields. Notice that both the CS term and the anomalous magnetic moment introduce a violation of the time reversal (T) and parity (P) symmetries.

Suppose that the potential given in Eq. (3) is selected to have a symmetry-breaking minimum at $|\phi| = v$. The spontaneous symmetry breaking can be easily implemented if we write the scalar field as $\phi = (v+\eta)e^{(i\alpha)}$. The mass term for the gauge field is generated by this mechanism. But we also find that the spontaneous symmetry breaking will generate (among many others) a term of the form $(egv^2/4)\epsilon^{\mu\nu\alpha}A_\mu F_{\nu\alpha}$. Which can be interpretated as a C-S term, induced by the spontaneous symmetry breaking mechanism [11].

The equations of motion for the lagrangian in Eq. (1) are

$$\frac{1}{2}D_\mu D^\mu\phi = -\frac{\partial V}{\partial\phi^*}\,, \tag{4a}$$

$$\epsilon_{\mu\nu\alpha}\partial^\mu\left[F^\alpha + \frac{g}{2e}J^\alpha\right] = J_\nu - \kappa F_\nu\,. \tag{4b}$$

The last equation has been written in terms of the dual field, $F_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha}F^{\nu\alpha}$, and the conserved current J_μ is given by

$$J_\mu = -\frac{ie}{2}\left[\phi^* D_\mu\phi - \phi(D_\mu\phi)^*\right]\,. \tag{5}$$

The energy momentum tensor is obtained by varying the curved-space form of the action with respect to the metric

$$T_{\mu\nu} = \left(1-\frac{g^2}{4}|\phi|^2\right)\left(F_\mu F_\nu - \frac{1}{2}g_{\mu\nu}F_\alpha F^\alpha\right) + \frac{1}{2}\left(\nabla_\mu\phi(\nabla_\nu\phi)^* - g_{\mu\nu}\left[\frac{1}{2}|\nabla_\lambda\phi|^2 - V(|\phi|)\right] + H.c.\right), \tag{6}$$

where $\nabla_\mu = \partial_\mu - ieA_\mu$ includes only the gauge potential contribution.

There is a particular limit in which the gauge field equations (4b) reduces from second to first order differential equations. To obtain this limit, notice that the solutions of the first order equations

$$F_\mu = \frac{1}{\kappa} J_\mu , \tag{7}$$

are also solutions of the gauge field equations (4b) provided that the following relation holds:

$$\kappa = -\frac{2e}{g} . \tag{8}$$

Equation (7) coincides with the gauge field equations of the pure CS theory, *i. e.*, a theory without the Maxwell term [12]. The Gauss law (zero component of eq. 7) implies that solutions with charge Q are also tubes of magnetic flux Φ, with $\Phi = -Q/\kappa$. Vortex solutions for gauge field equations of type (7) have been considered in several papers [8-9], but in all the previous work, the anomalous magnetic contribution and the Maxwel term were not explicitely included into the lagrangian. Here Eq. (7) arises when these two terms are included and the constraint given in Eq. (8) holds. Furthermore, the explicit expression for J_μ differs from previous expressions, because according to Eqs. (5) and (3), J_μ receives contributions from the anomalous magnetic term.

III. PROPAGATING MODES

The model described in the previous section has in general three propagating modes in the broken symmetry phase. The Higgs-field has one propagating excitation with mass $m_s = \sqrt{2a_2}$

In order to describe the particle content of the gauge field degrees of freedom consider that the vacuum configuration in the broken phase is selected in a gauge such that $\phi = v$. In this case the conserved current reduces to

$$J_\mu = -ev^2\left[eA_\mu + \frac{g}{2}F_\mu\right]. \tag{9}$$

Consider an plane wave solution for the gauge potential $A_\mu = \epsilon_\mu e^{ik\cdot x}$, where $k^\mu = (\omega, \vec{k})$, and we choose the x axis along the direction of the vector $\vec{k}$. Inserting this expression and J_μ given by Eq. (9) in the field equation (4b) we find

$$\left[e^2v^2g_{\mu\nu} + (1 - \frac{v^2g^2}{4})(k_\mu k_\nu - k\cdot kg_{\mu\nu}) + i(ev^2g + \kappa)\epsilon_{\mu\lambda\nu}k^\lambda\right]\epsilon^\nu = 0. \tag{10}$$

From this equation we find the dispertion relation $\omega = \sqrt{|\vec{k}|^2 + m_\pm^2}$ where the two photon masses are given by

$$m_\pm = \frac{\pm(\kappa + ev^2g) + \sqrt{(\kappa + ev^2g)^2 + 4e^2v^2(1 - v^2g^2/4)}}{2(1 - v^2g^2/4)}. \tag{11}$$

Therefore, the gauge field acquieres two propagating modes with distinct masses m_+ and m_- [13]. From Eq. (11) we observe that in order to have two distinct masses it is requiered to have both spontaneous symmetry breaking (v is the symmetry-breaking minimim) and at least one of the P and T violating terms; i.e. the CS term or the anomalous magnetic term.

The two values for the photon mass are related with two different polarizations of the electromagnetig wave. From Eqs. (10) and (11) we find that the electric field ($E^i = F^{0i}$) is determined as

$$\vec{E} = \vec{E}_0 \left(\pm \frac{im_\pm}{\omega} , 1 \right) e^{ik \cdot x} . \tag{12}$$

Therefore, there are two different ellipitical polarized solutions. The difference of the two solutions is a consequence of the P, T breaking properties of the κ and g terms.

IV. TOPOLOGICAL VORTICES

Looking for rotationally symmetric solutions of vorticity n, we consider the ansatz

$$\begin{aligned} \vec{A}(\vec{\rho}) &= -\hat{\theta} \frac{\left(a(\rho) - n\right)}{e\rho} , \qquad A_0(\vec{\rho}) = \frac{\kappa}{e} h(\rho), \\ \phi(\vec{\rho}) &= \frac{\kappa}{e} f(\rho)\, e^{-in\theta} . \end{aligned} \tag{13}$$

When this ansatz is substituted into the equations of motion (4), one gets a system of second order differential equations. Although it is a very complicated system of equations one can demonstrate, that for an appropiated selection of the potential $V(|\phi|)$, the system has both topological and nontopological soliton solutions [14]. We leave the analysis of these for elsewhere.

In what follows, we restrict ourselves to the pure CS limit. We assume that eq. (8) holds and that the system is described by the equations of motion ($4a$) and (7). After a lengthly calculation it can be shown that for the ansatz (13) the equations of motion ($4a$) and (7) reduce to

$$\frac{1}{\rho}(1 - f^2)\frac{da}{d\rho} + \kappa^2 f^2 h = 0 , \tag{14a}$$

$$\rho(1-f^2)\frac{dh}{d\rho}+f^2a=0\,, \tag{14b}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{df}{d\rho}\right]-\frac{\kappa^2 f}{(1-f^2)^2}\left(\frac{a^2}{\kappa^2\rho^2}-h^2\right)=\frac{e^2}{\kappa^2}\frac{\partial V}{\partial f}\,. \tag{14c}$$

We assume that the parameters in the potential are selected in such a way that $V(|\phi|)$ has a minimum at $|\phi|=v$. The appropiated boundary conditions at the origin follow from the requierement that the fields be nonsingular. This implies that $a(0)=n$ and $nf(0)=0$. At infinity, the condition for finite energy implies $f(\infty)=ev/\kappa$ and $a(\infty)=0$. We do not have exact analytical solution for the coupled system of equations (14). However, it is not difficult to find asymptotic solutions. In fact for large distances the fields approach their asymptotic values exponentially:

$$a(\rho)=\kappa C\rho K_1(\mu\rho)\,, \tag{15a}$$

$$h(\rho)=CK_0(\mu\rho)\,, \tag{15b}$$

where K_0 and K_1 are the modified Bessel functions, C is a constant and $\mu=\kappa\frac{e^2v^2}{\kappa^2-e^2v^2}$ is the vector meson mass. Notice that in the present limit, only one propagating mode for the gauge field survivies. In the limit $\kappa\gg ev$ this mass reduces to $\mu=\frac{e^2v^2}{\kappa}$, which is the result obtained by Hong *et.al.*, and Jackiw and Weinberg [8]. In fact, we only consider the case in which $\kappa>ev$, otherwise the energy density is not positive definite. For the small ρ behavior we obtain

$$f(\rho)=f_n(\kappa\rho)^{|n|}+\frac{(2a_2/\kappa^2-h_0^2)f_n}{4(1+|n|)}(\kappa\rho)^{|n|+2}+O\big((\kappa\rho)^{|n|+4}\big)\,, \tag{16a}$$

$$a(\rho)=n-\frac{h_0 f_n^2}{2(|n|+1)}(\kappa\rho)^{2|n|+2}+O\big((\kappa\rho)^{2|n|+4}\big)\,, \tag{16b}$$

$$h(\rho) = h_0 - \frac{f_n^2}{2}(\kappa\rho)^{2|n|} + O((\kappa\rho)^{2|n|+2}), \tag{16c}$$

where f_n and h_0 are constants and a_2 is the coefficient of the $|\phi|^2$ term in the potential.

Some of the physical properties of the vortices can be explicitely calculated. Using the equations of motion and the boundary conditions for a vortex with winding number n we find that the magnetic flux is quantized

$$\Phi = \int F^{12} d^2x = 2\pi n/e, \tag{17}$$

similarly the electric charge is given by $Q = \int J_0 d^2x = -\kappa\Phi$.

The spin (total angular momentum) is obtained from the gauge invariant, symmetric energy-momentum tensor

$$L = \int d^2x \left(\epsilon^{ij} x_i T_{0j} \right) = -\pi\kappa \left(\frac{n}{e} \right)^2 = \frac{Q\Phi}{2\pi}. \tag{18}$$

Notice that the spin is related with the Bohm-Aharanov factor $Q\Phi$, that will appear when one vortice goes around another one, according to what is expected from the spin-statistic relation.

The magnetic moment that will appear as the linear coupling of the vortex solution to an external magnetic field is given by

$$M = \int d^2x \left(\epsilon^{ij} x_i J_j \right) = \frac{2\pi\kappa^2}{e^2} \int \rho h(\rho) d\rho. \tag{19}$$

Notice that the magnetic flux, electric charge and spin can be considered topological properties of the vortices, in the sense that they only depend on the boundary conditions and they are not sensitive to the parameter of the model. On the other hand the magnetic moment depends on the details of the solution, and therefore it is not a topological quantity.

Further properties of the topological vortices and a detailed numerical analysis of the solutions are discussed in reference [14].

V. BOGOMOL'NYI LIMIT FOR NONTOPOLOGICAL VORTICES

Let us consider now the possibility of obtaining a Bogomol'nyi-type limit for the equations of motion. The energy functional that is obtained from eq. (6) for the ansatz (13) is

$$E = \frac{\kappa^2}{2e^2}\int d^2x\Big[(1-f^2)\big[(h')^2 + (\frac{a'}{\kappa\rho})^2\big] + (\kappa f h)^2 + (f')^2 + (\frac{fa}{\rho})^2 + \frac{2e^2}{\kappa^2}V(f)\Big], \quad (20)$$

where primes denote differentiation with respect to ρ. This energy functional will be positive definite for soliton configurations that fulfill the condition $max(f) < 1$. We can rearrange the energy functional *à la* Bogomol'nyi, to get

$$E = \int d^2x\Big[\frac{(1-f^2)}{2e^2f^2}\Big(\frac{a'}{\rho}\mp\frac{\kappa^2f^2}{(1-f^2)^{\frac{1}{2}}}\Big)^2 + \frac{\kappa^2}{2e^2}\Big(f'\pm\frac{fa}{\rho(1-f^2)^{\frac{1}{2}}}\Big)^2 + V(f) - \frac{\kappa^4}{2e^2}f^2\Big] \pm \frac{\kappa^2}{e}\Phi, \quad (21)$$

where the field h has been eliminated by using the equations of motion (14a) and (14b).

We consider only those configurations that satisfies the condition $max(f) < 1$. Then, there is a lower bound on the energy $E \geq \frac{\kappa^2}{e}|\Phi|$, provided that the potential is chosen as $V(|\phi|) = \frac{m^2}{2}|\phi|^2$ with the critical value $m = \kappa$, *i.e.* when the scalar and topological masses are equal. Therefore, in this limit we are necessarily in symmetric phase of the theory. The lower bound for the energy

$$E = \frac{\kappa^2}{e}|\Phi| = \frac{\kappa}{e}Q \quad (22)$$

is saturated when the following Bogomol'nyi equations are satisfied [16]:

$$\frac{a'}{\rho} = \pm\frac{\kappa^2 f^2}{(1-f^2)^{\frac{1}{2}}}, \quad (23a)$$

$$f' = \mp \frac{fa}{\rho(1-f^2)^{\frac{1}{2}}}\,. \tag{23b}$$

Notice that, in this Bogomol'nyi limit the $T_{\rho\rho}$ and $T_{\theta\theta}$ components of the energy-momentum tensor vanish. In this symmetric phase, topological solitons certainly do not exist, but the theory allows nontopological solitons [15]. We now analyse these solutions.

Combining eqs. (23a)and (23b), we get

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{df}{d\rho}\right] = \frac{1}{f(1-f^2)}\left[\left(\frac{df}{d\rho}\right)^2 - \kappa^2 f^4\right]. \tag{24}$$

If we consider the case of small f we can approximate $(1-f^2)^{-1} \approx 1$ in Eq. (24). Then, Eq. (24) reduces to the rotationally symmetric form of the Liouville's equation, which has the following solution

$$f(\rho) = \frac{2N}{\kappa\rho}\left[\left(\frac{\rho}{\rho_0}\right)^N + \left(\frac{\rho_0}{\rho}\right)^N\right]^{-1}, \tag{25}$$

where N and ρ_0 are arbitrary constants.

For arbitrary f, we have that at spatially infinity finiteness of the energy implies that $f(\infty) = 0$, and therefore the value $a(\infty) = -\alpha$ is not constrained. The large distance behavior of the solution yields

$$f(\rho) = \frac{C}{(\kappa\rho)^\alpha} - \frac{C^3}{4(\alpha-1)^2(\kappa\rho)^{3\alpha-2}} + O\big((\kappa\rho)^{-5\alpha+4}\big)\,, \tag{26a}$$

$$a(\rho) = -\alpha + \frac{C^2}{2(\alpha-1)(\kappa\rho)^{2\alpha-2}} - O\big((\kappa\rho)^{-4\alpha+4}\big)\,. \tag{26b}$$

We have two categories of solutions:

i) $n = 0$, so that $a(0)$ must vanish in order for the solution be nonsingular, but $f(0) = f_0$ is not so constrained. These are nontopological solitons that are characterized by the value of the magnetic flux $\Phi = \frac{2\pi}{e}|\alpha|$. The large-distance behavior is given by eqs. (26), while as $\rho \to 0$ we obtain a power-series solution

$$f(\rho) = f_0 - \frac{f_0^3}{4(1-f_0^2)}(\kappa\rho)^2 + \frac{f_0^5(4-f_0^2)}{64(1-f_0^2)^3}(\kappa\rho)^4 + O((\kappa\rho)^6)\,, \tag{27a}$$

$$a(\rho) = -\frac{f_0^2}{2(1-f_0^2)^{\frac{1}{2}}}(\kappa\rho)^2 + \frac{f_0^4(2-f_0^2)}{16(1-f_0^2)^{\frac{5}{2}}}(\kappa\rho)^4 + O((\kappa\rho)^6)\,. \tag{27b}$$

Acceptable soliton solutions exist for values of f_0 in the range $0 < f_0 < 1$. The short- and large-distance behavior of the solutions are related, since α is a function of f_0. When $f_0 \ll 1$ the Liouville's solution becomes exact, and one finds that $\alpha \to 2$. While, as $f_0 \to 1$ we find by numerical integration that $\alpha \to 1.755$. Therefore, the magnetic flux varies continuously between $\Phi = 0.877(4\pi/e)$ and $\Phi = 4\pi/e$.

ii) $n \neq 0$, so $f(0)$ must vanish and $a(0) = n$. At large distances they behave in the same way as the nontopological solitons Eqs. (26). The short-distance behavior is similar to the vortex solutions, and can be obtained from Eqs. (16) with the substitutions: $a_2 = \kappa^2/2$ and $h_0 = 1$. Following Jackiw *et.al.* [9] we refer to these solutions as nontopological vortices. These vortices are characterized by the value of the magnetic flux $\Phi = \frac{2\pi}{e}(n+\alpha)$, which need not be quantized. For each integer n there will be a continuous set of solutions corresponding to the range $0 < f_n < f_n^{max}$. For values such that $f_n > f_n^{max}$ there are no real solutions to the field equations (23), because the condition $f < 1$ is not satisfied for all ρ. For $f_n \ll 1$, $f(\rho)$ is small for all ρ and can therefore be approximated by the solution (25) of the Liouville equation, in particular we can determine the large ρ behavior of the solution (Eq. 26) with $\alpha \to n+2$. In fact, this value is an upper bound, since $\alpha < n+2$. On the other hand, as $f_n \to f_n^{max}$ we find that α tends to a minimum value α^{min}. Therefore, for each integer n the flux varies continuously between $\Phi_n^{min} = \frac{2\pi}{e}\left[n + \alpha_n^{min}\right]$ and $\Phi_n^{max} = \frac{4\pi}{e}\left[n+1\right]$.

Notice that in the present limit the vortices become noninteracting. This can be easily understood using Eq. (22). Let us consider two solitons of charges Q_1 and Q_2 of the same sign that are far apart. According to Eq. (22) their total energy is $E = \frac{\kappa}{e}(Q_1 + Q_2)$. If we now consider that the two vortices are superimposed at the same point, due to charge conservation, the resulting configuration will represent a vortex solution of charge Q_1+Q_2. Then according to Eq. (22) the total energy will be again $E = \frac{\kappa}{e}(Q_1 + Q_2)$. Therefore, we conclude that the vortices are noninteracting.

The condition $m = \kappa$ represents a transition between a phase in which vortices atract and a phase in which they repel each other, similar to the transition between type I and type II superconductors. In fact, what we have demonstrated is that in the present theory for an scalar field potential of the form $V(|\phi|) = \frac{m^2}{2}|\phi|^2$ the atraction between vortices due to the interaction through the scalar field has the same strength as the repulsion due to the interaction through the vector field. Therefore when the range of the two interactions is the same ($m = \kappa$) the vortices become noninteracting. On the other hand if the range of the scalar interaction is smaller than the range of the vector interaction ($m > \kappa$) the inter-vortex potential is repulsive; while for $m < \kappa$ the potential is atractive.

VI. CONCLUSIONS

In conclusion , we presented a $(2+1)$ dimensional Abelian Chern-Simons model which includes and anomalous magnetic coupling between the scalar field and the gauge field. It was demonstrated that for a particular relation between the CS coupling and the anomalous magnetic moment (Eq. 8) the gauge field equations reduce from second- to first-order differential equations, of the pure CS type.

Soliton solutions of the vortex type were found both in the broken and the symmetric phase of the model. These vortices are characterized by their "quantum" numbers: electric charge, magnetic flux and spin, that are of topological origen, and that in the broken phase are quantized. Furthermore the vortices carry a magnetic moment that depends on the precise form of the solutions.

A Bogomol'nyi limit was obtained for nontopological solitons , for a ϕ^2 potential when the scalar and the topological masses are equal. The flux spectrum consist of bands of finite width where the value $\Phi = 4\pi(n+1)/e$ corresponds to an upper bound. Similarly the energy spectrum consist of bands of finite width. According to Eq. (22) the soliton energy is proportional to the charge of the vortices, this fact is reminiscent of the Coleman's Q-ball solitons [15].

Further properties of topological and nontopological vortices and a detailed numerical analysis of the solutions are discussed in references [14] and [16]. The stability of the nontopological solitons and the description of the multisolitonic solutions, besides many other aspects that deserves further investigation are currently under study.

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Quantum field theory in toroidal spacetime

Klaus Kirsten*
Universität Kaiserslautern
Fachbereich Physik
6750 Kaiserslautern
Federal Republic of Germany

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Abstract

First we consider the finite temperature quantum field theory of a conformally noninvariant spin-0 gas in an arbitrary ultrastatic spacetime $\mathcal{M}$ with a nonvanishing background charge. In the Euclidean formulation this means doing quantum field theory in $S^1 \times \mathcal{M}_S$, where S^1 represents the imaginary time compactified to a circle of size β (β is the inverse temperature) and $\mathcal{M}_S$ is some $n-1$-dimensional Riemannian manifold. We are especially interested in the high temperature regime of the theory. Using zeta function regularization and heat kernel techniques we calculate the high temperature expansion of the grand thermodynamic potential to any power of the inverse temperature. As an application the phenomenon of Bose Einstein condensation is considered.

Choosing (mathematically) imaginary background charge one is effectively doing Abelian gauge field theory of a massless gauge field. In generalization to previous considerations we consider the gauge field theory in $T^N \times \mathcal{M}_S$ (with arbitrary compactification lenghts of the toroidal components). Due to the nontrivial topology, the interaction of the quantum fluctuations with the gauge field generates a gauge field mass and the dependence on the involved parameters is given.

*Address after October 1, 1992: Università degli Studi di Trento, Dipartimento di Fisica, 38050 Povo (Trento), Italia

1 Introduction

Quantum field theory in partially compactified spacetime plays a fundamental role in various contexts. Let us just mention
i.) finite temperature quantum field theory in the Euclidean formulation, where the imaginary time is compactified to a circle of size β (β is the inverse temperature) (see e.g. [1-5]),
ii.) Casimir energy calculations, where the sign of the energy strongly depends on the number of compactified dimensions (see e.g. [6-17]),
iii.) topological symmetry breaking or restoration and topological mass generation (see e.g. [18-20] and references therein).
Some of these aspects will be considered in this contribution. First we consider finite temperature quantum field theory of a massive noninteracting non-conformally invariant spin-0 gas in an arbitrary ultrastatic spacetime, where the spatial section may has a boundary. We will assume nonvanishing background charge. This has been introduced first by Haber and Weldon [21], [22] (see also [23]) in the context of a free relativistic bose gas in Minkowski spacetime. They found, that Bose Einstein condensation may occur at high temperature, where high means large compared to the mass of the field (we use units $\hbar = c = k = 1$). This remains true in a general static spacetime $S^1 \times \mathcal{M}_S$, where $\mathcal{M}_S$ is a $n-1$-dimensional compact Riemannian manifold, possibly with a boundary [24], [25].
To show this we determine the high temperature expansion of the grand thermodynamic potential of the spin-0 gas. Using zeta function regularization and heat kernel techniques, the expansion is given in any power of the inverse temperature in terms of the Minakshisundaraum-Pleijel coefficients of the heat kernel [26-34] (for related work see also [35-41]). After some introductory remarks concerning finite temperature quantum field theory in ultrastatic spacetime, this is done in section 2. Analogous results may be found for a spin-$\frac{1}{2}$ gas [28], [34]. Furthermore using conformal transformation techniques [27], [28], [42-48], the result may be generalized to a static spacetime with boundary [28], [34].
Based on the high temperature expansion of the grand thermodynamic potential the phenomenon of Bose-Einstein condensation is considered in section 3 (for several interesting references on that subject see [24]). Let us mention that an interpretation of Bose-Einstein condensation in terms of symmetry breaking to give a non-constant scalar field vacuum expectation value was provided very recently by Toms [24], [25].
Choosing imaginary background charge one is led to Abelian gauge field theory of a massless gauge field. As is well known a quantum field ϕ, whose fluctuations are constrained by boundaries or by nontrivial spacetime topo-

logy, when coupled to a massless gauge field A, can give this field A a quantum mass m_T. This quantum mass may be real or imaginary depending on the nature of spacetime as seen by the constrained field ϕ. In generalization to Actor [18], in section 4 we will choose a massive complex scalar field ϕ defined on $T^N \times \mathcal{M}_S$ with periodic boundary conditions in each of the toroidal components with compactification lengths $L_1, ..., L_N$, minimally coupled to a constant gauge potential.
In that context it is shown that for a massive field ϕ the topologically generated mass is always real. But as we will also see in section 5, for a massless quantum field ϕ the situation changes and generation of imaginary mass is possible.

2 Finite temperature quantum field theory in static spacetime

We shall first concern ourselves with the finite-temperature behaviour of a field theory in the n-dimensional ultrastatic spacetime $\mathcal{M}$

$$ds^2 = -dx_0^2 + g_{ab}(\vec{x})dx^a dx^b, \tag{2.1}$$

where $\vec{x} = (x_1, ..., x_n)$. The action of the field theory we consider is [49], [50],

$$S = -\frac{1}{2}\int d^n x |g|^{\frac{1}{2}} \phi(x)\left(\Box - \xi R - m^2\right)\phi(x), \tag{2.2}$$

with the Laplace-Beltrami operator $\Box$ of the ultrastatic spacetime. Variation of equation (2.2) subject to the constraints

$$\begin{aligned} \delta\phi(x') &= 0, & (2.3)\\ n^{\mu'}\nabla_{\mu'}\delta\phi(x') &= 0, & (2.4)\end{aligned}$$

where the prime refers to quantities defined on the boundary $\partial\mathcal{M}$, yields the equation of motion

$$\left(\Box - \xi R - m^2\right)\phi(x) = 0. \tag{2.5}$$

The following discussion will be quite general, so the boundary condition need not to be specified at this point. But a unique boundary value problem is for example posed by assuming Dirichlet- or Robin-boundary conditions for the field.
In a static spacetime the positive frequency part of the field is determined by the timelike Killing vector and the Hamiltonian without normal ordering

is

$$\begin{aligned}\hat{H} &= \sum_j E_j \left[a_j^\dagger a_j + b_j^\dagger b_j + 1\right] \\ &= \sum_j E_j \left[N_{+,j} + N_{-,j} + 1\right].\end{aligned} \tag{2.6}$$

Here the a operators annihilate particles, the b operators annihilate antiparticles, N_+ (respectively N_-) are the particle (respectively antiparticle) number operators and E_j are the energy eigenvalues determined by

$$\left(\Delta - \xi R - m^2\right)\psi_j(\vec{x}) = -E_j^2 \psi_j(\vec{x}). \tag{2.7}$$

The charge number operator is

$$\hat{Q} = \sum_j \left[N_{+,j} - N_{-,j}\right]. \tag{2.8}$$

As usual the particle-antiparticle system in thermal equilibrium at a finite temperature T is described through the grand partition function

$$Z = Tr \exp\left[-\beta\left(\hat{H} - \mu\hat{Q}\right)\right], \tag{2.9}$$

where μ is the chemical potential associated with the conserved charge. Formal manipulation yields the thermodynamic potential

$$\begin{aligned}\Psi[\beta,\mu] &= -\frac{1}{\beta}\ln Z \\ &= \frac{1}{\beta}\sum_j \left[\ln\left(1 - e^{-\beta[E_j+\mu]}\right) + \ln\left(1 - e^{-\beta[E_j-\mu]}\right)\right] \\ &\quad + \sum_j E_j.\end{aligned} \tag{2.10}$$

Expression (2.10) contains the divergent zero point energy of the field, which has to be regularized (see the following discussion). From (2.10) we find the thermal average of the charge density to be

$$\begin{aligned}Q &= -\frac{1}{V}\left(\frac{\partial\Psi[\beta,\mu]}{\partial\mu}\right)_{T,V} \\ &= \frac{1}{V}\sum_j \left[\frac{1}{\exp\{\beta[E_j-\mu]\} - 1} - \frac{1}{\exp\{\beta[E_j+\mu]\} - 1}\right].\end{aligned} \tag{2.11}$$

The realization that the particle density in any energy level has to be non-negative leads to the important conclusion $|\mu| \leq E_0$, where E_0 denotes the smallest energy eigenvalue.

Having briefly developed the general formalism of the quantum field theory in an ultrastatic spacetime, the problem now is of course to extract some information out of equations (2.10) and (2.11). For highly symmetric configurations the energy eigenvalues of equation (2.7) are known explicitely and an analysis of equations (2.10) and (2.11) is possible due to that knowledge [21], [51], [52], [53] . In these considerations generalized Epstein zeta functions

$$E_N^{M^2}(s;a_1,\dots,a_N;c_1,\dots,c_N) = \sum_{n_1,\dots,n_N=\left\{\substack{1\\0\\-\infty}\right.}^{\infty} \left[a_1(n_1-c_1)^2+\dots+a_N(n_N-c_N)^2+M^2\right]^{-s}$$

play an essential role (the range of the summation indices depending on the boundary conditions imposed on the field) and nowadays a very good knowledge of these functions is attained [54], [55], [18] and references therein (see also the contributions of Elizalde and of Mostepanenko in this workshop).
But some properties of the field theory may be found without refering to specific spacetimes and boundary conditions. For motivation let us remind you of the considerations by Stewardson and Waechter [56], [57] . They showed, that the frequency spectrum of a stretched membrane contains for example the information about the area, the perimeter and the number of holes of the membrane.
In the same way, the energy spectrum of the gas (which determines the grand thermodynamic potential) contains some information about the volume of the spacetime, the boundary and the curvature tensors of the manifold, or vice versa the grand thermodynamic potential is determined by specific geometrical properties of the spacetime.
To show this, we will regularize equation (2.10) using the zeta function prescribtion [58], [59]. In that scheme, the finite thermodynamic potential is defined by

$$\psi[\beta,\mu] = \frac{1}{\beta}\left\{\zeta_D(0,\beta,\mu)\ln\lambda^2 - \zeta_D'(0,\beta,\mu)\right\}, \tag{2.12}$$

where λ is a scaling length, the prime denotes differentiation with respect to the first argument (this means with respect to s, see equation (2.14)) and $\zeta_D(s,\beta,\mu)$ is the zeta function associated with the operator (τ is imaginary time)

$$D = -\left(\frac{\partial}{\partial\tau}-\mu\right)^2 - \Delta + \xi R + m^2. \tag{2.13}$$

That means

$$\zeta_D(s,\beta,\mu)=\sum_m \nu_m^{-s}=\frac{1}{\Gamma(s)}\sum_m\int_0^\infty dt\, t^{s-1}\exp(-\nu_m t) \tag{2.14}$$

valid for $\Re s>\frac{n}{2}$, with

$$\nu_{l,j}=-\left(\frac{2\pi i l}{\beta}-\mu\right)^2+E_j^2 \qquad ,l\in\mathbb{Z}. \tag{2.15}$$

It may be shown, that definition (2.12) agrees with equation (2.10), once the zero point energy has been regularized in the way proposed in [9].
In order to analyze the phenomenon of Bose-Einstein condensation we concentrate now on the high temperature expansion of the theory. At high temperature, this means small β, only small values of the parameter t essentially contribute to the integral (apart from the summation index $l=0$, which gives only a zero temperature contribution) and the use of the short-time asymptotic expansion of the integrated proper-time propagator [60], [61], [62],

$$\begin{aligned} K(t) &= \sum_j \exp(-E_j^2 t) \\ &\underset{t\to 0}{\sim} \left(\frac{1}{4\pi t}\right)^{\frac{n-1}{2}} \sum_{j=0,\frac{1}{2},1,\dots} C_j t^j \end{aligned} \tag{2.16}$$

yields the expansion we are looking for. Details of the calculation and the full result may be found at several places [26-28], [33], [34].
Here we will just concentrate on the dominant terms of the result, which for $n\geq 4$ reads

$$\begin{aligned} \Psi[\beta,\mu] &= -\beta^{-n}\pi^{-\frac{n}{2}}\times \\ &\left\{2C_0\Gamma\left(\frac{n}{2}\right)\zeta_R(n)+\beta C_{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)\zeta_R(n-1)\right. \\ &+\beta^2\zeta_R(n-2)\left[C_0\Gamma\left(\frac{n}{2}\right)\mu^2+\frac{1}{2}C_1\Gamma\left(\frac{n}{2}-1\right)\right] \\ &\left.+\dots\right\}. \end{aligned} \tag{2.17}$$

But in principle this expansion is known explicitly up to the order containing the heat-kernel coefficient C_2, which has been determined recently for several boundary conditions by Branson, Gilkey [63] and Dettki, Wipf [64] (see also [65], [66], [67]).
It is important to mention, that the expansion in equation (2.17) is only consistent, if in addition to the flat spacetime condition $T\gg m$ also $T\gg |R|^{\frac{1}{2}}$ holds, where $|R|$ is the magnitude of a typical curvature of the spacetime [24].

3 Bose-Einstein condensation in static space-time

Using equation (2.17) it is now easy to study the phenomenon of Bose-Einstein condensation [24], [25].
First one finds

$$Q = 2\pi^{-\frac{n}{2}}\zeta_R(n-2)\Gamma\left(\frac{n}{2}\right)\mu\beta^{-n+2} + \dots \tag{3.1}$$

Fixing the total charge, for T high enough it is always possible to satisfy equation (3.1) with $\mu^2 < E_0^2$. But as T decreases μ must increase until we reach the temperature at which $\mu^2 = E_0^2$. This defines the critical temperature T_C at which Bose-Einstein condensation takes place. The critical temperature T_C is easily found to be [24]

$$T_C = \left(\frac{\pi^{\frac{n}{2}}Q}{2\zeta_R(n-2)\Gamma\left(\frac{n}{2}\right)E_0}\right)^{\frac{1}{n-2}} \tag{3.2}$$

and for $T \leq T_C$ the charge density Q_0 in the ground state is given by [24]

$$Q_0 = Q\left[1 - \left(\frac{T}{T_C}\right)^{n-2}\right]. \tag{3.3}$$

More explicit results for several contributions are attainable and have been given for example by Toms [24] and in [33].

4 Abelian gauge field theory on $T^N \times \mathcal{M}_S$

The transition from finite temperature field theory in the Euclidean formulation with nonvanishing background charge to Abelian gauge field theory on $S^1 \times \mathcal{M}_S$ is formally obtained by the replacement $\mu \to iA$ [51]. Due to the nontrivial topology constant values of the gauge potential A are physical parameters and the effective potential of the gauge theory will depend on these parameters [1]. Because the main results remain valid, we will consider an Abelian gauge field theory on $T^N \times \mathcal{M}_S$, with compactification lengths $L_1, \dots, L_N$ of the N-dimensional torus and with constant gauge potentials $A_1, \dots, A_N$ of the toroidal components.
In generalization to equation (2.12) the effective potential is defined by

$$V_{T^N} V_{\mathcal{M}_S} V_{eff} = \zeta_A(0)\ln\lambda^2 - \zeta_A'(0). \tag{4.1}$$

[1]To ensure gauge invariance, constant values of A have to be interpreted as $\oint A$ [68]

Here $\mathcal{V}_{T^N}$ (respectively $\mathcal{V}_{\mathcal{M}_S}$) is the volume of the torus (respectively the manifold $\mathcal{M}_S$) and $\zeta_A(s)$ is the zeta function associated with the operator

$$D_A = -\sum_{i=1}^{N}\left(\frac{\partial}{\partial x_i} - iA_i\right)^2 - \Delta + \xi R + m^2 \tag{4.2}$$

with eigenvalues

$$\nu_{l_1,\ldots,l_N j} = \left(\frac{2\pi l_1}{L_1} - A_1\right)^2 + \ldots + \left(\frac{2\pi l_N}{L_N} - A_N\right)^2 + E_j^2, \tag{4.3}$$

$l_1, \ldots, l_N \in \mathbb{Z}$. Using the approach described in section 2 it is again possible to derive an approximate expression for the effective potential (4.1) valid for small compactification lenghts (for $\mathcal{M}_S = \mathbb{R}^n$ an exact treatment has been given in [69]). But we are interested in the topologically generated masses m_{T_i} of the toroidal components A_i of the gauge potential, defined as the coefficients of the quadratic terms in the expansion

$$\begin{aligned} V_{eff} &= \sum C_{k_1\ldots k_N} A_1^{k_1}\ldots A_N^{k_N} \\ &= \frac{1}{2g^2}\left[m_{T_1}^2 A_1^2 + \ldots + m_{T_N}^2 A_N^2\right] + \text{non quadratic terms} \end{aligned} \tag{4.4}$$

of the effective potential. Here g is the gauge coupling with dimension $(mass)^{2-\frac{N+n}{2}}$.

The most important property of the masses is the sign of their square. Restricting our attention from the beginning of the calculation on this sign, we will provide a very elegant way of its determination.

Let us first concentrate on $E_0^2 > 0$. Introducing dimensionless parameters

$$w_i = \left(\frac{L_1}{L_i}\right)^2, \qquad u_i = \frac{A_i L_i}{2\pi},$$

the quadratic terms in the gauge potentials A_i of the zeta function $\zeta_A(s)$ are easily derived,

$$\begin{aligned} \zeta_A(s) &= \left(\frac{L_1}{2\pi}\right)^{2s} \sum_{l_1,\ldots,l_N=-\infty}^{\infty} \sum_j \frac{1}{\Gamma(s)} \times \\ & \int_0^\infty dt\, t^{s-1} \exp\left\{-\left[(\sqrt{w_1}l_1 - u_1)^2 + \ldots + (\sqrt{w_N}l_N - u_N)^2 + \left(\frac{E_j L_1}{2\pi}\right)^2\right] t\right\} \\ &= -s\left(\frac{L_1}{2\pi}\right)^{2s} \sum_{i=1}^{N} u_i^2 \left\{1 + 2w_i \frac{\partial}{\partial w_i}\right\} \zeta_0(s+1) \\ & +\text{non quadratic terms,} \end{aligned} \tag{4.5}$$

with

$$\zeta_0(s) = \sum_{l_1,\ldots,l_N=-\infty}^{\infty} \sum_j \left[w_1 l_1^2 + \ldots + w_N l_N^2 + \left(\frac{E_j L_1}{2\pi}\right)^2 \right]^{-s}. \tag{4.6}$$

As a result, the topologically generated mass may be given in the form

$$\begin{aligned} m_{T_j}^2 &= \frac{2g^2}{\mathcal{V}_{T^N} \mathcal{V}_{\mathcal{M}_S}} \left(\frac{L_j}{2\pi}\right)^2 \left\{ 1 + 2w_j \frac{\partial}{\partial w_j} \right\} \times \\ & \left[PP\ \zeta_0(1) - \ln\left(\frac{2\pi\lambda}{L_1}\right)^2 Res\ \zeta_0(1) \right] \end{aligned} \tag{4.7}$$

where PP (respectively Res) denotes the finite part (respectively the residuum) of $\zeta_0(1)$. So we have already reduced the determination of the topologically masses to an understanding of the zeta function $\zeta_0(s)$, equation (4.6). To find the essential information about $\zeta_0(s)$ relevant for equation (4.7), we have to construct an analytical continuation of equation (4.6) valid at $s = 1$. But this continuation is easily found by employing for $t \in \mathbb{R}_+$,

$$\sum_{n=-\infty}^{\infty} \exp(-tn^2) = \left(\frac{\pi}{t}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi^2 n^2}{t}\right), \tag{4.8}$$

which is due to Jacobi's relation between theta functions [70].
The result is

$$\begin{aligned} \zeta_0(s) &= \frac{\pi^{\frac{N}{2}}}{\sqrt{w_1 \ldots w_N}} \frac{\Gamma\left(s - \frac{N}{2}\right)}{\Gamma(s)} \left(\frac{L_1}{2\pi}\right)^{N-2s} \zeta_{\mathcal{M}_S}\left(s - \frac{N}{2}\right) \\ &+ \frac{\pi^s}{\sqrt{w_1 \ldots w_N}} \frac{2}{\Gamma(s)} \sum_{l_1,\ldots,l_N=-\infty}^{\infty} \sum_j \left(\frac{E_j L_1}{2\pi}\right)^{\frac{N}{2}-s} \times \\ & \left[\frac{l_1^2}{w_1} + \ldots + \frac{l_N^2}{w_N} \right]^{\frac{1}{2}\left(s-\frac{N}{2}\right)} K_{\frac{N}{2}-s}\left(E_j L_1 \left[\frac{l_1^2}{w_1} + \ldots + \frac{l_N^2}{w_N} \right]^{\frac{1}{2}} \right) \end{aligned} \tag{4.9}$$

with the zeta functions $\zeta_{\mathcal{M}_S}(s)$ of the manifold $\mathcal{M}_S$ defined by

$$\zeta_{\mathcal{M}_S}(s) = \sum_j E_j^{-2s}, \tag{4.10}$$

valid for $\Re s > \frac{n}{2}$. But then, obviously

$$\begin{aligned} m_{T_j}^2 &= \frac{2g^2 L_1^{2-N}}{\pi \mathcal{V}_{\mathcal{M}_S}} \sum_{l_1,\ldots,l_N=-\infty}^{\infty} \sum_j \left(\frac{E_j L_1}{2\pi}\right)^{\frac{N}{2}-1} \times \\ & \frac{\partial}{\partial w_j} \left\{ \left[\frac{l_1^2}{w_1} + \ldots + \frac{l_N^2}{w_N} \right]^{\frac{1}{2}\left(1-\frac{N}{2}\right)} K_{\frac{N}{2}-1}\left(E_j L_1 \left[\frac{l_1^2}{w_1} + \ldots + \frac{l_N^2}{w_N} \right]^{\frac{1}{2}} \right) \right\} \\ &> 0. \end{aligned} \tag{4.11}$$

To show the inequality we use [71]

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty dt\, \frac{e^{-t-\frac{z^2}{4t}}}{t^{\nu+1}},$$

valid for $|arg\ z| < \frac{\pi}{2}$, $\Re z^2 > 0$.
So we found, that the topologically generated mass does not depend on the scaling length λ and that it is exponentially damped if the mass of the field $m \to \infty$.
The essential assumption for this result is $E_0^2 > 0$. Otherwise zero modes are present in equation (4.6) and $m_{T_j}^2$ will not be positive definite any longer.

5 Imaginary gauge field mass generation

To proof our last statement let us consider a massless quantum field ϕ and let us choose $\mathcal{M}_S = \mathbb{R}^n$. Then the relevant operator (4.2) for this configuration reads

$$D_A = -\sum_{i=1}^{N}\left(\frac{\partial}{\partial x_i} - iA_i\right)^2 - \Delta \tag{5.1}$$

with eigenvalues

$$\nu_{l_1,\ldots,l_N,\vec{k}} = \left(\frac{2\pi l_1}{L_1} - A_1\right)^2 + \ldots + \left(\frac{2\pi l_N}{L_N} - A_N\right)^2 + \vec{k}^2, \tag{5.2}$$

$l_1, \ldots, l_N \in \mathbb{Z}$, $\vec{k} \in \mathbb{R}^n$. Paying special attention to the summation index $l_1 = \ldots = l_N = 0$, similar calculations as done in section 4 show, that the relevant zeta function in the given context is

$$E_N(s; w_1, \ldots, w_N) = \sum_{l_1,\ldots,l_N=-\infty}^{\infty}{}' \left[w_1 l_1^2 + \ldots + w_N l_N^2\right]^{-s}, \tag{5.3}$$

where the prime means omission of the summation index $l_1 = \ldots = l_N = 0$. In terms of E_N, the topologically generated mass reads [69]

i.) $n \geq 3$

$$\begin{aligned} m_{T_j}^2 &= g^2 \pi^{-\frac{n+N}{2}} L_1^{-n-N+2} \Gamma\left(\frac{n+N}{2} - 1\right) \times \\ &\quad \frac{\partial}{\partial w_j} E_N\left(\frac{n+N}{2} - 1; \frac{1}{w_1}, \ldots, \frac{1}{w_N}\right) \end{aligned} \tag{5.4}$$

ii.) $n = 2$

$$m_{T_j}^2 = \frac{g^2}{2\pi \mathcal{V}_{T^N}} \left(\frac{L_j}{L_1}\right)^2 \left[\left(1 + 2w_j \frac{\partial}{\partial w_j}\right) E_N'(0; w_1, ..., w_N) + 1\right] \tag{5.5}$$

iii.) $n = 1$

$$m_{T_j}^2 = \frac{g^2 L_j^2}{2\pi L_1 \mathcal{V}_{T^N}} \left(1 + 2w_j \frac{\partial}{\partial w_j}\right) E_N\left(\frac{1}{2}; w_1, ..., w_N\right) \tag{5.6}$$

iv.) $n = 0$

$$m_{T_j}^2 = \frac{g^2 L_j}{2\pi^2 \mathcal{V}_{T^N}} \left(1 + 2w_j \frac{\partial}{\partial w_j}\right) E_N(1; w_1, ..., w_N) \tag{5.7}$$

Using equation (5.3), equation (5.4) is always seen to be positive for arbitrary values of $w_1, ..., w_N$. So for $n \geq 3$ the limit $m \to 0$ is smooth. This fails to be true for $n \leq 2$ and depending on the compactification lengths $L_1, ..., L_N$, real and imaginary mass generation is possible. To exemplify the remarks let us consider the case $n = 0$, $N = 2$. Using the resummation formula (4.8), one may find

$$\begin{aligned} m_{T_1}^2 &= 2g^2\sqrt{w_2}\left\{-\frac{1}{12} + 4w_2^{\frac{1}{4}} \sum_{l_1, l_2 = 1}^{\infty} l_1^{\frac{3}{2}} l_2^{\frac{1}{2}} K_{\frac{1}{2}}\left(2\pi l_1 l_2 \sqrt{w_2}\right)\right\} \\ &= 2g^2\sqrt{w_2}\left\{-\frac{1}{12} + 2\sum_{l=1}^{\infty} \frac{l}{\exp(2\pi l\sqrt{w_2}) - 1}\right\}, \end{aligned} \tag{5.8}$$

where in the last equality $K_{\frac{1}{2}}(z) = \sqrt{\pi/2z}\exp(-z)$ [71] has been used. But the limiting behaviour

$$\sum_{l=1}^{\infty} \frac{l}{\exp(2\pi l\sqrt{w_2}) - 1} \to 0 \text{ for } w_2 \to \infty$$

and

$$\sum_{l=1}^{\infty} \frac{l}{\exp(2\pi l\sqrt{w_2}) - 1} \to \infty \text{ for } w_2 \to 0$$

shows, that at some critical value of $\sqrt{w_2} = L_1/L_2$, determined by

$$\sum_{l=1}^{\infty} \frac{l}{\exp(2\pi l\sqrt{w_2}) - 1} = \frac{1}{24} \tag{5.9}$$

a transition from real to imaginary mass will take place.
Similar considerations show, that for $n = 0, 1, 2$, and arbitrary N in given ranges of the compactification lenghts $L_1, ..., L_N$ the generated gauge field mass is real or imaginary.

6 Conclusions

In this contribution we have studied different aspects of quantum field theory of a spin-0 field in partially compactified spacetime.
First we studied finite temperature quantum field theory of a conformally noninvariant spin-0 gas in an arbitrary static spacetime. Using zeta function regularization and heat kernel techniques a high-temperature expansion of the thermodynamic potential has been found and used to consider the phenomenon of Bose-Einstein condensation.
In generalization to finite temperature field theory we then considered an Abelian gauge field theory in an arbitrary number of compactified dimensions. In this analysis we concentrated only on specific terms in the effective potential defining the topologically generated mass m_{T_j} of the gauge field. We have seen that even in a quite general context it was possible to determine the sign of $m_{T_j}^2$.
It is hoped, that the presented computational approach will be helpful in considerations of the topological mass generation in non-Abelian gauge field theories and in the context of symmetry breaking in a selfinteracting scalar field theory in spacetimes with an arbitrary number of compactified dimensions.
Along the same lines it is possible to generate gauge field masses by quantum fluctuation constrained by boundaries with given boundary conditions.
Furthermore, one may deal just as well with a spin-$\frac{1}{2}$ field to analyse the mentioned problems.

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ZETA-FUNCTION REGULARIZATION TECHNIQUES FOR SERIES SUMMATION AND APPLICATIONS

E. ELIZALDE
Department E.C.M., Faculty of Physics, University of Barcelona
Diagonal 647, E-08028 Barcelona, Spain

Abstract

The zeta-function regularization procedure is one of the most elegant tools of quantum field theory. It comprises a whole set of different techniques, of increasing difficulty, to treat the several degrees of complexity of the physical problems to be handled. Here, the question of the regularization of multi-series of the general type

$$\sum_{n_1,\ldots,n_N} [a_1(n_1+c_1)^{\alpha_1}+\cdots+a_N(n_N+c_N)^{\alpha_N}+c]^{-s},$$

with $a_1,\ldots,a_N,\alpha_1,\ldots,\alpha_N > 0$, $c_1,\ldots,c_N$ arbitrary reals and $c \geq 0$, is addressed. Only the most simple cases have been properly studied in the literature (e.g., $a_1 = \cdots = a_N$, $c_1 = \cdots = c_N = 0$ or $\pm 1/2$, $\alpha_1 = \cdots = \alpha_N = 2$, $c = 0$, etc.). The zeta function regularization theorem in its most general form leads to an asymptotic expansion valid for arbitrary a's and α's, which is very convenient for numerical computations. In particular, useful expressions can be derived from it for the analytical continuation of Riemann, Hurwitz and Epstein zeta functions and their generalizations, and for their asymptotic expansions —including those of derivatives and integrals. Physical applications of the zeta-regularization procedure include the proper definition of the vacuum energy, the Casimir effect, spontaneous compacification in quantum gravity, stability analysis of strings and membranes, etc., and embrace also very recent experiments of solid state and condense matter physics employing liquid helium.

1. The zeta-function regularization theorem

The method of zeta-function regularization has a rather long history. There are precedents in the use of Riemann and Epstein zeta functions as summation (i.e., regularization) procedures in the late sixties [1]. However, the zeta-function regularization method as such was introduced in the middle seventies [2]. The paper of Hawking (of 1977) is considered as the first systematic description of the zeta function procedure as a useful technique for providing the finite values corresponding to path integrals over fields in curved backgrounds and for the evaluation of determinants of quadratic differential operators [3]. This is a basic, multipurpose need in theoretical physics and in several branches of mathematics (such as analysis and number theory).[1]

[1]In words of the great David Hilbert, the most relevant of all the (by now famous) problems that he proposed was that of proving if the *"extremely important statement by Riemann"* (the so called

In the last 15 years the zeta-regularization procedure has been used more and more by the leading physicists and mathematicians and we can definitely say, in particular, that it is now one of the most elegant tools in quantum field theory. At the begining the method was rather simply minded, but nowadays it comprises a whole set of different techniques, of increasing difficulty, to treat the several degrees of complexity of the physical (and corresponding mathematical) problems to be solved.

The list of people who have been dealing with zeta functions at one instance or other would be just non-ending. Maybe Al Actor is the one who has devoted more years to this subject (at least among those of the mathematical-physicists squad). According to Actor himself [5], a milestone in the field of regularization of discrete sums of the general type

$$\sum_{n_1,\ldots,n_N} [a_1(n_1+c_1)^{\alpha_1}+\cdots+a_N(n_N+c_N)^{\alpha_N}+c]^{-s}, \tag{1}$$

with $a_1,\ldots,a_N,\alpha_1,\ldots,\alpha_N>0$, $c_1,\ldots,c_N$ arbitrary reals, and $c\geq 0$, has been the proof of the so-called ***zeta-function regularization theorem***. In its final formulation, it is the result of hard work of A. Actor, H.A. Weldon, A. Romeo and myself [6-8]. The uses and applications of the theorem in its most general form [8] —for discrete series of the type (1)— are very far reaching. In particular it leads to asymptotic expansions, valid for arbitrary a's and α's, of the multi-series of this general kind, which are very well suited for numerical computations. These expansions are unchallenged in its usefulness for such purposes.

Let me just recall for comparison, that only the most simple cases of series of that kind have been properly studied in the literature (e.g., $a_1=\cdots=a_N$, $c_1=\cdots=c_N=0$ or $\pm 1/2$, $\alpha_1=\cdots=\alpha_N=2$, $c=0$, etc.). The zeta function regularization theorem actually provides a method for the computation of expressions like (1) —and even more involved ones— for $\mathrm{Re}\,(s)$ big enough, and of their analytic (usually meromorphic) continuation to other values of s. In the zeta-function procedure they are given in terms of the ordinary Riemann and Hurwitz zeta-functions.

A very simple case corresponds to the Hamiltonian zeta-function $\zeta(s)\equiv\sum_i E_i^{-s}$, with E_i eigenvalues of H [9]. For a system of N non-interacting harmonic oscillators, one has $\alpha_j=1$, $j=1,2,\ldots,N$, and the a_j are their eigenfrequencies ω_j. Another interesting case is partial toroidal compactification (spacetime $T^p\times R^{q+1}$). Then $\alpha_j=2$ and, usually, $c_j=0,\pm 1/2$. One is thus led to the Epstein zeta-functions [10]

$$\begin{aligned} Z_N(s) &= \sum_{n_1,\ldots,n_N=-\infty}^{\infty} (n_1^2+\cdots+n_N^2)^{-s}, \\ Y_N(s) &= \sum_{n_1,\ldots,n_N=-\infty}^{\infty}{}' \left[\left(n_1+\frac{1}{2}\right)^2+\cdots+\left(n_N+\frac{1}{2}\right)^2\right]^{-s} \end{aligned} \tag{2}$$

Riemann conjecture) is true [4]. It asserts that the real part of all the non-real zeros of the Riemann zeta function is the same, and equal to 1/2.

(the prime means omission of the term $n_1 = n_2 = \ldots = n_N = 0$). Other powers α_j appear when one deals with the spherical compactification (spacetime $S^p \times R^{q+1}$) and with more involved ones arising, e.g., in superstring theory and their membrane and p-brane generalizations. Hence the general expression (1). The only precedents in the literature (to my knowledge) of this kind of evaluations have been restricted to few special cases other than $a_1 = a_2 = \ldots = a_N$ and $c_1 = c_2 = \ldots = c_N = 0$. Very famous is the expression due to Hardy [11] (particular case of our final formula).

An interesting result concerning the interchange of the order of summation of infinite series appearing in zeta-function regularization is due to Weldon [7]. His investigation originated in some difficulties which appeared in a paper by Actor [6] when he tried to obtain the value of the thermodynamical potential corresponding to a relativistic Bose gas by using the zeta-function regularization procedure. Unfortunately, Weldon's proof has its own limitations, and the statements in [7] concerning the extent of its validity are actually not right. This is quite easy to check in some particular cases, and was stressed in [12].

Let me briefly summarize the proof due to Weldon of the validity of the zeta-function regularization procedure [7] and point out its shortcomings. Using the same notation as in [7], let us consider the four series

$$S_F = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{s+1}} \sum_{a=0}^{\infty} m^a f(a), \tag{3}$$

$$S_B = \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \sum_{a=0}^{\infty} m^a f(a), \tag{4}$$

$$S_{AF} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{s+1}} \sum_{a=0}^{\infty} (-1)^a m^a f(a), \tag{5}$$

$$S_{AB} = \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \sum_{a=0}^{\infty} (-1)^a m^a f(a), \tag{6}$$

where $f(a) \geq 0$ for positive integer a. They are assumed to be convergent, as they stand. The idea of the zeta-function regularization procedure begins with the interchange of the order of the summation of the two infinite series involved in each case. As was proven in [7], provided that $f(a)$ can be defined in the complex a-plane, satisfying:

(1) $f(a)$ is regular for $\mathrm{Re}\, a \geq 0$,
and, in the case of (3) and (4),
(2a) $a m^a f(a) \to 0$, as $|a| \to \infty$, for $\mathrm{Re}\, a \geq 0$ and fixed m,
and, in the case of (5) and (6),
(2b) $a m^a f(a) e^{-\pi |\mathrm{Im}\, a|} \to 0$, as $|a| \to \infty$, for $\mathrm{Re}\, a \geq 0$ and fixed m,

it turns out that in the fermionic cases, (3) and (5), one can naïvely interchange the order of the summations, to get

$$S_F = \sum_{a=0}^{\infty} \eta(s+1-a)f(a), \quad S_{AF} = \sum_{a=0}^{\infty} (-1)^a \eta(s+1-a)f(a), \tag{7}$$

while in the bosonic cases, (4) and (6), one obtains the additional contributions

$$\begin{aligned} S_B &= \sum_{a=0}^{\infty} \zeta(s+1-a)f(a) - \pi \mathrm{ctg}(\pi s) f(s), \ s \notin N, \\ S_B &= \sum_{\substack{a=0 \\ a\neq s}}^{\infty} \zeta(s+1-a)f(a) + \gamma f(s) - f'(s), \ s \in N, \end{aligned} \tag{8}$$

and

$$\begin{aligned} S_{AB} &= \sum_{a=0}^{\infty} (-1)^a \zeta(s+1-a)f(a) - \pi \csc(\pi s) f(s), \ s \notin N, \\ S_{AB} &= \sum_{\substack{a=0 \\ a\neq s}}^{\infty} (-1)^a \zeta(s+1-a)f(a) + (-1)^s[\gamma f(s) - f'(s)], \ s \in N, \end{aligned} \tag{9}$$

respectively. Here $\zeta(s)$ and $\eta(s)$ are the Riemann ordinary and alternating zeta functions:

$$\zeta(s) = \sum_{m=1}^{\infty} m^{-s}, \ \mathrm{Re}\, s > 1, \quad \eta(s) = \sum_{m=1}^{\infty} (-1)^{m+1} m^{-s}, \ \mathrm{Re}\, s > 0, \quad \eta(s) = (1-2^{1-s})\zeta(s), \tag{10}$$

γ is Euler-Mascheroni's constant, and $f'(s)$ means derivative of f with respect to s.

The proof of the preceding theorem proceeds by integration in the complex a-plane. One writes (3) to (6) under the form of contour integrals

$$S_F = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{s+1}} \oint_C \frac{da}{2i} m^a f(a) \mathrm{ctg}(\pi a), \tag{11}$$

$$S_B = \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \oint_C \frac{da}{2i} m^a f(a) \mathrm{ctg}(\pi a), \tag{12}$$

$$S_{AF} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{s+1}} \oint_C \frac{da}{2i} m^a f(a) \csc(\pi a), \tag{13}$$

$$S_{AB} = \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \oint_C \frac{da}{2i} m^a f(a) \csc(\pi a), \tag{14}$$

where C is the closed contour defined by the straight line $\mathrm{Re}\, a = -a_0$ —for fixed a_0 such that $0 < a_0 < 1$— and by the semicircumference at infinity on the right. The contribution from the semicircumference is zero in every case, due to the asymptotic behaviour of $f(a)$ and, as long as $\mathrm{Re}\, s > -1$, the integral extended to the line $\mathrm{Re}\, a =$

$-a_0$ can be interchanged with the remaining sum over m. The final step is to close the contour C again with the semicircumference at infinity. In the cases (12) and (14) there comes then an additional contribution from the pole of the zeta function $\zeta(s+1-a)$ at $a=s$. On the contrary, in the cases (11) and (13) the alternating zeta function $\eta(s+1-a)$ has no pole in the region enclosed by C. All the steps in this procedure are simple and one obtains eqs. (7) to (9).

However it was further explicitly stated by Weldon in [7] that the results for the alternating fermionic and for the alternating bosonic cases, S_{AF} and S_{AB}, respectively, could be naïvely extended to the following types of series

$$S_{AF}^{(N)} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^{s+1}} \sum_{a=0}^{\infty}(-1)^a m^{Na} f(a), \quad S_{AB}^{(N)} = \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \sum_{a=0}^{\infty}(-1)^a m^{Na} f(a), \quad (15)$$

with N *any* positive integer. By going over the same proof once more, he just obtained a trivial modification of the above results. That this generalization of (9) and for any positive integer N is *not right* is very easy to check. In particular, it was noticed by Actor in [12]. As an easy example let me study the simplest case after the (only correct) one $N=1$ (explicitly considered in [7]), i.e. $N=2$. Let

$$S \equiv \sum_{m=1}^{\infty} e^{-m^2} = \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \sum_{a=0}^{\infty}(-1)^a \left.\frac{m^{2a}}{a!}\right|_{s=-1}, \quad (16)$$

where the last operation consists in making the analytic continuation of the resulting series to $s=-1$. The function $f(a)$ is here $f(a)=\frac{1}{\Gamma(a+1)}$ and all the hypotheses of the theorem are fulfilled. Use of Weldon's formula gives

$$S = \sum_{a=0}^{\infty} \frac{(-1)^a}{a!}\zeta(-2a) - \frac{\frac{\pi}{2}\csc(-\frac{\pi}{2})}{\Gamma(1-\frac{1}{2})} = -\frac{1}{2} + \frac{\sqrt{\pi}}{2}, \quad (17)$$

which is false, though numerically almost undetectable, because

$$S = 0.3863186, \quad \frac{\sqrt{\pi}-1}{2} = 0.3862269, \quad \Delta \equiv \frac{\sqrt{\pi}-1}{2} - S = -9.17\times 10^{-5}. \quad (18)$$

Going on to $N=2,3,4,\ldots$, it is not difficult to see that, if N is constrained to be a positive integer, Weldon's formula is true *only for* $N=1$ (eqs. (8) and (9)).

As we managed to demonstrate [8], the step which fails to be correct in Weldon's proof for general N is *the last one*, namely, even if the asymptotic behaviour (2b) of the function $f(a)$ allows us to supress the contribution from the curved contour in the second step, this will be no longer true when we try to close again the circuit C in the last step. *There is in fact a contribution coming from the integral of* $\zeta(s+1-Na)f(a)$ *over the semicircumference at infinity* (due to the asymptotic behaviour of the zeta-function). And this is so whatever it be the value we choose for s. The study of the asymptotic behaviour of $\zeta(s+1-Na)$ immediately distinguishes the case $N\le 1$ from $N>1$. It is, however, misleading in some sense, because the fact that the zeta-function diverges for $N > 1$ does *not* necessarily mean that the contour actually provides a non-zero contribution invalidating Weldon's proof (that had been conjectured by Actor, at a first instance). Things must be done with very great care due to the presence of highly oscillating factors.

Let me restrict the argument to the case $f(a) = \frac{1}{\Gamma(a+1)}$. This is enough for many applications and the generalization to other cases proceeds by analogy. In this case, the fact that the poles of Γ are the non-positive integers and a suitable application of the zeta function reflection formula allow us to write the additional contribution as a contour integral over a curved path in the complex *left* half-plane. Besides, we use the relation

$$\Gamma\left(\frac{z}{2}\right)\zeta(z) = \int_0^\infty dt\ t^{z/2-1}S_2(t), \quad \mathrm{Re}\, z > 0, \tag{19}$$

where

$$S_\alpha(t) \equiv \sum_{m=1}^{\infty} e^{-m^\alpha t}, \tag{20}$$

and owing to the behaviour of the complex function $\Gamma(z)$ which has simple poles at $z = -n$ for $n = 0, 1, 2, \dots$, with residues

$$\mathrm{Res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}, \tag{21}$$

and with the aid of

$$\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z), \tag{22}$$

we can write

$$S_{AB}^{(\alpha)} \equiv \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \sum_{a=0}^{\infty} (-1)^a \frac{m^{\alpha a}}{\Gamma(a+1)}, \quad \alpha \in R, \tag{23}$$

as

$$S_{AB}^{(\alpha)} \equiv \sum_{m=1}^{\infty} \frac{1}{m^{s+1}} \oint_{\bar{C}} \frac{da}{2\pi i} m^{-\alpha a}\Gamma(a), \tag{24}$$

where now the contour $\bar{C}$ consists of the line $\mathrm{Re}\, a = a_0$, with a_0 fixed, $0 < a_0 < 1$, and of the semicircumference at infinity on the left. For $s = -1$,

$$S_{AB}^{(\alpha)}(s=-1) = \sum_{m=1}^{\infty}\sum_{a=0}^{\infty}(-1)^a \frac{m^{\alpha a}}{a!} = \sum_{m=1}^{\infty} e^{-m^\alpha} = S_\alpha(1). \tag{25}$$

By correctly making the last step in the above proof, we end up with

$$S_{AB}^{(\alpha)} = \sum_{a=0}^{\infty} \frac{(-1)^a}{a!}\zeta(s+1-\alpha a) + \frac{1}{\alpha}\Gamma\left(-\frac{s}{\alpha}\right) - \Delta_{AB}^{(\alpha)}, \quad \frac{s}{\alpha} \notin N, \tag{26}$$

$$S_{AB}^{(\alpha)} = \sum_{\substack{a=0 \\ a\neq s/\alpha}}^{\infty} \frac{(-1)^a}{a!}\zeta(s+1-\alpha a) + (-1)^{\frac{s}{\alpha}}\left[\frac{\gamma}{\Gamma(\frac{s}{\alpha}+1)} + \frac{\Gamma'(\frac{s}{\alpha}+1)}{\alpha\Gamma^2(\frac{s}{\alpha}+1)}\right] - \Delta_{AB}^{(\alpha)}, \quad \frac{s}{\alpha} \in N, \tag{27}$$

where $\Delta_{AB}^{(\alpha)}$ is the contribution of the curved part K of the contour $\bar{C}$:

$$\Delta_{AB}^{(\alpha)} \equiv \int_K \frac{da}{2\pi i}\zeta(s+1+\alpha a)\Gamma(a). \tag{28}$$

This contribution is *not* zero for any value of s. We can check that it actually provides the term missing from (17). Before proceeding to the actual calculation of (28), one can, as an illustrating exercise, reclose the contour on the right instead of the left, and check that the same series is obtained.

Coming back to eq. (28) and doing the same for $s=-1$ and $\alpha=2$, we must use first the reflection formula

$$\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{z-1/2}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z), \tag{29}$$

what yields

$$\Delta_{AB}^{(2)}(s=-1) = \int_K \frac{da}{2i\sqrt{\pi}}\int_0^\infty dt\; t^{-a-1/2}S_2(\pi^2 t) = -\sqrt{\pi}S_2(\pi^2), \tag{30}$$

that is

$$S_2(1) = -\frac{1}{2} + \frac{\sqrt{\pi}}{2} + \sqrt{\pi}S_2(\pi^2). \tag{31}$$

This result happens to be just a particular case of the famous theta function identity

$$\theta(z,\tau) = \tau^{-1/2}e^{\pi z^2/\tau}\theta\left(\frac{z}{i\tau},\frac{1}{\tau}\right), \tag{32}$$

θ being the elliptic function

$$\theta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2\tau+2\pi n z}, \quad z\in C, \tau\in R^+. \tag{33}$$

Notice that $S_2(\pi t) = \frac{1}{2}(\theta(0,t)-1)$. Eq. (30) is an *exact* expression. Once more, we observe that the contribution of the contour provides, in fact, the missing term.

Let us now again consider (21) for *general* α and, $s=-1$. Eqs. (19) and (20) read, in this case,

$$\Gamma(z)\zeta(\alpha z) = \int_0^\infty dt\; t^{z-1}S_\alpha(t), \tag{34}$$

$S_\alpha(t)$ being the function given in (25). No simple reflection formula like (29) exists for $\alpha \neq 2$. We have, instead,

$$\zeta(\alpha z) = \frac{2\Gamma(1-\alpha z)}{(2\pi)^{1-\alpha z}}\sin(\frac{\pi\alpha z}{2})\zeta(1-\alpha z), \tag{35}$$

and we get

$$S_\alpha \equiv S_\alpha(1) = \sum_{m=1}^\infty e^{-m^\alpha} = \sum_{a=0}^\infty \frac{(-1)^a}{a!}\zeta(-\alpha a) + \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right) - \Delta_\alpha, \tag{36}$$

being the contribution of the contour

$$\Delta_\alpha = \int_K \frac{da}{2\pi i}\zeta(\alpha a)\Gamma(a). \tag{37}$$

After some work, we obtain

$$\Delta_\alpha = \int_K \frac{da}{i\sqrt{2\pi\alpha}}\varphi_\alpha(a)\int_0^\infty dt\; t^{\frac{1}{\alpha}-a-1}S_\alpha\left[\left(\frac{2\pi}{\alpha}\right)^\alpha t\right], \tag{38}$$

where the function $\varphi_\alpha(a)$ comes from the asymptotic behaviour of the integrand (37) for $|a| \to \infty$,

$$\begin{aligned}\varphi_\alpha(a) \;&=\; \exp\left\{\left[(2-\alpha)+\left(\frac{1}{2}-\frac{1}{\alpha}\right)\right]\ln(-a)+(\alpha-2)a\right.\\ &\left.+\left(\frac{\alpha}{2}-1\right)\pi|\mathrm{Im}\ a|+\mathrm{sign}(\mathrm{Im}\ a)\ i\pi\left(\frac{\alpha}{2}-1\right)|\mathrm{Re}\,a|\right\}.\end{aligned} \tag{39}$$

It is immediate that, for $\alpha = 2$,

$$\varphi_2(a) = 1, \tag{40}$$

so (38) is in agreement with (30). Note also that for $\alpha < 2$ we have

$$\varphi_\alpha(a) \to 0\ ,\ |a| \to \infty. \tag{41}$$

For the sake of completeness, we quote also the following result. When, putting $\varphi_\alpha(a) = 1$ in (38), the remaining integral is *finite* and yields (see the proof for $\alpha = 2$ above)

$$\tilde{\Delta}_\alpha \equiv \Delta_\alpha\{\varphi_\alpha = 1\} = -\sqrt{\frac{2\pi}{\alpha}}S_\alpha\left[\left(\frac{2\pi}{\alpha}\right)^\alpha\right]. \tag{42}$$

Collecting everything together, we have proven the following

<u>Theorem</u> (zeta function regularization theorem).

Under the hypothesis above, (1), (2a), (2b), we have that:

i) For $-\infty < \alpha < 2$, the contribution of the semicircumference at infinity is zero, i.e.

$$\Delta_\alpha = 0,\ \alpha < 2. \tag{43}$$

ii) For $\alpha = 2$, the contribution of the semicircumference at infinity is given by

$$\Delta_2 = -\sqrt{\pi}S_2(\pi^2). \tag{44}$$

The result for $\alpha \leq 1$ was known already and constitutes Weldon's proof of zeta-function regularization. The result for $\alpha = 2$ shows, on the contrary, that the statements in [7] about the validity of the proof for any positive integer α were false, the reason being that the semicircumference at infinity does not provide a zero contribution. It was precisely the last step of the proof in [7] that was wrong. The fact

that the numerical value of Δ_α is so small (it can be thought of as an infinitesimal correction, see (44)) as compared with the rest of the terms in eqs. (26) and (27) gives sense to the whole procedure of zeta-function regularization. However, this is strictly true only for small α.
iii) For large α, Δ_α, is given by

$$\Delta_\alpha = -\left(1 - \frac{2}{e}\right) \frac{\pi}{\alpha} S_\alpha \left[\left(\frac{2\pi}{\alpha}\right)^\alpha\right]. \tag{45}$$

We observe that, for large α, Δ_α ceases to be an infinitesimal contribution. Actually,

$$\begin{aligned} \Delta_\alpha &= 0, \ \alpha < 2; \\ \Delta_2 &= 9.17 \times 10^{-5}; \ \ \Delta_4 = 0.04; \ \ \Delta_6 = 0.07; \\ \Delta_\alpha &\to 0.13, \ \alpha \to \infty, \end{aligned} \tag{46}$$

which represent, respectively, contributions of the 0%, 0.02%, 11%, 19%, and 36% on the whole value of $S_\alpha(1)$.

This theorem has been extended to situations of the kind [13]

$$S_c^{(\alpha)}(s) \equiv \sum_{m=0}^{\infty} (m+c)^{-s-1} \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} (m+c)^{\alpha a}. \tag{47}$$

The supplementary contributions for $\alpha \geq 2$ (always with $c = 1$) have been obtained in [13]. Write

$$S_c^{(\alpha)}(s) = \sum_{m=0}^{\infty} (m+c)^{-s-1} \oint_C \frac{da}{2\pi i} (m+c)^{-\alpha a} \Gamma(a), \tag{48}$$

with C a contour in the complex plane a, $C = L \cup K$, being L the straight line $\mathrm{Re}\,(a) = a_0$, $0 < a_0 < 1$, and K a curved part, the semicircumference at infinity on the left of this line. For $\mathrm{Re}\,(s)$ big enough,

$$S_c^{(\alpha)}(s) = \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} \zeta(s+1-\alpha a, c) + \begin{cases} \frac{1}{\alpha}\Gamma(-\frac{s}{\alpha}) + \Delta_c^{(\alpha)}(s), & \frac{s}{\alpha} \notin N, \\ (-1)^{s/\alpha} \left[\frac{\gamma}{\Gamma(\frac{s}{\alpha}+1)} + \frac{\Gamma'(\frac{s}{\alpha}+1)}{\alpha \Gamma^2(\frac{s}{\alpha}+1)}\right] + \Delta_c^{(\alpha)}(s), & \frac{s}{\alpha} \in N, \end{cases} \tag{49}$$

where $\Delta_c^{(\alpha)}(s)$ is the following integral over the curved part K of the contour C

$$\Delta_c^{(\alpha)}(s) \equiv \int_K \frac{da}{2\pi i} \zeta(s+1+\alpha a, c) \Gamma(a). \tag{50}$$

Summing up, this is the main content of the zeta-function regularization theorem (for more details see refs. [8,13] and the papers mentioned therein).

2. Expressions for multi-series involving arbitrary constants and positive exponents

We shall now make use of the zeta function regularization theorem in order to obtain expressions for the most general multi-series of the type presented in the introduction, which would be impossible to derive by other means (at least with comparable easiness and universality). The same notation which has commonly been used in other papers will be employed:

$$M_N^c(s;\vec{a};\vec{\alpha};\vec{c}) \equiv M_N^c(s;a_1,\ldots,a_N;\alpha_1,\ldots,\alpha_N;c_1,\ldots,c_N)$$
$$\equiv \sum_{n_1,\ldots,n_N=0}^{\infty} \left[a_1(n_1+c_1)^{\alpha_1}+\cdots+a_N(n_N+c_N)^{\alpha_N}+c\right]^{-s}, \quad (51)$$

and for the generalized Epstein-like case:

$$E_N^c(s;\vec{a};\vec{c}) \equiv M_N^c(s;a_1,\ldots,a_N;2,\ldots,2;c_1,\ldots,c_N)$$
$$\equiv \sum_{n_1,\ldots,n_N=0}^{\infty} \left[a_1(n_1+c_1)^2+\cdots+a_N(n_N+c_N)^2+c\right]^{-s}. \quad (52)$$

Consider the case of M_2^c. We need the result of the regularization theorem as applied to the double series

$$S_\alpha(t,s) = \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} n^{\alpha k}, \qquad \alpha \in R, \quad (53)$$

which converges for Re $(s) > 0$ large enough. We can write

$$S_\alpha(t,s) = \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} \oint_C \frac{dk}{2\pi i} t^k n^{-\alpha k}\Gamma(k), \quad (54)$$

where the contour C consists of the straight line Re $(k) = k_0$, with k_0 fixed, $0 < k_0 < 1$, and the semicircumference at infinity on the left of this line. The regularization theorem tells me in this case that [13]

$$S_\alpha(t,s) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!}\zeta(s+1-\alpha k) + \frac{1}{\alpha}\Gamma\left(-\frac{s}{\alpha}\right)t^{-1/\alpha} - \Delta_\alpha(t,s), \qquad \frac{s}{\alpha} \notin N, \quad (55)$$

where $\Delta_\alpha(t,s)$ is the contribution of the curved part K of the contour C:

$$\Delta_\alpha(t,s) \equiv \int_K \frac{dk}{2\pi i}\zeta(s+1+\alpha k)\Gamma(k)t^k. \quad (56)$$

With this, we obtain

$$\begin{aligned} M_2^c(s;\vec{a};\vec{\alpha};\vec{c}) &= \frac{a_2^{-s}}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m\Gamma(s+m)}{m!} \left(\frac{a_1}{a_2}\right)^m \zeta(-\alpha_1 m, c_1) M_1^{c/a_2}(s+m;1;\alpha_2;c_2) \\ &+ \frac{a_2^{-s}}{\alpha_1}\Gamma\left(\frac{1}{\alpha_1}\right)\left(\frac{a_2}{a_1}\right)^{1/\alpha_1} \frac{\Gamma\left(s-\frac{1}{\alpha_1}\right)}{\Gamma(s)} M_1^{c/a_2}(s-1/\alpha_1;1;\alpha_2;c_2) \\ &- \frac{a_2^{-s}}{\Gamma(s)}\left(\frac{a_2}{a_1}\right)^{1/\alpha_1} \int_K \frac{da}{2\pi i}\zeta(s+1+\alpha_1 a, c_1) M_1^{c/a_2}(s+a;1;\alpha_2;c_2)\Gamma(a)\Gamma(s+a), \end{aligned} \quad (57)$$

and also

$$
\begin{aligned}
M_1^c(s;a_1;\alpha_1;c_1) &= \frac{c^{-s}}{\Gamma(s)}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(s+m)}{m!}\left(\frac{a_1}{c}\right)^m\zeta(-\alpha_1 m,c_1) \\
&+ \frac{c^{-s}}{\alpha_1}\Gamma\left(\frac{1}{\alpha_1}\right)\left(\frac{c}{a_1}\right)^{1/\alpha_1}\frac{\Gamma\left(s-\frac{1}{\alpha_1}\right)}{\Gamma(s)} \\
&- \frac{c^{-s}}{\Gamma(s)}\left(\frac{c}{a_1}\right)^{1/\alpha_1}\int_K\frac{da}{2\pi i}\zeta(s+1+\alpha_1 a,c_1)\Gamma(a)\Gamma(s+a).
\end{aligned}
\tag{58}
$$

It is not difficult to build, from these two expressions, a recurrence leading to the calculation of M_N^c from the knowledge of M_{N-1}^c, and starting with the formula for M_1^c. At each step, this involves a complex integration over a curved contour at infinity, a term which is in general very small compared with the rest. Forgetting about their actual expressions, the recurrence can be solved explicitly, the result being [13] (corrected)

$$
\begin{aligned}
M_N^c(s;\vec{a};\vec{\alpha};\vec{c}) &= \frac{a_N^{-s}}{\Gamma(s)}\sum_{p=0}^{N-1}\sum_{C_{N-1,p}}\prod_{r=1}^{p}\frac{b_{i_r}^{-1/\alpha_{i_r}}}{\alpha_{i_r}}\Gamma\left(\frac{1}{\alpha_{i_r}}\right) \\
\times & \sum_{k_{j_1},\ldots,k_{j_{N-p-1}}=0}^{\infty}\Gamma\left(s+\sum_{l=1}^{N-p-1}k_{j_l}-\sum_{r=1}^{p}\frac{1}{\alpha_{i_r}}\right) \\
\times & \prod_{l=1}^{N-p-1}\frac{(-b_{j_l})^{k_{j_l}}}{k_{j_l}!}\zeta(-\alpha_{j_l}k_{j_l},c_{j_l})\,M_1^{c/a_N}\left(\alpha_N\left(s+\sum_{l=1}^{N-p-1}k_{j_l}-\sum_{r=1}^{p}\frac{1}{\alpha_{i_r}}\right);1;\alpha_N;c_N\right) \\
+ & \Delta_{ER},
\end{aligned}
\tag{59}
$$

with $b_{i_r}\equiv a_{i_r}/a_N$ (notice the errata in eqs. (3.22) and (3.23) of my ref. J. Phys. A [13]) and $1\le i_1<\cdots<i_p\le N-1$, $1\le j_1<\cdots<j_{N-p-1}\le N-1$, being $i_1,\ldots,i_p,j_1,\ldots,j_{N-p-1}$ a permutation of $1,2,\ldots,N-1$. The sum on $C_{N-1,p}$ means sum over the $\binom{N-1}{p}$ choices of the indices $i_1,\ldots,i_p$ among the $1,2,\ldots,N-1$, and the term Δ_{ER} includes all the Δ corrections which appear at each step of the recurrence.

Going down to the particular case (52) things become more concrete. As mentioned before, then the expression giving our additional corrections to the series commutation reduces to a theta function identity [13]

$$
\sum_{m=0}^{\infty}\exp[-a(m+c)^2]=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a^m\zeta(-2m,c)+\frac{1}{2}\sqrt{\frac{\pi}{a}}+\sqrt{\frac{\pi}{a}}\cos(2\pi c)S\left(\frac{\pi^2}{a^2}\right), \tag{60}
$$

and this yields the recurrence

$$
E_N^c(s;\vec{a};\vec{c}) = \frac{1}{\Gamma(s)}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}a_1^m\zeta(-2m,c_1)\Gamma(s+m)E_{N-1}^c(s+m;a_2,\ldots,a_N;c_2,\ldots,c_N)
$$

$$
\begin{aligned}
&+ \ \frac{1}{2}\sqrt{\frac{\pi}{a_1}}\frac{\Gamma(s-1/2)}{\Gamma(s)}E_{N-1}^c(s-1/2;a_2,\ldots,a_N;c_2,\ldots,c_N) \\
&+ \ \frac{2\pi^s}{\Gamma(s)}\cos(2\pi c_1)a_1^{-s/2-1/4}\sum_{n_1=1}^\infty \sum_{n_2,\ldots,n_N=0}^\infty n_1^{s-1/2}\left[c+\sum_{j=2}^N a_j(n_j+c_j)^2\right]^{-s/2+1/4} \\
&\times \ K_{s-1/2}\left(\frac{2\pi n_1}{\sqrt{a_1}}\sqrt{c+\sum_{j=2}^N a_j(n_j+c_j)^2}\right),
\end{aligned} \tag{61}
$$

where K_ν is the modified Bessel function of the second kind. The recurrence starts with

$$
\begin{aligned}
E_1^c(s;a_1;c_1) &= \frac{c^{-s}}{\Gamma(s)}\sum_{m=0}^\infty \frac{(-1)^m\Gamma(s+m)}{m!}\left(\frac{a_1}{c}\right)^m \zeta(-2m,c_1)+\frac{c^{1/2-s}}{2}\sqrt{\frac{\pi}{a_1}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \\
&+ \ \frac{2\pi^s}{\Gamma(s)}\cos(2\pi c_1)a_1^{-s/2-1/4}c^{-s/2+1/4}\sum_{n_1=1}^\infty n_1^{s-1/2}K_{s-1/2}\left(2\pi n_1\sqrt{\frac{c}{a_1}}\right).
\end{aligned} \tag{62}
$$

Then

$$
\begin{aligned}
E_2^c(s;a_1,a_2;c_1,c_2) &= \frac{a_2^{-s}}{\Gamma(s)}\sum_{m=0}^\infty \frac{(-1)^m\Gamma(s+m)}{m!}\left(\frac{a_1}{a_2}\right)^m\zeta(-2m,c_1) \\
&\times \ E_1^{c/a_2}(s+m;1;c_2)+\frac{a_2^{1/2-s}}{2}\sqrt{\frac{\pi}{a_1}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}E_1^{c/a_2}(s-1/2;1;c_2) \\
&+ \ \frac{2\pi^s}{\Gamma(s)}\cos(2\pi c_1)a_1^{-s/2-1/4}a_2^{-s/2+1/4}\sum_{n_1=1}^\infty\sum_{n_2=0}^\infty n_1^{s-1/2}\left[a_2(n_2+c_2)^2+c\right]^{-s/2+1/4} \\
&\times \ K_{s-1/2}\left(\frac{2\pi n_1}{\sqrt{a_1}}\sqrt{a_2(n_2+c_2)^2+c}\right),
\end{aligned} \tag{63}
$$

and so on. Expressions for the special case $c=0$ are given in ref. [13] (see also eqs. (93) and (94) below).

The very particular case, $a_1=\cdots=a_N=1$, $c_1=\cdots=c_N=1$ and $\alpha_1=\cdots=\alpha_N=2$, simplifies considerably. For $c=0$, we get

$$
E_N(s)=\frac{(-1)^{N-1}}{2^{N-1}}\frac{1}{\Gamma(s)}\sum_{j=0}^{N-1}(-1)^j\binom{N-1}{j}\Gamma(2s-j)\zeta\left(s-\frac{j}{2}\right)+\Delta_{ER}, \tag{64}
$$

and, for $c\neq 0$,

$$
E_N^c(s)=\frac{(-1)^{N-1}}{2^{N-1}}\frac{1}{\Gamma(s)}\sum_{j=0}^{N-1}(-1)^j\binom{N-1}{j}\Gamma\left(s-\frac{j}{2}\right)E_1^c\left(s-\frac{j}{2}\right)+\Delta_{ER}. \tag{65}
$$

The poles of this last function arise from those of $E_1^c(s-j/2)$, which are obtained for the values of s such that $s-j/2=1/2,-1/2,-3/2,\ldots$. They are poles of order one at $s = N/2, (N-1)/2, N/2-1, \ldots$, except for $s = 0, -1, -2, \ldots$, then the function is finite (owing to the $\Gamma(s)$ in the denominator). These poles are removed by zeta-function regularization [14,15].

3. Application: spontaneous compactification in 2D quantum gravity

Let us consider induced 2D gravity, with the action

$$S = \int d^2x \sqrt{g} \left(R \frac{1}{\Delta} R + \Lambda \right), \tag{66}$$

on the background $R^1 \times S^1$. On such a background —which is not the solution of the classical equations of motion— the convenient effective action is always gauge dependent. However, the S-matrix (the effective action *on shell*, i.e., at the stationary points) is independent of the gauge condition choice. Actually, working in the loop expansion, one is led to an explicit gauge dependence even on shell (perturbatively). This is why it is preferable to work with the gauge-independent effective action.

Using the standard background field method,

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + h_{\mu\nu}, \tag{67}$$

where $g_{\mu\nu}$ is the metric of flat space $R^1 \times S^1$ and $h_{\mu\nu}$ is the quantum gravitational field, choosing the gauge fixing action as

$$S_{GF} = \frac{1}{\alpha} \int d^2x \sqrt{g} \left(\nabla_\mu h^\mu_\rho - \beta \nabla_\rho h \right)^2, \tag{68}$$

where α and β are the gauge parameters and $h = h^\mu_\mu$, and defining the configuration-space metric in accordance with Vilkoviski [17]

$$\gamma_{ij} \equiv \gamma_{g_{\mu\alpha} g_{\nu\beta}} = \frac{1}{2} \sqrt{g} \left(g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - a g^{\mu\nu} g^{\alpha\beta} \right), \tag{69}$$

where a is a constant parameter, after some work, one obtains the following result for the one-loop effective action [18]

$$\Gamma^{(1)} = 2\pi R S \Lambda + \frac{1}{2} \left[\mathrm{Tr}\, \ln \left(\Delta + \frac{\Lambda}{4(2-a)} \right) - 2\, \mathrm{Tr}\, \ln \Delta \right]. \tag{70}$$

Here $2\pi R$ is the length of the compactified dimension while $S = \int dx$ is the 'volume' of the space R^1. As we see, the dependence on the gauge parameters α and β has disappeared. However, an explicit dependence on the parameter a remains.

The trace calculations involved in expression (70) for the one-loop effective action are not easy. Non-trivial commutations of series have to be carried out. Using the techniques developed above (already specified to $R^1 \times S^1$), we have

$$\begin{aligned}
\zeta_{-\Delta+m^2}\left(\frac{s}{2}\right) &= -S\int_0^\infty \frac{dk}{\pi}\sum_{n=-\infty}^{+\infty}\left[k^2+\left(\frac{2\pi n}{\beta}\right)+m^2\right]^{-s/2} \\
&= -\frac{S}{\sqrt{\pi}}m^{1-s}\left\{\frac{-\Gamma\left(\frac{s-1}{2}\right)}{2\Gamma\left(\frac{s}{2}\right)}+\frac{\beta m}{2\sqrt{\pi}}\frac{1}{s-2}+\frac{\left(\frac{\beta m}{2}\right)^{(s-1)/2}}{\Gamma\left(\frac{s}{2}\right)}\sum_{k=0}^{\infty}\frac{(16\pi)^{-k}}{k!}\right. \\
&\times \left.\left(\frac{2\pi}{\beta m}\right)^k\prod_{j=1}^{k}\left[(s-2)^2-(2j-1)^2\right]\sum_{n=1}^{\infty}n^{\frac{s-3}{2}-k}e^{-\beta m n}\right\}, \qquad (71)
\end{aligned}$$

wherefrom we get

$$\begin{aligned}
V &= \frac{\Gamma^{(1)}}{S} = 2\pi R\Lambda + \frac{R\Lambda}{32(2-a)}\left[1-\ln\left(\frac{\Lambda}{4(2-a)}\right)\right]-\frac{1}{8}\sqrt{\frac{\Lambda}{2-a}}+\frac{1}{24R} \\
&- \frac{1}{4\pi\sqrt{2R}}\left(\frac{\Lambda}{2-a}\right)^{1/4}\sum_{k=0}^{\infty}\frac{(16\pi)^{-k}}{k!}\left(\frac{R}{2}\sqrt{\frac{\Lambda}{2-a}}\right)^{-k} \qquad (72) \\
&\times \prod_{j=1}^{k}\left[4-(2j-1)^2\right]\sum_{n=1}^{\infty}n^{-(k+3/2)}\exp\left(-\pi R\sqrt{\frac{\Lambda}{2-a}}\,n\right).
\end{aligned}$$

This expression can be simplified in terms of the basic variables of the problem:

$$x \equiv \frac{\Lambda}{4(2-a)}, \qquad y \equiv R\sqrt{x} = \frac{R}{2}\sqrt{\frac{\Lambda}{2-a}}. \qquad (73)$$

Then

$$V = \sqrt{x}\left[8\pi(2-a)y+\frac{y}{8}(1-\ln x)-\frac{1}{4}+\frac{1}{24y}-F(y)\right], \qquad (74)$$

$F(y)$ being given by

$$F(y) = \frac{1}{4\pi}\sum_{k=0}^{\infty}\frac{(16\pi)^{-k}}{k!}y^{-(k+1/2)}\prod_{j=1}^{k}\left[4-(2j-1)^2\right]\sum_{n=1}^{\infty}n^{-(k+3/2)}e^{-2\pi n y}. \qquad (75)$$

It is now clear that all the dependence of the action on R, Λ and a comes through the specific combination given by the variable y, but for a global factor, $\sqrt{x}$, and for the first term, which is just linear in a.

To proceed with the compactification program, one imposes (as is done in multidimensional gravity)

$$\left\{\begin{array}{l} V(R,\Lambda,a)=0, \\ \dfrac{\partial V(R,\Lambda,a)}{\partial R}=0. \end{array}\right. \qquad (76)$$

The explicit a dependence can be eliminated, and one gets

$$\sqrt{x}\left[F(y) - yF'(y) - \frac{1}{12y} + \frac{1}{4}\right] = 0. \tag{77}$$

This transcendent equation involves an asymptotic series, and must be solved aproximately. Fortunately, the decreasing exponentials come to rescue and, after an explicit calculation one obtains the (expected) result:

$$y_1 = 0.33. \tag{78}$$

This is the non-trivial stationary point of the effective action. The trivial one is reached for

$$x_0 = 0. \tag{79}$$

As for the second derivative,

$$\frac{\partial^2 V}{\partial y^2} = \sqrt{x}\left[\frac{1}{12y^3} - F''(y)\right], \tag{80}$$

where the explicit a-dependence has disappeared. Hence, this derivative has a definite sign (independent of a) at the stationary point

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_{y=y_1} \simeq 2 > 0. \tag{81}$$

The point is a minimum, obtained for the following combination of parameters

$$\frac{\Lambda R^2}{2-a} \simeq \left(\frac{2}{3}\right)^2. \tag{82}$$

4. Application: stability of the rigid membrane

Consider now the following action, which is multiplicatively renormalizable only in the string case ($p = 1$),

$$S = \int d^{p+1}\xi\, \sqrt{g}\left(k + \frac{1}{2\rho^2}\left[\Delta(g)X^i\right]^2\right), \tag{83}$$

where $g_{\alpha\beta} = \partial_\alpha X^i \partial_\beta X^i$, $\alpha = 0, 1, \ldots, p$, $i = 1, 2, \ldots, d$, $\Delta(g) = g^{-1/2}\partial_\alpha g^{1/2} g^{\alpha\beta}\partial_\beta$, the constant k is the analog of the usual string tension, and $1/\rho^2$ is the coupling constant corresponding to the rigid term. We take for background the classical solutions of the field equations (which are the same for the rigid as for the usual p-brane)

$$X^0_{cl} = \xi_0, \quad X^\perp_{cl} = 0, \quad X^{d-1}_{cl} = \xi_1, \ldots, X^{d-p}_{cl} = \xi_p, \tag{84}$$

with $X_{cl}^{\perp} = (X^1, \ldots, X^{d-p-1})$ and $(\xi_1, \ldots, \xi_p) \in \mathcal{R} \equiv [0, a_1] \times \cdots \times [0, a_p]$. We use the axial gauge

$$X^0 = X_{cl}^0, \quad X^{d-1} = X_{cl}^{d-1}, \ldots, X^{d-p} = X_{cl}^{d-p}, \tag{85}$$

where the Faddeev-Popov ghosts are absent. In the case of the toroidal rigid p-brane, the boundary conditions are

$$X^{\perp}(0, \xi_1, \ldots, \xi_p) = X^{\perp}(T, \xi_1, \ldots, \xi_p) = 0 \tag{86}$$

and

$$\begin{aligned} X^{\perp}(\xi_0, 0, \xi_2, \ldots, \xi_p) &= X^{\perp}(\xi_0, a_1, \xi_2, \ldots, \xi_p), \\ &\vdots \\ X^{\perp}(\xi_0, \xi_1, \ldots, \xi_{p-1}, 0) &= X^{\perp}(\xi_0, \xi_1, \ldots, \xi_{p-1}, a_p). \end{aligned} \tag{87}$$

On the other hand, for fixed-end boundary conditions the first equation is exactly the same, while the rest are replaced by the following (of Dirichlet type)

$$\begin{aligned} X^{\perp}(\xi_0, 0, \xi_2, \ldots, \xi_p) &= \cdots = X^{\perp}(\xi_0, \xi_1, \ldots, \xi_{p-1}, 0), \\ X^{\perp}(\xi_0, a_1, \xi_2, \ldots, \xi_p) &= \cdots = X^{\perp}(\xi_0, \xi_1, \ldots, \xi_{p-1}, a_p). \end{aligned} \tag{88}$$

The effective potential is

$$V = - \lim_{T \to \infty} \frac{1}{T} \ln \int \mathcal{D}X^{\perp} \exp(-S). \tag{89}$$

Restricting ourselves to the one-loop approximation, we need only take into account the terms which are quadratic in the quantum fields.

Integrating out $X^{\perp}$ and imposing the boundary conditions, one obtains

$$\begin{aligned} V_{\text{fixed end}} &= k \prod_{i=1}^{p} a_i + \frac{d-p-1}{2} \left[\sum_{n_1, \ldots, n_p = 1}^{\infty} \left(\frac{\pi^2 n_1^2}{a_1^2} + \cdots + \frac{\pi^2 n_p^2}{a_p^2} \right)^{1/2} \right. \\ &+ \left. \sum_{n_1, \ldots, n_p = 1}^{\infty} \left(\frac{\pi^2 n_1^2}{a_1^2} + \cdots + \frac{\pi^2 n_p^2}{a_p^2} + k\rho^2 \right)^{1/2} \right], \end{aligned} \tag{90}$$

and

$$\begin{aligned} V_{\text{toroidal}} &= k \prod_{i=1}^{p} a_i + \frac{d-p-1}{2} \left[\sum_{n_1, \ldots, n_p = -\infty}^{\infty} \left(\frac{4\pi^2 n_1^2}{a_1^2} + \cdots + \frac{4\pi^2 n_p^2}{a_p^2} \right)^{1/2} \right. \\ &+ \left. \sum_{n_1, \ldots, n_p = -\infty}^{\infty} \left(\frac{4\pi^2 n_1^2}{a_1^2} + \cdots + \frac{4\pi^2 n_p^2}{a_p^2} + k\rho^2 \right)^{1/2} \right]. \end{aligned} \tag{91}$$

First, we calculate the static potential —that is, the effective potential in the limit of large spacetime dimensionality. Such calculation gives [19]

$$\begin{aligned} S_{eff} &= kTR^p \left\{ (1+\sigma_0)^{1/2}(1+\sigma_1)^{p/2} - \frac{1}{2}(\sigma_0\lambda_0 + p\sigma_1\lambda_1) \right. \\ &+ \frac{d-p-1}{2kR^{p+1}} \left[\sum_{\vec{n}} \left(\pi^2\vec{n}^2 + \frac{k\rho^2\lambda_0 R^2}{2} - \sqrt{k\rho^2(\lambda_0-\lambda_1)\vec{n}^2R^2 + \frac{k^2\rho^4\lambda_0^2R^4}{4}} \right)^{1/2} \right. \\ &+ \left.\left. \sum_{\vec{n}} \left(\pi^2\vec{n}^2 + \frac{k\rho^2\lambda_0 R^2}{2} + \sqrt{k\rho^2(\lambda_0-\lambda_1)\vec{n}^2R^2 + \frac{k^2\rho^4\lambda_0^2R^4}{4}} \right)^{1/2} \right] \right\}, \end{aligned} \tag{92}$$

where for the fixed-end p-brane $\vec{n}^2 = n_1^2 + \cdots + n_p^2$ and $\sum_{\vec{n}}$ means $\sum_{n_1,\ldots,n_p=1}^{\infty}$, while for the toroidal p-brane $\vec{n}^2 = 4(n_1^2 + \cdots + n_p^2)$ and $\sum_{\vec{n}}$ means $\sum_{n_1,\ldots,n_p=-\infty}^{\infty}$. Here the expressions to be regularized involve $a_1 = \cdots = a_p = 1$ and $\alpha_1 = \cdots = \alpha_p = 2$, and the general formulas of section 2 are considerably simplified.

A very useful and exact recurrent formula is [13] ((61) with $c_1 = \cdots = c_N = 1$ and c^2 instead of c)

$$\begin{aligned} E_N^c(s;a_1,\ldots,a_N) &\equiv \sum_{n_1,\ldots,n_N=1}^{\infty} (a_1n_1^2 + \cdots + a_Nn_N^2 + c^2)^{-s} \\ &= -\frac{1}{2}E_{N-1}^c(s;a_2,\ldots,a_N) + \frac{1}{2}\sqrt{\frac{\pi}{a_1}}\frac{\Gamma(s-1/2)}{\Gamma(s)}E_{N-1}^c(s-1/2;a_2,\ldots,a_N) \\ &+ \frac{\pi^s}{\Gamma(s)}a_1^{-s/2}\sum_{k=0}^{\infty}\frac{a_1^{k/2}}{k!(16\pi)^k}\prod_{j=1}^{k}[(2s-1)^2-(2j-1)^2]\sum_{n_1,s,n_N=1}^{\infty} n_1^{s-k-1} \\ &\times (a_2n_2^2 + \cdots + a_Nn_N^2 + c^2)^{-(s+k)/2}\exp\left[-\frac{2\pi}{\sqrt{a_1}}n_1(a_2n_2^2 + \cdots + a_Nn_N^2 + c^2)^{1/2}\right]. \end{aligned} \tag{93}$$

The recurrence starts from expression

$$E_1^c(s;1) = -\frac{c^{-2s}}{2} + \frac{\sqrt{\pi}}{2}\frac{\Gamma(s-1/2)}{\Gamma(s)}c^{-2s+1} + \frac{2\pi^sc^{-s+1/2}}{\Gamma(s)}\sum_{n=1}^{\infty}n^{s-1/2}K_{s-1/2}(2\pi nc). \tag{94}$$

In order to deal with the derivative of the function K_ν above, one can follow two equivalent procedures: either do first the usual analytic continuation, and then take $s=-1/2$ and the derivative afterwards, or else take first the derivative, perform then the analytic continuation and put $s=+1/2$ at the end. The result is exactly the same. In either way, other non-trivial series commutations have to be performed. We get, in particular, for $c \neq 0$

$$\begin{aligned} E_2^c(s) &= -\frac{1}{2}E_1^c(s) + \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}E_1^c\left(s-\frac{1}{2}\right) + \Delta_{ER}, \\ E_3^c(s) &= \frac{1}{4}E_1^c(s) - \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}E_1^c\left(s-\frac{1}{2}\right) + \frac{\pi}{4(s-1)}E_1^c(s-1) + \Delta_{ER}, \end{aligned} \tag{95}$$

and similar expressions for $c = 0$.

For the sake of conciseness, we shall now restrict ourselves to $p = 2$ —but it is obvious that we could consider as well any other value of p. We rely on equations (93) and (94), which specialized to $p = 2$ yield

$$\sum_{n_1,n_2=1}^{\infty} \sqrt{\left(\frac{n_1}{a_1}\right)^2 + \left(\frac{n_2}{a_2}\right)^2} = \frac{1}{24}\left(\frac{1}{a_1}+\frac{1}{a_2}\right) - \frac{\zeta(3)}{8\pi^2}\left(\frac{a_1}{a_2^2}+\frac{a_2}{a_1^2}\right) - \frac{\pi^{3/2}}{2\sqrt{a_1a_2}} \exp\left(-2\pi\frac{a_1}{a_2}\right)\left[1+\mathcal{O}(10^{-3})\right], \quad (96)$$

and (this one after additional regularization)

$$\sum_{n_1,n_2=1}^{\infty} \sqrt{\left(\frac{n_1}{a_1}\right)^2 + \left(\frac{n_2}{a_2}\right)^2 + c^2} = \frac{c}{4} - \frac{\pi}{6}a_1a_2c^3 + \left(\frac{1}{4\pi}\sqrt{\frac{c}{a_2}} - \frac{ca_1}{4\pi a_2}\right) \times \exp\left(-2\pi c a_2\right)\left[1+\mathcal{O}(10^{-3})\right]. \quad (97)$$

In both cases we have assumed (it is, of course, no restriction) that $a_2 \le a_1$. These expressions are really valuable. They are asymptotic, the last term (already of exponential kind) being of order 10^{-3} with respect to the two first ones, and the not explicitly written contributions being of order 10^{-6}. To our knowledge, the second expression —which can be termed as of inhomogeneous Epstein type— has never been discussed in the literature [19].

For fixed-end boundary conditions and not taking into account exponentially-small terms, we obtain

$$V_{f.e.} \simeq ka_1a_2 + \frac{(d-3)\pi}{24}\left[\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right) - \frac{3\zeta(3)}{2\pi^2}\left(\frac{a_1}{a_2^2}+\frac{a_2}{a_1^2}\right) + \frac{3}{\pi}\sqrt{k}\rho - \frac{2}{\pi^2}k^{3/2}\rho^3a_1a_2\right]. \quad (98)$$

It is straightforward to perform the analysis of extrema of V [19].

In the case of toroidal boundary conditions, again neglecting exponentially-small contributions, we get (for a very detailed discussion of the relations between the different boundary conditions see [20])

$$V_{tor} \simeq ka_1a_2 + \frac{(d-3)\pi}{2}\left[-\frac{\zeta(3)}{\pi^2}\left(\frac{a_1}{a_2^2}+\frac{a_2}{a_1^2}\right)\frac{1}{\pi}\sqrt{k}\rho - \frac{1}{6\pi^2}k^{3/2}\rho^3a_1a_2\right]. \quad (99)$$

The particular extremum for $a_1 = a_2 \equiv a$ is a minimum of V provided that $\sqrt{k}\rho^3 > 12\pi$ (it is a maximum for $\sqrt{k}\rho^3 < 12\pi$). Consistency with the series expansion implies

$$\sqrt{k}\rho > \frac{12\pi}{\rho^2} >> 1. \quad (100)$$

which can be met typically for values of $\rho \simeq 3$, $k \simeq 9$, $2\pi c \simeq 9$ —but, of course, as in the former case, the range of allowed values is much wider.

5. Application to the new experiments involving the Casimir effect

The following references on experimental applications in very recent determinations of the contribution of the Casimir effect to remarkable wetting and nonwetting patterns of helium 4 adsorbed on alkali metals [21], to critical fluctuations within very narrow fluid films confined by rigid walls (for different boundary conditions and temperatures) [22], and to related cavity effects in laser physics [23], are highly interesting. A more classically minded, historical account of this subject of the Casimir effect in quantum field theory and condense matter physics can be found in ref. [24].

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Nonabelian Gauge Fields in the Background of Magnetic Strings*

E. Wieczorek
DESY-IfH Zeuthen

Quantized nonabelian gauge fields are studied in the external classical background of a linear magnetic string. The determination of the gauge field propagator demands a specification of the string by suitable physical limiting procedures.The vacuum energy density is obtained after transforming the background problem into a Casimir problem.

We consider a quantized nonabelian gauge theory (with gauge group SU(2) for simplicity) in the background of a linear magnetic string where gauge potential and field strength are given by

$$B_\mu(x) = -\beta\epsilon^{\perp}_{\mu\rho}x^{\perp}_{\rho}/r^2_{\perp} \tag{1}$$

$$F_{\mu\nu} = \beta\epsilon^{\perp}_{\mu\nu}\delta(r_\perp)/r_\perp, \tag{2}$$

respectively. This background field carries the magnetic flux

$$\int d\sigma_{\mu\nu}F_{\mu\nu} = 2\pi\beta. \tag{3}$$

Here and in the following we apply the notations $x_\perp = (x_1, x_2) = (r\cos\varphi, r\sin\varphi)$, $x_{\|} = (x_3, x_4)$

$$\epsilon^{\perp}_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and use euclidean metric as long as we are concerned with the determination of propagators. The wave equations for a scalar massless matter field transforming according to the fundamental representation of $SU(2)$ reads

$$-D^2\psi = \lambda\psi \tag{5}$$

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with $D_\mu = \partial_\mu \pm \frac{ig}{2} B_\mu$. After a separation of variables $\psi = \exp ik_{||}x_{||} \exp -ip\varphi f(r)$ the radial equation

$$-(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{(p-\beta')^2}{r^2})f = k^2 f \tag{6}$$

remains (with $\beta' = \pm\beta/2$) where, in general, $p - \beta'$ takes noninteger values. In such cases the scalar wave equation in the background of a magnetic string has the same structure as the corresponding equation for cosmic stings (i.e. in the background of a conical space-time). The equation for the ghost field

$$(-D^2)_{ab}\psi_b = \lambda\psi_a, \tag{7}$$
$$D_\mu^{ab} = \partial_\mu \delta^{ab} + gB_\mu \epsilon^{ab3}$$

leads after diagonalization to the same radial equation ($\beta \to \beta'$). The propagators for matter and ghost fields have been obtained from the eigenfunctions of (5) and (7) in straightforward manner or by a direct method, see [1].

The gauge field propagator

The wave equation for the gauge field

$$((-D^2)^{ab}\delta_{\mu\nu} + 2gF_{\mu\nu}\epsilon^{ab3})\psi_b = \lambda\psi_a \tag{8}$$

or the corresponding radial equation

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{(p-\beta)^2}{r^2} \pm \frac{2g\beta}{r}\delta(r) + k^2)\psi(r) = 0 \tag{9}$$

lead to a serious problem connected with the localization of $F_{\mu\nu}$.

It is well-known that 2 or 3 dimensional δ potentials in wave equations represent an ill-defined mathematical problem. The rigorous mathematical method to deal with such an equation would be to take into account the possibility of self-adjoint extensions of the differential operators under consideration. Let us refer to two instructive examples.

The one-dimensional equation $-\partial_x^2\psi = k^2\psi, x \in [0,\infty)$ can be supplemented by the boundary condition $\psi'(x) = \theta\psi(x)$ at $x = 0_+$ with an arbitrary real parameter θ. The solution $\psi_+ = \cos kx + \theta/k \sin kx$ is a generalization of the usual even solution $\cos kx$. More familiar is the equation $(-\partial_x^2 + 2a\delta(x))\psi = k^2\psi$ which exactly has the same solutions, if we identify the parameter of s.a.extension θ with the δ potential coefficient. Quite different appears the radial equation for $d = 3 : -(\partial_r^2 + 2/r\partial_r)\psi = k^2\psi$ together with the boundary condition

$$\lim_{r\to 0}(r\psi) = \theta \lim_{r\to 0}(\psi + r\psi').$$

It allows solutions $\psi = \frac{\sin kr}{r} + \theta\frac{k\cos kr}{r}, k \geq 0$, which although being singular are normalizeable. In this case, however, θ cannot be quantitatively related to a potential $\delta(\vec{x})$,

see[2].
The same situation shows up in the radial equation for a magnetic string. The differential operator

$$K = -(\partial_r^2 + 1/r\partial_r - \nu^2/r^2) \tag{10}$$

(compare (9)) allows, if $0 < \nu = |p - \beta| < 1$, a s.a.extension defined in terms of the boundary condition

$$\lim_{r\to 0} r^\nu \psi(r) = \theta \lim_{r\to 0} 1/r^{2\nu}(r^\nu\psi(r) - \lim_{r'\to 0}(r'^\nu\psi(r')) \tag{11}$$

which leads to solutions

$$\psi_k(r) = J_\nu(kr) + \theta J_{-\nu}(kr). \tag{12}$$

Wave equations with s.a extensions have been studied in conical space-time [3] where the parameter θ is considered as to describe in a unspecified manner the coupling of the wave function to curvature localized at the tip of the cone. For other authors [4] θ characterizes inner properties of the magnetic string.
From our point of view it would be inadequate to abandon the relation between the gauge potential (entering via the parameter ν) and the field strength which would be done by keeping the parameter θ arbitrary. We therefore prefer to apply a physical regularization in which the string is treated as a limiting case of a flux tube of finite radius keeping the flux (3) fixed.

As a special model we have studied the homogeneous finite flux tube of radius a

$$\begin{aligned} gB_\mu &= -\beta\epsilon^\perp_{\mu\nu}\frac{x_\rho}{a^2} \qquad \text{for} \qquad 0 < r < a \\ &= -\beta\epsilon^\perp_{\mu\nu}\frac{x_\nu}{r^2} \qquad \text{for} \qquad r > a \end{aligned} \tag{13}$$

(for details see [1]). Outside the tube the radial solution to (8) has the form (12) where now, however, θ is not a number but a function $\theta = \theta(ka)$ which is determined by matching ψ'/ψ at $r = a$. We observe that for the model (13) as well as for the other models studied by us $\theta(ka) = \theta(\xi)$ has the properties

$$\begin{aligned} &\lim_{\xi\to 0} \theta(\xi) = 0, \\ &\lim_{\xi\to\infty} \theta(\xi) = (-1)^p \end{aligned} \tag{14}$$

for almost all values of the flux parameter β consistent with the condition $0 < |p - \beta| < 1$. There is a finite number of flux parameters β_i such that $\lim_{\xi\to 0}\theta(\xi) = \infty$ for $\beta = \beta_i$. The values β_i as well as the number of such "exceptional" flux parameters depend on the partial wave number p ,on the sign of the $F_{\mu\nu}$ term in (8) and (9) as well as on the special model for the string [1]. Up to now we have not found a model so that $\lim_{\xi\to 0}\theta(\xi) \neq 0$. An infinite limit for $\theta(\xi)$ yields, of course, the solution $\psi = J_{-\nu}(kr)$ for $r > a$.

From the property (14) one concludes at once, that the singular part of the outer solution tends to zero with vanishing tube radius for all fixed values of quasimomentum k.

To decide wether the singular part influences the propagator demands more care: in the construction of propagators from eigenfunctions

$$G(x,x') = \frac{1}{(2\pi)^3}\sum_{-\infty}^{+\infty} \exp ip(\varphi-\varphi')\int dk_{\|}\exp ik_{\|}(x-x')_{\|}\int_0^\infty dkk\frac{\psi_k(r)\psi_k^*(r')}{k_{|}^2+k^2} \qquad (15)$$

the infinite range of k integration forbids a naive application of (14). Let us therefore discuss the propagator in the outer region $r > a, r' > a$. Here the radial eigenfunction is

$$\psi_k(r) = \frac{1}{\sqrt{1+\theta^2}}J_\nu(kr) + \frac{\theta}{\sqrt{1+\theta^2}}J_{-\nu} \qquad (16)$$

which, together with its part in $0 < r < a$, is properly normalized and complete

$$\int_0^\infty dkk\psi_k(r)\psi_k^*(r') = \frac{1}{r}\delta(r-r').$$

Then G receives in the partial wave p the additional term ΔG^p originating from $\theta J_{-\nu}$

$$\begin{aligned}\Delta G^p &= \frac{1}{(2\pi)^3}\exp ip(\varphi-\varphi')\int_0^\infty dkk\int d^2k_{\|}\frac{\exp iq_{\|}(x-x')_{\|}}{q_{\|}^2+k^2}F(kr,kr',ka)\\ &= \frac{1}{4\pi^2}\exp ip(\varphi-\varphi')\int_0^\infty dkkK_0(kz)F(kr,kr',ka)\end{aligned} \qquad (17)$$

with

$$\begin{aligned}F &= \frac{\theta^2(ka)}{1+\theta^2}\left(J_{-\nu(kr)}J_{-\nu}(kr') - (\nu\to-\nu)\right)\\ &+ \frac{\theta}{1+\theta^2}\left(J_\nu(kr)J_{-\nu}(kr') + (\nu\to-\nu)\right)\end{aligned} \qquad (18)$$

and

$$z^2 = (x-x')_{\|}^2. \qquad (19)$$

A closer inspection of (17) shows the following properties of ΔG^p valid for non-exceptional values of the flux parameter:

1. ΔG^p tends to zero for $a \to 0$ if $z/(r+r')$ is kept fixed and different from zero.
2. The coefficient of $\log(z/(r+r'))$ which is the leading short distance term and the finite part $\Delta G^p(x,x)$ vanish in the limit $a \to 0$.

The second point is important with respect to the evaluation of the vacuum energy density: it is the short distance limit of propagators which determines this quantity.
A further point to be discussed is the contribution of bound state wave functions to the propagator. Indeed eq.(9) together with the regularization (13) has bound state solutions

$$\psi_i = \left(\frac{2\eta_i^2\sin\nu\pi}{a\nu\pi}\right)^{1/2}K_\nu(\eta_i r/a) \qquad \text{for} \qquad r > a \qquad (20)$$

corresponding to eigenvalues $k^2 = -\eta_i^2/a^2$. Here η_i are real numbers to be determined from the matching condition at $r = a$. Obviously bound states can occur for the eq.(9) with the upper sign only. Inserting the eigenfunctions (20) into the construction of the propagator we remark that this contribution vanishes exponentially in the limit $a \to 0$ for all fixed $r > 0, r' > 0$. The discussion of the gauge propagator leads finally to the following conclusion:

There are no remnants of the $F_{\mu\nu}$ term in (8) for $a \to 0$ as long as the flux parameter is non-exceptional, or, in other words, the mathematical possibility of self-adjoint extensions is in general not realized in our model. Consequently, the gauge propagator $G_{\mu\nu}^{ab}$ is simply related to the ghost propagator G^{ab} by

$$G_{\mu\nu}^{ab} = \delta_{\mu\nu}G^{ab} = \delta_{\mu\nu}(-D^2)_{ab}^{-1}. \tag{21}$$

The vacuum energy in external backgrounds

A rather elegant method to evaluate vacuum energies in 1-loop approximation is provided by the ζ function method. Be K the kinetic kernel of a boson field with normalized eigenfunctions ψ_{qn} corresponding to eigenvalues λ_{qn}. Defining the kernel

$$[K]^{-s}(x,y) = \sum_n \int dq \frac{\psi_{qn}(x)\psi_{qn}^*(y)}{\lambda_{qn}^s} \quad \text{for} \quad \Re s > s_0 \tag{22}$$

the 1-loop expressions for the vacuum functionals

$$Z = (Det K)^{-1/2} \qquad \text{and} \qquad W = \frac{1}{2} Tr \log K = \int dx \rho_{vac}$$

can be determined from the ζ function of K

$$\begin{aligned} \zeta_K(s) &= Tr K^{-s} \\ &= \int dx \int dy \delta(x-y)[K]^{-s}(x,y) \end{aligned} \tag{23}$$

by the formula

$$W = -\frac{1}{2}\frac{\partial}{\partial s}\zeta_k(s) \quad \text{at} \quad s = 0. \tag{24}$$

For a massive charged scalar field in the background of a magnetic string the kinetic kernel has a pure continuous spectrum $\lambda(q) = q^2 + m^2, q \in R_4$ which leads to the ζ_K function

$$\zeta_K(s) = 2V_4 \int d^4q(q^2 + m^2)^{-s}$$

which is well-defined for $\Re s > 2$ but independent of the flux parameter and therefore obviously useless. The appropriate method in such a situation is to enclose the system into a box rendering the spectrum at least partly discrete [5]. If in the present case one would enclose the string into a cylinder of radius R with Dirichlet boundary conditions at $r = R$ this discretizes the quasimomentum $k_\perp$ according to the equation $J_\nu(k_\perp r) = 0$. However, due to wellknown properties of the zeros of Bessel functions this leads to $\lim_{R\to 0} W(R) = 0$!

We therefore have to make recourse to a direct calculation which relies on the energy-momentum tensor. Let us start from a Lagrangian for a gauge theory in an external background restricted to quadratic terms (this corresponds to 1-loop approximation)

$$\begin{aligned} L = &- 1/2 D_\lambda^{cd} a_\mu^d D_{cb}^\lambda a^{\mu b} + g\epsilon_{bc3} a_\mu^b a_\nu^c F^{\mu\nu} \\ &+ D_\lambda^{cd} \bar\eta^d D_{cb}^\lambda \eta^b - (D\chi)^* D\chi. \end{aligned} \tag{25}$$

Here $a_\mu^b, \bar\eta^b, \eta^b, \chi$ denote the gauge field fluctuations, ghost and matter fields, respectively. As in (8) the background gauge fixing term $-1/2(D_\mu a^\mu)_b(D_\nu a^\nu)_b$ with $\alpha = 1$ has been chosen. From (25) the canonical energy-momentum tensor can be derived by conventional methods, especially

$$\begin{aligned} T_{00} = -1/2(&\sum_{i=0,3} \partial_i a_b^\mu \partial_i a_\mu^b + \sum_{i=1,2} (D_i a_\mu)_c (D_i a^\mu)_c \\ &- g F_{bc}^{\mu\nu} a_\mu^b a_\nu^c \\ &+ \sum_{i=0,3} \partial_i \bar\eta_a \partial_i \eta_a + \sum_{i=1,2} (D_i \bar\eta)_a (D_i \eta)_a \\ &+ \sum_{i=0,3} \partial_i \chi_r^* \partial_i \chi_r + \sum_{i=1,2} (D_i\chi)_r^* (D_i \chi)_r) \end{aligned} \tag{26}$$

where special attention has been paid to the ordering of ghost fields. $F_{bc}^{\mu\nu}$ is the external background field supposed to be time- independent.Furthermore, point splitting technique is applied in order to express $< 0|T_{00}|0 >$ in terms of the propagators

$$\begin{aligned} &< 0|T a_\mu^b(x) a_\nu^c(y)|0 > = G_{\mu\nu}^{bc}(x,y), \\ &< 0|T\bar\eta^b(x)\eta^c(y)|0 > = - < 0|T\eta^c(y)\bar\eta^b(x)|0 >= -G^{cb}(y,x) \\ &< 0|T\chi_r(x)\chi_s^*(y)|0 > = G_{rs}(x,y). \end{aligned} \tag{27}$$

One should notice the minus sign in the ghost part ! This method has been tested in the familiar case of a homogeneous $SU(2)$ color magnetic background. Combining eqs.(26), (27) and the explicitely known structure of the background propagators [6] one obtains, apart from renormalization terms $\sim B^2$, the wellknown expression

$$\begin{aligned} \rho_{vac} &= (-\frac{14}{3} + 8 + \frac{1}{3})\frac{g^2 B^2}{16\pi^2} \log \frac{gB}{\mu^2} \\ &= \frac{11}{48\pi^2} g^2 B^2 \log \frac{gB}{\mu^2} \end{aligned} \tag{28}$$

see[7]. It is instructive to indicate the origin of the single terms in the bracket of (28): -14/3, +8, and +1/3 correspond to the gauge field contribution with derivatives, the $F_{\mu\nu}$

part (second line in(26)) and the ghost contribution, respectively. The same method will be applied to the vacuum energy density in the background of a magnetic string. Before doing so the background problem is transformed into a Casimir problem.

Reduction to a Casimir problem

For reasons to become clear later it is convenient to transform the background problem of an ideal singular magnetic string into a generalized Casimir problem by applying appropriate gauge transformations for the matter field in fundamental representation

$$\begin{pmatrix} \psi'_+ \\ \psi'_- \end{pmatrix} = \begin{pmatrix} \exp \frac{i\beta\varphi}{2} \psi_+ \\ \exp \frac{-i\beta\varphi}{2} \psi_- \end{pmatrix} \tag{29}$$

and for the ghost and gauge field fluctuations

$$\chi'_i = R_{ij}(x)\chi_j \tag{30}$$

with

$$R_{ij} = \begin{pmatrix} \cos\beta\varphi & \sin\beta\varphi & 0 \\ -\sin\beta\varphi & \cos\beta\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{31}$$

respectively. Thereby the background field B_μ is locally transformed away. The transformed fields obey for $r > 0$ the field equation without a background field

$$-\Delta\psi = k^2\psi$$

together with boundary conditions (e.g. along the axis $\varphi = \pi$ in the $x_\perp$ plane)

$$\psi'_\pm(\varphi = \pi + 0) = \exp(\pm i\beta\pi)\psi'_\pm(\varphi = \pi - 0) \tag{32}$$

and

$$\begin{pmatrix} \chi'_1(\varphi = \pi + 0) \\ \chi'_2(\varphi = \pi + 0) \end{pmatrix} = \begin{pmatrix} \cos 2\pi\beta & \sin 2\pi\beta \\ -\sin 2\pi\beta & \cos 2\pi\beta \end{pmatrix} \begin{pmatrix} \chi'_1(\varphi = \pi - 0) \\ \chi'_2(\varphi = \pi - 0) \end{pmatrix}. \tag{33}$$

Of course, the boundary conditions (32) and (33) have their origin in the continuity and uniqueness of the unprimed fields as functions of φ. It is in this equivalent Casimir formulation, that it becomes obvious immediately, that an increase for β by an integer (or an even integer) is irrelevant as long as the $F_{\mu\nu}$ term in the original eq.(8) does not matter. Corresponding with the wave functions the propagators are transformed as follows

$$G'^{\pm}_{\beta/2} = \exp \frac{\pm i\beta\Delta}{2} G^{\pm}_{\beta/2} \tag{34}$$

$$G'^{ab}(x, x') = R^{ac}(x)R^{bd}(x')G^{cd}(x, x') \tag{35}$$

The unprimed matter field propagator has the structure

$$G^{+}_{\beta/2} = \exp(-i(n+1)\Delta)\frac{G_0}{\sinh\phi}\{(\sinh(\beta_0\phi/2) + \exp(i\Delta)\sinh((1-\beta_0/2)\phi))\} \quad (36)$$

$$G^{-}_{\beta/2} = \exp(in\Delta)\frac{G_0}{\sinh\phi}\{(\sinh((1-\beta_0/2)\phi) + \exp(i\Delta)\sinh(\beta_0\phi/2))\} \quad (37)$$

for the flux parameter $\beta = \beta_0 + 2n$ with $0 \le \beta_0 < 1$. The variables Δ and ϕ are defined by

$$\cosh\phi = \frac{(x-x')^2_{\parallel} + r^2 + r'^2}{2rr'} \quad (38)$$
$$\Delta = \varphi - \varphi'.$$

Furthermore, G_0 denotes the usual massless free propagator

$$G_0(x-x') = \frac{1}{4\pi^2(x-x')^2}. \quad (39)$$

The propagators in adjoint representation (i.e. for ghost and gauge fields, compare (21)) read

$$\begin{aligned} G^{11}_{\beta} &= G^{22}_{\beta} \\ &= \frac{G_0}{\sinh\phi}\{\cos((m+1)\Delta)\sinh\beta_0\phi + \cos(m\Delta)\sinh(1-\beta_0)\phi\} \quad (40) \\ G^{12}_{\beta} &= -G^{21}_{\beta} \\ &= -\frac{G_0}{\sinh\phi}\{\sin((m+1)\Delta)\sinh\beta_0\phi + \sin(m\Delta)\sinh(1-\beta_0)\phi\} \quad (41) \end{aligned}$$

for flux parameter $\beta = \beta_0 + m$ and $0 \le \beta_0 < 1$. Applying the gauge transformations (34) and (35) we obtain

$$\begin{aligned} G'^{+}_{\beta/2} &= \frac{G_0}{\sinh\phi}\{\exp(-i(1-\beta_0/2)\Delta)\sinh(\beta_0\phi/2) + \exp(i\beta_0\Delta/2)\sinh(1-\beta_0/2)\phi\} \quad (42) \\ G'^{-}_{\beta/2} &= \frac{G_0}{\sinh\phi}\{\exp(-i\beta_0\Delta/2)\sinh((1-\beta_0/2)\phi) + \exp(i(1-\beta_0/2)\Delta)\sinh\beta_0\phi/2\} \quad (43) \\ G'^{11}_{\beta} &= G'^{22}_{\beta} \\ &= \frac{G_0}{\sinh\phi}\{\cos((1-\beta_0)\Delta)\sinh(\beta_0\phi) + \cos(\beta_0\Delta)\sinh(1-\beta_0)\Delta\} \quad (44) \\ G'^{12}_{\beta} &= -G'^{21}_{\beta} \\ &= \frac{G_0}{\sinh\phi}\{\sin(\beta_0\Delta)\sinh((1-\beta_0)\phi) - \sin((1-\beta_0)\Delta)\sinh\beta_0\phi\}. \quad (45) \end{aligned}$$

The transformed propagators show the symmetry properties

$$G'^{\pm}_{\beta+2n} = G'^{\pm}_{\beta} \quad (46)$$
$$G'^{ab}_{\beta+n} = G'^{ab}_{\beta} \quad (47)$$

not shared with the unprimed propagators. Here one should keep in mind, that for the gauge propagator the symmetry (47) is not generally valid. For the string models studied by us exceptional flux parameters β_i with the corresponding modifications of the gauge propagator (compare (17) and (18) in the limit $\theta \to \infty$) appear only correlated with special partial waves and there is only a finite number of them. Clearly this restricts, in the case of the gauge propagator,the symmetry (47) to non-exceptional values of $\beta + n$.

Very important is another property: The gauge transformed propagators can be decomposed

$$G'^{\pm}_{\beta/2}(x,x') = G_0(x-x') + \bar{G}^{\pm}_{\beta/2}(x,x') \tag{48}$$

$$G'^{ab}_{\beta}(x,x') = G_0(x-x')\delta^{ab} + \bar{G}^{ab}_{\beta}(x,x') \tag{49}$$

where G_0 is the free Feynman propagator (39) and the $\bar{G}$ are finite for $x = x.'$ Such a decomposition is typical for Casimir problems [8] but in general not valid for propagators in backgrounds and especially it is not valid for the original propagators (36) - (41).

Let us now determine the vacuum energy density. For this aim the theory has to be considered in Minkowski space-time which demands the following replacements

$$x_{\|}^2 = x_4^2 + x_3^2 \to -x_0^2 + x_3^2$$
$$\delta_{\mu\nu} \to -g_{\mu\nu}$$

whereas $\epsilon^{\perp}_{\mu\nu}$ remains untouched, of course.The calculation is performed most conveniently in the gauge transformed theory making use of the advantage, provided by the decomposition (48) and (49), to subtract the usual infinite vacuum energy of free space-time. A straightforward calculation gives for non-exceptional flux parameters

$$\begin{aligned}< 0|T_{00}|0 >_{matter} &= \left(\partial_r\partial_{r'} + \frac{1}{rr'}\partial_\varphi\partial_{\varphi'}\right)\left(\bar{G}^{+}_{\beta/2} + \bar{G}^{-}_{\beta/2}\right) \quad \text{for} \quad r = r', \varphi = \varphi' \\ &= -\frac{1}{3\pi^2 r^4}\frac{\beta_0}{2}(1-\frac{\beta_0}{2})(1-\frac{\beta_0}{4}(1-\frac{\beta_0}{2}))\end{aligned} \tag{50}$$

if $1 \le \beta_0 \le 2$,

$$< 0|T_{00}|0 >_{ghost} = -2(\partial_r\partial_{r'} + \frac{1}{rr'}\partial_\varphi\partial_{\varphi'})\bar{G}^{11} \quad \text{for} \quad r = r', \varphi = \varphi' \tag{51}$$

$$\begin{aligned}< 0|T_{00}|0 >_{YM} &= 4(\partial_r\partial_{r'} + \frac{1}{rr'}\partial_\varphi\partial_{\varphi'})\bar{G}^{11} \quad \text{for} \quad r = r', \varphi = \varphi' \\ &= -\frac{2}{3\pi^2 r^4}\beta_0(1-\beta_0)(1-\frac{\beta_0}{2}(1-\beta_0))\end{aligned} \tag{52}$$

if $0 \le \beta_0 \le 1$.

The total vacuum energy density is obtained as the sum of the contributions (50)-(52). A symmetry

$$\rho_{vac}(r, \beta + 2n) = \rho_{vac}(r, \beta) \tag{53}$$

is valid as long as $\beta + n$ is non-exceptional for all integers n.

Discussion and Conclusions

In view of the kernel (8) of the gauge field the non-abelian magnetic string has to be regularized physically, i.e. the string has to be taken as the limit of a flux tube of finite radius a keeping the flux fixed. Depending on the model chosen and the partial wave considered there are discrete "exceptional" values of the flux parameter β such that in the limit $a \to 0$ the radial wave function for the gauge field is singular but normalizeable at $r = 0$. The propagators and the vacuum energy density have been determined for non-exceptional values of β with the symmetry properties (46), (47) and (53). It is to be expected that the expressions are discontinuous at the exceptional values $\beta = \beta_i$.

The vacuum energy density in the non-abelian theory is negative as it is in the abelian case (same as matter contribution (50)). The absence of the usual sign effect connected with non-abelian gauge theory can be traced back to vanishing of the field strength $F_{\mu\nu}$ outside the string (compare the discussion following (28)).

Our transformation of a background into a Casimir problem may, to some extent, answer the question wether the vacuum energy density is a locally determined quantity. Of course, ρ_{vac} is determined by the propagators at coinciding arguments. The propagators themselves, however, may be essentially be determined by global or topological conditions, as it is in case of ideal magnetic strings. This remark should throw some doubts onto naive applications of the local Schwinger -DeWitt expansion for propagators.

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PROPAGATORS IN MAGNETIC STRING BACKGROUND AND THE PROBLEM OF SELF-ADJOINT EXTENSIONS †

Hans Jürgen KAISER
DESY - IfH Zeuthen
Platanenallee 6, D-O-1615 Zeuthen, Germany

Ghost and gluon propagators of a non-Abelian gauge theory in the background of a magnetic string are calculated. A simple technique to derive the ghost propagator is presented which makes use of the fact that the presence of a magnetic string of strength β shifts the differential operator $\frac{\partial}{\partial\varphi}$ to $\frac{\partial}{\partial\varphi} - i\beta$. In the case of a gluon propagator in the magnetic string background a difficulty arises from the presence of the magnetic field strength term involving a δ function. Here the ambiguities of a self-adjoint extension of the differential operator must be met. A proper treatment demands the specification of a limiting process starting from a string of finite thickness and well-defined structure and leading to the δ function string. Without this additional structure information about the background string the gauge field propagator is undetermined.

We consider a non-Abelian gauge theory in the background of a classical magnetic field with the vector potential (for simplicity of notation we take the coupling constant - which enters only as factor of B or F - equal to 1)

$$B^a_\mu(x) = \delta^{a3}\, \epsilon^\perp_{\mu\rho} x^\perp_\rho\, \tilde{\beta}(r_\perp). \tag{1}$$

corresponding to the field strength

$$F^a_{\mu\nu}(x) = -\delta^{a3}\, \epsilon^\perp_{\mu\nu}\, [2\tilde{\beta}(r_\perp) + r_\perp\, d\tilde{\beta}(r_\perp)/dr_\perp\}. \tag{2}$$

We use the notation

$$\begin{aligned} x_\perp &= (x_1, x_2), \\ x_\parallel &= (x_3, x_4), \\ r_\perp &= |x_\perp|, \end{aligned} \qquad \epsilon^\perp_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

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In the case of a magnetic string we have

$$\tilde{\beta}(r_\perp) = \beta/r_\perp^2, \qquad 2\tilde{\beta} + r_\perp d\tilde{\beta}/dr_\perp = \beta\delta(r_\perp)/r_\perp. \tag{4}$$

$B(x)$ enters the covariant derivative

$$D_\mu^{ab} = \delta^{ab}\partial_\mu + \epsilon^{abc}B_\mu^c(x). \tag{5}$$

involved in the ghost and gluon kernels respectively

$$\begin{aligned} K^{ab} &= -(D^2)^{ab}, \\ K_{\mu\nu}^{ab} &= K^{ab}\delta_{\mu\nu} - 2\epsilon^{ab3}\beta\delta(r_\perp)/r_\perp. \end{aligned} \tag{6}$$

We take the Euclidean 4-dimensional case and the gauge group $SU(2)$ in adjoint representation. We use polar coordinates for (x_1, x_2) to express the kernels in the form

$$\begin{aligned} K^{ab} &= -\delta^{ab}\Delta + 2\epsilon^{ab3}\frac{\beta}{r_\perp^2}\frac{\partial}{\partial\varphi_\perp} + (\delta^{a1}\delta^{b1} + \delta^{a2}\delta^{b2})\frac{\beta^2}{r_\perp^2}, \\ \Delta &= \Delta_\parallel + \frac{\partial^2}{\partial r_\perp^2} + \frac{1}{r_\perp}\frac{\partial}{\partial r_\perp} + \frac{1}{r_\perp^2}\frac{\partial^2}{\partial\varphi_\perp^2}, \\ K_{\mu\nu}^{ab} &= K^{ab}\delta_{\mu\nu} - 2\epsilon^{ab3}\epsilon_{\mu\nu}\beta\delta(r_\perp)/r_\perp. \end{aligned} \tag{7}$$

We intend to construct the propagators G^{ab} and $G_{\mu\nu}^{ab}$. They fulfil

$$\begin{aligned} K^{ab}(x)G^{bc}(x,x') &= \delta^{ac}\delta(x-x'), \\ K_{\lambda\mu}^{ab}(x)G_{\mu\nu}^{bc}(x,x') &= \delta^{ac}\delta_{\lambda\nu}\delta(x-x') - \sum_{\text{zero modes}} \chi_\lambda^a(x)\chi_\nu^c(x')^*. \end{aligned} \tag{8}$$

After diagonalization and separation in $x_\parallel$ and $x_\perp$ we find the eigenvalue equations in $x_\perp$

$$\left[\frac{\partial^2}{\partial r_\perp^2} + \frac{1}{r_\perp}\frac{\partial}{\partial r_\perp} + \frac{1}{r_\perp^2}\left(\frac{\partial}{\partial\varphi_\perp} - i\beta\right)^2 - \frac{2g\beta}{r_\perp}\delta(r_\perp) + k_\perp^2\right]\psi(r_\perp,\varphi_\perp) = 0 \tag{9}$$

where the parameter g serves to embrace the various equations involved in the ghost and gluon cases:

$$\text{ghost}: \quad g = 0, \qquad \text{gluon}: \quad g = -1, 0, +1. \tag{10}$$

With

$$\psi_{p,k_\perp}(r_\perp,\varphi_\perp) = e^{ip\varphi_\perp}R_{p,k_\perp}(r_\perp) \tag{11}$$

the radial equation reads

$$\left[\frac{d^2}{dr_\perp^2} + \frac{1}{r_\perp}\frac{d}{dr_\perp} + \frac{1}{r_\perp^2}(p-\beta)^2 - \frac{2g\beta}{r_\perp}\delta(r_\perp) + k_\perp^2\right]R_{p,k_\perp}(r_\perp) = 0. \tag{12}$$

For $g = 0$ the treatment is straightforward:
We take the orthonormal solutions

$$\psi_{p,k}(r,\varphi) = \frac{1}{\sqrt{2\pi}} e^{ip\varphi} J_{|p-\beta|}(kr) \tag{13}$$

of the twodimensional problem and calculate

$$G(x,x') = \sum^{(\lambda_n \neq 0)} \frac{\psi_n(x)\psi_n(x')^*}{\lambda_n} \tag{14}$$

by combining the heath kernels of the two twodimensional problems

$$\begin{aligned} G_\beta(x,x') &= \int_0^\infty dt\, \tilde{G}_\perp(x_\perp, x'_\perp, t)\tilde{G}_\parallel(x_\parallel, x'_\parallel, t), \\ \tilde{G}_\perp(x_\perp, x'_\perp, t) &= \sum \psi_n^\perp(x_\perp)\psi_n^\perp(x_\perp)^* e^{-\lambda_n^\perp t}, \\ \tilde{G}_\parallel(x_\parallel, x'_\parallel, t) &= \sum \psi_n^\parallel(x_\parallel)\psi_n^\parallel(x_\parallel)^* e^{-\lambda_n^\parallel t}. \end{aligned} \tag{15}$$

In the case of the magnetic string we obtain

$$\begin{aligned} \tilde{G}_\parallel(x_\parallel, x'_\parallel, t) &= \frac{1}{4\pi t} e^{-(x_\parallel - x'_\parallel)^2/4t}, \\ \tilde{G}_\perp(r_\perp, r'_\perp, \Delta\varphi_\perp, t) &= \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} e^{ip\Delta\varphi_\perp} \int_0^\infty dk_\perp\, k_\perp J_{|p-\beta|}(k_\perp r_\perp) J_{|p-\beta|}(k_\perp r'_\perp) e^{-k_\perp^2 t}. \end{aligned} \tag{16}$$

which leads after a somewhat lengthy calculation to the remarkably simple result

$$\begin{aligned} G_\beta(x,x') &= \frac{1}{4\pi^2(x-x')^2} \frac{\sinh(1-\beta)\Phi + e^{i\Delta\varphi_\perp}\sinh\beta\Phi}{\sinh\Phi}, \qquad (0 \le \beta \le 1), \\ \cosh\Phi &= \frac{(x_\parallel - x'_\parallel)^2 + r_\perp^2 + r_\perp'^2}{2r_\perp r'_\perp}, \qquad G_{\beta+m} = e^{im\Delta\varphi_\perp} G_\beta \end{aligned} \tag{17}$$

entering the full ghost propagator

$$G_\beta^{ab}(x,x') = (\delta^{a1}\delta^{b1} + \delta^{a2}\delta^{b2})\Re G_\beta + \epsilon^{ab3}\Im G_\beta + \delta^{a3}\delta^{b3} G_0. \tag{18}$$

There exists a very simple derivation of (17) which avoids the construction of eigenfunctions and their tedious use in (14). We look again at the differential operator in cylindrical coordinates

$$D_\beta = \Delta_\parallel + \frac{\partial^2}{\partial r_\perp^2} + \frac{1}{r_\perp}\frac{\partial}{\partial r_\perp} + \frac{1}{r_\perp^2}\left(\frac{\partial}{\partial\varphi_\perp} - i\beta\right)^2. \tag{19}$$

It is obtained from $D_0 = \Delta$ by the substitution

$$\frac{\partial}{\partial\varphi_\perp} \to \frac{\partial}{\partial\varphi_\perp} - i\beta. \tag{20}$$

We know the propagator for $\beta = 0$

$$G_0(x,x') = \frac{1}{4\pi^2(x-x')^2}. \tag{21}$$

How to transform G_0 into G_β ?
The claim is that

$$G_\beta(\dots,\Delta\varphi) = \sum_{p=-\infty}^{\infty} e^{ip\Delta\varphi}\int_{C_p}\frac{d\chi}{2\pi}e^{-i(p-\beta)\chi}G_0(\dots,\chi) \tag{22}$$

will do it. Indeed, by partial integration,

$$\left(\frac{\partial}{\partial\Delta\varphi}-i\beta\right)^2 G_\beta(\dots,\Delta\varphi) = \sum_{p=-\infty}^{\infty} e^{ip\Delta\varphi}\int_{C_p}\frac{d\chi}{2\pi}e^{-i(p-\beta)\chi}\frac{\partial^2}{\partial\chi^2}G_0(\dots,\chi) \tag{23}$$

provided we choose the path C_p in a manner to avoid boundary terms. Note in particular that $\int_0^{2\pi} d\chi\dots$ would contribute the boundary terms $(1-\exp(2\pi i\beta))[G_0\delta'(\Delta\varphi) + (G_0' - i\beta G_0)\delta(\Delta\varphi)]$.

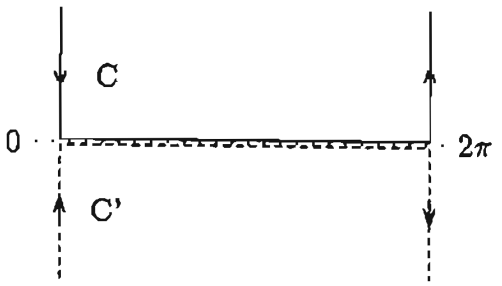

Fig. 1

The appropriate choice of the path is (see Fig. 1)

$$C_p = \begin{cases} C & \text{for } p<\beta \\ C' & \text{for } p>\beta \end{cases} \tag{24}$$

In application to the magnetic string background we express the free propagator in cylindrical coordinates

$$G_0(x,x') = G_0(x_\parallel, x'_\parallel, r_\perp, r'_\perp, \Delta\varphi) = \frac{q}{\cosh\Phi - \cos\Delta\varphi} \tag{25}$$

where

$$q = \frac{1}{8\pi^2 r_\perp r'_\perp}, \quad \cosh\Phi = \frac{(x_\parallel - x'_\parallel)^2 + r_\perp^2 + r_\perp'^2}{2r_\perp r'_\perp}, \quad \Delta\varphi = \varphi_\perp - \varphi'_\perp. \tag{26}$$

Furthermore we select again $0\le\beta\le 1$.
The transformation (22), (24) applied to (25) leads to

$$\begin{aligned} G_\beta = &\sum_{p=-\infty}^{0} e^{ip\varphi}\int_C \frac{d\chi}{2\pi}e^{-i(p-\beta)\chi}\frac{q}{\cosh\Phi-\cos\chi} \\ &+\sum_{p=1}^{\infty} e^{ip\varphi}\int_{C'} \frac{d\chi}{2\pi}e^{-i(p-\beta)\chi}\frac{q}{\cosh\Phi-\cos\chi}. \end{aligned} \tag{27}$$

We can close the integration path at infinity (Fig. 2) and get

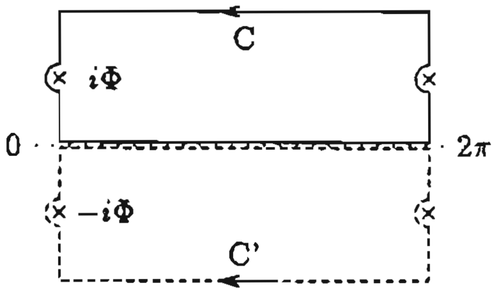

Fig. 2

$$
\begin{aligned}
G_\beta &= \frac{q}{\sinh\Phi}\left[e^{-\beta\Phi}\sum_{p=0}^{\infty} e^{-(\Phi+i\varphi)p} \right. \\
&\qquad \left. + e^{\beta\Phi}\sum_{p=1}^{\infty} e^{-(\Phi-i\varphi)p}\right] \\
&= \frac{q}{\sinh\Phi}\left[\frac{e^{-\beta\Phi}}{1-e^{-\Phi-i\varphi}} \right. \\
&\qquad \left. + \frac{e^{-(1-\beta)\Phi+i\varphi}}{1-e^{-\Phi+i\varphi}}\right]
\end{aligned}
\tag{28}
$$

and regain (17).

An alternative approach is to interchange in (22) summation and integration. Note that the sum

$$\sum_{p=0}^{\infty} e^{ip\varphi} = \pi\sum_{k=-\infty}^{\infty}\delta(\varphi+2\pi k)+\frac{1}{2}+\frac{i}{2}\widetilde{\cot}(\frac{\varphi}{2}) \tag{29}$$

involves besides the δ distribution the distribution $\widetilde{\cot}(\varphi/2)$ oscillating violently around $\cot(\varphi/2)$ within the limits 0 and $2\cot(\varphi/2)$. Fig. 3 shows approximations to $\frac{1}{2}\widetilde{\cot}(x/2)$, i.e. $\sum_{n=0}^{N}\sin nx$ for $N = 50, 1000, 20000$.

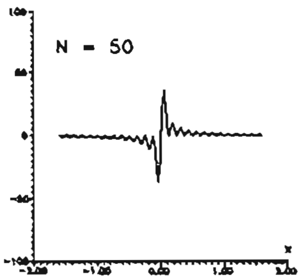

Fig. 3a

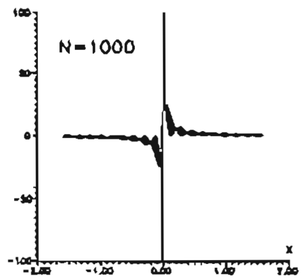

Fig. 3b

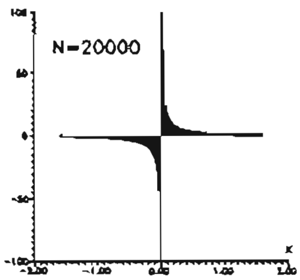

Fig. 3c

Under an integral with a smooth testfunction we can replace $\widetilde{\cot}$ with the usual cot function.

Then the transformation connecting G_β to G_0 reads

$$
\begin{aligned}
G_\beta(\ldots,\varphi) &= \int_C \frac{d\chi}{2\pi}e^{i\beta\chi}G_0(\ldots,\chi)\left(\pi\delta(\varphi-\chi)+\frac{1}{2}-\frac{i}{2}\widetilde{\cot}(\frac{\varphi-\chi}{2})\right) \\
&+\int_{C'} \frac{d\chi}{2\pi}e^{i\beta\chi}G_0(\ldots,\chi)\left(\pi\delta(\varphi-\chi)-\frac{1}{2}+\frac{i}{2}\widetilde{\cot}(\frac{\varphi-\chi}{2})\right).
\end{aligned}
\tag{30}
$$

Noting that

$$1\mp i\cot\frac{\varphi-\chi}{2} = \frac{e^{\mp i\chi}-e^{\pm i\varphi}}{\cos\chi-\cos\varphi} \tag{31}$$

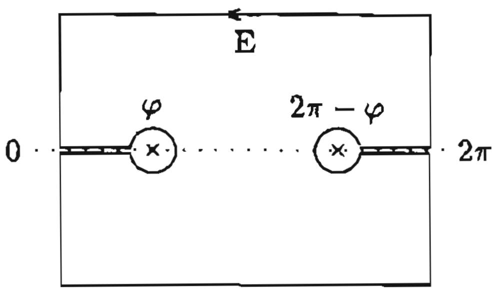

Fig. 4

we recognize that the contributions from the poles at $\chi=\varphi$ and $\chi=2\pi-\varphi$ compensate those from $\delta(\varphi-\chi)$. There remain the contributions from the poles of $G_0(\ldots,\chi)$ inside the path E (see Fig. 4)

$$G_\beta(\ldots,\varphi)=\int_E\frac{d\chi}{4\pi}e^{i\beta\chi}\times G_0(\ldots,\chi)\frac{e^{-i\chi}-e^{i\varphi}}{\cos\chi-\cos\varphi}. \tag{32}$$

As a check we insert (25) into (32)

$$\begin{aligned}G_\beta&=\int_E\frac{d\chi}{4\pi}e^{i\beta\chi}\frac{q}{\cosh\Phi-\cos\chi}\frac{e^{-i\chi}-e^{i\varphi}}{\cos\chi-\cos\varphi}\\&=\frac{iq}{2}\operatorname*{Res}_{\chi=\pm i\Phi}\frac{e^{i\beta\chi}}{\cosh\Phi-\cos\chi}\frac{e^{-i\chi}-e^{i\varphi}}{\cos\chi-\cos\varphi}\\&=\frac{q}{2\sinh\Phi}\frac{1}{\cosh\Phi-\cos\varphi}\left[e^{-\beta\Phi}(e^{\Phi}-e^{i\varphi})-e^{\beta\Phi}(e^{-\Phi}-e^{i\varphi})\right]\end{aligned} \tag{33}$$

and reproduce again (17).

Now we pass over to the cases $g=\pm1$ where the radial eigenvalue equation (12) contains a δ function. Naively one would expect that the $\delta(r)$ term could change nothing since $J_{|p-\beta|}$ starts as $(kr)^{|p-\beta|}$ and would not feel the δ function. But besides $J_{|p-\beta|}$ we must accept the normalizable solutions $J_{-|p-\beta|}$, $(0<|p-\beta|<1$, i.e. we could choose instead of the complete system

$$J_{|p-\beta|},\qquad(-\infty<p<\infty) \tag{34}$$

any one of the complete systems (parametrized by Θ)

$$\begin{aligned}&\frac{J_{|p-\beta|}+\Theta J_{-|p-\beta|}}{\sqrt{1+\Theta^2}},\qquad&(0<|p-\beta|<1)\\&J_{|p-\beta|},\qquad&(|p-\beta|>1).\end{aligned} \tag{35}$$

This amounts to a self-adjoint extension (parametrized by Θ) of the differential operator in (12).

To fix Θ we consider a special limiting process. We start with a magnetic string of finite radius a and let $a\to0$. We consider three models of this sort. In all cases the total flux is $2\pi\beta$.

Model I

$$\tilde{\beta}(r) = \begin{cases} \beta/a^2 & (r < a) \\ \beta/r^2 & (r > a \end{cases} \qquad 2\tilde{\beta} + r\frac{d\tilde{\beta}}{dr} = \begin{cases} 2\beta/a^2 & (r < a) \\ 0 & (r > a). \end{cases} \tag{36}$$

The interior equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(\frac{p}{r} - \frac{\beta r}{a^2}\right)^2 - \frac{4g\beta}{a^2} + k^2\right] R_{\rm int}(r) = 0, \quad (r < a) \tag{37}$$

is of the Laguerre type and has (for $p \geq 0$) the regular solution

$$R_{\rm int}(r) = r^p e^{-\beta r^2/2a^2}\, {}_1F_1(g + \tfrac{1}{2} - \tfrac{a^2k^2}{4\beta}, p+1, \tfrac{\beta r^2}{a^2}) \tag{38}$$

whereas the exterior equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(p-\beta)^2}{r^2} + k^2\right] R_{\rm ext}(r) = 0, \quad (r > a) \tag{39}$$

is of Bessel type and has the general solution

$$R_{\rm ext} = J_{|p-\beta|}(kr) + \Theta_p J_{-|p-\beta|}(kr). \tag{40}$$

The matching condition at $r = a$

$$\begin{aligned} &\frac{\partial}{\partial r}\ln R_{\rm int}(r)|_{r=a} = \frac{\partial}{\partial r}\ln R_{\rm ext}(r)|_{r=a} \\ &\frac{\beta}{a^2}\left[\frac{p}{\beta} - 1 + \frac{1 + 2g - \frac{a^2k^2}{2\beta}}{p+1}\frac{{}_1F_1\left(\frac{3}{2} + g - \frac{a^2k^2}{4\beta}, p+2, \beta\right)}{{}_1F_1\left(\frac{1}{2} + g - \frac{a^2k^2}{4\beta}, p+1, \beta\right)}\right] \\ &= \frac{k}{4a}\frac{J_{|p-\beta|-1}(ak) - J_{|p-\beta|+1}(ak) + \Theta_p(J_{-|p-\beta|-1}(ak) - J_{-|p-\beta|+1}(ak))}{J_{|p-\beta|}(ak) + \Theta_p J_{-|p-\beta|}(ak)} \end{aligned} \tag{41}$$

fixes the parameter Θ

$$\begin{aligned} \Theta_p &= -\frac{CJ_{|p-\beta|}(ak) + \frac{ak}{2\beta}[J_{|p-\beta|-1}(ak) - J_{|p-\beta|+1}(ak)]}{CJ_{-|p-\beta|}(ak) + \frac{ak}{2\beta}[J_{-|p-\beta|-1}(ak) - J_{-|p-\beta|+1}(ak)]}, \\ C &= 1 - \frac{p}{\beta} - \frac{1 + 2g - \frac{a^2k^2}{2\beta}}{p+1}\frac{{}_1F_1\left(\frac{3}{2} + g - \frac{a^2k^2}{4\beta}, p+2, \beta\right)}{{}_1F_1\left(\frac{1}{2} + g - \frac{a^2k^2}{4\beta}, p+1, \beta\right)}. \end{aligned} \tag{42}$$

We are interested in the limit of a very thin string, i.e. $ak \to 0$ where the Bessel function behaves as

$$J_\nu(ak) = \frac{1}{\Gamma(\nu+1)}\left(\frac{ak}{2}\right)^\nu \left[1 - \frac{1}{\nu+1}\left(\frac{ak}{2}\right)^2 + O\left((ak)^4\right)\right] \tag{43}$$

and Θ tends to

$$\Theta_p \to -\left(\frac{ak}{2}\right)^{2|p-\beta|} \frac{\Gamma(-|p-\beta|)}{\Gamma(|p-\beta|)} \frac{\left(\frac{1}{\beta}+\frac{C}{|p-\beta|}\right)-\left(\frac{ak}{2}\right)^2 \frac{C+(2+|p-\beta|)/\beta}{|p-\beta|(|p-\beta|+1)}}{\left(\frac{1}{\beta}-\frac{C}{|p-\beta|}\right)-\left(\frac{ak}{2}\right)^2 \frac{C+(2-|p-\beta|)/\beta}{|p-\beta|(|p-\beta|-1)}}. \tag{44}$$

We recognize that Θ almost always vanishes for $ak \to 0$ except if the leading term in the denominator is zero

$$\frac{1}{\beta_{\mathrm{exc}}} - \frac{C}{|p-\beta|} = 0, \tag{45a}$$

$$0 < |p - \beta_{\mathrm{exc}}| < 1. \tag{45b}$$

Inserting C from (42) we can rewrite the condition of (45a) as

$$(1+2g)\,{}_1F_1(\tfrac{3}{2}+g, p+2, \beta_{\mathrm{exc}}) = 2(p+1)(1-p/\beta_{\mathrm{exc}})\,{}_1F_1(\tfrac{1}{2}+g, p+1, \beta_{\mathrm{exc}}). \tag{46}$$

For $\beta = \beta_{\mathrm{exc}}$ the parameter Θ tends to infinity

$$\Theta \to \mathrm{const.}\,(ak)^{2(|p-\beta_{\mathrm{exc}}|-1)} \to \infty. \tag{47}$$

<table>
<tr><th>p</th><th>β</th><th></th></tr>
<tr><td>0</td><td>2.90433</td><td></td></tr>
<tr><td>1</td><td>0.73768</td><td rowspan="6">$\left.\right\} \beta_{\mathrm{exc}}$</td></tr>
<tr><td>2</td><td>1.54747</td></tr>
<tr><td>3</td><td>2.39087</td></tr>
<tr><td>4</td><td>3.25471</td></tr>
<tr><td>5</td><td>4.13263</td></tr>
<tr><td>6</td><td>5.02102</td></tr>
<tr><td>7</td><td>5.91755</td><td></td></tr>
<tr><td>8</td><td>6.82070</td><td></td></tr>
<tr><td>9</td><td>7.72933</td><td></td></tr>
<tr><td>...</td><td>......</td><td></td></tr>
</table>

The Model I has for $g = -1$ such exceptional values of β. They are listed in the table together with the first β which satisfy only (45a). A further set of β_{exc} is obtained by the replacement $g \to -g$, $p \to -p$, $\beta \to -\beta$. At the exceptional values of β where $\Theta_p \to \infty$ the contribution of the p^{th} partial wave to the propagator is changed by the amount

$$\begin{aligned} &\int_0^\infty dk_\perp\, k_\perp K_0(k_\perp|x_\parallel - x'_\parallel|) \\ &\qquad \times \left[J_{-|p-\beta|}(k_\perp r_\perp)J_{-|p-\beta|}(k_\perp r'_\perp) - J_{|p-\beta|}(k_\perp r_\perp)J_{|p-\beta|}(k_\perp r'_\perp)\right] \\ &\qquad = \frac{(b_+ + b_-)^{2|p-\beta|} - (b_+ - b_-)^{2|p-\beta|}}{(4r_\perp r'_\perp)^{|p-\beta|}\,b_+ b_-}, \\ &b_\pm = \sqrt{(x_\parallel - x'_\parallel)^2 + (r_\perp \pm r'_\perp)^2}. \end{aligned} \tag{48}$$

One could refine the analysis and look for the behaviour of the solutions in the immediate vicinity of β_{exc} (where Θ is finite).

Model II

Evidently the values β_{exc} depend on the model. Our second model is a hollow flux tube

$$\tilde{\beta}(r) = \begin{cases} 0 & r < a \\ \beta/r^2 & r > a \end{cases} \qquad 2\tilde{\beta} + r\frac{d\tilde{\beta}}{dr} = \frac{\beta}{a}\delta(r-a). \tag{49}$$

The matching condition is now

$$\frac{akJ'_{|p|}(ak) + 2g\beta J_{|p|}(ak)}{J_{|p|}(ak)} = ak\frac{J'_{|p-\beta|}(ak) + \Theta J'_{-|p-\beta|}(ak)}{J_{|p-\beta|}(ak) + \Theta J_{-|p-\beta|}(ak)}. \tag{50}$$

We find

$$\Theta = -\frac{(|p-\beta| - p - 2g\beta)J_{|p|}J_{|p-\beta|} + ak[J_{|p|+1}J_{|p-\beta|} - J_{|p|}J_{|p-\beta|+1}]}{(-|p-\beta| - p - 2g\beta)J_{|p|}J_{-|p-\beta|} + ak[J_{|p|+1}J_{-|p-\beta|} - J_{|p|}J_{-|p-\beta|+1}]} \tag{51}$$

(all J_ν with argument ak)

p	β
0	0
1	2/3
2	4/3
3	2
4	8/3
...	...

$\left.\begin{matrix}\\ \\ \end{matrix}\right\} \beta_{exc}$ (rows $p=1,2$)

and get β_{exc} from

$$|p-\beta| + p + 2g\beta = 0 \quad \text{and} \quad 0 < |p-\beta| < 1 \tag{52}$$

which is for $g = -1$ satisfied by the values of β listed in the table.

Model III

We can define a slightly generalized model by combining the two previous ones. Keeping the total flux at $2\pi\beta$ we introduce a free parameter η and write

$$\tilde{\beta}(r) = \begin{cases} \eta\beta/a^2 & r < a \\ \beta/r^2 & r > a \end{cases} \qquad 2\tilde{\beta} + r\frac{d\tilde{\beta}}{dr} = \begin{cases} 2\eta\beta/a^2 \\ 0 \end{cases} + \beta\frac{1-\eta}{a}\delta(r-a). \tag{53}$$

The interior and exterior solutions are those of Model I, only the matching condition is changed. In performing the limit $a \to 0$ we hold η fixed. Now β_{exc} is dependent on the parameter η. For $|\eta| \gg ak$ the condition for a β_{exc} is

$$\frac{1+2g}{p+1}\frac{{}_1F_1(\frac{3}{2}+g, p+2, \beta\eta)}{{}_1F_1(\frac{1}{2}+g, p+1, \beta\eta)} = 2\left(1-\frac{p}{\beta}\right) + \frac{1+\eta}{\eta}\left(1-2g-\frac{2p}{\beta}\right) \tag{54}$$

supplemented by $\quad 0 < |p-\beta| < 1.$

By changing η (i.e. the composition of the string) we can shift β_{exc}. The variation of η between 0 and 1 varies for example the β_{exc} belonging to the partial wave $p = 1$ between 0.667 and 0.738. Fig.5 shows for the partial waves involved a plot of β_{exc} as function of

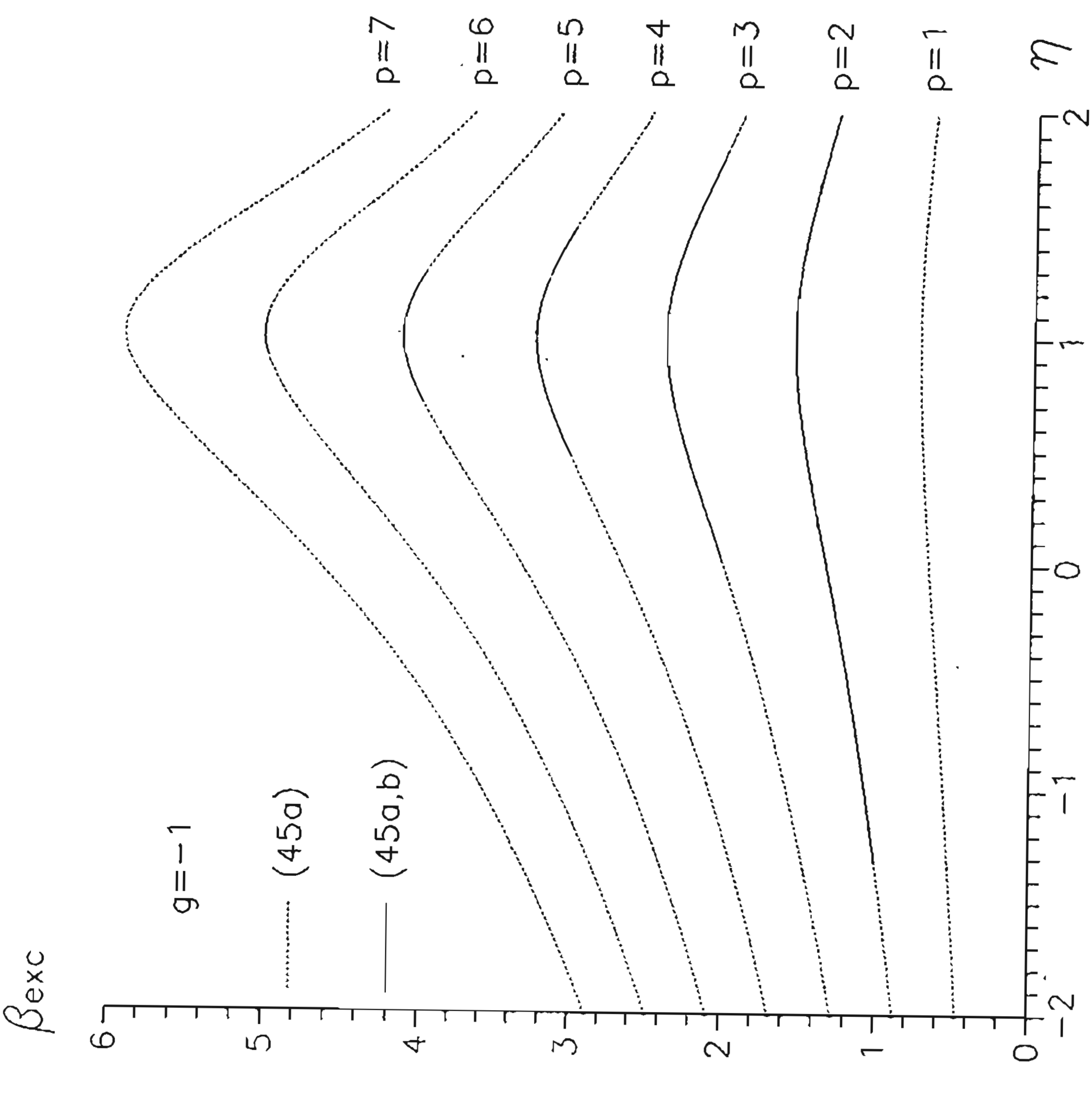

Fig. 5

η. By further complicating the structure of the string it should be possible to make any desired β an β_{exc} in one of the partial waves where $0 < |p - \beta| < 1$.

In conclusion it can be stated that for a non-Abelian gauge theory in the background of a magnetic string the gauge field propagator is undetermined unless a limiting process is specified how the δ-string is obtained from a string of finite thickness.

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Radiative Corrections for a Non–Abelian Gauge Theory in a Homogeneous Self–Dual Background*

U. Müller
Humboldt–Universität zu Berlin
Invalidenstr. 110, D-O-1040 Berlin, Germany

E. Wieczorek
DESY — IfH Zeuthen
Platanenallee 6, D-O-1615 Zeuthen, Germany

The counter terms for the polarization tensor were determined. It was shown that a modified Ward–Takahashi identity is valid for an arbitrary value of the gauge fixing parameter if all contributions except for some tadpole terms are considered. These tadpole terms are discussed.

1 Introduction

Here we consider a quantum field theory within a given external field as it is usual in quantum electrodynamics. There are several motivations for doing that. For example we can think of the fact that the Salam–Weinberg theory as a non–Abelian gauge theory includes quantum electrodynamics. Therefore every external electromagnetic field has to be considered as a background for a non–Abelian gauge theory. There are also physically interesting cases such as magnetic strings for which one wants to determine radiative corrections. Finally, gluon condensates can be investigated by means of background fields. This can be done by the calculation of the effective action for such condensates, at least approximately. In order to determine the 2–loop approximation [1, 2] it is necessary to find out what divergencies appear and how renormalization shall be done at the polarization tensor in 1–loop approximation. To get an entry into these problems it is helpful to investigate at first simple background fields in order to get to know what mathematical structures and peculiarities arise.

Very simple cases are homogeneous self–dual background fields [3, 4].

*Talk given by U. Müller at the workshop "Quantum Field Theory under the Influence of External Conditions", Leipzig (Sept. 1992)

2 Yang–Mills Theory in a Self–Dual Homogeneous Background: Propagators

Starting from the well–known Yang–Mills Lagrangian we obtain the Lagrangian of a gauge field in a classical background if we replace the original gauge field by a sum of this background B^a_μ and the new gauge field a^a_μ. Within this approach the gauge covariant derivative formed with the external field B^a_μ

$$D^{ab}_\mu = \delta^{ab}\partial_\mu + gf^{abc}B^c_\mu \tag{1}$$

plays a special role. f^{abc} are the structure constants of the gauge group, here $SU(2)$ with $f^{abc} = \varepsilon^{abc}$. We will always consider Euclidean space with four dimensions. If we choose the gauge fixing term according to the covariant gauge

$$\mathcal{L}_{g.f.} = \frac{1}{2\alpha}\left(D^{ab}_\mu a^b_\mu\right)^2 \tag{2}$$

and introduce the Grassmann valued ghost fields c^a, $\bar{c}^a$ in order to compensate the longitudinal degrees of freedom of the gauge field we obtain finally as Lagrangian

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}a^a_\mu K^{ab}_{\mu\nu}a^b_\nu + \bar{c}^a\tilde{K}^{ab}c^b + \\ & + gf^{abc}\left(D_\mu\bar{c}\right)^a c^b a^c_\mu - gf^{abc}D^{ad}_\mu a^d_\nu a^b_\mu a^c_\nu + \frac{1}{4}g^2 f^{abc}f^{ade}a^b_\mu a^c_\nu a^d_\mu a^e_\nu\end{aligned} \tag{3}$$

where $K^{ab}_{\mu\nu}$ and $\tilde{K}^{ab}$ are the kinetic kernels of the gauge and the ghost fields respectively,

$$K^{ab}_{\mu\nu} = -\left(\delta_{\mu\nu}D^2 - \left(1 - \frac{1}{\alpha}\right)D_\mu D_\nu\right)^{ab} - 2gf^{abc}F^c_{\mu\nu}, \tag{4}$$

$$\tilde{K}^{ab} = -\left(D^2\right)^{ab}. \tag{5}$$

$F^a_{\mu\nu}$ is the field strength tensor of the background in the adjoint representation,

$$F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - gf^{abc}B^b_\mu B^c_\nu. \tag{6}$$

In the derivation of the Lagrangian (3) the assumption was made that the classical background B^a_μ fulfils an equation of motion without external currents

$$D_\mu F_{\mu\nu} = 0. \tag{7}$$

Otherwise the Lagrangian would contain also a term depending linearly on the gauge field a^a_μ.

In our case we consider the gauge group $SU(2)$ and as background a homogeneous self–dual field:

$$\begin{aligned} B^a_\mu(x) &= -\tfrac{1}{2}B\varepsilon_{\mu\nu}x_\nu\delta^{a3}, \\ F^a_{\mu\nu} &= B\varepsilon_{\mu\nu}\delta^{a3}, \qquad \text{with} \qquad \varepsilon_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \\ f^{abc} &= \varepsilon^{abc}, \end{aligned} \tag{8}$$

The propagators for the gauge and the ghost fields respectively fulfil the following relations:

$$K^{ab}_{\mu\nu}G^{bc}_{\nu\lambda}(x,y;\alpha) = \delta_{\mu\lambda}\delta^{ac}\delta(x-y) - \sum_{\substack{\text{zero modes}\\ \text{of } K^{ab}_{\mu\nu}}} \chi^{a}_{\mu}(x)\,(\chi^{c}_{\lambda}(y))^{*} \tag{9}$$

$$\tilde{K}^{ab}\tilde{G}^{bc}(x,y) = \delta^{ac}\delta(x-y). \tag{10}$$

The kinetic kernel of the gauge field $K^{ab}_{\mu\nu}$ has zero modes, they have to be excluded when the propagators are constructed.

The calculation of the propagators is possible for arbitrary values of the gauge fixing parameter α. It can be traced back to the determination of the propagators at $\alpha = 1$ by the use of the rather general formula [5, 6]

$$G^{ab}_{\mu\nu}(x,y;\alpha) = G^{ab}_{\mu\nu}(x,y;1) - (1-\alpha)\int du\, D^{cd}_{\kappa}G^{da}_{\kappa\mu}(u,x;1)D^{ce}_{\lambda}G^{eb}_{\lambda\nu}(u,y;1) \tag{11}$$

$$= G^{ab}_{\mu\nu}(x,y;1) - (1-\alpha)\int du\, D^{ad}_{\mu}\tilde{G}^{dc}(x,u)D^{be}_{\nu}\tilde{G}^{ec}(y,u)$$

$$= G^{ab}_{\mu\nu}(x,y;1) - (1-\alpha)D^{ad}_{\mu}D^{be}_{\nu}\int du\, \tilde{G}^{dc}(x,u)\tilde{G}^{ec}(y,u). \tag{12}$$

Diagonalizing the kernels for $\alpha = 1$ [5] and using equation (12) we obtain finally the following expressions for the propagators ($z = x - y$) [7]:

$$\tilde{G}^{ab}(x,y) = \Phi^{ab}(x,y)D(z) + \delta^{ab}_{\parallel}D^{0}(z), \tag{13}$$

$$G^{ab}_{\mu\nu}(x,y;\alpha) = \Phi^{ab}(x,y)D^{\mathrm{C}}_{\mu\nu}(z) + \tilde{\Phi}^{ab}(x,y)D^{\mathrm{S}}_{\mu\nu}(z) + \delta^{ab}_{\parallel}D^{0}_{\mu\nu}(z) \tag{14}$$

with the phase factors

$$\Phi^{ab}(x,y) = \delta^{ab}_{\perp}\cos\varrho(x,y) + \varepsilon^{ab}_{\perp}\sin\varrho(x,y), \tag{15}$$

$$\tilde{\Phi}^{ab}(x,y) = -\delta^{ab}_{\perp}\sin\varrho(x,y) + \varepsilon^{ab}_{\perp}\cos\varrho(x,y) = \varepsilon^{ac}_{\perp}\Phi^{cb}(x,y), \tag{16}$$

where

$$\delta^{ab}_{\parallel} = \delta^{a3}\delta^{b3}, \qquad \delta^{ab}_{\perp} = \delta^{ab} - \delta^{a3}\delta^{b3}, \qquad \varepsilon^{ab}_{\perp} = \varepsilon^{ab3}, \tag{17}$$

$$\varrho(x,y) = \frac{gB}{2}\varepsilon_{\mu\nu}x_{\mu}y_{\nu}. \tag{18}$$

The translation invariant functions occuring in (13, 14) are

$$D(z) = \frac{1}{4\pi^2 z^2}e^{-\frac{1}{4}gBz^2}, \tag{19}$$

$$D^{0}(z) = \frac{1}{4\pi^2 z^2}, \tag{20}$$

$$D^{\mathrm{C}}_{\mu\nu}(z) = \delta_{\mu\nu}D^{\mathrm{C}}(z) - \frac{1-\alpha}{2}\left[\left(\frac{gB}{2}\right)^{2}\tilde{z}_{\mu}\tilde{z}_{\nu} + \partial_{\mu}\partial_{\nu}\right]\Delta(z), \tag{21}$$

$$D^{\mathrm{S}}_{\mu\nu}(z) = \varepsilon_{\mu\nu}D^{\mathrm{S}}(z) - \frac{1-\alpha}{2}\left[-\frac{gB}{2}\varepsilon_{\mu\nu} + \frac{gB}{2}\tilde{z}_{\nu}\partial_{\mu} - \frac{gB}{2}\tilde{z}_{\mu}\partial_{\nu}\right]\Delta(z), \tag{22}$$

$$D^{0}_{\mu\nu}(z) = \delta_{\mu\nu}D^{0}(z) - \frac{1-\alpha}{2}\left[\frac{\delta_{\mu\nu}}{4\pi^2 z^2} - \frac{2z_{\mu}z_{\nu}}{4\pi^2 z^4}\right] \tag{23}$$

where

$$D^{\mathrm{C}}(z) = \frac{gB}{16\pi^2}\int_0^\infty d\tau \left[\frac{\cosh 2\tau}{\sinh^2\tau} e^{-\frac{1}{4}gBz^2\coth\tau} - 2e^{-\frac{1}{4}gBz^2}\right] \tag{24}$$

$$= \frac{e^{-\frac{gBz^2}{4}}}{4\pi^2 z^2} + \frac{gBC}{8\pi^2}\sinh\frac{gBz^2}{4} + \frac{gB}{16\pi^2}e^{\frac{gBz^2}{4}}\int_0^{\frac{gBz^2}{2}} \ln x\, e^{-x}\,dx, \tag{25}$$

$$D^{\mathrm{S}}(z) = \frac{gB}{16\pi^2}\int_0^\infty d\tau \left[\frac{\sinh 2\tau}{\sinh^2\tau} e^{-\frac{1}{4}gBz^2\coth\tau} - 2e^{-\frac{1}{4}gBz^2}\right] \tag{26}$$

$$= -\frac{gB}{8\pi^2}e^{-\frac{gBz^2}{4}}\ln\frac{gBz^2}{2} - \frac{gBC}{8\pi^2}\cosh\frac{gBz^2}{4} - \frac{gB}{16\pi^2}e^{\frac{gBz^2}{4}}\int_0^{\frac{gBz^2}{2}} \ln x\, e^{-x}\,dx, \tag{27}$$

$$\Delta(z) = \left(\frac{2}{gB}\right)^2 \frac{1}{z^2}\left(D^{\mathrm{C}}(z) - D(z)\right) \tag{28}$$

$$\tilde{z}_\mu = \varepsilon_{\mu\nu} z_\nu, \qquad C \text{ is Euler's constant, } C = 0.5772\ldots. \tag{29}$$

The propagators show a characteristic structure of a sum of three phase factors multiplied by translation invariant functions. This structure gives rise to a modified dimensional regularization.

3 The Polarization Tensor in 1–Loop Approximation

In functional integral formulation the generating functional of the Green's functions reads

$$Z[j] = \int \mathcal{D}a\,\mathcal{D}c\,\mathcal{D}\bar{c}\,\exp\left(-\int \mathcal{L}[a,c,\bar{c}](x)d^4x + aj\right). \tag{30}$$

For shortness we use here and in the following the notation

$$aj = \int a_\mu^a(x) j_\mu^a(x) d^4x. \tag{31}$$

If we proceed to the generating functional of the connected Green's functions $\ln Z[j]$ and then perform a Legendre transform we obtain the effective action

$$\Gamma[A] = -\ln Z[j] + Aj \tag{32}$$

where

$$j_\mu^a(x) = \frac{\delta\Gamma[A]}{\delta A_\mu^a(x)}, \qquad A_\mu^a(x) = \frac{\delta}{\delta j_\mu^a(x)}\ln Z[j] = \frac{1}{Z[j]}\frac{\delta Z[j]}{\delta j_\mu^a(x)}. \tag{33}$$

Then the polarization tensor $\Pi_{\mu\nu}^{ad}(x,y)$ may be defined by the second functional derivative of the effective action as follows

$$\left.\frac{\delta^2\Gamma[A]}{\delta A_\mu^a(x)\delta A_\nu^d(y)}\right|_{j=0} = K_{\mu\nu}^{ad}\delta(x-y) - \Pi_{\mu\nu}^{ad}(x,y). \tag{34}$$

The generating functional $Z[j]$ can be expressed by the propagators of the gauge and the ghost fields if we decompose the Lagrangian (3) according to

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{I}} \tag{35}$$

into a free part

$$\mathcal{L}_0 = \frac{1}{2} a^a_\mu K^{ab}_{\mu\nu} a^b_\nu + \bar{c}^a \tilde{K}^{ab} c^b \tag{36}$$

and an interaction part

$$\mathcal{L}_{\mathrm{I}}[a,c,\bar{c}] = g f^{abc} \left(D_\mu \bar{c}\right)^a c^b a^c_\mu - g f^{abc} D^{ad}_\mu a^d_\nu a^b_\mu a^c_\nu + \frac{1}{4} g^2 f^{abc} f^{ade} a^b_\mu a^c_\nu a^d_\mu a^e_\nu . \tag{37}$$

The latter can be represented by variational derivatives acting on the generating functional where only $\mathcal{L}_0$ enters

$$Z_0[j,\eta,\bar{\eta}] = \int \mathcal{D}a\, \mathcal{D}c\, \mathcal{D}\bar{c} \, \exp\left(- \int \mathcal{L}_0[a,c,\bar{c}](x) d^4x + aj + \eta c + \bar{\eta}\bar{c} \right). \tag{38}$$

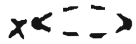
Like the fields c^a and $\bar{c}^a$ the variables η^a and $\bar{\eta}^a$ are Grassmann valued. The functional $Z_0[j,\eta,\bar{\eta}]$ can be evaluated and we obtain

$$Z[j] = \exp\left(- \int \mathcal{L}_{\mathrm{I}} \left[\frac{\delta}{\delta j}, \frac{\delta}{\delta \eta}, \frac{\delta}{\delta \bar{\eta}} \right] (x) d^4x \right) \left(\mathrm{const} \exp\left(\tfrac{1}{2} j G j\right) \exp\left(\bar{\eta} \tilde{G} \eta\right) \right) \Big|_{\eta=\bar{\eta}=0} \tag{39}$$

The power expansion of the exponential functions leads to a perturbation theoretical series, where the polarization tensor is represented by the propagators.

The final result in lowest order of perturbation theory is [7]

$$\Pi^{ad}_{\mu\nu}(x,y) = -g^2 f^{abc} f^{def} D^{cg}_\mu \tilde{G}^{ge}(x,y) D^{fh}_\nu \tilde{G}^{hb}(y,x) +$$

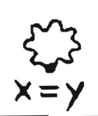


A (40)

$$\begin{aligned} &+ g^2 f^{abc} f^{def} D^{cg}_\mu D^{fh}_\nu \left[G^{be}_{\kappa\lambda}(x,y) G^{gh}_{\varrho\sigma}(x,y) + G^{bh}_{\kappa\sigma}(x,y) G^{ge}_{\varrho\lambda}(x,y) \right] \cdot \\ &\cdot [1 + (\nu\lambda) + (\mu\kappa) + (\nu\lambda)(\mu\kappa) - 2(\nu\sigma) - 2(\mu\varrho) - \\ &\quad - 2(\mu\varrho)(\nu\lambda) - 2(\mu\kappa)(\nu\sigma) + 4(\mu\varrho)(\nu\sigma)] \Big|_{\substack{\varrho=\kappa \\ \sigma=\lambda}} + \end{aligned}$$

A (41)

$$\begin{aligned} &+ g^2 \left[f^{abc} f^{bde} \delta_{\mu\nu} G^{ce}_{\kappa\kappa}(x,x) - \right. \\ &\quad \left. - \left(f^{adb} f^{bce} + f^{abe} f^{bdc} \right) G^{ce}_{\mu\nu}(x,x) \right] \delta(x-y) + \end{aligned}$$

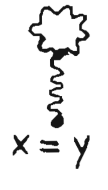


A (42)

$$\begin{aligned} &+ g f^{adc} \left[D^{cg}_\mu \tilde{g}^g_\nu(x) - D^{cg}_\nu \tilde{g}^g_\mu(y) \right] \delta(x-y) + \\ &+ g f^{abc} \left[\tilde{g}^b_\nu(x) D^{cd}_{x_\mu} - \delta_{\mu\nu} \tilde{g}^b_\kappa(x) D^{cd}_{x_\kappa} \right] \delta(x-y) + \\ &+ g f^{dbc} \left[\tilde{g}^b_\mu(y) D^{ca}_{y_\nu} - \delta_{\mu\nu} \tilde{g}^b_\kappa(y) D^{ca}_{y_\kappa} \right] \delta(y-x) + \end{aligned}$$

B (43)

$$\begin{aligned} &+ g f^{adc} \left[D^{cg}_\mu g^g_\nu(x) - D^{cg}_\nu g^g_\mu(y) \right] \delta(x-y) + \\ &+ g f^{abc} \left[g^b_\nu(x) D^{cd}_{x_\mu} - \delta_{\mu\nu} g^b_\kappa(x) D^{cd}_{x_\kappa} \right] \delta(x-y) + \\ &+ g f^{dbc} \left[g^b_\mu(y) D^{ca}_{y_\nu} - \delta_{\mu\nu} g^b_\kappa(y) D^{ca}_{y_\kappa} \right] \delta(y-x) \end{aligned}$$



B (44)

with

$$\tilde{g}^g_\nu(x) = -\int d^4y\, g f^{def} D^{fh}_\kappa \tilde{G}^{eh}(y,y) G^{gd}_{\nu\kappa}(x,y) \tag{45}$$

$$\begin{aligned} g^g_\kappa(x) = \int d^4y\, g f^{def} D^{fh}_\nu \big[& G^{gh}_{\kappa\lambda}(x,y) G^{de}_{\nu\lambda}(y,y) + \\ & + G^{ge}_{\kappa\lambda}(x,y) G^{dh}_{\nu\lambda}(y,y) + G^{gd}_{\kappa\nu}(x,y) G^{eh}_{\lambda\lambda}(y,y)\big]. \end{aligned} \tag{46}$$

Up to now no use at all was made of the special form of the background so that this result is quite general. The notation $(\alpha\beta)$ symbolizes an exchange of α and β. The Feynman diagrams are shown next to the corresponding parts of the formula. Later we will see that the last two contributions, denoted with letter B, have a special structure. Therefore they are omitted at the moment and are treated in section 5. Hence we look now at the contributions denoted by the letter A.

If we replace in (40 – 42) the propagators by their explicit representations (13, 14) we see that the structure of the propagators is reproduced and we obtain

$$\Pi^{ad}_{\mu\nu}(x,y)(\mathrm{A}) = \Phi^{ad}(x,y)\Pi^{(1)}_{\mu\nu}(z) + \tilde{\Phi}^{ad}(x,y)\Pi^{(2)}_{\mu\nu}(z) + \delta^{ad}_{\parallel}\Pi^{(3)}_{\mu\nu}(z) \tag{47}$$

with

$$\begin{aligned} \Pi^{(1)}_{\mu\nu}(z) = -g^2 \Big\{ & \partial_\mu D(z)\partial_\nu D^0(z) + \partial_\mu D^0(z)\partial_\nu D(z) + \\ & + \Big[D^{\mathrm{C}}_{\kappa\lambda}\partial_\mu\partial_\nu D^0_{\varrho\sigma} + D^0_{\kappa\lambda}\partial_\mu\partial_\nu D^{\mathrm{C}}_{\varrho\sigma} + \frac{gB}{2} D^0_{\kappa\lambda}\tilde{z}_\mu\partial_\nu D^{\mathrm{S}}_{\varrho\sigma} - \frac{gB}{2} D^0_{\kappa\lambda}\tilde{z}_\nu\partial_\mu D^{\mathrm{S}}_{\varrho\sigma} + \\ & \quad + \frac{gB}{2}\varepsilon_{\mu\nu} D^0_{\kappa\lambda} D^{\mathrm{S}}_{\varrho\sigma} + \left(\frac{gB}{2}\right)^2 D^0_{\kappa\lambda}\tilde{z}_\mu\tilde{z}_\nu D^{\mathrm{C}}_{\varrho\sigma} - \partial_\mu D^0_{\varrho\lambda}\partial_\nu D^{\mathrm{C}}_{\kappa\sigma} - \\ & \quad - \partial_\mu D^{\mathrm{C}}_{\varrho\lambda}\partial_\nu D^0_{\kappa\sigma} + \frac{gB}{2}\partial_\mu D^0_{\varrho\lambda}\tilde{z}_\nu D^{\mathrm{S}}_{\kappa\sigma} - \frac{gB}{2} D^{\mathrm{S}}_{\varrho\lambda}\tilde{z}_\mu\partial_\nu D^0_{\kappa\sigma} \Big] \cdot \\ & \cdot [1 + (\nu\lambda) + (\mu\kappa) + (\nu\lambda)(\mu\kappa) - 2(\nu\sigma) - 2(\mu\varrho) - \\ & \quad - 2(\mu\varrho)(\nu\lambda) - 2(\mu\kappa)(\nu\sigma) + 4(\mu\varrho)(\nu\sigma)]\big|_{\substack{\varrho=\kappa \\ \sigma=\lambda}} + \\ & + \left[\delta_{\mu\nu} D^{\mathrm{C}}_{\kappa\kappa}(0) + \delta_{\mu\nu} D^0_{\kappa\kappa}(0) - D^{\mathrm{C}}_{\mu\nu}(0) - D^0_{\mu\nu}(0)\right]\delta(z) \Big\} \end{aligned} \tag{48}$$

$$\begin{aligned} \Pi^{(2)}_{\mu\nu}(z) = -g^2 \Big\{ & \frac{gB}{2} D(z)\tilde{z}_\nu\partial_\mu D^0(z) - \frac{gB}{2} D(z)\tilde{z}_\mu\partial_\nu D^0(z) + \\ & + \Big[D^{\mathrm{S}}_{\kappa\lambda}\partial_\mu\partial_\nu D^0_{\varrho\sigma} + D^0_{\kappa\lambda}\partial_\mu\partial_\nu D^{\mathrm{S}}_{\varrho\sigma} - \frac{gB}{2} D^0_{\kappa\lambda}\tilde{z}_\mu\partial_\nu D^{\mathrm{C}}_{\varrho\sigma} + \frac{gB}{2} D^0_{\kappa\lambda}\tilde{z}_\nu\partial_\mu D^{\mathrm{C}}_{\varrho\sigma} - \\ & \quad - \frac{gB}{2}\varepsilon_{\mu\nu} D^0_{\kappa\lambda} D^{\mathrm{C}}_{\varrho\sigma} + \left(\frac{gB}{2}\right)^2 D^0_{\kappa\lambda}\tilde{z}_\mu\tilde{z}_\nu D^{\mathrm{S}}_{\varrho\sigma} - \partial_\mu D^0_{\varrho\lambda}\partial_\nu D^{\mathrm{S}}_{\kappa\sigma} - \\ & \quad - \partial_\mu D^{\mathrm{S}}_{\varrho\lambda}\partial_\nu D^0_{\kappa\sigma} - \frac{gB}{2}\partial_\mu D^0_{\varrho\lambda}\tilde{z}_\nu D^{\mathrm{C}}_{\kappa\sigma} + \frac{gB}{2} D^{\mathrm{C}}_{\varrho\lambda}\tilde{z}_\mu\partial_\nu D^0_{\kappa\sigma} \Big] \cdot \\ & \cdot [1 + (\nu\lambda) + (\mu\kappa) + (\nu\lambda)(\mu\kappa) - 2(\nu\sigma) - 2(\mu\varrho) - \end{aligned} \tag{49}$$

$$
\begin{aligned}
&\qquad - 2(\mu\varrho)(\nu\lambda) - 2(\mu\kappa)(\nu\sigma) + 4(\mu\varrho)(\nu\sigma)]\Big|_{\substack{\varrho=\kappa\\ \sigma=\lambda}} + \\
&+ 3D^{\mathrm{S}}_{\mu\nu}(0)\delta(z)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\Pi^{(3)}_{\mu\nu}(z) = -2g^2\Bigg\{ &\partial_\mu D(z)\partial_\nu D(z) - \left(\frac{gB}{2}\right)^2 D(z)\tilde{z}_\mu\tilde{z}_\nu D(z) + \\
&+ \Bigg[D^{\mathrm{C}}_{\kappa\lambda}\partial_\mu\partial_\nu D^{\mathrm{C}}_{\varrho\sigma} + \frac{gB}{2}D^{\mathrm{C}}_{\kappa\lambda}\tilde{z}_\mu\partial_\nu D^{\mathrm{S}}_{\varrho\sigma} - \frac{gB}{2}D^{\mathrm{C}}_{\kappa\lambda}\tilde{z}_\nu\partial_\mu D^{\mathrm{S}}_{\varrho\sigma} + \\
&\quad + \frac{gB}{2}\varepsilon_{\mu\nu}D^{\mathrm{C}}_{\kappa\lambda}D^{\mathrm{S}}_{\varrho\sigma} + \left(\frac{gB}{2}\right)^2 D^{\mathrm{C}}_{\kappa\lambda}\tilde{z}_\mu\tilde{z}_\nu D^{\mathrm{C}}_{\varrho\sigma} + \left(\frac{gB}{2}\right)^2 D^{\mathrm{S}}_{\kappa\lambda}\tilde{z}_\mu\tilde{z}_\nu D^{\mathrm{S}}_{\varrho\sigma} + \\
&\quad + D^{\mathrm{S}}_{\kappa\lambda}\partial_\mu\partial_\nu D^{\mathrm{S}}_{\varrho\sigma} - \frac{gB}{2}D^{\mathrm{S}}_{\kappa\lambda}\tilde{z}_\mu\partial_\nu D^{\mathrm{C}}_{\varrho\sigma} + \frac{gB}{2}D^{\mathrm{S}}_{\kappa\lambda}\tilde{z}_\nu\partial_\mu D^{\mathrm{C}}_{\varrho\sigma} - \\
&\quad - \frac{gB}{2}\varepsilon_{\mu\nu}D^{\mathrm{S}}_{\kappa\lambda}D^{\mathrm{C}}_{\varrho\sigma} - \partial_\mu D^{\mathrm{C}}_{\varrho\lambda}\partial_\nu D^{\mathrm{C}}_{\kappa\sigma} - \frac{gB}{2}D^{\mathrm{S}}_{\varrho\lambda}\tilde{z}_\mu\partial_\nu D^{\mathrm{C}}_{\kappa\sigma} + \\
&\quad + \frac{gB}{2}D^{\mathrm{S}}_{\kappa\sigma}\tilde{z}_\nu\partial_\mu D^{\mathrm{C}}_{\varrho\lambda} + \left(\frac{gB}{2}\right)^2 D^{\mathrm{S}}_{\varrho\lambda}\tilde{z}_\mu\tilde{z}_\nu D^{\mathrm{S}}_{\kappa\sigma} - \partial_\mu D^{\mathrm{S}}_{\varrho\lambda}\partial_\nu D^{\mathrm{S}}_{\kappa\sigma} + \\
&\quad + \frac{gB}{2}D^{\mathrm{C}}_{\varrho\lambda}\tilde{z}_\mu\partial_\nu D^{\mathrm{S}}_{\kappa\sigma} - \frac{gB}{2}D^{\mathrm{C}}_{\kappa\sigma}\tilde{z}_\nu\partial_\mu D^{\mathrm{S}}_{\varrho\lambda} + \left(\frac{gB}{2}\right)^2 D^{\mathrm{C}}_{\varrho\lambda}\tilde{z}_\mu\tilde{z}_\nu D^{\mathrm{C}}_{\kappa\alpha}\Bigg] \cdot \\
&\quad \cdot [1 + (\nu\lambda) + (\mu\kappa) + (\nu\lambda)(\mu\kappa) - 2(\nu\sigma) - 2(\mu\varrho) - \\
&\qquad - 2(\mu\varrho)(\nu\lambda) - 2(\mu\kappa)(\nu\sigma) + 4(\mu\varrho)(\nu\sigma)]\Big|_{\substack{\varrho=\kappa\\ \sigma=\lambda}} + \\
&+ \left(\delta_{\mu\nu}D^{\mathrm{C}}_{\kappa\kappa}(0) - D^{\mathrm{C}}_{\mu\nu}(0)\right)\delta(z)\Bigg\}
\end{aligned} \tag{50}
$$

In order to regularize this expression for the polarization tensor dimensional regularization is performed on the translation invariant functions $\Pi^{(1)}_{\mu\nu}(z)$, $\Pi^{(2)}_{\mu\nu}(z)$ and $\Pi^{(3)}_{\mu\nu}(z)$.

4 The Counter Terms for the Polarisation Tensor

The special structure of the polarization tensor (47) gives rise to a modification of dimensional regularization in that way that it is applied only to the translation invariant functions $\Pi^{(1)}_{\mu\nu}(z)$, $\Pi^{(2)}_{\mu\nu}(z)$ and $\Pi^{(3)}_{\mu\nu}(z)$.

We use the n–dimensional Fourier transform

$$
\mathcal{F}_n[f(z)](k) = \mu^{n-4}\int f(z)e^{ikz}d^nz, \qquad \mathcal{F}_n^{-1}[g(k)](z) = \frac{\mu^{4-n}}{(2\pi)^n}\int g(k)e^{-ikz}d^nk \tag{51}
$$

to generalize all distributions into n dimensions

$$
\overset{\mathrm{reg}}{f}(z) = \mathcal{F}_n^{-1}\left[\mathcal{F}_4[f]\right]. \tag{52}
$$

μ is an arbitrary parameter with the dimension of a mass. The product of distributions is then defined as follows

$$
\begin{aligned}
(f\cdot g)_{\mathrm{reg}}(z) = \overset{\mathrm{reg}}{f}(z)\cdot\overset{\mathrm{reg}}{g}(z) &= \mathcal{F}_n^{-1}\left[\mathcal{F}_4[f]\right]\cdot\mathcal{F}_n^{-1}\left[\mathcal{F}_4[g]\right] \\
&= \mathcal{F}_n^{-1}\left[\mathcal{F}_4[f] * \mathcal{F}_4[g]\right]
\end{aligned} \tag{53}
$$

where

$$(f * g)(k) = f(k) * g(k) = \frac{\mu^{4-n}}{(2\pi)^n} \int f(q) g(k-q)\, d^n q. \tag{54}$$

For the divergent parts of the polarization tensor only the short distance behaviour of the functions D, D^0, $D^{\mathrm{C}}_{\mu\nu}$, $D^{\mathrm{S}}_{\mu\nu}$ and $D^0_{\mu\nu}$ is important:

$$\begin{aligned}
D(z) &= \frac{1}{4\pi^2 z^2} + \mathcal{O}(1), \qquad D^0(z) = \frac{1}{4\pi^2 z^2}, \\
D^{\mathrm{C}}_{\mu\nu}(z) &= \frac{\delta_{\mu\nu}}{4\pi^2 z^2} - \frac{1-\alpha}{2}\left[\frac{\delta_{\mu\nu}}{4\pi^2 z^2} - \frac{2 z_\mu z_\nu}{4\pi^2 z^4}\right] + \mathcal{O}(1), \\
D^0_{\mu\nu}(z) &= \frac{\delta_{\mu\nu}}{4\pi^2 z^2} - \frac{1-\alpha}{2}\left[\frac{\delta_{\mu\nu}}{4\pi^2 z^2} - \frac{2 z_\mu z_\nu}{4\pi^2 z^4}\right], \\
D^{\mathrm{S}}_{\mu\nu}(z) &= -\frac{gB}{8\pi^2}\varepsilon_{\mu\nu} \ln\frac{gBz^2}{2}\left(1 - \frac{1-\alpha}{4}\right) - \frac{1-\alpha}{2}\frac{gB}{2}\frac{z_\mu \tilde{z}_\nu - \tilde{z}_\mu z_\nu}{4\pi^2 z^2} + \mathcal{O}(1),
\end{aligned} \tag{55}$$

Hence it follows that for the calculation of the counter terms we may use the approximations

$$D(z) \approx D^0(z), \qquad D^{\mathrm{C}}_{\mu\nu}(z) \approx D^0_{\mu\nu}(z). \tag{56}$$

Furthermore it can be shown that for our purposes it is possible to set [7]

$$\mathcal{F}_4\left[-\frac{gB}{8\pi^2}\ln\frac{gBz^2}{2}\right] \approx 2gB \int_0^\infty \tau e^{-\tau k^2}\, d\tau. \tag{57}$$

With that we find the following representations:

$$\begin{aligned}
\mathcal{F}_4[D^{\mathrm{S}}_{\mu\nu}] &\approx gB \int_0^\infty \left(2\sigma\varepsilon_{\mu\nu} + \frac{1-\alpha}{2}\left(2\sigma^2 k_\mu \tilde{k}_\nu - 2\sigma^2 \tilde{k}_\mu k_\nu + \sigma\varepsilon_{\mu\nu}\right)\right) e^{-\sigma k^2}\, d\sigma, \\
\mathcal{F}_4[D^0_{\mu\nu}] &= \int_0^\infty \left(\delta_{\mu\nu} - (1-\alpha)\sigma k_\mu k_\nu\right) e^{-\sigma k^2}\, d\sigma, \\
\mathcal{F}_4[D^0] &= \int_0^\infty e^{-\sigma k^2}\, d\sigma.
\end{aligned} \tag{58}$$

Finally we obtain for the counter terms

$$\begin{aligned}
\overset{\mathrm{div}}{\Pi}{}^{ab}_{\mu\nu}(x,y)(\mathrm{A}) &= -\frac{5}{24\pi^2}\frac{g^2}{2-\frac{n}{2}}\left(1 + \frac{3}{10}(1-\alpha)\right) \cdot \\
&\quad \cdot \left[\left(\Phi^{ab}(x,y) + \delta^{ab}_{\parallel}\right)\left(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right) + \frac{3}{2} gB\varepsilon_{\mu\nu}\right]\delta(x-y)
\end{aligned} \tag{59}$$

This expression can be written as

$$\begin{aligned}
\overset{\mathrm{div}}{\Pi}{}^{ab}_{\mu\nu}(x,y)(\mathrm{A}) &= -\frac{5}{24\pi^2}\frac{g^2}{2-\frac{n}{2}}\left(1 + \frac{3}{10}(1-\alpha)\right) \cdot \\
&\quad \cdot \left[\left(\delta_{\mu\nu}D^2 - D_\mu D_\nu\right)^{ab} + 2g f^{abc} F^c_{\mu\nu}\right]\delta(x-y)
\end{aligned} \tag{60}$$

and fulfils the modified Ward identity

$$D_\mu^{ab} \overset{\mathrm{div}}{\Pi}{}^{bc}_{\mu\nu}(x,y)(\mathrm{A}) = 0 \tag{61}$$

for all values of the gauge fixing parameter α.

5 Contributions of the Tadpole Terms with Three Gluon Vertices

Now the two contributions B of the polarization tensor (43, 44) are treated. Since they have the same structure they can be united as

$$\begin{aligned}\Pi^{ad}_{\mu\nu}(x,y)(\mathrm{B}) = g f^{adc}\left[D^{cg}_\mu \bar{g}^g_\nu(x) - D^{cg}_\nu \bar{g}^g_\mu(y)\right]\delta(x-y) + \\ + g f^{abc}\left[\bar{g}^b_\nu(x) D^{cd}_{x_\mu} - \delta_{\mu\nu}\bar{g}^b_\kappa(x) D^{cd}_{x_\kappa}\right]\delta(x-y) + \\ + g f^{dbc}\left[\bar{g}^b_\mu(y) D^{ca}_{y_\nu} - \delta_{\mu\nu}\bar{g}^b_\kappa(y) D^{ca}_{y_\kappa}\right]\delta(y-x)\end{aligned} \tag{62}$$

with

$$\begin{aligned}\bar{g}^a_\mu(x) &= g^a_\mu(x) + \tilde{g}^a_\mu(x) \qquad (g^a_\mu,\ \tilde{g}^a_\mu \text{ defined in } (45, 46)) \\ &= g f^{def}\int d^4u\, D^{fh}_\kappa\left[G^{ah}_{\mu\lambda}(x,u) G^{de}_{\kappa\lambda}(u,u) + G^{ae}_{\mu\lambda}(x,u) G^{dh}_{\kappa\lambda}(u,u) + \right. \\ &\qquad \left. + G^{ad}_{\mu\kappa}(x,u) G^{eh}_{\lambda\lambda}(u,u) - G^{ad}_{\mu\kappa}(x,u)\tilde{G}^{eh}(u,u)\right].\end{aligned} \tag{63}$$

From the functional integral representation we obtain the following relation in lowest order of perturbation theory:

$$\langle a^a_\mu(x)\rangle = \frac{1}{Z[j]}\left.\frac{\delta Z[j]}{j^a_\mu(x)}\right|_{j=0} = \bar{g}^a_\mu(x). \tag{64}$$

Dimensional regularization yields

$$D^{fh}_\kappa G^{dh}_{\kappa\lambda}(u,u) = \text{finite}, \qquad D^{fh}_\kappa G^{eh}_{\lambda\lambda}(u,u) = \text{finite}, \qquad D^{fh}_\kappa \tilde{G}^{eh}(u,u) = \text{finite}$$

and

$$\begin{aligned}G^{de}_{\kappa\lambda}(u,u) &= \varepsilon^{de}_\perp D^{\mathrm{S}}_{\kappa\lambda}(0) + \text{finite terms} \\ &= \frac{gB}{8\pi^2}\frac{1}{2-\frac{n}{2}}\varepsilon_{\kappa\lambda}\frac{3+\alpha}{4}\varepsilon^{de}_\perp + \text{finite terms}.\end{aligned}$$

Further calculation leads to ('$\simeq$' means that finite terms are neglected)

$$\bar{g}^a_\mu(x) \simeq -\frac{g^2 B}{8\pi^2}\frac{1}{2-\frac{n}{2}}\frac{3+\alpha}{2}\delta^{a3}\varepsilon_{\kappa\mu}\partial_\kappa\int D^0(x-u)d^4u. \tag{65}$$

The integral does not exist, it has to be regularized. Dimensional regularization does not work here because $(D^0)_{\mathrm{reg}}(z) \sim |z|^{2-n}$ and therefore

$$\int (D^0)_{\mathrm{reg}}(x-u)d^n u \sim \int |x-u|^{2-n} d^n u \tag{66}$$

diverges for all values of the dimension n, namely for large u. This is not so astonishing because the type of divergency is an infrared one. If we use other regularizations the integral can be calculated. Two possibilities are

$$\int D^0(x-u)e^{-\frac{1}{2}\varepsilon(u-a)^2}d^4u = -\frac{1}{8}x^2 + \frac{1}{4}a_\mu x_\mu + \frac{1}{2\varepsilon} - \frac{1}{8}a^2 + O(\varepsilon) \tag{67}$$

$$\int\limits_{|u-a|<R} D^0(x-u)d^4u = -\frac{1}{8}x^2 + \frac{1}{4}a_\mu x_\mu + \frac{R^2}{4} - \frac{1}{8}a^2 + O\left(\frac{1}{R}\right) \tag{68}$$

These two examples have the general form

$$\mathrm{reg}\int D^0(x-u)d^4u = -\frac{1}{8}x^2 + \frac{1}{4}p_\mu x_\mu + q \tag{69}$$

with constants p_μ and q which depend on the regularization. Unfortunately the regularized expression (69) is not unique in the stronger sense that it can obtain a totally different structure as a result of other regularizations. Let us e. g. introduce a gluon mass m into the gluon propagator in (65). This results in the replacement $D^0(z) \Rightarrow (m/4\pi^2\sqrt{z^2})K_1(m\sqrt{z^2})$ which makes the integral (69) finite and independent from x. Within the latter regularization the vacuum expectation value $\langle a_\mu^a(x)\rangle$ as well as the contribution (62) to $\Pi_{\mu\nu}^{ab}(x,y)$ would vanish. If on the other hand one chooses regularizations like (67) or (68) with the general result (69) then one obtains finally

$$\bar{g}_\mu^a(x) \simeq -\frac{g^2 B}{32\pi^2}\frac{1}{2-\frac{n}{2}}\frac{3+\alpha}{2}\delta^{a3}(\tilde{x}_\mu - \tilde{p}_\mu) = \frac{g^2}{16\pi^2}\frac{1}{2-\frac{n}{2}}\frac{3+\alpha}{2}B_\mu^a(x-p). \tag{70}$$

If we substitute this expression in equation (62) for the part B of the polarization tensor we find for the corresponding counter terms

$$\begin{aligned}\overset{\mathrm{div}}{\Pi}{}_{\mu\nu}^{ad}(x,y)(\mathrm{B}) = \frac{g^2}{16\pi^2}\frac{3+\alpha}{2}\frac{1}{2-\frac{n}{2}}\Big\{&gf^{adc}\left[D_\mu^{cg}B_\nu^g(x-p) - D_\nu^{cg}B_\mu^g(y-p)\right]\delta(x-y) + \\ &+gf^{abc}\left[B_\nu^b(x-p)D_{x_\mu}^{cd} - \delta_{\mu\nu}B_\kappa^b(x-p)D_{x_\kappa}^{cd}\right]\delta(x-y) + \\ &+gf^{dbc}\left[B_\mu^b(y-p)D_{y_\nu}^{ca} - \delta_{\mu\nu}B_\kappa^b(y-p)D_{y_\kappa}^{ca}\right]\delta(y-x).\end{aligned} \tag{71}$$

Since the background field is homogeneous the constant p represents not more than a gauge transformation. These counter terms do not obey the generalized Ward–Takahashi identity (61). For the special value $\alpha = -3$ the divergencies (70) disappear as well as the counter terms (71).

There is another interesting relation between the structure (62) and a suitable variation of the kinetic kernel of the gauge field. This kernel depends by the covariant derivative on the product of the coupling constant g and the background field $B_\mu^a(x)$. The coupling constant has to be renormalized. Let us ask now how counter terms have to look like in order to renormalize $gB_\mu^a(x)$. Therefore we consider the variation of the kinetic kernel if the product gB is replaced by the renormalized one $(gB)_{\mathrm{ren}} = gB + \delta(gB)$

$$\delta K_{\mu\nu}^{ab}(gB)\delta(x-y) \stackrel{\mathrm{def}}{=} K_{\mu\nu}^{ab}(gB+\delta(gB))\delta(x-y) - K_{\mu\nu}^{ab}(gB)\delta(x-y). \tag{72}$$

The parenthesis after the $K_{\mu\nu}^{ab}$ give the respective dependencies of the kinetic kernel. If we look only at the lowest order of perturbation theory we have to neglect all products of $\delta(gB)$ with itself. Then we obtain after some transformations

$$\begin{aligned}\delta K^{ab}_{\mu\nu}\delta(x-y) = &-gf^{abc}\left[(D_\mu\delta(gB)_\nu(x))^c - (D_\nu\delta(gB)_\mu(y))^c\right]\delta(x-y) - \\ &-gf^{adc}\left[\left(1-\frac{1}{\alpha}\right)\delta(gB)^d_\nu(x)D^{cb}_{x_\mu} - \delta_{\mu\nu}\delta(gB)^d_\kappa(x)D^{cb}_{x_\kappa}\right]\delta(x-y) - \\ &-gf^{bdc}\left[\left(1-\frac{1}{\alpha}\right)\delta(gB)^d_\mu(y)D^{ca}_{y_\nu} - \delta_{\mu\nu}\delta(gB)^b_\kappa(y)D^{ca}_{y_\kappa}\right]\delta(y-x). \quad (73)\end{aligned}$$

In comparison with formula (62) and (71) we can observe that these structures are almost the same as in (73). They would be identical if $\alpha \to \infty$, i. e. if the kinetic kernel would not contain the gauge fixing term. In this case we could identify

$$\delta(gB)^a_\mu(x) = -\frac{g^2}{16\pi^2}\frac{3+\alpha}{2}\frac{1}{2-\frac{n}{2}}gB^a_\mu(x-p). \quad (74)$$

This suggests for $p=0$ a multiplicative renormalization of the product gB^a_μ

$$\left(gB^a_\mu\right)_{\text{ren}} = \left(1-\frac{1}{16\pi^2}\frac{3+\alpha}{2}\frac{g^2}{2-\frac{n}{2}}\right)gB^a_\mu. \quad (75)$$

But since we have $\alpha \neq \infty$ the counter terms (71) do not have the right structure. There are two possibilities now.

On the one hand we can want to keep the multiplicative renormalization scheme. Then we must demand for the counter terms (71) to be zero. This is the case if $\alpha = -3$ so that we obtain a restriction for the gauge fixing parameter.

On the other hand we can admit arbitrary values of α. Then the counter terms (71) bring new terms into the Lagrangian. Since these new structures have the same dimension as already existing terms of the Lagrangian the renormalizability is guaranteed though we give up multiplicative renormalization.

There is another argumentation [8] which yields $\alpha = -3$. It is also based on multiplicative renormalization. From the way of the introduction of the external field one should expect that the background field is renormalized by the same Z factor as the gauge field. Furthermore the product gB^a_μ also enters the covariant derivative in the gauge fixing term (2) which is independent from the rest of the kinetic kernel for the gauge field (4). Since the partial derivative ∂_μ cannot be renormalized the same must be valid for the product gB^a_μ in order to keep the covariant derivative unchanged. If we now assume that the renormalization factors are the same as in a gauge theory without a background then we have

$$g_{\text{ren}} = \left(1-\frac{11}{48\pi^2}\frac{g^2}{2-\frac{n}{2}}\right)g, \qquad \left(B^a_\mu\right)_{\text{ren}} = \left(1+\frac{1}{48\pi^2}\frac{13-3\alpha}{2}\frac{g^2}{2-\frac{n}{2}}\right)B^a_\mu \quad (76)$$

and we obtain (75) again. Because of $(gB^a_\mu)_{\text{ren}} = gB^a_\mu$ we find once more $\alpha = -3$.

6 Conclusions

The radiative corrections to the polarization tensor $\Pi^{ab}_{\mu\nu}(x,y)$ have been studied in 1–loop approximation for arbitrary values of the gauge fixing parameter α. The counter terms have been explicitly calculated using a modified dimensional regularization. Apart from the contribution of some gluonic tadpole diagrams the counter terms obey a generalized Ward–Takahashi identity for arbitrary values of α. The just mentioned tadpole terms, which are related to the expectation value of the gauge field fluctuation $\langle 0|a_\mu^a(x)|0\rangle$, are divergent both ultraviolet and infrared. The latter divergency has been treated by different regularization procedures. If one does not choose a procedure which regularizes this contribution to zero (as e. g. introducing a gluon mass into the final formulas) the remaining counter term is proportional to $3+\alpha$. In this case the simple multiplicative renormalization scheme is lost for all $\alpha \neq -3$. At the same time the polarization tensor $\Pi_{\mu\nu}^{ab}(x,y)$ needs the counter term (71) and the expectation value $\langle a_\mu^a(x)\rangle$ should be renormalized by the counter term (70).

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QED between parallel mirrors: light signals faster than c, or amplified by the vacuum

G. Barton*

Physics and Astronomy Division, University of Sussex, Brighton BN1 9QH, England

K. Scharnhorst†

INTSEM and Fachbereich Physik, University of Leipzig, Augustusplatz 10, D - O-7010 Leipzig, Federal Republic of Germany

Abstract

Because it is scattered by the zero-point oscillations of the quantized fields, light of frequency ω travelling normally to two parallel mirrors experiences the vacuum between them as a dispersive medium with refractive index $n(\omega)$. Our earlier low-frequency result that $n(0) < 1$ is combined with the Kramers-Kronig dispersion relation for n and with the classic Sommerfeld-Brillouin argument to show (under certain physically reasonable assumptions) that either $n(\infty) < 1$, in which case the signal velocity $c/n(\infty)$ exceeds c; or that the imaginary part of n is negative at least for some ranges of frequency, in which case the vacuum between the fixed mirrors fails to respond to a light probe like a normal passive medium. Further, the optical theorem suggests that n exhibits no dispersion to order e^4, i.e. that $n(\infty) = n(0)$ up to corrections of order e^6 at most.

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*e-mail: mpfk7 @ central.sussex.ac.uk

†e-mail: kjsch @ qft.physik.uni-leipzig.dbp.de

Quantum Processes in Cosmic-String Spacetimes

Jürgen Audretsch and Athanasios Economou
Fakultät für Physik der Universität Konstanz
Postfach 5560, D 7750 Konstanz, Germany

Abstract

The effect of the conical topological structure of cosmic strings on quantum interactions is studied in local terms and illustrated in the case of pair creation and bremsstrahlung. We find that the influence of the cosmic-string is localized: Only such processes are significantly affected that take place at distances from where they can 'sense' the cosmic string with their intrinsic quantum extensions. This extension is for pair creation the Compton wavelength of the created particles while for bremsstrahlung it is the wavelength of the emitted photon, which can be arbitrarily large. This results to a finite pair creation cross section proportional to the Compton wavelength, and to the infrared catastrophe for the bremsstrahlung process. Quantitative results for the pair creation cross section and the bremsstrahlung energy loss are finally given.

1. Introduction

One of the most important features in the cosmic-string physics is the conical structure of the spacetime around the cosmic string. The corresponding angle deficit is $\Delta\varphi = 8\pi G\mu$ where G is the gravitational constant and μ is the mass per unit length of the cosmic-string. Of astrophysical interest are cosmic-strings formed at the grand unification scale having $G\mu \approx 10^{-6}$. The spacetime of an infinite straight static cosmic string, taken to lie along the z axis, is described in the zero thickness approximation by the metric[1]

$$ds^2 = -dt^2 + dz^2 + dr^2 + b^2r^2d\varphi^2. \tag{1.1}$$

Here the coordinates are the usual polar coordinates in Minkowski spacetime and $b = 1 - 4G\mu$. Note that the only difference with the Minkowski spacetime is in the constant b which is slightly less than unity. This topological modification, although small, is responsible for several important physical effects. Indeed, let us remember (see e.g. Refs. 1, 2)

- the classical gravitational analog of Aharonov-Bohm effect: test particles moving in the spacetime (1.1) are deflected off the cosmic-string although they feel no gravitational forces (the curvature is localized along the z axis).
- the characteristic gravitational lensing field.
- and the step-like temperature anisotropies in the microwave background produced by moving strings. All these effects provide a basis for the detection of cosmic strings.

- Also the formation of wakes behind moving cosmic-strings provides a seeding mechanism for the formation of sheet-like large scale structure.

The conical topology can also affect classical and quantum fields.

- A static electric charge feels a repulsive self force[3] while similarly a massive particle feels an attractive gravitational force[4] of second order in G.
- The quantum vacuum is polarized in a very similar way as in the well known Casimir effect,[5] while free quantum fields scatter in a non trivial way.[6]
- Quantum interactions are modified.

This last effect will be the subject of the present talk. To isolate effects due to the conical topology we will consider no coupling of external fields to the fields that make up the string core. Consequently, we shall not study the interesting Callan-Rubakov effect which has been discussed e.g. in Ref. 7. Discussions on Aharonov-Bohm type of interactions due to the magnetic flux of the cosmic-string core can be found e.g. in Refs. 8, 9. For interesting reviews and further references on physical effects around cosmic-strings see Ref. 10.

Starting out, let us look at particular quantum processes as e.g. the quantum electrodynamical (QED) pair creation (PC), and bremsstrahlung (BS). These processes are *not allowed* in empty Minkowski spacetime due to momentum conservation constraints. They could however *take place* in the presence of a cosmic string because now there is a breakdown of the translational invariance on the $x - y$ plane, the plane perpendicular to the cosmic-string.

This argument of *momentum non conservation* seems strong enough to explain why such processes are expected to take place in the presence of a cosmic string. However, one should note that it is a *non local statement*. Therefore it is not so useful in analyzing our intuitive expectation that the influence of the cosmic string on the quantum processes is a spatially dependent one (processes close to the string are certainly expected to be modified more than others far away). The question that now arises is whether we could give within a S-matrix scheme a local analysis of quantum processes around the cosmic string.

Let us mention some of the difficulties that one faces trying to answer this question. The usual S-matrix quantities (plain-wave particle states, transition probability amplitudes, ...) are non local. Of course the *cross section* gives, if it can be defined, a measure of the range of the influence of the scattering centre, identified in our case with the cosmic string. However it is not so useful when it is infinite. Furthermore, in general it cannot be defined satisfactorily, as e.g. in the case of the collision of two particles in the presence of the cosmic string. There, to each collision event should correspond something like a spatially varying cross section, depending on the distance from the cosmic string. Such a notion, however, cannot come out from a usual S-matrix scheme. Finally, it should be mentioned that local information can in principle be obtained using wave-packet states but one soon runs into technical problems.

So besides the calculation of transition probabilities, cross sections and so on, we are also interested here in extracting within an S-matrix scheme qualitative or, even better, quantitative information about the local influence of the cosmic-string. As we shall explain below, a way to do this is to use angular momentum eigenstates which on the one hand have interesting local properties and on the other allow for analytic calculations. Calculational details will not be given here but can be found in the Refs. 11, 12, 13 which are also the basic references for what follows.

2. Cylindrical modes and their local behavior

Let us consider a Klein-Gordon field with mass M. The field equation in the background (1.1) can be solved in terms of the *cylindrical modes*

$$u_j = u_{\kappa\ell\zeta} = \frac{1}{\sqrt{2b}} e^{-iE_j t} e^{i\kappa z} e^{i\ell\varphi} J_{\frac{|\ell|}{b}}(\zeta r). \tag{2.1}$$

j is the collective quantum label $j = \{\kappa, \ell, \zeta\}$ with $\ell \in Z, \kappa \in (-\infty, +\infty)$ and $\zeta \in (0, \infty)$. Here $\kappa, \frac{\ell}{b}, \zeta, E_j = (\kappa^2 + \zeta^2 + M^2)^{1/2}$ denote respectively the z momentum, z angular momentum, measure of the $x - y$ momentum and the energy. Note that the topological influence of the cosmic-string on the free fields consists of producing non integer angular momentum.

Let us now look at the particle density of the u_j modes. This is a quantity proportional to the square of the Bessel function that appears in Eq.(2.1). As it is well known, the Bessel function $J_\nu(x)$ goes to zero as $\propto x^\nu$ for $x < \nu$ while for $x \gg \nu$ it oscillates with an amplitude that falls off as $1/\sqrt{r}$ and is independent of ν. This behavior implies for the u_j modes the existence of a characteristic radial scale $r_\ell \equiv \frac{|\ell|}{b\zeta}$ and the following important properties

- *Localized absence* for radial distances $r \lesssim r_\ell \equiv \frac{|\ell|}{b\zeta}$: a quantum particle with $j = \{\kappa, \ell, \zeta\}$ is not very likely to be found in the cylindrical region $r \lesssim r_\ell$. Note that, keeping ζ momentum fixed, we can enlarge the region of localized absence by increasing the angular momentum label $|\ell|$.
- ℓ *- independence* for large r : Two modes differing only in their angular momentum numbers tend to coincide in the average for sufficiently large distances.

Note that r_ℓ is the classical radius of minimum approach from the cosmic string for a particle with angular momentum $\frac{\ell}{b}$ and $x - y$ momentum measure ζ.

3. Three particle interactions at tree level

Having in mind the QED interaction that involves at tree-level three-particle processes, we will consider here for simplicity a structurally similar scalar interaction

$$\mathcal{L}_I = -\lambda \phi \psi^2, \tag{3.1}$$

where ϕ is a massless and ψ a massive scalar field with mass M. The coupling constant λ has units of mass and corresponds to the tree-level QED quantity $2Mq$, where q is the electric charge.

The transition amplitude from a one particle state, with quantum numbers $j = \{\kappa, \ell, \zeta\}$ and energy E, to a two particle one, with $j_1 = \{\kappa_1, \ell_1, \zeta_1\}$, $j_2 = \{\kappa_2, \ell_2, \zeta_2\}$ and energies E_1, E_2, is

$$P \propto \delta(E - E_1 - E_2)\delta(\kappa - \kappa_1 - \kappa_2)\delta_{\ell, \ell_1 + \ell_2} P_1$$

$$P_1 := \int_0^\infty d\tau \; \tau J_{\frac{|\ell|}{b}}(\zeta\tau) J_{\frac{|\ell_1|}{b}}(\zeta_1\tau) J_{\frac{|\ell_2|}{b}}(\zeta_2\tau), \tag{3.2}$$

Here, besides the easily recognized terms expressing energy, z momentum and z angular momentum conservation, we have also the P_1 term, the importance of which is not immediately evident. It turns out that we can analyze the behavior of P_1 when $|\ell| = |\ell_1 \pm \ell_2|$, a condition that is satisfied in our case because of z-angular momentum conservation. We find that P_1 has different angular momentum dependence, depending on whether the $x - y$ momentum measures ζ, ζ_1, ζ_2 satisfy or not the triangle inequalities. Specifically we have for the functional form of P_1

(i) $|\zeta_1 - \zeta_2| < \zeta < \zeta_1 + \zeta_2$ $\qquad P_1 \propto \cos(f_1|\ell_1| + f_2|\ell_2|)$

(ii) $\zeta > \zeta_1 + \zeta_2$ $\qquad P_1 \propto \exp(-g_1|\ell_1| - g_2|\ell_2|)$

(iii) $\zeta < |\zeta_1 - \zeta_2|$ $\qquad P_1 \propto \exp(-h_1|\ell_1| - h_2|\ell_2|)$.

The arguments of cos and exp are linear in $(\ell), \ell_1, \ell_2$ with coefficients $f_1, f_2, g_1, g_2, h_1, h_2$ that are some functions of ζ, ζ_1, ζ_2. In particular g_1, g_2, h_1, h_2 are non negative.

Combining now this behavior of P_1 with the localized absence property of the cylindrical modes we conclude that:

- Type (i) processes can take place with significant probability all over the space since the transition probability amplitude is not sensitive to whether the involved angular momentum numbers, and correspondingly the regions of localized absence, are small or large.

- On the other hand, processes of type (ii) and (iii) are more likely to take place for smaller angular momenta and therefore within a finite distance from the cosmic-string.

It is interesting to remark that momentum conserving processes are necessarily of type (i). This happens because for such processes the momentum vectors in the $x-y$ plane should fit and form a triangle and consequently the corresponding measures should satisfy the triangle inequalities.

4. Local behavior of pair creation and bremsstrahlung

Now we are ready to discuss the local behavior of the pair creation and bremsstrahlung processes :

$$(\mathrm{PC}): \quad |1_j^\phi\rangle \rightarrow |1_{j_1}^\psi 1_{j_2}^\psi\rangle \qquad \text{and} \qquad (\mathrm{BS}): \quad |1_j^\psi\rangle \rightarrow |1_{j_1}^\psi 1_{j_2}^\phi\rangle$$

The parameter space of the quantum numbers ζ, ζ_1, ζ_2 is restricted for kinematical reasons namely the demand of energy and z-momentum conservation. It is not very difficult to see that for PC there exists a mass theshold at $\zeta = 2M$ and that both PC and BS processes are restricted to be of type (ii) in the classification of the previous section. However, for BS we find processes that are arbitrarily close to the process with $\zeta_1 = \zeta, \zeta_2 = 0$ which lies on the border line between type (i) and type (ii, iii) processes. Having in mind the different local behavior of type (i) and (ii) processes it is now very interesting to compare PC and BS.

In momentum space we easily find a difference. Summing over the final states we obtain the total probabilities $w(\kappa, \ell, \zeta)$ and it turns out that for PC the $w_{\mathrm{PC}}(\kappa, \ell, \zeta)=$ finite while for BS the $w_{\mathrm{BS}}(\kappa, \ell, \zeta)=$ infinite.

To understand this result we may pass over to the configuration space using the basic properties of the cylindrical modes. The quantity

$$\delta w_\ell := w(\kappa, \ell, \zeta) - w(\kappa, \ell + 1, \zeta). \tag{4.1}$$

can be roughly interpreted as the average probability for an initial particle with quantum numbers $j = \{\kappa, \ell, \zeta\}$ to make the corresponding transition within the cylindrical ring $r \in (r_\ell, r_{\ell+1})$.

For PC we find

$$\delta w_\ell \propto \exp(-\alpha r_\ell M) \tag{4.2}$$

where α is some function of ζ of the order of unity. The behavior (4.2) implies that the PC process is well localized within a Compton wavelength $r \lesssim M^{-1}$ from the cosmic-string. Thus the total probability, and of course the respective cross section, are finite and proportional to the Compton wavelength of the created particles.

On the other hand, the BS transition probability in a cylindrical ring of radius r_ℓ is

$$\delta w_\ell \propto 1/r_\ell \quad \text{for} \quad r_\ell \gg \frac{1}{M} \tag{4.3}$$

which has a slow falloff rate giving finally a logarithmically divergent total probability when the contribution of all the rings is taken into account. This infinity is one side of the well known *infrared catastrophe* which is typically expected to accompany bremsstrahlung processes. The other side, the more familiar one, is most easily seen by looking at the differential transition probability $dw_{\mathrm{BS}}/d\omega$, where ω is the photon's energy. It has the characteristic bremsstrahlung low frequency spectrum

$$\frac{dw_{\mathrm{BS}}}{d\omega} \propto \frac{1}{\omega}. \qquad \omega \to 0 \tag{4.4}$$

which gives again a logarithmic divergence in the total probability.

Concluding we arrive at the following intuitive physical picture for the local behavior of PC and BS processes. Each individual process has an intrinsic quantum extension which is of the order of the wavelength (Compton wavelength) of the massless (massive) product particles. The probability for this process is significant only if it takes place at a point from where it can reach the cosmic string with its extension. Thus the extension of a PC process is approximately equall to Compton wavelength M^{-1} of the created particles and therefore it can take place with significant probability only very close to the cosmic string at distances $r \lesssim M^{-1}$. On the other hand, the extension of a bremsstrahlung process can be arbitrarily large since the emitted massless particles have available arbitrarily large wavelengths. Thus, with 'softer' and 'softer' photons the cosmic string can be sensed from any distance. As explained above, the detailed calculation for the local BS transition probability at large distances shows a $1/r$ radial dependence, a falloff rate which finally leads to the infrared catastrophe.

5. Quantitative results

Here we would like to quote the results for the pair creation cross section and the amount of emitted bremsstrahlung radiation. As already mentioned above, calculational details can be found in Refs. 11,12,13. See also Ref. 14 where the PC process was treated for the first time.

5.1. Pair creation cross section

For an ingoing plane wave massless state with momentum $\vec{p} = \{p_x, p_y, p_z = \kappa\}$, the PC cross section per unit z-length is

$$\sigma_{\rm PC}(\vec{p}) = \begin{cases} \frac{3\pi(1-b)^2}{17920} q^2 \left(\frac{\zeta}{M}\right)^2 \frac{1}{M}, & \text{for } 1 \ll \frac{\zeta}{2M} \ll \frac{1}{\pi(1-b)} \\ \frac{1}{192\pi} q^2 \frac{1}{M}, & \text{for } \frac{\zeta}{2M} \gg \frac{1}{\pi(1-b)} \end{cases} \tag{5.1}$$

where $\zeta^2 = p_x^2 + p_y^2$. Also by q we denote the quantity $\lambda/2M$ which corresponds, as we mentioned in Sec. 3, to the charge of the created particles.

From Eq.(5.1) we see that the σ_{PC} is proportional to the Compton wavelength $1/M$ of the created particles. Furtheremore, for energies well above the mass threshold $\zeta \gg 2M$, the cross section σ_{PC} increases as ζ^2 and is proportional to the small quantity $(1-b)^2$. This behavior is however valid up to the characteristic energy scale $\zeta \approx 2M/(\pi(1-b))$, after which the $\sigma_{\rm PC}(\vec{p})$ looses the dependence on ζ and interestingly enough also the dependence on the cosmic string strength parameter $(1-b)$.

5.2. Bremsstrahlung energy loss

For a massive particle state with quantum label $j = \{\kappa, \ell, \zeta\}$ and mass M, the total emitted bremsstrahlung energy per unit time and per unit z length is

$$\mathcal{E}_{\rm rad}(\kappa, \ell, \zeta) = \frac{1}{TL} \int_0^\infty \omega \frac{dw_{\rm BS}(\kappa, \ell, \zeta)}{d\omega} d\omega \tag{5.2}$$

where T, L are usual time and z length normalization intervals. Note that $\mathcal{E}_{\rm rad}$ is finite even though the total amount of emitted soft photons is infinite, a consequence of the particular $1/\omega$ low energy behavior of bremsstrahlung spectrum in Eq.(4.4).

Thanks to the Lorentz boost invariance in the z direction we may restrict ourselves, without loss of generality, to the computation of $\mathcal{E}_{\rm rad}$ in a frame where $\kappa = 0$. We find that

For $r_\ell = |\ell|/(b\zeta) \gg M^{-1}$

$$\mathcal{E}_{\rm rad}(0, \ell, \zeta) = \begin{cases} \frac{q^2(1-b)^2}{12 r_\ell} \mathcal{V}^3, & \text{if } \mathcal{V} \ll 1 \\ \frac{3\pi q^2 (1-b)^2}{128 r_\ell} \gamma^3, & \text{if } \mathcal{V} \to 1, \quad \gamma \ll [\pi(1-b)]^{-1} \\ \frac{q^2}{32\pi r_\ell} \gamma, & \text{if } \gamma \gg [\pi(1-b)]^{-1} \end{cases} \tag{5.3}$$

where $\mathcal{V} = \zeta/(\zeta^2 + M^2)^{1/2}$ is the velocity of the incoming charged particle and γ is the corresponding Lorentz factor $\gamma = (1 - \mathcal{V}^2)^{-1/2}$.

For $r_\ell \ll M^{-1}$

$$\mathcal{E}_{\rm rad}(0,\ell,\zeta) = \begin{cases} \frac{q^2(1-b)^2 M}{12}\gamma^3, & \text{if } \mathcal{V} \to 1, \quad \gamma \ll [\pi(1-b)]^{-1} \\ \frac{q^2}{4\pi^2} M\gamma, & \text{if } \gamma \gg [\pi(1-b)]^{-1} \end{cases} \tag{5.4}$$

Note here the dependence of $\mathcal{E}_{\rm rad}$ on the velocity of a relativistic charged particle and the cosmic-string strength parameter $(1-b)$. $\mathcal{E}_{\rm rad}$ is proportional to $(1-b)^2$ and increases as the third power of the respective Lorentz factor as long $\gamma \ll 1/(\pi(1-b))$. For larger velocities $\mathcal{E}_{\rm rad}$ scales only as $\propto \gamma$ and is independent of $(1-b)$. This reminds the energy behavior of the PC cross section.

In Eqs.(5.3,5.4) we see that the emitted BS energy has an $1/r_\ell$ dependence when the charged particle remains far away from the cosmic-string, at distances r_ℓ quite larger than its Compton wavelength M^{-1}. Notice that, according to the discussion of the previous section, the BS process takes place at such distances thanks to the emitted massless particle, which can sense the cosmic string from arbitrarily large distances. However, as the particle comes closer at distances comparable to M^{-1}, the cosmic-string can be directly sensed by the massive particle. This is reflected in the expressions for $\mathcal{E}_{\rm rad}$ as a smooth transition from the $1/r_\ell$ dependence to an r_ℓ independence.

Closing, let us remark that there exists a *classical analogue* to the quantum BS process just considered. It is the radiation from a classical charged particle, with charge q, freely moving in the cosmic-string spacetime, starting from radial infinity and scattering off back to radial infinity, having $x-y$ velocity measure ζ and radius of minimum approach $r_{min} = r_\ell$. This classical problem has been considered in Refs. 15, 16 for scalar, vector and tensorial moving charges. The respective classical expressions for the total energy emitted during this motion are, in the scalar case, exactly those given in Eq.(5.3) above. Consequently, the classical treatment is adequate provided that the charged particle remains at distances from the cosmic-string that are quite larger than its Compton wavelength. For smaller distances one must treat the process quantum field theoretically.

6. Acknowlegment

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Creation of Mutually Interacting Particles in Anisotropically Expanding Universes

Karl-Heinz Lotze
Friedrich-Schiller-Universität Jena
Theoretisch-Physikalisches Institut
Max-Wien-Platz 1
O-6900 Jena, Germany

Abstract

This paper is aimed at developing a general framework for the calculation of the creation of pions and photons mutually interacting via the $\pi^0 - 2\gamma$ model interaction in anisotropically expanding universes of Bianchi type I. We give explicit results at least for the interaction - free contributions to particle creation and particular expansion laws.

1 Introduction: Conformal invariance properties of field theories and particle creation effects in isotropic cosmology

It is well known that gravitational fields are capable of producing particles out of the vacuum. This occurs already for particles which are free of any non-gravitational interaction with other particles or themselves (see e. g. Parker 1971). Thus the question naturally arises how self - or mutual interactions change the numbers of created particles in particular in expanding universes (Birrell and Ford 1979, Birrell et al. 1980).

One typical result is that because of their interaction with other particles even conformal ones like photons can be produced by conformally flat expanding universes (Lotze 1985) while they cannot be created as free particles.

Another one is a spin-statistics effect: In modes with spontaneous creation the interaction gives rise to an amplification (attenuation) of the creation of bosons (fermions) (Audretsch and Spangehl 1985, Lotze 1985).

If we turn to anisotropically expanding universes, we have in general three different mechanisms of conformal symmetry breaking: (i) by at least one particle species being non-conformal, (ii) by the interaction, and (iii) by the anisotropy of the background. It is the goal of this paper to investigate the interplay among, and the relative importance of, these three effects. To be more specific, we investigate the $\pi^0 - 2\gamma$ interaction between pions and photons (Birrell et al. 1980) in anisotropically expanding universes of Bianchi type I.

Before doing so, we compare four particular field theories in order to study the influence of conformal symmetry breaking on particle creation effects these theories give rise to. For that purpose we consider these field theories in the background of a conformally flat universe with isotrpic expansion and the line element [1]

$$ds^2 = C^2(\eta)(dx^2 + dy^2 + dz^2 - d\eta^2). \tag{1.1}$$

Our *first example* is a self-interacting scalar field with the Lagrangian

$$L = \sqrt{-g}\left(\frac{1}{2}\left[g^{\mu\nu}\tilde{\Phi}_{,\mu}\tilde{\Phi}_{,\nu} + (m^2 + \xi R)\tilde{\Phi}^2\right] + \lambda_0\tilde{\Phi}^4\right) , \tag{1.2}$$

ξ and λ_0 being dimensionless constants. Performing the conformal transformation

$$g_{\mu\nu} = C^2(\eta)\,\eta_{\mu\nu} \tag{1.3}$$

of the metric (1.1) onto the Minkowski-space metric $\eta_{\mu\nu}$ together with

$$\tilde{\Phi} = \frac{1}{C}\Phi \tag{1.4}$$

[1] Notations and conventions: $c = \hbar = 1$, metric signature $(+ + + -)$; Greek indices run from 1 to 4, Latin ones from 1 to 3. An overdot denotes the derivative with respect to the conformal time parameter η. In the figures a solid line represents a massive and/or non-conformally coupled scalar particle ((KG): Klein-Gordon) while a broken line denotes a conformal ($m = 0$ and $\xi = 1/6$) scalar particle. A photon ((M): Maxwell) is characterized by a wavy line and electrons and positrons ((D): Dirac) by solid lines with an arrow.

we end up with

$$L = \frac{1}{2}\left[\eta^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} + C^2m^2\Phi^2 + (6\xi - 1)\frac{\ddot{C}}{C}\Phi^2\right] + \lambda_0\Phi^4 .$$

Obviously, the conformal invariance is broken by L_{free}, that part of the Lagrangian which is free of the non-gravitational interaction. As is well known, this gives rise to free creation of pairs of scalar particles if only they are not conformal ones ($m \neq 0$ and/or $\xi \neq 1/6$; fig. 1a).
Furthermore, because of the self-interaction, pairs and quartets of interacting particles may come out of the background (fig. 1.b; Birrell and Ford 1979).

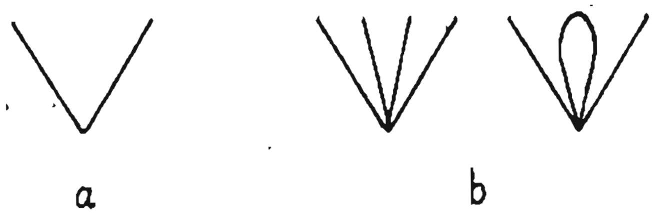

Figure 1: Creation of non-conformal pions as free particles (a) and due to their self-interaction (b). The corresponding processes for conformal particles are not allowed.

As a *second example* we consider the mutual interaction between Fermi and Bose fields according to quantum electrodynamics with the Lagrangian

$$L = \sqrt{-g}\left[-\frac{1}{4}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} - \frac{1}{2}\left(\bar{\tilde{\psi}}\tilde{\gamma}^{\mu}\tilde{\psi}_{;\mu} - \bar{\tilde{\psi}}_{;\mu}\tilde{\gamma}^{\mu}\tilde{\psi} + 2m\bar{\tilde{\psi}}\tilde{\psi}\right) + ie_0\bar{\tilde{\psi}}\tilde{\gamma}^{\mu}\tilde{A}_{\mu}\tilde{\psi}\right] . \tag{1.5}$$

Because of (1.3) the Dirac matrices are to be transformed according to

$$\tilde{\gamma}^{\mu} = \frac{1}{C}\gamma^{\mu}.$$

If we transform the bispinor field as

$$\tilde{\psi} = \frac{1}{C^{\frac{3}{2}}}\psi$$

and the electromagnetic field quantities as

$$\tilde{A}_\mu = A_\mu$$

and consequently

$$\tilde{F}_{\mu\nu} = F_{\mu\nu} \tag{1.6}$$

we get from (1.5) the transformad Lagrangian

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\left(\overline{\psi}\gamma^\mu\psi_{,\mu} - \overline{\psi}_{,\mu}\gamma^\mu\psi + 2mC\overline{\psi}\psi\right) + ie_0\overline{\psi}\gamma^\mu A_\mu\psi.$$

Since the coupling constant $e_0^2 \approx \frac{4\pi}{137}$ is dimensionless conformal invariance is violated by L_{free} only, as in the case of the previous example.
Consequently, we expect that not only free electron-positron pairs can be created from vacuum (fig. 2a) but also photons together with them (fig. 2b; Lotze 1985, 1992).

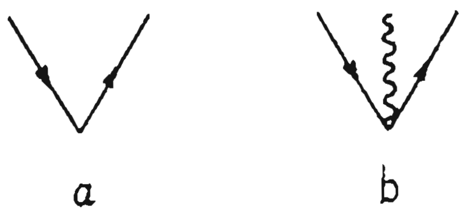

Figure 2: Creation of free electron-positron pairs (a) and simultaneous creation of electron-positron pairs together with photons (b) due to their interaction.

The *third example* is the self-interacting electromagnetic field with the Lagrangian

$$L = \sqrt{-g}\left(-\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + b_1(\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu})^2 + b_2({}^*\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu})^2\right). \tag{1.7}$$

Applying again the conformal transformation (1.3) and transforming the electromagnetic field strength tensor according to (1.6) leads finally to

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{C^4}\left[b_1(F_{\mu\nu}F^{\mu\nu})^2 + b_2({}^*F_{\mu\nu}F^{\mu\nu})^2\right].$$

In this case the conformal invariance is broken by the interacting part of the Lagrangian, L_{int}, only. On one hand, this reproduces the well-known result that photons cannot be produced from vacuum as free particles. On the other hand, a quartet creation of interacting photons is allowed due to the self-interaction (fig. 3; Birrell and Ford 1979).

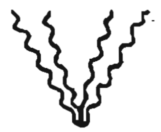

Figure 3: Creation of photon quartets (even in conformally flat space-times!) due to their self-interaction.

Our *fourth example* is the $\pi^0 - 2\gamma$ model interaction between scalar and electromagnetic fields (Schwinger 1951, Birrell et al. 1980) the Lagrangian of which is

$$
\begin{aligned}
L &= \sqrt{-g}\Big(-\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}\left[g^{\mu\nu}\tilde{\Phi}_{,\mu}\tilde{\Phi}_{,\nu} + (m^2+\xi R)\tilde{\Phi}^2\right] \\
&\quad + \beta_0\, {}^*\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\tilde{\Phi}\Big).
\end{aligned} \tag{1.8}
$$

The coupling constant β_0 has the dimension of length. Performing the transformations (1.3), (1.4) and (1.6) we get

$$
\begin{aligned}
L &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[\eta^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} + C^2m^2\Phi^2 + (6\xi-1)\frac{\ddot{C}}{C}\Phi^2\right] \\
&\quad + \frac{1}{C}\beta_0\, {}^*F_{\mu\nu}F^{\mu\nu}\Phi.
\end{aligned}
$$

Now the conformal invariance is violated by both L_{free} and L_{int}. As a consequence, photons may simultaneously be created not only together with non-conformal scalar particles (fig. 4a) but even together with conformal ones ($m = 0$ and $\xi = 1/6$; fig. 4b).

a b

Figure 4: Simultaneous creation of photon pairs and non-conformal (a) or conformal (b) pions as a consequence of their $\pi^0 - 2\gamma$ interaction.

2 The $\pi^0 - 2\gamma$ interaction in Bianchi type I universes

We want to establish a general scheme for the calculation of the creation of photons and and pions interacting according to the $\pi^0 - 2\gamma$ model (1.8) in a spatially flat spacetime

$$ds^2 = C_1{}^2(\eta)dx^2 + C_2{}^2(\eta)dy^2 + C_3{}^2(\eta)dz^2 - C^2(\eta)d\eta^2 \qquad (2.1)$$

with anisotropic expansion.

In this situation we have two mechanisms of conformal symmetry breaking. Conformal symmetry may be violated by

(i) $L^{(KG)}_{\text{free}}$ if $m \neq 0$ and/or $\xi \neq 1/6$ and L_{int} since β_0 has the dimension of length ($\beta_0 = 1.21 \cdot 10^{-16} cm$),

(ii) the anisotropy of the expansion.

Therefore, in general we expect not only the processes of fig. 5 to be allowed but also those shown in fig. 6.

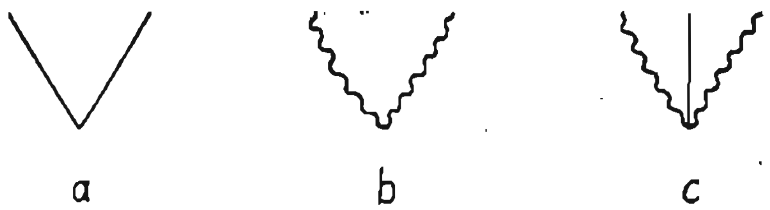

Figure 5: Creation of non-conformal pions (a) and photons ((b), only for anisotropic expansion) as free particles and their simultaneous creation (c) due to the $\pi^0 - 2\gamma$ interaction.

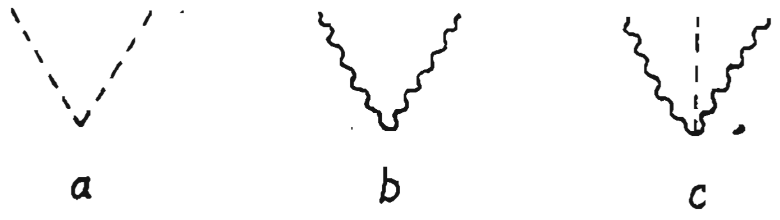

Figure 6: The same processes as in fig. 5 but with conformal pions. The processes (a) and (b) are allowed for anisotropic expansion only.

The Klein-Gordon equation

The solutions of the Klein-Gordon equation

$$g^{\mu\nu}\,\tilde{\Phi}_{,\mu;\nu} + (m^2 + \xi R)\,\tilde{\Phi} = 0$$

may be written as

$$\tilde{\Phi} = \frac{1}{(2\pi)^{3/2}\,C}\int d^3\vec{p}\;\left[b(\vec{p})\,\chi(\vec{p};\eta)\,e^{i\vec{p}\vec{r}} + \text{h.c.}\right] \tag{2.2}$$

where $\chi(\vec{p};\eta)$ is a solution of the equation

$$\ddot{\chi} + \Delta\dot{\chi} + \left[(\Omega_p{}^2 + m^2 + \xi R)\,C^2 + \Delta\frac{\dot{C}}{C} - \frac{\ddot{C}}{C}\right]\chi = 0. \tag{2.3}$$

Here we introduced

$$\Omega_p{}^2 = \sum_i \left(\frac{p_i}{C_i}\right)^2, \tag{2.4}$$

the "damping parameter"

$$\Delta = \frac{1}{2}\frac{\dot{g}}{g} - 4\frac{\dot{C}}{C} = \sum_i \frac{\dot{C}_i}{C_i} - 3\frac{\dot{C}}{C} \tag{2.5}$$

and the curvature scalar

$$R = \frac{2}{C^2}\sum_k \frac{1}{C_k}\left(\ddot{C}_k - \frac{\dot{C}}{C}\dot{C}_k\right) + \frac{1}{C^2}\sum_k \frac{\dot{C}_k}{C_k}\sum_{i\neq k}\frac{\dot{C}_i}{C_i} \,. \tag{2.6}$$

The Maxwell equations

The Maxwell equations

$$\tilde{F}^{\mu\nu}_{;\nu} = 0 \qquad \text{and} \qquad {}^{*}\tilde{F}^{\mu\nu}_{;\nu} = 0$$

are non-trivial in the metric (2.1) which is no longer conformally flat. In terms of certain time dependent, orthogonal unit vectors $\vec{e}_\Theta$ and $\vec{e}_\Phi$ they may be solved for the electric and magnetic fields $\vec{E}$ and $\vec{B}$ in the rather simple-looking form (Sagnotti and Zwiebach 1981, Lotze 1990)

$$\begin{aligned}
\vec{E} &= (2\pi)^{-3/2}\int \mathrm{d}^3\vec{k}\sum_\sigma \left[a(\vec{k},\sigma)\,\vec{f}^{(\sigma)}(\vec{k},\eta)\,e^{i\vec{k}\vec{r}} + \text{h.c.}\right] \\
\text{and}\quad \vec{B} &= -i(2\pi)^{-3/2}\int \mathrm{d}^3\vec{k}\sum_\sigma \sigma\left[a(\vec{k},\sigma)\,\vec{f}^{(\sigma)}(\vec{k},\eta)\,e^{i\vec{k}\vec{r}} + \text{h.c.}\right] \\
\text{where}\quad \vec{f}^{(\sigma)}(\vec{k},\eta) &= \frac{N^{(\sigma)}}{2k\beta^{1/2}(-g)^{1/4}}\left[\Omega_k C F^{(\sigma)}(\eta)\vec{e}_\Theta - \sigma\dot{F}^{(\sigma)}(\eta)\vec{e}_\Phi\right] .
\end{aligned}$$

Now the pendant to (2.3) reads

$$\ddot{F}^{(\sigma)} - \frac{\dot{\beta}}{\beta}\dot{F}^{(\sigma)}\left[C^2\Omega_k{}^2 + \sigma k\beta\left(\frac{\alpha}{\beta}\right)^{\bullet}\right] F^{(\sigma)} = 0 \tag{2.7}$$

where Ω_k is defined by (2.4) with k instead of p, and α and β are the "anisotropy parameters" ($k_\perp{}^2 = k_1{}^2 + k_2{}^2$)

$$\alpha = \frac{C^2}{\sqrt{-g}}\,\frac{k_1k_2k_3}{k\,k_\perp{}^2}\,(C_2{}^2 - C_1{}^2) \tag{2.8}$$

$$\text{and}\quad \beta = \frac{C^2}{\sqrt{-g}}\,\frac{1}{k_\perp{}^2}\,(C_1{}^2k_2{}^2 + C_2{}^2k_1{}^2) \,. \tag{2.9}$$

The connection between the time dependent unit vectors $\vec{e}_\Theta$ and $\vec{e}_\Phi$ and time-independent cartesian basis vectors $\vec{e}_{(i)}$ is the same as for spherical coordinates but with time dependent coefficients,

$$\begin{aligned}\vec{e}_\Theta &= \frac{k}{\Omega_k}\frac{C}{\beta^{1/2}(-g)^{1/4}}\left[\frac{C_2}{C_3}\cos\vartheta\cos\varphi\,\vec{e}_{(1)} + \frac{C_1}{C_3}\cos\vartheta\sin\varphi\,\vec{e}_{(2)}\right. \\ &\quad \left. - \frac{\beta C_3}{C}\sin\vartheta\,\vec{e}_{(3)}\right] , \\ \vec{e}_\Phi &= \frac{C}{\beta^{1/2}(-g)^{1/4}}\left[-C_1\sin\varphi\,\vec{e}_{(1)} + C_2\cos\varphi\,\vec{e}_{(2)}\right] .\end{aligned}$$

The main advantage of this method of solving Maxwell's equations is that one gets the polarization degrees of freedom decoupled.

Axisymmetric expansion laws

In what follows we restrict ourselves to axisymmetric expansions ($C_1 = C_2$) such that the anisotropy parameters (2.8) and (2.9) substantially simplify to $\alpha = 0$ and $\beta = C/C_3$. As a consequence, $F^{(\sigma)}$ of (2.7) becomes independent of the polarization σ.

With this restriction in mind we consider four particular expansion laws (ν: number of expanding directions):

(i) expansion in one direction ($\nu = 1$)

$$C_1 = C_2 = 1 \ , \quad C_3 = C(\eta) \ , \tag{2.10}$$

(ii) expansion in two directions ($\nu = 2$)

$$C_1 = C_2 = C(\eta) \ , \quad C_3 = 1 \ , \tag{2.11}$$

(iii) isotropic expansion ($\nu = 3$) as a limiting case

$$C_1 = C_2 = C_3 = C(\eta) \ , \tag{2.12}$$

(iv) weak anisotropy

$$C_i^2(\eta) = C^2(\eta)\,[1 + h_i(\eta)] \ , \quad |h_i(\eta)| \ll 1 \tag{2.13}$$

where $h_1 = h_2 = h(\eta)$

and, for convenience, $\sum_i h_i = 0$ (Birrell and Davies 1980). (2.14)

The first two examples may be looked at as originally isotropic expansions where one resp. two directions are kept fixed. For these and the isotropic expansion the parameters (2.4), (2.5) and (2.6) of the Klein-Gordon equation can be given in terms of the number ν of the expanding directions,

$$C^2{\Omega_p}^2 = \nu_\perp {p_\perp}^2 + \nu_3 {p_3}^2 \,, \tag{2.15}$$

$$\Delta = (\nu - 3)\,\frac{\dot{C}}{C} \tag{2.16}$$

$$\text{and} \quad R = 2\nu\,\frac{\ddot{C}}{C^3} + \nu(\nu-3)\,\frac{\dot{C}^2}{C^4}$$

$$\text{where} \quad \nu_\perp = \frac{1}{2}(4-\nu)(\nu-1) + \frac{1}{2}(\nu-3)(\nu-2)C^2$$

$$\text{and} \quad \nu_3 = (\nu-2)^2 + (3-\nu)(\nu-1)C^2 \,.$$

In the isotropic case we get in particular $\nu_\perp = \nu_3 = 1, \Delta = 0$ and the well known $R = 6\ddot{C}/C^3$.

Equ. (2.7) which is what essentially remains from the Maxwell equations simply reads

$$\ddot{F} + (\nu - 1)\,\Delta\dot{F} + C^2{\Omega_k}^2 F = 0$$

with Ω_k and Δ given by (2.15) resp. (2.16) with the only modification that $p_\perp$ and p_3 are to be replaced by $k_\perp$ and k_3.
For weak anisotropy (2.13), (2.14) the equations (2.3) and (2.7) may be solved by perturbative methods.

3 A general scheme for the calculation of particle creation

Particle number desities

In order to calculate the number density of created particles we again choose the in state to be the vacuum in the remote past,

$$|\,0,\mathrm{in}\,\rangle = |\,0^{(KG)},\mathrm{in}\,\rangle\;|\,0^{(M)},\mathrm{in}\,\rangle \;,$$

where we have to distinguish between in in and out states for the eletromagnetic field as well. With this in state the particle number density is

$$n = \frac{\langle \text{in}, 0 \mid S^{+} N_{\text{out}} S \mid 0, \text{in} \rangle}{(2\pi)^3 C_1 C_2 C_3} \; . \tag{3.1}$$

Number operator and S matrix are

$$N_{\text{out}} = N_{\text{out}}^{(KG)} + N_{\text{out}}^{(M)}$$

and $$S = T \exp\left\{ 4i\beta_0 \int \mathrm{d}^4 x \, \sqrt{-g} (\tilde{\vec{E}} \tilde{\vec{B}}) \tilde{\Phi} \right\} = 1 + \beta_0 S^{(1)} + \cdots \; .$$

Inserting these into (3.1) yields

$$n = n_{00}^{(KG)} + n_{00}^{(M)} + \left(n_{11}^{(KG)} + n_{11}^{(M)} \right) + \cdots \; .$$

The notation is chosen in such a way that the sum of the subscripts at each contribution to n indicates the power of the coupling constant this term is proportional to.

The pure background effects

The number density of free pion pairs created out of the vacuum is (figs. 5a, 6a)

$$\begin{aligned} n_{00}^{(KG)} &= \frac{1}{(2\pi)^3 C_1 C_2 C_3} \int \mathrm{d}^3 \vec{p} \, N^{(KG)}(\vec{p}) \; , \\ \text{where} \quad N^{(KG)}(\vec{p}) &= |\beta(-\vec{p})|^2 \end{aligned} \tag{3.2}$$

denotes the mode dependent number of out pions in the in vacuum. The Bogoljubov coefficient turns out to be

$$\beta(\vec{p}) = i \frac{\sqrt{-g}}{C^4} N_{\text{in}} N_{\text{out}} (\chi_{\text{out}} \dot{\chi}_{\text{in}} - \chi_{\text{in}} \dot{\chi}_{\text{out}}) \; .$$

The corrsponding quantities for the creation of free photons (figs. 5b, 6b) are

$$\begin{aligned} n_{00}^{(M)} &= \frac{1}{(2\pi)^3 C_1 C_2 C_3} \int \mathrm{d}^3 \vec{k} \, N^{(M)}(\vec{k}) \; , \\ N^{(M)}(\vec{k}) &= \sum_{\sigma} N^{(M)}(\vec{k}, \sigma) = \sum_{\sigma} |\beta^{(\sigma)}(-\vec{k})|^2 \\ \text{and} \quad \beta^{(\sigma)}(\vec{k}) &= \frac{i}{2\beta k^2} N_{\text{in}}^{(\sigma)} N_{\text{out}}^{(\sigma)} \left[F_{\text{out}}^{(\sigma)} \dot{F}_{\text{in}}^{(\sigma)} - F_{\text{in}}^{(\sigma)} \dot{F}_{\text{out}}^{(\sigma)} \right] \; . \end{aligned} \tag{3.3}$$

Simultaneous creation of pions and photons: The amplification effect

Now we turn to the simultaneous creation of pions and photons due to their mutual interaction (figs. 5c, 6c).

The number density of pions is

$$n_{11}^{(KG)} = \frac{1}{(2\pi)^3 C_1 C_2 C_3} \int d^3\vec{p}\, d^3\vec{k}\, d^3\vec{k}'\, \delta(\vec{k} + \vec{k}' + \vec{p}) \times \left[1 + 2N^{(KG)}(\vec{p})\right] w(\vec{k}, \vec{k}', \vec{p}) \tag{3.4}$$

and that of photons

$$n_{11}^{(M)} = \frac{1}{(2\pi)^3 C_1 C_2 C_3} \int d^3\vec{p}\, d^3\vec{k}\, d^3\vec{k}'\, \delta(\vec{k} + \vec{k}' + \vec{p}) \times \left\{\left[1 + \frac{1}{2} N^{(M)}(\vec{k})\right] + \left[1 + \frac{1}{2} N^{(M)}(\vec{k}')\right]\right\} w(\vec{k}, \vec{k}', \vec{p}) \,. \tag{3.5}$$

A first glance at these results reveals that the spontaneous creation of pions and photons amplifies their creation stimulated by interaction. This is because the commuting character of the Bose operators of the Klein-Gordon and Maxwell fields gives rise to the plus signs in (3.4) and (3.5). Therefore a simple counting of the lines in figs. 5c, 6c does not yield the relative number of photon pairs per one pion.

Moreover, because of the non-vanishing background effect for photons in anisotropic space-times even the photons are no good indicators of the non-gravitational interaction. None of the two particle species is a good indicator because any detected particle could have been produced freely and by mutual interaction as well.

The particle creation probability occurring in (3.4) and (3.5) is ($k_1 = k_\perp \cos\varphi$, $k_2 = k_\perp \sin\varphi$)

$$
\begin{aligned}
w(\vec{k},\vec{k}',\vec{p}) &= \beta_0{}^2 \frac{|N_{\rm in}(\vec{k})|^2\,|N_{\rm in}(\vec{k}')|^2\,|N_{\rm in}(\vec{p})|^2}{2\pi^3(kk')^2} \left\{ |J_1|^2 \cos^2(\varphi-\varphi') \right. \\
&+ \left| k_3 k_3' \cos(\varphi-\varphi') J_2{}^{(1)} + k_\perp k_\perp' J_2{}^{(2)} \right|^2 \\
&+ 2\cos(\varphi-\varphi) \,\mathrm{Re}\left[J_1{}^* \left(k_3 k_3' \cos(\varphi-\varphi') J_2{}^{(1)} + k_\perp k_\perp' J_2{}^{(2)} \right) \right] \\
&+ \left. \sin^2(\varphi-\varphi') |k_3 J_3 - k_3' J_4|^2 \right\} .
\end{aligned}
$$

The time integrals read

$$
\begin{aligned}
J_1 &= \int_{-\infty}^{\infty} d\eta\, \frac{C_3}{C^2}\, \chi_{\rm in}(\vec{p},\eta)\, \dot{F}_{\rm in}(\vec{k},\eta)\, \dot{F}_{\rm in}(\vec{k}',\eta) , \\
J_2{}^{(1)} &= \int_{-\infty}^{\infty} d\eta\, \frac{1}{C_3}\, \chi_{\rm in}(\vec{p},\eta)\, F_{\rm in}(\vec{k},\eta)\, F_{\rm in}(\vec{k}',\eta) , \\
J_2{}^{(2)} &= \int_{-\infty}^{\infty} d\eta\, \frac{C_3}{C_1{}^2}\, \chi_{\rm in}(\vec{p},\eta)\, F_{\rm in}(\vec{k},\eta)\, F_{\rm in}(\vec{k}',\eta) , \\
J_3 &= \int_{-\infty}^{\infty} d\eta\, \frac{1}{C}\, \chi_{\rm in}(\vec{p},\eta)\, F_{\rm in}(\vec{k},\eta)\, \dot{F}_{\rm in}(\vec{k}',\eta) , \\
J_4 &= \int_{-\infty}^{\infty} d\eta\, \frac{1}{C}\, \chi_{\rm in}(\vec{p},\eta)\, \dot{F}_{\rm in}(\vec{k},\eta)\, F_{\rm in}(\vec{k}',\eta) .
\end{aligned}
$$

There are no plane-wave exponentials in the integrands because there are no conformal particles. Moreover, certain powers of the scale factors $C_1 = C_2, C_3$ and C appear explicitly. In the isotropic limit they all reduce to $1/C$, the conformal factor of the scalar field (cf. (1.4) and (2.2)).

Test of the formalism: Isotropic expansion

We want to test our general scheme for the caculation of particle creation developed so far by applying it first to isotropic expansion. From (3.4) we get

$$n_{11}^{(KG)} = \frac{1}{(2\pi C)^3} \int d^3\vec{p}\, d^3\vec{k}\, d^3\vec{k}'\, \delta(\vec{k}+\vec{k}'+\vec{p}) \times \left[1+2N^{(KG)}(\vec{p})\right] w(\vec{k},\vec{k}',\vec{p}) \,. \tag{3.6}$$

With isotropic expansion the photons become conformal particles such that $N^{(M)}(k) = 0$. Therefore (3.5) simply reduces to

$$n_{11}^{(M)} = \frac{2}{(2\pi C)^3} \int d^3\vec{p}\, d^3\vec{k}\, d^3\vec{k}' \delta(\vec{k}+\vec{k}'+\vec{p}) w(\vec{k},\vec{k}',\vec{p}) \tag{3.7}$$

where now

$$w_{isotr}(\vec{k},\vec{k}',\vec{p}) = {\beta_0}^2 \frac{|N_{in}(\vec{p})|^2}{2\pi^3} kk' \left(1 - \frac{\vec{k}\vec{k}'}{kk'}\right)^2 |J|^2$$

and

$$J = \int\limits_{-\infty}^{\infty} d\eta\, \frac{1}{C}\, \chi_{in}(\vec{p},\eta) \exp\left\{-i(\omega_{\vec{k}} + \omega_{\vec{k}'})\eta\right\} \,. \tag{3.8}$$

If we take the scalar particles to be conformal ones too, we simply get $|N_{in}(\vec{p})|^2 = 1/2\omega_p$, $\chi_{in}(\vec{p},\eta) = \exp(-i\omega_p\eta)$ and consequently $N^{(KG)}(\vec{p}) = 0$. Then the only remaining mechanism for conformal symmetry breaking is that by L_{int} (Birell et al. 1980). Only in that case we get from (3.6) and (3.7) $n_{11}^{(M)} = 2n_{11}^{(KG)}$ as suggested by fig. 6c. That there is a non-vanishing effect at all *if both particle species are conformal*, crucially depends on the expansion of the background. For, if we set $C = C_{in} = const$ in (3.8), we get energy conservation, $J \sim \delta(\omega_k + \omega_{k'} + \omega_p)$, which together with the momentum conservation expressed by the delta functions in (3.6) and (3.7) would kinematically forbid the creation process.

The pure background effects: An explicitly solvable example

We conclude by investigating the pure background effects (3.2) and (3.3). For all expanding directions in (2.10) - (2.12) we choose the expansion law $C^2(\eta) = b^2\eta^2 \quad (b = const > 0, -\infty < \eta < +\infty)$.

What we have to do is essentially to solve (4.3) and (4.7). This can be done in terms of Whittaker functions,

$$\left.\begin{array}{l}\chi(\eta)\\ F(\eta)\end{array}\right\} = \eta^{a} W_{\kappa,l}(c\eta^{2}) \,. \tag{3.9}$$

The parameters a, κ, l and c are given in tabs. 1 and 2 together with the mode-dependent numbers of created particles.

ν	a	c	κ	l	$N^{(KG)}(\vec{p})$
1	$\frac{1}{2}$	$ib\sqrt{m^2+p_\perp{}^2}$	$\frac{p_3{}^2}{4c}$	$\frac{1}{4}\sqrt{9+8(1+\xi)}$	$\exp\left(\frac{-\pi p_3{}^2}{b\sqrt{m^2+p_\perp{}^2}}\right)$
2	0	$ib\sqrt{m^2+p_3{}^2}$	$\frac{p_\perp{}^2}{4c}$	$\frac{1}{2}\sqrt{2(1+\xi)}$	$\exp\left(\frac{-\pi p_\perp{}^2}{2b\sqrt{m^2+p_3{}^2}}\right)$
3	$-\frac{1}{2}$	ibm	$\frac{p^2}{4c}$	$\frac{1}{4}$	$\exp\left(-\pi p^2/bm\right)$

Tab. 1: Parameters of the solutions (3.9) of the Klein-Gordon equation and the mode-dependent numbers of pion pairs created as free particles in a metric (2.1) expanding according to (2.10) - (2.12)

ν	a	c	κ	l	$N^{(M)}(\vec{k})$
1	$-1/2$	$ibk_\perp$	$k_3{}^2/4c$	$1/4$	$2\exp\left(-\pi k_3{}^2/bk_\perp\right)$
2	0	ibk_3	$k_\perp{}^2/4c$	$1/2$	$2\exp\left(-\pi k_\perp{}^2/2b\,\lvert k_3\rvert\right)$
3		conformally trivial			0

Tab. 2: Parameters of the solutions (3.9) of the Maxwell equations and the mode-dependent numbers of photon pairs as free particles in a metric (2.1) expanding according to (2.10) - (2.12).

Let us comment on the results. For isotropic expansion only the amounts of the momenta enter the mode-dependent numbers of created articles. In contrast to this, for anisotropic expansion the momentum components appear individually in the results. In case of two isotropically expanding directions the components p_1 an p_2 resp. k_1 and k_2 do so symmetrically such that the square of the momentum is replaced by the arithmetic means of ${p_2}^2$ and ${p_2}^2$ resp. ${k_1}^2$ and ${k_2}^2$. The momentum components p_3 and k_3 corresponding to that direction in which no expansion takes place appear together with, resp. instead of, the mass of the particles. For one expanding direction $p_\perp$ and p_3 resp. $k_\perp$ and k_3 simply change their roles.

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Functional Integrals in Curved Spacetime for Arbitrary Densities

M. Basler*
Friedrich-Schiller-Universität Jena
Institut für Theoretische Physik
Max-Wien-Platz 1
O-6902 Jena

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Abstract

We examine the method of functional integration in quantum field theory in curved spacetime exploiting tensorial densities of arbitrary weight as integration variables. Special attention is drawn to the correct choice of functional integral measure. After reviewing the method for free fields being based on a paper by Toms, we present a survey on the functional measures for arbitrary scalar, spinorial or vectorial densities as integration variables. Afterwards, we describe the generalization to interacting fields, taking a self-interacting scalar field as an example. As a result, we derive the weight-dependent Feynman graphs.

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1 Introduction

There are two approaches for investigating quantum field theory in curved spacetime [1, 2] . One approach is based on canonical quantization, the other one on functional integration. Investigating such processes as particle creation or scattering under the influence of an external gravitational field in the first approach one often exploits the method of Bogoljubov transformation [3, 4], whereas in the second approach in most cases the quantity of interest is the effective action [5, 6]. Here, we will report on a technical detail in the second approach.

If one wants to generalize the functional integral

$$Z = \int d\mu[\phi]e^{iS[\phi]} \tag{1}$$

to curved (Riemannian or Euclidean) spacetime often people refer to $d\mu[\phi]$ as to "a suitably choosen measure" which leads to the well-known determinant formula. However, in certain cases, for instance in exploiting the lattice approximation to the functional integral [7], for evaluating anomalies [8, 9, 10, 11, 12, 13] etc., one would like to have an explicit expression for the measure. Thus, the question arizes, how to generalize the functional integral and its measure to a curved spacetime.

Naively one would expect the measure to be the same as in flat space,

$$d\mu = \prod_x d\phi(x). \tag{2}$$

In this expression the product is to be understood as the limes of the lattice approximation defining the functional integral. However, as has been shown by Hawking [14], Fujikawa [8, 9, 10, 11, 12, 13] and De Witt [15] the naive measure (2) is not generally covariant (which means that the functional integral defined on the basis of (2) is not a scalar. For obtaining a scalar the measure (2) has to be modified as follows:

$$d\mu = \prod_x d\left(\sqrt[4]{-g(x)}\ {}^0\phi(x)\right). \tag{3}$$

Here and in the following the upper index on the field indicates its weight, i.e. ${}^0\phi(x)$ is a true scalar field (scalar density of weight zero). Since then, most authors have implicitely understood the functional integral measure as defined in (3). But this is a bit strange: Shouldn't it be possible to allow for changes of the integration variable, as they are possible in ordinary integrals? Perhaps, this would allow transforming the measure to the form

$$d\mu = \prod_x d\phi(x), \tag{4}$$

but with $\phi(x)$ possibly being a scalar density now. Based on a paper by Toms [16], who considered this question for a free scalar field, we will investigate this possibility in more detail here. Toms has shown, that it is indeed possible to make a transformation of the integration variable changing to an arbitrary scalar density; moreover, he has solved the question what is special with the measure choosen by Fujikawa. Here, we concentrate on the consideration of higher spin fields and on the effect, the special choice of measure has on interacting fields. The result will not be a new calculational scheme, but merely a kind of consistency check of the formalism of functional integration. This is part of a review on this topic [17]. As anyone knows, there is a series of subtleties in the subject of functional integration present in flat space already, being mainly connected to the question of convergence. We will not go into the details of these general problems, but will restrict ourselves to the special problems appearing in passing over to curved spacetime. To allow for considerations in higher dimensional spaces as well as for dimensional regularization we consider an N-dimensional spacetime.

2 The Method

The method is heuristic, but results will be checked afterwards by explicit calculation [16]. It is based on a finite-dimensional analogy. Consider a n-dimensional real vector space with the inner product

$$(u,v)=\sum_{i=1}^{n}\sum_{j=1}^{n}G_{ij}u^iv^j. \tag{5}$$

The invariant volume element (measure) has the form

$$d\mu[v]=\sqrt{\det G_{ij}}\prod_{i=1}^{n}dv^i. \tag{6}$$

Only in an orthonormal coordinate system the measure takes on the form

$$d\mu[v]=\prod_{i=1}^{n}dv^i. \tag{7}$$

This consideration can be generalized to the case of an infinite-dimensional space, where, instead of the vectors v^i, there is a scalar field $\phi(x)$. The inner product can be taken as

$$(\phi,\psi)=\int d^Nx\sqrt{-g_x}\,{}^0\phi(x)^0\psi(x). \tag{8}$$

If the true scalar fields ${}^0\phi(x)$ and ${}^0\psi(x)$ are substituted in favour of scalar densities of weight w being defined as

$$\phi(x) = {}^0\phi(x)(-g(x))^{-w/2} \tag{9}$$

then the inner product can be re-written as

$$(\phi, \psi) = \int d^N x \int d^N y \; \rho(x, y) \; \phi(x)\psi(y) \tag{10}$$

with

$$\rho(x, y) = (-g)_x^{w+(1/2)} \; \delta(x - y). \tag{11}$$

From (5) and (10) one can read off the correspondence

$$G_{ij} \leftrightarrow \rho(x, y). \tag{12}$$

This suggests defining the measure as

$$d\mu[\phi] = \prod_x (-g)^{(w/2)+(1/4)} \; d\phi(x). \tag{13}$$

Thus, indeed one can take an arbitrary scalar density as integration variable, but in this case one has to include the pre-factor

$$\prod_x (-g)^{(w/2)+(1/4)} . \tag{14}$$

On the other hand, one recovers the choice taken by Fujikawa and Hawking by taking $w = -1/2$:

$$\rho(x, y, w = -1/2) = \delta(x - y). \tag{15}$$

Thus, this choice corresponds to choosing an orthonormal basis in the space of scalar functions.

For integrating over the corresponding density the integrand, i.e. the action

$$S = \frac{1}{2} \int d^N x \sqrt{(-g)} \; {}^0\phi \Delta {}^0\phi \tag{16}$$

with

$$\Delta = -g^{\mu\nu} \nabla_\mu \nabla_\nu + \zeta R \tag{17}$$

must be brought into an appropriate form. Introducing scalar densities one arrives at

$$S = \frac{1}{2} \int d^N x \sqrt{-g} \; (-g)^{w/2} \; \phi(x) \Delta \left((-g)^{w/2} \; \phi(x) \right). \tag{18}$$

At first sight, this has not the form of a scalar product necessary for performing the functional integral, but it can be re-written into this form by exploiting the covariant derivative of tensorial (scalar) densities, to be found for instance in [18],

$$T^{\cdots}_{\cdots;\rho} = (-g)^{-w/2}\left((-g)^{w/2}T^{\cdots}_{\cdots}\right)_{;\rho}. \tag{19}$$

It has the special property, that the scalar density g is covariantly constant w.r.t. this derivative. This allows re-writing (18) as

$$S = \frac{1}{2}\int d^N x(-g)^{w+(1/2)}\,\phi(x)\Delta\phi(x) = (\phi,\Delta\phi). \tag{20}$$

Thus, the functional integral written in general densities becomes

$$Z = \int \prod_x (-g)_x^{(w/2)+(1/4)} d\phi(x)\; e^{\frac{i}{2}\int d^N x(-g)^{w+(1/2)}\phi\Delta\phi}. \tag{21}$$

Of course one has to check, that the functional integral defined this way indeed is independent of w and leads to the well-known determinant formula. For doing so the measure has to be written as

$$d\mu = \prod_x (-g)_x^{(w/2)+(1/4)} d\phi(x) = \prod_x d\left((-g)_x^{(w/2)+(1/4)}\phi(x)\right). \tag{22}$$

(This is not trivial, because one has to evaluate the functional determinant, which in the case of a scalar field reduces to an infinite constant. From this point, the further analysis proceeds as usual. The scalar density $\phi(x)$ has to be expanded into eigenfunctions of the operator Δ,

$$\phi(x) = \sum_n \phi_n f_n(x), \tag{23}$$

$$\Delta f_n = \lambda_n f_n, \tag{24}$$

where the eigenfunctions f_n are taken as densities of the same weight as the scalar fields are. Then, the measure reduces to a product over the numbers ϕ_n (the pre-factor $1/\sqrt{2\pi i}$ being choosen for convenience)

$$d\mu = \prod_n \frac{1}{\sqrt{2\pi i}}\, d\phi_n, \tag{25}$$

whereas the action becomes

$$S = \frac{1}{2}\, l^2 \sum_n \lambda_n \phi_n \phi_n. \tag{26}$$

Finally, one obtaines the desired formula

$$Z = \frac{1}{\sqrt{\mathrm{Det}(l^2\Delta)}}. \tag{27}$$

As it should be, the result is indeed independent of w and leads to the usual functional determinant. Of course, this expression is only of formal value because it is divergent and has to be renormalized.

At the end of this section, we would like to stress the point, that the measure inself is independent of the weight choosen for defining it, whereas the explicit form of the measure, expressed through a special density of weight w does depend on w. This can be compared to a vector, being an invariant object itself but having components, which depend on the choice of the coordinate system.

3 Functional Measures

The method described above developed by Toms [16] can be applied to other types of fields as well. In this section we only present the results (explicit form of the measures based on tensorial densities with arbitrary weight).

Real Scalar Field: $\phi = (-g)^{-w/2}\, {}^0\phi$

$$d\mu = \prod_x (-g)_x^{(w/2)+(1/4)} d\phi(x) \tag{28}$$

Charged Scalar Field:

$$d\mu = \prod_x (-g)_x^{w+(1/2)} d\phi^*(x) d\phi(x) \tag{29}$$

Charged Dirac Spinor Field: $\psi = (-g)^{-w/2}\, {}^0\psi$

$$d\mu = \prod_x (-g)_x^{-r_N(w+1/2)} \prod_\alpha d\bar\psi_\alpha(x) d\psi_\alpha(x) \tag{30}$$

Here, r_N denotes the number of components of a Dirac spinor in N dimensions,

$$r_N = \begin{cases} 2^{N/2} & N \text{ even} \\ 2^{(N-1)/2} & N \text{ odd.} \end{cases} \tag{31}$$

Real Vector Field: $A_\mu = (-g)^{-w/2}\, {}^0A_\mu$

$$d\mu = \prod_x (-g)_x^{(Nw+(N/2)-1)/2} \prod_\mu dA_\mu(x) \tag{32}$$

There is one point worth mentioning. Contrary to the scalar field, the explicit structure of the measure for the spinor field does not only depend on the weight, but on the dimension of spacetime too. This fact could not be recovered by Fujikawa [10], who set $w = -1/2$ from the beginning. A similar N-dependence occurs for the (covariant) vector field too.

4 Interacting fields

The case of interacting fields is a bit more difficult to handle (and was not treated before), because the fields enter the Lagrangian to different powers. As a result, we will see that contrary to the functional integral itself propagators and vertices will depend on the weight w. Here, we will restrict ourselves to perturbation theory and to a $\lambda\phi^4$-theory. The results for a pure Yang-Mills-theory can be found elsewere [17]. Perturbation theory is based on the vacuum functional including an external source,

$$Z[J, g_{\mu\nu}] = \int \prod_x (-g)^{(w+(1/2))/2} d\phi(x) e^{iS[\phi, J, g_{\mu\nu}]} = e^{iW}. \tag{33}$$

Investigations of external field problems are often based on the effective action Γ, which is obtained by a Legendre transformation from J to ϕ_c, in flat space being defined as

$$\phi_c(x) =< 0|\ \phi(x)\ |0 >= \frac{\delta W[J]}{\delta J(x)}. \tag{34}$$

In a curved spacetime, ϕ_c should become a density of the same weight w as is ϕ itself. Because W and J are true scalars and the generalization of the functional derivative to curved spacetime

$$\frac{\delta W[J]}{\delta J(x)} := \lim_{\epsilon \to 0} \frac{W[{}^0J(y) + \epsilon\ {}^0\delta(y,x)] - W[{}^0J(y)]}{\epsilon} \tag{35}$$

(${}^0\delta(y, x)$ is the covariant delta function) leads to a true scalar again, this can only be achieved by replacing the ordinary functional derivative by a weighted one,

$$\phi_c(x) =< 0|\ \phi(x)\ |0 >= \frac{{}^w\delta W[J]}{\delta J(x)} := (-g)^{-w/2} \frac{\delta W[J]}{\delta J(x)}. \tag{36}$$

On the basis of a similar argument it can be shown, that the 2-point Green function or propagator has to be generalized to

$$G(x, y) =< 0|T[\phi(x)\phi(y)]|0 >= \frac{1}{i} \frac{{}^w\delta^2 W}{\delta J(x)\delta J(y)}. \tag{37}$$

This definition ensures that the propagator becomes a density of weight w w.r.t. x and y. Inserting for W only the free functional integral leads to the free propagator for the scalar field density.

Next, we are interested in vertices, appearing in the expansion of the effective action,

$$\Gamma[\phi_c] = \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^N x_1 \ldots \int d^N x_n \, \Gamma^n(x_1 \ldots x_n) \, \phi_c(x_1) \ldots \phi_c(x_n). \quad (38)$$

The effective action should be a true scalar, as the action itself is. Because ϕ is a scalar density of weight w, the four-vertex of the ϕ^4-theory, Γ^4 must become a density of weight $-4w-4$. This is achieved by the definition

$$i\Gamma^4(x,y,z,v) = i\left(\frac{1}{i}\right)^4 \frac{\delta}{\delta\phi_c(x)} \frac{\delta}{\delta\phi_c(y)} \frac{\delta}{\delta\phi_c(z)} \frac{\delta}{\delta\phi_c(v)} \Gamma[\phi_c] \,|_{\phi_c=0}, \quad (39)$$

where the functional derivative is the ordinary one in curved spacetime, as defined in (35).

After these definitions we are in a position to derive the perturbation expansion from the functional integral, which for the ϕ^4 -theory has the structure

$$Z[J, g_{\mu\nu}] = \int \prod_x (-g)^{(w+(1/2))/2} d\phi \, e^{iS}, \quad (40)$$

$$S = \int d^N x \left(\frac{1}{2}(-g)^{w+(1/2)} \phi \Delta \phi - \frac{\lambda}{4!}(-g)^{2w+(1/2)} \phi^4 + (-g)^{(w+1)2} \phi J \right). \quad (41)$$

For deriving the perturbation expansion the term ϕ^4 in the perturbation Lagrangian has to be replaced by derivatives after the current. By taking the ordinary functional derivatives, this would lead to an additional prefactor involving powers of $(-g)$. These can be compensated by exploiting the weighted functional derivative once more and performing the substitution

$$\phi(x) \to \frac{1}{i} \frac{{}^w\delta}{\delta J(x)}. \quad (42)$$

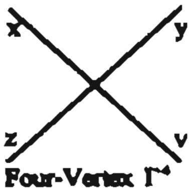
As a result, this leads to

$$Z = \exp\left(i \int d^N z \frac{\lambda}{4!} (-g)^{2w+(1/2)} \left[\frac{1}{i} \frac{{}^w\delta}{\delta J(z)} \right] \right) *$$
$$* \exp\left(i \int d^N x \left(\tfrac{1}{2}(-g)^{w+(1/2)} \phi \Delta \phi + (-g)^{(w+1)2} \phi J \right) \right) \quad . \quad (43)$$

Integrating over the free part of Z leads to the free propagator via (37), while the vertex is obtained from (39). This results in

Propagator $G(x,y)$

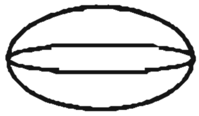
$\Delta_s G(x,y) = i\, {}^{w}\delta(x,y) = i\, (-g)_x^{-w/2}\, {}^{0}\delta(x,y)\, (-g)_y^{-w/2}$ for the propagator and in the vertex



Four-Vertex Γ^4

$i\Gamma_4(x,y,z,v) = i\, \lambda(-g)_v^{2w+2}\, {}^{0}\delta(v,x)\, {}^{0}\delta(v,y)\, {}^{0}\delta(v,z)$.

One might worry about the correctness of these results, especially because of the fact that some of the definitions taken could also be choosen in a different way. A simple check can be done by considering a vacuum graph of the theory, for instance the following:



Example Vacuum Graph

Graphs like this without external legs must not depend on the weight choosen for defining the functional integral. In rough outlines this graph contributes to the effective action a term

$$\begin{aligned}
\sim \lambda^2 \int & dx dy dz dv dx' dy' dz' dv' (-g)_x^{2w+2}\, {}^{0}\delta(x,y)\, {}^{0}\delta(x,z)\, {}^{0}\delta(x,v) \\
& \cdot \frac{1}{\Delta_x}(-g)_x^{-w/2}\, {}^{0}\delta(x,x')\, (-g)_{x'}^{-w/2} \cdot \frac{1}{\Delta_y}(-g)_y^{-w/2}\, {}^{0}\delta(y,y')\, (-g)_{y'}^{-w/2} \\
& \cdot \frac{1}{\Delta_z}(-g)_z^{-w/2}\, {}^{0}\delta(z,z')\, (-g)_{z'}^{-w/2} \cdot \frac{1}{\Delta_v}(-g)_v^{-w/2}\, {}^{0}\delta(v,v')\, (-g)_{v'}^{-w/2} \\
& \cdot (-g)_{x'}^{2w+2}\, {}^{0}\delta(x',y')\, {}^{0}\delta(x',z')\, {}^{0}\delta(x',v').
\end{aligned} \tag{44}$$

Of course, this expression is highly divergent and has to be renormalized. Moreover, it will not be possible to evaluate it for a general spacetime.

Despite these difficulties we go on by integrating over y, z, v and y', z', v' to obtain

$$
\begin{aligned}
&\sim \lambda^2 \int dx dx' (-g)_x^{2w+2} \left(\frac{1}{\sqrt{-g_x}}\right)^3 \\
&\quad \left[\frac{1}{\Delta_x}(-g)_x^{-w/2}\, {}^0\delta(x,x')\, (-g)_{x'}^{-w/2}\right]^4 \\
&\quad (-g)_{x'}^{2w+2} \left(\frac{1}{\sqrt{-g_{x'}}}\right)^3
\end{aligned} \tag{45}
$$

$$
\sim \lambda^2 \int dx dx' (-g_x)^{1/2} (-g_x)^{1/2} \left[\frac{1}{\Delta_x}\, {}^0\delta(x,x')\right]^4 . \tag{46}
$$

Obviously, this is independent on the weight w chooseen for defining the functional integral. Moreover, it has the desired structure explicitely indicating the covariance of the expression.

5 Conclusions

As a result we have shown, that not only the measure, but the whole formalism of functional integration can be consistently formulated for free as well as for interacting scalar, spinorial and vector fields independently on the weight chooeen for defining the functional integral. The formalism obtained this way leads to the well known results for the free field case, but to weight dependent Feynman rules for interacting fields. Pure vacuum graphs contributing to the effective action without external legs are indendent on the weight; however, graphs with external legs do depend on the weight chooeen because of the weight dependence of the external field.

The formalism has the advantage that one understands the various factors of $-g$ appearing in the formula; it reduces to the usual formalism by considering the special case $w = -1/2$ corresponding to choosing an orthonormal basis in the space of functions. Thus, up to now this is merely a consistency check of the formalism. It might find more applications in considering non-flat function spaces, which do not allow introducing a global orthogonal basis as it is the case in the examples considered here.

6 Acknowledgements

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ON THE EFFECT OF GLUON CONFINEMENT AND QUARK ACCELERATION IN HADRON PRODUCTION

G. Cocho, S. Hacyan, F. Soto and C. Villarreal

Instituto de Física, U.N.A.M.

Apartado Postal 20-364, C.U.

México 01000, D.F.

and

A. Sarmiento

Instituto de Astronomía, U.N.A.M.

Apartado Postal 70-264, C.U.

México 04510, D.F.

Summary.- The energy-density spectrum of vacuum according to an accelerated observer inside a prismatic cavity presents two conspicuous contributions: A strongly oscillating component associated to confinement, and a pseudothermal one associated to acceleration. We propose that some features of hadron production by electron positron anihilation may be understood in terms of the product of this modified gluon phase space times the $e^+e^- \rightarrow q\bar{q}g \rightarrow$ hadron matrix element.

1.- **Introduction**

Over the last two decades QCD has emerged as the standard theory of hadronic interactions. In particular, perturbative QCD has been very useful in the analysis of scattering and bound state problems where the distances are small and the momentum transfer is large. However, it is more difficult to have specific predictions in regimes where the momentum transfer is small and perturbative QCD is not valid. That is why it might be useful to look for predictions which do not depend on the details of the theory but rather on general features. Within this context, bag models have been able to predict in reasonable fashion experimental quantities such as hadron masses, magnetic moment ratios, etc. In general, the bag energy may be written as $E=E_B+E_0$, where E_B is the energy associated to occupied modes in the bag, and E_0 the corresponding vacuum energy. Taking into account results obtained for the vacuum enrgy between parallel plates, DeGrand et al.[1] obtained their results by parametrising E_0 as $-Z_0/a$, where a is the hadron radius and $Z_0=1.84$. However, Milton showed in a more detailed calculation [2] that the finite contribution to the vacuum energy of non-interacting gluon and quark fields confined inside a spherical bag is $E^G_0 \sim 1/\pi a$ and $E^Q_0 \sim -N/48\pi a$, respectively, where N is the number of effective massless quarks. The gluon contribution is dominant and this leads to $E_0 \sim 0.3/a$. Note that in this approach the energy changes its sign, and that its value is consistent with the equivalent QED result for a spherical cavity multiplied by eight (the number of gluons), $E_0 \sim 0.36/a$. Baacke and Igarashi [3] calculted the Casimir energy for confined masive quarks. In the massless limit they obtained $E^Q_0 \sim -N/144\pi a$, about $\frac{1}{3}$ of Milton's result. In a similar trend, Ambjorn and Wolframm [4] calculated the vacuum energy for scalar and vector fields in hyper-parallelipedal cavities in arbitrary dimensions. They found that the Casimir energy is minimised (and frequently becomes negative) when some of the dimensions are very long compared to the others. They pointed out that this could provide a mechanism for the spontaneous reduction of the dimensionality of a field theory. In particular, in three dimensions, the Casimir energy leads to a force that tends to deform bubbles of fixed volume into long tubes.

On the other hand, opposed to bag elongations, the color interaction tries to restore the original bubble configuration by exerting a constant tension among the bag constituents. This should be manifested, among other things, as a constant deceleration of the valence quarks inside. Guided by the results obtained in field theories in non-inertial systems, Harrington and Tabb [5] proposed that associated to a deceleration α there exists a fluctuation dissipation mechanism that induces the transformation of a Minkowskian vacuum into a heat bath with an effective temperature T_a given by $T_a^{-1}=2\pi\alpha^{-1}=2\pi mk^{-1}$, where m is the reduced constituent mass of the quark and $(2\pi k)^{-1}$ is the slope of the linear potential. They got a value of 108 MeV for T_a, consistent with empirical "temperatures" found in hadronic collision experiments [6], where the dominant p_T dependence in the CM frame is described by $E=exp[-(p_T^{\,2}+M^2)^{\frac{1}{2}}/T]$, with M the hadron mass. Barshay and Throost [7] arrived to a similar acceleration temperature by different arguments.

According to previous estimations, the gluon contribution to the vacuum energy of a bag is dominant. We thus study in this work the vacuum energy spectral density associated to both gluon confinement and quark acceleration in the gluon vacuum. For simplicity, we consider an uniformly decelerating observer immersed in a scalar vacuum confined by a long prismatic cavity with quadrangular cross section. We then study the possible effects of this bag spectrum on the energy spectrum of hadronic jets produced in e^+e^- anihilations.

2.- Vacuum spectral density.

In a previous paper [8], we calculated the explicit form of the energy density spectrum of a scalar vacuum as detected by an uniformly accelerated observer travelling in the axial direction of a prismatic cavity. The explicit formula for the spectral density is

$$\frac{dE}{d\omega} = \frac{\omega^2}{\pi^2}\left\{ \frac{1}{2} + \frac{1}{e^{2\pi\omega/\alpha} - 1} \right\}\left\{ ab + \frac{b}{\omega} F(\alpha a) + \right.$$

$$\left. + \frac{a}{\omega} F(\alpha b) + \frac{2ab}{\omega} G(\alpha\rho) \right\} , \qquad (2a)$$

where

$$F(x) = \sum_{-\infty}^{\infty} \frac{\sin(2\omega\alpha^{-1}\operatorname{arcsinh}(x))}{n(1 + (nx)^2)^{\frac{1}{2}}} , \qquad (2b)$$

and

$$G(x) = \sum_{-\infty}^{\infty} \frac{\sin(2\omega\alpha^{-1}\operatorname{arcsinh}(x))}{\rho(1 + (\rho x)^2)^{\frac{1}{2}}} . \qquad (2c)$$

a and b are the sides of the section of the prism, ω is the frequency, α is the observer acceleration, $\rho=(a^2n^2+b^2m^2)^{\frac{1}{2}}$, and n, m are integer numbers. The first parenthesis is a "Planckian" pseudothermal distribution with an effective temperature given by $T_\alpha=\alpha/2\pi$, including a zero-point term. The second one represents the Casimir energy associated to each pair of the walls that form the cavity. The last term im this parenthesis is an "interference" term between the effects of confinement and acceleration. The graph of the Casimir and interference terms $F(\omega)$ and $G(\omega)$ divided by the mode density ($\sim \tilde{\omega}^{-2}$) appear in Figs. 1a and 1b in terms of adimensional variables $\tilde{\omega}=a\omega$, $\tilde{\alpha}=a\alpha$, $\tilde{E}=a^2E$, and $\beta=b/a$. The total spectral density (also divided by the mode density) without the divergent zero-point term appears in Fig. 1c. Figs. 2a-2c represent the same spectrum for a larger frequency range. We see a strongly oscillating behaviour, with peaks more or less equally spaced. On the other hand, the Planckian term dies rapidly and influences low frequency modes only.

As we will show now, acceleration and confinement of the same order ($\simeq$1 fermi) seem to be present in the positron-electron annihilation with hadron production. As it was calculated by Harrington and Tabb, one derives a string tension $k=m_q\alpha_k$ from the slope of the Regge-trajectories ($1/2\pi k \sim 0.8-0.9\,(\mathrm{GeV}/c^2)^{-2}$) and by assuming a mass of $m_q=300$ MeV for the non-strange constituent quarks, the resulting acceleration is $\alpha_k=.6$ GeV. An acceleration temperature $T_\alpha = \alpha_k/(2\pi) \simeq$ 100 MeV is thus obtained. This value implies an 'acceleration length' $\ell_\alpha=1/\alpha_k \simeq 0.3$ fermis and an 'interference parameter' $\alpha_k a \simeq 10/3$. The interference term is larger when the confinement length (a or b) and the 'acceleration length' are of the same order of magnitude. As mentioned before, a Planckian distribution with a 'temperature' $\simeq$ 100 MeV is expected after hadronization for low energy gluons. However, 'thermal jets' showing an oscillatory tail like that of the energy spectrum of Figs. 1-2 might exist for high energy gluons. This effect seems to be present in the e^-e^+ annihilation experimental data at 29 GeV [9].

3.-Possible effect on $e^+e^- \rightarrow$ hadrons.

The $e^+e^- \rightarrow$ hadrons process can be considered as a virtual time-like photon decaying into a quark-antiquark pair. The quark and antiquark move in opposite directions with large momenta, but remain united by the action of the gluonic string. Assuming a constant tension in the string, both particle and antiparticle will move with a constant deceleration within an approximately cylindric hadronic bag or cavity having a diameter of the order of 1 fermi. The excited, uniformly accelerated quark and antiquark will emit gluons with a difference of energy E between excited levels, and in proportion to the energy density of vacuum (as seen from the confined accelerated system) corresponding to such an energy E.

In order to analyse this multiproduction process, the matrix

$$Q^{\alpha\beta}=\sum_{l=1}^{n} p_l^{\alpha} p_l^{\beta} \qquad (\alpha,\beta=1,2,3) \qquad (2.1)$$

is defined for every event [10], where n is the number of secondary particles in the event and p_l^{α} is the three momentum of the lth particle in the center of mass system. Diagonalisation of this matrix gives the three-eigenvector $\hat{z}_k$ with the eigenvalues in the order $Q_1>Q_2>Q_3$. The value Q_1 measures the length along the principal axis of the event $\hat{z}_1$, Q_2 corresponds to the 'width', Q_3 to the 'height', and the sphericity Δ is defined as $\Delta=\frac{3}{2}(Q_2+Q_3)=\frac{3}{2}(1-Q_1)$. The region with $\Delta>0.25$, and $Q_3<0.10$ (a flat disk) is usually considered as the quark-antiquark-gluon three-jet region. It is also customary to define

$$\langle p_T^2\rangle_{out} \cong \frac{1}{n}\sum_{l=1}^{n}(\vec{p}_l\cdot\hat{z}_3)^2, \qquad (2.2)$$

and

$$\langle p_T^2\rangle_{in} \equiv \frac{1}{n}\sum_{i=1}^{n} (\vec{p}_i.\hat{z}_2)^2. \tag{2.3}$$

The quantity $\langle p_T^2\rangle_{out}$ measures the transversal momentum in the direction perpendicular to the reaction plane, and $\langle p_T^2\rangle_{in}$ the transversal momentum in the reaction plane.

For three-jets $q\bar{q}g$ events, one expects relatively large values of $\langle p_T^2\rangle_{in}$, while $\langle p_T^2\rangle_{out}$ reflects the low-energy gluon distribution. Two important features of the 29 GeV distributions are:

i) The $\langle p_T^2\rangle_{out}$ distribution may be fitted by 40 $exp(-p_T/T_\alpha)$ with $T_\alpha \simeq 80$ MeV, in good agreement with our previous estimate (see Fig. 3*a*).

ii) The $\langle p_T^{\,2}\rangle_{in}$ distribution seems to show small oscillations (see Fig.3*b*). These oscillations could arise from the gluon mode distribution associated to accelerated sources inside the quasi-cylindrical cavity. This modified gluon phase space would multiply the $e^+e^- \rightarrow q\bar{q}g \rightarrow$ hadrons matrix element. In figure 3*b*, the local minima in the distribution, whose separation is $\Delta p_{Tin} = \Delta[\langle p_T^{\,2}\rangle_{in}]^{\frac{1}{2}} \sim 0.15$ GeV, have been marked by arrows. The influence of confined gluons may be estimated by taking into account the contribution to the $\langle p_T\rangle_{in}$ distribution of the particles generated by hard gluons. The corresponding energy difference at neighboring minima may be evaluated as follows:

a) In three-jets events, it is assumed than each jet is provenient from a parent quark, an antiquark, and a gluon, respectively. Thus the total number of particles arising from the gluon may be estimated from

$$\langle n_g\rangle = (\langle n_{ch}\rangle + \langle n_n\rangle)/n_j, \tag{2.4}$$

where $\langle n_{ch}\rangle$ is the mean number of detected charged particles, $\langle n_n\rangle$ corresponds to the neutral particles, and n_j is the number of jets. By assuming that the neutral particles are produced with equal probability than positive and negative ones, we may add a 50% to take them into account. From the fact that $\langle n_{ch}\rangle=13.1\pm0.05\pm0.6$ in the experiment [9], we obtain a total number of 18 particles. Dividing by $n_j=3$, we find about 6 particles generated by a gluon.

b) From Eq. (2.3), it turns out that

$$\Delta p_{T_{ln}} = \frac{1}{\langle n_g\rangle} \langle \Delta p_{gluon} \cos\theta_{z_2} \rangle, \tag{2.5}$$

where Δp_{gluon} is the r.m.s momentum of the gluonic particles. The mean $\cos\theta_{z_2}$ factor may be approximated by $1/\sqrt{2}$. The resulting momentum difference between neighboring minima is:

$$\langle \Delta p_{gluon}\rangle \propto \Delta p_{T_{ln}} \cdot \langle n_g\rangle \cdot \langle \cos\theta_{z_2}\rangle^{-1} = 1.3 \text{ GeV}.$$

For a comparison of these difference with the oscillations of figure 1, we take the $\tilde{\omega}$ separation between local minima. The mean minima spacing is $\Delta\tilde{\omega} = 2\alpha^{-1}\Delta\omega \equiv 2\ell_a \Delta E \simeq 4.1$. We thus obtain

$$\Delta E \simeq \tfrac{1}{2} \times 4.1 \times 0.6 \text{ GeV} = 1.2 \text{ GeV},$$

in good agreement with the estimated $\langle \Delta p_{gluon}\rangle$.

This comparison shows that even although the theoretical model we present is too rough and much more precise experimental measurements are needed to confirm the oscillations of figure 3*b*, there seems to be consistency between the model and the available experimental data.

3.- Discusion and conclusions.

Although the details of the results of QCD, a non abelian gauge theory, certainly will differ with respect to our free scalar field calculations, we hope that the general features associated to confinement and acceleration will remain. Current calculations [11] show that the spectral density associated to vector fields in prismatic cavities is similar to that encountered for scalar fields.

Of course, real hadrons are confined in an irregular and fluctuating quasi-cylindrical bag rather than in a prismatic cavity. Nevertheless, the prismatic approximation is mathematically simple and we expect that the qualitative results do not differ drastically from the more complex cylindrical case. The effect of radial oscillations can be estimated by averaging, in general with different weights, the results of computations for prismatic cavities with different edge values . When this averaging is practised for a=0.75 fermis with weight g=0.25, a=1.0 fermis with weight g=0.50, and a=1.25 fermis and weight g=0.25, the process tends to erase all the minima except for the nearest ones.

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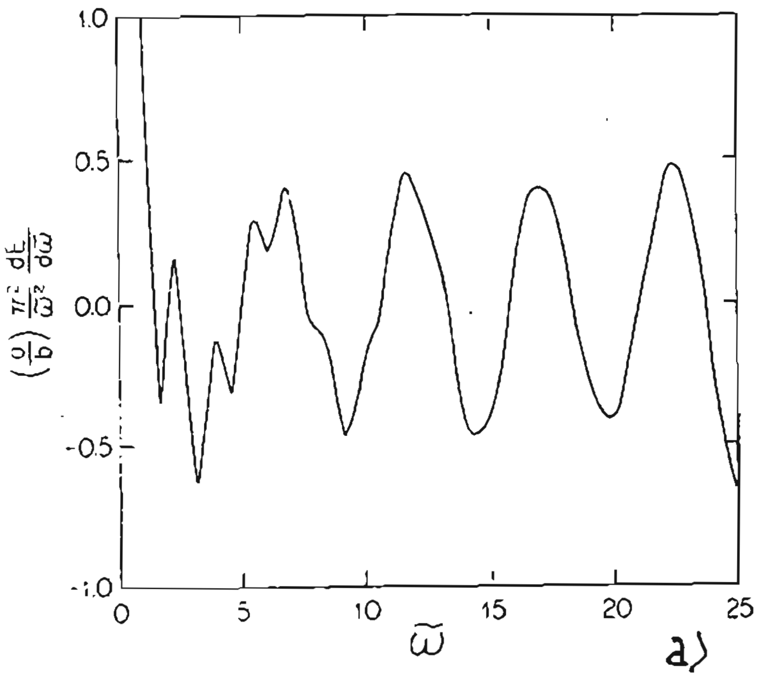

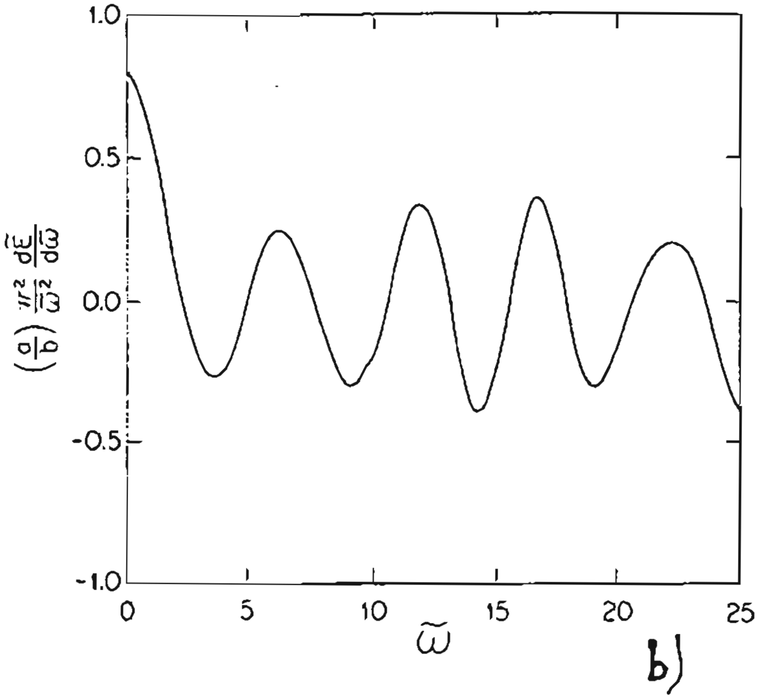

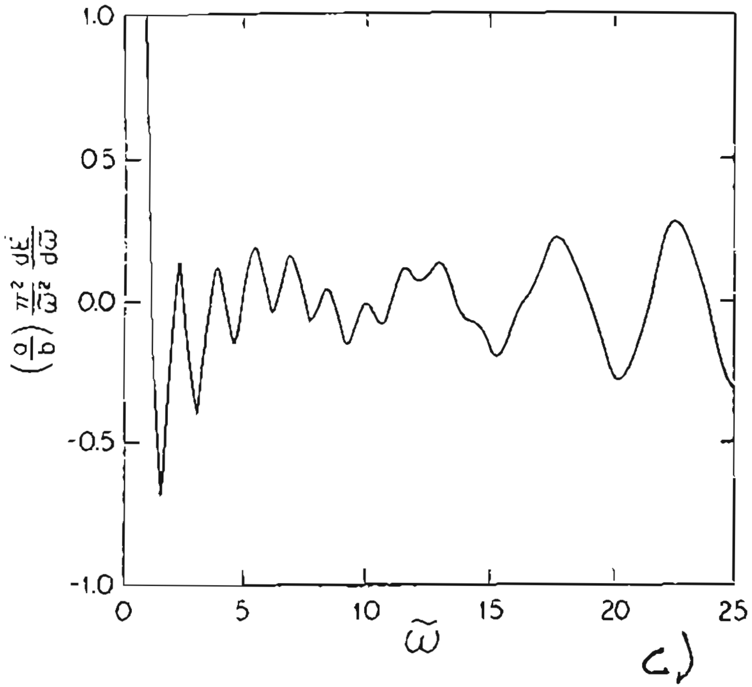

Figure 1. *a)* Vacuum energy spectral density per unit length (divided by the mode density~ $\tilde{\omega}^{-2}$), in terms of adimensional variables $\tilde{\omega}=a\omega$, $\tilde{\alpha}=a\alpha$, $\tilde{E}=a^2E$, and $\beta=b/a$. Here, $\beta=1$, and $\tilde{\alpha}=3.3$. *b)* Contribution to the vacuum energy density associated to parallel plates. c) Contribution associated to "interference" between confinement and acceleration.

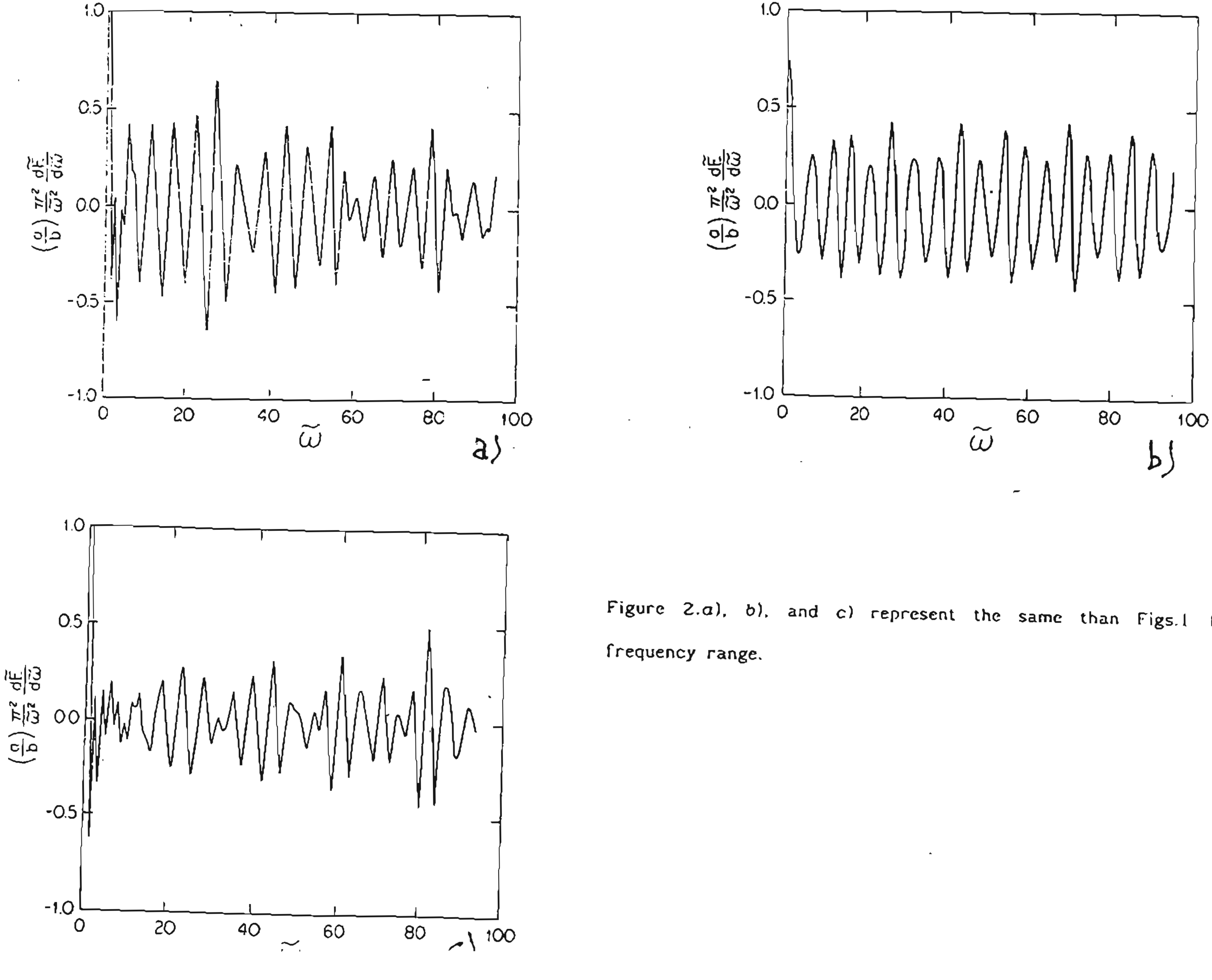

Figure 2.a), b), and c) represent the same than Figs.1 for a larger frequency range.

Figure 3.

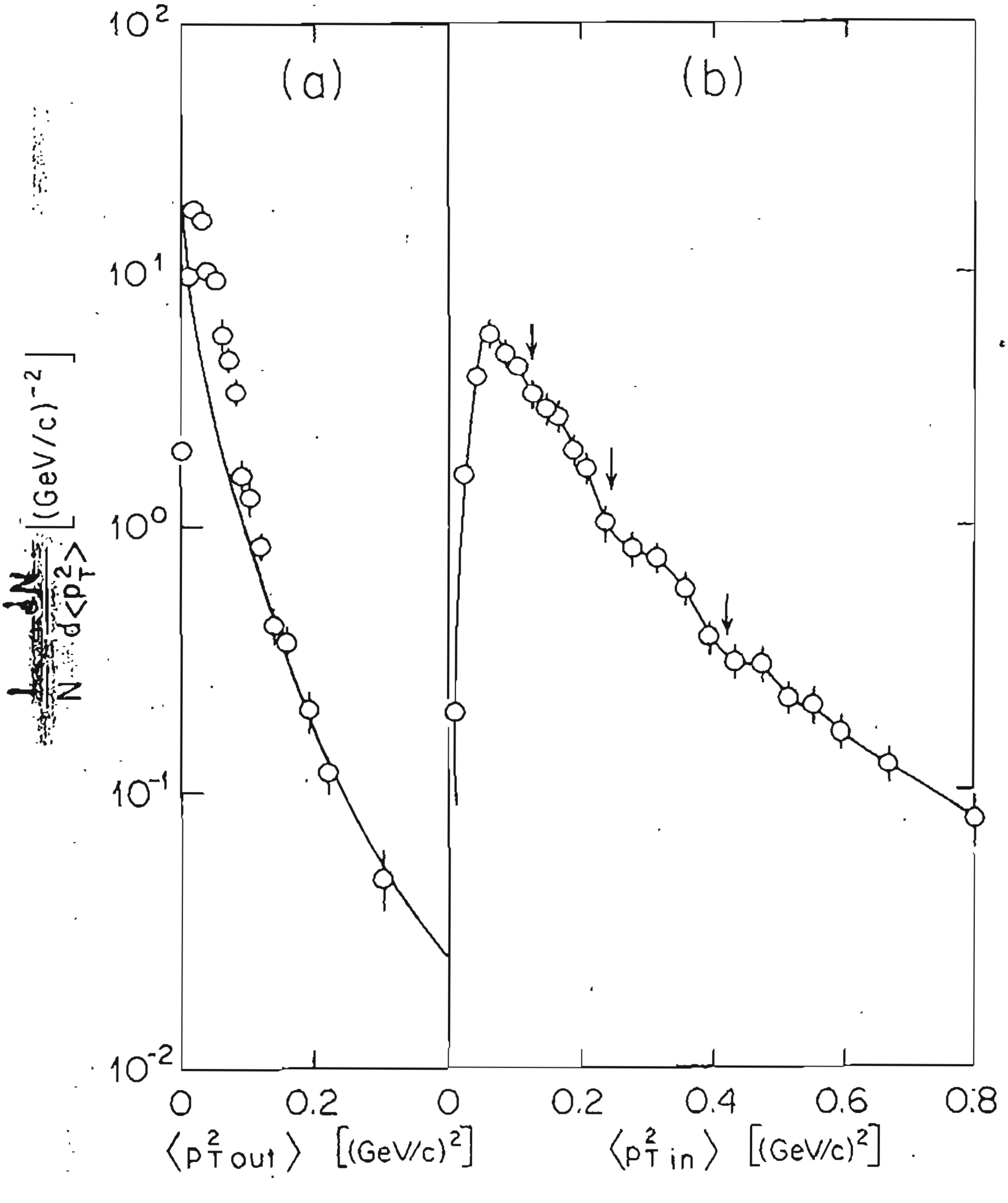

Figure 3. Transversal momentum distribution: *a)* Distribution off the event-plane; as a function of $\langle p_T^2\rangle_{out}$ the jet is approximately described by $\frac{1}{N}\frac{dN}{d\langle p_T^2\rangle} = 40\, e^{-p_\perp/b}$ $\cdot[\mathrm{GeV/c}]^2$ with b = 80 MeV. *b)* Distribution in the event plane; the solid line connects the experimental points and the arrows mark the place of the possible local minima.

A RINDLER-DIRAC ELECTRON IN THE ROLE OF A QUANTUM DETECTOR

R. Jáuregui

Instituto de Física, UNAM

Apdo. Post. 20-364, 01000 México D.F., MEXICO

Abstract

The interaction of a Rindler-Dirac electron with the electromagnetic Minkowski vacuum is studied to first order in perturbation theory from the point of view of the changes that it may induce in the electron quantum numbers. The presence of terms modulated by a thermal factor is recognized as part of the possible quantum effects. However, we find that the electron has a very high probability of remaining in its original state with the simultaneous emission of Minkowski photons. This behaviour resembles the classical first order result according to which a uniformily accelerated charge does not has a radiation reaction force. Possible spin flips and changes in the perpendicular momentum are also studied.

INTRODUCTION

Quantum field theory in uniformily accelerated systems contains many of the special features of curved-space quantum field theory [1,2]. Most of the works published so far on quantum field theory in accelerated systems deal with scalar fields or massless fermion fields [3-5], since these are the simplest cases.

From these studies, it is now well established that there are thermal like effects associated to the vacuum in a uniformily accelerated frame, although their physical interpretation is not enterely clear. These effects should manifest by the interaction of a detector with the corresponding quantum field. A detector is a system with internal degrees of freedom which may change due to the interaction with the quantized field.The temperature associated to these effects is

$$T = \frac{\hbar a}{2\pi k_B c}$$

So that it is necessary to have a very large acceleration a to get observable temperatures. Thus, the most realistic objects for use as accelerated detectors are elementary particles.

In 1983, Bell and Leinaas [6] proposed the use of electrons as detectors of Rindler photons. They made interesting predictions on the polarization effects that could arise from the coupling of accelerated electrons with the electromagnetic field. It is the purpose of this work to perform calculations that take into account the spin 1 nature of the electromagnetic quantum field and the spinorial character of the electron wavefunction in order to achieve a better understanding of the efficiency of the electron as a detector.

By an accelerated quantum electron we shall understand a particle represented by a wavefunction which is a stationary solution of Dirac equation in Rindler coordinates. We recognize from the begining that an electron in spite of its apparent simplicity is quite a rich system. The interaction may lead to changes in any of the four quantum numbers: energy ϵ, perpendicular momentum $\vec{p}_\perp$ and spin s.

Another aim of our calculations is to achieve a better understanding of the radiative processes of an accelerated quantum particle. In this context, we want to find possible links between our results and those concerning the radiation of an accelerated charged classical particle.

The method of the calculation is the standard one. That is, we calculate the response function per unit proper time of a Rindler-Dirac electron interacting with Minkowski vacuum. By making the integration over the proper time difference two different kind of terms arise. One is modulated by a thermal-like function and the other is in fact proportional to a $\delta(\epsilon' - \epsilon)$ distribution. This result is independent of the specific form of the stationary electron wavefunctions.

Then we consider the particular case of a Rindler-Dirac electron without external fields. The general expression for the transition probability is quite complicated. However, it is possible to obtain a closed expression for spin flip without changes in the perpendicular momentum. This transition is in fact completely modulated by a thermal like factor but has extra dependence on the difference of Rindler energies involved in the transition.

BASIC EQUATIONS

RINDLER COORDINATES

Define the Rindler coordinates (τ, ξ) :

$$t = \pm\xi \sinh \tau, \qquad z = \pm\xi \cosh \tau.$$

The two signs correspond to the disconnected regions with $z^2 - t^2 \geq 0 : z > 0$ (Region I) and $z < 0$ (Region II). The two other regions with $t^2 - z^2 \geq 0$ correspond to the future and past spacetime sectors and the corresponding coordinates are Milne coordinates:

$$t = \pm\xi \cosh \tau, \qquad z = \pm\xi \sinh \tau.$$

The paths $\xi = constant$ are wordlines of constant proper acceleration $a = \xi^{-1}$. We shall work in Region I.

ELECTROMAGNETIC WIGHTMAN FUNCTIONS

In Minkowski coordinates and Feynman gauge, the electromagnetic Wightman functions $D^{\pm}_{\mu\nu}(x,x')$ are given by

$$D^{\pm}_{\mu\nu}(x,x') = \eta_{\mu\nu} D^{\pm}(x,x')$$

where

$$D^{\pm}(x,x') = \frac{1}{(t-t' \mp i\epsilon)^2 - (x-x')^2 - (y-y')^2 - (z-z')^2}$$

So the Wightman functions for the Rindler electromagnetic field in Minkowski vacuum are

$$\mathcal{D}^{\pm}_{\mu\nu}(x,x') = \mathbf{g}_{\mu\nu}(x,x')\mathcal{D}^{\pm}(x,x')$$

with

$$\mathbf{g}_{\mu\nu}(x,x') = \begin{pmatrix} \xi\xi' cosh(\tau-\tau') & 0 & 0 & -\xi senh(\tau-\tau') \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \xi' senh(\tau-\tau') & 0 & 0 & -cosh(\tau-\tau') \end{pmatrix}$$

and

$$\mathcal{D}^{\pm}(x,x') = \frac{1}{2\xi\xi' cosh(\tau-\tau' \mp i\epsilon) - (\xi'^2+\xi^2) - (x-x')^2 - (y-y')^2}$$

RINDLER-DIRAC SPINORS AND TRANSITION CURRENT [7]

The generally covariant Dirac equation in Rindler metric has the standard form

$$(\gamma^{\mu} D_{\mu} + im)\Phi = 0$$

where covariant γ matrices satisfy the usual relation

$$\{\gamma^{\alpha}, \gamma^{\beta}\} = g^{\alpha\beta}$$

so that they can be written in terms of tetrad vectors e^{μ}_{n} as

$$\gamma^{\mu} = e^{\mu}_{n}\gamma^{n}$$

with γ^n the standard Dirac matrices in Cartesian coordinates and

$$e_0^\mu = (\xi^{-1}, 0, 0, 0)$$

$$e_i^\mu = \delta_i^\mu, i = 1, 2, 3.$$

In the following, latin and nummerical indices refer to tetrad components.

The covariant derivative of Dirac-Rindler equation includes the affine connection Γ_μ,

$$\Gamma_\mu = (-\frac{1}{2}\gamma_0\gamma_3, 0, 0, 0)$$

and could include an external electromagnetic field $A_\mu(x, y)$:

$$D_\mu = \partial_\mu + \Gamma_\mu - ieA_\mu$$

The conserved current J^α is given by

$$J^\alpha = \Phi^\dagger \gamma^0 \gamma^\alpha \Phi$$

The positive energy solutions of Dirac equation in Rindler coordinates (sector I) have the form

$$\Phi = Ne^{-i\epsilon\tau} K_{\frac{1}{2} - i\epsilon\gamma^0\gamma^3} \mathcal{X}_k(x, y).$$

where, in general, for any function of $\gamma^0\gamma^3$ matrices

$$f(\gamma^0\gamma^3) = \mathbf{P}_+ f(1) + \mathbf{P}_- f(-1)$$

$$\mathbf{P}_\pm = \frac{1}{2}(1 \pm \gamma^0\gamma^3),$$

$K(x)$ is the modified Bessel functions, which is regular at $\xi \to \infty$, and the normalization factor is

$$N = \frac{2}{\pi^2} cosh(\pi\epsilon).$$

so that

$$\int dx dy d\xi \Phi^\dagger_{\epsilon k} \Phi_{\epsilon' k'} = \delta(\epsilon - \epsilon')\delta_{kk'}$$

whenever

$$\int dx dy \mathcal{X}_k^\dagger \mathcal{X}_{k'} = \delta_{kk'}$$

The bispinor $\mathcal{X}$ for a free particle ($A_\mu = 0$) in the spinorial representation is given by

$$\mathcal{X}_{\vec{k}_\perp}(x,y) = \frac{1}{2\pi}\left[\frac{k_\perp + m}{4k_\perp}\right]^{\frac{1}{2}} e^{ik_x x + ik_y y} \begin{pmatrix} \Upsilon_\pm \\ \pm i\Upsilon_\pm \end{pmatrix}$$

with

$$\Upsilon_+ = \begin{pmatrix} 1 \\ \frac{-i(k_x+ik_y)}{k_\perp+m} \end{pmatrix}$$

$$\Upsilon_- = \begin{pmatrix} \frac{-i(k_x-ik_y)}{k_\perp+m} \\ 1 \end{pmatrix}$$

and

$$k_\perp = (k_x^2 + k_y^2 + m^2)^{\frac{1}{2}}.$$

METHOD OF CALCULATION

The probability of transition of a Rindler electron in an initial state "a" to a final state "b" independently of the emited possitive energy Minkowski photon per unit proper time per unit perpendicular area is given by

$$F^+_{a\to b} =$$

$$\int_{-\infty}^{\infty} d(\tau-\tau') \int_{-\infty}^{\infty} d(x-x') \int_{-\infty}^{\infty} d(y-y') \int_0^\infty d\xi\xi \int_0^\infty d\xi'\xi' j^\mu_{ab}(x)\mathcal{D}^+_{\mu\nu}(x,x')j^\nu_{ba}(x')$$

The transition current is

$$j^\mu_{ab}(x) = \Phi_a^\dagger \gamma^0\gamma^\mu\Phi_b.$$

By direct substitution of the expressions given above one obtains

$$j^\mu_{ab}(x)g_{\mu\nu}(x,x')j^\nu_{ba}(x') =$$

$$(\xi\xi' j^\tau_{ab}j^{\tau'}_{ba} - j^\xi_{ab}j^{\xi'}_{ba})cosh(\tau-\tau') + (\xi' j^\xi_{ab}j^{\tau'}_{ba} - \xi j^\tau_{ab}j^{\xi'}_{ba})sinh(\tau-\tau') - j^x_{ab}j^x_{ba} - j^y_{ab}j^y_{ba}$$

$$= (j^0_{ab} j^{0'}_{ba} - j^3_{ab} j^{3'}_{ba}) cosh(\tau - \tau') + (j^3_{ab} j^{0'}_{ba} - j^0_{ab} j^{3'}_{ba}) sinh(\tau - \tau')$$

$$- j^2_{ab} j^2_{ba} - j^1_{ab} j^1_{ba}$$

Notice that the vectorial character of the electromagnetic field is responsable of the presence of the hyperbolic sine and cosine factors and gives rise to the presence of two different kinds of integrals with respect to the proper time difference:

$$I_1(x - x', y - y', \xi, \xi') = \frac{1}{2\xi\xi'} \int_0^\infty \frac{e^{i(\epsilon - \epsilon')(\tau - \tau')} d(\tau - \tau')}{cosh(\tau - \tau' - i\eta) - cosh\lambda - r^2}$$

$$I_2(x - x', y - y', \xi, \xi') = \frac{1}{2\xi\xi'} \int_0^\infty \frac{e^{i(\epsilon - \epsilon')(\tau - \tau')} e^{(\tau - \tau')} d(\tau - \tau')}{cosh(\tau - \tau' - i\eta) - cosh\lambda - r^2}$$

$$\lambda = log(\frac{\xi}{\xi'}); r = \frac{(x - x')^2 + (y - y')^2}{2\xi\xi'}$$

The first integral is similar to that appearing when studying the interaction of two scalar fields and is proportional to a thermal like factor

$$I_1(x - x', y - y', \xi, \xi') = [\frac{1}{1 - e^{2\pi|\epsilon - \epsilon'|}}] \frac{1}{4\pi\xi\xi'} \frac{sen(2|\epsilon - \epsilon'|\Theta_{\lambda,r})}{[senh^2(\lambda/2) + r^2/2]^{\frac{1}{2}} [1 + senh^2(\lambda/2) + r^2]^{\frac{1}{2}}}$$

with

$$\Theta_{\lambda,r} = sinh^{-1}([senh^2(\lambda/2) + r^2/2]^{\frac{1}{2}})$$

The second integral is not divergent and can be worked out as

$$I_2(x - x', y - y', \xi, \xi') = \frac{1}{2\xi\xi'} [2\delta^+(\epsilon - \epsilon') - 2 \int_0^\infty d\sigma \frac{isen((\epsilon - \epsilon')\sigma) e^{-\sigma}}{cosh\sigma - cosh\lambda - r^2 - i\eta}$$
$$+ 2 \int_0^\infty d\sigma \frac{[cosh\lambda + r^2] e^{i\omega\sigma}}{cosh\sigma - cosh\lambda - r^2 - i\eta}]$$

Let us define

$$I_c(x - x', y - y', \xi, \xi') = \frac{1}{2\xi\xi'} \int_0^\infty \frac{e^{i(\epsilon - \epsilon')(\tau - \tau')} cosh(\tau - \tau') d(\tau - \tau')}{cosh(\tau - \tau' - i\eta) - cosh\lambda - r^2}$$

$$I_s(x-x',y-y',\xi,\xi') = \frac{1}{2\xi\xi'}\int_0^\infty \frac{e^{i(\epsilon-\epsilon')(\tau-\tau')}sinh(\tau-\tau')d(\tau-\tau')}{cosh(\tau-\tau'-i\eta)-cosh\lambda-r^2}$$

It results that

$$I_c(x-x',y-y',\xi,\xi') = \frac{1}{\xi\xi'}\delta(\omega)+(cosh\lambda+r^2)I_1(x-x',y-y',\xi,\xi')$$

while, it can be shown that

$$\begin{aligned}
I_s(x-x',y-y',\xi,\xi') =& \frac{1}{\xi\xi'}\Big[\mathcal{P}(\frac{1}{\omega})-\int_0^\infty \frac{sin\omega\sigma e^{-\sigma}d\sigma}{cosh(\sigma-i\eta)-cosh\lambda-r^2} \\
&+\int_0^\infty \frac{sin\omega\sigma(cosh\lambda+r^2)d\sigma}{cosh(\sigma-i\eta)-cosh\lambda-r^2}\Big] \\
=& \frac{1}{\xi\xi'}\Big[\mathcal{P}(\frac{1}{\omega})-\frac{1}{1-e^{-2\pi\omega}}\big[\int_0^{2\pi}\frac{sinxe^{\omega x}}{cosx-cosh\lambda-r^2}- \\
&\frac{cos(\omega\Theta_{\lambda,r})[e^{-\Theta_{\lambda,r}}+[cosh\lambda+r^2]]}{4\pi[senh^2(\lambda/2)+r^2/2]^{\frac{1}{2}}[1+senh^2(\lambda/2)+r^2/2]^{\frac{1}{2}}}\big]\Big]
\end{aligned}$$

Let us focus our attention on the first terms of these equations *i.e.* the δ and principal value terms. They depend on the spacelike variables just through the $1/\xi\xi'$ factor which cancels with the $\sqrt{g}$ factors appearing in $F^+_{a\to b}$. Thus, even in the presence of an external magnetic field in the acceleration direction, the space integration of the j^0 factors is given directly by the orthonormalization conditions.The net result for a free particle case is that the contribution of δ factor is of the form

$$\delta(\epsilon-\epsilon')\delta^{(2)}(\vec{k}_\perp-\vec{k}'_\perp)\delta_{s,s'}\mathcal{G}$$

where $\mathcal{G}$ is a divergent factor given by the product of $\delta(\epsilon-\epsilon')$ due to the continuum normalization, while the contribution of the principal value term is zero. Therefore, contrary to other detector-quantum field systems [1,4], the probability of emission of Minkowski photons without changes in the electron quantum numbers is different from zero to first order in perturbation theory. This phenomenum is analogous to the approximate classical result of nonradiative reaction force for uniformily accelerated charges. It is also similar to that obtained by De Witt [8] who describes a monopole detector which also emits Minkowski particles when no detector transition occurs.

Notice that this probability is asociated with the $D_{\mu\nu}, \mu, \nu = \tau, \xi$ terms of the photon Wightman functions. Thus the emited Minkowski photons would be described in terms of electromagnetic potentials with zeroth or third component different from zero. Classically, it is known [9] that there is a singularity in the electromagnetic field produced by a classical uniformily accelerated particle. This singularity manifest like a *delta* restricted to the null surface z + t = 0 and it is the "original Lorentz transformed Coulomb field of the charge 'before' it started its acceleration". This field has just components $F^{t\vec{r}_\perp}$ and $F^{z\vec{r}_\perp}$ as the electromagnetic field one expect to be produced from the A_0 and A_ξ quantum fields.

The integration over the spacelike variables associated with the I_1 cannot be written in closed form in general. However, it has a different behaviour from that expected for the corresponding integrals appearing in I_2. While the former are well defined for any well behaved wavefunction, the space integration of I_2 is well defined just when the wavefunction is localized in $X - Y$ plane. This could be achieved for instance by the application of a magnetic field parallel to the acceleration direction. In this case we can make a multipole expansion and obtain some general results about the response function. The second possibility corresponds to a free Rindler Dirac electron. In that case, a multipole expansion is not well justified and thus, it is necessary to apply different approximation technics. The results are quite complicated and will be reported elsewhere. However there is a particular case of physical interest for which we can give a simple and closed expression. This is the case of spin flip without change in the perpendicular momentum $\vec{k}_\perp$ for which the transition probability is completely given in terms of I_1 (see the appendix) with the result

$$F(\epsilon, \vec{k}_\perp, s_3, \epsilon', \vec{k}_\perp, s'_3 = -s_3) =$$

$$\frac{3}{16\pi^2} \frac{1}{1 - e^{-2\pi|\epsilon-\epsilon'|}} \frac{1}{|\epsilon - \epsilon'|} \frac{1}{|2 + i|\epsilon - \epsilon'||^2} \frac{k_\perp + m}{k_\perp^5} [|1 - k_+ k_+|^2 (\epsilon + \epsilon') + |1 + k_+ k_+|^2$$

where

$$k_{\pm} = \frac{k_x \pm ik_y}{k_{\perp} + m}$$

Notice that even in this simple case, the dependence on the difference of energies is not completely contained in the thermal like factor. However, transitions with small changes in the Rindler energy are the most probable ones. Once again, soft Rindler photons are dominant.

DISCUSSION

The most important result of this work is the recognition that a Rindler Dirac free electron is highly stable and therefore has a high probability of conserving its original quantum numbers even when it couples to the vacuum electromagnetic field. This behavior is manifest not only in the presence of a $\delta(\epsilon-\epsilon')$ term, but it already present in the thermal-like terms. Thus the Minkowski photons emited by the electron are mainly a superposition of low frequency Rindler photons.

Although I have reported here mainly the results for a free Dirac-Rindler electron, some properties of this transition amplitudes, such as the δ term, are directly related to the quantum electromagnetic field. However, for a given detector the importance of each term may be different. This was illustrated by the calculation of the probability of spin flip without changes in the perpendicular momentum where just the thermal-like modulated factor is present.

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APPENDIX

In the free particle case the Rindler-Dirac bispinors are proportional to the matrix

$$\Upsilon_{\pm\vec{k}_\perp} = \begin{pmatrix} \Upsilon_\pm \\ \pm i\Upsilon_\pm \end{pmatrix}$$

where the $\pm$ signs determine the sign of spin component S_3. When calculating transition probabilities which involve spin flip it results convinient to define the algebraic coefficients

$$d^{\mu\nu}_{\sigma\sigma'} = \Upsilon^\dagger_{s_3\vec{k}_\perp} \mathbf{P}_\sigma \gamma^0 \gamma^\mu \Upsilon^{-s_3\vec{k}'_\perp} \Upsilon^\dagger_{-s_3\vec{k}'_\perp} \mathbf{P}_{\sigma'} \gamma^0 \gamma^\nu \Upsilon_{s_3\vec{k}_\perp}.$$

The explicit values of these coefficients are

$$d^{00}_{++} = d^{00}_{--} = -d^{00}_{+-} = -d^{00}_{-+} = |k'_+ - k_+|^2$$

$$d^{11}_{++} = d^{11}_{--} = d^{11}_{+-} = d^{11}_{-+} = |1 - k'_+ k_+|^2$$

$$d^{22}_{++} = d^{22}_{--} = d^{22}_{+-} = d^{22}_{-+} = |1 + k'_+ k_+|^2$$

$$d^{33}_{++} = d^{33}_{--} = d^{33}_{+-} = d^{33}_{-+} = |k'_+ - k_+|^2$$

$$d^{03}_{++} = -d^{03}_{--} = d^{03}_{+-} = -d^{03}_{-+} = |k'_+ - k_+|^2$$

$$d^{30}_{\sigma\sigma'} = d^{03}_{\sigma\sigma'}$$

where

$$k_\pm = \frac{k_x \pm i k_y}{k_\perp + m}$$

If the perpendicular momentum does not change during the transition $d^{\mu\nu}, \mu, \nu = 0, 3$ is zero. Thus, they do not contribute to spin flip.

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Inertial and accelerated particle detectors with back-reaction in flat space-time

F. Hinterleitner

Institut für theoretische Physik der Universität Wien, Boltzmanngasse 5, A-1090 Wien, Austria

An inertial and a uniformly accelerated harmonic oscillator are coupled to a scalar, massless field as models for particle detectors. The behaviour of these detectors in several Minkowskian and Rindler quantum states and the back-reaction to the field are investigated. From the Bogolubov transformation between in- and outgoing particle states it is seen that almost everywhere in space-time neither the inertial nor the accelerated model gives rise to an energy flux from the detector to the field or energy absorption from the field by the detector, although the accelerated one behaves as if it were immersed in a thermal state of the field.

1 Introduction

In the area of quantum field theory in non-inertial frames there has been recently some revival of the discussion about the radiation of uniformly accelerated objects [1,2,3]. There seems to be some controversy whether or not uniformly accelerated model detectors do radiate. As very different tools like the reduction of wave packets, stochastic techniques, or different notions of particles are involved, an answer to the question of radiation is non-trivial.

The approach to these problems chosen in this paper is - in principle - very simple, as only a scalar, massless field in two-dimensional flat space-time and a harmonic oscillator are employed. The concept of particles is confined to the convential one, defined by the notion of positive and negative frequency field modes. As frequencies of field modes can be defined with respect to various (inertial or non-inertial) global time coordinates - here "global" refers also (and especially) to open submanifolds of the whole Minkowski space-time - there are different formal concepts of "particles", if no further decisions are made, e.g. singling out the quantum field theory basing on Minkowski modes by virtue of the translation symmetry of its vacuum state, a symmetry which the other vacua do not possess [4].

An example of a non-inertial frame admitting a global space - time decomposition is provided by the Rindler coordinates in the two domains $x < -|t|$ and $x > |t|$ of flat space-time, where t and x are the usual Minkowski coordinates.

The canonical quantization of a scalar field in Rindler space-time was first studied by Fulling [5]; in the Fulling-Rindler quantum field theory the Minkowski vacuum looks like a thermal state. Detector models of Unruh and Wald [1,2] which are at rest in Rindler space - that means, they are uniformly accelerated - "measure" a temperature T in Minkowski vacuum, given by the Unruh-Davies formula

$$k_B T = \frac{\hbar}{c} \frac{a}{2\pi}, \tag{1}$$

where k_B is the Boltzmann constant and a is the acceleration of the detector. Thus, these models confirm the predictions of Fulling-Rindler quantum field theory.

In contrast to these examples, which are calculated in first order perturbation theory, an exactly soluble model of a detector in two-dimensional space-time will be considered here, including its back-reaction to the field (eventual bremsstrahlung). As the two-dimensional model does not allow for transversal motions, the acceleration of the detector is always parallel or antiparallel to the direction of propagation of the considered field. So restriction to two dimensions has the advantage to make the physical effects of acceleration easier to calculate and to understand. The crucial point will be whether or not the (uniformly) accelerated detector itself generates the particles "seen" in Minkowski vacuum. Padmanabhan argues in this sense, especially for nonuniformly accelerated detectors [6,7].

In section 2 the detector properties of a harmonic oscillator at rest in two-dimensional flat space-time are considered. The equations of motion of the coupled system are derived from a Lagrangian and solved; afterwards the system is quantized.

In section 3 a similar Lagrangian is formulated for the case of a uniformly accelerated harmonic oscillator, which behaves in the same way towards the quanta of the Fulling-Rindler quantum field theory as the inertial one does towards Minkowski quanta.

In section 4 the behaviour of the uniformly accelerated detector in Minkowski quantum states is studied. In the vacuum the result of Unruh and Davies (eq. (1)) is reproduced.

At last, in section 5, considerations about the Fulling-Rindler quantum states are made by a look at the behaviour of the inertial detector in Rindler vacuum.

2 A model of an inertial detector

2.1 Classical treatment

The coupling of the system is supposed to be described by the following Lagrangian:

$$L(t) = \frac{1}{2}\int_{-\infty}^{\infty} dx \left\{ \left(\frac{\partial \Phi}{\partial t}\right)^2 - \left(\frac{\partial \Phi}{\partial x}\right)^2 + 2\lambda\rho Q \Phi \right\} + \frac{1}{2}\left[\left(\frac{dQ}{dt}\right)^2 - \omega^2 Q^2\right]. \tag{2}$$

$\Phi(t,x)$ ... scalar field,

$Q(t)$ ... elongation of the harmonic oscillator, some "inner" degree of freedom,

λ ... coupling constant,

$\rho(x)$ ... interaction density, describes the spatial extension of the detector. It is taken to be an odd function of x to guarantee finite total energy.

The two-dimensional metric is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Variation of the action integral $\int_{-\infty}^{\infty} L(t)dt$ yields the coupled equations of motion:

$$\frac{d^2Q(t)}{dt^2} + \omega^2 Q(t) = \lambda \int_{-\infty}^{\infty} dx\, \rho(x)\Phi(t,x), \tag{3}$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\Phi(t,x) = \lambda\rho(x)Q(t). \tag{4}$$

This kind of model and its solution were established by Aichelburg-Beig [8] and Schwabl-Thirring [9]. The solutions can be given with the aid of retarded or advanced Green's functions of the differential operators on the left hand side of (3) and (4) in terms of ingoing or outgoing variables, respectively.

As the harmonic oscillator will be used as a model for a detector which is to react to the influences of the field acting on it, it is reasonable to assume $Q^{in} = 0$. In [8] it is shown in detail that in the solution of an analogous 4-dimensional problem, calculated from initial values of Q, $\dot{Q}$, Φ, $\dot{\Phi}$ at some time $t = t_0$, $Q(t)$ always vanishes in the asymptotic limits $t \to \pm\infty$ if the energy stored in the field at $t = t_0$ is finite. This calculation can be adapted to 2 dimensions to give the same result.

For solving (3) and (4) it is convenient to go to Fourier space, where the variables are defined by

$$Q(k_0) := \int_{-\infty}^{\infty} dt\, e^{ik_0 t} Q(t)$$

$$\Phi(k_0,k) := \int_{-\infty}^{\infty} dt\, dx\, e^{ik_0 t - ikx}\Phi(t,x)$$

$$\rho(k) := \int_{-\infty}^{\infty} dx\, e^{-ikx}\rho(x), \qquad \rho(-k) = -\rho(k) = \rho^*(k).$$

$\Phi(k_0,k)$ is given by a solution of the homogenous equation obtained from (4), $\Phi^{in}(k_0,k)$, plus the retarded solution of the inhomogenous equation. $Q(k_0)$ does not contain a homogenous term, as Q^{in} was assumed to be zero.

$$\Phi(k_0,k) = \Phi^{in}(k_0,k) + \lim_{\varepsilon\to 0}\frac{\lambda\,\rho(k)}{k^2 - (k_0 + i\varepsilon)^2} Q(k_0), \tag{5}$$

$$Q(k_0) = \lim_{\varepsilon\to 0}\frac{\lambda}{2\pi\left(\omega^2 - (k_0 + i\varepsilon)^2\right)}\int_{-\infty}^{\infty} dk\, \rho(k)\Phi(k_0,k). \tag{6}$$

Inserting (6) into (5) yields a Fredholm integral equation of the second kind with a degenerate kernel,

$$\Phi(k_0,k) = \Phi^{in}(k_0,k) + \lim_{\varepsilon\to 0}\frac{\lambda^2}{2\pi}\frac{1}{\omega^2 - (k_0 + i\varepsilon)^2}\int_{-\infty}^{\infty} dk'\, \frac{\rho(k)\,\rho(k')}{k^2 - (k_0 + i\varepsilon)^2}\,\Phi(k_0,k'). \tag{7}$$

Its solution can be given in closed form:

$$\Phi(k_0,k) = \Phi^{in}(k_0,k) + \frac{\lambda^2}{2\pi}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{dk'}{k^2-(k_0+i\varepsilon)^2}\,\frac{\rho(k)\,\rho(-k')\,\Phi^{in}(k_0,k')}{\omega^2-(k_0+i\varepsilon)^2-\frac{\lambda^2}{2\pi}\int_{-\infty}^{\infty}\frac{dq\,|\rho(q)|^2}{q^2-(k_0+i\varepsilon)^2}} \tag{8}$$

With the advanced Green's functions, which are complex conjugate to the retarded ones, one gets the corresponding relations between the interacting system and the outgoing field.

In view of a later quantization it is convenient to decompose the ingoing field variables into their respective positive and negative frequency parts:

$$\Phi^{in}(t,x) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{|k|}}\left\{a^{in}(k)e^{ikx-i|k|t} + a^{in*}(k)e^{-ikx+i|k|t}\right\}. \tag{9}$$

For a real, classical field $a^{in}(k)$ and $a^{in*}(k)$ are complex conjugate functions. Inserting

$$\Phi^{in}(k_0,k) = 2\sqrt{\frac{\pi^3}{|k|}}\left\{a^{in}(k)\delta(k_0-|k|) + a^{in*}(-k)\delta(k_0+|k|)\right\}$$

into (8) one obtains $\Phi(k_0,k)$ in terms of the ingoing field coefficients.

If $\rho(x)$ is an odd function, only the parts of the field antisymmetric in the momentum space variable k interact with the detector, because of $\int_{-\infty}^{\infty}dk'\,\rho(-k')\Phi^{in}(k_0,k')$ in (8). Therefore it is convenient to split the free field coefficients in the following manner:

$$a^{in/out}_{s,a}(k) := \frac{1}{\sqrt{2}}[a^{in/out}(k) \pm a^{in/out}(-k)]. \tag{10}$$

This will simplify the further calculations. a^{in}_s, a^{in}_a will be referred to as symmetric, respectively antisymmetric (classical) field variables; k can now be assumed to be positive. Inserting $\Phi^{in}(k_0,k)$ into (8) and taking into account this decomposition we arive at

$$\begin{aligned}\Phi(k_0,k) =& \sqrt{\frac{2\pi^3}{k}}a^{in}_s(k)\delta(k_0-k) + \sqrt{\frac{2\pi^3}{k}}a^{in*}_s(k)\delta(k_0+k) + \\ &+\sqrt{\frac{2\pi^3}{k}}a^{in}_a(k)\delta(k_0-k) - \sqrt{\frac{2\pi^3}{k}}a^{in*}_a(k)\delta(k_0+k) + \\ &+\sqrt{2\pi}\lambda^2\lim_{\varepsilon\to 0}\int_0^{\infty}\frac{dk'}{\sqrt{k'}}\frac{\rho(k)\rho(-k')a^{in}_a(k')\delta(k_0-k)}{(k^2-k_0^2-i\varepsilon)\,D_+(k_0)} - \\ &-\sqrt{2\pi}\lambda^2\lim_{\varepsilon\to 0}\int_0^{\infty}\frac{dk'}{\sqrt{k'}}\frac{\rho(k)\rho(-k')a^{in*}_a(k')\delta(k_0+k')}{(k^2-k_0^2+i\varepsilon)\,D_-(k_0)}\end{aligned} \tag{11}$$

with

$$D_{\pm}(k) = \omega^2 - k^2 + \frac{\lambda^2}{\pi}\int_0^{\infty}\frac{dk'\,|\rho(k')|^2}{k^2-k'^2} \mp \frac{i\lambda^2}{2|k|}|\rho(k)|^2. \tag{12}$$

Now the relations

$$\lim_{\varepsilon\to 0}\frac{1}{x\pm i\varepsilon} = P\left(\frac{1}{x}\right) \mp i\pi\delta(x), \tag{13}$$

P denoting the Cauchy principal part and

$$\delta(k^2 - k_0^2) = \frac{1}{2|k|}(\delta(k - k_0) + \delta(k + k_0))$$

is applied to (11):

$$\begin{aligned}
\Phi(k_0, k) = & \sqrt{\frac{2\pi^3}{k}}\delta(k_0 - k)\mathbf{a}_s^{in}(k) + \sqrt{\frac{2\pi^3}{k}}\delta(k_0 + k)\mathbf{a}_s^{in*}(k) + \\
& + \sqrt{\frac{2\pi}{k}} D_0(k)\delta(k_0 - k)\frac{\mathbf{a}_a^{in}(k)}{D_+(k)} - \\
& - \sqrt{2\pi}\lambda^2\rho(k)\int_0^\infty \frac{dk'}{\sqrt{k'}}\frac{\rho(k')}{k^2 - k'^2}\delta(k_0 - k')\frac{\mathbf{a}_a^{in}(k')}{D_+(k')} - \\
& - \sqrt{\frac{2\pi^3}{k}} D_0(k)\delta(k_0 + k)\frac{\mathbf{a}_a^{in*}(k')}{D_-(k)} + \\
& + \sqrt{2\pi}\lambda^2\rho(k)\int_0^\infty \frac{dk'}{\sqrt{k'}}\frac{\rho(k')}{k^2 + k'^2}\delta(k_0 + k')\frac{\mathbf{a}_a^{in*}(k')}{D_-(k')},
\end{aligned} \tag{14}$$

D_0 denoting the real part of $D_\pm$.

Q may be obtained from (6) or the original equation (4):

$$Q(t) = -\frac{\lambda}{\sqrt{2\pi}}\int_0^\infty \frac{dk}{\sqrt{k}}\frac{\rho(k)}{D_+(k)}\mathbf{a}_a^{in}(k)e^{-i|k|t} + \frac{\lambda}{\sqrt{2\pi}}\int_0^\infty \frac{dk}{\sqrt{k}}\frac{\rho(k)}{D_-(k)}\mathbf{a}_a^{in*}(k)e^{i|k|t}. \tag{15}$$

The corresponding expressions in terms of outgoing field coefficients are obtained by the the exchange $D_+ \leftrightarrow D_-$, which comes from employing the advanced Green's functions rather than the retarded ones. Comparation of the retarded and the advanced solution leads to the relations between in- and outgoing field coefficients,

$$\mathbf{a}_s^{out}(k) = \mathbf{a}_s^{in}(k), \qquad \mathbf{a}_a^{out}(k) = \frac{D_-(k)}{D_+(k)}\mathbf{a}_a^{in}(k). \tag{16}$$

The transformation of the symmetrized variables is trivial, also the antisymmetric $\mathbf{a}$'s are not mixed with the $\mathbf{a}^*$'s, they are only endowed with a phase factor. – Ingoing waves are are merely phase-shifted and transmitted or scattered back by the detector.

2.2 Quantization of the system

The solutions (14,17,20) are the basis of quantization. The quantum analogs of the classical variables $\mathbf{a}^{in}$, $\mathbf{a}^{in*}$, will be denoted by $\hat{\mathbf{a}}^{in}$, $\hat{\mathbf{a}}^{in\dagger}$. They are stated to be elements of an algebra with the canonical commutation relations:

$$\begin{aligned}
&[\hat{\mathbf{a}}^{in/out}(k), \hat{\mathbf{a}}^{in/out}(k')] = [\hat{\mathbf{a}}^{in/out\dagger}(k), \hat{\mathbf{a}}^{in/out\dagger}(k')] = 0, \\
&[\hat{\mathbf{a}}^{in/out}(k), \hat{\mathbf{a}}^{in/out\dagger}(k')] = \delta(k - k').
\end{aligned} \tag{17}$$

The commutation relations for the interacting variables $Q(t)$ and $\Phi(t, x)$ and their time derivatives can be derived from those of the ingoing ones, they turn out to be canonical. If the field variables above are replaced by the symmetric and antisymmetric ones (10), k can be constrained to values ≥ 0. Then the same commutation relations as the original ones (17) are valid for $\hat{a}_s^{in}(k)$, $\hat{a}_a^{in}(k)$,...

The representation space of the field algebra is the conventional Fock space of the ingoing free field. The ingoing vacuum is defined by

$$\hat{a}^{in}(k) \mid 0 \rangle^{in} = 0 \ \forall k \quad \text{und} \quad {}^{in}\langle 0 \mid 0 \rangle^{in} = 1. \tag{18}$$

One can define two orthogonal Fock (sub-)spaces of ingoing states, spanned by n-particle states which are obtained by application of general symmetrized or antisymmetrized creation operators, constructed from $\hat{a}_s^\dagger$, respectively $\hat{a}_a^\dagger$, to the ingoing vacuum,

$$\mid n \rangle_{s,a}^{in} = \int_0^\infty d^n k \, f_n(k_1, ..., k_n) \, \hat{a}_{s,a}^{in\dagger}(k_1) \dots \hat{a}_{s,a}^{in\dagger}(k_n) \mid 0 \rangle^{in}, \tag{19}$$

with the normalization

$$\int_0^\infty d^n k \, |f_n(k_1, ..., k_n)|^2 = \frac{1}{n!} \, .$$

(Note that the indices s and a of the states mean *symmetric* or *antisymmetric in the momentum space-variable k;* this should not be confused with bosonic, respectively fermionic states, all this paper is dealing only with bosonic fields!

The operator version of (16) is a "trivial" Bogolubov transformation, from which follows

$$\mid 0 \rangle^{in} = \mid 0 \rangle^{out}. \tag{20}$$

Eq. (20) shows that in the asymptotic limit the harmonic oscillator does not create or absorb particles; nor does it change the magnitude of their momentum, at most their direction. This is a desirable feature of a detector model.

The total Hamiltonian of the system,

$$\begin{aligned} H(t) = \int_{-\infty}^{\infty} dx \, \mathcal{H}(t,x) &= \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \{ |\dot{\Phi}(t,k)|^2 + k^2 |\Phi(t,k)|^2 + \lambda \rho(k) Q(t) \Phi(t,k) \\ &+ \lambda \rho(k) \Phi(t,k) Q(t) \} + \frac{1}{2} [\dot{Q}^2(t) + \omega^2 Q^2(t)] = \int_{-\infty}^{\infty} dk \, |k| \, \hat{a}^{in\dagger}(k) \hat{a}^{in}(k) + const, \end{aligned} \tag{21}$$

is equal to the Hamiltonian of the ingoing field plus a constant consisting of the (infinite) vacuum energy of the field, the ground state energy of the harmonic oscillator, and the negative interaction energy (which is finite, if $\rho(x)$ is odd). Its (stationary) eigenstates, the physical states of the system, are the normalizeable ingoing states.

2.3 The harmonic oscillator as a particle detector

When used as a particle detector, the harmonic oscillator is coupled to the field. For a measuring process, it is decoupled at some later instant t; afterwards it will be found in one of its energy eigenstates, which are clearly not eigenstates of the total Hamiltonian. The probabilities of the different excitation states depend on the ingoing state of the field and the instant of measurement. According to these probabilities the energy expectation value of the detector varies with time. The time-dependent Hamiltonian of the detector is

$$
\begin{aligned}
H_D(t) = \frac{1}{2}[\dot{Q}^2(t) + \omega^2 Q^2(t)] = & \\
-\frac{\lambda^2}{2\pi}\int_0^\infty \frac{dkdk'}{\sqrt{kk'}}(\omega^2 + kk')\frac{\rho(k)\rho(k')}{D_-(k)D_+(k')}\hat{a}_a^{in\dagger}(k)\hat{a}_a^{in}(k')e^{i(k-k')t} & \\
+\frac{\lambda^2}{4\pi}\int_0^\infty \frac{dkdk'}{\sqrt{kk'}}(\omega^2 - kk')\frac{\rho(k)\rho(k')}{D_+(k)D_+(k')}\hat{a}_a^{in}(k)\hat{a}_a^{in}(k')e^{-i(k+k')t} & \\
+\frac{\lambda^2}{4\pi}\int_0^\infty \frac{dkdk'}{\sqrt{kk'}}(\omega^2 - kk')\frac{\rho(k)\rho(k')}{D_-(k)D_-(k')}\hat{a}_a^{in\dagger}(k)\hat{a}_a^{in\dagger}(k')e^{i(k+k')t} + E_0; &
\end{aligned}
\tag{22}
$$

the first integral has a minus sign, because $\rho(k)$ is imaginary, and

$$
E_0 = {}^{in}\langle\, 0 \mid H_D(t) \mid 0\,\rangle^{in} = \frac{\lambda^2}{4\pi}\int_0^\infty \frac{dk}{k}(\omega^2 + k^2)\frac{|\rho(k)|^2}{D_+(k)D_-(k)}\,. \tag{23}
$$

$\rho(0) = 0$ and $\lim_{k\to\infty}\rho(k) = 0$ are essential for the integral to converge.

Having one ingoing particle, one gets

$$
\begin{aligned}
{}^{in}\langle\, 1 \mid H_D(t) \mid 1\,\rangle^{in} = E_0 + \frac{\lambda^2\omega^2}{2\pi}\left|\int_0^\infty \frac{dk}{\sqrt{k}}\frac{\rho(k)}{D_+(k)}f(k)e^{-ikt}\right|^2 & \\
+\frac{\lambda^2}{2\pi}\left|\int_0^\infty dk\sqrt{k}\frac{\rho(k)}{D_+(k)}f(k)e^{-ikt}\right|^2 . &
\end{aligned}
\tag{24}
$$

The vacuum expectation value is enlarged by two positive definite expressions, similar terms are added in other igoing particle states. From that it can be seen that the harmonic oscillator has its lowest energy expectation value in the ingoing vacuum, an essential detector property. If $f(k) \neq \delta(k-k')$ (ingoing momentum eigenstate), then in the limits $t \to \pm\infty$ the additional terms, which are due to temporary absorption of field quanta, tend to zero.

If one takes an ingoing thermal state with temperature T, given by a density operator

$$
\varrho_T = \prod_k \left(1 - e^{-\beta k}\right)\sum_{n_k} e^{-n_k\beta|k|} \mid n_k\,\rangle^{in}\, {}^{in}\langle\, n_k \mid, \tag{25}
$$

$\beta = 1/k_B T$,

$\mid n_k\,\rangle^{in}$ ... state with n_k ingoing particles with momentum k,

one gets a time-independent energy expectation value

$$
\mathrm{Tr}(\varrho_T H_D(t)) = E_0 + \frac{\lambda^2}{2\pi}\int_0^\infty \frac{dk}{k}\frac{|\rho(k)|^2}{D_+(k)D_-(k)}(\omega^2 + k^2)\frac{1}{e^{\beta k} - 1}\,. \tag{26}
$$

3 The uniformly accelerated detector

3.1 The classical system

Rindler coordinates τ, ζ with the distance between infinitesimally seperated points in two-dimensional space-time given by

$$ds^2 = \zeta^2 d\tau^2 - d\zeta^2 \tag{27}$$

are comoving with uniformly accelerated objects in two-dimensional flat space-time. The metric is static, the Killing field $\partial/\partial\tau$ is future-directed in the domain $x > |t|$. Here the coordinates transform in the following way:

$$t = \zeta \mathrm{sh}\tau, \quad x = \zeta \mathrm{ch}\tau; \qquad \zeta = {}_{+}\sqrt{x^2 - t^2}, \quad \tau = \mathrm{arth}\frac{t}{x}. \tag{28}$$

The uniformly accelerated detector is supposed to be at rest in Rindler coordinates at ζ_0. For its proper time one has $ds^2 = \zeta_0^2\, d\tau^2 \;\Rightarrow s = \zeta_0\tau$, thus the Lagrangian of the oscillator, harmonic in its proper time, is

$$L_Q(s) = \frac{1}{2}\left[\left(\frac{dQ(s)}{ds}\right)^2 - \omega^2 Q^2(s)\right] = \frac{1}{2}\left[\frac{1}{\zeta_0^2}\left(\frac{dQ(\tau)}{d\tau}\right)^2 - \omega^2 Q^2(\tau)\right]. \tag{29}$$

The Lagrange density of the massless scalar field can be transformed covariantly to arbitrary coordinates:

$$\mathcal{L}_\Phi(\tau,\zeta) = \frac{1}{2}\sqrt{-g}\,\partial_\mu\Phi\,\partial^\mu\Phi, \tag{30}$$

$g = -\zeta^2$ is the determinant of the metric.

In the inertial case the spatial extension of the detector is described by a function $\rho(x)$. Supposing that a detector behaves like a rigid body under uniform acceleration, it will be described by a function $\rho(\zeta,\zeta_0)$ in Rindler coordinates. ζ^0 denotes one distinguished point of the detector, the proper time of which is taken as the proper time of the whole detector. The resulting total Lagrangian is

$$\begin{aligned} L(\tau) = {} & \frac{1}{2}\int_0^\infty d\zeta \left\{ \frac{1}{\zeta}\left(\frac{\partial\Phi(\tau,\zeta)}{\partial\tau}\right)^2 - \zeta\left(\frac{\partial\Phi(\tau,\zeta)}{\partial\zeta}\right)^2 + 2\lambda\frac{1}{\zeta}\rho(\zeta,\zeta_0)\Phi(\tau,\zeta)Q(\tau)\right\} \\ & + \frac{1}{2}\left[\frac{1}{\zeta_0^2}\left(\frac{dQ(\tau)}{d\tau}\right)^2 - \omega^2 Q^2(\tau)\right], \end{aligned} \tag{31}$$

and the equations of motion are

$$\begin{aligned} \left(\frac{\partial^2}{\partial\tau^2} - \zeta\frac{\partial}{\partial\zeta}\zeta\frac{\partial}{\partial\zeta}\right)\Phi(\tau,\zeta) &= \lambda\rho(\zeta,\zeta_0)Q(\tau), \\ \frac{1}{\zeta_0^2}\frac{d^2Q(\tau)}{d\tau^2} + \omega^2 Q(\tau) &= \lambda\int_0^\infty \frac{d\zeta}{\zeta}\rho(\zeta,\zeta_0)\Phi(\tau,\zeta). \end{aligned} \tag{32}$$

Before solving these equations it is of advantage to consider the mode decomposition of the field in Rindler coordinates, which is the basis for Fulling-Rindler quantization,

$$\Phi^{\mathrm{in}}(\tau,\zeta) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{d\kappa}{\sqrt{|\kappa|}}\left\{\mathrm{b}^{\mathrm{in}}(\kappa)\zeta^{i\kappa}e^{-i|\kappa|\tau} + \mathrm{b}^{\mathrm{in}*}(\kappa)\zeta^{-i\kappa}e^{i|\kappa|\tau}\right\}. \tag{33}$$

The logarithm of ζ plays formally the same role as the Minkowski coordinate x in (9). For this reason, $\rho(\zeta,\zeta_0)$ is assumed to be odd in $(\ln\zeta - \ln\zeta_0)$. The spatial Fourier transform is replaced by the following integral tranformations to the κ-space:

$$\rho(\kappa,\zeta_0) := \int_0^\infty \frac{d\zeta}{\zeta}\zeta^{-i\kappa}\rho(\zeta,\zeta_0), \quad \rho(\zeta,\zeta_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\kappa\, \zeta^{i\kappa}\rho(\kappa,\zeta_0). \tag{34}$$

It follows that

$$\rho(-\kappa,\zeta_0) = -\zeta_0^{2i\kappa}\rho(\kappa,\zeta_0).$$

The decomposition of the ingoing field coefficients into interacting ("antisymmetric") and undisturbed ("symmetric") parts corresponds to this property of ρ:

$$\mathrm{b}^{in}_{s,a}(\kappa) := \frac{1}{\sqrt{2}}[\mathrm{b}^{in}(\kappa) \pm \zeta_0^{-2i\kappa}\mathrm{b}^{in}(-\kappa)]. \tag{35}$$

The frequency of the harmonic oscillator corresponds to its proper time $\zeta_0\tau$, therefore $Q(\tau)$ must contain the functions $\exp(\pm i\omega\zeta_0\tau)$.

The functions $D_\pm$ (12) are changed to

$$D_\pm(\kappa) = (\zeta_0\omega)^2 - \kappa^2 + \frac{(\lambda\zeta_0)^2}{\pi}\int_0^\infty \frac{d\kappa'\,|\rho(\kappa')|^2}{\kappa^2-\kappa'^2} \mp i\frac{(\lambda\zeta_0)^2}{2|\kappa|}|\rho(\kappa)|^2 \tag{36}$$

in the accelerated case.

Now all the formal requirements for a solution corresponding to that of the inertial detector are given (it will be formally very similar to the former one (14,15)); the asymptotic variables obey the equations:

$$\mathrm{b}^{out}_s(\kappa) = \mathrm{b}^{in}_s(\kappa), \qquad \mathrm{b}^{out}_a(\kappa) = \frac{D_-(\kappa)}{D_+(\kappa)}\mathrm{b}^{in}_a(\kappa). \tag{37}$$

3.2 Quantization

Quantization is achieved by postulating canonical commutation relations like (17) for $\mathrm{b}^{in}(\kappa)$ and $\mathrm{b}^{in\dagger}$ and construction of Fock spaces for the ingoing field. An ingoing n-quanta Rindler state will be denoted by $|\ n\ \rangle^{in}_R$. The Fulling-Rindler quanta defined here are built up from waves of the form $\zeta^{i\kappa}e^{-i|\kappa|\tau}$ and play the same role as Minkowski quanta in the usual quantum field theory with the field coupled to an inertial detector. In particular one has

$$|\,0\,\rangle^{out}_R = |\,0\,\rangle^{in}_R. \tag{38}$$

The Hamiltonian of the harmonic oscillator can again be expressed by the operators of the ingoing field. The resulting statements are analogous to those of the inertial model: The energy expectation value in some Rindler state of the field is equal to a constant vacuum expectation value plus a positive definite function of time, vanishing for $\tau \to \pm\infty$. Together with (38), this shows that the uniformly accelerated model detector has the same detector properties for Fulling-Rindler quanta as the inertial one has for Minkowski quanta.

4 Minkowski states and the accelerated detector

4.1 Bogolubov transformations

The transformation of the ingoing field operators can be calculated with the aid of Φ^{in} and its time derivative at the instant $t = \tau = 0$. On this line one has

$$x = \zeta, \quad \frac{\partial}{\partial \tau} = x\frac{\partial}{\partial t}.$$

Therefore $\hat{b}^{in}$ can be written in the following manner:

$$\hat{b}^{in}(\kappa) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{dx}{x} x^{-i\kappa}\left\{\sqrt{|\kappa|}\Phi^{in}(0,x) + \frac{i}{\sqrt{|\kappa|}}\, x\, \frac{\partial \Phi^{in}(t,x)}{\partial t}\bigg|_{t=0}\right\}.$$

When Φ^{in} and $\partial\Phi^{in}/\partial t$ are expressed in the form (9) the transformation from Minkowski to Rindler field operators is as follows:

$$\hat{b}^{in}(\kappa) = \frac{\Theta(\kappa)}{2\pi}\sqrt{\kappa}\,\Gamma(-i\kappa)\int_0^\infty \frac{dk}{\sqrt{k}}\left\{\hat{a}^{in}(k)e^{\frac{\kappa\pi}{2}}k^{i\kappa} + \hat{a}^{in\dagger}(k)e^{-\frac{\kappa\pi}{2}}k^{i\kappa}\right\} \tag{39}$$
$$+\frac{\Theta(-\kappa)}{2\pi}\sqrt{-\kappa}\,\Gamma(-i\kappa)\int_{-\infty}^0 \frac{dk}{\sqrt{-k}}\left\{\hat{a}^{in}(k)e^{-\frac{\kappa\pi}{2}}(-k)^{i\kappa} + \hat{a}^{in\dagger}(k)e^{\frac{\kappa\pi}{2}}(-k)^{i\kappa}\right\}.$$

The signs of the k's and κ's coincide, left- (right-) moving Rindler modes consist only of left- (right-) moving Minkowski modes.

The transformation (39) uses the line $t = 0$, $x > 0$. The inverse transformation from Rindler to Minkowski modes is impossible without including the field outside the Rindler space considered so far. If the operator coefficients of the field in the other Rindler space ($x < -|t|$, here $\partial/\partial\tau$ is past-directed) are denoted by $\hat{c}$ and $\hat{c}^\dagger$,

$$\Phi^{in}(\tau,\zeta) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{d\kappa}{\sqrt{|\kappa|}}\left\{\hat{c}^{in}(\kappa)(-\zeta)^{i\kappa}e^{-i|\kappa|\tau} + \hat{c}^{in\dagger}(\kappa)(-\zeta)^{-i\kappa}e^{i|\kappa|\tau}\right\}, \tag{40}$$

then initial values of the field on the whole line $t = 0$ can be formulated in terms of $\hat{b}$, $\hat{b}^\dagger$, $\hat{c}$ and $\hat{c}^\dagger$, and

$$\hat{a}^{in}(k) =$$
$$\frac{\Theta(-k)}{2\pi\sqrt{-k}}\int_{-\infty}^0 d\kappa\sqrt{-\kappa}\left\{\hat{b}^{in}(\kappa)e^{-\frac{\kappa\pi}{2}}(-k)^{-i\kappa}\Gamma(i\kappa) - \hat{b}^{in\dagger}(\kappa)e^{\frac{\kappa\pi}{2}}(-k)^{i\kappa}\Gamma(-i\kappa)\right.$$
$$\left.-\hat{c}^{in}(\kappa)e^{\frac{\kappa\pi}{2}}(-k)^{-i\kappa}\Gamma(i\kappa) + \hat{c}^{in\dagger}(\kappa)e^{-\frac{\kappa\pi}{2}}(-k)^{i\kappa}\Gamma(-i\kappa)\right\}$$
$$+\frac{\Theta(k)}{2\pi\sqrt{k}}\int_0^\infty d\kappa\sqrt{\kappa}\left\{\hat{b}^{in}(\kappa)e^{\frac{\kappa\pi}{2}}k^{-i\kappa}\Gamma(i\kappa) - \hat{b}^{in\dagger}(\kappa)e^{-\frac{\kappa\pi}{2}}k^{i\kappa}\Gamma(-i\kappa)\right.$$
$$\left.-\hat{c}^{in}(\kappa)e^{-\frac{\kappa\pi}{2}}k^{-i\kappa}\Gamma(i\kappa) + \hat{c}^{in\dagger}(\kappa)e^{\frac{\kappa\pi}{2}}k^{i\kappa}\Gamma(-i\kappa)\right\}. \tag{41}$$

4.2 Behaviour of the accelerated detector in Minkowski vacuum

With the aid of the Bogolubov transformations (39) the Hamiltonian of the detector,

$$H_D(\tau) = \frac{1}{2}\left[\frac{1}{\zeta_0^2}\dot{Q}^2(\tau) + \omega^2 Q^2(\tau)\right], \tag{42}$$

can be expressed by ingoing Minkowski field operators, so that an expectation value in the Minkowski vacuum $|\,0\,\rangle^{in}$ can easily be calculated:

$$^{in}\langle\, 0 \mid H_D(\tau) \mid 0\,\rangle^{in} = \frac{(\lambda\zeta_0)^2}{2\pi}\int_0^\infty \frac{d\kappa}{\kappa}[\kappa^2 + (\omega\zeta_0)^2]\left|\frac{\rho(\kappa)}{D_+(\kappa)}\right|^2 \frac{1}{e^{2\kappa\pi}-1} + E_0,$$

$$E_0 = \frac{(\lambda\zeta_0)^2}{4\pi}\int_0^\infty \frac{d\kappa}{\kappa}[\kappa^2 + (\omega\zeta_0)^2]\left|\frac{\rho(\kappa)}{D_+(\kappa)}\right|^2. \tag{43}$$

The energy (frequency) of a field mode according to the proper time of the detector being $|\kappa|/\zeta_0$, eq. (43) is the energy expectation value in a thermal Rindler state with temperature $T = (2\pi k_B \zeta_0)^{-1}$ (compare (26)). Thus the behaviour of the detector is in accordance with the well-known fact that the Minkowski vacuum is a thermal Rindler state with a temperature proportional to the acceleration of the observer [1,2].

In his work on accelerated particle detectors [7], Padmanabhan suggests that the excitations in Minkowski vacuum should not be interpreted as absorption of Rindler quanta but rather as direct effects of the acceleration, which manifest themselves also by the emission of radiation. In addition to that, Unruh and Wald [2] argue that in first order perturbation theory the event which looks like the absorption of a Rindler quantum by a DeWitt detector model [10] for an accelerated observer is the emission of a Minkowski quantum from the point of view of an inertial observer.

In order to investigate the radiation of the detector the outgoing energy density of the ingoing vacuum state in the presence of a uniformly accelerated detector will be calculated. The Bogolubov transformations from the ingoing to the outgoing Minkowski field operators are obtained by first replacing "in" by "out" in (41). Then for the $\hat{b}$'s the following relation can be derived from (37):

$$\hat{b}^{out}(\kappa) = \frac{1}{2}\left(1 + \frac{D_-(\kappa)}{D_+(\kappa)}\right)\hat{b}^{in}(\kappa) + \frac{1}{2}\zeta_0^{-2i\kappa}\left(1 - \frac{D_-(\kappa)}{D_+(\kappa)}\right)\hat{b}^{in}(-\kappa),$$

whereas the $\hat{c}$'s are transformed trivially. Finally $\hat{b}^{in}$ and $\hat{b}^{in\dagger}$ are expressed by $\hat{a}^{in}$ and $\hat{a}^{in\dagger}$ (39); $\hat{c}^{in}$ and $\hat{c}^{in\dagger}$ are treated analogously. The result is

$$\hat{a}^{out}(k) = \hat{a}^{in}(k) + i\frac{(\lambda\zeta_0)^2}{8\pi}\frac{\Theta(-k)}{\sqrt{-k}}\left\{\int_{-\infty}^0 \frac{dk'}{\sqrt{-k'}}\times\right.$$

$$\left(\hat{a}^{in}(k')\int_{-\infty}^\infty \frac{d\kappa\, |\rho(\kappa)|^2\, e^{\kappa\pi}\, (-k)^{i\kappa}\, (-k')^{-i\kappa}}{\kappa\, \mathrm{sh}(\kappa\pi) D(\kappa)}\right.$$

$$\left. + \hat{a}^{in\dagger}(k')\int_{-\infty}^\infty \frac{d\kappa\, |\rho(\kappa)|^2\, (-k)^{i\kappa}\, (-k')^{-i\kappa}}{\kappa\, \mathrm{sh}(\kappa\pi) D(\kappa)}\right)$$

$$-\frac{1}{\pi}\int_0^\infty \frac{dk'}{\sqrt{k'}}\left(\hat{a}^{in}(k')\int_{-\infty}^\infty \frac{d\kappa\, |\rho(\kappa)|^2\, \Gamma(-i\kappa)^2\, \zeta_0^{-2i\kappa}\, e^{\kappa\pi}\, (-k)^{i\kappa}\, k'^{\, i\kappa}}{D(\kappa)}\right.$$

$$\left. + \hat{a}^{in\dagger}(k')\int_{-\infty}^\infty \frac{d\kappa\, |\rho(\kappa)|^2\, \Gamma(-i\kappa)^2\, \zeta_0^{-2i\kappa}\, (-k)^{i\kappa}\, k'^{\, i\kappa}}{D(\kappa)}\right)\Bigg\}$$

$$+i\frac{(\lambda\zeta_0)^2}{8\pi}\frac{\Theta(k)}{\sqrt{k}}\left\{\int_0^\infty \frac{dk'}{\sqrt{k'}}\times\right. \tag{44}$$

$$\left(\hat{a}^{in}(k')\int_{-\infty}^\infty \frac{d\kappa\, |\rho(\kappa)|^2\, e^{\kappa\pi}\, k^{-i\kappa}\, k'^{\, i\kappa}}{\kappa\, \mathrm{sh}(\kappa\pi) D(\kappa)} + \hat{a}^{in\dagger}(k')\int_{-\infty}^\infty \frac{d\kappa\, |\rho(\kappa)|^2\, k^{-i\kappa}\, k'^{\, i\kappa}}{\kappa\, \mathrm{sh}(\kappa\pi) D(\kappa)}\right)$$

$$-\frac{1}{\pi}\int_{-\infty}^0 \frac{dk'}{\sqrt{-k'}}\left(\hat{a}^{in}(k')\int_{-\infty}^\infty \frac{d\kappa\, |\rho(\kappa)|^2\, \Gamma(i\kappa)^2\, \zeta_0^{2i\kappa}\, e^{\kappa\pi}\, k^{-i\kappa}\, (-k')^{-i\kappa}}{D(\kappa)}\right.$$

$$\left. + \hat{a}^{in\dagger}(k')\int_{-\infty}^\infty \frac{d\kappa\, |\rho(\kappa)|^2\, \Gamma(i\kappa)^2\, \zeta_0^{2i\kappa}\, k^{-i\kappa}\, (-k')^{-i\kappa}}{D(\kappa)}\right)\Bigg\};$$

$$D(\kappa) := (\zeta_0\omega)^2 - \kappa^2 + \frac{(\lambda\zeta_0)^2}{\pi}\int_0^\infty \frac{dq\, |\, \rho(q)\, |^2}{\kappa^2 - q^2} - i\frac{(\lambda\zeta_0)^2}{2\kappa}|\rho(\kappa)|^2,$$

$$D(\pm|\kappa|) = D_\pm(\kappa).$$

Although the Bogolubov transformation between the ingoing and the outgoing field in presence of an accelerated detector does not mix Rindler creation and annihilation operators, it does so with Minkowski operators; therefore the ingoing and the outgoing vacua do not coincide. It seems that a uniformly accelerated detector creates particles. The expectation value of the outgoing Hamiltonian in the ingoing vacuum is to show where to the detector will radiate energy:

$$E^{out}(t,x) := {}^{in}\langle\, 0\, |: \mathcal{H}^{out}(t,x) :|\, 0\, \rangle^{in} = \frac{1}{2}\, {}^{in}\!\left\langle 0\, \middle|\, :\left(\frac{\partial\Phi^{out}}{\partial t}\right)^2 + \left(\frac{\partial\Phi^{out}}{\partial x}\right)^2 :\, \middle|\, 0\right\rangle^{in}, \tag{45}$$

where : : means normal ordering in the sense of the outgoing field algebra, $\hat{a}^{out\dagger}$ stands left from $\hat{a}^{out}$.

For $x \neq \pm t$ one has

$${}^{in}\langle\, 0\, |: \mathcal{H}^{out}(t,x) :|\, 0\, \rangle^{in} = 0, \tag{46}$$

for $x = \pm t$ the momentum space-integrals in (45) diverge. Accordingly, the ingoing vacuum cannot be represented as a normalizeable outgoing state.

The phase transformation in the domain of dependence of the detector described by (37) causes a discontinuity of the field at the boundary of this domain; that is the reason for the singularity of the field energy on the light cone of the coordinate origin.

As uniform acceleration lasting infinitely long cannot be realized physically, the transformation (44) should not be taken too literally in a physical sense. What remains are predictions for the limit of very long finite time intervals: (46) can be interpreted in the way that the detector, when accelerated uniformly for a sufficiently long time, approaches a state of equilibrium with the field vacuum at a higher energy level than before the acceleration.

Assume, for example, that the acceleration of the detector is given by a step function,

$$a(t) = \frac{1}{\zeta_0}[\Theta(t - t_1) - \Theta(t_2 - t)].$$

The detector moves inertially until the time t_1, being at equilibrium with the vacuum, respectively with a cloud of virtual particles. If we assume that the influence of the acceleration on the material of the detector can be described by a transition from $\rho(x)$ to some $\rho(\zeta)$, then for some time interval after t_1 the Fulling-Rindler quantum field theory will become the valid one for the detector. Now it is immersed in a bath of Rindler quanta and will absorb them, until a new equilibrium at a higher energy level, corresponding to the temperature $T = (2\pi k_B \zeta_0)^{-1}$, is established (or approximated asymptotically). During this transition the detector may radiate, but from (46) it follows that it will not radiate when it has reached the new equilibrium.

After the end of the acceleration there is some $t > t_2$ when ρ has changed to its original form $\rho(x)$. Then Minkowski quantum field theory is valid again and the detector will be radiating away its excitation energy by emitting Minkowski quanta and so approximating the same state as before t_1. The repeated sudden change of the state of motion may cause radiation, which is fading in the following periods of constant acceleration.

5 The Fulling-Rindler vacuum

The energy expectation values of a uniformly accelerated detector can make one think that the Fulling-Rindler quantum field theory is "the right one" for accelerated observers. What is the physical significance of it? Its vacuum state (and all its states with fewer quanta than the Minkowski-vacuum) cannot be realized as a state of the free field. Accordingly, energy expectation values below ${}^{in}\langle\ 0 \mid H_D(\tau) \mid 0\ \rangle^{in}$, especially the Rindler vacuum expectation value E_0, are not accesible for the accelerated detector in physical particle states.

The negative energy density of the Rindler vacuum has been calculated in various manners [4,10,11], it arises also directly from the Bogolubov transformation (41).

According to Gerlach [13] it is the fluctuations of the Minkowski vacuum which appear to have a thermal spectrum for accelerated observers, and the Rindler vacuum could be approximated in parts of the Rindler space-time by freezing out some vacuum fluctuations in the interior of a large accelerated "refrigerator". One can even try to calculate the energy expectation value of an inertial detector in Rindler vacuum. A detector on a world line (t, x_0) is immersed in Rindler vacuum for a time $-x_0 < t < x_0$. The decomposition of the field into an undisturbed part and a part interacting with the detector is somewhat different from the decomposition when the detector is placed in the spatial coordinate origin: In analogy to (10,35) one has

$$\hat{\mathbf{a}}^{in}_{s,a}(k) := \frac{1}{\sqrt{2}}[e^{ikx_0}\hat{\mathbf{a}}^{in}(k) \pm e^{-ikx_0}\hat{\mathbf{a}}^{in}(-k)]. \tag{47}$$

$\mathbf{a}^{in}_a$ and $\mathbf{a}^{in\dagger}_a$ are expressed in terms of Rindler field operators (41) and inserted into the detector Hamiltonian (22). The resulting energy expectation value of the detector is lower than in Minkowski vacuum:

$$\begin{aligned} {}^{in}_{R}\langle\, 0,0' \mid H_D(t) \mid 0,0' \,\rangle^{in}_{R} = E_0 - \frac{\lambda^2}{8\pi^2}\int_0^\infty d\kappa \times \\ \left\{ \omega^2 \left| \int_0^\infty \frac{dk}{k} \frac{\rho(k)}{D_-(k)} k^{-i\kappa} e^{ik(t-x_0)} \right|^2 + \left| \int_0^\infty dk \frac{\rho(k)}{D_-(k)} k^{-i\kappa} e^{ik(t-x_0)} \right|^2 \right. \\ \left. +\omega^2 \left| \int_0^\infty \frac{dk}{k} \frac{\rho(k)}{D_-(k)} k^{i\kappa} e^{ik(t+x_0)} \right|^2 + \left| \int_0^\infty dk \frac{\rho(k)}{D_-(k)} k^{i\kappa} e^{ik(t+x_0)} \right|^2 \right\}. \end{aligned} \tag{48}$$

Far away from the horizons of the Rindler spaces the integrals behave like $(t+x_0)^{-1}$. Consequently, the negative energy contribution goes like $1/x_0^2$ for large $x_0 \gg t$, according to the negative energy density of the Rindler vacuum [4,10,11]. At large distances from the horizons the detector behaves asymptotically like in Minkowski vacuum.

In contrast to an investigation made by Candelas and Sciama [12], here the detector behaves differently in Rindler vacuum and in Minkowski vacuum, even if it is never accelerated. The expectation value (48) is in accordance with Gerlach's interpretation of the Rindler vacuum as a state which is gained from ordinary Minkowski vacuum by removing some vacuum fluctuations [13].

The reasons for the different canonical field quantizations in Rindler and Minkowski coordinates are on the one hand the existence of horizons of the Rindler space, and on the other hand the fact that the Rindler time τ is no inertial time. Because of these differences, only measurements by inertial detectors of the type considered here are actually in accordance with the state of the field in flat space. Indeed, the effects of the acceleration on the excitation of a detector is far below measurability. To measure a temperature of $1K$ in the vacuum, the acceleration would have to amount to $\approx 2.5 \times 10^{20} ms^{-2}$.

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ELECTROMAGNETIC FLUCTUATIONS AND MOVING DETECTORS IN A MEDIUM.

V. P. Frolov

P. N. Lebedev Physics Institute,

117924, Moscow, Russia.

and

Copenhagen University Observatory,

Oster Voldgade 3, DK-135, Copenhagen, Denmark.

and

S. Hacyan

Instituto de Física

Universidad Nacional Autónoma de México

Apartado Postal 20-364, México, D. F. , Mexico.

Abstract

We study the quantum fluctuations of the electromagnetic field in a medium, and calculate the spectrum of the stress-energy tensor in a moving frame. The possibility of superluminal motion introduces some new features which have an origin similar to that of the vacuum fluctuations in noninertial systems.

Quantum field theory predicts the existence of a zero-point field due to fluctuations even in vacuum state. The spectrum of these quantum fluctuations is Lorentz invariant, and, therefore, the zero point field cannot be detected in an inertial frame in free space. However, the situation is altogether different when the Lorentz invariance is broken. This is the case in a noninertial frame or in the presence of boundaries. For instance, the Casimir effect [1] can be interpreted as due to the presence of a privileged frame of reference, that of the boundaries, which breaks the Lorentz invariance.

A somewhat similar situation happens in the presence of a dielectric medium. Since the medium defines a preferred reference frame, the quantum fluctuations of the electromagnetic field should produce some observable effects. Furthermore, the possibility of superluminal motion in a medium has some interesting effects which are related to the Cherenkov radiation and to the so called anomalous Doppler effect (ADE) (Ginzburg [2]). That there exists a connection between the Cherenkov radiation and the ADE, on the one hand, and the radiation in an accelerated frame, on the other hand, was already pointed out by Ginzburg and Frolov [3].

The aim of the present work is to further explore the phenomenon of quantum fluctuations in moving frames. The general theory of electromagnetic fluctuations in a frame moving with respect to a material medium is outlined, and the stress-energy tensor is evaluated for the case of uniform motion. Already in this simple case, some interesting effects appear when the motion is superluminal.

The present formalism is based on the Green or Wightman two-point correlation functions of the electromagnetic field in a medium. These functions can be written in the form (see, e. g., Lifshitz and Pitaevski [4]):

$$\langle A(x_1),\ B(x_2)\rangle =$$
$$(8\pi^2)^{-1}\int_0^\infty d\omega \int d\mathbf{k}\ e^{-i\omega\Delta t + i\mathbf{k}\cdot\Delta\mathbf{x}}\ k^{-1}\ \mathcal{F}(A,B)\ \delta(n\omega - k) \tag{1}$$

Here, A and B represent any component of the electric or magnetic field, E_i and H_i, and

$$\mathcal{F}(E_i, E_j) = \omega^2\delta_{ij} - k_i k_j n^{-2} \tag{2a}$$

$$\mathcal{F}(E_i, H_j) = -\mathcal{F}(H_i, E_i) = \varepsilon_{ijk} k_k \omega \tag{2b}$$

$$\mathcal{F}(H_i, H_j) = n^2(\omega^2\delta_{ij} - k_i k_j n^{-2}) \tag{2c}$$

The above correlation functions involve the electric and magnetic fields referred to the rest frame of a homogeneous medium with no spatial dispersion. It is assumed that the permeability $\mu = 1$, and that the refractive index $n = \sqrt{\varepsilon}$ is real and depends on the frequency ω measured in that rest frame.

Now, in general, the energy density u, the Maxwell stress tensor σ_{ab}, and the Poynting vector **S** in a medium are given by [5]:

$$u = (8\ \pi)^{-1}\ (\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}) \tag{3}$$

$$\mathbf{S} = (4\ \pi)^{-1}\ \mathbf{E} \times \mathbf{H} \tag{4}$$

$$\sigma_{ab} = (8\ \pi)^{-1} \left[(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})\ \delta_{ab} - 2\ (E_a D_b + B_a H_b) \right]. \tag{5}$$

Under a Lorentz transformation, the pairs (**E**, **B**) and (**D**, **H**) transform as the components of four-dimensional antisymmetric tensors. If the standard relations $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ (with arbitrary ε and μ) are assumed , then a transformation to a frame moving with velocity V in the direction x_3 yields the following transformed energy, stress and flux densities:

$$\tilde{u} = \gamma^2 \left[u - V\ (1 + n^2)\ S_3 + V^2\ \sigma_{33} \right] \tag{6}$$

$$\tilde{S}_3 = \gamma^2 \left[(1 + n^2\ V^2)\ S_3 - V\ (u + \sigma_{33}) \right], \tag{7}$$

$$\tilde{\sigma}_{33} = \gamma^2 \left[\sigma_{33} - V\ (1 + n^2)\ S_3 + V^2\ u \right], \tag{8}$$

where $n^2 = \varepsilon\mu$ and $\gamma = (\ 1 - V^2\ )^{\frac{1}{2}}$, and tildes refer to the moving frame.

A convenient procedure to calculate the spectrum of the stress-energy tensor in an arbitrarily moving frame [6] is outlined in the following. If the world line of the detector is given by the equation $x^\alpha = x^\alpha\ (\tau)$, where τ is the proper time of the detector, then one calculates the Green function at two points $x^\alpha = x^\alpha\ (\tau \pm \sigma)$ on the world line, and Fourier transform it with respect to σ. In this way, one obtains the spectrum as a function of $\tilde{\omega}$, which is precisely the frequency measured in the rest frame of the detector.

We apply this procedure to the case of a detector moving with constant velocity. The world line for motion along the x_3 axis is:

$$x^{\mu} = (\gamma \tau, 0, 0, \gamma V \tau). \tag{9}$$

Let us illustrate the method by calculating the spectrum of the electric field magnitude. Other relevant quantities can be calculated in an exactly similar way. We start from the correlation of the electric field in the medium frame:

$$\langle \mathbf{E}(\tau+\tfrac{1}{2}\sigma)\cdot \mathbf{E}(\tau-\tfrac{1}{2}\sigma)\rangle = (8\pi^2)^{-1} \int_0^{\infty} d\omega \int d\mathbf{k}\ e^{-i(\omega - \mathbf{k}\cdot\mathbf{V})\gamma\sigma}$$

$$\times\ k^{-1} (3\ \omega^2 - k^2/n^2)\ \delta(n\omega - k). \tag{10}$$

Then, the Fourier transform of this expression,

$$\langle \mathbf{E}^2\rangle_{\tilde{\omega}} = \int_{-\infty}^{\infty} d\sigma\ e^{i\,\tilde{\omega}\,\sigma} \langle \mathbf{E}(\tau+\tfrac{1}{2}\sigma)\cdot \mathbf{E}(\tau-\tfrac{1}{2}\sigma)\rangle, \tag{11}$$

can be presented in the form

$$\langle\ \mathbf{E}^2\ \rangle_{\tilde{\omega}} = \int_0^{\infty} d\omega \int_{-1}^{1} d\mu\ \delta[\tilde{\omega} - \gamma\ \omega\ (1 - n\ V\ \mu)]\ n\ \omega^3. \tag{12}$$

Here we have used spherical coordinates in order to perform the $\mathbf{k}$ integration ($d\mathbf{k} = k^2\ dk\ d\mu\ d\varphi$), together with the formula

$$\delta\ (x) = (2\pi)^{-1} \int_{-\infty}^{\infty} dy\ e^{ixy}. \tag{13}$$

Finally, one finds

$$\langle\ \mathbf{E}^2\ \rangle_{\tilde{\omega}} = (\gamma\ V\ )^{-1} \int_0^{\infty} d\ \omega\ \omega^2\ \theta(n\ \omega\ \gamma\ V - |\tilde{\omega} - \gamma\ \omega|\ ). \tag{14}$$

dependent of the cutoff.

The simplest choice for the refractive index is [2]:

$$n(\omega) = \begin{cases} n & \text{for } \omega \le \omega_c \\ 1 & \text{for } \omega > \omega_c, \end{cases} \tag{21}$$

which reproduces all the qualitative properties of the general case in which the refractive index is larger than 1 except for high frequencies.

Given the step function form of the refractive index, there are essentially six ranges of values for the frequency $\tilde{\omega}$ which must be taken into account when performing the integrals in Eqs. (15) and (17). These ranges are:

I: $-\infty < \tilde{\omega} < -(nV-1)\gamma\omega_c$

II: $-(nV-1)\gamma\omega_c < \tilde{\omega} < 0$

III: $0 < \tilde{\omega} < (1-V)\gamma\omega_c$

IV: $(1-V)\gamma\omega_c < \tilde{\omega} < (1+V)\gamma\omega_c$

V: $(1+V)\gamma\omega_c < \tilde{\omega} < (1+nV)\gamma\omega_c$

VI: $(1+nV)\gamma\omega_c < \tilde{\omega} < \infty$.

The functions $F(\tilde{\omega})$, $G(\tilde{\omega})$ and $H(\tilde{\omega})$ are zero in the range I, and $G(\tilde{\omega})$ is also zero in the range IV. The integrals can be calculated with some simple but tedious algebra. For our purposes, it is enough to give explicitly the following results. First, the Poynting flux vanishes identically for $\tilde{\omega} > (1+nV)\gamma\omega_c$ and is mainly concentrated in the range IV, namely around the value $\gamma\omega_c$. The exact value in this range of frequencies is

$$\tilde{S}_3 = - \frac{\tilde{\omega}}{8\pi} \omega_c^2 (n^2 - 1).$$

As for the energy density and stress, they have the exact values

$$\tilde{u} = 3\,\tilde{\sigma}_{33} = \frac{\tilde{\omega}^3}{2\pi} \gamma$$

just as in vacuum, for frequencies $\tilde{\omega}$ greater than $(1 + nV)\gamma\omega_c$.

The fact that the Poynting flux $\tilde{S}_3$ does not vanish for $vn > 1$ is an indication that there is a vacuum energy flux in a moving frame, which can be detected in principle. A somewhat similar situation happens in a rotating frame [6, 7], where this energy flux can be related to the synchrotron radiation [8] In general, a possible manifestation of this flux is through its "friction" with a moving charged particle. This point will be investigated in the future.

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QUANTUM ELECTRODYNAMICAL CORRECTIONS IN CRITICAL FIELDS

G. Soff

Gesellschaft für Schwerionenforschung (GSI), Planckstraße 1
Postfach 110 552, D-6100 Darmstadt, West Germany

1. INTRODUCTION

We investigate field-theoretical corrections, such as vacuum polarization and self energy to study their influence on strongly bound electrons in heavy and superheavy atoms. In critical fields ($Z \simeq 170$) for spontaneous e^+e^- pair creation the coupling constant of the external field $Z\alpha$ exceeds 1 thereby preventing the ordinary perturbative approach of quantum electrodynamical corrections which employs an expansion in $Z\alpha$. For heavy and superheavy elements radiative corrections have to be treated to all orders in $Z\alpha$. The Feynman diagrams for the lowest-order (a) vacuum polarization and (b) self energy are displayed in Fig. 1. The double lines indicate the exact propagators and wave functions in the Coulomb field of an extended nucleus. Fig. 2 shows an $Z\alpha$-expansion of the vacuum polarization graph. The dominant effect is provided by the Uehling contribution being visualized by the first diagram on the right hand side. It is linear in the external field and thus of order $\alpha(Z\alpha)$.

2. VACUUM POLARIZATION OF ORDER $\alpha(Z\alpha)^n$

The influence of the attractive Uehling potential on electronic binding energies for $Z > 100$ has been calculated already by Werner and Wheeler [1]. For the critical nuclear charge number Z_{cr} at which $E_{1s} = -mc^2$ the Uehling potential leads to an 1s-energy shift of $\Delta E_{vp}^{(n=1)} = -11.8$ keV [2], which decreases Z_{cr} by one third of a unit. The remaining vacuum-polarization part in lowest order of the fine-structure constant α but to all orders in $(Z\alpha)^n$ with $n \geq 3$ is more difficult to elaborate. First evaluations of this contribution were presented by Gyulassy [3-5] and by Rinker and Wilets [6-8]. These authors made use of the angular momentum decomposition of the electron propagator in spherical symmetric potentials that was developed by Wichmann and Kroll [9].

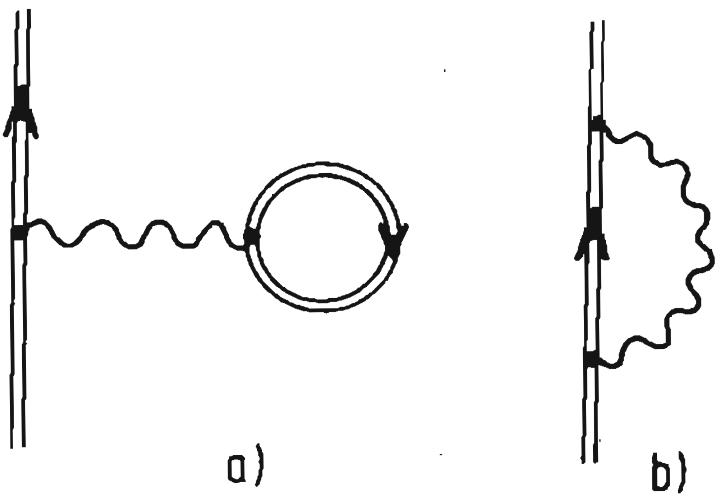


Figure 1: Feynman diagrams for the lowest-order (a) vacuum polarization and (b) self energy.

From bound state QED [10] the energy shift corresponding to the total vacuum polarization is given by

$$\Delta E = 4\pi i\alpha \int d(t_2 - t_1) \int d\vec{x}_2 \int d\vec{x}_1\, \bar{\phi}_n(x_2)\gamma^\mu \phi_n(x_2) D_F(x_2 - x_1)\, Tr[\gamma_\mu S_F(x_1, x_1)] \tag{1}$$

where the photon propagator is

$$D_F(x_2 - x_1) = \frac{-i}{(2\pi)^4} \int d^4k\, \frac{e^{-ik(x_2-x_1)}}{k^2 + i\epsilon} \tag{2}$$

and the Feynman propagator for the electron can be represented by

$$\begin{aligned} S_F(x_2, x_1) &= \frac{1}{2\pi i} \int_{C_F} dz \sum_n \frac{\phi_n(\vec{x}_2)\bar{\phi}_n(\vec{x}_1)}{E_n - z}\, e^{-iz(t_2-t_1)} \\ &= \frac{1}{2\pi i} \int_{C_F} dz\, \mathcal{G}(\vec{x}_2, \vec{x}_1, z)\gamma^0\, e^{-iz(t_2-t_1)}. \end{aligned} \tag{3}$$

It obeys the equation

$$[\gamma^\mu(i\partial_\mu - eA_\mu(x_1)) - m]\, S_F(x_1, x_2) = \delta^4(x_1 - x_2) \tag{4}$$

which implies that external field effects are included to all orders. ϕ_n denotes the electron wave function.

The level shift can be expressed as an expectation value of an effective potential U with

$$\begin{aligned} U(\vec{x}_2) &= 4\pi i\alpha \int d(t_2 - t_1) \int d\vec{x}_1\, D_F(x_2 - x_1)\; Tr[\gamma_0\, S_F(x_1, x_1)] \\ &= \frac{i\alpha}{2\pi} \int d\vec{x}_1\, \frac{1}{|\vec{x}_2 - \vec{x}_1|} \int_{C_F} dz\, Tr\, \mathcal{G}(\vec{x}_1, \vec{x}_1, z). \end{aligned} \tag{5}$$

With the vacuum polarization charge density ρ

$$\rho(\vec{x}) = \frac{e}{2\pi i} \int_{C_F} dz\, Tr\, \mathcal{G}(\vec{x}, \vec{x}, z) \tag{6}$$

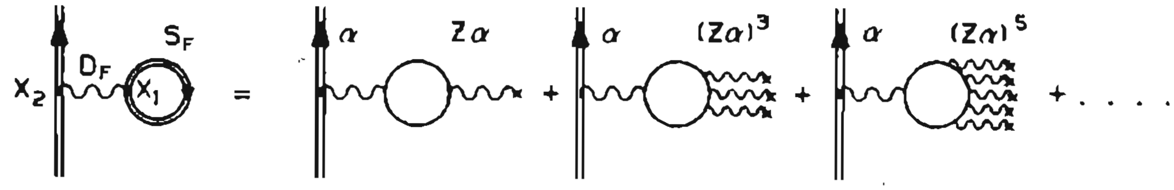


Figure 2: $Z\alpha$**-expansion of the vacuum polarization diagram.**

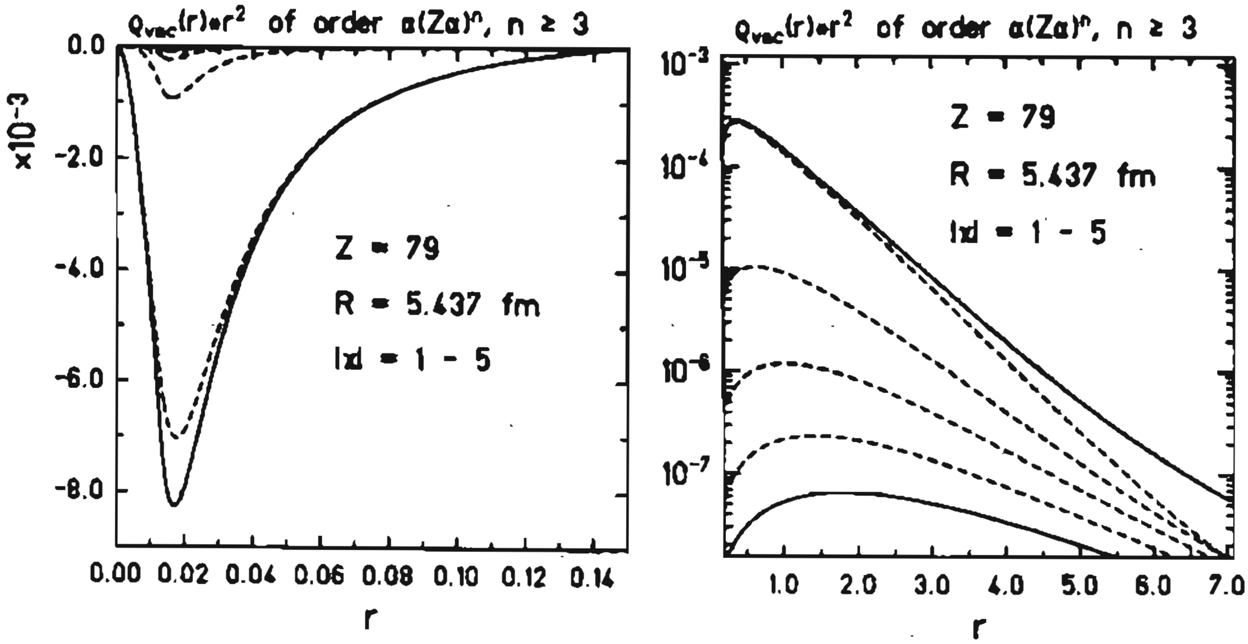
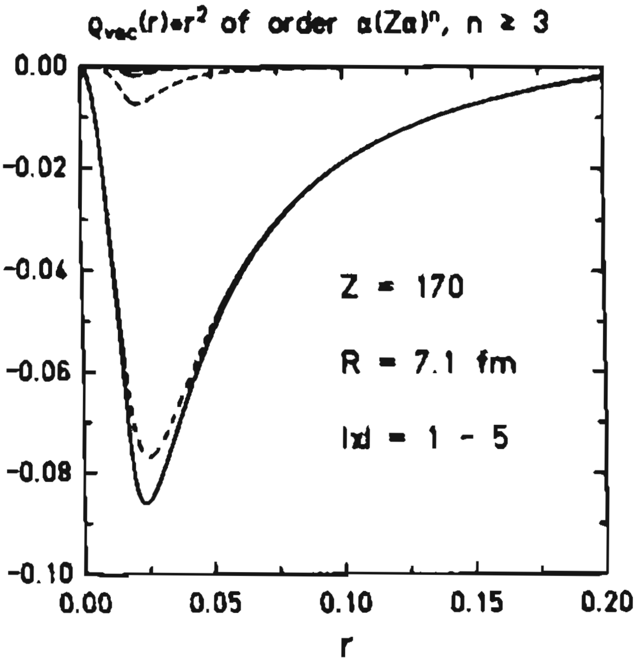


Figure 3: Radial vacuum polarization charge density $\rho_{|\kappa|} \cdot r^2$ of order $\alpha(Z\alpha)^n$ with $n \geq 3$ for the system $Z = 79$ with a nuclear radius $R = 5.437$ fm versus the radial coordinate r in natural units. The various contributions for $|\kappa| = 1, 2, 3, 4$ and 5 are shown separately by the dashed lines. $\rho_{|\kappa|} \cdot r^2$ is given in units of the elementary charge e. The solid line indicates the sum $\rho \cdot r^2$ of the various angular momentum components. a) Linear scale for the range in which the charge density is negative. b) Logarithmic scale to demonstrate the large distance behaviour of $\rho_{|\kappa|} \cdot r^2$. Here the vacuum polarization charge density is positive.



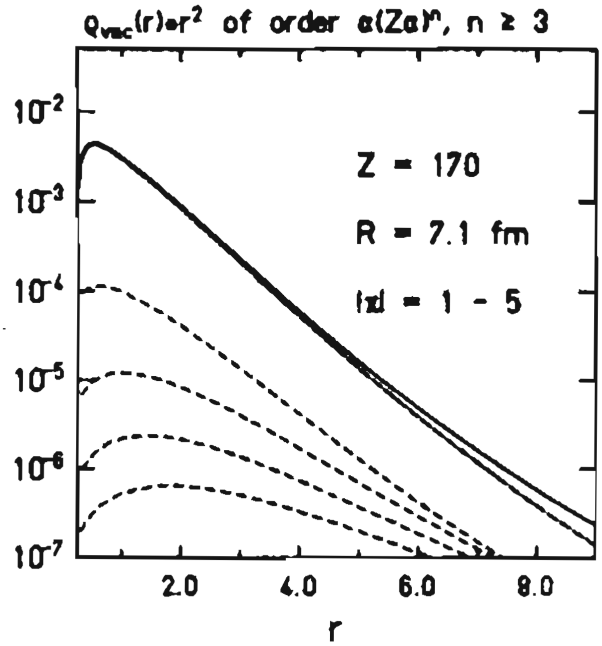


Figure 4: The same as in figure 3 for the almost critical system $Z = 170$ with a nuclear radius $R = 7.1$ fm.

it simply follows

$$U(\vec{x}_2) = -e \int d\vec{x}_1 \frac{\rho(\vec{x}_1)}{|\vec{x}_2 - \vec{x}_1|} \tag{7}$$

The formal expression for ρ still contains the infinite unrenormalized charge. A regularization procedure [5,11] for the total vacuum polarisation charge density is to subtract the Uehling contribution which can then be renormalized separately. Expansion of the Green function in eigenfunctions of angular momentum yields for the vacuum polarisation charge density of order $\alpha(Z\alpha)^n$ with $n \geq 3$

$$\begin{aligned}\rho(x) - \rho^{(1)}(x) &= \frac{e}{2\pi^2} \int_0^\infty du \left(\sum_{\kappa=\pm 1}^{\pm\infty} |\kappa| \, Re \left\{ \sum_{i=1}^{2} \mathcal{G}_\kappa^{ii}(x, x, iu) + \right.\right. \\ &\quad \left.\left. \int_0^\infty dy\, y^2\, V(y) \sum_{i,j=1}^{2} [\mathcal{F}_\kappa^{ij}(x, y, iu)]^2 \right\} \right) + \\ &\quad \frac{e}{2\pi} \sum_{\substack{\kappa=\pm 1 \\ -m<E<0}}^{\pm\infty} |\kappa| \left\{ f_1^2(x) + f_2^2(x) \right\}.\end{aligned} \tag{8}$$

This equation includes terms from any bound-state pole on the negative real z axis. These terms are picked up as residues in the rotation of the contour of integration. Such terms only appear for superheavy systems where the binding energy of the electron exceeds the electron rest mass. $f_1(x)$ and $f_2(x)$ denote components of the radial Dirac wave function, normalized according to

$$\int_0^\infty dx\, x^2 \left[f_1^2(x) + f_2^2(x) \right] = 1. \tag{9}$$

$\mathcal{F}_\kappa^{ij}$ are components of the free Dirac Green functions [12,13]. According to Wichmann and Kroll [9] the radial Coulomb Green function components $\mathcal{G}_\kappa^{ij}$ may be represented by solutions of the radial Dirac equation. Expression (8) has been solved numerically.

We have examined the vacuum polarization in the field of a high-Z finite size nucleus. The polarization charge density in coordinate space of order $\alpha(Z\alpha)^n$ with $n \geq 3$ is calculated [11,16]. Energy level shifts of K- and L-shell electrons in hydrogen-like systems are derived. Vacuum polarization corrections to energy eigenvalues of bound leptons have been examined extensively in the past. However, the corresponding influence of vacuum polarization effects on electron levels in atoms has not been completely calculated. One purpose of our work was to carry out a complete calculation of the vacuum polarization of order α in order to provide improved electron binding energy values and to investigate more closely radiative corrections for strongly bound electrons in superheavy systems.

The energy shifts of K- and L-shell electrons in various hydrogen-like systems due to the vacuum polarization of order $\alpha(Z\alpha)^n$ with $n \geq 3$ may be deduced from table 1. The energy correction ΔE usually is expressed via a function ΔF with

$$\Delta E = \frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3}\Delta F\, mc^2 .$$

n denotes the principal quantum number of the electron state. In table 1 we present the dimensionless quantities ΔF.

$\Delta F(Z\alpha)$

System	R[fm]	$1s_{1/2}$	$2s_{1/2}$	$2p_{1/2}$	$2p_{3/2}$
$_{30}$Zn	3.955	0.0020	0.0020	0.0000	0.0000
$_{54}$Xe	4.826	0.0059	0.0064	0.0004	0.0001
$_{82}$Pb	5.500	0.0150	0.0185	0.0035	0.0005
$_{92}$U	5.751	0.0207	0.0272	0.0068	0.0007
$_{100}$Fm	5.886	0.0269	0.0377	0.0118	0.0010
Z = 170	7.100	0.518	0.764	3.75	0.017

Table 1: Higher-order vacuum polarisation contributions [11] to the Lamb-shift of K- and L-shell electrons in various hydrogen-like systems. For the nuclear charge distribution we assumed a homogeneously charged spherical shell with a radius R.

For fermium we get a noticeable energy shift of about 9 eV for the K_α-line. In conclusion we have computed the vacuum polarization charge density of order $\alpha(Z\alpha)^n$ with $n \geq 3$ for various hydrogen-like systems of the known periodic table of elements. Employing the developped computer code more accurate numbers for the electron Lamb-shift [14,15] in hydrogen-like atoms can be provided.

The computed vacuum polarization charge density for $Z = 79$ (Au) and for the almost critical system $Z = 170$ is depicted in fig. 3 and 4, respectively. The nuclear radius R is indicated. The various contributions for the angular momentum components $|\kappa|$ = 1 - 5 are shown separately. The radial distance r is given in units of the electron Compton wavelength. $\rho_{|\kappa|} \cdot r^2$ is measured in units of the elementary charge e and the inverse Compton wavelength. Part a) shows on a linear scale $\rho_{|\kappa|} \cdot r^2$ in the range where the charge density is negative. The large distance behaviour of $\rho_{|\kappa|} \cdot r^2$ can be taken from figs. 3b and 4b. Please note the logarithmic scale. Here the radial charge density is positive and displays almost an exponential decline in the depicted range. Obviously the ($\kappa = \pm 1$) - contribution to the vacuum polarization charge density dominates by about an order of magnitude. A rapid convergence in the κ-summation is indicated. For large distances ($2 \leq r \leq 7$) $\rho_{|\kappa|} \cdot r^2$ decreases rapidly with different decline constants for the various κ-components. For $Z = 170$ the binding energy of the strongest bound electron state amounts to $E_{1s} = -1020.895$ keV. The effect of the higher-order vacuum polarization on a K-shell electron in the superheavy system $Z = 170$ results in $\Delta E_{1s} \simeq 1.46$ keV [11,16], which is completely negligible.

3. VACUUM POLARIZATION CORRECTIONS OF HIGHER ORDER IN MUONIC ATOMS

We also computed the energy shifts in muonic atoms caused by the vacuum polarization of order $\alpha(Z\alpha)^n$ with $n \geq 3$. Nuclear size corrections are taken into account. The calculations are performed for all muonic levels from the $1s_{1/2}$-state up to the $5g_{9/2}$-state in various atoms between $Z = 70$ and $Z = 100$.

The Bohr radius of a bound particle in an atomic orbit is inversely proportional to its rest mass m. Thus to test any deviation from the Coulomb potential at small radial distances it is favorable to measure precisely transition energies of bound muons or pions in heavy atoms. In particular high-lying muonic states, e.g. the $5g_{9/2}$- and the $4f_{7/2}$-state in $_{82}$Pb, are best suited to explore quantum electrodynamical corrections in strong external fields [8]. Despite their small radial expectation values of $< r > \approx 50$ fm these states are hardly influenced by the nuclear extension or by intrinsic degrees of freedom of the nucleus, e.g., nuclear polarization. In addition electron screening corrections play a minor role in these exotic atoms. The vacuum polarisation charge density is concentrated close to the nucleus which can be verified by measuring muonic transition energies.

Fig. 5 displays radial probability densities $|\psi r|^2$ in muonic lead. ρ_N indicates the nuclear charge distribution being described by a two-parameter Fermi distribution. The $1s_{1/2}$-muon obviously exhibits a striking overlap with the nuclear interior. The binding energy is extremely sensitive concerning any modification of the nuclear charge distribution. In consequence this state may not be utilized for precision tests of QED. For comparison the dashed line shows $|\psi r|^2$ for a K-shell electron with a radius of about 800 fm. This state has been computed using a Thomas–Fermi potential to account for electron screening. Please note the logarithmic scale for the radial coordinate r. Most important for QED tests are the two muonic states in between, the $5g_{9/2}$- and the $4f_{7/2}$-state. The maximum of their radial probability distribution is located at about 50 fm. The measured transition energy amounts to $\Delta E^{exp} = 431.353$ keV ± 14 eV. This accuracy allows for high-precision tests of QED in strong fields.

The various QED processes in the interaction of a muon with a nucleus are visualized in fig. 6. The first graph on the right hand side is the ordinary Coulomb interaction. The second graph again represents the Uehling part. The last diagram on the right hand side of the first line as well as the diagrams on the third line are summarized as Källén–Sabry contributions. They are of order $\alpha^2(Z\alpha)$. Their influence on electronic binding energies will be discussed later. The last diagram in fig. 6 of order $\alpha^2(Z\alpha)^2$ represents a Delbrück scattering. For a review of the various contributions we refer to ref. 8. Here we concentrate again on the diagrams in the second line, i.e. on the vacuum polarization of order $\alpha(Z\alpha)^n$ with $n \geq 3$.

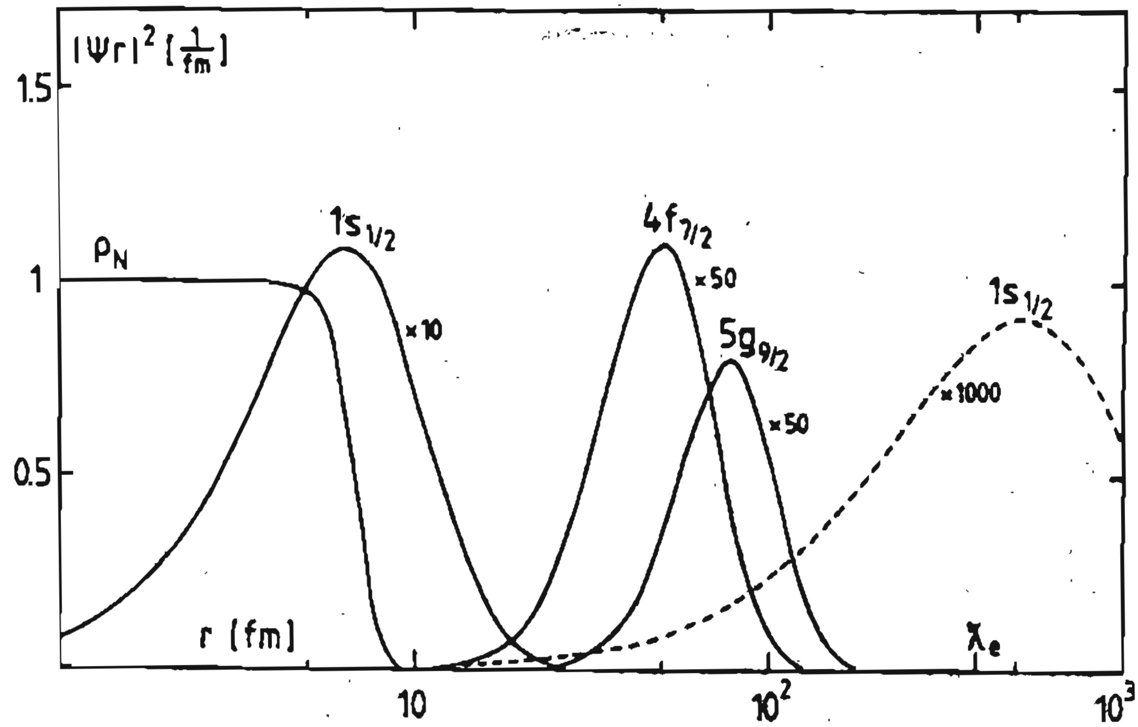


Figure 5: Radial density distributions $|\psi r|^2$ for the muonic $1s_{1/2}-$, $4f_{7/2}-$ and $5g_{9/2}-$state in muonic lead ($Z = 82$) as function of the radial coordinate r in units of fermi. The dashed line plots $|\psi r|^2$ for a K-shell electron. ρ_N indicates the nuclear charge distribution which is described by a two-parameter Fermi distribution.

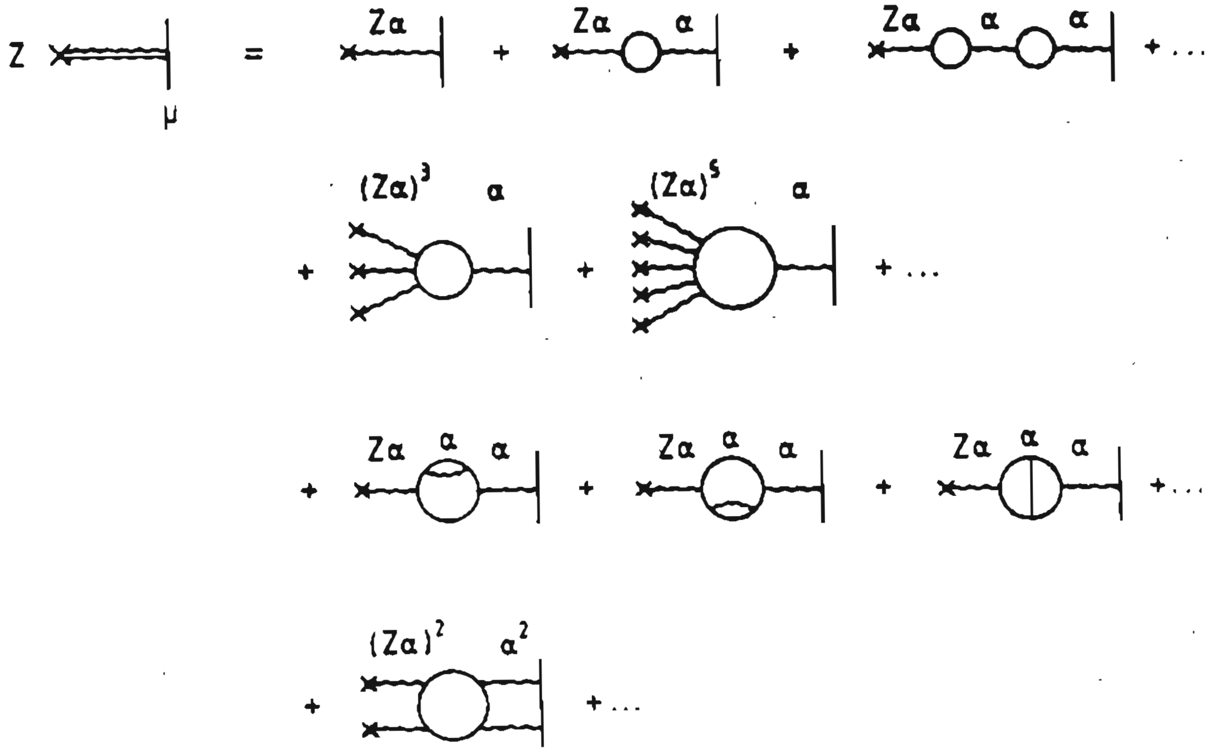


Figure 6: Feynman diagrams describing the vacuum polarization interaction in muonic atoms.

ΔE (eV)

State	$_{70}$Yb	$_{74}$W	$_{79}$Au	$_{82}$Pb	$_{86}$Rn	$_{92}$U	$_{96}$Cm	$_{100}$Fm
$1s_{1/2}$	299.9	354.4	436.0	489.2	562.2	692.8	792.3	900.7
$2s_{1/2}$	130.8	161.9	208.2	239.5	274.1	356.4	420.0	449.0
$2p_{1/2}$	194.4	229.1	296.8	342.3	407.3	521.7	609.3	705.7
$2p_{3/2}$	186.5	230.1	284.0	328.4	392.3	504.3	590.1	684.8
$3s_{1/2}$	65.3	80.4	105.2	122.2	147.3	190.9	227.5	268.8
$3p_{1/2}$	87.1	107.6	141.1	164.1	198.0	256.4	305.1	359.8
$3p_{3/2}$	85.0	106.3	137.7	160.3	193.7	250.8	298.9	352.9
$3d_{3/2}$	96.8	121.9	161.2	188.8	230.4	305.3	365.6	434.3
$3d_{5/2}$	94.0	118.0	155.8	182.2	221.9	293.1	350.6	416.0
$4s_{1/2}$	34.5	43.6	56.9	66.8	81.6	107.1	129.0	154.0
$4p_{1/2}$	43.8	55.7	73.7	85.5	104.7	139.4	165.1	197.2
$4p_{3/2}$	42.9	54.6	72.4	83.9	102.9	137.3	164.6	194.3
$4d_{3/2}$	47.8	61.2	81.9	96.1	118.6	159.7	192.0	230.4
$4d_{5/2}$	46.7	59.7	79.6	93.3	115.1	154.8	185.7	222.8
$4f_{5/2}$	48.1	61.6	82.6	97.5	120.6	163.2	197.5	237.6
$4f_{7/2}$	47.4	60.7	81.2	95.7	118.3	159.7	192.9	231.7
$5s_{1/2}$	19.8	25.0	33.4	39.0	48.1	64.7	77.6	93.3
$5p_{1/2}$	24.5	31.0	41.6	49.0	59.9	80.7	97.5	116.0
$5p_{3/2}$	24.1	30.5	40.9	48.3	59.0	79.7	96.3	114.6
$5d_{3/2}$	26.3	33.7	45.6	54.0	66.8	90.9	109.9	132.7
$5d_{5/2}$	25.8	33.0	44.5	52.7	65.0	88.5	107.4	129.0
$5f_{5/2}$	26.2	33.7	45.7	54.3	67.5	92.3	112.5	135.9
$5f_{7/2}$	25.8	33.2	45.0	53.4	66.3	90.5	110.2	132.9
$5g_{7/2}$	25.5	33.0	44.9	53.4	66.6	91.4	111.6	135.2
$5g_{9/2}$	25.3	32.7	44.4	52.8	65.9	90.2	110.1	133.3

Table 2: Energy corrections in units of eV of muonic bound states caused by the vacuum polarization of order $\alpha(Z\alpha)^n$ with $n \geq 3$. We performed the calculations [17] for all muonic bound states between the $1s_{1/2}$- and the $5g_{9/2}$-state. Muonic atoms ranging from $_{70}$Yb up to $_{100}$Fm are considered.

Employing first-order perturbation theory,

$$\Delta E \;=\; \int_0^\infty \left(f_1^2(r) + f_2^2(r)\right) U(r)\, r^2\, dr\,, \tag{10}$$

we evaluated the energy correction [17] of muons bound in heavy atoms ranging from $_{70}$Yb up to $_{100}$Fm. In (10) $f_1(r)$ and $f_2(r)$ denote radial components of the bound-state wave function of the muon. Table 2 includes the final energy shifts [17] in units of eV for all muonic levels between the $1s_{1/2}$- and the $5g_{9/2}$-state. The corresponding nuclear radii can be deduced from the table in ref. 14. In muonic lead ($Z = 82$) we obtained as transition energy corrections $\Delta E(5g_{9/2} - 4f_{7/2}) = 42.9$ eV and $\Delta E(5g_{7/2} - 4f_{5/2}) = 44.1$ eV which agree within the quoted uncertainties with the corresponding results published by Borie and Rinker [8] and by Gyulassy [5] . We also investigated modifications of binding energies in pionic atoms caused by the higher-order vacuum polarization potential $U(r)$. Here we have to solve the Klein-Gordon equation incorporating the nuclear Coulomb potential $V(r)$ as well as $U(r)$. For pionic xenon ($Z = 54$) and pionic lead ($Z = 82$) we found ultimately $\Delta E(5g - 4f) = 10.2$ eV and 56.4 eV, respectively.

In conclusion we have presented energy shifts in various exotic atoms caused by higher-order vacuum polarization processes. By comparison with precision experimental data the tabulated numbers may be utilized to test quantum electrodynamics in strong Coulomb fields.

4. THE INFLUENCE OF VACUUM POLARIZATION CORRECTIONS OF ORDER $\alpha(Z\alpha)$ AND $\alpha(Z\alpha)^3$ IN HYDROGEN-LIKE URANIUM

Energy shifts of a bound electron in hydrogen-like uranium caused by vacuum polarization corrections of order $\alpha(Z\alpha)$ and $\alpha(Z\alpha)^3$ are calculated [18]. It is demonstrated that the Wichmann-Kroll correction of order $\alpha(Z\alpha)^3$ dominates for higher electron shells compared with the Uehling contribution.

Usually the Uehling potential provides the dominant vacuum polarization contribution to the Lamb-shift of inner-shell electrons in ordinary atoms as well as in muonic atoms. For large radial distances r from the nuclear charge centre this vacuum polarization potential $V_{11}(r)$ of order $\alpha(Z\alpha)$ displays an exponential decline on a length scale being determined by the electron Compton wavelength ($\lambda_e \approx 386$ fm). Higher-order vacuum polarization corrections were originally discussed by Wichmann and Kroll [9] and later evaluated e.g. in refs. 3 - 8. A striking feature is the asymptotic behaviour of the vacuum polarization charge density of order $\alpha(Z\alpha)^3$, which displays a r^{-7}-dependence at large distances ($r \to \infty$). The corresponding vacuum polarization potential $V_{13}(r)$ declines as [19]

$$V_{13}(r) \overset{r\to\infty}{=} \frac{\alpha(Z\alpha)^3}{\pi r}\frac{32}{225}\frac{1}{(2r)^4}\,. \tag{11}$$

Thus one may obviously expect that the Wichmann-Kroll corrections surpass the Uehling corrections for bound electrons in higher shells. To quantify this insight we computed the corresponding energy shifts for a bound electron in hydrogen-like uranium. The nucleus was assumed to be point-like. The bound-state wave functions have been computed according to Rose [20]. The Uehling potential was evaluated using a representation in terms of modified Bessel functions by Klarsfeld [21]. For the calculation of the vacuum polarization potential of order $\alpha(Z\alpha)^3$ we utilised expressions presented by Blomqvist [22]. Some related technical ingredients are discussed in ref. 14. For radial distances $r > 20\,\lambda_e$ we used the asymptotic form (11).

The computed energy shifts in units of eV are given in table 3. The considered electron levels are signified by the principal quantum number n, the orbital angular momentum quantum number l and by the total angular momentum quantum number j. Already for the 4f-shell the striking long-distance dependence (11) leads to a dominance of the Wichmann-Kroll correction of order $\alpha(Z\alpha)^3$ over the Uehling correction of order $\alpha(Z\alpha)$. However, the tiny absolute value of the energy shifts represents a severe challenge for a possible experimental verification of this exciting QED phenomenon.

n	l	j	$\Delta E_{VP}[\alpha(Z\alpha)]$	$\Delta E_{VP}[\alpha(Z\alpha)^3]$
1	0	1/2	-9.800E+01	4.674E+00
2	0	1/2	-1.731E+01	7.748E-01
2	1	1/2	-2.995E+00	1.894E-01
2	1	3/2	-1.265E-01	2.056E-02
3	0	1/2	-5.130E+00	2.272E-01
3	1	1/2	-1.035E+00	6.367E-02
3	1	3/2	-4.831E-02	7.521E-03
3	2	3/2	-1.436E-03	3.748E-04
3	2	5/2	-2.573E-04	1.416E-04
4	0	1/2	-2.117E+00	9.344E-02
4	1	1/2	-4.467E-01	2.726E-02
4	1	3/2	-2.192E-02	3.367E-03
4	2	3/2	-8.775E-04	2.183E-04
4	2	5/2	-1.603E-04	8.118E-05
4	3	5/2	-1.666E-06	3.507E-06
4	3	7/2	-4.336E-07	2.533E-06
5	0	1/2	-1.063E+00	4.683E-02
5	1	1/2	-2.287E-01	1.390E-02
5	1	3/2	-1.155E-02	1.763E-03
5	2	3/2	-5.168E-04	1.262E-04
5	2	5/2	-9.538E-05	4.683E-05
5	3	5/2	-1.460E-06	2.521E-06
5	3	7/2	-3.799E-07	1.753E-06
5	4	7/2	-1.837E-09	1.747E-07
5	4	9/2	-5.696E-10	1.592E-07

Table 3: Energy shifts ΔE_{VP} in units of eV for electron states with quantum numbers n, l and j in hydrogen-like uranium caused by vacuum polarization corrections of order $\alpha(Z\alpha)$ and $\alpha(Z\alpha)^3$.

5. THE VACUUM POLARIZATION POTENTIAL OF ORDER $\alpha^2(Z\alpha)$

The theoretical values [14,15] for the electron Lamb-shift in hydrogen-like atoms contain uncertainties of various types. The major motivation for our investigations is provided by a possible improvement in the accuracy of these theoretical data. Our investigations dealt with a higher-order vacuum polarization correction which was originally investigated by Källén and Sabry [23] . These authors studied the vacuum polarization process of order $\alpha^2(Z\alpha)$. The corresponding Feynman-diagrams either contain two electron-positron loops or one additional photon line within the ordinary electron-positron loop (cf. fig. 6). The analytical expression for the related vacuum polarization potential was presented by Blomqvist [22] . It yields

$$V_{21}(r) \;=\; (Z\alpha)\, g_2(r)\,. \tag{12}$$

For $r \gg 1$ the potential $g_2(r)$ decreases exponentially [24]. At $r = 20$ a value of 10^{-19} eV is already reached. In first-order perturbation theory the associated energy shifts follow from

$$\Delta E = \int_0^\infty (f^2(r) + g^2(r))\, V_{21}(r)\, r^2\, dr\,, \tag{13}$$

in which $f(r)$ and $g(r)$ denote the small and large component of the relativistic radial wave function, respectively. The energy shift can be represented as

$$\Delta E_{VP} = \frac{\alpha}{\pi}\frac{(Z\alpha)^4}{n^3}\,\Delta F_{VP}\, mc^2 \tag{14}$$

The calculated energy shifts of the $1s_{1/2}$-, $2s_{1/2}$-, $2p_{1/2}$- and $2p_{3/2}$-state caused by the vacuum polarization potential of order $\alpha^2(Z\alpha)$ can be deduced from table 1 in ref. 24, in which we tabulate the shift ΔF. As consequence of the attractive interaction the corresponding energy shifts are negative. In particular, we obtain for hydrogen $\Delta F(1s_{1/2}) = -\, 0.00232705$, $\Delta F(2s_{1/2}) = -\, 0.00232713$, $\Delta F(2p_{1/2}) = -\, 3.248\cdot 10^{-6}$ and $\Delta F(2p_{3/2}) = -\, 9.074\cdot 10^{-9}$. The vacuum polarization contribution of order $\alpha^2(Z\alpha)$ to the traditional Lamb-shift $E(2s_{1/2}) - E(2p_{1/2})$ in hydrogen amounts to about 236.7 kHz. For the $1s_{1/2}$-state in hydrogen-like uranium we find $\Delta E(1s_{1/2}) \simeq 0.75$ eV, which already represents a sizable binding energy variation. We note, that the energy shift of the $2p_{1/2}$-state is not negligible compared with that of the $2s_{1/2}$-state.

Furthermore we estimated the influence of the vacuum polarization potential of order $\alpha^2(Z\alpha)$ on the energy eigenvalues of the strongest bound electron states in superheavy atoms [24]. To simulate nuclear-size corrections we computed the vacuum polarization potential at about the nuclear radius R and employed this constant value also inside the nucleus. For the nuclear charge distribution we assumed a spherical shell. For the almost critical system Z = 170 with R = 7.1 fm, in which the $1s_{1/2}$-state almost reaches the negative energy continuum, we calculated $\Delta E(1s_{1/2}) = -\, 88.9$ eV. This small value can be completely omitted compared with the huge binding energy of E^b_{1s} $= -\, 1020.895$ keV .

6. SELF ENERGY

Electronic self energy corrections for high-Z systems were first studied in the pioneering work by Brown and co-workers [25-27]. Their method was further refined and successfully applied in computations of electron energy shifts in high-Z elements by Desiderio and Johnson [28] as well as by Cheng and Johnson [29]. In our calculations we employed these methods, which may be slightly simplified by restriction to K-shell electrons. The energy shift of a $1s_{1/2}$-electron due to the quantum electrodynamical self energy formally is given by

$$\Delta E = 4\pi i\alpha \int d(t_2 - t_1) \int d\vec{x}_2 \int d\vec{x}_1\; \bar{\phi}_n(x_2)\gamma^\mu S_F(x_1, x_1)\gamma^\nu \phi_n(x_2) D^F_{\mu\nu}(x_2 - x_1) \tag{15}$$

The self-energy correction to be calculated is represented by the Feynman diagram b) in fig. 1. Again the double line indicates the exact electron propagator in the Coulomb field of a nucleus. The next step is to transform propagators and wave functions into momentum space. This admits a decomposition of the self-energy diagram, so enabling infinite mass terms to be identified and removed, leaving the finite observable part of

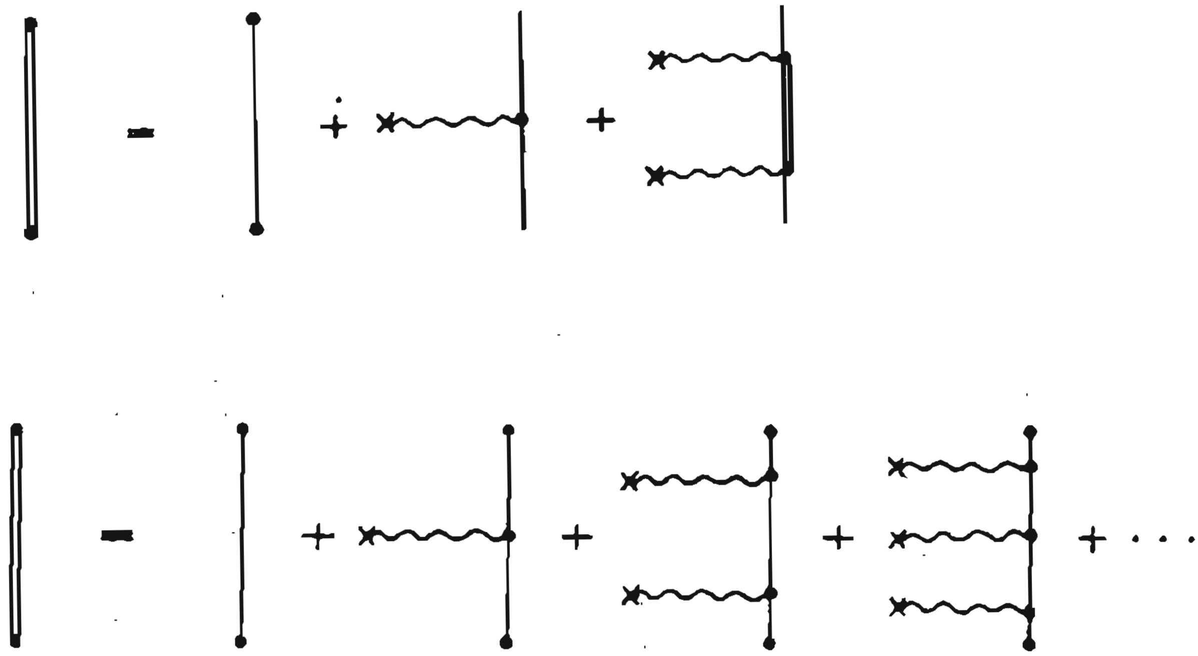


Figure 7: Electron propagator in the external field. Graphical representation of the integral equation for the Coulomb propagator of the electron and the iterated form.

the self energy. We introduce the following Fourier transformations

$$\phi(p) = \int d^4x\, \phi(x)\, e^{ipx} \tag{16}$$

$$A_\mu^{ex}(p) = \int d^4x\, A_\mu^{ex}(x)\, e^{ipx} \tag{17}$$

$$S_F(p_2, p_1) = \int d^4x_2 \int d^4x_1\, S_F(x_2, x_1)\, e^{-i(p_1x_1 - p_2x_2)} \tag{18}$$

The full Feynman propagator in momentum space obeys an integral equation. The result may be represented graphically, where the double line denotes S_F and a single line the free propagator $S_F^{(0)}$. The decomposition of the Feynman propagator may be inserted into the self-energy graph. Calculating the various terms is rather lengthy and not very enlightening. Details of the calculations may be taken from Schlüter [30].

The various existing calculations [12-15,28-32] on the self-energy of K-shell electrons in high-Z systems may be directly compared for mercury ($Z = 80$). The self-energy contribution on the binding energy amounts to about 206 eV. The relative deviation between the different calculations was found to be less than 1%. The obtained energy shifts caused by the self-energy of the strongest bound electron are summarized in fig. 9, where ΔE is plotted versus the nuclear charge number Z. The apparent discrepancy between Mohr's calculation ($\Delta E = 2.586 \pm 0.156$ keV) and our result ($\Delta E = 1.896$ keV) for $Z = 130$ is caused by the neglection of nuclear size effects in ref. [13].

Our most important result was the self-energy shift for 1s-electrons in the superheavy atom with the critical nuclear charge number $Z = 170$. Here the nuclear radius was adjusted so that the K-electron energy eigenvalue differed only by 10^{-3} eV from the borderline of the negative energy continuum. Our numerical calculations [31] for $Z = 170$ yielded $\Delta E_{se} = 10.989$ keV, which still represents only a 1% correction to the total

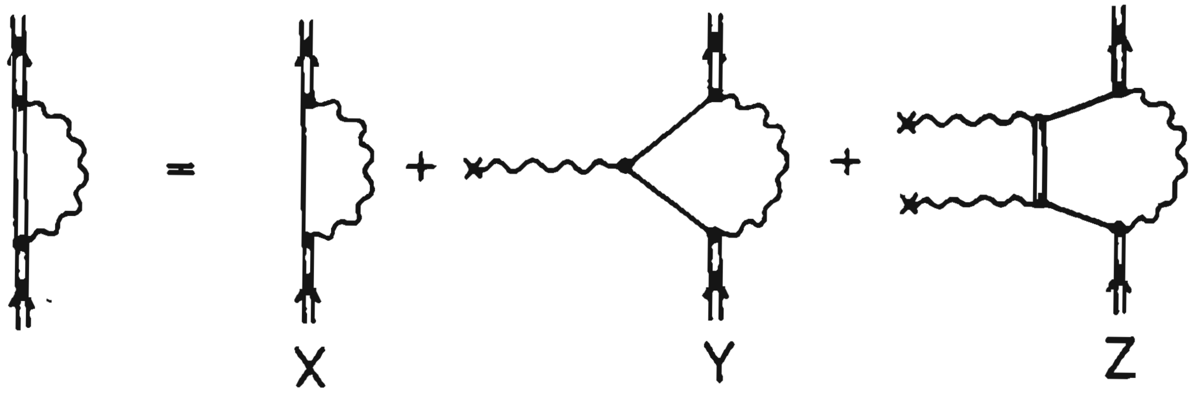


Figure 8: Graphical representation of the self energy in fig. 1b. The upper part of fig. 7 has been inserted. The various terms are denoted by X, Y and Z.

K-electron binding energy. The sum of all radiative corrections thus almost cancels completely at the continuum boundaries.

We conclude that radiative corrections such as vacuum polarization or self energy may not prevent the K-shell binding energy from exceeding $2mc^2$ in superheavy systems with $Z > Z_{cr} \sim 170$.

7. THE LAMB SHIFT IN HYDROGEN-LIKE ATOMS

With the new GSI – SIS facility it will be posible to produce hydrogen-like high-Z atoms. As a contribution to precision atomic spectroscopy and as a test of quantum electrodynamics in strong fields we evaluated higher-order radiative corrections to the binding energy of inner-shell electrons. In fig. 10 we summarize graphically the contributions to the $1s_{1/2}$ Lamb shift [14]. This figure illustrates the well-known fact that the point-nucleus self energy and the Uehling potential yield the dominant contributions to the Lamb shift for low and intermediate values of Z. The figure also illustrates the fact that nuclear finite size corrections become as important as the self energy toward the end of the periodic table.

Fig. 11 displays a comparison of our theoretical results for the total 1s Lamb shift in hydrogen-like atoms with available experimental data (cf. e.g. refs. 33-55). The same units as in fig. 10 are employed. The finger points to a very precise measurement [48] of the 1s Lamb shift in a high-Z system. Employing a recoil-ion technique the 1s Lamb shift for hydrogen-like argon ($Z = 18$) could be determined with a relative accuracy of about 1% in fair agreement with the theoretical predictions [14,15].

The ultimate aim of these QED tests for strong Coulomb fields will be a precise determination of the 1s Lamb shift in hydrogen-like uranium. Various considered deviations from ordinary QED corrections e.g. nonlinear extensions of the Dirac equation [56] are expected to be most pronounced in atoms with strong electric fields and high electron densities. However, it was demonstrated [57] that QED tests aiming at utmost

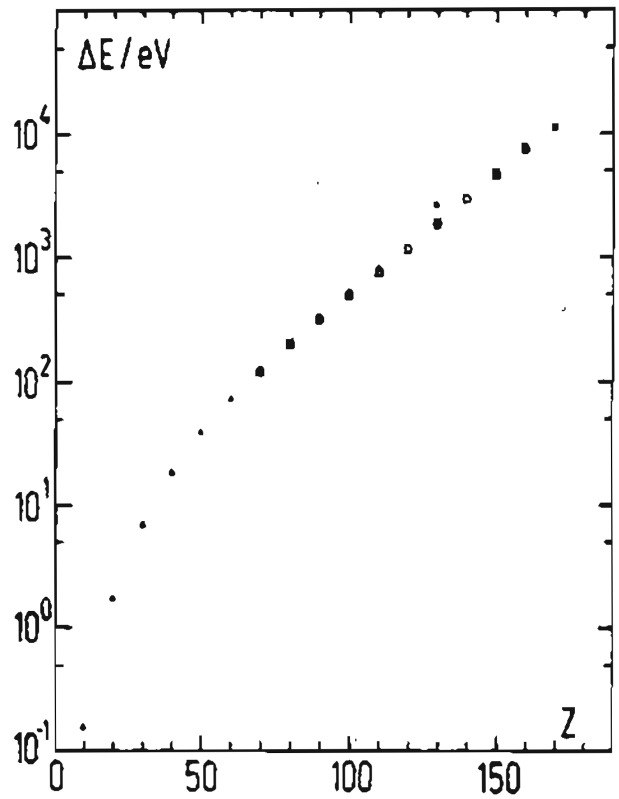


Figure 9: The self-energy shift of K-shell electrons is plotted versus the nuclear charge number Z. $(\bullet\ \bullet\ \bullet)$ = numerical result of Mohr [13] for 1s-electrons in the Coulomb field of point-like nuclei; $(\square\square\square)$ = the computed values of Cheng and Johnson [29]; $(\times\ \times\ \times)$ = our result [31].

precision are limited by nuclear polarization corrections which amount for the 1s state to about 1 eV in $^{238}_{92}$U compared with a total Lamb shift of about 450 eV.

The precise knowledge of radiative corrections on electron levels is also of definite interest in connection with the prospects of an atomic parity violation experiment in helium-like uranium which ultimately could lead to a more accurate determination of the Weinberg angle. In this system electron wavefunctions display a relatively strong overlap with the nuclear interior which causes a considerable parity violation effect [58] on the almost degenerated electron states $^1S_0$ and $^3P_0$ with opposite parity.

Finally we discuss briefly as a side-remark the influence of the vacuum polarization on nuclear fusion cross sections at astrophysical energies. In heavy-ion scattering the vacuum polarization potential leads to an additional contribution to the Coulomb potential which ultimately results in deviations from ordinary Rutherford or Mott scattering. Subbarrier fusion is extremely sensitive to any correction of the Coulomb potential. Due to the exponential dependence of the tunneling probability on the relative separation between the charge centers tiny changes may cause considerable drifts of the fusion cross section. This may even modify the element synthesis in the universe. As example we show in fig. 12 the ratio $R(E) = \sigma_{stan}(E)/\sigma_{vac}(E)$ where σ_{stan} refers to the ^{16}O + ^{16}O subbarrier fusion cross section calculated in the standard approach without consideration of vacuum polarization effects. The latter are included in σ_{vac}. The calculations are performed as outlined in ref. 59. For subbarier energies being most relevant for the element synthesis in the universe we obtained modifications of the nuclear fusion cros section in the order of typically 10% - 20%.

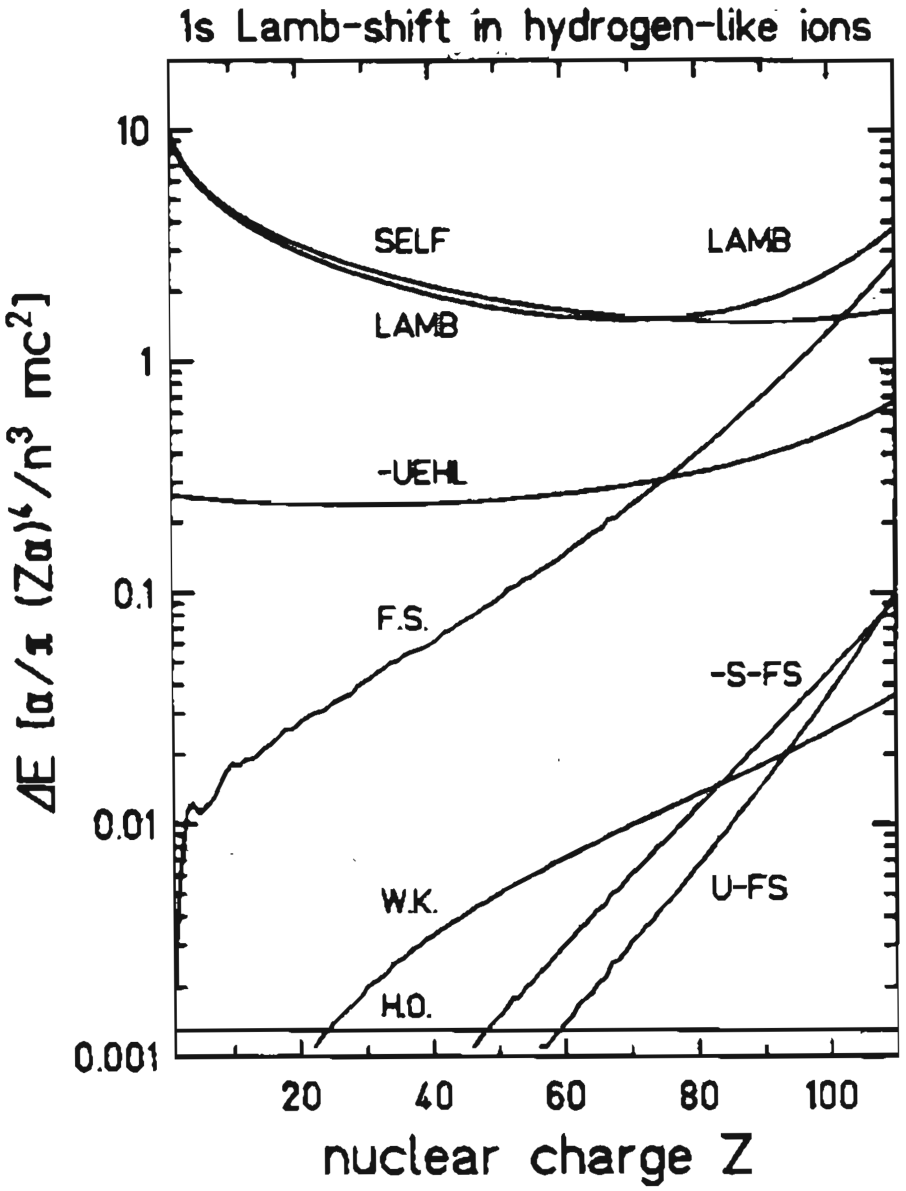


Figure 10: Contributions to the Lamb shift [14] of $1s_{1/2}$ electrons in hydrogen-like atoms versus the nuclear charge number Z. The energy shift ΔE is presented in units of $(\alpha/\pi)\,(Z\alpha)^4/n^3\,mc^2$. LAMB indicates the sum of all contributions. The dominant term (SELF) is provided by the point-nucleus self-energy shift. UEHL denotes the level shift caused by the Uehling potential for point-like nuclei. The energy correction F.S. results from the finite size of the nucleus. The slight irregularities reflect the noncontinuous dependence of the nuclear radius R on the charge number Z. The finite nuclear size correction to the self energy and to the Uehling potential lead to energy shifts S-FS and U-FS, respectively. W.K. denotes the Wichmann-Kroll term and H.O. signifies higher-order corrections incorporating the exchange of two photons. Most of the contributions as well as the total Lamb shift are repulsive. Attractive contributions are indicated by a minus sign.

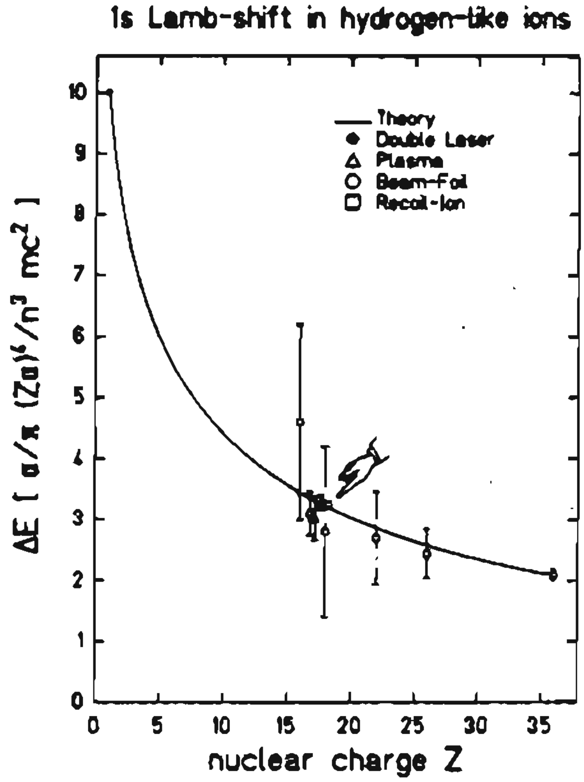


Figure 11: Comparison of theoretical results for the total 1s Lamb shift with available experimental data. The finger points to a precise experimental result of ref. 48.

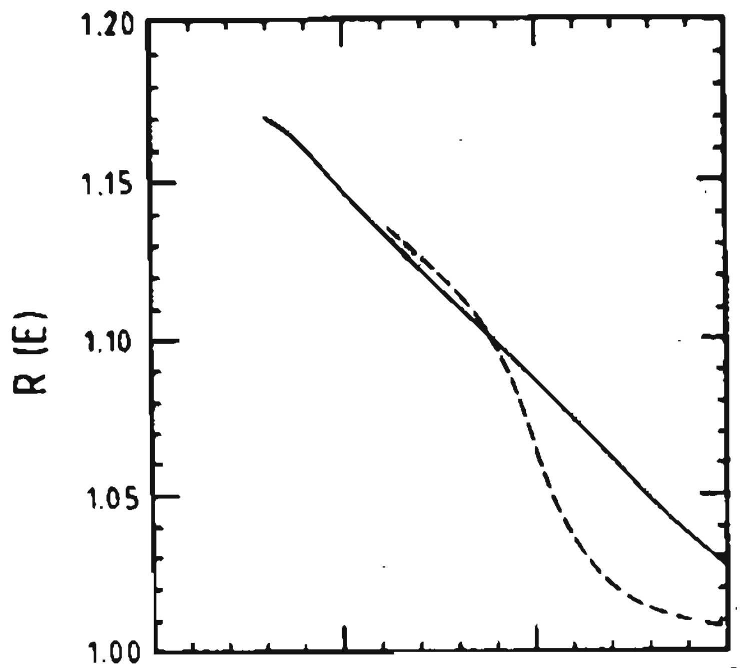


Figure 12: Influence of the vacuum polarization potential on nuclear fusion cross sections for the system $^{16}O + {}^{16}O$. Comparison of the ratio of barrier penetrabilities (solid line) with the ratio of cross sections $R(E) = \sigma_{stan}(E)/\sigma_{vac}(E)$ as calculated within the incoming-wave boundary condition model [59].

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NON-LOCAL EFFECTS OF CLASSICAL ELECTRODYNAMICS AND THEIR APPLICATION IN QED.

S.L.Lebedev
Chuvash state pedagogical institute
428000 Cheboksary

1.INTRODUCTION.

The definition of the self-field energy of an electron is as much fundamental problem as ancient one. This communication is devoted to the determination of the combined influence of the external electromagnetic field and the boundary on the self-energy of the classical charge. We use two models-of an ideally conducting planar surface and of a plane resonator,-which enable one to find out unambiguously the boundary effects and to preserve at that time the Lorentz covariance with respect to the boosts parallel to the boundary. From a more general point of view, those models give an opportunity to construct a relativistic theory of a charge interecting with a conducting "medium". Contrary to the problems of transition and Cherenkov radiations, there is a need here to examine an accelerated motion of the charge, because in this case an infinite self - field energy experiences a finite shift nontrivially depending on acceleration and relative position of the charge and the boundary. Similar shifts take place also for the charge moving in an inhomogeneous medium or located in the thermal "bath" or gravitational field [1-3].

Self-action effects in the classical electrodynamics of the point particle are observed locally as a radiation force and non-locally as a final additions to the mass and magnetic moment of an electron. The dependence of such attributes of the particle on parameters determining its surroundings, just reveals non-locality of self-field corrections [4]. Finiteness of the velocity of the light means that non-locality in time is also present: the formation time should pass, making it possible to observe above-mentioned corrections. During this time interval a self-field reconstruction takes place, giving a finite addition to the particle's mass, which does not subsequently depend on time, if constant fields and the trajectories parrallel (in the "mean") to the boundary are considered (see Sec.3 below).

Following to [5-8] let us define correction to the classical charge action, which stems from the self-action change, caused by the external field (F) and the boundary (B)[1]:

$$\Delta W = \frac{e^2}{2} \int\int \dot{x}_\alpha(\tau)\, \dot{x}_\beta(\tau') D^{(B)}_{\alpha\beta}(x,x') d\tau\, d\tau' \Big|_0^{F,B} . \qquad (1)$$

Here $x=x_\alpha(\tau)$ is the world-line of the particle, τ is its proper time and $D^{(B)}_{\alpha\beta}(x, x')$ is the causal photon Green function (GF) incorporating the boundary conditions imposed on the self-field of the charge. The causality of GF in (1) is relevant, for it makes possible to interpret $\exp(i\Delta W)$ as a classical limit of the electron's elastic scattering amplitude, so that the $\mathrm{Im}\Delta W$ determines the probability of the radiation and should be positive [5, 9]. The real part of ΔW could be also calculated through the retarded (and real) GF. A causal connection between the reactive change of the self-field of the charge and its radiation is expressed by means of dispersion relations (DR) between real and imaginary parts of ΔW. In Sec. 2 for the motion parallel to the boundary we prove the validity of DR on a basis of a rather general properties of the world-lines of the particle. This result in a sense copies the one in [9], where however there was no boundary but instead nonzero mass (μ) was attributed to "photon". General formulae of Sec.2 are used further in three possible particular cases. In addition to a simple analytical expressions, we give a numerical results for the mass shifts as well. In Sec.3 method is generalized for the periodic trajectories wich are not parallel to the boundary. Therein we discuss a previously established [8] connection between the mass shift conception and the boundary-induced cyclotron frequency shift (CFS), which was demonstrated as having an appreciable role in (g-2)-experiments with a single electron, confined in a resonator [10-

[1] We use a system of units with $c=1$, $h=1$, and $\alpha = e^2/4\pi hc$; components of 4-vectors are denoted by $a_\alpha=(\bar{a}, ia_0)$.

-13]. The link is found with the help of an effective Darwin-type Lagrangian incorporating self-interaction and CFS. We give also comparison with the known up to now (particular) results.The last section concerns CFS in a plane resonator. Some specific features make this second problem interesting, despite both are rather a qualitative approximations to the real situation in Penning trap [10]. These are: i)the appearance of resonances, i.e. infinite growth of CFS [11,13,14[2]] with respect to the magnetic field; ii) the existance of an "antiresonant" points between plates, where resonances do not occur; iii) some differencies of present results from the previous works on that problem.

2. MASS SHIFT FOR THE MOTION PARALLEL TO THE MIRROR.

2a) GENERAL FORMULAS AND DR

Causal GF in a presence of the conducting plane could be found by separation of space and Lorentz variables in the form [15]

$$D^{(B)}_{\alpha\beta}(x,\ x')=\delta_{\alpha\beta}D^{(\alpha)}(x,\ x'), \tag{2}$$

where

$$D^{(1)}=D^{(2)}=D^{(0)}=(i/4\pi^2)[(x-x')^{-2}-(x-\tilde{x}')^{-2}] \tag{3}$$

and satisfy the Dirichlet boundary condition at $x_3=0$, whereas

$$D^{(3)}=(i/4\pi^2)[(x-x')^{-2}+(x-\tilde{x}')^{-2}] \tag{4}$$

is the Neumann boundary problem solution. x_α and $\tilde{x}_\alpha$ in (3) , (4) are 4-vectors of the charge and its image correspondingly . The world trajectory of the charge, moving in a constant electromagnatic field, has a property of isometry[3] [5]:

$$(x(\tau)-x(\tau'))^2=f(\tau-\tau'). \tag{5}$$

If performing a motion parallel to the plane, the interval corres-

[2] The main result of this paper was critisized [11] as related to the spin-precession frequency shift. We show, however, its similarity to ours (63), though the latter relating to CFS.

[3] This term in the problem at hand was introduced in [16].

ponding to (5) for the image charge [whose world line is $\tilde{x}_\alpha(\tau)$] is

$$(x(\tau)-\tilde{x}(\tau'))^2=f(\tau-\tau')+R^2 \ , \tag{6}$$

where R is twice the distance from the charge to the plane.

The isometry of world lines entails [5]

$$\Delta W = -\Delta m\cdot\tau \tag{7}$$

(the translational divergence in τ being extracted as a factor) . With the help of (1)-(7) we, finally, obtain

$$\Delta m = \frac{-i\alpha}{2\pi}\int_0^\infty dx\left[\frac{f''(x)}{f(x)}-\frac{2}{x^2}-\frac{f''(x)}{f(x)+R^2+i0}\right]. \tag{8}$$

The main properties of the $f(x)$ obvious from its definition (5), are: i) $f(x) = f(-x)<0$ if $x \neq 0$, ii) $f'(\tau-\tau')= 2\dot{x}_\alpha(\tau)(x(\tau)-x(\tau'))_\alpha \neq 0$ if $\tau \neq \tau'$ (causality of world line); iii) being entire function $f(x)= =-x^2+ O(x^4)$ when x or F (field) are approaching zero. As a consequence, the equation

$$f(\tau)+R^2=0 \tag{9}$$

has only two (non-multiple) roots $\tau_\pm$, $\tau_+=-\tau_-$; τ_+ corresponds to retarded proper time interval between the emission of the photon and its absorption after reflection from the mirror . The real part of Δm, determined by the +i0-prescription in the denominator in (8), is

$$\mathrm{Re}\ \Delta m = \frac{\alpha}{2}\,\frac{f''(\tau_+)}{|f'(\tau_+)|}<0, \tag{10}$$

and could be seen as having a purely geometric nature. It is essential that extraction of the integral corresponding to third term in a brackets of (8) is possible only for infrared-finite case, i.e. when integral (8) exists as $R^2\to\infty$ [4].

Analitical properties of $\Delta m(R^2)$ are easily deduced from the repre-

[4] This is not the case for the motion in the electric-type field ($F^2_{\mu\nu}>0$). Similar behaviour of Δm has been observed at μ^2-0, [5,9].

sentation (8): i) Δm is analytical function in complex R^2-plane with the cut along the positive real axis; ii) $\Delta m\,(z)^* = -\Delta m\,(z^*)$ (Riemann-Schwarz symmetry).

Those properties should be accompanied by asymptotic constraints at $z \to 0$:

$$\Delta m \simeq \frac{-\alpha}{2\sqrt{z}} + O(\sqrt{z}) \,, \tag{11}$$

and at $|z| \to \infty$:

$$|\Delta m| \leqslant \mathrm{const}\cdot \ln|z| \,. \tag{12}$$

The first constraint (11) could be readily recognised as a Coulomb asymptotic (if $Z = R^2 + i0$); the second one (12) is infrared behaviour constraint (in particular case of purely electric field both were discussed in [7]). Making use of Cauchy theorem with a contour depicted at Fig. 1, after some standard mathematics [17] we obtain

$$\mathrm{Re}\,\Delta m_a(R^2) = \frac{2R}{\pi}\int_0^\infty \frac{\mathrm{Im}\Delta m_a(u^2)du}{(u^2-R^2)} \,, \tag{13}$$

$$\mathrm{Im}\,\Delta m_a(R^2) = \frac{-2R^2}{\pi}\int_0^\infty \frac{\mathrm{Re}\Delta m_a(u^2)du}{u(u^2-R^2)} \,, \tag{14}$$

where $\Delta m_a = \Delta m + \alpha/2R$, -mass shift (8) with the Coloumb term $(-\alpha/2R)$ subtracted, see (11).

2b) PARTICULAR CASES

Three different examples should be discussed: i) the crossed EM - field ($F^2_{\mu\nu}=0$, $\tilde{F}_{\mu\nu}.F_{\mu\nu}=0$); ii) the purely electric field ($F^2_{\mu\nu} = \mathcal{E}^2 > 0$); iii) the purely magnetic field ($F^2_{\mu\nu} = -\eta^2 < 0$). Another combinations could be considered by making use of corresponding Lorenz transformation. For the crossed field case we have [5]:

$$f(\tau) = -\tau^2 - a^2\frac{\tau^4}{12} \,, \quad a^2 = (eF_{\mu\nu}\dot{x}_\nu)^2 \,, \tag{15}$$

so that according to (9), (10)

$$\tau_+ = a^{-1}\left[6\,(\sqrt{1+R^2a^2/3}-1)\right]^{1/2} , \tag{16}$$

$$\mathrm{Re}\Delta m^{cr} = -\frac{\alpha a}{\sqrt{6}} \cdot \frac{1{,}5(1+R^2a^2/3)^{1/2}-1}{(1+R^2a^2/3)^{1/2}\,[(1+R^2a^2/3)^{1/2}-1]^{1/2}} \ . \tag{17}$$

The weak-field ($Ra \to 0$) limit of (17) obviously coinsides with (11), and ultrarelativistic (UR) limit ($Ra \to \infty$) is

$$\mathrm{Re}\Delta m^{cr} = -\frac{3\alpha}{2R}(Ra)^{1/2}\, 6^{-1/4} \ . \tag{18}$$

For the planar motion in electric field $\mathcal{E}$ ($\nu = e\mathcal{E}$) or magnetic field η ($\ae = e\eta$) a corresponding formulae were derived elsewhere [7]:

$$\mathrm{Re}\Delta m^{el} = -\frac{\alpha\nu}{2m}\,\frac{\mathrm{ch}\theta - V_{\perp o}^2}{\mathrm{sh}\theta - V_{\perp o}^2\theta} \ ; \qquad \theta = \frac{\nu\tau_+}{m} \tag{20}$$

$$V_{\perp o}^2 = 2(\tilde{F}_{\mu\nu}\dot{x}_\nu)^2/(2(\tilde{F}_{\mu\nu}\dot{x}_\nu)^2 + \tilde{F}_{\mu\nu}^2) \ , \tag{21}$$

$$\mathrm{Re}\Delta m^{mag} = -\frac{\alpha\ae}{2m}\,\frac{1 - V_\perp^2\cos\theta}{\theta - V_\perp^2\sin\theta} \ , \qquad \theta = \frac{\ae\,\tau_+}{m} \ , \tag{22}$$

$$V_\perp^2 = 2(F_{\mu\nu}\dot{x}_\nu)^2/(2(F_{\mu\nu}\dot{x}_\nu)^2 + F_{\mu\nu}^2) \ . \tag{23}$$

$V_{\perp o}^2$ (21) and $V_\perp^2$ (23) are integrals of the motion in electric and magnetic fields respectively. Dimensionless parameter θ is the root of corresponding transcendential equation:

$$4\mathrm{sh}^2\frac{\theta}{2} = V_{\perp o}^2\,\theta^2 + \left[\frac{\nu R}{m}\right]^2(1 - V_{\perp o}^2) \qquad \text{(electric field)}, \tag{24}$$

$$\theta^2 = 4V_\perp^2\,\sin^2\frac{\theta}{2} + \left[\frac{\ae R}{m}\right]^2(1 - V_\perp^2) \qquad \text{(magnetic field)}. \tag{25}$$

It is worthy of note that UR asymtotics of $\mathrm{Re}\Delta m$ in all cases considered here coinside[5] [7]:

[5] Dr. V I Ritus first obtained formulae (17),(18) (private communication) and called author's attention to the (wanted) universal character of the $\mathrm{Re}\Delta m$ UR-asymptotic: UR particle in its proper reference system

$$\mathrm{Re}\Delta m^{el} \simeq \frac{-3\alpha}{2R}\left[\frac{\nu R}{m}\,\gamma_{\perp o}\right]^{1/2} 6^{-1/4} \qquad (\gamma_{\perp o}=(1-V_{\perp o}^2)^{-1/2} \to \infty\ ), \quad (26)$$

$$\mathrm{Re}\Delta m^{mag} \simeq \frac{-3\alpha}{2R}\left[\frac{\varkappa R}{m}\,\gamma_{\perp}\right]^{1/2} 6^{-1/4} \qquad (\gamma_{\perp}=(1-V_{\perp}^2)^{1/2} \to \infty\ ), \quad (27)$$

and correspondence amongst (18),(26) and (27) is as following:

$$a \sim \frac{\nu}{m}\,\gamma_{\perp o} \sim \frac{\varkappa}{m}\,\gamma_{\perp} \quad . \qquad (28)$$

Boundary-induced mass shift of the charge, moving in electric field, has remarkable property: it does not fall to zero in no-boundary limit $R \to \infty$ (the answer well known from [5, 9]); in that time its manifestly non-local nature, seeing from (10), becomes hidden.This classical mass shift determines lowest-order radiation correction to the pair production rate in QED [21].

2c) NUMERICAL RESULTS

Mass shifts (20) and (22) are viewed in the units of $-\alpha/2R$ at the Figs 2,3 ($B=10V_{\perp}^2$). Most interesting seems an oscillatory behaviour of $\mathrm{Re}\Delta m^{mag}$ which increases with the parameters $\frac{\varkappa R}{m}$ and $V_{\perp}^2$ growing. The extreme properties of $\mathrm{Re}\Delta m^{mag}$ may be of some interest for the accelerator physics. Figures 4-6 depict boundary-induced imaginary parts of Δm^{mag} and Δm^{el} respectively. It should be stressed, however, that

$$\mathrm{Im}\ \Delta m^{mag} = \mathrm{Im}\ \Delta m_b^{mag}(V_{\perp}^2,\ \varkappa R/m) + \mathrm{Im}\ \Delta m_{\infty}^{mag}(V_{\perp}^2)\ ,$$

where the first item is just the boundary contribution whereas $\mathrm{Im}\Delta m_{\infty}^{mag}$ corresponds to no-boundary limit of mass shift (it was derived in [6]). No such partitioning exists for the $\mathrm{Im}\Delta m^{el}$, since the latter becomes infinite at $R \to \infty$ (see footnote 4 on p.4).

Below we give basic formulae for $\mathrm{Im}\Delta m_b^{mag}$, $\mathrm{Im}\Delta m_{\infty}^{mag}$, $\mathrm{Im}\Delta m^{el}$, which are a direct consequences of (8) and of the explicite field-dependent forms (see [6]) of function $f(\tau)$ (5) ($\omega_c=\varkappa/m\gamma_{\perp}$):

"sees" either EM-field as a crossed one (See e.g. [18]).

$$\mathrm{Im}\ \Delta m_b^{mag} = \frac{\alpha}{2R}\cdot\frac{\text{æ} R}{\pi m}\int_0^\infty \frac{1-V_\perp^2\cos 2w}{w^2-V_\perp^2\sin^2 w-(\omega_c R/2)^2}\,dw\ ,$$

$$\mathrm{Im}\ \Delta m_\infty^{mag} = \frac{-\alpha}{2R}\cdot\frac{\text{æ} R}{\pi m}\int_0^\infty \left[\frac{1-V_\perp^2\cos 2w}{w^2-V_\perp^2\sin^2 w} - \frac{1}{w^2}\right]dw\ ,$$

$$\mathrm{Im}\ \Delta m^{el} = \frac{-\alpha}{2R}\cdot\frac{\nu R}{m\pi}\int_0^\infty \left[\frac{V_{10}^2-\mathrm{ch}2w}{V_{10}^2 w^2-\mathrm{sh}^2 w} - \frac{1}{w^2} - \frac{V_{10}^2-\mathrm{ch}2w}{V_{10}^2 w^2-\mathrm{sh}^2 w+(R\nu/2\gamma_{10}m)^2}\right]dw.$$

For convenience sake $\mathrm{Im}\Delta m_\infty^{mag}$ (Fig.6) as well as $\mathrm{Im}\Delta m_b^{mag}$ and $\mathrm{Im}\Delta m^{el}/4$ (Figs.4,5) are presented in the units of $\alpha/2R$ (despite $\mathrm{Im}\Delta m_\infty^{mag}$ does not really depend on R). A domain of values of æR/m is chosen to be qualitatively corresponding to the experiment (see below). Neither arguments were used to determine a domain of $\nu R/m$, but monotonic character of $\mathrm{Im}\Delta m^{el}$ as a function of $\nu R/m$ and $V_\perp^2$ is beyond question.

3.MASS SHIFT AND CFS FOR THE MOTION IN A PROXIMITY TO THE MIRROR.

3a) DIMENSIONLESS PARAMETERS.

Let me remind an "experimental" situation, which forms the basis of full QED calculations [11,12,19,20] and is governed by interrelations amongst following parameters:

$1/m \equiv \hbar/mc$ - Compton length;

$R_c^q = (e\eta)^{-1/2} \equiv (e\eta/\hbar c)^{-1/2}$ - quantum cyclotron radius determinig the domain of diffusion of the ground state[6] WF of an electron;

l and/or R - the size of the resonator and/or the distance between centre of the orbit and (one of) the plate.

The main inequalities stemming from the analysis of the real experi-

[6] According to [22] the mean value of cyclotron quantum number ⟨n⟩ is about 1.2 , so that the excitement is small.

mental conditions [10], are as following

$$\frac{1}{ml} < \frac{R_c^q}{l} < 1 \,, \tag{29}$$

where the first inequality manifests non-relativizm, whereas the second implies that boundary restriction of the electron field is completely negligible. Quantum constraints (29) preserve a degree of freedom for the classical parameter $\omega_c l \sim \omega_c R$ (ω_c being a cyclotron frequency $æ/m\gamma_\perp$) since, along with (29), two possibilities

$$\frac{1}{ml} < \left[\frac{R_c^q}{l}\right]^2 \,, \quad \frac{1}{ml} > \left[\frac{R_c^q}{l}\right]^2 \tag{30a,b}$$

are equally admissible. Ineq. (30b) is realised in (g-2)-experiments, ($\omega_c l \sim 25$) [13]. QED-results for the rest-energy shift and CFS were obtained for this limiting cases [11,12,19,20] and turned out to be classical. So, the classical calculation, which embraces the whole domain of $\omega_c R$ is most likely to be useful in QED.

3b) EFFECTIVE LAGRANGIAN AND CFS.

Mass shift conception appears to be applicable, when evaluating the apparatus-dependent contributions to the mass and magnetic moment of the electron confined in a resonator ("Penning trap") [10]. The first quantitative estimates of this phenomenon were based on a simple models of (plane, cylindrical, spherical) resonators, within which electron was supposed to perform a motion in a strong (~ 50 kGs) homogeneous magnetic field. It was concluded [10 - 13], that main contribution in g-2 should stem from the boundary-induced CFS [7], determining through the time averaging of the classical Lorentz equation, incorporating the image charges' fields.

At first let us consider a simplest model of one-mirror "resonator" to demonstrate the mass-shift method [7]. Most important seems to be a distinct behaviour of the rest-energy shift and CFS when field intensity varying: the former being always Coulomb in nature (and independent

[7] but not from the spin-precession frequency shift.

[illegible] field), but the latter containing Coulomb interaction as well as the retardation effects. A further advantage of the method lies in its applicability to the whole interval of $R\omega_c$, the retarded ($R\omega_c \gg 1$) and non-retarded ($R\omega_c \ll 1$) regions included. There is no restriction also on [illegible] - the integral of the cyclotron motion, but below we shall consider [illegible] $\ll$ [illegible] to be relevant to the experiment [10].

As a function of $V_\perp^2$ $\mathrm{Re}\Delta m^{mag}$ changes the "dispertion law" $E(V_\perp^2)$, where E is electron energy taken at the rest system of the cyclotron orbit. Indeed, as we shall see from (40), the addition $\Delta L = -\mathrm{Re}\Delta m^{mag}/\gamma_\perp$ to the Lagrangian of the particle in that frame has a non-relativistic expansion

$$\Delta L = \frac{\alpha}{2R} + \frac{\delta m}{2} V_\perp^2 , \tag{31}$$

so that the energy of the particle

$$E(V_\perp^2) = m - \frac{\alpha}{2R} + \frac{m + \delta m}{2} V_\perp^2 . \tag{32}$$

Darwin-type expressions like (31), (32) [incorporating interaction of the charge and its image] entail the CFS

$$\delta\omega_c = -\omega_c \, \delta m/m \tag{33}$$

[note that δm in general is not equal to $-\alpha/2R$].

Let φ will be an angle between cyclotron trajectory plane and the boundary, disposed at $x_3 = 0$. Supposing the centre of the orbit being R/2 apart from the boundary, for the intervals (5), (6) we obtain respectively

$$(x - x')^2 = 4\omega_c^{-2} \; (-w^2 + V_\perp^2 \sin^2 w) \equiv f(w), \tag{34}$$

$$(x - \tilde{x}')^2 = 4\omega_c^{-2} \; [V_\perp^2 \sin^2 w + \tfrac{1}{2} V_\perp^2 \sin^2\varphi \,(\cos 2w + \cos 2u) -$$
$$- R\omega_c V_\perp \sin\varphi \cos u \cos w + (R\omega_c/2)^2 - w^2 \;] \equiv \tilde{f}(w,u). \tag{35}$$

Here the notations $w = \varkappa(\tau - \tau')/2m$ and $u = \varkappa(\tau + \tau')/2m$ have been introduced. Making use of eqs. (1)-(4) we thus obtain

$$\Delta W = \left[\frac{\varkappa}{2m}\right]^2 \int_{-\tau_0}^{\tau_0} d\tau \int_{-\tau_0}^{\tau_0} d\tau' \, \mathcal{F}(w,u), \tag{36}$$

where

$$\tau_0 = (2\pi m/\varkappa)N, \quad N \to \infty , \tag{37}$$

and the averaged over the proper time period $2\pi m/\varkappa$

$$\Delta m^{mag} = \frac{-1}{2\tau_0}\Delta W = \frac{-\varkappa}{2\pi m}\int_0^{2\pi} du \int_0^{\infty} dw \, \mathcal{F}(w,u) \; . \tag{38}$$

Here the function

$$\mathcal{F}(w,u) = \frac{i\alpha}{4\pi}\left[\frac{f''(w)}{f(w)} - \frac{2}{w^2} - \frac{\tilde{f}''_{ww}(w,u) - \tilde{f}''_{uu}(w,u)}{\tilde{f}(w,u)+i0}\right] \tag{39}$$

is periodic with respect to u ,and formula (38) is a direct generalization of (8) . The real part of Δm^{mag} could be found in a close analogy to (10),

$$\mathrm{Re}\Delta m^{mag} = \frac{\alpha\varkappa}{8\pi m}\int_0^{2\pi} du \, \frac{\tilde{f}''_{ww}(w_+,u) - \tilde{f}''_{uu}(w_+,u)}{|\tilde{f}'_w(w_+,u)|} , \tag{40}$$

w_+, being a positive root of the equation $\tilde{f}(w,u) = 0$, admits of power expansion in $V_\perp^2$, as well as the whole integrand of (40) does. Then term by term integration entails ($V_\perp^2 < 1$)

$$\mathrm{Re}\Delta m^{mag} = \frac{-\alpha}{2R}\gamma_\perp\left[1 + (V_\perp^2/2)\,(\Delta_1 + \sin^2\varphi\,\Delta_3)\right], \tag{41}$$

$$\Delta_1 = 2\left(\frac{\sin 2X}{2X} - \cos 2X - \frac{1-\cos 2X}{4X^2}\right), \tag{42}$$

$$\Delta_3 = \cos 2X + \frac{\sin 2X}{2X} + \frac{3+\cos 2X}{4X^2}\, , \; X = R\omega_c/2. \tag{43}$$

The final answer for the $\delta\omega_c$ obtained with the help of (31)-(33) is as following:

$$\delta\omega_c/\omega_c = \frac{-\alpha}{2Rm}\,(\Delta_1 + \sin^2\varphi\,\Delta_3). \tag{44}$$

3c) COMPARISON WITH EARLIER RESULTS.

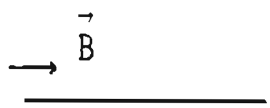
Formula (44) covers all known up to now special results and asymptotics of CFS in the vicinity of the conducting wall. The work [12] contains (in our notations) a following expression for CFS:

$$\frac{\delta\,\omega_c}{\omega_c} = \frac{\alpha}{2Rm}\,(1 + \cos^2\varphi)\cos\,\omega_c R\ , \tag{45}$$

which is an equation (44) at $\omega_c R > 1$.The authers of [11] recieved some more results: the table below is from their Table 1, and may be easily reconciled with (44).

$\varphi = 0$		$\varphi = \pi/2$	
$R\omega_c > 1$	$R\omega_c < 1$	$R\omega_c > 1$	$R\omega_c < 1$
$\frac{\delta\omega_c}{\omega_c} = \frac{\alpha}{Rm}\cos R\omega_c$	$\frac{\delta\omega_c}{\omega_c} = \frac{\alpha}{2Rm}$	$\frac{\delta\omega_c}{\omega_c} = \frac{\alpha}{2Rm}\cos R\omega_c$	$\frac{\delta\omega_c}{\omega_c} = \frac{-\alpha}{2Rm}\cdot\frac{4}{(R\omega_c)^2}$

(46a,b,c,d)



Note, that (46d) contains a singularity at $\omega_c \to 0$, corresponding to the fall of the charge on the boundary when

$$R_c^{cl} \equiv mV_\perp\gamma_\perp/e\eta \rightarrow R/2. \tag{47}$$

Of course, being (47) fullfilled, one could not neglect the perturbation of the trajectory caused by the mirror.

4. MASS SHIFT AND CFS FOR THE MOTION IN A PLANE RESONATOR.

4a) FORMULAS.

All calculations for the resonator repeat ones for the single mirror. We shall discuss them briefly. GF could be found by the image-charge method [15] and has the form of (2), where, in place of (3), we have

$$D^{(0)}(x,\ x') = \sum_{-\infty}^{\infty} (-1)^{N+1} \frac{1/4\pi^2}{(x-x'^{(N)})^2} = D^{(1)} = D^{(2)} \tag{48}$$

$$D^{(3)}(x,x') = \sum_{-\infty}^{\infty} \frac{1/4\pi^2}{(x-x'^{(N)})^2} \tag{49}$$

Here $x^{(N)}$ corresponds to 4-vector $(x_1,\ x_2,\ x_3^{(N)},\ ix_0)$

$$x_3^{(N)} = (-1)^N\left(x_3 - \frac{l}{2}\right) + l\left(N + \frac{1}{2}\right), \tag{50}$$

N being integer, and the origine is taken to be at one of the plate. With the help of (1), (2) and (48), (49), we arrive at the patterns (36),(38) for the self-action and mass shift, where, however, function $\mathcal{F}(w,\ u)$ redenoting here as $\mathcal{F}_1(w,u)$, has a more complicated structure[8]:

$$\mathcal{F}_1(w,\ u) = \mathcal{F}(w,\ u) + \frac{i\alpha}{4\pi}\sum_{k=1}^{\infty}\left[2\,\frac{g''_{ww}(w,u,Z_k) - g''_{uu}(w,u,Z_k)}{g(w,u,Z_k) + i0} - \right.$$

$$\left. - \frac{\tilde{f}''_{ww}(w,u,X_k^+) - \tilde{f}''_{uu}(w,u,X_k^+)}{\tilde{f}(w,u,X_k^+) + i0} - \frac{\tilde{f}''_{ww}(w,u,X_k^-) - \tilde{f}''_{uu}(w,u,X_k^-)}{\tilde{f}(w,u,X_k^-) + i0}\right] \tag{51}$$

The following notations have been introduced in (51):

$$Z_k = kl\omega_c = 2\Lambda k\ ;\ X_k^{\pm} = 2\Lambda(k \pm t)\ ;\ t = R/2l\ , \tag{52}$$

$$g(w,u,Z_k) = \frac{4}{\omega_c^2}\left[-w^2 + V_1^2\sin^2 w - 2Z_k V_1 \sin\varphi\ \sin u + Z_k^2\right], \tag{53}$$

and function $\tilde{f}(w,u,X_k^{\pm})$ is just the function (35) with $R\omega_c/2 \equiv X_0^+$ replaced by $X_k^{\pm}$. Note that first item of r.h.s. of (51) tally with (39) giving a one-mirror mass shift, but the second one is a resonator contribution disappearing at $l \to \infty$.

Non-relativistic expansion of $\mathrm{Re}\Delta m^{mag}$ is obtained in similarity to

8 On deriving (51) we have used partitioning of the sums (48), (49) into an odd and even summations and some apparent symetry properties of the integrand of (1).

one-mirror case, so that analogue of (41) takes the form

$$\mathrm{Re}\ \Delta m^{mag} = \gamma_{\perp}(\delta m_C - \frac{v^2}{2}\,\delta m\ ), \tag{54}$$

where

$$\delta m_C = \frac{\alpha}{4l}\left[\psi(t) + \psi(1-t) + 2\mathbf{C}\right] \tag{55}$$

is a well-known Coulomb rest energy shift discovered also in QED calculations [11, 19, 20] carried out under conditions (29) [$\mathbf{C}$ = 0,577...is Euler constant]. In accordance with eqs. (31)-(33) δm in (54) determines CFS

$$\frac{\delta\omega_c}{\omega_c} = \frac{\alpha}{4ml}\sum_{k=1}^{\infty}\Big[\,(\Delta_1(Z_k) + \sin^2\varphi\cdot\Delta_2(Z_k))\cdot\frac{2}{k} - (\Delta_1(X_k^-) +$$

$$+ \sin^2\varphi\cdot\Delta_3(X_k^-))\cdot\frac{1}{k-t} - (\Delta_1(X_{k-1}^+) + \sin^2\varphi\cdot\Delta_3(X_{k-1}^+))\cdot\frac{1}{k-1+t}\Big]\,. \tag{56}$$

This combersome expression includes, nevertheless, trivial functions $\Delta_1(X)$ and $\Delta_3(X)$, as they were introduced in (42), (43). Function

$$\Delta_2(X) = \cos 2X - \frac{3\sin 2X}{2X} + \frac{3\sin^2 X}{2X^2}\ . \tag{57}$$

Because of apparent property

$$\Delta_1(X_K^{\pm}) \to -1 \quad \text{when} \quad \omega_c \to 0 \tag{58}$$

we again arrive at coincidence between zero-field limit of $\delta m(\varphi{=}0)$ and δm_C (see (33), (55) and (56)).

4b) COMPARISONS.

When angle φ = 0 and R = l (t = 1/2) formula (56) leads to an expression known from [13] (eq.(3.18)):

$$\frac{\delta\omega_c}{\omega_c} = \frac{2\alpha}{ml} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Big[\frac{\sin 2\Lambda n}{2\Lambda n} - \frac{1 - \cos 2\Lambda n}{4n^2\Lambda^2} - \cos 2\Lambda n \Big]. \tag{59}$$

Note that correspondences between our notations and those of authers [13] are as following:

$$l \equiv 2L, \qquad \alpha/m \equiv r_o \tag{60}$$

$$2\Lambda = \omega_c l \equiv 2\pi\xi_c \simeq 2\pi\xi, \qquad \delta\omega_c \simeq R_p(\omega_c). \tag{61}$$

When $\varphi = \pi/2$, $R = l$ $(t = 1/2)$ formula (56) yields:

$$\frac{\delta\omega_c}{\omega_c} = \frac{-\alpha}{lm} \Big\{ \sum_{n=1}^{\infty} \frac{1}{n} \Big[(-1)^n \cos 2n\Lambda + \frac{\sin 2n\Lambda}{2n\Lambda} + \frac{\cos 2n\Lambda - (-1)^n}{(2n\Lambda)^2} \Big] +$$

$$+ 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \Big[\frac{\sin 2\Lambda(2k-1)}{2\Lambda} + \frac{\cos 2\Lambda(2k-1)}{4\Lambda^2(2k-1)} \Big] \Big\}. \tag{62}$$

Expression (62) contains a familiar from (46d) ($1/\Lambda^2$)-singularity when ω_c or Λ is approaching zero.

The Fourier series of (59), (60) could be summed up [23]. For CFS (62) the answer reads:

$$\frac{\delta\omega_c}{\omega_c} = \frac{-\alpha}{ml} \Big[- \frac{1}{2} \log\, (4\cos^2\Lambda) + \frac{7\zeta(3)}{8\Lambda^2} - \frac{1}{2} \int_0^1 y(3\log|\tan\Lambda y| +$$

$$+ \log|2\sin 2\Lambda y|)dy \Big]. \tag{63}$$

Both expressions (59) and (62) (or (63)) exhibit a resonance behaviour with respect to Λ stemming from the $1/n$ - terms in a summands of (59) and (62). Those resonances were extensively discussed in literature [11, 13, 14].

Some interesting point for us is comparison of the CFS (63) with the anomaly factor correction found in the paper of Bordag [14]. In the notations of [11, 12] anomaly

$$a_e = \frac{g-2}{2} = \frac{\omega_s - \omega_c}{\omega_c}, \tag{64}$$

where ω_s being spin-precession frequency. Boundary-induced contribution to a_e (denoting δa_e here) put forward by Bordag was attributed to $\delta\omega_s/\omega_c$, and, by contrast with another estimates, turned out to be more than expected [11, 12] by the factor $mc^2/\omega_c \gg 1$:

$$\delta a_e = \frac{\alpha}{ml}\left\{\frac{1}{4}\, \log(4\cos^2\Lambda) + \frac{41\zeta(3)}{64\Lambda^2} - \frac{1}{4}\int_0^1 dy \cdot\right.$$

$$\left. \cdot y\left[3\, \log|\tan \Lambda y| + \log|\sin 2\Lambda y|\,\right]\right\} \tag{65}$$

The apparent similarity of the results (63) and (65) is likely to be instructive for elimination of the discrepancy amongst [11, 12] and [14].

4c) "ANTIRESONANT" POINTS.

Formula (56) leads to one important consequence concerning above-mentioned resonances. For simplicity let us take $\varphi=0$ and put together the terms responsible for the logarithmic divergency of CFS (56). Then, denoting the resonance contribution as $(\delta\omega_c/\omega_c)_{res}$ (it would be proportional to $(1/l)$), we have

$$\left[\frac{\delta\omega_c}{\omega_c}\right]_{res} = \frac{-\alpha}{2lm}\sum_{k=1}^{\infty}\left[\frac{-\cos4\Lambda(k-t)}{(k-t)} - \frac{\cos4\Lambda(k-1+t)}{k-1+t} + \frac{2\cos4\Lambda k}{k}\right] \tag{66}$$

With the help of expression 5.4.3.(2) from [24] shift (66) is transformed to[9]

[9] $\beta(t)$-function is determined through an Euler ψ-function:

$$\left[\frac{\delta\omega_c}{\omega_c}\right]_{res} = \frac{\alpha}{2lm}\left[(\beta(t) - \beta(1-t))\cos\pi t + \frac{\cos 4\Lambda t}{t} + 2\log 2 + \right.$$

$$\left. + 2\int_{\pi/2}^{2\Lambda} dy\cdot(1-\cos 2ty)\cdot \mathrm{ctg}\, y\right] \qquad (0<t<1) \tag{67}$$

It is easy to check that resonance points are

$$2\Lambda = 2\Lambda_r = N\pi\ , \quad N = 1, 2, \ldots, \tag{68}$$

and not only $\Lambda=(2N+1)\pi/2$ as it follows from (59) or (63). This fact is already known but for the spin-precession frequency shift [11]. The loss of series $2\Lambda_r=2N\pi$ in previous works is due to existence of an "antiresonant" points.

One could easily find from representation (67), that resonance at $2\Lambda=N\pi$ do not occur for the positions characterizing by

$$t = M/N, \qquad M = 1, 2, \ldots, N-1. \tag{69}$$

This is rather interesting, that between plates being a resonance condition (68) fullfilled, there would be

$$\nu = \frac{2\Lambda_r}{\pi} - 1 \tag{70}$$

points with no logarithmic divergency.

5. CONCLUSION.

In coclusion I would like to give some qualitative remarks only. Sometimes, when deriving CFS and spin-precession frequency shift, one makes use (or tacitly implies) of substitution

$$m \rightarrow m + \delta m \tag{71}$$

into expressions

$$\omega_c = \frac{e\eta}{m\gamma_- c}\ , \qquad \mu = \frac{eh}{2mc}\ . \tag{72}$$

$$\beta(t) = \frac{1}{2}\left[\psi((1+t)/2) - \psi(t/2)\right]$$

δm is supposed to be δm_C or, may be, something else relating to the rest-energy addition. This is not completely true by virtue of different reasons. Firstly, we have shown δm_C differing from δm, so that substitution [10] $m \longrightarrow m + \delta m_C$ (as regards ω_c) is only possible under specific conditions ($\varphi = 0$, $\omega_c - 0$ in adove). Secondly, neither δm_C, nor δm are, strictly speaking a (self-)mass corrections because they are determined as such only for a distinguished reference system (in contrast to Δm (7),(38)).At last should not it be forgotten, that both are defined as a non-local quantities, so characterizing rather a "mean" values of the dynamical variables of moving particle.

As regards boundary-induced magnetic moment correction, I'd like to remind an example, where similar to (71) operation holds for cyclotron frequency, but not for spin-precession one. Magnetic susceptibility of electrons in a metal has two items: a well-known diamagnetic (Landau's) and paramagnetic (Pauli's) contributions. The former determining by the effective mass m_* and cyclotron frequency $\omega_c^* = e\eta/m_*$, but the latter containing a "bare" electron mass m. It is interesting to note, that here we also meet a non-locality though associated with a (Bloch) wave function of an electron.

I would like to thank Professor V.I.Ritus for many stimulating discussions.

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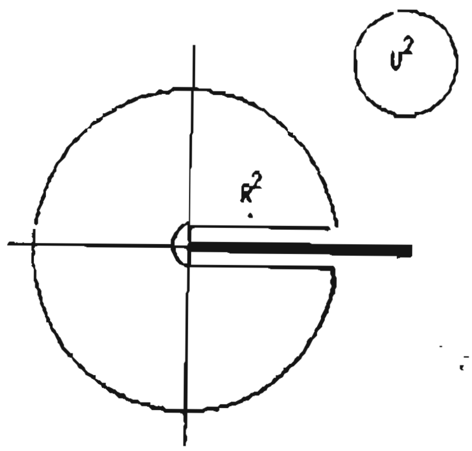


FIG 1

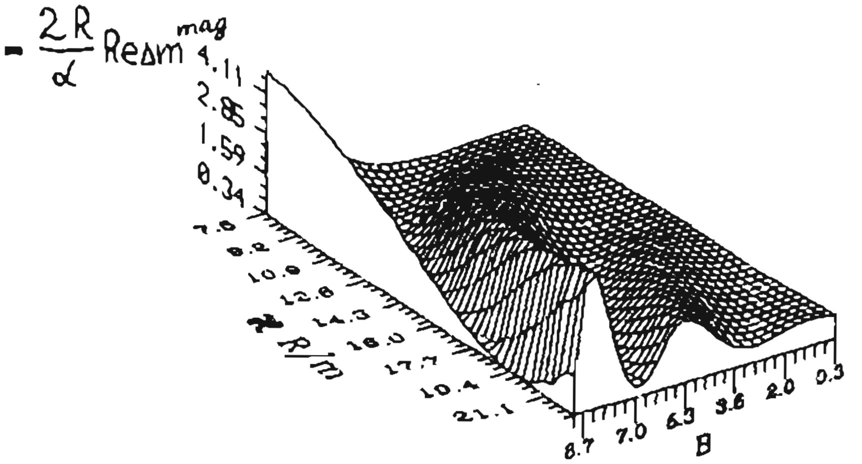


Fig 2

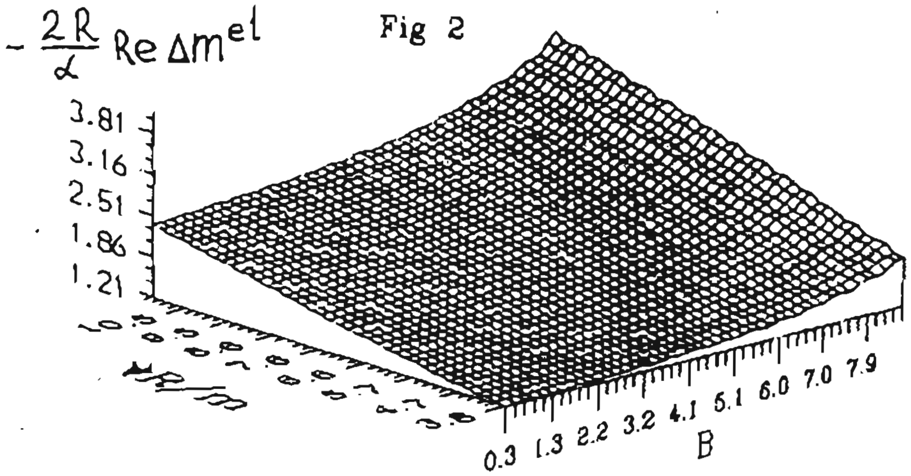
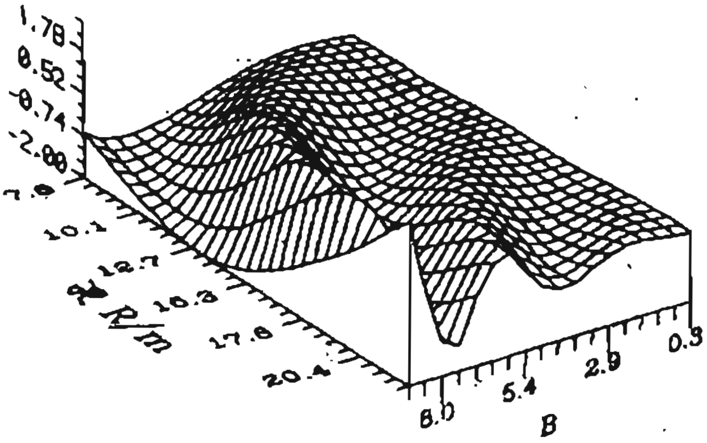


Fig 3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0.52
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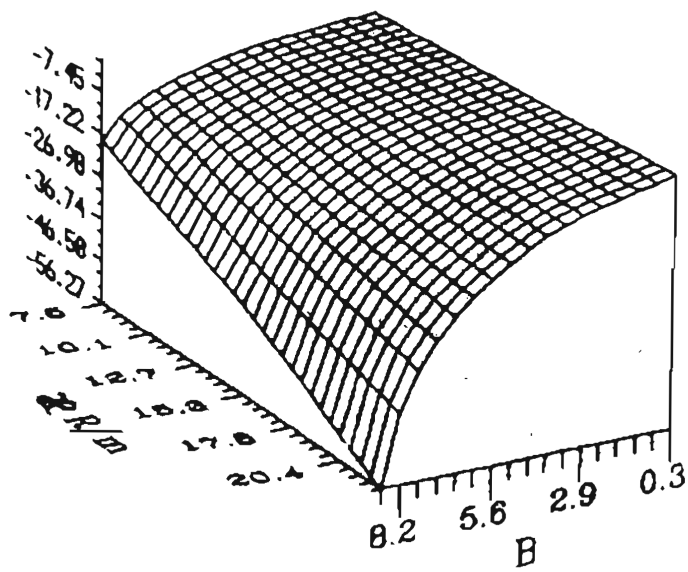
-2.00
7.0
10.1
12.7
15.3
17.8
20.4
R/m
0.3
2.9
5.4
8.0
B

Fig.4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-17.22
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-46.58
-56.27
7.0
10.1
12.7
15.3
17.8
20.4
R/m
0.3
2.9
5.6
8.2
B

Fig.6

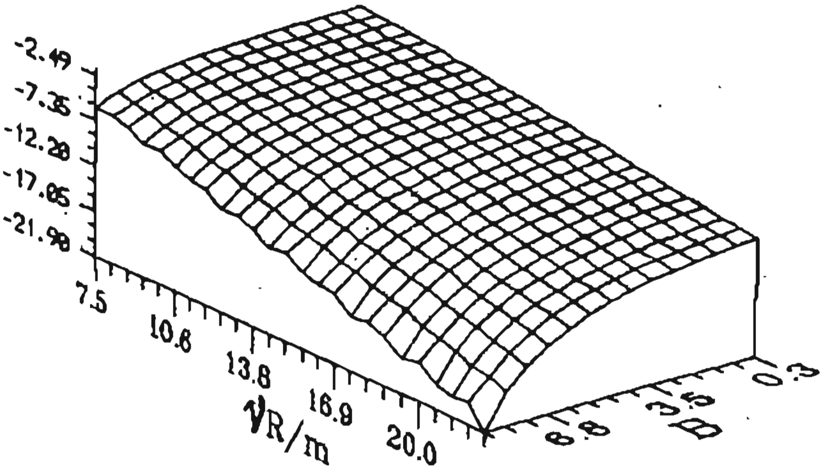

Fig.5

Hydrogen levels between plates

M. Bordag

Universität Leipzig, FB Physik
Augustusplatz 10
O-7010 Leipzig, Germany

Abstract

Progress in high precision measurements enforces growing interest on the influence of boundaries on radiation corrections. In this paper we consider a hydrogen atom inserted between two conducting plates separated by a distance L and calculate the distance dependent part of the levelshift. We include all one loop contributions in the QED perturbation theory and calculate them in a leading order apprximation. The methods employed are based on the full field theoretical formalism of QED. The main tool is the photon propagator in the presence of boundary conditions, which was derived earlier. Explicit numerical results are given for the parameter values relevant in the current experiment.

1 Introduction

Progress in high precision measurements enforces growing interest on the influence of boundaries on radiation corrections. After the discussion of apparatus dependent corrections to the anomalous magnetic moment of the electron several years ago, the interest is focussed on level shifts of the hydrogen [1]. In this paper we consider a hydrogen atom inserted in the middle of a cavity formed by two conducting plates of distance L perpendicular to the axis x_3 (intersecting them at $x_3 = \pm L/2$) and calculate the distance dependent part of the levelshift. Thereby we include all one loop contributions in the sense of QED perturbation theory and calculate them in leading order in the small parameters present here. The starting point is the quantization of electrodynamics with conductor boundary conditions

$$E_{\|} = H_{\perp} = 0 \quad \text{at} \quad x_3 = \pm L/2 \,. \tag{1}$$

Although this is a quite old topic it still requires some clarifications. The conditions (1) are defined in terms of the field strengths. The question arises what they do mean for the electromagnetic potentials $A_\mu(x)$. Consider the decomposition $A_\mu(x) = \sum_{s=0}^{3} e^s_\mu a_s$ with the photon polarizations

$$e^0_\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e^1_\mu = \begin{pmatrix} 0 \\ \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \frac{1}{N_1}, \quad e^3_\mu = \begin{pmatrix} 0 \\ -\partial_2 \\ \partial_1 \\ 0 \end{pmatrix} \frac{1}{N_2}, \quad e^3_\mu = \begin{pmatrix} 0 \\ \partial_1\partial_3 \\ \partial_2\partial_3 \\ -\partial_1^2-\partial_2^2 \end{pmatrix} \frac{1}{N_3}$$

($\partial_s \equiv \partial/\partial x^s$, N_i are normalization factors). The amplitudes a_s describe the two transversal photons ($s = 1,2$), the timelike photon ($s = 0$), and the longitudinal photon ($s = 3$). Application of the boundary conditions (1) yields

$$a_0 + \partial_0\partial_3 \frac{1}{N_3} a_3 - \partial_0 \frac{1}{N_1} a_1 = 0, \; a_2 = 0, \quad \text{at} \quad x_3 = \pm L/2. \tag{2}$$

These are two conditions for four photon amplitudes. The usual treatment, which can be found nearly throughout in literature, is to put $a_3 = 0$ which is equivalent to $\vec{\nabla}\vec{A} = 0$ and means the use of Coulomb gauge and, after that, to require the boundary conditions

$$a_0 = 0, \; a_1 = 0, \; \partial_3 a_2 = 0, \quad \text{at} \quad x_3 = \pm L/2 \,. \tag{3}$$

This is, of course, sufficient to fulfill the initial conditions (1), however not necesary. The condition $a_0 = 0$ leads to the instantaneous electrostatic interaction with Dirichlet boundary conditions, the other two conditions determine the boundary conditions to the radiation field. In the present paper we use the approach proposed earlier in [2]. There the boundary conditions to the field strength are considered as constraints to the potentials, i.e. they are taken into account in a 'minimal' way. Then the quantization yields an new, different photon propagator in covariant gauge which is given in the next section. In the simple case of parallel plates it is possible to explain this in termini of new photon polarizations [3]

$$\tilde{e}^0_\mu = \begin{pmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \\ 0 \end{pmatrix}, \quad \tilde{e}^1_\mu = \begin{pmatrix} \partial_1^2+\partial_2^2 \\ \partial_0\partial_1 \\ \partial_0\partial_2 \\ 0 \end{pmatrix} \frac{1}{N_1}, \quad \tilde{e}^3_\mu = \begin{pmatrix} 0 \\ -\partial_2 \\ \partial_1 \\ 0 \end{pmatrix} \frac{1}{N_2}, \quad \tilde{e}^3_\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \frac{1}{N_3}. \tag{4}$$

Expanding the potential in these polarizations $A_\mu(x) = \sum_{s=0}^{3} \tilde{e}^s_\mu \tilde{a}_s$, the conditions (1) become diagonal, i.e. they yield

$$\tilde{a}_1 = 0, \; \tilde{a}_2 = 0 \quad \text{at} \quad x_3 = \pm L/2 \,, \tag{5}$$

whereas $\tilde{a}_0$ and $\tilde{a}_3$ remain free from conditions. These conditions to the potentials are necessary and sufficient for (1) and in this sense 'minimal'.

An essential difference to the conditions (3) is that the electrostatic interaction, entering the $\mu = 0$ component of the polarization $\tilde{e}^1_\mu$ does not decouple from the space components $\mu = 1,2$ and that therefore the corresponding $\mu = \nu = 0$ component of the propagator is not instantaneous (see formulae (6) below). We remark that, due to $e^2_\mu = \tilde{e}^2_\mu$, the corresponding amplitudes are the same in both approaches.

A general restriction to the validity of the present calculations is that the electron wave function should be small at the plates. This is the case for not to high principal quantum numbers n of the electron state and $L \gg a_0$ (a_0 - Bohr radius). The distance scale is determined by the Bohr radius a_0, the wavelenght λ of transition to the nearest principal quantum number and the wavelenght λ_{fs} of the fine structure transitions. For the experiment of actual interest are $L \sim 1\text{mm}$ and $n \sim 30$. This is clearly an intermediate region far from asymptotics (i.e. from $L \gg \lambda_{fs}$). The parameters combine into two small constants

$$\frac{\alpha}{L}\left(\frac{a_0}{L}\right)^2 = 0.974\ 10^{-6}\left(\frac{L}{1\text{mm}}\right)^{-3}\ \text{Hz} \tag{6}$$

and

$$\frac{\alpha}{L}\alpha^2 = 18.6\left(\frac{L}{1\text{mm}}\right)^{-1}\ \text{kHz}\,. \tag{7}$$

Remark, that these relations by means of $\alpha = 1/(a_0 m_e)$ (m_e is the electron mass) may be represented in another form too. Expression (6) is the coefficient in front of the electrostatic contribution and (7) appears in front of the relativistic correction. Both contributions depend on the quantum numbers of the considered state of the electron and, therefore, give a shift of the hydrogen lines. It should be remarked, that (6) enters the level shift multiplied with a expression proportional to n^4 and is of order 1Hz for Rydberg states. The relativistic contribution contains the factor (7) and is for large n proportional to n. So it goes down to ~ 1 for $n \sim 30$ (for $L \sim 1\text{mm}$). In general, the dependence on the parameters is quite complicated. The level shifts are calculated numerically and the results are given in several plots.

The level shift of the hydrogen between plates shows a familiar oscillating behaviour. The physical reason is that the transitions between two hydrogen states come in resonance with cavity eigenmodes for special values for the external parameters (distance L especially). If the oscillations were undamped the frequency shift could diverge in general. In the present problem the energy shift calculated in perturbation theory diverges in fact when the cavity eigenmodes coincide with some hydrogen transition frequencies. However, this divergence is due to the perturbational solution of the Dirac equation used here. When the perturbations becomes large near a resonance this equation must be solved in an other way. In ref.[4] it was shown, that in doing so in the resonance case a factor of almost 20 appear. We use another possibility to handle the resonances. In fact, divergences may occur for ideal oscillators only. Any real cavity has a finit quality Q which removes the divergences in a natural way. So we introduce the quality factor in the photon propagator. We do this in the leading order for large Q. The effect of Q is that the energy shift in the resonance case becomes proportional to $\log Q$. So the quality confines the energy shift in the resonance case for $q \leq e^{20}$. A second effect of the quality is that the contribution from an intermediate state with wavelength λ much larger than L decreases as $\exp(-L/(\lambda Q))$ so that Q dampes the higher order resonances. In all other contributions Q gives a small correction only and may be neglegted.

In section 2 we introduce the necessary notations and give the field theoretic formulation of the problem of level shifts in a cavity. In section 3 we specify the general formulas for the one loop correction and calculate the corresponding contributions to the energy shift ΔE of a electron state in the discrete part of the spectrum. Thereby we consider principal quantum numbers n from 1 up to 30 and a distance L about 1mm. Conclusions are given in the last section. We use units $\hbar = c = 1$ and $e^2/4\pi = \alpha = 1/137$.

2 Basic Notations and Field Theoretical Approach

In this section we introduce the necessary notations and give the field theoretic formulation of the problem of level shifts in a cavity.

The first ingredient needed is the photon propagator in the presence of boundary conditions. We use the representation of the photon propagator ${}^{s}D^{c}_{\mu\nu}(x,y)$ in the presence of two parallel plates (perpendicular to the x_3-axis intersecting them at $x_3 = \pm(L/2)$) given in ref.[2] and change the notations slightly. This propagator consists of two parts

$$ {}^{s}D^{c}_{\mu\nu}(x,y) = D^{c}_{\mu\nu}(x-y) + \overline{D^{c}_{\mu\nu}}(x,y)\,, \tag{8}$$

where

$$ D^{c}_{\mu\nu}(x-y) = \int \frac{d^4k_\mu}{(2\pi)^4} \frac{e^{-ik_\mu(x^\mu-y^\mu)}}{-k^2-i\epsilon} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}(1-\alpha) \right) \tag{9}$$

($\mu = 0,1,2,3$) is the usual free space propagator in covariant gauge. Here $\epsilon > 0$ defines it to be the causal propagator and α is the gauge parameter. The boundary dependent part reads

$$ \overline{D^{c}_{\mu\nu}}(x,y) = \int \frac{d^3k_\alpha}{(2\pi)^3} \sum_{s,\sigma} \frac{1-\sigma e^{i\Gamma L}}{8\Gamma \sin(\Gamma L)} F_{\mu s\sigma}(k,x) F_{\nu s\sigma}(-k,y) \tag{10}$$

($\alpha = 0, 1, 2;\ \sigma = \pm 1;\ s = 1, 2$) with the function

$$F_{\mu s \sigma}(k, x) = \tilde{e}^s_\mu e^{-ik_\alpha x^\alpha}\left(e^{i\Gamma x_3} + \sigma e^{-i\Gamma x_3}\right) \tag{11}$$

and the photon polarization vectors (they are the same as in (4) in momentum space now)

$$\tilde{e}^1_\mu = \begin{pmatrix} k_\perp^2 \\ k_0 k_1 \\ k_0 k_2 \\ 0 \end{pmatrix} \frac{1}{k_\perp \Gamma}, \quad \tilde{e}^2_\mu = \begin{pmatrix} 0 \\ k_2 \\ -k_1 \\ 0 \end{pmatrix} \frac{1}{k_\perp} \tag{12}$$

with $\Gamma = \sqrt{k_0^2 - k_1^2 - k_2^2 + i\epsilon}$ and $k_\perp = \sqrt{k_1^2 + k_2^2}$. This representation is valid in between the mirrors.

In order to take into account the real structure of the mirrors we assume that the cavity has a finite quality (due to imperfect conductivity of the mirrors, for instance) with a quality factor Q which will be assumed to be large. Than the cavity eigenmodes decay and the time dependence of the photon wave function is proportional to $\exp\left(-ik_0 x^0 (1 - 2i/Q)\right)$. This can be incorporated into the mirror dependent part of the photon propagator by substituting

$$k_0 \to k_0(1 + 2i/Q) \tag{13}$$

in the integrand of (10).

The proton, located in the origin of coordinates (we use spherical coordinates (r, θ, ϕ) for $\vec{x}$), is described by the current

$$j_\mu(x) = e\, g_{\mu 0}\, \delta^3(\vec{x}) \,. \tag{14}$$

It produces the potential

$$A_\mu(x) = \int d^4y\ {}^s\!D^c_{\mu\nu}(x, y)\, j_\nu(y) \,, \tag{15}$$

which can be represented in accordance with (8) as a sum of the two parts $A_\mu(x) = A^0_\mu(x) + A^1_\mu(x)$, where

$$A^0_\mu(x) = \int d^4y\ D^c_{\mu\nu}(x, y)\, j_\nu(y) = \frac{e}{4\pi |r|}\, g_{\mu 0} \tag{16}$$

is the pure Coulomb potential of a point like charge and

$$A^1_\mu(x) = \int d^4y\ \overline{D^c_{\mu\nu}}(x, y)\, j_\nu(y) \tag{17}$$

can in the simple case of parallel mirrors be viewed as the potential of the image charges of the proton.

The electron is described by a 4-component Dirac spinor $\Psi(x)$. It satisfies the generalized Dirac equation

$$(i\gamma^\mu\partial_{x^\mu} - m - e\gamma^\mu A_\mu(x))\ \Psi(x) \ = \ \int d^4y\Sigma(x,y)\ \Psi(y)\,, \tag{18}$$

where $\Sigma(x,y)$ is the electron self energy operator and m_e is the electron mass. The one loop contribution to $\Sigma(x,y)$ takes the form

$$\Sigma(x,y) \ = \ -ie^2\gamma^\mu S^c(x,y)\gamma^\nu\ {}^*D^c(x,y)\,, \tag{19}$$

where $S^c(x,y)$ is the propagator of the electron in the field $A_\mu(x)$ (15). It obeys the equation

$$(i\gamma^\mu\partial_{x^\mu} - m - e\gamma^\mu A_\mu(x))\ S^c(x,y) \ = \ \delta^4(x-y)\,. \tag{20}$$

By these formulae the problem is completely described. Since we are, of course, not able to solve them, we are left with perturbation theory. As for the unperturbated problem we define the electron in the pure Coulomb potential (16). This is just the Dirac theory of hydrogen. The solution is well-known, see e.g.[3]. The equation can be written in the form

$$(i\partial_{x^0} - H)\ \Psi(x) \ = \ 0 \tag{21}$$

with the Hamiltonian

$$H = -i\gamma^0\gamma^s\partial_{x^s} + \gamma^0 m_e + e^2/(4\pi r) \tag{22}$$

$(s = 1,2,3)$. It has a complete set of eigenfunctions

$$H\ \Psi_{(n)}(\vec{x}) \ = \ E_{(n)}\ \Psi_{(n)}(\vec{x})\ , \tag{23}$$

where the multiindex (n) denotes the relevant quantum numbers. In general, (n) includes the discrete levels and the continuous states (for both, electron and proton) as well.

The wavefunctions are

$$\Psi_{(n)}(\vec{x}) \ = \ \frac{1}{r}\begin{pmatrix} iG_{n,\kappa}(r)\Omega_{\kappa,m}(\theta,\phi) \\ F_{n,\kappa}(r)\Omega_{-\kappa,m}(\theta,\phi)\end{pmatrix}\,.$$

The upper and the lower components of the 4 component spinor $\Psi_{(n)}(\vec{x})$ are the large and the small 2 component spinors. Therefore $G_{n,\kappa}(r)$ is of order one, whereas $F_{n,\kappa(r)}$ is of order α in the non relativistic limit.

Now, by means of eq. (21) the time dependent wavefunctions are $\Psi(x) = e^{-iE_{(n)}x^0}\ \Psi(\vec{x})$. Using these formulae we write down a representation for the electron propagator $S_0^c(x,y)$ of the unperturbed problem. It safisfies equation (20) with $A_\mu^0(x)$ (16) instead of $A_\mu(x)$ (15) and reads

$$S_0^c(x,y) \ = \ \int\frac{dp_0}{2\pi}e^{-ip_0(x^0-y^0)}\sum_{(n)}\frac{\Psi_{(n)}(\vec{x})\overline{\Psi_{(n)}(\vec{y})}}{p_0 - E_{(n)}(1-i\epsilon)} \tag{24}$$

($\epsilon > 0$). The sum in this formulae includes the discrete states and the continuous states. For the discrete states the formulae given above have to be used, the continuous states will not be specified here.

Eq. (18) can be solved in perturbation theory

$$\Delta E_{(n)} = \int d^3\vec{x}\, d^3\vec{y}\ \Psi^*_{(n)}(\vec{x})\, V_{E_{(n)}}(x,y)\, \Psi_{(n)}(\vec{y}), \tag{25}$$

where

$$V(x,y) = e\gamma^0\gamma_\mu A^1_\mu(x)\delta^4(x-y) + \gamma^0\Sigma(x,y) \tag{26}$$

collects the perturbations. It divides into

$$V_{es} = e\gamma^0\gamma^\mu \int d^4z\, \overline{D^c_{\mu\nu}}(x,z)\, j^\nu(z) = e\gamma^0\gamma^\mu A^1_\mu(x) \tag{27}$$

and

$$V^{\text{loop}}_{\text{bound}}(x,y) = -ie^2\gamma^0\gamma^\mu S^c_0(x,y)\gamma^\nu\, \overline{D^c_{\mu\nu}}(x,y). \tag{28}$$

The boundary dependent contributions to the energy to be calculated in the next section are correspondingly

$$\Delta E^{\text{bound}}_{(n)} = \Delta E^{es}_{(n)} + \Delta E^{\text{loop}}_{(n)\,\text{bound}}. \tag{29}$$

3 The Calculation of the Energyshift

The perturbation potential $V_{es}(x)$ (27), i.e. the electrostatic potential of the mirror images of the proton, and the corresponding energy shift are part of the electrostatic contribution which was calculated by several authors (see Barton [4] or Eberlein [5] for example). In the field theoretical approach it appears as a zero loop contribution. The calculation is simple and several techniques are possible, especially the summation over mirror images. We give the calculation here within our formalism. This can be considered as an instructive introduction to it.

We start with the potential $A^1_\mu(x)$ (17). Using (10) and (25) we get

$$A^1_\mu(x) = eg_{\mu 0}\int \frac{d^2k_s}{(2\pi)^2}\frac{1-\sigma e^{-k_\perp L}}{4k_\perp \sinh(k_\perp L)} e^{-ik_s x_s}\left(e^{k_\perp x_3} + \sigma e^{k_\perp x_3}\right)$$

($s = 1,2$). Apart from some differences in the notations this is the same representation as in ref. [5] where the equivalence to the sum over mirror images was shown explicitely. For the calculation of the energy we substitute $V_{es}(x)$ (2.26a) using (2.27) into (2.24) and get

$$\Delta E^{es}_{(n)} = e^2\int \frac{d^2k_s}{(2\pi)^2}\frac{1-\sigma e^{-k_\perp L}}{4k_\perp \sinh(k_\perp L)}\int d^3\vec{x}\Psi^*_{(n)}(x)e^{-ik_t x_t}\left(e^{-k_\perp x_3} + \sigma e^{k_\perp x_3}\right)\Psi_{(n)}(x),$$

where the sum over $\sigma = \pm 1$ is assumed. For small a_0/L (a_0 is the Bohr radius) due to the absolute convergence of the k-integration it can be approximated by expanding the exponentials

$$e^{-\imath k_t x_t}\left(e^{-k_\perp x_3} + \sigma e^{k_\perp x_3}\right) = (1+\sigma)\left(1 - \imath k_t x_t - \frac{(k_t x_t)^2}{2} + \frac{k_\perp^2 x_3^2}{2}\right) - (1-\sigma)k_\perp x_3 + \dots .$$

The zero order term in this expansion gives a contribution to $E^{\text{es}}_{(n)}$ which is independent on (n). The first order terms gives a contribution vanishing after the k-integration and the second order terms group into the quadrupol operator $Q = 2x_3^2 - x_1^2 - x_2^2$. So we get

$$\Delta E^{\text{es}}_{(n)} = \frac{e^2}{4\pi L}\left(\frac{a_0}{L}\right)^2 \frac{3\zeta(3)}{4}\int d^3\vec{x}\,\Psi^*_{(n)}(x) Q \Psi_{(n)}(x)\,. \tag{30}$$

Now we calculate the mirror dependent loop correction to the energy levels $\Delta E^{\text{loop}}_{(n)\text{bound}}$. Consider the perturbation potential $V^{\text{loop}}_{\text{bound}}(x,y)$ (28). Inserting repr. (24) for $S^c_0(x,y)$ and repr. (10) for $\overline{D^c_{\mu\nu}}(x,y)$ we obtain

$$\begin{aligned} V^{\text{loop}}_{\text{bound}}(x,y) &= \imath e^2 \int_{-\infty}^{\infty}\frac{dp_0}{2\pi} e^{-\imath p_0(x^0-y^0)} \int \frac{d^3k_\alpha}{(2\pi)^3}\sum_{s,\sigma}\frac{1-\sigma e^{\imath\Gamma L}}{8\Gamma\sin(\Gamma L)} \\ &\quad \gamma^\mu F_{\mu s\sigma}(k,x)\sum_{(n)}\frac{\Psi_{(n)}(\vec{x})\overline{\Psi_{(n)}(\vec{y})}}{p_0 - E_{(n)}(1-\imath\epsilon)}\gamma^\nu F_{\nu s\sigma}(-k,y)\,. \end{aligned} \tag{31}$$

Taking the matrix elements (25) of (31) we get

$$\Delta E^{\text{loop}}_{(n)\text{bound}} = \imath e^2 \int \frac{d^3k_\alpha}{(2\pi)^3}\sum_{s,\sigma}\frac{1-\sigma e^{\imath\Gamma L}}{8\Gamma\sin(\Gamma L)}\sum_{(n')}\frac{M^{s,\sigma}_{(n),(n')}(k) M^{s,\sigma}_{(n'),(n)}(-k)}{k_0(1-2\imath/Q) - \left(E_{(n)} - E_{(n')}(1-\imath\epsilon)\right)} \tag{32}$$

with the matrix elements

$$M^{s,\sigma}_{(n),(n')}(k) = \int d^3x\,\Psi_{(n)}(\vec{x})^*\gamma^0\gamma^\mu \vec{e}^{\,s}_\mu e^{-\imath k_t x_t}\left(e^{\imath\Gamma x_3} + \sigma e^{-\imath\Gamma x_3}\right)\Psi_{(n')}(\vec{x}) \tag{33}$$

(s=1,2). In formula (32) the quality factor Q of the cavity is mentioned explicitely in the denominator resulting from the electron propagator.

In order to evaluate eq.(4) it is useful to rotate the k_0-integration to the imaginary axis $k_0 \to k_4 = -\imath k_0$, i.e. to perform the Wick rotation. There occur additional contributions from the poles on the right of the imaginary axis, i.e. for $k_0 = E_{(n)} - E_{(n')} > 0$. They may be interpreted as originating from a on-shell process, where the electron is in the intermediate state (n') on a lower level ($E_{(n')} < E_{(n)}$) . If one evaluates the k_0-integration as the residuum in this pol then the photon can be seen to carry just the difference energy: $k_0 = E_{(n)} - E_{(n')}$.

Now, the energy shift (32) $\Delta E^{\rm loop}_{(n)\rm bound}$ can be calculated in an approximation with respect to two small parameters. The first one comes from the mirror distance L. From dimensional grands it occurs in the combination a_0/L (a_0 is the Bohr radius). The second is the fine structure constant α which enters the electron wave functions. An expansion with respect to a_0/L arises from the expansion of the matrix elements $M^{s,\sigma}_{(n),(n')}(k)$ (5) into powers of k. Such a expansion is possible due to the convergence of the k-integration and by dimensional reasons each power of k gives a power of a_0/L. So we use in (33)

$$e^{-ik_t x^t}\left(e^{i\Gamma x_3}+\sigma e^{-i\Gamma x_3}\right) = \left(1 - ik_t x^t - \frac{(k_t x^t)^2}{2} - \frac{\Gamma^2 x_3^2}{2}\right)(1+\sigma) + i\Gamma x_3(1-\sigma) + \dots ,$$

where terms up to second order have to be included.

The polarisation vectors $\vec{e}^{\,s}_\mu$ (12) of the photon propagator, multiplied with the γ-matrices, can be represented by

$$\gamma^\mu \vec{e}^{\,1}_\mu = \gamma^0 \frac{k_\perp}{\Gamma} + \frac{k_0}{k_\perp \Gamma}\left(k_1\gamma^1 + k_2\gamma^2\right) , \; \gamma^\mu \vec{e}^{\,2}_\mu = \frac{1}{k_\perp}\left(k_2\gamma^1 - k_1\gamma^2\right) .$$

The term which containes γ_0 corresponds to the $\mu = 0$ component of the electromagnetic potential and gives rise to the electrostatic contribution whereas the terms with γ^1 and γ^2 give rise to the relativistic contribution.

Now we write down the expansion of the matrix elements $M^{s,\sigma}_{(n),(n')}(k)$ (33) . After some steps we get for the product of two such matrix elements

$$\sum_{s=1,2} M^{s,\sigma}_{(n),(n')}(k) M^{s,\sigma}_{(n'),(n)}(-k) \tag{34}$$

$$\begin{aligned} &= \frac{k_\perp^2}{\Gamma^2}\delta_{(n),(n')}(1+\sigma)^2 + \frac{k_\perp^4}{\Gamma^2}(1+\sigma)^2\mathcal{N}_1\left((n),(n')\right) - k_\perp^2(1-\sigma)^2\mathcal{N}_2\left((n),(n')\right) \\ &\quad + \left(\frac{k_0^2}{\Gamma^2}+1\right)(1+\sigma)^2\mathcal{N}_3\left((n),(n')\right) \\ &\quad - \frac{k_\perp^2}{\Gamma^2}(1+\sigma)^2\delta_{(n),(n')}\int d^3\vec{x}\,\Psi^*_{(n)}(\vec{x})\left(\frac{1}{2}k_\perp^2\left(x_1^2+x_2^2\right)+\Gamma^2 x_3^2\right)\Psi_{(n)}(\vec{x}) + \dots \end{aligned}$$

with the notations

$$\mathcal{N}_1\left((n),(n')\right) \;=\; \frac{1}{2}\sum_{t=1}^{2}\left|\int d^3\vec{x}\,\Psi^*_{(n)}(\vec{x})x_t\Psi_{(n')}(\vec{x})\right|^2 , \tag{35}$$

$$\mathcal{N}_2\left((n),(n')\right) = \left|\int d^3\vec{x}\Psi^*_{(n)}(\vec{x})x_3\Psi_{(n')}(\vec{x})\right|^2 , \tag{36}$$

$$\begin{aligned}\mathcal{N}_3\left((n),(n')\right) &= \frac{1}{2}\sum_{t=1}^{2}\left|\int d^3\vec{x}\Psi^*_{(n)}(\vec{x})\gamma^0\gamma^t\Psi_{(n')}(\vec{x})\right|^2 \\ &= \left(E_{(n)} - E_{(n')}\right)^2 \frac{1}{2}\sum_{t=1}^{2}\left|\int d^3\vec{x}\Psi^*_{(n)}(\vec{x})x_t\Psi_{(n')}(\vec{x})\right|^2 .\end{aligned} \tag{37}$$

The last line is a consequence of

$$\int d^3\vec{x}\overline{\Psi_{(n)}(\vec{x})}\left(\gamma^t - \imath E_{(n),(n')}x_t\right)\Psi_{(n')}(\vec{x}) = 0$$

and follows from the Dirac equation (23).

The contribution from the last term in rhs. of (34) to the energy shift (32) can be shown to cancel in (29) the contribution of the electrostatic part (30). The matrix elements $\mathcal{N}_1$ and $\mathcal{N}_2$ are of order $(a_0/L)^2$ and $\mathcal{N}_3$ is of order α^2 (note $a_0 = 1/(\alpha m_e)$). For large n, $\mathcal{N}_1$ and $\mathcal{N}_2$ behave like n^4, whereas $\mathcal{N}_3$ is proportional to $1/n^2$.

After defining the matrix elements $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$ one can combine the k-integration in (32) into the following functions:

$$\mathcal{R}_1(L,\mu) = -\imath\int\frac{d^3k_\alpha}{(2\pi)^3}\sum_\sigma\frac{1-\sigma e^{\imath\Gamma L}}{8\Gamma\sin(\Gamma L)}\frac{-k_\perp^4}{\Gamma^2}\frac{(1+\sigma)^2}{k_0-\mu} , \tag{38}$$

$$\mathcal{R}_2(L,\mu) = -\imath\int\frac{d^3k_\alpha}{(2\pi)^3}\sum_\sigma\frac{1-\sigma e^{\imath\Gamma L}}{8\Gamma\sin(\Gamma L)}(-k_\perp^2)\frac{(1-\sigma)^2}{k_0-\mu} , \tag{39}$$

$$\mathcal{R}_3(L,\mu) = -\imath\int\frac{d^3k_\alpha}{(2\pi)^3}\sum_\sigma\frac{1-\sigma e^{\imath\Gamma L}}{8\Gamma\sin(\Gamma L)}\left(\frac{k_0^2}{\Gamma^2}+1\right)\frac{(1+\sigma)^2}{k_0-\mu} , \tag{40}$$

where μ stands for the energy difference $\left(E_{(n)} - E_{(n')}(1-\imath\epsilon)\right)(1+2\imath/Q)$. Special values are

$$\mathcal{R}_1(L,0) = -3\zeta(3)/\left(8\pi L^3\right), \ \mathcal{R}_2(L,0) = -\zeta(3)/\left(2\pi L^3\right), \ \mathcal{R}_3(L,0) = \ln 2/\left(4\pi L\right) , \tag{41}$$

and for $L|\mu| \to \infty$ we note

$$\mathcal{R}_1(L,\mu) \approx \frac{7\pi^2}{240\,L^4\,\mu} , \quad \mathcal{R}_2(L,\mu) \approx \frac{\pi^2}{30\,L^4\,\mu} , \quad \mathcal{R}_3(L,\mu) \approx \frac{-1}{24\,L^2\,\mu} . \tag{42}$$

The reason for introducing these functions is, apart from a more compact notation, that they correspond to the contributions from the different intermediate states $E_{(n')}$ of the electron. The behaviour of the real part of this functions is shown in fig.1 for a quality factor $Q = 100$.

So we get for the distance dependent and level dependent part of the levelshift

$$\Delta E^{\text{bound}}_{(n)} = \Delta E_{es} + \Delta E^{\text{loop}}_{\text{bound}} = e^2 \sum_{i=1}^{3} \sum_{(n')} \mathcal{N}_i\left((n),(n')\right) \mathcal{R}_i\left(L, E_{(n)} - E_{(n')}\right) . \quad (43)$$

This formula is one of the main results in this paper. All quantities entering it are well defined and can be calculated numerically. The results are represented in the figures 2 and 3.

The order of magnitude of the terms in (43) is given in the following.

For $i = 1,2$, because of $\mathcal{R}_i(L,\mu) = L^{-3}\mathcal{R}_i(1,L\mu)$ and $\mathcal{N}_i$ being proportional to $(a_0/L)^2$, the contribution to the energy shift is of order (6) $\alpha a_0^2 L^{-3} = 0.974\ 10^{-6}\,(L/1\text{mm})^{-3}$ Hz. This is at once the order of magnitude of the result for low n. For Rydberg states it should be multiplied by $\mathcal{N}_i \approx n^4$ ($30^4 = 0.81\ 10^6, \mathcal{N}_1|_{n=n'=30,\kappa=-1} = 4.8\ 10^6$ for example) and gives a shift of roughly 1Hz for a distance of 1mm.

For $i = 3$, because of $\mathcal{R}_3(L,\mu) = L^{-1}\mathcal{R}_3(1,L\mu)$ and $\mathcal{N}_3$ is proportional to α^2, the contribution to the energy shift is of order (7) $\alpha^3/L = 18.6\,(L/1\text{mm})^{-1}$ kHz . For low n, we have for the argument of the function $\mathcal{R}_3$, $L\left(E_{(n)} - E_{(n')}\right) \gg 1$, especially for $n = 1, n' = 2$ we note $L\left(E_{(1)} - E_{(2)}\right) = -1.6\ 10^4\ \pi\ (L/1\text{mm})$. The matrix element $\mathcal{N}_3$ is of order one ($\mathcal{N}_3(1,2) = 0.62$) so that using (42) a contribution of order 1Hz appears. Especially, the contribution from $n' = 2$ to the levelshift of the groundstate ($n = 1$) is

$$e^2 \sum_{n'=2} \mathcal{N}_3(1,2)\mathcal{R}_3\left(L, E_{(1)} - E_{(2)}\right) = -0.12\ \text{Hz}\,(L/1\text{mm})^{-2} .$$

The contribution from all n' is $-0.16\,\text{Hz}(L/1\text{mm})^{-2}$. For high n, $\mathcal{N}_3$ behaves like n^{-2} ($30^{-2} = 1.1\ 10^{-3}$, $\mathcal{N}_{3}|_{n=30,n'=29,\kappa=-1} = 1.42\ 10^{-4}$ for instance) and the corresponding contribution goes down to roughly 1Hz.

In the present calculation the contineous intermediate states are neglected. It can be shown that their contribution is less than 1% in the parameter region considered here. This is in contrast to the Lamb shift calculation in free space. A heuristic argument states that the photon spectrum is different: whereas in free space the integral over the photon momentum converges power like (after substracting the UV divergences) it goes like $\exp(-kL)$ in between the mirrors as can be shown from (38)-(40) after Wick rotation.

4 Conclusions

The boundary dependent level shift, derived here, differs in two features from the known results. Consider the electrostatic part, defined to be the contribution from the $\mu = \nu = 0$ component of the photon propagator (10) with the polarizations (12) . It containes the sum over the intermediate electron states (n')(see formulas (32),(33)), whereas in the quantum mechanical approach of [4] it is

$$\Delta_1 = \int d^3\vec{x}\Psi^*_{(n)}(x)H_{es}(x)\Psi_{(n)}(x) ,$$

where $\Psi_{(n)}(x)$ is the hydrogen wave function and $H_{es}(x)$ is the classical Hamiltonian of the boundary dependent part of the electrostatic energy. As beeing a first order perturbation it does not contain intermediate electron states. Remark, that a second order perturbation would contain a sum over different intermediate states, however, it would contain an additional order of smallness, i.e. a factor e^2, too. So we see, that in the considered order of smallness (i.e. $e^2a_0^2/L^3$) field theory and quantum mechanics yield different results. The question, in what case field theory turns to quantum mechanics in this example, can be partly answered by the following considerations. Suppose, that the distance L is so small, that $L\left(E_{(n)} - E_{(n')}\right) \ll 1$, i.e. that no retardation is possible. In that case, using (40), the k_0-integration can be done and the photon propagator becomes the instantaneous electrostatic one. The sum over the intermediate states (n') decouple. Using ($\Psi_{(n)}$ is a complete set of states)

$$\sum_{(n')} \mathcal{N}_1\left((n),(n')\right) = \int d^3\vec{x}\Psi^*_n(\vec{x})\frac{x_1^2+x_2^2}{2}\Psi_x(\vec{x}) , \tag{44}$$

$$\sum_{(n')} \mathcal{N}_2\left((n),(n')\right) = \int d^3\vec{x}\Psi^*_n(\vec{x})x_3^2\Psi_x(\vec{x}) , \tag{45}$$

we get

$$\sum_{i=1,2} \mathcal{R}_i(L,0)\mathcal{N}_i\left((n),(n')\right) = \frac{e^2}{4\pi L}\left(\frac{a_0}{L}\right)^2\zeta(3)\int d^3\vec{x}\Psi^*_n(\vec{x})\left(\frac{3}{4}Q - \frac{7}{2}x_3^2\right)\Psi_n(\vec{x}) . \tag{46}$$

In the nonrelativistic limit the Dirac spinors $\Psi_{(n)}(x)$ can be substituted by the usual hydrogen wavefunctions $|n,l\rangle$ and we get the same result as in quantum mechanics, see [4] for instance.

Let's remark, that the sum over the intermediate states for the electrostatic contributions is dominated by the nearby terms. This can be seen from the corresponding formulas for the radial matrix elements in the case $n \gg l$:

$$\frac{|\langle n,l|r|n,l-1\rangle|^2}{\langle n,l|r^2|n,l\rangle} = \frac{9}{10}\left(1 + O((l/n)^2)\right) ,. \tag{47}$$

where the first term ($n\prime = n$) contributes already 90% in formulas like (44),(45(). Therefore, including the intermediate electron states in the electrostatic interaction gives no dramatic changes.

The second difference to the usual treatment consists in the time dependence of the electrostatic (i.e. $\mu = \nu = 0$) part of the photon propagator used here. It results in a retardation and an oscillating behaviour of the electrostatic part of the levelshift like that of the relativistic part (i.e. the $i = 3$ contribution in formula (3.13) which corresponds to the retarded contribution in [4]). Its influence is numerically small as can be seen from fig.2, however present.

Acknowledgement
I thank G. Barton for essential discussions and remarks. Further I thank C. Eberlein and D. Robaschik for useful discussions.

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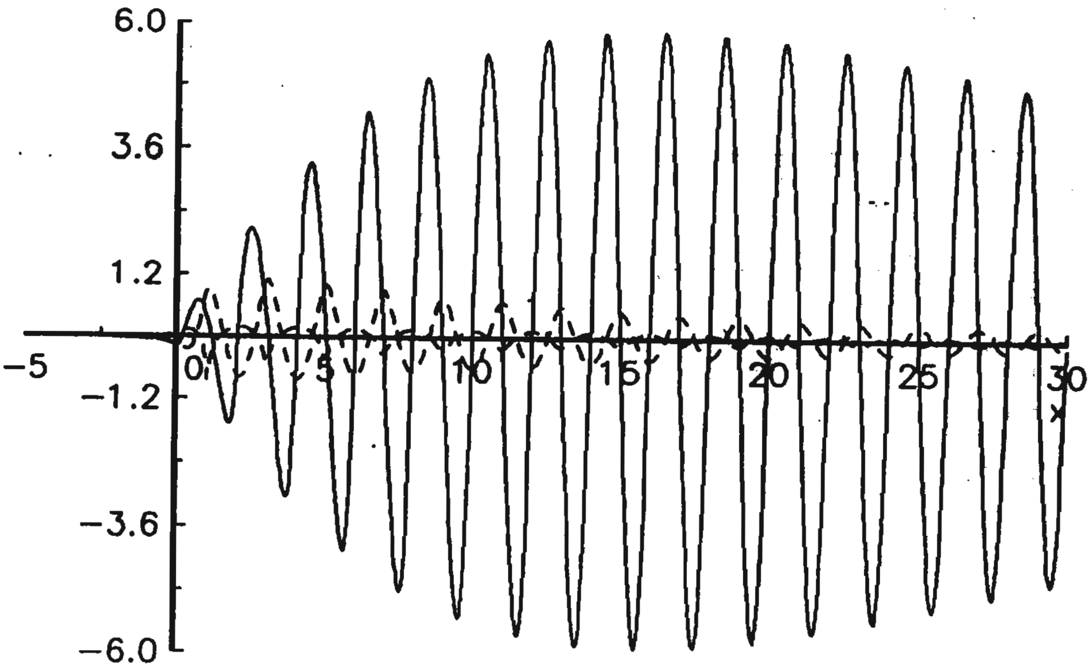

Fig.1a The functions $\mathcal{R}_i(\pi x)$, $\mathcal{R}_2$ – solid line, $\mathcal{R}_1$ and $\mathcal{R}_3$ – dashed lines

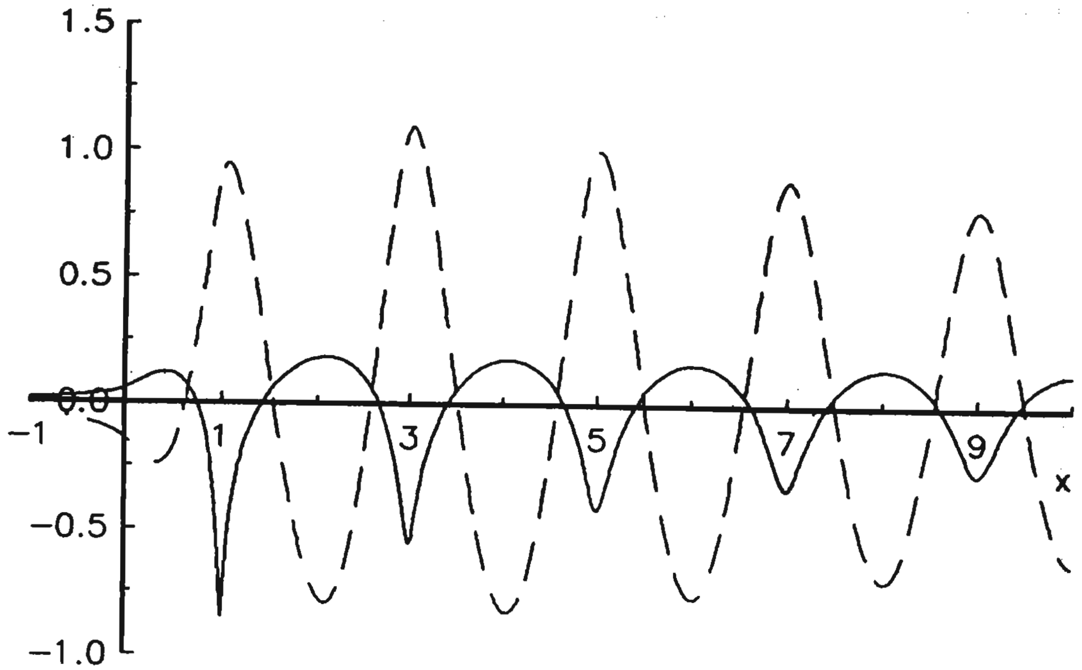

Fig.1b The functions $\mathcal{R}_i(\pi x)$, $\mathcal{R}_3$ – solid line, $\mathcal{R}_1$ – dashed line

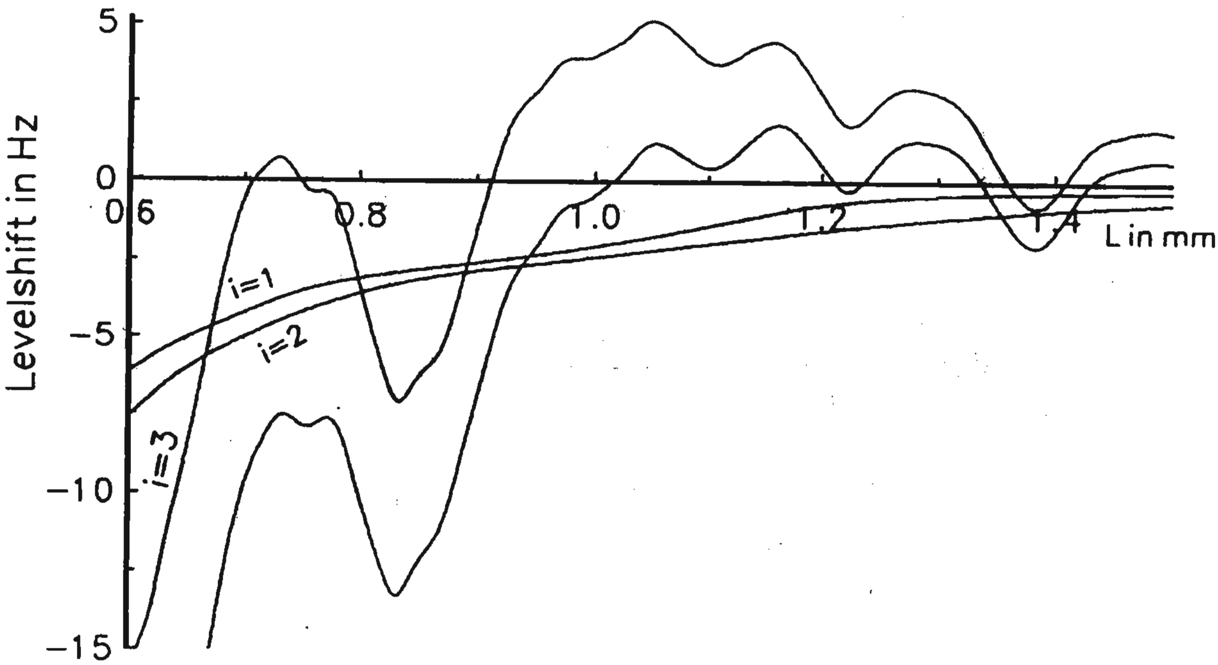

Fig.2 The levelshift for $n = 30$ and its contributions from $i = 1, 2, 3$ in eq.(43)

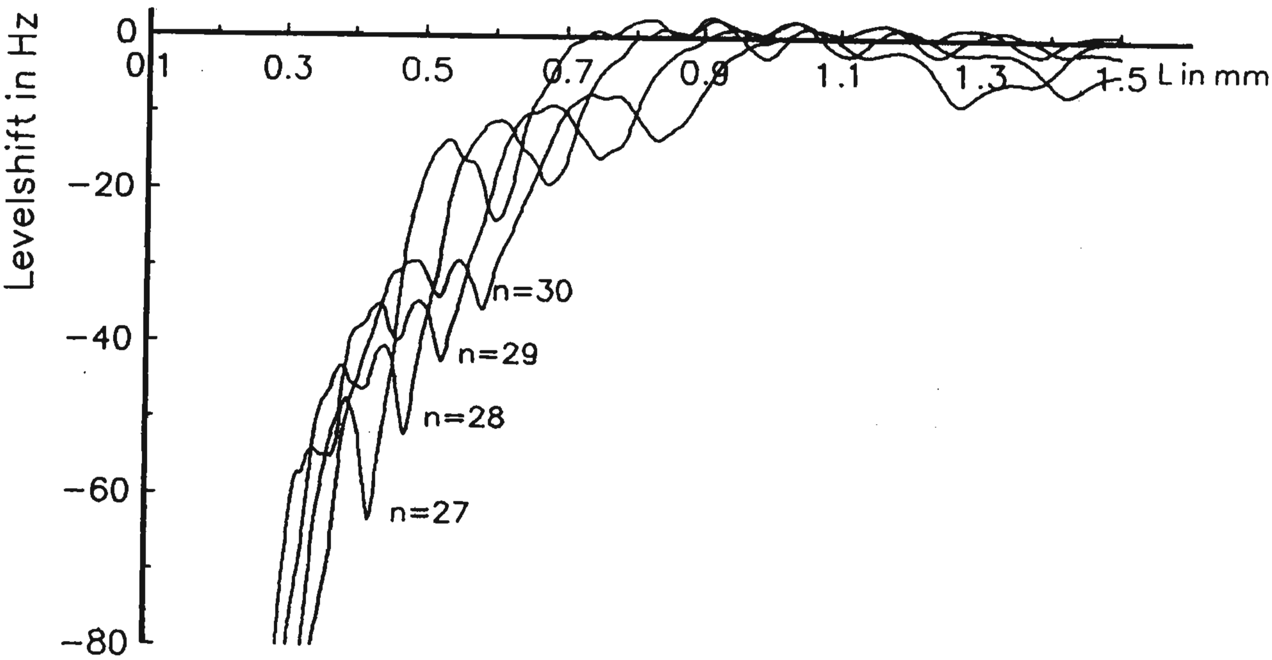

Fig.3 The levelshift for $n = 27, 28, 29, 30$

HYPOTHETICAL LONG - RANGE FORCES AND MODERN EXPERIMENTS

V.M.Mostepanenko and I.Yu.Sokolov

Dept. of Mathematics, St.Petersburg Technological Institute

26 Moskowsky pr., St.Petersburg 198013, Russia

Summary

Manifestations of new hypothetical long-range forces, which are predicted by modern quantum theories, are explored in terrestrial experiments, mainly, on force measurements. The perspective experiments for the search of the new hypothetical forces are considered. Up today restrictions and perspective ones are obtained and summarized. The limits on the masses of spin-1 antigraviton and dilaton from perspective experiments are shown to be greater than 10^{-2}eV and 2×10^{-2}eV respectively and the restriction on axion mass can be achieved $m<10^{-6}$ eV ($\theta=10^{-9}$).

1 Introduction

Many modern quantum theories predict the existence in nature a number of new light (or massless) hypothetical particles. The consequences of it can be manifested in astrophysical phenomena (anisotropy of microwave background, connection between the age and radiation of a star, baryosynthesis *etc* .) and in appearing of the additional chanals of elementary particles decays. Moreover, the hypothetical particles predicted by quantum gravity theories can lead to the shift of frequency of atomic radiation and to additional forces between macrobodies. It is the latter problems that we shall discuss in the present paper. It should be noted that the search of such forces in terrestrial conditions allows to obtain the most model independent information about the restrictions on new force parameters. The experiments which will be considered here are of Eötvös and Galileo type, Cavendish type, on verification of Casimir effect, measuring the van der Waals forces, atomic force microscopy *etc* . As a consequence the nonobservation of mentioned manifestations leads to obtaining of the restrictions upon the parameters of corresponding hypothetical particles.

One of the most interesting particles which leads to additional long-range interaction is spin-1 atigraviton (Scherk 1979, Zachos 1978). It appears in gravitational supermultiplets of all extended broken supergravity schemes and gives rise to gravity-like forces, but repulsive. The potential of Yukawa type between atoms arises as a result of the exchange by the particles of that sort

$$V(r)=\alpha N^2\frac{e^{-r/\lambda}}{r} \tag{1}$$

where $\lambda\equiv m^{-1}, m$ is the mass of exchange particle, r is the distance between atoms, N is the number of nucleons per one atom, α is a dimensionless interaction constant.

As has been shown by Scherk (1979) the constant $\alpha=8\pi Gm_0^2\sim10^{-40}$, where m_0 was the sum of current quark masses of nucleon.

Another particle resulted in (1) is a pseudo-Goldstone boson which appears due to scale symmetry breaking. It is a so-called dilaton (Fujii 1972, O'Honlon 1972). The potential (1) arises because of mixing with graviton. The constant α for dilaton was shown to be equal to $\frac{1}{3}Gm_N^2\simeq2\times10^{-39}$, where G is the gravitational constant, m_N is the mass of nucleon.

One more particle which we shall consider is axion (see, e.g. Moody and Wilczek 1984). Axions appear both in modern unified theories and for solving the strong CP problem in chromodynamics. This particle can play an important role in solving the Dark Matter problem. Likewise the particles mentioned above the axions are able to bring about a long-range interaction of Yukawa type (1). In the work by Moody and Wilczek (1984) the constant α and the mass of the axion were calculated. They turned out to be equal to

$$\alpha \simeq \frac{G_N^2}{4\pi}\left(\frac{\Theta}{F}\right)^2\left[\frac{2m_u m_d}{(m_u+m_d)^2}\right]^2, \qquad m \simeq \frac{m_\pi f_\pi}{F}\frac{\sqrt{m_u m_d}}{m_u+m_d}, \tag{2}$$

where $m_{u,d}$ are current masses of u, d quarks, m_π is the mass of π-meson, $f_\pi \simeq 90 MeV$ is the constant of π-meson decay, G_N is the pion-nucleon σ-term $\simeq 60 MeV$, Θ is the parameter of CP-violation ($|\Theta| < 10^{-9}$, Kim (1987)), F is the parameter of Peccei-Quinn symmetry breaking. As was shown in, e.g., the paper by Turner (1990) the limit on the axion mass $m \leq 10^{-3} eV$ followed from supernova SN1987A. Hence, one can conclude from (2) that $F > 10^{10} GeV$.

For the case of the exchange of massless particles the effective potential between atoms has a power law (Feinberg and Sucher 1979, Mostepanenko and Sokolov 1987):

$$V_n(r) = \frac{\lambda_n \times N^2}{r}\left(\frac{r_0}{r}\right)^{n-1}, \tag{3}$$

where λ_n is a dimensionless constant, $r_0 \equiv 10^{-15} m$.

The potentials of such type appear due to exchange by such particles as arions ($n = 3$)(Mostepanenko and Sokolov 1987), goldstinos ($n = 7$) (Radesk 1984), even by usual massless neutrinos ($n = 5$) (Feinberg and Sucher 1968) *etc* . The search of such particles is a good way for verification of the modern unified and gravity theories.

The velocity of light c and the Planck constant $\hbar$ will be chosen equal to unity.

2 Up today terrestrial experiments and restrictions upon hypothetical interaction

2.1 Eötvös and Galileo types experiments

In the experiments of these types the possible difference between the gravitational and inertial masses was measured. In Eötvös type experiments (Stubbs *et al* 1987, 1989, Braginsky and Panov 1972, Heckel *et al* 1989) two bodies of approximately equal gravitational masses but of different materials were suspended on a torsion balance. For the case of being the difference between two types of masses the turning moment would appear.

In the experiments of Galileo type by Cavasini *et al* (1986), Niebauer *et al* (1987), Kuroda and Mio (1990) the test bodies were falling down and the falling time for each body was registered. The great attention was attracted to these experiments after the work of Fishbach *et al* (1986) in which the violation of the equivalence principle was announced to be registered in Eötvös experiments really.

The typical result of both these experiments is that relative difference of accelerations a_1/a_2 imported by the Earth to various substances

$$\left|\frac{a_1}{a_2}-1\right| < A, \tag{4}$$

where $A \sim 10^{-8}$ for the typical Eötvös experiments and $\sim 10^{-10}$ for Galileo type.

The existence of new long-range force of (1)-type must result in effective difference between inertia and gravitational masses. Therefore, following the work by Mostepanenko and Sokolov (1988), let us show which restrictions upon α, λ (1) are obtained from (4). One has

$$a_i = -\frac{G}{r^2}M_\oplus + \alpha N_\oplus \frac{N_i}{M_i}\left(1+\frac{r}{\lambda}\right)\frac{e^{-r/\lambda}}{r^2} f\left(\frac{R_\oplus}{\lambda}\right), i=1,2 \tag{5}$$

where i is the number of test bodies, M_i is its mass, N_i is the number of nucleons in the body. $f(x) = \frac{3}{x^3}(x\cosh(x) - \sinh(x))$, $R_\oplus, M_\oplus$ are the radius and the mass of the Earth.

Thence one can write

$$\left|\frac{a_1}{a_2}-1\right| = \frac{\alpha}{G^{m_N}} e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda} f\left(\frac{R_\oplus}{\lambda}\right)\left|\frac{N_2}{m_2}-\frac{N_1}{m_1}\right| \simeq$$

$$\frac{\alpha}{Gm_N^2}\left(1+\frac{r}{\lambda}\right)e^{-r/\lambda} f\left(\frac{R_\oplus}{\lambda}\right)\left|\frac{N_1}{N_2}-1\right| \tag{6}$$

The difference $|N_1/N_2 - 1|$ in the experiments of Eötvös type was $\sim 10^{-2}$ (the test bodies were of water and cuprum). As to Galileo type experiment $|N_1/N_2 - 1| \sim 10^{-3}$. Therefore one obtains from (6), (4)

$$\alpha < ce^{l/\lambda}\frac{R_\oplus}{\lambda} \tag{7}$$

where $l \simeq 1m$ is the distance between the Earth and the test body. The constant $c \simeq 4 \times 10^{-46}$ for Eötvös type experiments and for Galileo ones. These restrictions are shown in Fig.1. The region of allowed values for α, λ lies below the curve 1. In the same figure the curve 3 represents the restrictions obtained by **Heckel *et al*** (1989), where the experiment practically of Eötvös type was carried out. In difference from other experiment of Eötvös type instead of the Earth the massive laboratory body was used in these experiments.

If additional long-range force arises due to exchange of spin-1 antigraviton or dilaton then the following restrictions upon their masses are obtained: $m^{-1} = \lambda < 5m$ and $m^{-1} < 0.3m$ for spin-1 antigraviton and dilaton respectively.

Likewise one can obtain the restrictions upon λ_n for power law long-range interactions (3). So, from Eötvös experiments one has $\lambda_n < 10^{-45}, 10^{-23}, 10^{-2}$ with $n = 1, 2, 3$ respectively (Feinberg and Sucher 1979). The modified Eötvös type experiment was made in 1971 by Braginsky and Panov where the Sun was as an attractive body. Corresponding restrictions were the following: $\lambda_n < 10^{-47}, 10^{-20}, 10^{-7}$ with $n = 1, 2, 3$.

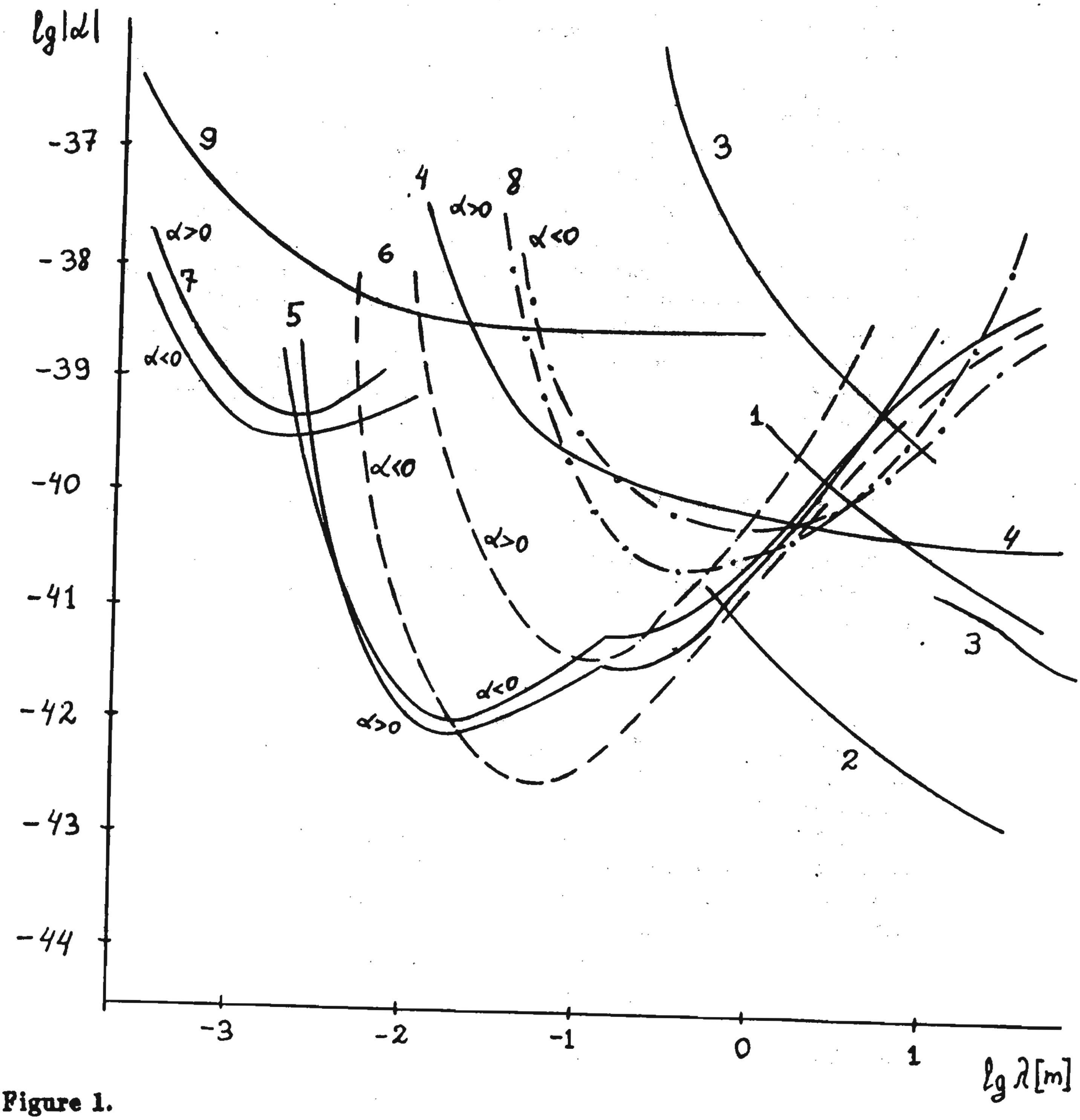

Figure 1.

The restrictions on Yukawa parameters α, λ obtained by Heckel (1989) — the curve 2, Niebauer *et al* (1987) — 1, Stubbs *et al* (1987), (1989) — 3, Mostepanenko and Sokolov (1988), (1989) — 4, Hoskins *et al* (1985) — 5, Chen *et al* (1984) — 6, Mitrofanov and Ponomarjova (1988) — 7, Panov and Frontov (1979) — 8, Kusmin *et al* (1984) — 9.

2.2 Cavendish type experiments

The earlest experiments dealt with direct measurement of gravitational force. The value of gravitational constant G was as a result of these experiments. Nowadays experiments on G measurement shown the value of G was the same up to $\sim 1\%$ for different type experiments (cf. geophysical experiments by Holding *et al* (1986), Stacey *et al* (1987)). As a consequence the restrictions on α, λ were obtained by Mostepanenko and Sokolov (1988), (1989) (the curve 4 in Fig.1). For dilaton and spin-1 antigraviton one has $\lambda < 0.3m$ and $< 2 \times 10^{-2}m$ respectively.

In modern experiments of Cavendish type the deviation from Newtonian gravitational law is measured by Holding *et al* (1986), Stacey *et al* (1987), Hoskins *et al* (1985), Chen *et al* (1984), Mitrofanov and Ponomarjova (1988), Panov and Frontov (1979). In order to measure the deviation it is not necessary to know the exact value of G (which is known up to 1%). This fact allows to do these experiments with great sensitivity to that deviation. Furthermore the interest in these experiments was stimulated by Long's communication of 1974 about experimental discovery of deviation from the inverse square law. For the time being there is no evidence, however, of the exostence of such deviations.

Usually, the characteristic value of the deviation from the inverse square law is written as follows

$$\varepsilon = \frac{1}{rF}\frac{d}{dr}\left(r^2F\right) \tag{8}$$

where r is the distance between point-like bodies, F is the force acting between them.

The typical value of ε is $\pm 10^{-4}$ for $r \sim 1cm \div 1m$ (e.g., Hoskins *et al* (1985), Chen *et al* (1984)). The ε value due to Yukawa type long-range interaction (1) is given by

$$\varepsilon = \frac{\alpha}{Gm_N^2}\frac{r^2}{\lambda^2}e^{-r/\lambda}. \tag{9}$$

The restrictions which are obtained from these equations are shown in the Fig.1 (the curves 5—8). The allowed region for α, λ lies below the corresponding curves. The difference between the cases of $\alpha > 0$ and $\alpha < 0$ is explained by the fact that the mean value of ε is not equal to zero.

From the Figure it is easy to obtain the following restrictions on the masses of spin-1 antigraviton $m > 6 \times 10^{-5}eV(\lambda < 4 \times 10^{-4}m)$.

By the same way the restrictions upon the constants λ_n of the power-law interaction can be derived. The best restrictions upon λ_n from the experiments of this type were obtained by Mostepanenko and Sokolov (1990) and turned out to be $\lambda_2 < 7 \times 10^{-30}$, $\lambda_3 < 7 \times 10^{-17}$, $\lambda_4 < 1 \times 10^{-3}$.

2.3 Verification of Casimir effect

In the experiments on verification of the Casimir effect by Derjaguin *et al* (1968), Hunklinger *et al* (1972) the theoretical value of Casimir force F_{theor} was compared with the experimental one F_{expr}. There was no difference between them in the region of experimental error δ. Therefore one can obtain the restriction needed from obvious inequality $F_{add} < \delta F_{expr}$.

In the experiments mentioned above the Casimir force was measured between a plate and a spherical lens of the radius R ($R \gg \ell$ which is the distance between the plate and lens) with $\ell \leq 0.05\mu m \div 2\mu m$. An approximate theoretical Casimir force for this configuration is given by

$$F = -\frac{2\pi R}{3}\frac{B}{\ell^3} \tag{10}$$

where B is a constant depending on dielectric properties of test bodies.

Hypothetical force for such configuration was derived by Kusmin *et al* (1984) and by Mostepanenko and Sokolov (1987a,b). The force, for instance, due to the potential of Yukawa type (1) results in

$$F_{add} = \alpha n_1 n_2 4\pi^2 \lambda^3 R \left(1 - e^{-D/\lambda}\right) e^{-\ell/\lambda} \times$$

$$\left[1 - \frac{\lambda}{R} + e^{-\frac{H}{\lambda}}\left(\frac{H}{R} - 1 + \frac{\lambda}{R} + \frac{H^2}{2R\lambda} - \frac{H}{\lambda}\right)\right], \tag{11}$$

where n_1, n_2 are atoms densities of test body materials, H, D are the thickness of the lens and plate correspondingly.

In the experiments (e.g., by Hunklinger *et al* (1979)) the theoretical value of Casimir force was confirmed with experimental error $\delta \sim 10\%$ for $\ell \sim 0.8 \div 1\mu m$. Thence one has the sought restrictions. Allowed region for α, λ lies below the curve 9 in Fig.1.

Unfortunately, the insufficient accuracy of the experiments and an ambiguity of the data for D and H in publications do not allow to obtain the reliable restrictions on the masses of dilaton and of spin-1 antigraviton. However, in the next section it will be shown that well restriction can be obtained in optimised experiment of such type in the nearest future.

In order to obtain the restrictions upon the power law interaction constants λ_n it is necessary to derive the expression for corresponding additional force. It has been done in the paper by Mostepanenko and Sokolov (1987 a,b) and the restrictions were found to be $\lambda_1 < 10^{-40}, \lambda_2 < 10^{-27}, \lambda_3 < 5 \times 10^{-15}, \lambda_4 < 3 \times 10^{-3}$.

2.4 Measurements of van der Waals forces between crossed cylinders, the atomic force microscopy and the spectroscopy of exotic atoms

The experiments of these types deal with rather small distances between test bodies — from $1\mathring{A}$ to $1000\mathring{A}$. So-called nonretarded van der Waals forces act between bodies at such distance. In (Israelachvili and Tabor 1972) the force between crossed cylinders was measured for the distance between them from 15 to $1300\mathring{A}$. The force of the same type was measured in atomic force microscope (AFM) acting between a tip of AFM and plane sample (Moiseev *et al* 1989). A typical distance in this experiment was 4—$15\mathring{A}$.

6

The concent between experimental and theoretical data in the limits of experimental error allows to obtain the restrictions upon $\alpha, \lambda, \lambda_n$ by the same way as in the previous section. It should be noted that restrictions upon λ_n turn out badly to be improved in this way. Therefore the restrictions on Yukawa type interaction will be considered here.

As was shown by Moiseev *et al* (1989) the restrictions upon α, λ from crossed cylinders and AFM are as follow

$$\alpha \leq \frac{A}{n_1 n_2 \ell^2 \lambda^3} e^{\ell/\lambda}, \quad 10^{-10} m \leq \lambda < 10^{-8} m \tag{12}$$

where the constant $A \sim 1.1 \times 10^4 m^{-1}$ for crossed cylinders and $A \sim 2.6 \times 10^3 m^{-1}$ for AFM, ℓ is the distance between test bodies, $n_{1,2}$ are atom densities of test bodies materials (mica for crossed cylinders and sapfire for AFM).

The restrictions (12) are shown in Fig.2 (curves 1,2) where the limits from Casimir effect are also represented (the curve 3). As it is seen from the Figure, the best restrictions upon α with $\lambda > 60 Å$ are obtained from the verification of the Casimir effect, with $10 Å \leq \lambda \leq 60 Å$ — from crossed cylinders and with $1 Å \leq \lambda \leq 10 Å$ — from AFM.

Another type of experiments which can give the information about new hypothetical forces is the measuring of frequency of transitions in exotic atoms(of types nucleus-antiproton, meson *etc* .)(Tausher 1977). The restrictions upon the power law interactions were obtained by Feinberg and Sucher (1979). The restrictions on Yukawa type interactions can be derived in the same way. These restrictions are represented in Fig.2 (the curve 4). It is the best for $\lambda < 1 Å$.

3 The perspectives for experiments with interacting bodies

3.1 The experiments on Casimir effect verification

The increasing of the sensitivity to additional hypothetical force can be achieved in the experiments of this type by means of changing the configuration of test-bodies, the distance between them and their density. As was shown by Mostepanenko and Sokolov (1991a) the configurations of test bodies leading to maximal F_{add}/F, where F is experimental force between test bodies, is "a small ball in the centre of a thin spherical shell" and "two plane plates". And the latter configuration was shown to be more availiable for the experiments under consideration. The optimal distance between macrobodies turned out to be $\sim 60 \div 200 \mu m$. For such distance it is essential to take into consideration the forces of gravitation between bodies. Moreover, the Casimir forces become the temperature ones. So, the experiment on Casimir effect verification turns into the experiment on direct measurement of the gravitational and temperature Casimir forces between plane plates. As to materials of test-bodies, the heavy metals were shown to be the best ones. After all, in order to obtain, e.g., the restrictions upon the power law interaction constants it is suggested to measure the force $\sim 10^{-13} N/cm^2$, with the error of δ, between plane plates of thickness $D \sim 100 \mu m$ with the

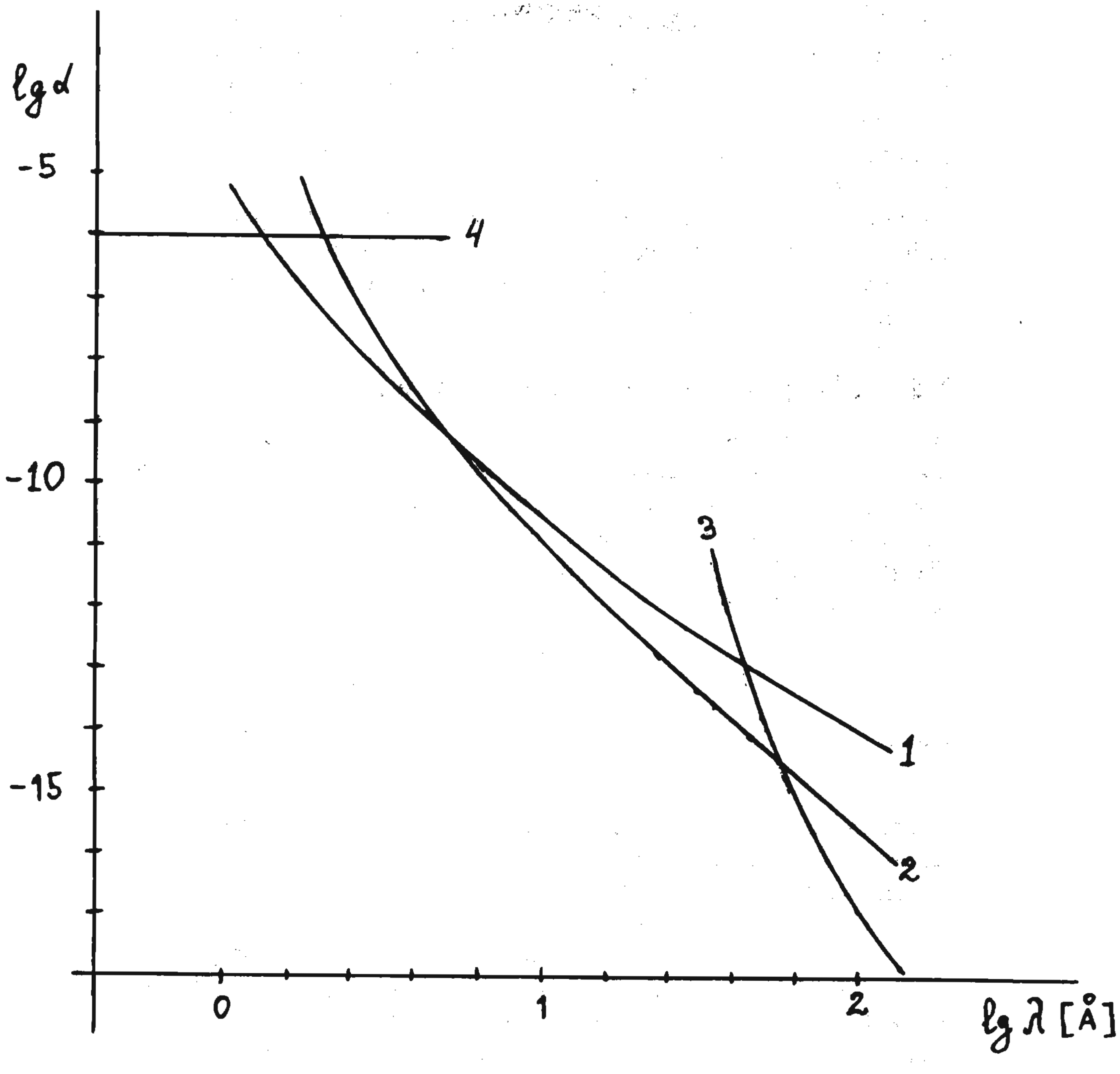

Figure 2.

The restrictions on α, λ from atomic force microscopy (1), measurements of van der Waals force between crossed cylinders (2), Casimir effect (3), exotic atoms (4).

distance between them $\sim 100\mu m$. Then one has (Mostepanenko and Sokolov 1991a) $\lambda_2 < 2\times10^{-27}\frac{\delta}{100\%}, \lambda_3 < 3\times10^{-16}\frac{\delta}{100\%}, \lambda_4 < 4\times10^{-5}\frac{\delta}{100\%}$. The value of δ is limited by the accuracy of gravitational constant measurement which is $\sim 1\%$. Therefore, assuming $\delta \sim 1\%$ one has that perspectives for power law interaction constants λ_n are: $\lambda_2 < 2\times10^{-29}, \lambda_3 < 3\times10^{-18}, \lambda_4 < 4\times10^{-7}$. It means that nowadays restrictions (from Cavendish type experiments) can be strengthened in 30 times with $n = 3$ and in 2,500 times with $n = 4$. The perspectives for the restrictions on Yukawa long-range interaction in the experiments of the type under consideration are represented in Fig.3 (allowed region for α, λ lies below the curve 2). In the Fig.3 all the best nowadays restrictions are shown (the curve 1). The forbidden region for α, λ is hatched.

3.2 The perspectives for Cavendish type experiments

Such perspectives were found by Mostepanenko and Sokolov (1991b,c). Likewise the previous types of experiments the Cavendish experiment on measuring the deviation from gravitational law transfers into measurements of the deviation from total force which can consist both of gravitational and Casimir forces. Then the generalisation of characteristic value of the deviation from known force law (8) will be given by

$$\varepsilon = \frac{1}{(m-2)}\frac{1}{r^{m-1}F(r)}\frac{d}{dr}\left(r^{m-1}\frac{d}{dr}r^2F(r)\right) , \tag{13}$$

where

$$F(r) = \frac{Gm_1m_2}{r^2} + \frac{C_m}{r^m} + F_{add}(r) \tag{14}$$

is a force acting between point-like bidies of m_1, m_2 masses of the same materials as test-bodies, C_m is a constant of Casimir force ($m = 8$ for retarded force, $m = 7$ for temperature and non-retarded ones).

The purpose of the experiment suggested in (Mostepanenko and Sokolov 1991 b, c) was the quest of such configuration of test-bodies that possible deviation ε (13) would be maximal. Such a problem was solved as variational task and showed that optimal configuration is the same as in the previous section, i.e. "a small ball in the centre of a thin spherical shell" or "two plane plates". Let us see the case of two plane plates at first.

Plane plates. In (Mostepanenko and Sokolov 1991b) the optimal parameters of the plates (the thickness D and the distance between them ℓ) were found, so that ε (13) were to be maximal. For instance, in the case of power law hypothetical interaction, the deviation from known force law must be measured inside the region $\ell \sim 50 \div 500\mu m, D \sim 100\mu m$. The corresponding force displacement for the ℓ-region mentioned above is $\sim 10^{-11} \div 10^{-12}N$. Putting the sensitivity to such displacements $\sim 2.5\times10^{-14}N$ (Panov and Frontov 1979, Braginsky *et al* 1981) and the area of the plates $S = 100cm^2$, one has $\delta\varepsilon \sim 10^{-4}$. It corresponds to the following restrictions: $\lambda_2 < 5\times10^{-29}, \lambda_3 < 1\times10^{-18}, \lambda_4 < 4\times10^{-7}$. These restrictions are practically the same as the perspectives for the experiments on

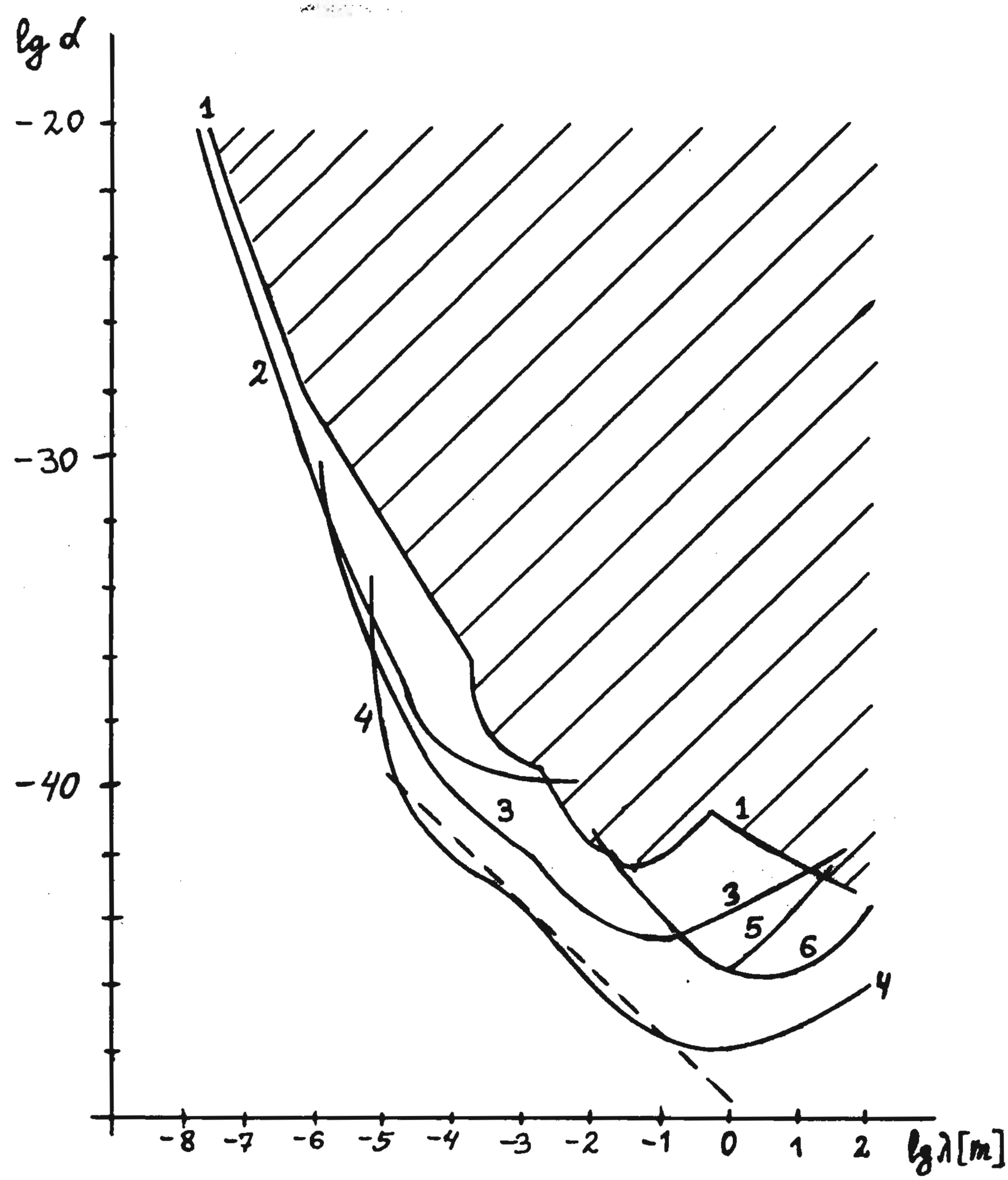

Figure 3.

The best up today restrictions on α, λ (curve 1) and the perspective ones for direct force measuring (Mostepanenko and Sokolov 1991a) — 2, for Cavendish type experiments (Mostepanenko and Sokolov 1991b, c) — 3, 4, 5, 6.

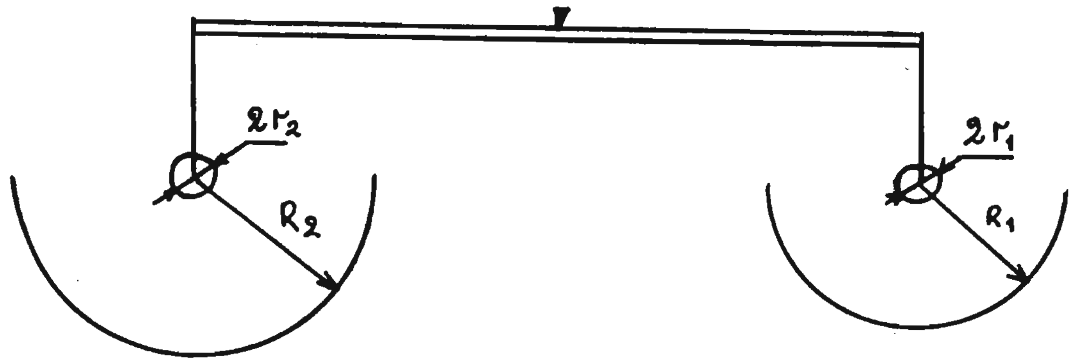

Figure 4.

The scheme of Cavendish type experiment with optimal configuration of test-bodies.

Casimir effects, but the formers are more perspective because the increasing, e.g., of the area of plates up to $1000cm^2$ leads to strengthening of the restrictions in 10 times.

For obtaining the restrictions on Yukawa long-range interaction parameters it was suggested to take plane plates with $D = 1cm, S = 10^{-2}m^2$ and to measure the deviation from known (for this case, practically gravitational) force law into the region of $\ell \sim 10^2 \div 10^5 \mu m$. The modern experimental facilities was shown allow to achieve the sensitivity to ϵ up to 2×10^{-7} for this configuration. From this fact and from eq.(1ɔ) one can obtain the sought restrictions on α as a function of λ. These restrictions are shown in Fig.3 (the curve 3). Choosing the values of $D = 10cm$ and of $S = 1m^2$ one has the restrictions represented in Fig.3, the curve 4.

A small ball in the centre of a thin spherical shell. The calculations for such configuration were made by Mostepanenko and Sokolov (1991c). A scheme of the experiment suggested is shown in Fig.4. The forces acting between test-bodies are compared for two pairs of "a small ball in the centre of a thin spherical shell" by means of the scales (of balance or torsion types). The force for each pair of test-bodies is given by (for R_i we are interested in the Casimir forces are negligible here)

$$F_i = G\frac{m_i M_i}{2R_i^2} + F_{add}(R_i) \quad , i = 1, 2, \tag{15}$$

where $m_{1,2}(M_{1,2}$ are the masses of small balls (of thin spherical shell), $R_{1,2}$ are the radii of the shells, F_{add} id a hypothetical force. For instance, in the case of Yukawa type hypothetical interactions (1), one has

$$F_{add} = \frac{\alpha}{2}\frac{m_i M_i}{m_N^2} e^{-R_i/\lambda}\left(\frac{1}{R_i^2} + \frac{1}{R_i \lambda}\right). \tag{16}$$

Choosing $m_1 = m_2$ one can obtain the equality of gravitational forces acting between test-bidies for both configurations when the thicknesses of spherical shells $D_1 = D_2$. Then the difference between F_i (15) which can be registered by the scales results in the only difference of additional forces $F_{add}(R_i)$. If the approaching of the spherical shells to the scales does not violate the state of equilibrium then in the region under investigation the hypothetical additional force is absent up to experimental error.

The parameters of the ball and spherical shell were suggested to take as follows:$R_1 = 1$ or $5m, D_{1,2} = R_1/10, r_{1,2} = R_1/10, R_2 = R_1 + R_1/3$. The materials were supposed to be of metal with density $\rho = 10^4 kg/m^3$. As was shown by Mostepanenko and Sokolov (1991c), the equality of gravitational forces of the configurations of Fig.4 can be maintained up to $\Delta F \simeq 1.4 \times 10^{-12}N$. Therefore, unless the equilibrium of the scales violates, then one concludes that $\Delta F_{add} = F_{add}(R_2) - F_{add}(R_1) < \Delta F \simeq 1.4 \times 10^{-12}N$. The restrictions on Yukawa potential parameters α, λ obtained from this inequality are shown in Fig.3 (the curve 5 with $R_1 = 1m$ and the curve 6 with $R_1 = 5m$).

4 Conclusion

In the oresent paper the manifistations of new hypothetical forces, predicted by the modern quantum theories, in the terrestrial experiments are exploried. It was shown that the modern experiments considered allowed to obtain new restrictions upon the parameters of hypothetical particles such as spin-1 antigraviton, dilaton.

The perspective experiments on Casimir force measurement and Cavendish-type experiments are considered in the framework of the modern experimental facilities. It was demonstrated that the present restrictions on power law constants λ_n could be strengthened in 70 and 2,500 times with n=3,4 respectively. The limits on Yukawa constants α, λ were shown in the Fig.3. The improving of the restrictions we have now can achieve 10^7 times. Moreover, provided the experiments described were fulfiled the restrictions on the masses of spin-1 antigraviton and dilaton would be less than 10^{-2}eV and 2×10^{-2}eV respectively.

The suggested experiment of Cavendish type (the curve 4 in Fig.3) allows to achieve the level of axion forces (2), which is shown in Fig.3 by dashed line (the parameter $\theta = 10^9$). If such forces are not registered then one will have the following restrictions upon the axion mass $m < 10^{-6}\mathrm{eV}(10^{-9}/\theta)$.

Thus, being compact and comparatively inexpensive, the terrestrial experiments are a good instrument for verification of the modern quantum theories and for receiving a new information about elementary particles and their interactions.

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Address of the authors:
Dept. of Mathematics,
Technological Institute,
26 Moskowsky pr.,
St.Peterburg 198013,
Russia

RADIATIVE EFFECTS IN THE STANDARD MODEL UNDER THE INFLUENCE OF EXTERNAL FIELDS

V.Ch. Zhukovsky

Physics Faculty, Moscow State University, Moscow, 117234, Russian Federation

Abstract

A review of recent results of calculations of various vacuum effects, influenced by external gauge fields and finite temperature , in the Weinberg - Salam model and in QCD is given.

1. Introduction

The influence of external electromagnetic field on the QED processes is extensively studied in literature (see, for example [1, 2, 3]). New problems arise when one takes into account the effect of external fields in non-abelian gauge theories. In the Weinberg-Salam (W-S) model external electromagnetic fields are introduced in the same way as in QED. In QCD "external fields" are understood as certain vacuum gluon condensate fields, introduced as mean fields, averaged over some model configurations.

Strong laser fields can find application in future e^+e^- colliders, involving real (beamstrahlung and backscattered laser) or quasireal (bremstrahlung) photons in initial state and hadrons in the final state, $\gamma\gamma$-colliders.

Future colliders such as SSC can be used due to strong electromagnetic fields between colliding particles to ensure revealing the nature of electroweak symmetry breaking.

High-energy processes such as deep inelastic scattering or e^+e^- annihilation in hadrons, spin-flip processes and others are influenced by the QCD condensate fields.

In astrophysics the problem of solar neutrino deficit can find its resolution in possible interaction of the magnetic moment of neutrino with the magnetic field at the solar surface (see e.g. [4]).

The behaviour of symmetries at finite temperatures and densities is one of the most outstanding and relevant problems in many current areas of particle physics: e.g. cosmology, relativistic heavy - ion collisions, and the quark -gluon plasma.

The problem of symmetry breaking and its restoration is intrinsically nonperturbative. Most of our knowledge comes from lattice simulations. On the other hand influence of external conditions (temperature, density and gauge fields) is also mostly nonperturbative.

The influence of external fields can be considered by the method of exact solutions of relativistic equations for some solvable model field configurations [1, 2, 3].

2. Anomalous moments of fermions and bosons

Explanation of the solar neutrino puzzle can lie in three possible directions:
1) nonstandard solar neutrino model,
2) MSW effect (Mikheev, Smirnov, Wolfenstein [5]) - matter-enhanced oscillations: coherent forward scattering of ν in e - matter accompanied by an almost complete conversion $\nu_e \to \nu_f$ $(f \neq e)$,
3) flip of the ν spin $\nu_L \to \nu_R$ in the magnetic field of the convective zone of the Sun [4].

Unfortunately the value of the neutrino magnetic moment in W-S model with additional right-handed neutrino ($m_\nu \neq 0$) is too small to be astrophysically important [6]

$$\mu_{\nu 0} = 3eG_F \, m_{\nu_e} / (8 \cdot 2^{1/2} \pi^2) = 3 \cdot 10^{-19} \, \mu_B \, (m_{\nu_e} \, / \, 1\mathrm{eV}), \qquad (2.1)$$

where $\mu_B = 5.8 \cdot 10^{-15} \mathrm{MeV} \; G^{-1}$.

This value was obtained for an electron Dirac neutrino in the so called static limit, that is for weak fields and low energies. In connection with astrophysical problems, such as solar neutrino deficit, it is of interest to investigate the dynamical character of the neutrino magnetic moment, incorporating the dependence of it on the intensity of the external field F and on the neutrino energy ε

$$\mu_\nu = f \, (F, \, \varepsilon). \qquad (2.2)$$

This problem for the whole range of energies and electric and magnetic field strengths was solved (see [7] and references therein) by the use of the Dirac-Schwinger equation for the neutrino wave function

$$(i\gamma\partial - m_\nu)\psi(x) = \int d^4x' M(x,x')\psi(x') \tag{2.3}$$

where $M(x,x')$ - the mass operator in an external electromagnetic field $F_{\mu\nu}$. This method was successfully used in ref. [8] for investigation of the dynamical nature of the vacuum magnetic moment of an electron.

Radiative corrections to the neutrino mass

$$\Delta m = \langle M_R \rangle \tag{2.4}$$

(M_R - the renormalized value of M) include two terms

$$\Delta m_M = \mu_\nu \,(s^\mu \tilde{F}_{\mu\alpha} p^\alpha)/m_\nu \,, \tag{2.5}$$

$$\Delta m_D = d_\nu (s^\mu F_{\mu\alpha} p^\alpha)/m_\nu \,, \tag{2.6}$$

linear in the neutrino spin 4-vector s^μ. Coefficients μ_ν and d_ν can be interpreted as the anomalous magnetic and electric moments of neutrino, d_ν being proportional to (EH).

For our purposes in this talk it is sufficient to present only two special cases of the general result of ref. [7]. In the weak feald limit $H \ll H_0 = m_e^2/e = 4.4 \cdot 10^{13}$G, E=0 for the longitudinal motion, $p \parallel H$, we have

$$\mu_\nu = \mu_{\nu 0} \,(1 + (4/9)(H^2 \ln\Lambda/(H_0^2\Lambda^2)), \tag{2.7}$$

where $\Lambda = M_W^2/m_e^2$, M_W - the mass of the W-boson.

The magnetic moment increases quadratically with H but with a very small coefficient.

In the strong field limit

$$H \to H_C = M_W^2/e = 1.1 \cdot 10^{24} G$$

the value of μ_ν goes to infinity

$$\mu_\nu=(2/3)\mu_{\nu 0}(M_W^2/m_e^2)\,ln[H_0/(H_c-H)] \,. \qquad (2.8)$$

This divergence is due to a tachyonic mode in the W-boson energy spectrum and a resulting instability of the perturbative W-boson field vacuum state (see below).

The experimental discovery of the W-boson, predicted by the W-S model, was the first triumph of this model. Further investigations of the properties of W-boson are essential to prove that it is an elementary gauge particle. Specific characteristics of W as a gauge particle are necessary to guarantee the renormalizability of the W-S theory. In particular gyromagnetic ratio $\gamma_W=2$ is characteristic of the gauge nature of the theory and of the structure of the 3-linear vector vertex. Hence, mesurement of the γ_W is a test of the gauge nature of the W-boson.

The equation for W-particle, linearized near the perturbative vacuum in the magnetic field H

$$(D_\nu D^\nu + M_W^2)W^\mu + 2ieF^{\mu\nu}W_\nu = 0 \,, \qquad (2.9)$$

$$(D_\nu = \partial_\nu + ieA_\nu)$$

can be solved to give the energy spectrum

$$\varepsilon_n^2 = M_W^2 + eH(2n-1) + p_z^2, \; n = 0, 1, 2, 3,\ldots \qquad (2.10)$$

The W in the ground state $n=0$, $p_z=0$ has the energy

$$\varepsilon_0^2=M_W^2-eH. \qquad (2.11)$$

For $H\to H_c=M_W^2/e$ we have $\varepsilon_0\to 0$, manifesting the restoration of gauge symmetry.

In the region $H>H_c$ the linear theory of the W-boson is not valid and we have to find new nonperturbative vacuum for the W-field. In the region $H<H_c$ the linear theory is applicable and we have to complement eq.(2.9) with the equation

$$S_\nu^{\ \mu} W^\nu = \zeta W^\mu \qquad (\zeta = 0, \pm 1) \tag{2.12}$$

for the polarization states of W-boson, where $S_\nu^{\ \mu}$ is the spin-operator of the particle. In the rest frame of the W-boson we have

$$S_\nu^{\ \mu} = -iF_\nu^{\ \mu}/H. \tag{2.13}$$

The lepton contribution to polarization operator of the W-boson was calculated in the 1-loop approximation in [9]. For comparatively weak fields $H \ll H_c = M_W/e$ the leptonic contribution to the vacuum anomalous magnetic moment of the W-boson was found to be [9]

$$\Delta k_l = -g^2/(32\pi^2)(1/3 + 1/\Lambda - 2/\Lambda^2), \tag{2.14}$$

$$\Lambda = M_W^2/m_e^2.$$

The bosonic contribution is of the same order of magnitude. It was obtained after some controversy in [10]

$$\Delta k_W = 7e^2/(16\pi^2). \tag{2.15}$$

3. W - condensate in the Weinberg-Salam model.

The effective mass of the W-boson can be defined as follows

$$M_{eff}^{2(0)} = \varepsilon_0^2 = M_W^2 - eH. \tag{3.1}$$

When $H \to H_c$ we have $M_{eff}^{2(0)} \to 0$ and gauge symmetry is restored. For $H > H_c$ the effective mass squared is negative, corresponding to the tachyonic mode in the energy spectrum of the W-boson. When

$H \to H_c$ the perturbative vacuum becomes unstable, which manifests itself in the divergence of the vacuum correction to the mass

$$M_{eff}^{2(0)}(H) = M_W^2 - eH + \Delta M_{vac}^2(H) \tag{3.2}$$

and to the neutrino mass

$$m_{\nu\ eff}(H) = m_\nu + \Delta m_{\nu\ vac}(H). \tag{3.3}$$

For the W-boson and charged ϕ-scalar contributions we have [11]

$$\Delta M_W^2 = -e^2/(8\pi^2) M_W^2 ln[\, M_W^2/(eH) - 1], \tag{3.4}$$

$$\Delta M_\phi^2 = e^2/(4\pi^2) M_W^2\ exp(m_\phi^2/(2eH))|Ei(-m_\phi^2/(eH))|\,ln(M_W^2/(eH) - 1).$$

As was shown earlier (see (2.8)) the spin dependent part of $\Delta m_{\nu vac}$ - the anomalous magnetic moment of the neutrino divergies when $H \to H_c$. For such huge magnetic fields the new, nonperturbative, vacuum state - W condensate has to be constructed.

Recently it was conjectured that such W-condensate can be formed in high energy collisions [12]. Transient magnetic fields of sufficient strength to induce a W condensation will be present in a large number of high energy collisions. At the moment of collision for a short time large magnetic fields are formed

$$H \sim H_c = (M_W^2/e) \sim 10^{24} G,$$

exceeding the average magnetic field in a neutron star $H \sim H_0 = 4.4 \cdot 10^{13} G$ by 11 orders of magnitude. For this to occure an impact parameters of the size $r_\perp \sim 1/M_W$ should be attained. This can be easily done at supercolliders like LHC and SSC. The charged particles in this collision can be valence quarks of the same charge inside each of the colliding protons.

The formation of W-condensate may lead to a spatial anysotropy in electromagnetic field distribution. Let us take for simplicity the Georgi-Glashow SU(2) model

$$L_{G-G} = -(1/2)G^{a}_{\mu\nu}G^{\mu\nu}_{a} + (1/2)D_{\mu}\chi^{a}D^{\mu}\chi_{a} - m_{\chi}^{2}\chi^{2}/2 + \lambda(\chi^{a}\chi_{a})^{2}/8. \quad (3.5)$$

where χ -scalar field. The bosonic part of the Lagrangian

$$L_{B} = -P_{\mu}^{*}W_{\nu}^{+}P^{\mu}W^{-\nu} + P_{\mu}^{*}W_{\nu}^{+}P^{\nu}W^{-\mu} + ieF^{\mu\nu}W^{+}_{\mu}W^{-}_{\nu} + M_{W}^{2}W^{+}_{\nu}W^{-\nu} - (e^{2}/2)[(W_{\mu}^{+}W^{-\mu})^{2} - W_{\mu}^{-2}W_{\nu}^{+2}], \quad (3.6)$$

where $P_{\mu} = i\partial_{\mu} - eA_{\mu}$, leads to field equations for W in linear approximation

$$(D_{\nu}D^{\nu} + M_{W}^{2})W^{\mu} + ieF^{\mu\nu}W_{\nu} - D_{\nu}(D^{\mu}W^{\nu}) = 0. \quad (3.7)$$

For arbitrary sourceless Maxwell field, $\partial_{\mu}F^{\mu\nu}=0$, we have

$$D_{\mu}W^{\mu} = 0 \quad (3.8)$$

and equation (3.7) goes over to more simple one (2.9).

The ground state $n=0$, $\zeta=-1$ is described by the solution

$$W^{1} = (1+i)e^{i\delta}\Omega/2^{1/2}, \quad W^{2} = (1-i)e^{i\delta}\Omega/2^{1/2}, \quad (3.9)$$

where δ and Ω are constant phase and amplitude of the solution. For $H > H_{c} = M_{W}^{2}/e$ the one-particle solution of the problem is no longer valid, and condensate of the W-field is formed. Naively, ignoring the dependence of the condensate on spatial coordinates, Ω=const, we find by variation of the energy of the field

$$\delta \int H\, d^{3}x = \delta\int d^{3}x[\,2\Omega^{2}(\varepsilon^{2} + \varepsilon_{0}^{2} + e^{2}\Omega^{2})]_{\varepsilon=\varepsilon_{0}} = 0,$$

the value

$$e^2\Omega^2 = -\varepsilon_0^2 = eH - M_W^2. \qquad (3.10)$$

We can extract Maxwell field mass term from the bosonic Lagrangian (3.6), which arises due to W-condensate

$$L_{Maxwell} = e^2\Omega^2 (A_0^2 - A_3^2).$$

The conclusion is that a W-condensate is formed and a photon acquires a mass .

$$m_\gamma^2 = const(H - H_c).$$

A distinctive feature of this Higgs mechanism is spatial anisotropy [13]. The external magnetic field is not affected in contrast to ordinary abelian case - isotropic Meissner effect.

4. QCD -condensate and its role in high energy collisions.

As is well known the ground state of QCD is characterized by nonezero expectation value of gauge fields forming a vacuum condensate [14]. The simplest model of nonperturbative vacuum is a state with a uniform color magnetic field, considered as a classical external (background) field.

Although the one-loop energy density (effective potential) of this "color ferromagnetic state" is lower than that of perturbative vacuum [15], it has the nonzero imaginary part, indicating that this state is unstable [16]. Various attempts were made in order to find a stable configuration with lower energy, such as models with domain-like structure of the magnetic field [17].

Statistical properties of systems interacting with non-abelian gauge fields have recently attracted much attention in connection with the description of the early Universe and investigations of the behaviour of the hadronic matter in heavy-ion collisions.

In the high temperature limit the effective coupling constant $g^2(T)$ is small due to asymptotic freedom and at certain temperature a deconfinement phase transition occurs.

The question of the color ferromagnetic state in QCD at finite temperature was investigated in ref.[18]. The simple ansatz of Savvidy [15] for the vacuum state with the uniform color magnetic field was used. Calculation of the one-loop effective potential showed that the imaginary part is nonzero, demonstrating that the instability of the Savvidy ansatz was not only cured by the temperature but instead is growing indefinitely with temperature. Recently in a number of papers (see e.g. [19, 20, 21], dealing with infrared properties of the Yang-Mills theory at high temperature, possibility of formation of a new condensate was discussed. The two-loop effective potential of gluons at finite temperature in the presence of an imaginary chemical potential μ was considered. The effective potential has a nontrivial minimum at $i\mu \sim gT$, indicating formation of condensate of the A_0 field. This background field $A_0 = const$ cannot be gauged away preserving the periodicity conditions for the spatial components A_1 in the Euclidean gauge theory with compact time coordinate x_0

$$0 \leqslant x_0 \leqslant \beta = 1/T .$$

In ref.[22] it was shown that a superposition of a constant A_0 and a uniform magnetic color field condensates eliminates the imaginary part of the effective potential, stabilizing the nonperturbative color ferromagnetic vacuum state in QCD.

In [22] for the superposition

$$A^a_\mu = \delta_{\mu 2}\delta^a_3 Hx_1 + \delta_{\mu o}\delta^a_3 A_o, \ (H = const, \ -i\mu = A_o = const) \tag{4.1}$$

the thermodynamic potential was calculated in the one-loop approximation

$$\Omega^{(1)} = (gHV/2\pi\beta) \sum_{l=-\infty}^{\infty} \int (dq_3/2\pi) \sum_{n=o}^{\infty} ln[(2\pi l/\beta + gA_o)^2 + $$
$$+ 2gH(n + 1/2 + \sigma) + q_3{}^2], \ \beta = 1/T. \tag{4.2}$$

The real part of $\Omega^{(1)}$ has a nontrivial minimum at H_{min}

$$(gH_{min})^{1/2} \sim g^2(T)T. \qquad (4.3)$$

The imaginary part is due to the presence in the spectrum of gluons

$$\varepsilon_{n\sigma q}^2 = 2gH\ (n + 1/2 + \sigma\ ) + q_3^2\ ,\ n = 0,1,2,\dots \qquad (4.4)$$

of the tachyonic mode $n=0$, $\sigma=-1$, $q_3^2=0$

$$\varepsilon_0^2 = -gH. \qquad (4.5)$$

At high temperature $T \gg (gH)^{1/2}$ we have

$$Im\ \Omega^{(1)} = -(gHTV/2\pi)[\ gH - (gA_0)^2\ ]^{1/2}. \qquad (4.6)$$

When the condensate gA_0, which is of the order $g^2(T)T$ becomes sufficiently large

$$gA_0 > gA_{0c} = (gH_{min})^{1/2} \qquad (4.7)$$

the imaginary part disappears

$$Im\ \Omega^{(1)} = 0. \qquad (4.8)$$

This stabilization of the color ferromagnetic vacuum is due to interaction of gluons with constant condensate A_0, appearing as a consequence of the infra-red behaviour of the theory.

Another possibility of a constant chromomagnetic vacuum is a non-Abelian configuration of the type

$$A_1^a = (\ \alpha^2/3\ )^{1/2}\delta_1^a\ ,\quad A_4^a = 0. \qquad (4.9)$$

One-loop thermodynamic potential in high temperature limit

$T \gg a$ has a nontrivial minimum at $a_{min} \sim gT$, demonstrating a possibility of formation of a spherically symmetric condensate. As the imaginary part of the thermodynamic potential is rather small

$$Im\,\Omega\ /\ Re\,\Omega \sim (\ a/T\ )^3 \ll 1$$

the new vacuum configuration can be considered as quasistable [23].

This simple model of a vacuum condensate can be used to estimate the contribution of the gluonic condensate to characteristics of high energy collisions.

Consider e^+e^- annihilation to hadrons. The method of sum rules is effective in this problem. For example vacuum polarization by heavy C-quark is described by various moments

$$M_n = \int (R_c(s)/\ s^{n+1})ds \qquad (4.10)$$

of the polarization operator P(s)

$$R_c(s) = 4\pi\ Im\ P(s)/s\ . \qquad (4.11)$$

The polarization operator of the photon in gluonic condensate field was calculated for the abovementioned model of vacuum (4.9) in ref. [24]. For its imaginary part we have

$$Im\ P(t) = e^2/(8\pi^2)[((t - t_1)/t)^{1/2}(t - t_1 + \lambda)\theta(t-t_1) +$$
$$+ (2\lambda t/(t-\lambda)^{3/2})(t - t_2)^{1/2}\theta(t-t_2)], \qquad (4.12)$$

where the thresholds are

$$t_1 = 4(\ m^2 + \lambda/2\ ),\ t_2 = 4(\ m^2 + 3\lambda/4\ )$$

and $\lambda = (a^2/3)^{1/2}$, m is the heavy quark mass.

To estimate the influence of the gluonic condensate $\langle F_{\mu\nu}{}^{a}F_{\mu\nu}{}^{a}\rangle$ on the $e^{+}e^{-}$ annihilation to hadrons we calculate the integral (4.10) for n=1 in the sum rules method (the simple case of spin zero quarks is considered)

$$\int_0^\infty (R(s)/s^2)ds = 3(Q_c{}^2/\pi)[\ 1/5 + (2/15)(\lambda/t_1) - 3(t_2/\lambda\ )^{3/2} +$$

$$+2(t_2/\lambda)(\ 2(t_2/\lambda) - 1)\ ln((\ \lambda^{1/2} + t_2{}^{1/2})/(t_2 - \lambda)^{1/2})]. \quad (4.13)$$

In the weak field limit $\lambda/m^2 \ll 1$, after averaging over different field configurations

$$\lambda^2 \rightarrow 1/6\ \langle F_{\mu\nu}{}^{a}F_{\mu\nu}{}^{a}\rangle$$

we obtain

$$\int(R(s)/s^2)ds = 3/(20\pi\ m^2)(\ 1 - (29\alpha_s\pi)/(126m^4)\langle F_{\mu\nu}{}^{a}F_{\mu\nu}{}^{a}\rangle).$$

For the SU(2) model [25]

$$\langle\ (\alpha_s/\pi)F_{\mu\nu}{}^{a}F_{\mu\nu}{}^{a}\ \rangle \approx 0.07\ \mathrm{Gev}^4$$

and for the C-quark (m_c=1.26 Gev) the condensate correction is 6%, which corresponds to the value obtained in the operator product expansion method [14, 26]. Our result (4.13) gives the explicit dependence on the condensate field strength, including the nonperturbative nonanalytic terms, which are not described by the operator expansion method. The analogous phenomenon of insufficiency of operator expansion method for the full description of the external field influenced effects in the deep inelastic scattering was lately demonstrated in ref.[25].

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WORKSHOP

Quantum Field Theory under the Influence of External Conditions

Leipzig, Germany
September 14 -20, 1992.

Program

Monday, September 14

G. Barton (Brighton)	Casimir Effect: Stress Fluctuations and Damping Force
C. Eberlein (Brighton)	Photon Pair Production by Moving Dielectrics
D. Robaschik (Leipzig), E. Wieczorek (Zeuthen)	Fluctuations of the Vacuum Stress in the Presence of Boundaries
A. Wipf (Zurich)	Soluble Models at Finite Temperature or Q(F)T under the Influence of External Conditions
M. Torres (Mexico)	Vortices in Chern-Simons Theory with Anomalous Magnetic Moment
K. Kirsten (Kaiserslautern)	Quantum Field Theory in Toroidal Space-Time

Tuesday, September 15

E. Elizalde (Barcelona)	Zeta-Function Regularization Techniques for Series Summation and Applications
H. Reeh (Göttingen)	1-particle Hamiltonians with Contact Interaction
A. Actor (Fogelsville)	Quantum Fields near Boundaries
E. Wieczorek (Zeuthen)	Non-abelian Gauge Fields in the Background of Magnetic Strings
H.-J. Kaiser (Zeuthen)	Propagators in Magnetic String Background and the Problem of Self-Adjoint Extensions
U. Günther (Kiev/Leipzig)	On QED with the External Potential $V(z,t) = \gamma\delta(z) - \theta(t)e_0Ez$

Wednesday, September 16

V.Ch. Zhukovsky (Moscow)	Radiative Effects in the Standard Model under the Influence of External Fields
B. Geyer (Leipzig), D. Mülsch (Leipzig)	Contribution of Instanton Sector to the Z–Functional in Yang–Mills–Dirac Theory
U. Müller (Berlin)	Radiative Corrections for a Non-Abelian Gauge Theory in a Homogeneous Background
V.N. Frolov (Moscow)	Topology, Causality and Chronology Protection
R. Müller (Konstanz), J. Audretsch (Konstanz) R.	Nonlocal Corellations and Stimulated Emission in the Unruh Effect
G. Barton (Sussex), K. Scharnhorst (Leipzig)	QED between parallel mirrors and the Signal Velocity of Light

Thursday, September 17

V.D. Skarzhinsky (Moscow)	QED in a Conic Spacetime
A. Economu (Konstanz), J. Audretsch (Konstanz)	Quantum Processes in Cosmic String Spacetimes
K.H. Lotze (Jena)	Creation of Mutually Interacting Particles in Anisotropically Expanding Universe
M. Basler (Jena)	Functional Integral in Curved Spacetime for Arbitrary Densities
G. Cocho (Mexico), S. Hacyan (Mexico), F. Soto (Mexico), C. Villarreal (Mexico)	Gluon Confinement and Quark Acceleration in Hadron Production
R. Jauregui (Mexico)	A Rindler–Dirac Electron in the Role of a Quantum Detector
F. Hinterleitner (Wien)	An Inertial and an Accelerated Model Detector in 2–dimensional Flat Spacetime
S. Hacyan (Mexico), V.N. Frolov (Moscow)	Electromagnetic Fluctuations and Moving Detectors in a Medium

Friday, September 18

V.I. Ritus (Moscow)	Anomalous Magnetic Moment of Electron in Intense Electric Field
G. Soff (Darmstadt)	Nuclear Size Effects on QED Corrections in Heavy Ions
V.M. Mostepanenko (St. Petersburg)	Casimir Effect and its Application
S. Graf (Frankfurt/M.)	Pair Creation and Transport in Classical Fields
S.L. Lebedev (Cheboksary)	Non-Local Effects of Classical Electrodynamics and their Application in QED
M. Bordag (Leipzig)	Levelshift in the Hydrogen Atom between Plates

C. Lämmerzahl (Konstanz), J. Audretsch (Konstanz)	QED between Plates: Measurement by Atom Beam Interferometry

Saturday, September 19

S.A. Voropaev (Moscow)	On the Point Interaction for the Fermions in the Gauge Aharonov-Bohm Effect
S. Abe (Erlangen), R.E. Ehrhardt (Erlangen)	Method of Hermitian Invariant in Non-Stationary Field Theory
V.I. Ritus (Moscow)	The Width of the Stokes Lines
V.M. Mostepanenko (St. Petersburg), I.Y. Sokolov (St. Petersburg)	The Restrictions on the Constants of Hypothetical Long Range Interactions from Casimir Force Measurements

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List of Participants

Abe, S.A.
ehrhardt @ theorie3.physik.uni-erlangen.dbp.de

Universität Erlangen-Nürnberg
Institut für Theoretische Physik III
Staudtstr. 7
D-W-8520 Erlangen, Germany

Actor, A.

Penn State University
Academic Building
8400 Mohr Lane
Fogelsville, PA 18051-9999, USA

Audretsch, J.
phlaemm @ dknkurz1.bitnet

Universität Konstanz
Fakultät für Physik
Postfach 5560
D-~~W-7750-~~ 78434 Konstanz 1, Germany

Barton, G.
mpfk7 @ central.sussex.ac.uk.

University of Sussex
Physics Building
Brighton, BN1 9QH, England

Basler, M.
michael.basler @ physik.uni-jena.dbp.de

Friedrich-Schiller-Universität
Institut für Theoretische Physik
Max-Wien-Platz 1
D-O-6900 Jena, Germany

Bordag, M.

bordag @ qft.physik.uni-leipzig.dbp.de

Universität Leipzig
Fachbereich Physik
Augustusplatz 10
D-O-7010 Leipzig, Germany

Cocho, G.

cocho @ unamvm1.bitnet

Universidad Nacional Autonoma de Mexico
Instituto de Fisica
Apartado Postal 20-364
Mexico, DF 01000, Mexico

Eberlein, C.

mpfa9 @ central.sussex.ac.uk
University of Sussex
MAPS
Brighton, BN1 9QH, England

Economu, A.

phecono @ dknkurz1.bitnet

Universität Konstanz
Fakultät für Physik
Postfach 5560
D-~~W-7750~~ 78434 Konstanz 1, Germany

Ehrhardt, R.E.

ehrhardt @ theorie3.physik.uni-erlangen.dbp.de

Universität Erlangen-Nürnberg
Institut für Theoretische Physik III
Staudtstr. 7
D-W-8520 Erlangen, Germany

Elizalde, E.

eli @ ebubecm1.bitnet

Universitat de Barcelona
Facultat de Fisica
Department E.C.M.
Diagonal 647
08028 Barcelona, Spain

Frolov, V.P.

frolov @ astro.ku.minerva.dk400.dk

Copenhagen University Observatory
Oster Voldgade 3
DK-2100 Copenhagen, Denmark

Lebedev Physical Institute
Lenin Prospect 53
Moscow, 117 924, Russia

Fütterer, C.

fuetterer @ physik.uni-karlsruhe.dbp.de

Universität Karlsruhe
Institut für Theoretische Teilchenphysik
D-W-7500 Karlsruhe, Germany

Geyer,B.

geyer @ qft.physik.uni-leipzig.dbp.de

Universität Leipzig
Fachbereich Physik
Augustusplatz 10
D-O-7010 Leipzig, Germany

Graf, S.

uf41 @ ddagsi3.bitnet

Universität Frankfurt/M.
Institut für Theoretische Physik
Robert-Mayer-Str. 10
D-W-6000 Frankfurt/M., Germany

Günther, U.

Kiev Polytechnic Institute
Kiev, Ukraine

Universität Leipzig
Fachbereich Physik
Augustusplatz 10
D-O-7010 Leipzig, Germany

Hacyan, S.

teorica @ ifunam.ifisicacu.unam.mx

Universidad Nacional Autonoma de Mexico
Instituto de Fisica
Apartado Postal 20-364
Mexico, DF 01000, Mexico

Hennig, D.

Humboldt-Universität zu Berlin
Fachbereich Physik
Institut für Theoretische Physik
Invalidenstr. 110
D-0-1040 Berlin, Germany

Hinterleitner, F.

Universität Wien
Institut für Theoretische Physik
Boltzmanngasse 5
A-1090 Wien, Austria

Hitzer, E.

Universität Konstanz
Fakultät für Physik
LS Dehnen
Postfach 5560
D-~~W-7750~~ Konstanz 1, Germany
78434

Jauregui, R. teorica @ ifunam.ifisicacu.unam.mx

Universidad Nacional Autonoma de Mexico
Instituto de Fisica
Apartado Postal 20-364
Mexico, DF 01000, Mexico

Kaiser, H.J. t00kai @ dhhdesy3.bitnet

DESY
Institut für Hochenergiephysik
Platanenallee 6
D-O-1615 Zeuthen, Germany

Kirsten, K.

Universität Kaiserslautern
Fachbereich Physik
Postfach 3049
D-W-6750 Kaiserslautern, Germany

Lämmerzahl, C. phlaemm @ dknkurz1.bitnet

Universität Konstanz
Fakultät für Physik
Postfach 5560
D-~~W-7750~~ Konstanz 1, Germany
78434

Lebedev, S. L.

Chuvash State Pedagogical Institute
Ph.-M. Faculty
Department of Theoretical Physics
K. Marx St. 38
Cheboksary, 428000, Russia

Lotze, K.-H.

Friedrich-Schiller-Universität
Institut für Theoretische Physik
Max-Wien-Platz 1
D-O-6900 Jena, Germany

Mostepanenko, V. M.

St. Petersburg Technological Institute
Department of Higher Mathematics
Moskovsky pr. 26
St. Petersburg, 198013, Russia

Müller, R.

phmuellr @ dknkurzl.bitnet

Universität Konstanz
Fakultät für Physik, AG Audretsch
Postfach 5560
D-~~W-7750~~ Konstanz 1, Germany
78434

Müller, U.

Humboldt-Universität zu Berlin
Fachbereich Physik
Institut für Elementarteilchenphysik
Invalidenstr. 110
D-0-1040 Berlin, Germany

Reeh, H.

Universität Göttingen
Institut für Theoretische Physik
Bunsenstr. 9
D-W-3400 Göttingen, Germany

Ritus, V. I.

theordep @ sci.fian.msk.su

Lebedev Physical Institute
Lenin Prospect 53
Moscow, 117 924, Russia

Robaschik, D.

robaschik @ qft.physik.uni-leipzig.dbp.de

Universität Leipzig
Fachbereich Physik
Augustusplatz 10
D-O-7010 Leipzig, Germany

Scharnhorst, K.

kjsch @ qft.physik.uni-leipzig.dbp.de

Universität Leipzig
Fachbereich Physik
Augustusplatz 10
D-O-7010 Leipzig, Germany

Skarzhinsky, V. D. **skarzh @ sci.fian.msk.su**

Lebedev Physical Institute
Lenin Prospect 53
Moscow, 117 924, Russia

Soff, G. tp06 @ ddagsi3.bitnet

Gesellschaft für Schwerionenforschung
Planckstr.1
Postfach 110 552
D-W-6100 Darmstadt, Germany

Torres, M. mtorres @ unamvin1.bitnet

Universidad Nacional Autonoma de Mexico
Instituto de Fisica
Apartado Postal 20-364
Mexico, DF 01000, Mexico

Voropaev, S. A. venus @ glas.apc.org

Moscow Physics and Technical Institute
Department of Theoretical Physics
Moscow, Russia

Wieczorek, E. wieczorek @ ifh.de

DESY
Institut für Hochenergiephysik
Platanenallee 6
D-O-1615 Zeuthen, Germany

Wipf, A. wipf @ czheth5a.bitnet

ETH Zürich
Institut für Theoretische Physik
Hönggerberg
CH-8093 Zürich, Switzerland

Zhukovsky, V. Ch.

Moscow State University
Physics Department
Lenin Hills
Moscow, 117234, Russia