Physics through Extra Dimensions: On Dualities, Unification, and Pair Production

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A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF PHYSICS

November, 2003

Abstract

This thesis has two parts. In the first part we study M-theory compactifications on singular manifolds of G_2 holonomy which are asymptotic to quotients of cones over $\mathbf{S}^3 \times \mathbf{S}^3$. We investigate the moduli space of M-theory compactifications on these manifolds, and we discover smooth interpolations leading to a new kind of duality among four dimensional supersymmetric gauge theories with different gauge groups, with a fixed number of abelian factors. The quotients are such that the internal space has a finite, non-trivial fundamental group, and we construct a grand unified model in which the breaking of the gauge symmetry by Wilson lines to the standard model arises naturally. This model provides the basis for the study of unification from M-theory. Contrary to long-standing perception, we show that a grand unification scale exists even in the presence of threshold corrections. We obtain a precise relation between the unification scale and the mass of heavy gauge bosons in terms of topological invariants. This relation replaces the often-made assumption that the unification scale and the heavy masses are equal. We also find the relation between Newton's constant, the unification scale, and the unified couplings. We go on to investigate the lifetime of the proton and we find that, when compared with standard four dimensional GUT's, some modes of proton decay in M-theory compactifications are suppressed relative to others.

In the second part of the thesis, we combine two rather fundamental, yet previously disjoint, results: one is the Kaluza-Klein unification of gravity with electromagnetism via the introduction of an extra dimension; the other is Schwinger's production of electron-positron pairs from a constant electric field. The combination involves investigating the production of pairs of KK-charged particles from a KK electric field. While it has been well-accepted that Schwinger pair production occurs via a semi-classical tunneling mechanism, we find here that pair production takes place even though tunneling does not. Instead, the pair production occurs via a different mechanism which involves a combina-

tion of the Unruh effect and vacuum polarization.

Acknowledgements

I have been very fortunate and privileged to have had the opportunity to work with and learn from some of the greatest minds of physics today. I am grateful to Professor Edward Witten, my advisor, for stimulating discussions and for his time and dedication. I cherish every moment of our work together, especially the ups and downs, when we went down for coffee and hot chocolate and came back up for physics. I am grateful to Professor Herman Verlinde for producing particle pairs with me and for his support and encouragement. I am grateful to Professor Val Fitch for guiding me into a unique world in which experimental physics is magical and powerful, and for his constant encouragement. I am grateful to Professor Alexander Polyakov for sharing his clarity and beauty of thought with me throughout my days in Princeton.

I owe a special debt to the exceptionally unique faculty of the mathematics department in which I grew up as an undergraduate. I would like to thank Professors Conway, Gunning, Sarnak, Shimura, Stein, and Wiles. I would especially like to thank my senior thesis advisor, Professor MacPherson, for his continuous support and encouragement.

I enjoyed the time spent with the argentine tango and ballroom dancers, on and off the dance floor.

Finally, many thanks go to my friends, especially to CheeWhye Chin, Hulya Guler, Jonathan Hanke, Joanna Huang, and Yuan-Pin Lee, for their invaluable support and friendship.

The work presented in this thesis was supported in part by a National Science Foundation Graduate Research Fellowship, in part by a Paul and Daisy Soros Fellowship for New Americans, and in part by NSF Grant PHY-9802484. I would also like to thank the IAS at Princeton for hospitality.

To my super-mother

and to my father.

Nothing would have been possible

without them.

Contents

Prelude

In this thesis, we investigate three distinct features that can arise in four dimensional physics from the study of theories with extra dimensions: dualities, unification, and pair production.

The idea that theories which have more than four space-time dimensions can be used to describe the four dimensional world originates in the work of Kaluza and Klein in the 1920's [1, 2]. Motivated by the quest for unification, Kaluza and Klein introduced a fifth dimension to unify the forces of gravity and electromagnetism. Their argument was rather simple and elegant: they showed that when a five dimensional metric is decomposed into a four dimensional metric, a four-vector, and a scalar, the Einstein action for gravity in five dimensions contains terms involving the Einstein action for gravity in four dimensions as well as the Maxwell action for electromagnetism in four dimensions. Hence, one force in five dimensions – gravity – is enough for describing both gravity and electromagnetism in four dimensions.

This idea of introducing extra dimensions has had an enormous amount of influence on theoretical physics. Often, as in the case of Kaluza and Klein, extra dimensions are associated with unification of forces. For example, superstring theories, which require ten dimensions for their space-time, have emerged as candidates for unification of all the four forces: gravity as well as the three gauge interactions. Sometimes, extra dimensions may unify something other than forces. For example, recently a seventh extra dimension unified various string theories – which are believed to be dual, or equivalent, to each other – into one eleven-dimensional picture, known as M-theory; this theory is believed to be an underlying unified theory and a common origin of the various string theories and their dualities [3, 4, 5, 6].

Once the extra dimensions are included in a theory, one attempts to see how they affect four dimensional physics. The method of describing the effect of the extra dimensions on four dimensional physics is known as Kaluza-Klein reduction or compactification. The extra dimensions form a space known as the internal space or compactification manifold. Various quantities in four dimensions are given by properties of this space. For example, masses of particles as seen by the four dimensional observer are given by eigenvalues of the Laplacian operator of the internal space; electric charge as seen in four dimensions is given by momentum in an extra direction. Sometimes, it is not the case that the reduction arises in a way similar to that which appeared in Kaluza and Klein's work. One example which will be used here is geometric engineering, in which certain types of singularities in the internal space lead to the appearance of certain kinds of particles in four dimensions. Another example is preservation of supersymmetry in four dimensions, which is determined by the holonomy of the internal space.

In the first part of this thesis, we investigate dualities and unification as they arise from our study of M-theory, in which we take the seven dimensional internal manifold to be singular and have G_2 holonomy. In the second part of this thesis, we investigate pair production of Kaluza-Klein charged particles in an electric field which arises from one extra dimension. We now discuss each of these investigations in turn.

Part I: On dualities and unification from G_2 manifolds

Dualities

Dualities are equivalences among theories which may seem to be distinct but in fact describe the same physics. There is a close interrelation between unification of seemingly distinct theories and dualities between them, which dates back to the development of the theory of electromagnetism: Maxwell's theory of electromagnetism unifies electricity with magnetism, and at the same time makes manifest a symmetry between them known as electric-magnetic duality.

As the ultimate unifying theory, M-theory is believed to underlie dualities among the various ten dimensional string theories. It is also believed to be the origin of dualities among four dimensional theories. For example, modern manifestations of electric-magnetic duality in four dimensional supersymmetric gauge theories [7, 8, 9] have been rederived via smooth deformations in M-theory [10, 11, 12].

Here, we deal with dualities of four dimensional supersymmetric gauge theories as they arise from M-theory. We arrive at such theories by choosing internal spaces with appropriate properties. Supersymmetry in four dimensions is ensured by a choice of an internal space which has G_2 holonomy, and the gauge group that we obtain in four dimensions is encoded geometrically in the structure of the internal space via geometric engineering. When the internal space has, in addition to its G_2 holonomy, also a singularity of ADE type, it corresponds to a supersymmetric gauge theory in four dimensions with gauge group $SU(N)$ (corresponding to A), $SO(2N)$ (corresponding to D), or $E_{6,7,8}$ (corresponding to E) [13]; singularities of a type different from ADE can lead to chiral matter fields in four dimensions [14, 15, 16].

We consider the internal manifolds which are asymptotic to cones over quotients of $\mathbf{S}^3 \times \mathbf{S}^3$; they have both G_2 holonomy and ADE singularities. In Chapter 1 [17], we study the moduli space of M-theory compactifications on these manifolds, and we find that it is composed of several connected components. On each component, we discover smooth interpolations leading to a new kind of duality among four dimensional supersymmetric gauge theories with different gauge groups, all of which have the same number of abelian factors. The dual theories all lie on the same connected component of the moduli space.

An example of a set of theories smoothly connected to each other is theories whose low energy gauge group is of the form $\Pi_i SU(n_i) \times U(1)^s$, where s is a fixed integer, $\sum n_i$ is p, q, or r, where p, q, r are relatively prime integers, and the $\{n_i\}$ can be any set of integers satisfying these properties; due to confinement in non-abelian gauge theories, the gauge group that is actually observed on a given branch at low energies is given by the abelian factors, i.e. $U(1)^s$. Further examples include dualities among gauge groups which contain orthogonal, symplectic, and exceptional Lie groups. The smooth interpolations and dualities derived here from M-theory have been rederived and extended within a field theory point of view in [18, 19, 20] for the unitary gauge groups and in [21, 22, 23] for orthogonal and symplectic gauge groups.

The abelian factors arise when the gauge groups corresponding to the ADE singularities are broken by Wilson lines; the Wilson lines arise naturally due to the non-trivial fundamental group of the locus of the ADE singularity, which is a lens space. By counting the number of inequivalent Wilson lines, we find the number of dual theories and incidentally arrive at a certain symmetry between the Lie groups $SU(p)$ and $SU(q)$ where p and q are relatively prime. The idea presented here of counting the number of theories, or vacua, that one could get from M-theory has been further developed in [24].

Symmetry breaking by Wilson lines [25, 26, 27, 28, 29] is well-known to be an ingredient in models of grand unification; hence our internal spaces of [17] lead us directly to construct unified theories. In fact, choosing $p = 5$ and $s = 1$ above leads us to the well-known $SU(5)$ model of grand unification. This is the subject of the second chapter.

Unification

The quest for unification has motivated many of the great developments in physics in the last century. The goal of this quest has been to find a unique theory which describes all the fundamental forces of nature.

An early important step towards unification, which we already mentioned, was the one taken by Kaluza and Klein in the 1920's, in which the forces of gravity and electromagnetism were unified by introducing an extra dimension.

In addition to gravity and electromagnetism, there are two more forces which must be included in a unified theory, namely the strong force and the weak force. Of these four forces, three are well-known to be described by a gauge theory; electromagnetism is described by a $U(1)$ gauge group, the weak force by $SU(2)$, and the strong force by $SU(3)$.

A major step towards unifying these three forces was achieved in the 1970's [30, 31]. In what is known as four dimensional grand unified theories, or GUT's, the three gauge interactions are included in a simple gauge group at a high energy scale. This scale is called the gauge coupling unification scale M_{GUT} and is defined to be the scale at which all the three gauge coupling constants are equal. At energies below this scale, the simple gauge group, which is also known as the unified group and is denoted G_{GUT}, is broken into the product of the three individual gauge groups, which make up the group of the standard model of particle physics, namely $SU(3) \times SU(2) \times U(1)$.

It is believed that to obtain unification that includes gravity as well as the three gauge interactions, the gauge coupling unification scale M_{GUT} should somehow be close to the Planck scale $M_{Pl} = 10^{19}\,\mathrm{GeV}$, which is given in terms of the coupling constant G_N of gravity, also known as Newton's constant, by $G_N = 1/M_{Pl}^2$. In the original estimation of the running of the three gauge coupling constants in four-dimensional GUT's [31], it was found that the coupling constants can be approximately unified in the simple gauge group $SU(5)$ at an energy scale of about $10^{15}\,\mathrm{GeV}$. Subsequently, with more precise measurements of the couplings at low energies, it became clear that gauge coupling unification occurs much more precisely if the gauge theories are supersymmetric [32], in which case the unification scale is higher, around $10^{16}\,\mathrm{GeV}$. In this sense, supersymmetric GUT's are better since their unification scale is closer to the Planck scale.

It is interesting to note that the unifying theories of the 1970's did not invoke extra dimensions; this might explain why they largely left gravity out of the picture. In the next decade, however, the ten dimensional supersymmetric string theories, which naturally contain gravity, have emerged as candidates for unification of all the four forces. Especially with the development of heterotic string theory [33], it became feasible to include both GUT's and gravity in one framework [28], since in addition to containing gravity, the heterotic string theory also contains a gauge group which can give rise to a unified group as well as chiral matter fields which appear in the standard model. Other GUT models have since been constructed, not just from the heterotic string but also from other string theories (for some examples and reviews, see [34, 35, 36, 37, 38, 39, 40]).

Even though it is possible to include both GUT's and gravity in superstring theory, the appearance of superstring theories was not the end of the road to unification; it might have been if there were only one superstring theory, but there were five. Another major step in the search for unification was taken when the five superstring theories were unified into the eleven dimensional M-theory. Naturally, this theory should contain GUT's as well as gravity.

In Chapter 2, we study grand unified theories as they arise from M-theory. Our study of GUT's from M-theory is a direct continuation of our study of dualities, since the singular internal spaces of G_2 holonomy considered in our derivation of dualities have certain key properties which lead not just to supersymmetric gauge theories, but also to supersymmetric GUT models.

A GUT model must have a unified group and a mechanism for breaking this group to the standard model. In our internal spaces, the ADE singularity, which is determined by our choice of quotient of $\mathbf{S}^3 \times \mathbf{S}^3$, corresponds to the unified group, which we take to be $SU(5)$; the Wilson line provides the mechanism for breaking it down precisely to the group of the standard model. In addition, a GUT model should also include the chiral fermions of the standard model; we put them into our model by hand, or by assuming that in addition to the ADE singularities, the G_2 manifold has other singularities that can give rise to chiral fermions. (We should note that many of our results, specifically the formula that we discuss below, are independent of the inclusion of these fermions.)

One of our main tasks in investigating our GUT model is to study in detail the equations which govern the running of the gauge coupling constants, known as renormalization group equations (RGE's). These include threshold correction terms, which arise from Kaluza-Klein modes with momenta around the extra dimensions.

We find that the threshold correction terms, which as we show are determined by the topological invariant known as the Ray-Singer torsion, satisfy a property which leads to exact preservation of unification, and holds when $G_{GUT} = SU(5)$. The unification scale is in fact the same as the one obtained in standard four dimensional supersymmetric GUT's with no threshold effects, namely 2.2×10^{16} GeV. We derive a simple formula which provides a precise relation between the unification scale and the volume of a certain submanifold in the internal space, namely the three dimensional locus Q of the ADE singularity, which is a lens space. The formula implies two important results, both of which involve some historical perspective.

a. In 1981, when Weinberg [42] included threshold corrections in the RGE's of a four dimensional model, he found that in the presence of threshold corrections, there is no energy scale at which the three gauge coupling constants are equal. While he undertook the study of the RGE's for any unified group, he checked for the existence of a unification scale only for the example of a model in which the unified group was $SO(10)$. In his abstract, however, Weinberg stated this result as true for any unified group; it has since become standard lore that there is no unification scale in the presence of threshold corrections.

It is indeed true that, generically, in the presence of threshold corrections there is no energy scale at which all three gauge couplings are equal (see also explanation in Section 2.3). However, if one repeats Weinberg's calculation for the popular example of a model in which the unified group is $SU(5)$, one finds the opposite answer: there does exist a value of the energy scale for which all three gauge couplings are equal, and it is given by[1] $M_{GUT} = e^{1/21} M_V$, where M_V is the mass of a gauge boson. Hence, contrary to standard lore, threshold corrections do not necessarily preclude the existence of a unification scale: unification may be preserved in the presence of threshold corrections.

The fact that the unification scale actually exists in our model as well is therefore really an M-theory analog of a result about four dimensional $SU(5)$ theories.

b. A major theme in GUT's is the proton lifetime. In the original papers [30, 31] of the 1970's about GUT's, the authors noticed that the lifetime of the proton came out much too short unless they made the assumption that the mass of gauge bosons which appear

[1] While this fact appeared in [43], it seems to have gone unnoticed.

in GUT's and can mediate proton decay was of the same order as the gauge coupling unification scale. Since then, such an assumption is very often made in GUT models, and the gauge boson mass is sometimes used interchangeably with the unification scale. Sometimes [44], the relation between the unification scale and this mass is taken to be a matter of convention.

Since in our model, the mass of the heavy gauge bosons can be given in terms of the volume of the locus Q and the eigenvalues of its Laplacian, our formula can be translated into a precise relation between the unification scale and the mass of the heavy gauge bosons. It shows that, while indeed in many cases these two quantities are of the same order – proving that the assumption originally made in the 1970's was a good one – they are distinct and should not be used interchangeably.

For the possible cases in which the two quantities are not of the same order, our formula can be seen as a challenge to the common practice of using simple dimensional analysis in determining or estimating physical quantities.

Given the formula for the gauge coupling unification scale, we also obtain expressions for the gravitational coupling constant G_N, for the eleven-dimensional Planck mass M_{11}, and for the eleven-dimensional gravitational coupling κ_{11}. These expressions are given in terms of the unified coupling, the unification scale, and the Ray-Singer torsions mentioned above. In addition to these, Newton's constant also depends on a certain ratio of volumes in the G_2 manifold. We find that in our model, the value of Newton's constant is in a certain sense closer to its true experimental value than it was in models arising from heterotic strings.

We go on to investigate the classic question of the proton lifetime. We find that, when compared with standard four dimensional supersymmetric GUT's, some modes of proton decay in M-theory compactifications are suppressed relative to others.[2] If proton decay is ever found experimentally, and if such suppression is found as well, it could be interpreted

[2]This calculation was repeated in [45] for GUT models that arise from intersecting D-branes.

as a signature of M-theory.

Part II: Pair production of Kaluza-Klein particles

In this part of the thesis, we go back to the original Kaluza-Klein theory, with just one extra dimension. In this framework, we study the production of pairs of Kaluza-Klein charged particles from a static background electric field that arises from Kaluza-Klein reduction. Hence, our study combines two fundamental results: the unification of gravity and electromagnetism as achieved by Kaluza and Klein in the 1920's, and Schwinger's classic study [46] of production of electron-positron pairs from a constant electric field, carried out via quantum field theory methods in 1951.

It has been widely accepted and well-known that Schwinger pair production occurs via quantum mechanical barrier penetration, or tunneling. The intuition behind it is simple: the height of the potential barrier to create a pair of charged particles in a constant electric field is equal to their total mass, and the width of the barrier is given by a critical separation of the pair. By using their electrostatic energy, which increases with their separation, virtual pairs can overcome this barrier and materialize. Once a pair reaches the critical separation at which the total potential – the mass together with the electrostatic energy – is zero, and beyond which the potential is negative, the system has tunneled through the barrier and a pair has been created. The modern instanton method [47, 48] of computing the pair creation rate depends crucially on this intuition, as do other related studies [49, 50].

Schwinger's original calculation did not invoke tunneling. His result, however, was rederived as a tunneling effect in [51], and was understood as such much earlier (for example, see [52]). Another well–known example of particle production which is often considered to arise by a tunneling process is the Hawking–Unruh effect [53, 54], whose derivation via tunneling was carried out recently in [55]. Other contexts in which pair production by tunneling appears include an argument given by Dyson [56] about the divergence of perturbation theory in QED, and the Josephson effect in superconductors

[57].

However, we show here that tunneling is not a prerequisite for pair production from a static electric field. In fact, pair creation takes place even though the gravitational backreaction of the electric field on the geometry – which we incorporate via the electric Kaluza-Klein Melvin solution – prevents the electrostatic potential from overcoming the rest mass of the Kaluza-Klein particles, thus impeding the tunneling mechanism which is often thought of as responsible for the pair creation. We find that pair creation occurs with a finite rate formally similar to the classic Schwinger result, but via an apparently different mechanism, involving a combination of the Unruh effect and vacuum polarization due to the electric field.

Another case which shows that tunneling and pair production are not necessarily linked appeared in a study of a static magnetic field in [58], in which pair production does not occur even though the tunneling probability is non-zero.

It therefore seems plausible that, contrary to standard lore, the original Schwinger effect also does not necessarily arise from tunneling. This may be relevant to experimental searches for Schwinger pair production [59, 60].

This thesis is organized as follows. Part I is divided into two chapters and contains the study of M-theory compactified on G_2 manifolds, leading to dualities and unification. Chapter 1 has been published in [17]. Chapter 2 overlaps with [61], which was done in collaboration with E. Witten. Part II presents the study of pair production of Kaluza-Klein particles. It was done in collaboration with H. Verlinde and has appeared in [62].

I would like to thank E. Witten for discussions and guidance while the work in Chapter 1 was carried out, M. Goresky and P. Sarnak for helpful discussions related to Chapter 2, and E. Brezin, K.T. McDonald, and A.M. Polyakov for helpful discussions related to Part II.

Part I

M-Theory on G_2 Manifolds: Dualities and Unification

Chapter 1

On the Quantum Moduli Space of M-Theory Compactifications

1.1 Introduction

The study of M-theory compactifications on seven dimensional manifolds X of G_2 holonomy has been motivated by the fact that such compactifications result in unbroken supersymmetry in four dimensions. The properties of the compactification manifold X determine the particle spectrum of the corresponding four dimensional theory. It has been shown in recent years that compactifications on singular manifolds can result in low energy physics containing interesting massless spectra. Specifically, certain singular G_2 manifolds give rise to $\mathcal{N} = 1$ supersymmetric gauge theories at low energies, as shown for example in [13, 63, 64]. There, X was taken to be asymptotic to a quotient of a cone on $\mathbf{S}^3 \times \mathbf{S}^3$, and the singularities of X took the form of families of co-dimension four ADE singularities giving ADE gauge theories at low energies.

Subsequently, the quantum moduli space of M-theories on G_2 manifolds X which are asymptotic to a cone on $\mathbf{S}^3 \times \mathbf{S}^3$ or quotients thereof has been studied in [16]. It was shown that the moduli space is a Riemann surface of genus zero, which interpolates smoothly between different semiclassical spacetimes.

The purpose of this chapter is to generalize the construction of [16] to other quotients of $\mathbf{S}^3 \times \mathbf{S}^3$ and obtain the moduli spaces and smooth interpolations for those as well. Our quotients contain those in [16] as special cases. We propose that the moduli space for our quotients consists of several branches classified according to the number of massless $U(1)$ factors that appear in the low energy gauge theories corresponding to semiclassical points. Each branch of our moduli space interpolates smoothly among the different semiclassical points appearing on it; hence, we get smooth interpolation among supersymmetric gauge theories with different gauge groups.

This chapter is organized as follows. In Section 1.2 we review the M-theory dynamics on the cone on $Y = \mathbf{S}^3 \times \mathbf{S}^3$ given in [16]. In Section 1.3, we describe quotients of this cone by discrete groups of the form $\Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3$ where the Γ_i are ADE subgroups of $SU(2)$; these ADE groups must be chosen carefully in order to obtain known low energy gauge theories from the compactification. In Section 1.4, we turn to the description of the moduli space $\mathcal{N}_\Gamma$ of M-theories on these quotients, beginning with the classical moduli space and concluding with the quantum moduli space.

1.2 Dynamics of M-theory on the Cone over $\mathbf{S}^3 \times \mathbf{S}^3$

In this section, we review the M-theory dynamics on a manifold X of G_2 holonomy which is asymptotic at infinity to a cone over $Y = \mathbf{S}^3 \times \mathbf{S}^3$ [16]. The manifold Y can be described as a homogeneous space $Y = SU(2)^3/SU(2)$, where the equivalence relation is $(g_1, g_2, g_3) \sim (g_1 h, g_2 h, g_3 h)$, $g_i, h \in SU(2)$. Viewed this way, this manifold has $SU(2)^3$ symmetry via left action on each of the three factors in Y, as well as a "triality" symmetry S_3 permuting the three factors. Up to scaling, there is a unique metric with such symmetries given by

$$d\Omega^2 = da^2 + db^2 + dc^2, \tag{1.2.1}$$

where $a, b, c \in SU(2)$, $da^2 = -\mathrm{Tr}(a^{-1}da)^2$, the trace is taken in the fundamental representation of $SU(2)$, and a, b, c are related to g_1, g_2, g_3 by $a = g_2 g_3^{-1}$ and cyclic permutations thereof.

The metric for a cone on Y is

$$ds^2 = dr^2 + r^2 d\Omega^2, \tag{1.2.2}$$

where $d\Omega^2$ is the metric on Y. Such a cone can be constructed by filling in one of the three $SU(2) \sim \mathbf{S}^3$ factors of Y to a ball. We denote the manifold obtained by filling in a given g_i by X_i. The metric on a manifold X, asymptotic to X_i at infinity, can be written with a new radial variable y, which is related to r by

$$y = r - \frac{r_0^3}{4r^2} + O(1/r^5), \tag{1.2.3}$$

as

$$ds^2 = dy^2 + \frac{y^2}{36}\Big(da^2 + db^2 + dc^2 - \frac{r_0^3}{2y^3}\left(f_1\, da^2 + f_2\, db^2 + f_3\, dc^2\right) + O(r_0^6/y^6)\Big), \tag{1.2.4}$$

where r_0 is a parameter denoting the length scale of X_i, and $(f_{i-1}, f_i, f_{i+1}) = (1, -2, 1)$ (indices are understood mod 3). When $y \to \infty$ or $r \to \infty$, this becomes precisely the cone (1.2.2).

We will need to study the 3-cycles of Y in order to understand the relations between the periods of the M-theory C-field and the membrane instanton amplitudes, which we shall need in order to describe the moduli space.

The 3-cycles D_j of Y are given by projections of the j^{th} factor of $SU(2)^3$ to Y. Hence, $D_j \cong \mathbf{S}^3$. The third Betti number of Y is two, so the three D_j satisfy the relation

$$D_1 + D_2 + D_3 = 0. \tag{1.2.5}$$

The intersection numbers of the D_i are given by

$$D_i \cdot D_j = \delta_{j,i+1} - \delta_{j,i-1}. \tag{1.2.6}$$

At X_i, where the i^{th} factor is filled in, D_i shrinks to zero and the relation (1.2.5) reduces to $D_{i-1} + D_{i+1} = 0$ (where again the indices are understood mod 3).

At each X_i, there is a supersymmetric 3-cycle Q_i given by $g_i = 0$. It can be shown that Q_i is homologous to $\pm D_{i-1}$ and $\mp D_{i+1}$, where the sign depends on orientation.

A manifold still has G_2 holonomy up to third order in r_0/y if we take the f_j of (1.2.4) to be any linear combination of $(1, -2, 1)$ and its permutations – so we have G_2 holonomy as long as

$$f_1 + f_2 + f_3 = 0. \tag{1.2.7}$$

These f_j can be interpreted as volume defects of the cycle D_j at infinity: the volume of D_j depends linearly on a positive multiple of f_j. Furthermore, since at the classical manifold X_i, only one of the D_j vanishes, only one of the f_j (namely f_i) can be negative. So the classical moduli space may contain manifolds with the relation (1.2.7) as long as only one of the f_j is negative [16, 65].

The periods of the C-field along the cycles D_j are $\alpha_j = \int_{D_j} C$. We combine them with the f_j into holomorphic observables η_j where now the C-field period is a phase:

$$\eta_j = \exp\Big(\frac{2k}{3} f_{j-1} + \frac{k}{3} f_j + i\alpha_j\Big), \tag{1.2.8}$$

where k is a parameter. The relation (1.2.7) means that the η_j are not independent, but instead they obey

$$\eta_1\eta_2\eta_3 = \exp\Big(i \sum \alpha_j\Big). \tag{1.2.9}$$

(It can be shown that due to a global anomaly in the membrane effective action, the right hand side above is -1).

The moduli space at the classical approximation is given by three branches $\mathcal{N}_i$, each of which contains one of the points X_i with $r_0 \to \infty$. On X_i, α_i vanishes and the parameters f_j are such that $\eta_i = 1$. So on $\mathcal{N}_i$ the functions η_j obey

$$\eta_i \;=\; 1, \;\; \eta_{i-1}\eta_{i+1} = -1. \tag{1.2.10}$$

At the quantum level, there are corrections to this statement. It has been suggested in [64] that the different classical points X_i are continuously connected to one another.

Hence they should appear on the same branch of the moduli space $\mathcal{N}$. We proceed now with the assumption that the only classical points are the X_i, which are the points where some of the η_j have a zero or pole. As explained in [16], since a component of $\mathcal{N}$ which contains a zero of a holomorphic function η_j must also contain its pole, and since the only points at which the η_j are singular are associated with one of the X_i, it follows indeed that all X_i are contained on a single component of $\mathcal{N}$. Furthermore, each η_j has a simple zero and simple pole in $\mathcal{N}$. The existence of such functions on $\mathcal{N}$ means that the branch containing the zero and pole has genus zero. In addition, any of the η_j can be identified as a global coordinate of $\mathcal{N}$. Choosing any η_j gives a complete description for this branch of $\mathcal{N}$.

1.3 Quotients and Low Energy Gauge Groups

Here, we begin our study of manifolds which are asymptotic to a cone over quotients of Y. We shall consider a discrete group action of $\Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3$ on Y where the Γ_i will be chosen from ADE subgroups of $SU(2)$ in such a way that the low energy physics is known.

We begin with the simplest case where $\Gamma = \mathbf{Z}_p \times \mathbf{Z}_q \times \mathbf{Z}_r$. Each $\mathbf{Z}_n$ is embedded in $SU(2)$ via

$$\beta^k = \begin{pmatrix} e^{2\pi i k/n} & 0 \\ 0 & e^{-2\pi i k/n} \end{pmatrix}, \tag{1.3.1}$$

where β is the generator of $\mathbf{Z}_n$ and $k = 0, 1, \ldots, n-1$. The action of Γ on $Y = SU(2)^3/SU(2)$ is given by

$$(\gamma, \delta, \epsilon) \in \mathbf{Z}_p \times \mathbf{Z}_q \times \mathbf{Z}_r : (g_1, g_2, g_3) \mapsto (\gamma g_1, \delta g_2, \epsilon g_3), \tag{1.3.2}$$

and we denote the resulting quotient space by Y_Γ.

The spaces $X_{i,\Gamma}$, obtained by filling in the i^{th} $SU(2)$ factor of Y_Γ, are quotients of $\mathbf{R}^4 \times \mathbf{S}^3$ where the $\mathbf{R}^4$ corresponds to the filled-in factor. Choosing $i = 1$ and gauging g_2 away using the right diagonal $SU(2)$ action, the identification corresponding to $(\gamma^k, \delta^l, \epsilon^m) \in \Gamma$ is

$$(g_1, 1, g_3) \sim (\gamma^k g_1 \delta^{-l}, 1, \epsilon^m g_3 \delta^{-l}), \tag{1.3.3}$$

where $g_1 \in \mathbf{R}^4$ and $g_3 \in SU(2) \sim \mathbf{S}^3$. The set $(0, 1, g_3)$ with g_3 varying in $SU(2)$ is a fixed point of the action of the $\mathbf{Z}_p$ subgroup of Γ, and this singularity is identical to the standard A_{p-1} singularity of codimension four of the form $\mathbf{R}^4/\mathbf{Z}_p$ or $\mathbf{C}^2/\mathbf{Z}_p$, which gives an $SU(p)$ gauge theory at low energies.

Depending on the values of the integers q and r, there may be additional, unfamiliar singularities for which we do not know the low energy physics. Namely, there may be values of g_3 which are fixed under a non trivial subgroup of $\mathbf{Z}_q \times \mathbf{Z}_r$, i.e. where the following holds

$$\epsilon^m g_3 \delta^{-l} = g_3. \tag{1.3.4}$$

This is the same as looking for elements g_3 of $SU(2)$ which diagonalize δ^l:

$$\epsilon^m = g_3 \delta^l g_3^{-1}. \tag{1.3.5}$$

Choosing the orders q and r of δ and ϵ to be relatively prime, $(q, r) = 1$, ensures that there are no solutions of this equation (since then the orders of the left and right hand sides of (1.3.5) are relatively prime). Similarly, we choose $(p, q) = (p, r) = 1$, and so there are no singularities at $X_{i,\Gamma}$ other than the ADE singularities whose low energy physics is known: an A_{p-1} singularity on $\mathbf{S}^3/(\mathbf{Z}_q \times \mathbf{Z}_r)$ at $X_{1,\Gamma}$, an A_{q-1} singularity on $\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$ at $X_{2,\Gamma}$, and an A_{r-1} singularity on $\mathbf{S}^3/(\mathbf{Z}_p \times \mathbf{Z}_q)$ at $X_{3,\Gamma}$, with the discrete group action on $\mathbf{S}^3$ given by the appropriate cyclic permutation of the action on g_3 in (1.3.3).

Now consider also the non-abelian ADE groups. Again, we would like to choose Γ such that we will only get singularities whose physics at low energies we understand – namely, ADE singularities. For this purpose we review the relevant properties of the DE groups. For information about these groups, see [66].

As in the abelian case, we let $\Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3$ act on Y by

$$(\gamma, \delta, \epsilon) \in \Gamma_1 \times \Gamma_2 \times \Gamma_3 : (g_1, g_2, g_3) \mapsto (\gamma g_1, \delta g_2, \epsilon g_3), \tag{1.3.6}$$

from which equations (1.3.3) and (1.3.4) follow in the same way as before.

The binary dihedral groups $\mathbf{D}_q$ have order $4q-8$ and are generated in $SU(2)$ by two elements:

$$\mathbf{D}_q = \left\langle \begin{pmatrix} e^{\frac{\pi i}{q-2}} & 0 \\ 0 & e^{-\frac{\pi i}{q-2}} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle . \tag{1.3.7}$$

Since all $\mathbf{D}_q$ groups share an element of order 4, we cannot choose more than one of the Γ_i to be a dihedral group, since otherwise we would get solutions to (1.3.4). Hence we let $\Gamma = \mathbf{Z}_p \times \mathbf{D}_q \times \mathbf{Z}_r$ with $(p,r) = (p, 2(q-2)) = (r, 2(q-2)) = 1$.

We turn to the E series. A singularity $\mathbf{R}^4/G$ which gives at low energies $\mathbf{E}_6$, $\mathbf{E}_7$, or $\mathbf{E}_8$ gauge groups corresponds to G being the tetrahedral group $\mathbf{T}_{24}$, the octahedral group $\mathbf{O}_{48}$, or the icosahedral group $\mathbf{I}_{120}$. The orders of these groups are 24, 48, and 120 respectively, and each of them has elements of orders 3 and 4, so we cannot have more than one E group appearing in Γ. The group $\mathbf{I}_{120}$ also has elements of order 5. Hence, in addition to $(p,r) = 1$, for $\Gamma = \mathbf{Z}_p \times \mathbf{E}_6 \times \mathbf{Z}_r$ or $\Gamma = \mathbf{Z}_p \times \mathbf{E}_7 \times \mathbf{Z}_r$, we need also $(p, 2\cdot 3) = (r, 2\cdot 3) = 1$, and for $\Gamma = \mathbf{Z}_p \times \mathbf{E}_8 \times \mathbf{Z}_r$, we need $(p, 2\cdot 3\cdot 5) = (r, 2\cdot 3\cdot 5) = 1$.

Therefore, our group Γ is always chosen to be of the form $\Gamma = \mathbf{Z}_p \times \Gamma_2 \times \mathbf{Z}_r$ where Γ_2 is an A, D, or E group, and p, r, and Γ_2 satisfy the conditions noted above, which can be summarized by

$$(p, N) = (r, N) = (p, r) = 1, \tag{1.3.8}$$

where N is the order of the group Γ_2. At $X_{1,\Gamma}$ we have an A_{p-1} singularity on $\mathbf{S}^3/(\Gamma_2 \times \mathbf{Z}_r)$, at $X_{3,\Gamma}$ we have an A_{r-1} singularity on $\mathbf{S}^3/(\mathbf{Z}_p \times \Gamma_2)$, and at $X_{2,\Gamma}$ we have an A, D, or E singularity on $\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$, where here the discrete group action is given by the appropriate cyclic permutation of the action on g_3 in (1.3.3).

The low energy gauge theories obtained from compactifying M-theory on $\mathbf{R}^4 \times X_{i,\Gamma}$ are listed in Table (1.3.9). Each entry contains the gauge group and the compact 3-manifold which is the locus of the ADE singularity.

As we shall see below, for the cases where Γ_2 is a D or E group, there are additional semiclassical points where the low energy gauge group is different from those listed above.

Γ_2	$X_{1,\Gamma}$		$X_{2,\Gamma}$		$X_{3,\Gamma}$	
$\mathbf{Z}_q$	$SU(p)$	$\mathbf{S}^3/(\mathbf{Z}_q \times \mathbf{Z}_r)$	$SU(q)$	$\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$	$SU(r)$	$\mathbf{S}^3/(\mathbf{Z}_p \times \mathbf{Z}_q)$
$\mathbf{D}_q$	$SU(p)$	$\mathbf{S}^3/(\mathbf{D}_q \times \mathbf{Z}_r)$	$SO(2q)$	$\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$	$SU(r)$	$\mathbf{S}^3/(\mathbf{Z}_p \times \mathbf{D}_q)$
$\mathbf{T}_{24}$	$SU(p)$	$\mathbf{S}^3/(\mathbf{T}_{24} \times \mathbf{Z}_r)$	$\mathbf{E}_6$	$\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$	$SU(r)$	$\mathbf{S}^3/(\mathbf{Z}_p \times \mathbf{T}_{24})$
$\mathbf{O}_{48}$	$SU(p)$	$\mathbf{S}^3/(\mathbf{O}_{48} \times \mathbf{Z}_r)$	$\mathbf{E}_7$	$\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$	$SU(r)$	$\mathbf{S}^3/(\mathbf{Z}_p \times \mathbf{O}_{48})$
$\mathbf{I}_{120}$	$SU(p)$	$\mathbf{S}^3/(\mathbf{I}_{120} \times \mathbf{Z}_r)$	$\mathbf{E}_8$	$\mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$	$SU(r)$	$\mathbf{S}^3/(\mathbf{Z}_p \times \mathbf{I}_{120})$

(1.3.9)

We note that for the case with $r = 1$, $X_{3,\Gamma}$ is smooth and its low energy theory has no gauge symmetry. If also $p = 1$, $X_{1,\Gamma}$ is smooth as well (this is the case studied in [16]).

1.4 The Curve of M-theories on the Quotient

1.4.1 Classical geometry

The 3-cycles D'_i of Y_Γ are the projections of the i^{th} factor of $SU(2)^3$ to Y_Γ. Hence, for $\Gamma = \mathbf{Z}_p \times \Gamma_2 \times \mathbf{Z}_r$ we have

$$\begin{aligned} D'_1 &= \mathbf{S}^3/\mathbf{Z}_p, && (1.4.1) \\ D'_2 &= \mathbf{S}^3/\Gamma_2, && (1.4.2) \\ D'_3 &= \mathbf{S}^3/\mathbf{Z}_r. && (1.4.3) \end{aligned}$$

Using the relation (1.2.5) in Y and the fact that $D_1 \in Y$ projects to a p-fold cover of $D'_1 \in Y_\Gamma$, as well as cyclic permutations of this fact, we find

$$pD'_1 + ND'_2 + rD'_3 = 0, \tag{1.4.4}$$

where N is the order of the group Γ_2. To study the intersection numbers of the D'_i we note that $D'_1 \in Y_\Gamma$ lifts to $NrD_1 \in Y$, and similar statements are true for the other D'_i.

Counting the intersection numbers in Y and then dividing by pNr (since there are pNr points in Y which project to one point in Y_Γ), we get

$$D'_1 \cdot D'_2 = r, \qquad D'_2 \cdot D'_3 = p, \qquad D'_3 \cdot D'_1 = N. \tag{1.4.5}$$

Here we see that the D'_i generate the third homology group of Y_Γ: since $(r, N) = 1$, we can find integers m, n such that

$$D'_1 \cdot (mD'_2 + nD'_3) = mr - nN = 1, \tag{1.4.6}$$

and similarly for the other cycles.

We define the periods of the M-theory C-field at infinity by

$$\alpha'_j = \int_{D'_j} C \mod 2\pi. \tag{1.4.7}$$

Note that these are related to the α_j of Y by

$$\alpha_1 = p\alpha'_1, \qquad \alpha_2 = N\alpha'_2, \qquad \alpha_3 = r\alpha'_3. \tag{1.4.8}$$

1.4.2 Classical moduli space

We define our holomorphic observables to be the following functions of the periods α'_j and of the volumes f_j:

$$\begin{aligned}
\eta_1 &= \exp\Big(\frac{2k}{3p} f_3 + \frac{k}{3p} f_1 + i\alpha'_1\Big), \\
\eta_2 &= \exp\Big(\frac{2k}{3N} f_1 + \frac{k}{3N} f_2 + i\alpha'_2,\Big), \\
\eta_3 &= \exp\Big(\frac{2k}{3r} f_2 + \frac{k}{3r} f_3 + i\alpha'_3.\Big).
\end{aligned}$$

These functions are adopted from (1.2.8), where we substitute the expressions in (1.4.8) for the periods and then take the largest possible root that still leaves the η_i invariant under $\alpha'_j \mapsto \alpha'_j + 2\pi$.

The periods of the C-field are interpreted as the phases of the holomorphic observables. Due to (1.2.7), we have

$$\eta_1^p \eta_2^N \eta_3^r = \exp\Big(i \sum_j \alpha'_j\Big). \tag{1.4.9}$$

The η_j have zeros or poles at the semiclassical points $X_{i,\Gamma}$ with large r_0 in which the f_j diverge. As in Section 1.2, classically at the point $X_{1,\Gamma}$, $\eta_1 = 1$ and $\alpha'_1 = 0$. Hence, at this point

$$\eta_2^N \eta_3^r = \exp\Big(i(\alpha'_2 + \alpha'_3)\Big), \tag{1.4.10}$$

so when η_2 has a pole, η_3 has a zero and vice versa. In fact, the order of the zeros or poles of η_2 must be a multiple of r, and similarly the order of the zeros or poles of η_3 must be a multiple of N for this equation to hold. In the classical approximation, there are three branches $\mathcal{N}_i$ of the moduli space, on which we have $\eta_i = 1$ and $\eta_{i\pm1}$ obeying the relation (1.4.10) for $i = 1$ or cyclic permutations of it for $i = 2, 3$.

1.4.3 Quantum curve via membrane instantons

To study the quantum curve, we study the singularities, i.e. the zeros and poles of the holomorphic observables η_j, which correspond to the classical points $X_{i,\Gamma}$ with $r_0 \to \infty$. We shall use a relation between the η_j and the amplitude for membrane instantons which wrap on supersymmetric cycles Q in X. Using chiral symmetry breaking of the low energy gauge theories, we find a clear relation between the local parameter on the moduli space and our observables, and hence can describe the moduli space.

A supersymmetric cycle in $X_{i,\Gamma}$ is given by the 3-manifolds Q_i given by $g_i = 0$:

$$Q_1 = \mathbf{S}^3/(\Gamma_2 \times \mathbf{Z}_r), \tag{1.4.11}$$

$$Q_2 = \mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p), \tag{1.4.12}$$

$$Q_3 = \mathbf{S}^3/(\mathbf{Z}_p \times \Gamma_2). \tag{1.4.13}$$

At $X_{1,\Gamma}$, Q_1 is homologous (up to orientation) to the D'_j as follows:

$$rQ_1 \sim D'_2, \tag{1.4.14}$$

$$NQ_1 \sim D_3', \tag{1.4.15}$$

and cyclic permutations of that give the relations at $X_{2,\Gamma}$ to be $pQ_2 \sim D_3'$ and $rQ_2 \sim D_1'$, and at $X_{3,\Gamma}$ we have $NQ_3 \sim D_1'$ and $pQ_3 \sim D_2'$.

We now study the zeros and poles of the η_j. To understand the orders of the zeros and poles, we must compare the η_j to the true local parameter on $\mathcal{N}_\Gamma$ around each $X_{i,\Gamma}$ with large r_0.

One would expect at first that the membrane instanton amplitude u itself, given near $X_{i,\Gamma}$ by

$$u = \exp\Big(-TV(Q_i) + i\int_{Q_i} C\Big), \tag{1.4.16}$$

where T is the membrane tension and $V(Q_i)$ is the volume of Q_i, would be a good local parameter near $X_{i,\Gamma}$. However, at low energies we have a supersymmetric A, D, or E gauge theory in four dimensions, and due to chiral symmetry breaking, we expect the good local parameter – the gluino condensate – to be $u^{1/h}$ where h is the dual Coxeter number of the gauge group.

We now compare phases of the η_j to the phase of u. Let $P_{i,\Gamma}$ correspond to the manifolds $X_{i,\Gamma}$ with large r_0. For the case where $\Gamma_2 = \mathbf{Z}_q$, at $P_{1,\Gamma}$ equation (1.4.14) implies that the phase $\int_{D_2'} C$ of η_2 is related to the phase $\int_{Q_1} C$ of u by $\int_{D_2'} C \sim r\int_{Q_1} C$. Since the good local parameter is actually $u^{1/p}$ due to chiral symmetry breaking of the $SU(p)$ gauge theory at $P_{1,\Gamma}$, the true order of the zero of η_2 at $P_{1,\Gamma}$ is pr. The same calculation for the other η_j and $P_{i,\Gamma}$ gives the orders of zeros and poles shown in the following table:

$\Gamma_2 = \mathbf{Z}_q$	$P_{1,\Gamma}$	$P_{2,\Gamma}$	$P_{3,\Gamma}$
η_1	1	∞^{qr}	0^{qr}
η_2	0^{pr}	1	∞^{pr}
η_3	∞^{pq}	0^{pq}	1

(1.4.17)

The cases where Γ_2 is a D or E group give similar tables, except that in these cases we get extra semiclassical points in the same way as in [16]: for the case $\Gamma_2 = \mathbf{D}_q$, we have

$\Gamma_2 = \mathbf{D}_q$	$P_{1,\Gamma}$	$P_{2,\Gamma}$	$P_{2',\Gamma}$	$P_{3,\Gamma}$
η_1	1	∞^{rh}	$\infty^{2rh'}$	0^{rN}
η_2	0^{rp}	1	-1	∞^{rp}
η_3	∞^{Np}	0^{hp}	$0^{2h'p}$	1

, (1.4.18)

where $h = 2q - 2$, $h' = q - 3$, and $h + 2h' = N$. The low energy gauge theory at $P_{2',\Gamma}$ has gauge group $Sp(q - 4)$.

For Γ_2 in the E series, the table is

$\Gamma_2 = \mathbf{E}_a$	$P_{1,\Gamma}$	$P_{2,\Gamma}$	$P_{\mu t,\Gamma}$	$P_{3,\Gamma}$
η_1	1	∞^{rh}	∞^{rth_t}	0^{rN}
η_2	0^{rp}	1	$e^{2\pi i\mu/t}$	∞^{rp}
η_3	∞^{Np}	0^{hp}	0^{pth_t}	1

, (1.4.19)

where t, h_t, μ are given for each $\mathbf{E}_a$ as follows: let k_i be the Dynkin indices of $\mathbf{E}_a$, and let t be the positive integers which divide some of the k_i; μ runs over positive integers less than t that are prime to t, unless $t = 1$ in which case $\mu = 0$; h_t is the dual Coxeter number of the associated group K_t whose Dynkin indices are k_i/t where here the k_i run through the indices of $\mathbf{E}_a$ that divide t. The t and h_t obey the relation $\sum th_t = N$. The low energy gauge group is given by the ADE group corresponding to K_t.

From the relation $\sum th_t = N$ and the tables above, we see that for each η_j, the total number of zeros and poles is equal. Since the total number of zeros is the same as the total number of poles for each of the η_j, it seems reasonable to assume that we have found all the zeros and poles, and hence all the semiclassical limits in our moduli space. It would seem, therefore, that we can now proceed to describe the moduli space completely, by writing our functions η_i explicitly and identifying the points $P_{i,\Gamma}$ with values of a good coordinate on the moduli space. However, as we shall see, we run into a few puzzles.

The first question we ask is: what can be said about the genus of $\mathcal{N}_\Gamma$? For the cases $p = r = 1$, which are the cases considered in [16], the function η_2 has a simple zero and a simple pole, and hence can be identified with a global coordinate on the moduli space, which can then be claimed to have genus zero. If $p, r > 1$, this is not so: none of our η_j have just a simple zero and pole, so we cannot identify the moduli space with any of the η_j, and we do not know the genus.

However, the simplest result would be that the curve has genus zero, and we proceed with this assumption. Hence, we assign the curve a global coordinate z, write the η_j as holomorphic functions of z, and see how well we can describe the curve.

For the case $\Gamma_2 = \mathbf{Z}_q$, this turns out to be straightforward; we may fix $P_{1,\Gamma}$ at $z = 0$, $P_{2,\Gamma}$ at $z = 1$, and $P_{3,\Gamma}$ at $z = \infty$, and then write our functions:

$$\eta_1 = \frac{1}{(1-z)^{qr}}, \qquad (1.4.20)$$

$$\eta_2 = z^{pr}, \qquad (1.4.21)$$

$$\eta_3 = \frac{(1-z)^{pq}}{z^{pq}}. \qquad (1.4.22)$$

This description is unique up to possible overall factors which are related to an anomaly in the membrane effective action, analogous to the one described in Section 5 of [16].

For $\Gamma_2 = \mathbf{D}_q$, we run into a puzzle. Once we fix the first three points, we have to find at what value z_4 the fourth point $P_{2',\Gamma}$ sits: our functions in this case are

$$\eta_1 = \frac{z_4^{2rh'}}{(1-z)^{rh}(z_4-z)^{2rh'}}, \qquad (1.4.23)$$

$$\eta_2 = z^{rp}, \qquad (1.4.24)$$

$$\eta_3 = \frac{(1-z)^{ph}(z_4-z)^{2h'p}}{z^{Np}}, \qquad (1.4.25)$$

again up to overall factors. The forms of η_1 and η_3 do not constrain z_4, but to satisfy $\eta_2(z_4) = -1$, we need $z_4^{pr} = -1$ for which there are pr solutions. A similar situation arises for Γ_2 in the E series, where there are pr choices for each point beyond the first three.

The pr solutions, however, should correspond to the same point in the moduli space of M-theories, since they correspond to the same theory. Hence, it seems that we have a redundancy in our description of the moduli space; we should impose a symmetry on $\mathcal{N}_\Gamma$ which identifies the different values of z_4.

There is another, more serious puzzle which shows up, also involving possible extra classical points on $\mathcal{N}_\Gamma$: from table (1.3.9), we see that our low energy gauge theory is compactified on a manifold which is not simply connected, but rather is of the form $\mathbf{S}^3/H$ for some discrete group H. Hence its fundamental group is equal to H. Therefore, it is possible to construct theories which have gauge fields with non-trivial Wilson loops which break the gauge symmetry.[1] Where in $\mathcal{N}_\Gamma$ do these theories lie?

For the case $\Gamma_2 = \mathbf{Z}_q$, the point $P_{1,\Gamma}$ can have Wilson loops which are conjugacy classes of elements of $SU(p)$ of order qr. One can show that, when p, q, r are relatively prime, the number of inequivalent such elements is

$$\frac{1}{qr}\begin{pmatrix} p+qr-1 \\ qr-1 \end{pmatrix} = \frac{(p+qr-1)!}{p!(qr)!} \tag{1.4.26}$$

with cyclic permutations for $P_{2,\Gamma}$ and $P_{3,\Gamma}$. Furthermore, for Wilson loops that break $SU(p)$ in a way that leaves $s-1$ factors of $U(1)$, i.e.

$$SU(p) \longrightarrow \Pi_{i=1}^{s} SU(n_i) \times U(1)^{s-1},$$

where $\sum n_i = p$, the number of inequivalent Wilson loops is

$$\frac{s}{pqr}\begin{pmatrix} p \\ s \end{pmatrix}\begin{pmatrix} qr \\ s \end{pmatrix}. \tag{1.4.27}$$

Each set of theories with a given number $s-1$ of $U(1)$ factors should lie on a separate component $\mathcal{N}_{s,\Gamma}$ of the moduli space, since smooth interpolation means that the number of massless modes – which corresponds to $U(1)$ fields – is constant on each component. For $s > 1$, we know that the theories on $\mathcal{N}_{s,\Gamma}$ do not have a mass gap due to the massless $U(1)$ field. On the other hand, the theories corresponding to the points $P_{i,\Gamma}$ with no non trivial Wilson loops are believed to have a mass gap. Hence we claim that $\mathcal{N}_{1,\Gamma}$ contains theories with a mass gap.

[1]These theories are the basis for our GUT models in Chapter 2.

For the case $r = 1$, we obtain no singularity at $X_{3,\Gamma}$. Hence, for that case the mass gap of the theory at $X_{3,\Gamma}$ means that all of $\mathcal{N}_{1,\Gamma}$ has a mass gap.

Continuing with the case where $r = 1$, we note a manifest symmetry between p and q in the expression (1.4.27) for the number of possible Wilson loops at each level s. At first sight, this could support the assertion that these points lie on their own branch of the moduli space, which will interpolate smoothly among them and contain no other singular points. However, chiral symmetry breaking means that the number of vacua at each classical point is given by $\Pi_i n_i$ which is clearly not symmetric between p and q, and spoils the counting of the orders of zeros and poles.

Going back to general r and looking at $\mathcal{N}_{1,\Gamma}$ only, we see that we have smooth interpolation among theories with different gauge groups: $SU(p)$, $SU(q)$, and $SU(r)$ when $\Gamma_2 = \mathbf{Z}_q$; $SU(p)$, $SO(2q)$, $Sp(q-4)$, and $SU(r)$ when $\Gamma_2 = \mathbf{D}_q$; and analogously for Γ_2 in the E series, where we interpolate between $SU(p)$, K_t, and $SU(r)$, with K_t as described after table (1.4.19). Similarly, the other branches $\mathcal{N}_{s,\Gamma}$ smoothly interpolate among theories with these gauge groups broken by Wilson lines.

Chapter 2

Unification from M-Theory

2.1 Introduction

We construct a supersymmetric M–theory compactification on a G_2 manifold X in which a grand unified theory arises naturally. The unified group G_{GUT} appears due to a co-dimension four ADE singularity in the compactification manifold, so the unified gauge theory is seven-dimensional. The breaking of G_{GUT} to an effective low-energy gauge group G, which occurs around an energy scale M_{GUT} known as the grand unification scale, arises from a Wilson line which appears due to the non-trivial fundamental group of the 3–dimensional locus Q of the ADE singularity. The low energy effective gauge theory is four dimensional.

In our model so far, the spectrum of the four dimensional theory is composed entirely of Kaluza-Klein modes of the adjoint fields of G_{GUT}. This spectrum contains, for example, the gauge fields of the effective low energy gauge group as well as the superheavy vector bosons which have a mass M_V and mediate proton decay. It does not, however, include the full spectrum of the standard model, which contains the quarks, leptons, and Higgs bosons. To be as realistic as possible, we shall take as our main example $G_{GUT} = SU(5)$ broken to the standard model gauge group $G = SU(3) \times SU(2) \times U(1)$, and at one point we shall put in the quarks, leptons, and Higgs bosons. We can add these chiral fields either by hand, or by assuming that X has, in addition to its ADE singularities, certain other singularities which are different from the ADE ones and which can give rise to the chiral fermions; such singularities were studied in [16, 67, 14, 15].

The three effective gauge couplings in four dimensions obey renormalization group equations (RGE's). As in usual four dimensional GUT's [30, 31, 68, 43], the description of the effective theory in terms of these RGE's is valid up to an energy scale M_{GUT}. Above this scale, the three couplings are equal to each other and to the unified coupling g_{GUT}, and obey a single RGE.

We consider the threshold corrections (studied within four dimensional models originally by Weinberg [42] and by Ovrut and Schnitzer [69]) to the four dimensional RGE's due to the entire massive Kaluza-Klein modes (we do not include any winding modes).

We find that the threshold corrections are a topological invariant of the 3–manifold Q, given by linear combinations of Ray-Singer torsions. (For a related result within heterotic string theory, see section 8.2 of [70].) The fact that they are topological depends heavily on choosing the compactification to be supersymmetric.

Furthermore, we show that the threshold corrections due to Kaluza-Klein harmonics satisfy a property which leads to exact preservation of unification; that is, there is an energy scale M_{GUT} at which the gauge couplings are equal, even in the presence of threshold corrections. This property stems primarily from group theory factors, and holds when $G_{GUT} = SU(5)$ (but not when $G_{GUT} = SO(10)$). We find not only an exact unification scale, but also a precise relation between this scale and the mass M_V of the heavy vector bosons.

Generically, in the presence of threshold corrections [42, 71, 72, 73, 74] there is no energy scale at which all three gauge couplings are exactly equal. There are only a few cases in the literature in which threshold corrections do exactly preserve unification, as they do here. In one case, it is stated in [43] that for $G_{GUT} = SU(5)$, the model considered in [42] leads to preservation of unification in the presence of threshold corrections, along with the relation $M_{GUT} = e^{-1/21} M_V$ (in [42], only the case of $G_{GUT} = SO(10)$ which does not preserve unification was considered, and not the case $G_{GUT} = SU(5)$). [1]

An underlying assumption often made in GUT's is that, in order to explain the suppression of effects such as proton decay, the mass M_V is assumed to be superheavy and of the same order as the unification scale M_{GUT} [30, 31, 43, 79, 80]. The mass M_V and the unification scale M_{GUT} are sometimes used interchangeably. What we have here is a *derivation* of a *precise* relation between these two quantities , and in fact, it could be that – depending on Q – the above assumption may even be false. From here on, to avoid confusion, we will use E_{GUT} to denote the unification scale which we previously called M_{GUT}; when we say E_{GUT} we *strictly* mean the energy at which the couplings are equal, to be distinguished from the masses M_V of heavy vector bosons.

Our relation between E_{GUT} and M_V can be presented quite generally in terms of a relation between E_{GUT} and the volume V_Q of Q:

$$E_{GUT}^2 = \frac{D(Q)}{V_Q^{2/3}} \ , \tag{2.1.1}$$

where $D(Q)$ is a topological invariant to be defined in Section 2.6. This formula is the one we referred to in the Prelude.

The relation in equation (2.1.1) is valid whether or not we have included the chiral fermions. We get realistic predictions for the scale E_{GUT} as well as the weak mixing angle $\sin^2\theta_W$ by adding to our low-energy beta functions the contribution from the quarks, leptons, and Higgs bosons of the supersymmetric standard model (we assume that they have no Kaluza-Klein excitations). What we find is that the predictions we get are exactly the same as – and hence as good as – those coming from the standard field theoretic grand unified model: $E_{GUT} = 2.2 \times 10^{16} GeV$, $\sin^2\theta_W = 0.23$.

[1] See also [75, 76] for a mechanism in which gauge coupling unification is (essentially) preserved in the presence of extra dimensions, see [77] for a model in which stringy threshold corrections conserve gauge coupling unification, and see [78] for a case where the standard model spectrum is extended for the specific purpose of preserving unification.

We complete our unification picture by including the gravitational coupling G_N via dualities between M–theory and string theories. As in heterotic string theory GUT's (see [81] and references therein), where at tree level, the gravitational couplings automatically unify with the gauge couplings, here too unification of gauge couplings leads to unification with G_N, since by duality with string theories, all couplings are given in terms of the one string coupling constant g_s. We find an expression for G_N which may agree with its experimental value, and could be even in better agreement with the experimental value than the case of the strongly or weakly coupled heterotic string (see [82, 83, 84] and references therein).

Finally, we study the classic problem of proton decay in our model. To discuss proton decay in a sensible fashion, we need to assume a mechanism for doublet-triplet splitting; here, we assume the mechanism of [41] (which is a close cousin of the original mechanism for doublet-triplet splitting in Calabi-Yau compactification of the heterotic string [29]; this mechanism has been reconsidered recently from a bottom-up point of view [85, 86, 87, 88, 89]). One consequence of this particular doublet-triplet splitting mechanism is that dimension five (or four) operators contributing to proton decay are absent; proton decay is dominated, therefore, by dimension six operators, which are the ones we shall study.

In four-dimensional SUSY GUT's, the gauge boson exchange contributions to proton decay, if they dominate, lead to a proton lifetime that has been estimated recently as $5 \times 10^{36\pm1}$ years [90, 91]. (Four-dimensional GUT's also generally have a dimension five contribution that can be phenomenologically troublesome (but see [92, 93]); it is absent in the models we consider.) Relative to four-dimensional GUT's, we find a mechanism that enhances proton decay modes such as[2] $p \to \pi^0 e^+_L$ relative to $p \to \pi^0 e^+_R$ or $p \to \pi^+ \bar{\nu}_R$; these are typical modes that arise from gauge boson exchange. Given numerical uncertainties that will appear in Section 2.7, it is hard to say if in practice this mechanism enhances the

[2] And similar modes with π^0 replaced by K^0, or e^+_L by μ^+_L. As in most four-dimensional GUT's, we do not have precise knowledge of the flavor structure.

$p \to \pi^0 e_L^+$ modes relative to GUT's or suppresses the others. Because of these issues, to cite a very rough estimate of the proton lifetime, we somewhat arbitrarily keep the central value estimated in the four-dimensional models and double the logarithmic uncertainty, so that if gauge boson exchange dominates, the proton lifetime in these models might be $5 \times 10^{36\pm 2}$ years.[3] (In practice, the uncertainty could be much larger if because of additional light charged particles, the unification scale is modified; the same statement applies to four-dimensional GUT models.)

This Chapter is organized as follows. In Section 2.2 we present our M-theory GUT model. In Section 2.3 we write the standard expressions for the RGE's and for the threshold corrections which shall be used throughout. In Section 2.4 we describe the Kaluza-Klein spectrum obtained from our model. Section 2.5 is devoted to showing how to write the threshold corrections in terms of Ray-Singer torsions. In Section 2.6 we arrive at our main results: we show that the threshold corrections preserve unification, and we display the relation between E_{GUT} and the volume V_Q of Q in terms of Ray-Singer torsions. We discuss the mass of the heavy gauge bosons, briefly comment on proton decay, and then show how the chiral fermions are included. In Section 2.7 we add the gravitational coupling to the story. Section 2.8 contains our results about proton decay. Throughout our presentation, we derive our results for a general 3-manifold Q which has a finite fundamental group, and we also include completely explicit results for our case of the lens space. Appendix A contains the calculation of a Ray-Singer torsion which is used in Section 2.5, and Appendix B generalizes our results to different GUT groups.

2.2 GUT's from M-theory

We construct here our GUT model, based on a compactification of M-theory on a manifold X of G_2 holonomy, which preserves $\mathcal{N} = 1$ supersymmetry. In compactifying eleven-

[3] The current experimental bound on $p \to \pi^0 e^+$ is 4.4×10^{33} years (for a recent report, see [94]) and the next generation of experiments may improve this by a factor of 10 to 20 (for example, see [95]).

dimensional M-theory on a seven-manifold X, the properties of X determine what theory we get in four dimensions. For example, a singular manifold X leads to a gauge theory in four dimensions [13, 63, 64, 16]. In the model we are about to describe, which is based on results obtained in Chapter 1 [17], the choice of X is such that both a grand unified gauge group and its breaking into the standard model subgroup arise naturally. As we will show, the unified group arises from an ADE singularity in the compactification manifold, which translates into an ADE gauge theory at low energies, while the breaking of this group comes from Wilson loops which arise due to the non-trivial fundamental group of the locus Q of the ADE singularity.

As described in Chapter 1 [17], our compactification manifold X is asymptotic to a cone over quotients of $Y = \mathbf{S}^3 \times \mathbf{S}^3 = SU(2)^3/SU(2)$, where the equivalence relation is $(g_1, g_2, g_3) \sim (g_1 h, g_2 h, g_3 h)$, with $g_i, h \in SU(2)$. We consider a discrete group $\Gamma = \Gamma_1 \times \Gamma_2 \times \Gamma_3$, where the Γ_i are ADE subgroups of $SU(2)$, which acts on Y by

$$(\gamma, \delta, \epsilon) \in \Gamma_1 \times \Gamma_2 \times \Gamma_3 : (g_1, g_2, g_3) \mapsto (\gamma g_1, \delta g_2, \epsilon g_3), \tag{2.2.1}$$

and we denote the resulting quotient space by Y_Γ.

The compactification manifolds $X_{i,\Gamma}$ asymptotic to Y_Γ are singular spaces obtained by filling in the i^{th} $SU(2)$ factor of Y_Γ to a ball in $\mathbf{R}^4$; they are topologically quotients of $\mathbf{R}^4 \times \mathbf{S}^3$, where the $\mathbf{R}^4$ corresponds to the filled-in factor. For $i = 1$, gauging g_2 away using the right diagonal $SU(2)$ action in Y, the identification corresponding to $(\gamma^k, \delta^l, \epsilon^m) \in \Gamma$ becomes

$$(g_1, 1, g_3) \equiv (\gamma^k g_1 \delta^{-l}, 1, \epsilon^m g_3 \delta^{-l}), \tag{2.2.2}$$

where $g_1 \in \mathbf{R}^4$ and $g_3 \in SU(2) \sim \mathbf{S}^3$; the set $(0, 1, g_3)$ with g_3 varying in $SU(2)$ and

$$g_3 \equiv \epsilon^m g_3 \delta^{-l} \ , \qquad (\delta, \epsilon) \in \Gamma_2 \times \Gamma_3 \ , \tag{2.2.3}$$

is the fixed point locus of the action of the Γ_1 subgroup of Γ. A similar description holds when $i = 2, 3$. For these fixed points to be identical to a standard A, D, or E singularity of codimension four of the form $\mathbf{R}^4/\Gamma_i$ or $\mathbf{C}^2/\Gamma_i$, which gives a Γ_i gauge theory in seven dimensions, it was explained in Chapter 1 [17] that we must choose our group Γ to be of the form $\Gamma = \mathbf{Z}_p \times \Gamma_2 \times \mathbf{Z}_r$ where Γ_2 is any A, D, or E group, and where, if N is the order of the group Γ_2, then p, r, and N satisfy

$$(p, N) = (r, N) = (p, r) = 1. \tag{2.2.4}$$

At $X_{1,\Gamma}$ we have an $SU(p)$ gauge theory on $\mathbf{R}^4 \times Q$ where $Q = \mathbf{S}^3/(\Gamma_2 \times \mathbf{Z}_r)$, at $X_{3,\Gamma}$ we have an $SU(r)$ gauge theory on $\mathbf{R}^4 \times Q$ where $Q = \mathbf{S}^3/(\mathbf{Z}_p \times \Gamma_2)$, and at $X_{2,\Gamma}$ we have an A, D, or E gauge theory on $\mathbf{R}^4 \times Q$ where $Q = \mathbf{S}^3/(\mathbf{Z}_r \times \mathbf{Z}_p)$; in each of these three cases, the free discrete group action on $\mathbf{S}^3$ is given by the appropriate cyclic permutation of the action on g_3 in (2.2.2).

We can view $\mathbf{S}^3$ as embedded in $\mathbf{C}^2$ satisfying $|z_1|^2 + |z_2|^2 = 1$, and it will be useful for us later to rewrite the action of $\Gamma_2 \times \mathbf{Z}_r$ on $\mathbf{S}^3$ for the case $\Gamma_2 = \mathbf{Z}_q$ in terms of its action on $\mathbf{C}^2$:

$$(\delta, \epsilon) \cdot (z_1, z_2) = (e^{2\pi i/q - 2\pi i/r} z_1 \, , \, e^{-2\pi i/q + 2\pi i/r} z_2), \tag{2.2.5}$$

where δ and ϵ here generate $\mathbf{Z}_q$ and $\mathbf{Z}_r$, respectively.

We see that our low energy gauge theory lives on a manifold of the form $Q = \mathbf{S}^3/H$, where H is a finite group. This manifold is not simply connected – its fundamental group is equal to H. Therefore, it is natural to include in the low energy theory non-trivial Wilson loops which break the gauge symmetry. The remaining gauge group will contain abelian $U(1)$ factors, and, as was shown in Chapter 1 [17], the moduli space for theories corresponding to Γ consists of several branches classified according to the number of these abelian factors that appear in the low energy gauge theory. Within each branch, called $\mathcal{N}_{s,\Gamma}$, we get smooth interpolation between supersymmetric gauge theories with different gauge groups containing the same number $s-1$ of $U(1)$ factors.

The theories we will be studying here live on $\mathcal{N}_{2,\Gamma}$, the branch of moduli space which contains theories with Wilson lines that break the ADE gauge group into a subgroup containing one $U(1)$ factor. For example, one theory on this moduli space contains a

Wilson line in $SU(p)$ that breaks it as follows:

$$SU(p) \longrightarrow SU(n) \times SU(m) \times U(1), \qquad n+m=p. \tag{2.2.6}$$

This provides a natural model for symmetry breaking of a grand unified gauge group into the Standard Model: we can treat $SU(p)$ as a GUT group, and with $p=5$ we can choose a Wilson line which will break $SU(5)$ precisely into the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$. We shall use this theory as our main example.

Hence, we take the simplest example $\Gamma = \mathbf{Z}_5 \times \mathbf{Z}_q$ ($\Gamma_3 = I$) and fill in g_1. The co-dimension four $\mathbf{Z}_5$ singularity gives a gauge theory in seven dimensions with $G_{GUT} = SU(5)$ gauge group on $\mathbf{R}^{1,3} \times Q$ where $Q = \mathbf{S}^3/\mathbf{Z}_q$ is the locus in X of the $\mathbf{Z}_5$ singularity. Since $\pi_1(Q) = \mathbf{Z}_q$, we can take the Wilson loop to be

$$U = \begin{pmatrix} e^{4\pi i w/q} & & & & \\ & e^{4\pi i w/q} & & & \\ & & e^{4\pi i w/q} & & \\ & & & e^{-6\pi i w/q} & \\ & & & & e^{-6\pi i w/q} \end{pmatrix} \tag{2.2.7}$$

where we can choose[4] an integer $w = 1, \ldots, q-1$ with $5w$ not divisible by q. This breaks $SU(5)$ to the standard model group $SU(3) \times SU(2) \times U(1)$.

To have a realistic model, we need to include chiral fields. We shall include them by assuming that at some points P_i on Q, there are singularities in X which are worse than the ADE singularity and which support chiral fermions. At these points, Q itself is smooth but the normal directions to Q in X have a singularity more complicated than the orbifold singularity. This mechanism was determined in [14], and the relevant singularities have been studied in [16, 67, 14, 15]. Their details will not be important here.

[4]In Chapter 1 [17] we would take $(5,q)=1$ and $5w$ would be automatically indivisible by q; however, here the condition $(5,q)=1$ is not necessary since in this example with g_1 filled in and $\Gamma_3 = I$, the action on g_3 in (1.3.3) (which is (2.2.2) here) is smooth for any q.

2.3 Renormalization Group Equations

The renormalization group equations (RGE's) at the one-loop level for a supersymmetric gauge theory with gauge group $G = \Pi_a G_a$, including threshold corrections, are given by [44, 96, 81]:[5]

$$\frac{1}{k_a g_a^2(\mu)} = \frac{1}{g_M^2} + \frac{1}{16\pi^2}\frac{b_a}{k_a}\log\frac{\Lambda^2}{\mu^2} + \frac{1}{16\pi^2}\frac{\mathcal{S}_a}{k_a} , \tag{2.3.1}$$

where

$$b_a = 2\sum_{m=0} \mathrm{Tr}\, Q_a^2 (-1)^F (\frac{1}{12} - \chi^2) , \tag{2.3.2}$$

$$\mathcal{S}_a = 2\sum_{m\neq 0} \mathrm{Tr}\, Q_a^2 \, (-1)^F (\frac{1}{12} - \chi^2) \log\frac{\Lambda^2}{m^2} . \tag{2.3.3}$$

Here, a indexes the factors of G, $g_a(\mu)$ is the coupling associated with G_a, g_M is the underlying gauge coupling as deduced from M-theory in the supergravity approximation, χ is the helicity operator, μ is the renormalization scale, F is the fermion number, and the Q_a are generators of the a^{th} gauge group for the representation in which the massless (in (2.3.2)) or massive (in (2.3.3)) particles live. The trace over them is normalized such that for $SU(N)$, $\mathrm{Tr}\, Q_a^2 = N$ for the adjoint representation and $\mathrm{Tr}\, Q_a^2 = 1/2$ for the fundamental representation.

The k_a are normalization constants, which for the standard model with unified group $G_{GUT} = SU(5)$ are $k_Y = 5/3$, $k_2 = k_3 = 1$. They encode the tree-level relation among the three couplings which follows from the existence of an underlying G_{GUT} structure; the tree-level equation is (2.3.1) with $b_a = \mathcal{S}_a = 0$.

The second and third terms of (2.3.1) are the one-loop corrections to the tree-level relation. The second term, where the b_a are the one-loop beta function coefficients, gives the running of the couplings due to massless particles. The third term is the one-loop

[5] The threshold corrections are called Δ_a in [44, 96], but we will call them $\mathcal{S}_a$ to avoid confusion with the use of Δ for the Laplacian.

threshold correction and is a sum over contributions from heavy particles; it is independent of μ. The assumption of gauge coupling unification in the absence of threshold corrections, i.e. when $\mathcal{S}_a = 0$, is equivalent to the fact that Λ and g_M in (2.3.1) are independent of the index a: when $\mu = \Lambda$, we have $k_Y g_Y^2(\mu) = k_2 g_2^2(\mu) = k_3 g_3^2(\mu) = g_M^2$, so the energy scale of unification is $E_{GUT} = \Lambda$. In the presence of threshold corrections, i.e. when $\mathcal{S}_a \neq 0$, generically there is no longer a value of μ at which all the couplings are equal to each other (but in our case, we will find such a value of μ).

Combining the formulas above, we can write

$$\begin{aligned}\frac{1}{g_a^2(\mu)} = \frac{k_a}{g_M^2} \quad &+ \quad \frac{1}{16\pi^2} 2 \sum_{m=0} \mathrm{Tr} Q_a^2 (-1)^F \left(\frac{1}{12} - \chi^2\right) \log(\Lambda^2/\mu^2) \\ &+ \quad \frac{1}{16\pi^2} 2 \sum_{m\neq 0} \mathrm{Tr} Q_a^2 (-1)^F \left(\frac{1}{12} - \chi^2\right) \log(\Lambda^2/m^2). \end{aligned} \qquad (2.3.4)$$

2.4 Kaluza-Klein Reduction

In this section, we fit both the massive and massless Kaluza-Klein modes of our GUT model into four dimensional $\mathcal{N} = 1$ supermultiplets.

As described in Section 2.2, we have compactification on a manifold with a codimension four A_{p-1} singularity which leads to a seven dimensional supersymmetric $SU(p)$ gauge theory on $\mathbf{R}^4 \times Q$; $\mathbf{R}^4$ is four dimensional Minkowski space, and Q is a supersymmetric cycle in the G_2 manifold. The (massless) bosonic sector of the seven dimensional theory consists of the gauge field and 3 scalar fields, all in the adjoint representation of $SU(p)$. Under Kaluza-Klein reduction on Q, the seven dimensional gauge field decomposes into a four dimensional gauge field, which is at the same time a scalar on Q, and a four dimensional scalar, which is a 1–form on Q. As explained in [13], the identification of part of the structure group of the normal bundle of Q with the struture group of the tangent bundle of Q, which is a result of the fact that Q is a supersymmetric cycle in the G_2 manifold, means that the set of 3 seven dimensional scalars can be treated as a 1–form on Q which is at the same time a scalar in four dimensions. So the bosonic spectrum in

Table 2.1: Field content in four dimensions

<table>
<tr><td></td><td colspan="2">Form on Q</td><td>Supermultiplet</td><td>χ</td><td>Tr $(-1)^F(\frac{1}{12}-\chi^2)$</td></tr>
<tr><td rowspan="3">massive</td><td colspan="2">0–forms</td><td rowspan="2">vector</td><td>$1, 1/2, -1/2, -1$</td><td>$-3/2$</td></tr>
<tr><td rowspan="2">1–forms</td><td>closed</td><td>$1/2, 0, 0, -1/2$</td><td>$1/2$</td></tr>
<tr><td>co-closed</td><td>chiral</td><td>$1/2, 0, 0, -1/2$</td><td>$1/2$</td></tr>
<tr><td rowspan="2">massless</td><td>0–forms</td><td>harmonic</td><td>vector</td><td>$1, 1/2, -1/2, -1$</td><td>$-3/2$</td></tr>
<tr><td>1–forms</td><td>harmonic</td><td>chiral</td><td>$1/2, 0, 0, -1/2$</td><td>$1/2$</td></tr>
</table>

four dimensions consists of one gauge field which is a scalar on Q and two scalars which are two 1–forms on Q.

We would like to fit this spectrum into $\mathcal{N} = 1$ supermultiplets in four dimensions. We have taken into account only the massless fields in seven dimensions, but in four dimensions their mass is given by the eigenvalues of the Laplacian on Q. The spectrum of the four dimensional gauge fields is the spectrum of the Laplacian on 0–forms on Q. The spectrum of the four dimensional scalars is that of 1–forms on Q; the spectrum of 1–forms can be divided into closed 1–forms, which have the same spectrum as 0–forms, and co-closed 1–forms, which have a different spectrum.

The massive gauge fields in four dimensions fit into a massive vector multiplet with helicities $(1, 1/2, 1/2, 0, 0, -1/2, -1/2, -1)$. The modes of the two scalars, each of helicity 0, that have the same spectrum as the gauge field (namely the closed 1–forms on Q) fit into the same multiplet. The rest of the scalar modes, namely the co-closed 1–forms, fit into a different multiplet, namely a massive chiral multiplet with helicities $(1/2, 0, 0, -1/2)$.

As to the massless fields, the harmonic forms are the modes which correspond to massless fields in four dimensions. The value of the k^{th} betti number $b_k(Q)$ tells us how many harmonic k–forms exist on Q. There will be $b_0(Q)$ harmonic scalars on Q, which are massless four dimensional gauge fields, and there will be $b_1(Q)$ harmonic 1–forms on Q, which are simultaneously closed and co-closed, which are massless four dimensional scalars. The massless four dimensional gauge fields fit into a massless vector multiplet with helicities $(1, 1/2, -1/2, -1)$, and the massless four dimensional scalars fit into $b_1(Q)$ massless chiral multiplets with helicities $(1/2, 0, 0, -1, 2)$, corresponding to harmonic 1–forms. For the case in which $b_0(Q) = 1$ and $b_1(Q) = 0$, the massless spectrum is just one massless vector multiplet $(1, 1/2, -1/2, -1)$.

We summarize the results of this section in Table 2.1, where in the last column we also calculate the sum $\mathrm{Tr}\,(-1)^F(\frac{1}{12} - \chi^2)$ which appears in the RGE's via equations (2.3.2) and (2.3.3). For completeness, we include in the table the harmonic 1–forms even though they do not appear in our case.

2.5 Threshold Corrections as Ray-Singer Torsions

Here, we first describe the Ray–Singer torsion for a general closed, oriented manifold. Then, we derive an expression for the threshold corrections $\mathcal{S}_a$ in terms of Ray–Singer torsions of the 3–manifold Q. We conclude this section by computing them explicitly for the case $Q = \mathbf{S}^3/\mathbf{Z}_q$.

The Ray–Singer torsion is a topological invariant defined for any Riemannian or complex manifold. We focus on the Riemannian case [97, 98, 99], as it is the relevant one for our 3–manifold Q.

Let M^d be a closed, oriented d–dimensional Riemannian manifold with finite fundamental group $\pi_1(M^d)$, and let $\tilde{M}^d$ be its universal covering space on which $\pi_1(M^d)$ acts freely. Then, given a one dimensional complex representation ω of $\pi_1(M^d)$, the Ray–Singer torsion is expressed as a sum of two terms:

$$\mathcal{T}(M^d, \omega) = \left(\frac{1}{2}\sum_{k=0}^{d}(-1)^{k+1} k \log\det(\Delta'_k)\right) + \left(\sum_{k=0}^{d}(-1)^k \log\det A^k\right) \qquad (2.5.1)$$

$$\equiv \mathcal{K}(M^d, \omega) + \mathcal{A}(M^d, \omega)\ . \qquad (2.5.2)$$

In the first term, the Laplacian Δ'_k is taken to act on non-harmonic k–forms ϕ_k of $\tilde{M}^d$ which satisfy

$$\gamma \cdot \phi_k(\theta) \equiv \phi_k(\gamma \cdot \theta) = \omega(\gamma)\phi_k(\theta) \ , \qquad \gamma \in \pi_1(M^d) \ , \quad \theta \in \tilde{M}^d \ . \tag{2.5.3}$$

In the second term, A^k is the matrix of change of basis from an integral basis for the k^{th} integral cohomology to the basis of harmonic k–forms of norm 1 (there is also a dependence on the choice of basis for the cohomology, which we shall not need to consider here). This second term appears only for the representations ω for which the Laplacian has zero modes.

Hodge duality tells us that $\log\det(\Delta'_k) = \log\det(\Delta'_{d-k})$, so in particular for $d = 3$, which is the case we study here, the expression for the torsion for a 3–manifold Q becomes

$$\begin{aligned} \mathcal{T}(Q,\omega) &= \left(\frac{3}{2}\log\det(\Delta'_0) - \frac{1}{2}\log\det(\Delta'_1)\right) + \left(\log\frac{\det A^0 \det A^2}{\det A^1 \det A^3}\right) \\ &\equiv \mathcal{K}(Q,\omega) + \mathcal{A}(Q,\omega) \ , \end{aligned} \tag{2.5.4}$$

where the Laplacian again acts on the fields satisfying (2.5.3) with $M^d = Q$.

Since the Laplacian is the energy operator, its k–form eigenvalues $\lambda_{k,n}$ correspond to the squares of the masses of particles, $m^2_{k,n}$. The logarithm of the determinant of the Laplacian is formally

$$\log\det(\Delta'_k) = \sum_n \log\lambda_{k,n} = \sum_n \log m^2_{k,n} \ , \tag{2.5.5}$$

where the sum is over non-zero eigenvalues $\lambda_{k,n}$ of Δ'_k.[6] Hence, the sums in the expression (2.3.3) for $\mathcal{S}_a$ have the form of (2.5.5).

As a first step towards showing that the threshold corrections are actually given by the Ray-Singer torsions, we insert the values in the last column of the upper half of Table

[6]This sum as written formally is clearly infinite; it becomes well–defined via zeta function regularization, of which we shall have an explicit example in Appendix A.

2.1 into (2.3.3) to get

$$\mathcal{S}_a = 2\mathrm{Tr}\, Q_a^2\Big(-\frac{3}{2}\sum_n \log\frac{\Lambda^2}{m_{0,n}^2} + \frac{1}{2}\sum_n \log\frac{\Lambda^2}{m_{1,n}^2}\Big), \tag{2.5.6}$$

where $m_{0,n}^2$ are 0–form eigenvalues and $m_{1,n}^2$ are 1–form eigenvalues, both closed and co-closed.

There are two happy coincidences that have entered (2.5.6) as coefficients in front of the sums, and which at the end will allow us to rewrite $\mathcal{S}_a$ in terms of Ray-Singer torsions. The first is the fact that the coefficient for the closed 1–forms is equal to that for the co-closed 1–forms (namely, 1/2). This will allow us to write the threshold corrections as a linear combination of logarithms of determinants of the Laplacian on k–forms without separating the closed from the co-closed sectors. The other coincidence is that the ratio between the coefficient $-3/2$ for the 0–forms and $1/2$ for the 1–forms is the same ratio appearing in $\mathcal{K}(Q,\omega)$ in (2.5.4).

To show that the threshold corrections can indeed be given in terms of the Ray-Singer torsions, we need to be more precise about which 0–form and 1–form eigenvalues appear in the sums in (2.5.6). Namely, we need to find the analog in our model of (2.5.3). As we shall show, this analog is given by the twisting of the adjoint fields ϕ_k by the Wilson line.

Using the setup of Section 2.2, we take $SU(5)$ gauge theory[7] living on $\mathbf{R}^4 \times Q$, where Q has a finite, non-trivial fundamental group $\pi_1(Q)$. As explained in Section 2.2, it is natural to include Wilson lines that break the gauge symmetry. Choose such a Wilson line

$$U_\gamma = \exp\left(iv_\gamma \cdot \mathrm{diag}\{2,2,2,-3,-3\}\right), \tag{2.5.7}$$

where $\gamma \in \pi_1(Q)$ and $v_\gamma|\pi_1(Q)| \in 2\pi\mathbf{Z}$ so that we have an embedding of $\pi_1(Q)$ in $SU(5)$. This Wilson line breaks an $SU(5)$ gauge field into the following five representations of the

[7] Note that the constructions in the rest of this section and in parts of the next section can be presented more generally for any choice of gauge group; we present them here for $SU(5)$ both for concreteness and since this is the example we shall pursue later. We will make a few comments on a more general case later, and present the details in Appendix B.

standard model gauge group $SU(3) \times SU(2) \times U(1)$:

$$\underset{\rho_1}{(8,1,0)} \oplus \underset{\rho_2}{(1,3,0)} \oplus \underset{\rho_3}{(1,1,0)} \oplus \underset{\rho_4}{(3,2,-5/6)} \oplus \underset{\rho_5}{(\bar{3},2,5/6)}, \tag{2.5.8}$$

where in each parenthesis, the first two numbers denote dimensions of representations of $SU(3)$ and $SU(2)$, and the third number is hypercharge. In this context of symmetry breaking by Wilson lines [25, 26, 27, 28, 29], the fields ϕ_k which transform in these representations are k–forms on the universal cover $\tilde{Q}$ which obey the following boundary condition: the adjoint action of the Wilson line on the fields, given by

$$U \cdot \phi_k = U \phi_k \, U^{-1}, \tag{2.5.9}$$

translates into an action of the fundamental group Γ on these fields via

$$\phi_k(\gamma\theta) = U\phi_k(\theta)U^{-1}, \tag{2.5.10}$$

where γ is a generator of $\pi_1(Q)$ and $\theta \in \tilde{Q}$. This action is a self-intertwining map for each irreducible representation ρ_i and, by Schur's lemma, is just multiplication by a scalar depending only on i. This scalar is trivial for $i = 1, 2, 3$, but not for $i = 4, 5$:

$$\begin{aligned} \phi_k(\gamma\theta) &= \phi_k(\theta) & \phi_k &\in (8,1,0) \oplus (1,3,0) \oplus (1,1,0) \\ \phi_k(\gamma\theta) &= e^{5iv_\gamma}\phi_k(\theta) & \phi_k &\in (3,2,-5/6) \\ \phi_k(\gamma\theta) &= e^{-5iv_\gamma}\phi_k(\theta) & \phi_k &\in (\bar{3},2,5/6). \end{aligned} \tag{2.5.11}$$

The equations (2.5.11) are precisely analogs of (2.5.3): each representation ρ_i corresponds by (2.5.11) to a one dimensional complex representation of $\pi_1(Q)$, which we shall call ω_i. To simplify notation, if γ generates $\pi_1(Q)$, we shall write $\omega_i(\gamma) \equiv \omega_i$.

We need only one more point before expressing the threshold corrections in terms of Ray-Singer torsions. The eigenvalues $m_{k,n}^2 = \lambda_{k,n}$ of the Laplacian scale with the size of Q: they have units of $(\text{length})^{-2}$, and if we rewrite them in terms of the volume V_Q of Q

as $m_{k,n}^2/V_Q^{2/3}$, we see that the sums in (2.5.6) are of the form

$$\sum_n \log \frac{V_Q^{2/3}\Lambda^2}{m_{k,n}^2} \ . \tag{2.5.12}$$

Changing $V_Q^{2/3}\Lambda^2$ is equivalent to rescaling the masses, which is equivalent to changing the metric of Q. For representations of the fundamental group for which there are no zero modes, namely those that do not commute with the Wilson line and hence acquire a mass (they are ω_4 and ω_5), the sum in (2.5.12) is topological (the term $\mathcal{A}(Q,\omega)$ does not appear in the expression (2.5.4) for the torsion), and there can be no volume dependence. Therefore, we may set $V_Q^{2/3}\Lambda^2 = 1$, and we simply have

$$\mathcal{K}(Q,\omega_4) = \mathcal{T}(Q,\omega_4), \qquad\qquad \mathcal{K}(Q,\omega_5) = \mathcal{T}(Q,\omega_5). \tag{2.5.13}$$

For the representations ω_1, ω_2, and ω_3, which do have zero modes since they commute with the Wilson line and hence their lowest KK mode does not acquire a mass, we will get a dependence on $V_Q^{2/3}\Lambda^2$ which, for any 3–manifold Q with $b_1(Q) = b_2(Q) = 0$, is given by (see Appendix A.2 for a computation of $\mathcal{A}(Q,1)$)

$$\mathcal{K}(Q,\omega_i) = \mathcal{K}(Q,1) = \mathcal{T}(Q,1) + \frac{3}{2}\log V_Q^{2/3}\Lambda^2 \ , \qquad\qquad i = 1,2,3 \ . \tag{2.5.14}$$

Putting everything together, we have the following general expression for the threshold corrections to the RGE's in terms of Ray–Singer torsions:

$$\mathcal{S}_a = 2\sum_i f^a(\rho_i)\, C_2^a(\rho_i)\ \mathcal{K}(Q,\omega_i) \ . \tag{2.5.15}$$

The sum is over the representations ρ_i in the decomposition (2.5.8), $f^a(\rho_i)$ is the dimension of the representation ρ_i divided by the dimension of the representation of the a^{th} gauge group that is included in ρ_i, and $C_2^a(\rho_i)$ is the quadratic Casimir of the representation of the a^{th} gauge group that is included in ρ_i, normalized as explained after (2.3.2).

Our current form for the RGE's is:

$$
\begin{aligned}
\frac{1}{g_a^2(\mu)} = \frac{k_a}{g_M^2} + \quad & + \quad \frac{b_a}{16\pi^2} \log \frac{\Lambda^2}{\mu^2} \\
& + \quad \frac{1}{16\pi^2} \, 2 \sum_{\omega_i = 1} f^a(\rho_i) C_2^a(\rho_i) \left(\mathcal{T}(Q,1) + \frac{3}{2} \log V_Q^{2/3} \Lambda^2 \right) \\
& + \quad \frac{1}{16\pi^2} \, 2 \sum_{\omega_i \neq 1} f^a(\rho_i) C_2^a(\rho_i) \mathcal{T}(Q, \omega_i) \ . \qquad (2.5.16)
\end{aligned}
$$

Note that since the derivation of equation (2.5.16) does not depend on the choice of the unified group, it is true for any choice of a GUT group, with the ρ_i given by the appropriate analog of (2.5.8) and ω_i are the representations of $\pi_1(Q)$ corresponding to the ρ_i.

For $SU(5)$ broken to $SU(3) \times SU(2) \times U(1)$ via equations (2.2.7) and (2.5.8), the threshold corrections become

$$
\begin{aligned}
\mathcal{S}_Y &= \frac{25}{3} \left(\mathcal{T}(Q,\omega_4) + \mathcal{T}(Q,\omega_5) \right) , \\
\mathcal{S}_2 &= 4 \left(\mathcal{T}(Q,1) + \log V_Q \Lambda^3 \right) + 3 \left(\mathcal{T}(Q,\omega_4) + \mathcal{T}(Q,\omega_5) \right) , \\
\mathcal{S}_3 &= 6 \left(\mathcal{T}(Q,1) + \log V_Q \Lambda^3 \right) + 2 \left(\mathcal{T}(Q,\omega_4) + \mathcal{T}(Q,\omega_5) \right) . \qquad (2.5.17)
\end{aligned}
$$

The quantities that will become meaningful when we put the threshold corrections into the RGE's are the differences between the $\mathcal{S}_a/k_a$, since an overall additive constant can be absorbed into $1/g_M^2$ in equation (2.3.1). Also, since the representations ω_4 and ω_5 are complex conjugates, and complex conjugation of the eigenfunctions exchanges a representation with its complex conjugate without changing the eigenvalues of the Laplacian, we have $\mathcal{T}(Q,\omega_4) = \mathcal{T}(Q,\omega_5) \equiv \mathcal{T}(Q,\omega)$. Therefore, we can replace the $\mathcal{S}_a$ by $\mathcal{S}_a^-$ given by $\mathcal{S}_a^-/k_a = \mathcal{S}_a/k_a - \mathcal{S}_Y/k_Y$ to get

$$
\begin{aligned}
{\mathcal{S}_Y}^- &= 0 , \\
{\mathcal{S}_2}^- &= 4 \left(\mathcal{T}(Q,1) + \log V_Q \Lambda^3 \right) - 4 \, \mathcal{T}(Q,\omega) , \\
{\mathcal{S}_3}^- &= 6 \left(\mathcal{T}(Q,1) + \log V_Q \Lambda^3 \right) - 6 \, \mathcal{T}(Q,\omega) . \qquad (2.5.18)
\end{aligned}
$$

The ${\mathcal{S}_a}^-$ are proportional to each other:

$$
{\mathcal{S}_Y}^- : {\mathcal{S}_2}^- : {\mathcal{S}_3}^- = 0 : 2 : 3, \qquad (2.5.19)
$$

where the ratios originate from quadratic Casimirs of the adjoint representations of $SU(2)$ and $SU(3)$. This proportionality will become crucial in our study of unification in the next section: we will show there that it means the threshold corrections preserve unification. It would be interesting to find a deep underlying reason for this proportionality.[8]

As we show explicitly in Appendix B, this proportionality is still present in a more general case, namely not just for $G_{GUT} = SU(5)$ breaking to $SU(3) \times SU(2) \times U(1)$, but for any breaking of $G_{GUT} = SU(p)$ to a standard model-like group $SU(m) \times SU(n) \times U(1)$, where $m + n = p$. Just like the standard model, all these correspond to points on the branch $\mathcal{N}_{2,\mathrm{r}}$ of the moduli space.

The lens space

We specialize to the case of the lens space $Q = \mathbf{S}^3/\mathbf{Z}_q$. A general lens space is given with an action of its fundamental group

$$\gamma z_j = e^{2\pi i \nu_k / q} z_j \ , \qquad j = 1, 2 \tag{2.5.20}$$

where z_j are complex coordinates denoting the 4-dimensional space in which $\mathbf{S}^3$ is embedded, and ν_k are integers relatively prime to q. Taking equation (2.2.5) with $r = 1$ and replacing the notation for δ by γ, we see that in our case, $\nu_1 = 1$ and $\nu_2 = -1$ in (2.5.20).

Fixing γ to be a generator of $\mathbf{Z}_q$, we can take our Wilson line U_γ to be given by (2.2.7). The action of the Wilson line on the fields ϕ_k is

$$\begin{aligned} \phi_k(\gamma\theta) &= \phi_k(\theta) && \phi_k \in (8,1,0) \oplus (1,3,0) \oplus (1,1,0) \\ \phi_k(\gamma\theta) &= e^{10\pi i w/q} \phi_k(\theta) && \phi_k \in (3,2,-5/6) \\ \phi_k(\gamma\theta) &= e^{-10\pi i w/q} \phi_k(\theta) && \phi_k \in (\bar{3},2,5/6). \end{aligned} \tag{2.5.21}$$

Therefore, the threshold corrections are given by

$$\mathcal{S}_{Y^-} = 0 \ ,$$

[8] In string theory models, properties of the threshold corrections of a similar flavor have been attributed to modular invariance [77].

$$
\begin{aligned}
\mathcal{S}_2{}^- &= 4\ \mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, 1) - 4\ \mathcal{T}(\mathbf{S}^3/\mathbf{Z}_q, e^{10\pi i w/q})\ , \\
\mathcal{S}_3{}^- &= 6\ \mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, 1) - 6\ \mathcal{T}(\mathbf{S}^3/\mathbf{Z}_q, e^{10\pi i w/q})\ . \qquad (2.5.22)
\end{aligned}
$$

The torsion of the lens space has been explicitly calculated in [97, 98, 99] for non–trivial representations:

$$
\mathcal{T}(\mathbf{S}^3/\mathbf{Z}_q, \omega) = \mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, \omega) = \log|\omega - 1||\omega^{-1} - 1|\ , \qquad (2.5.23)
$$

from which we get

$$
\mathcal{T}(\mathbf{S}^3/\mathbf{Z}_q, e^{\pm 10\pi i w/q}) = \log|e^{10\pi i w/q} - 1||e^{-10\pi i w/q} - 1| = \log(4\sin^2 5\pi w/q) \qquad (2.5.24)
$$

For the trivial representation, the quantity $\mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, 1)$ is computed in Appendix A:

$$
\mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, 1) = \log\left(\frac{2\pi^2}{q}\right) + \frac{3}{2}\log R^2\Lambda^2 = -\log q + \frac{3}{2}\log V_Q^{2/3}\Lambda^2\ , \qquad (2.5.25)
$$

where $V_Q = 2\pi^2 R^3/q$ is the volume of the lens space. As expected, we see the term $\mathcal{A}(Q,1) = -\frac{3}{2}\log V_Q^{2/3}\Lambda^2$ which appeared in (2.5.14). Putting this into (2.5.22) and defining

$$
D(q,w) = \Big(4q\sin^2(5\pi w/q)\Big)^{2/3}\ , \qquad (2.5.26)
$$

we have

$$
\begin{aligned}
\mathcal{S}_Y^- &= 0\ , \\
\mathcal{S}_2^- &= 6\log\left(\frac{V_Q^{2/3}\Lambda^2}{D(q,w)}\right)\ , \\
\mathcal{S}_3^- &= 9\log\left(\frac{V_Q^{2/3}\Lambda^2}{D(q,w)}\right)\ . \qquad (2.5.27)
\end{aligned}
$$

If our lens space could have any value of ν_k in (2.5.20), where ν_k are relatively prime to q, this result would be slightly modified. By replacing γ by a power of itself, there is no loss of generality to take, say $\nu_2 = 1$; we then denote ν_1 simply as ν. In this more general case (see [99], pp. 168-9), $\mathcal{T}(Q,1)$ is unchanged, and $\mathcal{T}(Q,\omega)$ becomes $\log(4|\sin(5\pi w/q)\sin(5\pi jw/q)|)$ where $j\nu \equiv 1$ modulo q, so $D(q,w) = (4q|\sin(5\pi w/q)\sin(5\pi jw/q)|)^{2/3}$.

2.6 Unification

In this section we finally write the RGE's for the three gauge couplings, with the threshold corrections included. In most of this section, we consider the case in which the only singularity that we have is the ADE singularity, and the only particles which appear in the model are those we obtained in Section 2.4. First, we show that for any G_{GUT}, the RGE's are independent of the cutoff Λ. Then we show that for an $SU(5)$ GUT group, unification of the gauge couplings is preserved in the presence of the threshold corrections, and we derive a precise relation between the unification scale E_{GUT} and the volume V_Q of Q; we also show the relation between E_{GUT} and the masses M_V of heavy vector bosons, and use it to make some (preliminary) comments about the lifetime of the proton (though most of our discussion of proton decay will appear in Section 2.8). We then proceed to add the quarks, leptons, and Higgs bosons to our model, and show that the predictions for the unification scale E_{GUT} and the weak mixing angle $\sin^2\theta_W$ are the same as the well known ones from the standard four dimensional supersymmetric unified $SU(5)$ model. We conclude this section by writing all the results explicitly for the lens space.

2.6.1 Elimination of Λ

From the discussion in Section 2.3, we see that every helicity state, massless or massive, makes a contribution to the low energy couplings that has the same dependence on the cutoff Λ, independent of its mass.

In a unified four-dimensional GUT theory, the quantum numbers of the tree level particles are G-invariant, though the masses are not. Hence, the coefficient of $\log\Lambda$ in (2.3.4) is G-invariant. The precise value of Λ is therefore irrelevant in the sense that a change in Λ can be absorbed in redefining the unified coupling (which in a four-dimensional theory would usually be called g_{GUT} rather than g_M).

It seems from (2.5.16) that the RGE's depend on the cutoff Λ. However, for any 3–manifold with $b_1(Q) = b_2(Q) = 0$, a magical cancellation occurs: the coefficient 3/2 multiplying $\log \Lambda^2/\mu^2$ in the first line of (2.5.16) precisely cancels the 3/2 which is the coefficient of $\log V_Q^{2/3}\Lambda^2$ in the second line of equation (2.5.16). Hence there is no dependence on Λ and we can now rewrite the formula for the threshold corrections in a Λ-independent and useful form:

$$\frac{1}{g_a^2(\mu)} = \frac{k_a}{g_M^2} + \frac{b_a}{16\pi^2} \log\left(\frac{1}{\mu^2 V_Q^{2/3}}\right) + \frac{\mathcal{S}'_a}{16\pi^2}\ , \tag{2.6.1}$$

with

$$\mathcal{S}'_a = 2\sum_i f^a(\rho_i) C_2^a(\rho_i)\ \mathcal{T}(Q,\omega_i). \tag{2.6.2}$$

The fact that the same factor of 3/2 appears twice is rather intriguing, as it seems to arise from two completely different origins: one comes from a sum over chiralities (given in the last column of Table 2.1), and the other arises in the computation of a term in the Ray-Singer torsion (computed in Appendix A.2). They do have an intimate relation, though: the first comes from an expression for zero-modes, and the second arises from the term $\mathcal{A}$ in the Ray-Singer torsion which appears because massless contributions are omitted.

By using the relation to analytic torsion, we have defined the second sum in equation (2.3.4) with zeta-function regularization, and we found there are no divergences. With a different regularization, there would be divergences, but they would be G-invariant. Of course, if we really had a proper understanding of M-theory, we would use whatever regularization it gives.

2.6.2 Preservation of Unification

As we have emphasized in the introduction and in Section 2.3, generically in the presence of threshold corrections, there is no energy scale μ at which the gauge couplings exactly unify. However, taking our result above and evaluating for $SU(5)$, which just amounts to working out some Casimirs of representations of $SU(5)$ (which we in fact already did in deriving (2.5.17)–(2.5.19)), we can write the RGE's in the form

$$\frac{1}{g_a^2(\mu)} = \left(\frac{1}{g_M^2} + \frac{10\mathcal{T}(Q,\omega_4)}{16\pi^2}\right) k_a + \frac{b_a}{16\pi^2} \log \frac{E_{GUT}^2}{\mu^2} \ , \tag{2.6.3}$$

where

$$E_{GUT}^2 = \frac{D(Q)}{V_Q^{2/3}}, \tag{2.6.4}$$

$$D(Q) \equiv \exp\left(-\frac{2}{3}(\mathcal{T}(Q,1) - \mathcal{T}(Q,\omega_4))\right) \ . \tag{2.6.5}$$

Hence, there is an energy scale at which the gauge couplings unify, namely $\mu = E_{GUT}$. The quantity $D(Q)$ is topological; the only dependence of E_{GUT} on the metric of Q is through its dependence on the volume.

Even if Λ was not eliminated, we would find a unification scale E_{GUT} for an $SU(5)$ theory; however, in that case, E_{GUT} would be Λ-dependent. We were able to write our RGE's in the form (2.6.3) because it turned out that the threshold corrections were proportional, as shown in (2.5.19), and furthermore their ratios matched those of the beta functions $b_Y : b_2 : b_3 = 0 : 2 : 3$. We will show in Appendix B that this generalizes to some other gauge groups.

If we compare (2.6.3) to a naive one-loop renormalization group formula that we might write in a GUT theory, which would read

$$\frac{1}{g_a^2(\mu)} = \left(\frac{1}{g_{GUT}^2}\right) k_a + \frac{b_a}{16\pi^2} \log(E_{GUT}^2/\mu^2) \ , \tag{2.6.6}$$

then we also see that we can write

$$\frac{1}{g_{GUT}^2} = \frac{1}{g_M^2} + \frac{10\mathcal{T}(Q,\omega)}{16\pi^2} \ . \tag{2.6.7}$$

This formula tells us how the coupling g_M used in M-theory should be compared to the g_{GUT} that is inferred from low energy data. Our computation really only makes sense if the difference between g_M and g_{GUT} is much smaller than either, since otherwise higher order corrections would be important. Moreover, a different regularization (such as M-theory may supply) might have given a different answer for the shift in g_M, so this shift is unreliable.

A noteworthy fact – though a simple consequence of $SU(5)$ group theory – is that the massive Kaluza-Klein harmonics have made no correction at all to the prediction of the theory for $\sin^2\theta_W$.

2.6.3 Interlude on Masses of Heavy Bosons

For any manifold, the eigenvalues of the Laplacian scale with the volume. Hence, as we described in the Prelude, the relation (2.6.4) is also a relation between the unification scale and the masses of those Kaluza-Klein modes that correspond to the heavy gauge bosons that can mediate proton decay in four dimensional theories. That mass is

$$M_V = \frac{\sqrt{\lambda_{min}(Q)}}{V_Q^{1/3}} \ , \tag{2.6.8}$$

where $\lambda_{min}(Q)$ is the smallest $\lambda_{0,n}$ which corresponds to a vector boson ϕ_0 in either the $(3, 2, -5/6)$ or $(\overline{3}, 2, 5/6)$ representation. We take the smallest one here since we are assuming for this interlude that the matrix elements for proton decay in our model are the same as in usual GUT's, i.e. that they are roughly proportional to the fourth power of the mass of these gauge bosons and hence dominated by the lightest gauge boson. (However, we shall see in Section 2.8 that our situation is different due to the way in which the chiral fermions are localized.) The relation between M_V and E_{GUT}, in terms of $D(Q)$ and $\lambda_{min}(Q)$, is

$$E_{GUT} = M_V \sqrt{\frac{D(Q)}{\lambda_{min}(Q)}} \tag{2.6.9}$$

With our assumption, the proton lifetime would get a correction from its usual prediction in which M_V is taken to be equal to E_{GUT} by a factor

$$\left(\frac{M_V}{E_{GUT}}\right)^4 = \left(\frac{\lambda_{min}(Q)}{D(Q)}\right)^2 . \tag{2.6.10}$$

For many lens spaces, both $D(Q)$ and $\lambda_{min}(Q)$ are of order 1-10, so for those cases, this correction is close to 1.

2.6.4 Adding Quarks, Leptons, and Higgs Bosons

Here we discuss how the quarks, leptons, and Higgs bosons can be introduced into the model. The construction here will be important for the study of proton decay in Section 2.8.

So far we have solely considered the case that the normal space to Q has only the standard orbifold singularity, so that the only charged particles with masses $\leq 1/R_Q$ are the Kaluza-Klein harmonics. Now we want to introduce quarks, leptons, Higgs bosons, and possibly other charged light fields such as messengers of gauge-mediated supersymmetry breaking. We do this by assuming at points $P_i \in Q$ the existence of certain more complicated singularities of the normal bundle. These generate charged massless $SU(5)$ multiplets (which may ultimately get masses at a lower scale if a superpotential is generated or supersymmetry is spontaneously broken). If the singularities of the P_i are generic, each one contributes a new irreducible $SU(5)$ multiplet M_i of massless chiral superfields. Specific singularities that generate chiral multiplets transforming in the $\mathbf{5}$, $\mathbf{10}$, $\overline{\mathbf{10}}$, and $\overline{\mathbf{5}}$ of $SU(5)$ have been studied in [16, 67, 14, 15].

It is believed that these singularities are *conical.* This is definitely true in a few cases in which the relevant G_2 metrics are conical metrics that were constructed long ago [100, 101] (and found recently [16] to generate massless chiral multiplets). Since a conical metric introduces no new length scale that is positive but smaller than the eleven-dimensional Planck length, we expect that these singularities, apart from the massless multiplets M_i, introduce no new particles of masses $\leq 1/R_Q$ that need to be considered in evaluating the threshold corrections.

We can also argue, a little less rigorously, that the singularities of the normal bundle that produces massless chiral superfields in the **5**, **10**, etc., have no effect on the Kaluza-Klein harmonics of the seven-dimensional vector multiplet on $\mathbf{R}^4 \times Q$. To show this, we consider the construction in [14], where the association of massless chiral multiplets with singularities was argued using duality with the heterotic string. In this argument, the existence in the G_2 description of a conical singularity that generates massless chiral superfields was related, in a heterotic string description that uses a $\mathbf{T}^3$ fibration, to the existence of a certain zero mode of the Dirac equation on a special $\mathbf{T}^3$ fiber. The existence of this zero mode generates a localized massless multiplet in the **5**, **10**, etc., as shown in [14], but does nothing at all to the seven-dimensional vector multiplet (which has no exceptional zero mode on the $\mathbf{T}^3$ in question).

Granted these facts, to incorporate the effects of the multiplets M_i, all we have to do is add their contributions to the starting point (2.3.1) or to the final result (2.6.3). If all of the new multiplets are massless down to the scale of supersymmetry breaking, then, for μ greater than this scale, all we have to do is add a contribution to (2.3.1) due to the new light fields. Let Δb_a be the contribution of the new light fields to the beta function coefficients b_a – note that since the M_i form complete $SU(5)$ multiplets, they contribute to each b_a in proportion to k_a. The contribution of the new fields to (2.3.1) is to add to the right hand side

$$\Delta b_a \log(\widetilde{\Lambda}^2/\mu^2), \tag{2.6.11}$$

which is the contribution due to renormalization group running of the M_i from their cutoff $\widetilde{\Lambda}$ down to μ. We do not exactly know what effective cutoff $\widetilde{\Lambda}$ to use for the M_i, but it is of order M_{11}. Anyway, the exact value of $\widetilde{\Lambda}$ does not matter; it can be absorbed in a small correction to g_M (this correction is no bigger than other unknown corrections due for example to possible charged particles with masses of order M_{11}). In fact, up to a small shift in g_M, it would not matter if we replace $\widetilde{\Lambda}$ by the (presumably lower) scale $\exp\left(\frac{1}{3}(\mathcal{T}(Q,\omega)-\mathcal{T}(Q,1)\right)/V_Q^{1/3}$ that appears in (2.6.4). So if all components of the M_i are light, we can take our final answer to be simply that of (2.6.3), but with all b_a redefined (by the shift $b_a \to b_a + \Delta b_a$) to include the effects of the M_i. In other words, if all components of M_i are light, we simply have to take the b_a in (2.6.3) to be the exact β function coefficients of the low energy theory.

Then, our RGE's are precisely the same ones we would write for the standard model if we were not considering threshold corrections, except that we have replaced the expression for E_{GUT} by $D(Q)/V_Q^{2/3}$. Therefore, the predictions obtained from measurements of g_3 and g_{em} are the same as those obtained from the four dimensional SUSY standard model: $E_{GUT} = D(Q)/V_Q^{2/3} = 2.2 \times 10^{16}\, GeV$, $\alpha_{GUT}^{-1} = 4\pi/g_{GUT}^2 = 24.3$, $\sin^2\theta_W = .23$.

The assumption that all components of the M_i are light is inconsistent with the measured value of the weak mixing angle $\sin^2\theta_W$. That measured value (and the longevity of the proton) is instead compatible with the hypothesis that all components of the M_i are light except for the color triplet partners of the ordinary $SU(2)\times U(1)$ Higgs bosons; we call those triplets T and $\tilde{T}$. Let m_T be the mass of T and $\tilde{T}$ (we assume this mass comes from a superpotential term $T\tilde{T}$, in which case T and $\tilde{T}$ have equal masses), and let $\Delta b_a^{T,\widetilde{T}}$ be their contribution to the beta functions. These are not proportional to k_a since T and $\tilde{T}$ do not form a complete $SU(5)$ multiplet. Then (2.6.11) should be replaced by

$$(\Delta b_a - b_a^{T,\widetilde{T}})\log(\widetilde{\Lambda}^2/\mu^2) + b_a^{T,\widetilde{T}}\log(\widetilde{\Lambda}^2/m_T^2), \tag{2.6.12}$$

the idea being that the T, $\widetilde{T}$ contributions run only from $\widetilde{\Lambda}$ down to m_T, while the others run down to μ. Up to a small correction to g_M, we can again replace $\widetilde{\Lambda}$ in (2.6.12) by

$$\widetilde{\Lambda} \to \exp\left(\frac{1}{3}(\mathcal{T}(Q,\omega)-\mathcal{T}(Q,1))\right)V_Q^{-1/3}. \tag{2.6.13}$$

If we do this, then (2.6.3) is replaced by

$$\frac{1}{g_a^2(\mu)} = \left(\frac{1}{g_M^2}+\frac{10\mathcal{T}(Q,\omega)+\delta}{16\pi^2}\right)k_a+\frac{b_a}{16\pi^2}\log\left(\frac{\exp(\frac{2}{3}(\mathcal{T}(Q,\omega)-\mathcal{T}(Q,1))}{\mu^2 V_Q^{2/3}}\right)$$
$$+\frac{b_a^{T,\widetilde{T}}}{16\pi^2}\log\left(\frac{\exp(\frac{2}{3}(\mathcal{T}(Q,\omega)-\mathcal{T}(Q,1)))}{m_T^2 V_Q^{2/3}}\right). \qquad (2.6.14)$$

Here b_a are the full beta functions of the low energy theory below the mass m_T, and $b_a^{T,\tilde{T}}$ is the additional contribution to the beta functions from $T,\widetilde{T}$ between m_T and the effective GUT mass $E_{GUT}=\exp\left(\frac{1}{3}(\mathcal{T}(Q,\omega)-\mathcal{T}(Q,1))\right)/V_Q^{1/3}$. Finally, δ expresses an unknown shift in the effective value of g_M; this shift is presumably unimportant within the accuracy of the computation.

The lens space

The result (2.6.3) holds for any 3–manifold Q. For our example of the lens space, we can make our result completely explicit:

$$\begin{aligned}
\frac{1}{g_Y^2(\mu)} &= \frac{k_Y}{g_{GUT}^2}\ , \\
\frac{1}{g_2^2(\mu)} &= \frac{1}{g_{GUT}^2}-\frac{6}{16\pi^2}\log\frac{D(q,w)}{V_Q^2\mu^2}\ , \\
\frac{1}{g_3^2(\mu)} &= \frac{1}{g_{GUT}^2}-\frac{9}{16\pi^2}\log\frac{D(q,w)}{V_Q^2\mu^2}\ ,
\end{aligned} \qquad (2.6.15)$$

where $D(q,w)$ is given by (2.5.26) or its generalization at the end of Section 2.5; one can check, from the discussion of the lens space at the end of Section 2.5, that $D(q,w)$ is $D(Q)$ when Q is the lens space.

The value of μ at which all gauge couplings unify is given by

$$\mu^2 = E_{GUT}^2 = \frac{D(q,w)}{V_Q^{2/3}}. \qquad (2.6.16)$$

and the relation between E_{GUT} and V_Q in this model is

$$E_{GUT} = \left(\frac{4q \sin^2(5\pi w/q)}{V_Q} \right)^{1/3}, \quad V_Q = \frac{4q \sin^2(5\pi w/q)}{E^3_{GUT}}. \tag{2.6.17}$$

2.7 On Newton's Constant

One virtue of computing the threshold corrections is that we can make somewhat more precise the formulas for the parameters E_{GUT}, α_{GUT}, and G_N that are read off from the eleven-dimensional supergravity action.

Since M-theory on $\mathbf{C}^2/\mathbf{Z}_5$ is an $SU(5)$ gauge theory equivalent to a gauge theory coming from 5 wrapped D6 branes in Type IIA string theory, we can use the relations between the gauge couplings on the $D6$ branes in Type IIA string theory and the gravitational coupling in 11 dimensions to express the value of G_N in terms of α_{GUT}, V_Q, and E_{GUT}.

The gravitational action in 11 dimensions is given by

$$\frac{1}{16\pi G_N^{(11)}} \int d^{11}x \sqrt{g} R = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{g} R\,, \tag{2.7.1}$$

where κ_{11} is the 11 dimensional gravitational coupling and $G_N^{(11)}$ is Newton's constant in 11 dimensions. The Yang–Mills action in 7 dimensions is given by the coupling of Yang–Mills theory to a $D6$ brane in type IIA string theory (see eqns. (13.3.25) and (13.3.26) of [102]):

$$\frac{1}{4(2\pi)^4 g_s (\alpha')^{3/2}} \int d^7x \sqrt{g}\, \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}, \tag{2.7.2}$$

where g_s is the string coupling constant and Tr is the trace in the fundamental representation of $U(n)$. In the GUT literature, one usually writes $F_{\mu\nu} = \sum_a F^a_{\mu\nu} Q_a$, where Q_a are generators of $U(n)$ normalized to $\mathrm{Tr}\, Q_a Q_b = \frac{1}{2}\delta^{ab}$. So (2.7.2) can be written

$$\frac{1}{8(2\pi)^4 g_s (\alpha')^{3/2}} \int d^7x \sqrt{g} \sum_a F^a_{\mu\nu} F^{\mu\nu\, a}. \tag{2.7.3}$$

Going to M-theory, the relation between κ_{11}, g_s, and α' is (eqn. (14.4.5) of [102])

$$\kappa_{11}^2 = \frac{1}{2} (2\pi)^8 g_s^3 (\alpha')^{9/2}. \tag{2.7.4}$$

Combining these, we see that the Yang-Mills action in seven dimensions is

$$\frac{1}{4g_7^2}\int d^7x\sqrt{g}\sum_a F_{a\mu\nu}F^{a\mu\nu} = \frac{1}{8(2\pi)^{4/3}(2\kappa_{11}^2)^{1/3}}\int d^7x\sqrt{g}\sum_a F_{a\mu\nu}F^{a\mu\nu}\ , \tag{2.7.5}$$

where the trace is taken in the fundamental representation and the generators are normalized as described after (2.3.2).

Let X_7 be the volume of X. After reducing (2.7.1) and (2.7.5) to four dimensions and getting a vactor of V_X and V_Q respectively, the gravitational and gauge couplings in 4 dimensions can be written as

$$\begin{aligned} \frac{1}{16\pi G_N} &= \frac{1}{2\kappa_{11}^2}V_X\ , \\ \frac{1}{16\pi\alpha_{GUT}} &= \frac{1}{8(2\pi)^{4/3}(2\kappa_{11}^2)^{1/3}}V_Q\ . \end{aligned} \tag{2.7.6}$$

Here, G_N is Newton's constant in 4 dimensions. Eliminating κ_{11} and solving for G_N gives

$$G_N = \frac{1}{32\pi^2}\alpha_{GUT}^3\frac{V_Q^3}{V_X} = \frac{1}{32\pi^2 a}\alpha_{GUT}^3 V_Q^{2/3}\ , \tag{2.7.7}$$

where $V_X = aV_Q^{7/3}$. Using the relation between V_Q and E_{GUT} given in (2.6.4)

$$V_Q = \frac{1}{E_{GUT}^3}D(Q)^{3/2}, \tag{2.7.8}$$

we get

$$G_N = \frac{1}{32\pi^2 a}\frac{\alpha_{GUT}^3}{E_{GUT}^2}D(Q). \tag{2.7.9}$$

Here E_{GUT} is the unification scale as inferred from low energy data. For the simplest lens space, $D(Q) = (4q\sin^2(5\pi w/q))^{2/3}$.

We now discuss the quantity a. Since the 7-manifolds discussed here are not compact, in fact it is the case that a is infinite. However, one can also postulate the existence of G_2 manifolds which have similar properties (namely, an ADE singularity with locus which has a finite, non-trivial fundamental group, as well as extra singularities that support chiral fermions) and are at the same time compact. Such manifolds have not yet been explicitly constructed, but there has been much development lately in constructions of compact G_2 manifolds [103], so there is hope that the necessary manifolds will soon be constructed. Furthermore, such manifolds are believed to exist via dualities. For the cases when dualities can be used to deduce the existence of a compact G_2 manifold with the right properties, it is the case (see [61]) that $a \ll 1$.

If there is an upper bound on a, then (2.7.9) is a lower bound on G_N that depends on the values of the GUT parameters, the readily computed constant $D(Q)$, as well as the bound on a.

Within the uncertainties, such a bound on G_N may well be saturated in nature. If we use the often-quoted values $E_{GUT} = 2.2 \times 10^{16}$ GeV and $\alpha_{GUT}^{-1} = 24.3$, then, with $G_N = 6.7 \times 10^{-39}\,\text{GeV}^{-2}$, we need approximately $D(Q)/a = 14.7$. We recall, however, that these values correspond to a minimal three family plus Higgs boson spectrum below the GUT scale, and that the doublet-triplet splitting mechanism [41] that we are assuming in the present work really leads to extra light fields in vector-like $SU(5)$ multiplets. If we use the values $E_{GUT} \sim 8 \times 10^{16}$ GeV, $\alpha_{GUT} \sim .2$, which are typical values found in [104] (see also [105]) for certain models that have TeV-scale vector-like fields transforming as $\bar{\mathbf{5}} \oplus \mathbf{10}$ plus their conjugates, then we get[9] $D(Q)/a = 1.7$.

Ignoring for the moment the unknown value of a, the expression (2.7.9) gives a better agreement with the known value of G_N than the analogous one for the weakly or strongly coupled heterotic string; the main reason is the power of α_{GUT}: in the weak heterotic string, the power was $\alpha_{GUT}^{4/3}$ and in the strong it was α_{GUT}^2. Both gave lower bounds for G_N that were too large [82, 83]. Here, the power is α_{GUT}^3, allowing G_N to be smaller.

We can combine (2.7.6) and (2.7.9) to give

$$\kappa_{11}^2 = \frac{\alpha_{GUT}^3 D(Q)^{9/2}}{4\pi E_{GUT}^9}. \tag{2.7.10}$$

[9] To get a rough feel, note that if $a = 1$, we can take for the lens space $q = 14$ and $w = 7$ to get $D(Q) = 14.7$, or we can take $q = 17$ and $w = 7$ to get $D(Q) = 1.7$.

This formula is attractive because it is expresses the fundamental eleven-dimensional coupling κ_{11} in terms of quantities – α_{GUT} and E_{GUT} – about whose values we have at least some idea from experiment, and another quantity – $D(Q)$ – that is readily calculable in a given model. Furthermore, a does not appear here.

The eleven-dimensional Planck mass M_{11} has been defined ([102], p. 199) by $2\kappa_{11}^2 = (2\pi)^8 M_{11}^{-9}$. So we can express (2.7.10) as a formula for M_{11}:

$$M_{11} = \frac{2\pi E_{GUT}}{\alpha_{GUT}^{1/3} D(Q)^{1/2}}. \tag{2.7.11}$$

One important result here is that E_{GUT} is parametrically smaller than M_{11} – by a factor of $\alpha_{GUT}^{1/3}$. This factor of $\alpha_{GUT}^{1/3}$ is the reason that it makes sense to use perturbation theory – as we have done in computing threshold corrections in Section 2.5.

Regrettably, the precise factors in the definition of M_{11} have been chosen for convenience. We really do not know if the characteristic mass scale at which eleven-dimensional supergravity breaks down and quantum effects become large is M_{11}, or $2\pi M_{11}$, or for that matter $M_{11}/2\pi$.[10] This uncertainty will unfortunately be important in Section 2.8.

2.8 Proton Decay

In this section, we will analyze the gauge boson contribution to proton decay in the present class of models.

First, we recall how the analysis goes in four-dimensional GUT's. We will express the analysis in a way that is convenient for the generalization to $\mathbf{R}^4 \times Q$. The gauge boson contribution to proton decay comes from the matrix element of an operator product

$$g_{GUT}^2 \int d^4x \, J^\mu(x) \tilde{J}_\mu(0) D(x, 0), \tag{2.8.1}$$

[10] Just to get a feel for what M_{11} might be, note that for $(q, w) = (2, 1)$, we have $M_{11} = 2.0 \times 10^{17} GeV$ for the often-quoted values of E_{GUT} and α_{GUT}, and $M_{11} = 4.3 \times 10^{17} GeV$ for the values in [104].

where J and $\tilde{J}$ are the currents in emission and absorption of the color triplet gauge bosons. We have used translation invariance to place one current at the origin, and $D(x,0)$ is the propagator of the heavy gauge bosons that transform as $(3,2)^{-5/6}$ of $SU(3)\times SU(2)\times U(1)$. Because the proton is so large compared to the range of x that contributes appreciably in the integral, we can replace $J^\mu(x)$ by $J^\mu(0)$, and then use

$$\int d^4x\, D(x,0) = \frac{1}{M_V^2} \tag{2.8.2}$$

(with M_V the mass of the heavy gauge bosons) to reduce (2.8.1) to

$$\frac{g_{GUT}^2 J_\mu \tilde{J}^\mu(0)}{M_V^2}. \tag{2.8.3}$$

Equation (2.8.2) is a direct consequence of the equation for the propagator, which is

$$\left(\Delta + M_V^2\right) D(x,0) = \delta^4(x), \tag{2.8.4}$$

with $\Delta = -\eta^{\mu\nu}\partial_\mu\partial_\nu$ the Laplacian. Of course, in deriving (2.8.2), we should be careful in defining the operator product $J_\mu \tilde{J}^\mu$; doing so leads to some renormalization group corrections to the above tree level derivation. These can be treated the same way in four dimensions and in the G_2-based models, and hence need not concern us here.

In $\mathbf{R}^4\times Q$, the idea is similar, except that the currents are localized at specific points on Q, which we will call P_1 and P_2. The gauge boson propagator is a function $D(x,P;y,P')$ with $x,y\in\mathbf{R}^4$, and $P,P'\in Q$; the equation it obeys is

$$\left(\Delta_{\mathbf{R}^4} + \Delta_Q\right) D(x,P;y,P') = \delta^4(x-y)\delta(P,P'). \tag{2.8.5}$$

Here $\Delta_{\mathbf{R}^4}$ is the Laplacian on $\mathbf{R}^4$, acting on the x variable, and similarly Δ_Q is the Laplacian on Q, acting on P. Now we set P and P' to be two of the special points P_1 and P_2 on Q (with enhanced singularities in the normal directions) at which chiral matter fields are supported. The analog of (2.8.1) is

$$g_7^2 \int d^4x J_\mu(x,P_1)\tilde{J}^\mu(0,P_2)D(x,P_1;0,P_2), \tag{2.8.6}$$

where we have used translation invariance to set $y = 0$. Again, because the proton is so large compared to the range of x that contributes significantly to the integral, we can set x to 0 in $J_\mu(x, P_1)$, giving us

$$g_7^2 J_\mu(0; P_1) \tilde{J}^\mu(0; P_2) \int d^4x \, D(x, P_1; 0, P_2). \tag{2.8.7}$$

Now it follows from (2.8.5) that the function

$$F(P, P') = \int d^4x \, D(x, P; 0, P') \tag{2.8.8}$$

obeys

$$\Delta_Q F(P, P') = \delta(P, P'). \tag{2.8.9}$$

In other words, F is the Green's function of the scalar Laplacian on Q (for scalar fields valued in the $(3, 2)^{-5/6}$ representation). In particular, F is bounded for P away from P', and for $P \to P'$,

$$F(P, P') \to \frac{1}{4\pi|P - P'|}, \tag{2.8.10}$$

with $|P - P'|$ denoting the distance between these two points. The proton decay interaction is

$$g_7^2 J_\mu(0; P_1) \tilde{J}^\mu(0; P_2) \, F(P_1, P_2). \tag{2.8.11}$$

More exactly, this is the contribution for fermions living at the points P_1, P_2. It must be summed over possible P_i.

Given (2.8.10), if it is possible to have P_1 very close to P_2, this will give the dominant contribution. But how close will the P_i be? The smallest that the denominator in (2.8.10) will get is if $P_1 = P_2$, in other words if the currents J_μ and $\tilde{J}^\mu$ in the proton decay process act on the same $\mathbf{10}$ or $\bar{\mathbf{5}}$ of $SU(5)$, living at some point $P = P_1 = P_2$ on Q. If this is the case, then the result (2.8.7) is infinite. M-theory will cut off this infinity, but we do not know exactly how. The best we can say is that the cutoff will occur at a distance of order the eleven-dimensional Planck length, as this is the only scale that is relevant in studying the conical singularity at P. If we naively say that setting $P_1 = P_2$ means replacing $1/|P_1 - P_2|$ by $1/R_{11} = M_{11}$ (with R_{11} the eleven-dimensional Planck length; M_{11} was evaluated in (2.7.11)), then we would replace $F(P,P)$ by $M_{11}/4\pi$. Unfortunately, as noted at the end of Section 2.7, we have no idea whether M_{11} or some multiple of it is the natural cutoff in M-theory. This is an important uncertainty, since (for example) 4π is a relatively large number and the proton decay rate is proportional to the square of the amplitude. All we can say is that the effective value of $F(P,P)$, though uncalculable with the present understanding of M-theory, is model-independent; it does not depend on the details of X or Q, but is a universal property of M-theory with the conical singularity P. The effective interaction is

$$\sum_P C\frac{M_{11}g_7^2}{4\pi}J_\mu \tilde{J}^\mu(P), \tag{2.8.12}$$

where C is a constant that in principle depends only on M-theory and not the specific model. So C might conceivably be computed in the future (if better methods are discovered) without knowing how to pick the right model. We have made explicit the fact that the interaction is summed over all possibilities for $P = P_1 = P_2$. Subleading (and model-dependent) contributions with $P_1 \neq P_2$ have not been written.

Let us compare this to the situation in four-dimensional GUT's. The currents J and $\tilde{J}$ receive contributions from particles in the $\mathbf{10}$ and $\bar{\mathbf{5}}$ of $SU(5)$. Thus we can expand

$$J_\mu \tilde{J}^\mu = J_\mu^{\mathbf{10}}\tilde{J}^{\mu\,\mathbf{10}} + J_\mu^{\bar{\mathbf{5}}}\tilde{J}^{\mu\,\bar{\mathbf{5}}} + J_\mu^{\mathbf{10}}\tilde{J}^{\mu\,\bar{\mathbf{5}}} + J_\mu^{\bar{\mathbf{5}}}\tilde{J}^{\mu\,\mathbf{10}}. \tag{2.8.13}$$

Among these terms, the $\mathbf{10}\cdot\mathbf{10}$ operator product contributes to $p \to \pi^0 e_L^+$, $\bar{\mathbf{5}}\cdot\bar{\mathbf{5}}$ does not contribute to proton decay, and the cross terms contribute to $p \to \pi^0 e_R^+$ and $p \to \pi^+ \bar{\nu}_R$. Assuming that the points supporting $\mathbf{10}$'s are distinct from the points supporting $\bar{\mathbf{5}}$'s, the above mechanism, in comparison to four-dimensional GUT's, enhances the decay $p \to \pi^0 e_L^+$ relative to the others.

Using the formulas in Section 2.7, we can evaluate the product $g_7^2 M_{11}$. Reading off g_7^2 from (2.7.5) and M_{11} from (2.7.11), we find that the effective interaction is

$$\sum_{P_i} C J_\mu \tilde{J}^\mu(P_i)\cdot\frac{2\pi D(Q)\alpha_{GUT}^{2/3}}{E_{GUT}^2}. \tag{2.8.14}$$

The equivalent formula in four-dimensional GUT's is

$$J^T_\mu \tilde{J}^{T\,\mu}\frac{g^2_{GUT}}{M_V^2} = J^T_\mu \tilde{J}^{T\,\mu}\frac{4\pi\alpha_{GUT}}{M_V^2}, \tag{2.8.15}$$

where the superscript T refers to the total current for all fermion multiplets. Moreover, M_V is the mass of the color triplet gauge bosons, and so may not coincide with E_{GUT}, which is the unification scale as deduced from the low energy gauge couplings. The above formulas show that, in principle, the decay amplitude for $p \to \pi^0 e^+_L$ in the G_2-based theory is enhanced as $\alpha_{GUT} \to 0$ by a factor of $\alpha_{GUT}^{-1/3}$ relative to the corresponding GUT amplitude. The enhancement means that, in some sense, in the models considered here, proton decay is not purely a gauge theory phenomenon but a reflection of M-theory.

In practice, $\alpha_{GUT}^{-1/3}$ is not such a big number. Whether the effect we have described is really an enhancement of $p \to \pi^0 e^+_L$ or a suppression of the other decays depends largely on the unknown M-theory constant C. The factor $D(Q)$ can also be significant numerically. For example, for the simplest lens space, with the minimal choice $w = 1$, $q = 2$, we get $D(Q) = (4q\sin^2(5\pi w/q))^{2/3} = 4$.

Even if C and the other factors in (2.8.14) were all known, the proton lifetime would also depend on how the light quarks and leptons are distributed among the different P_i, or equivalently, how they are distributed among the different **10**'s of $SU(5)$. The same remark applies in four-dimensional GUT's: the proton decay rate can be reduced by mixing of quarks and leptons among themselves as well as with other multiplets, including multiplets that have GUT-scale masses.

The arrangement of the different quarks and leptons among the P_i will also affect the flavor structure of proton decay. What we have referred to as $p \to \pi^0 e^+_L$ will also contain an admixture of other modes with π^0 replaced by K^0, and/or e^+_L replaced by μ^+_L. As in four-dimensional GUT's, these relative decay rates are model-dependent.

Appendix

We have two appendices. In the first one, we compute the Ray–Singer torsion for the lens space for the trivial representation and show how its two parts $\mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, 1)$ and $\mathcal{A}(\mathbf{S}^3/\mathbf{Z}_q, 1)$ depend on the volume; we also show that the volume dependence obtained from it generalizes to any 3–manifold with $b_1(Q) = 0$. In the second appendix, we generalize our results to the breaking of $SU(p)$ to $SU(m) \times SU(n) \times U(1)$.

A The Ray-Singer torsion for the trivial representation

A.1 $\mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, 1)$

We will calculate $\mathcal{K}(\mathbf{S}^3, 1)$ first, and we shall use the relation

$$\mathcal{K}(\mathbf{S}^3, 1) = \mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, 1) + \sum_{\omega} \mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, \omega) \,, \tag{A.1}$$

where the sum is over non-trivial representations of $\mathbf{Z}_q$ for which the torsion is known and given by (2.5.23), to obtain $\mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, 1)$. The relation (A.1) follows from an analogous one for the logarithm of the determinant of each Laplacian: each eigenform on $\mathbf{S}^3$ is in some representation of the $\mathbf{Z}_q$ action on $\mathbf{S}^3$ that defined $\mathbf{S}^3/\mathbf{Z}_q$, so we can separate the sum over eigenforms on $\mathbf{S}^3$ into a sum over the eigenforms living in the trivial representation of $\mathbf{Z}_q$ and those living in the non trivial representations.

The quantity we are about to calculate is

$$\mathcal{K}(\mathbf{S}^3,1) = \frac{3}{2}\log\det\Delta_0'(\mathbf{S}^3) - \frac{1}{2}\log\det\Delta_1'(\mathbf{S}^3)\,,$$

so we need the eigenvalues of the Laplacian and their multiplicities for 0–forms and 1–forms on $\mathbf{S}^3$. We shall calculate everything first for a sphere of radius 1, and include the dependence on the radius later.

The 0–forms have eigenvalues $\lambda_{0,n} = n(n+2)$ with multiplicities $y_{0,n} = (n+1)^2$. There are two types of 1–forms: closed, which have the same eigenvalues and multiplicities as the 0–forms, and co-closed, which have eigenvalues $\lambda_{1,n} = (n+1)^2$ and multiplicities $y_{1,n} = 2n(n+2)$ (see (3.19) of [106]).

The logarithm of the determinant of the Laplacian is defined by analytic continuation using the Riemann zeta function (for more examples of the use fo zeta function regularization in studying Ray-Singer torsions, see [106]). We begin by writing the zeta function of the Laplacian on 0–forms or closed 1–forms, and the zeta function of the Laplacian on co-closed 1–forms:

$$\zeta_0(s) = \sum \frac{y_{0,n}}{\lambda_{0,n}^s} = \sum_{n=1}^{\infty} \frac{(n+1)^2}{(n(n+2))^s}\,, \tag{A.2}$$

$$\zeta_1(s) = \sum \frac{y_{1,n}}{\lambda_{1,n}^s} = \sum_{n=1}^{\infty} \frac{2n(n+2)}{(n+1)^{2s}}\,. \tag{A.3}$$

Then we have

$$-\zeta_0'(0) = \log\det\Delta_0' = \log\det{\Delta_1'}^{closed}\,, \tag{A.4}$$

$$-\zeta_1'(0) = \log\det{\Delta_1'}^{co-closed}\,. \tag{A.5}$$

In terms of these zeta functions, we have

$$\frac{1}{2}\Big[3\log\det\Delta_0' - (\log\det{\Delta_1'}^{closed} + \log\det{\Delta_1'}^{co-closed})\Big] = \frac{1}{2}\zeta_1'(0) - \zeta_0'(0)\,. \tag{A.6}$$

We wish to rewrite the sums $\zeta_0(s)$ and $\zeta_1(s)$ in terms of the well known Riemann zeta function $\zeta(s)$ so that they will be well-defined at $s = 0$. This is straightforward for $\zeta_1(s)$:

$$\zeta_1(s) = 2\sum_{n=1}^{\infty}\Big(\frac{(n+1)^2}{(n+1)^{2s}} - \frac{1}{(n+1)^{2s}}\Big)\,,$$

whose analytic continuation is

$$\zeta_1(s) = 2((\zeta(2s-2)-1)-(\zeta(2s)-1)) = 2(\zeta(2s-2)-\zeta(2s))\,. \qquad \text{(A.7)}$$

As for $\zeta_0(s)$, we rewrite it as follows:

$$\begin{aligned}\zeta_0(s) &= \sum_{n=1}^{\infty}(n+1)^2\Big[\Big(1-\frac{1}{(2s+1)}\Big)\frac{1}{(n+1)^{2s}}+\frac{1}{2(2s+1)}\Big(\frac{1}{n^{2s}}+\frac{1}{(n+2)^{2s}}\Big)\Big] \qquad \text{(A.8)}\\ &\quad+\sum_{n=1}^{\infty}(n+1)^2\Big[\frac{1}{(n(n+2))^s}-\Big(1-\frac{1}{(2s+1)}\Big)\frac{1}{(n+1)^{2s}}-\frac{1}{2(2s+1)}\Big(\frac{1}{n^{2s}}+\frac{1}{(n+2)^{2s}}\Big)\Big]\,.\end{aligned}$$

The sum on the second line converges absolutely for $\Re e(s) > -1/2$: with $u = \frac{1}{n+1}$, it becomes

$$\sum_{n=1}^{\infty} u^{2s-2}\Big[\frac{1}{(1-u^2)^s}-\Big(1-\frac{1}{(2s+1)}\Big)-\frac{1}{2(2s+1)}\Big(\frac{1}{(1-u)^{2s}}+\frac{1}{(1+u)^{2s}}\Big)\Big]\,,$$

and one can see that the leading order term for large n (small u) is $u^{2s-2}u^4 = u^{2s+2}$, hence it is bounded by $1/(n+1)^{2s+2}$ which converges absolutely for $\Re e(2s+2) > 1$ or $\Re e(s) > -1/2$. Hence for $\Re e(s) > -1/2$, we can do the sum term by term. At $s = 0$, each term in the sum vanishes, and furthermore the derivative of each term with respect to s at $s = 0$ vanishes. Therefore, for small s we can write

$$\zeta_0(s) = \sum_{n=1}^{\infty}(n+1)^2\Big[\Big(1-\frac{1}{(2s+1)}\Big)\frac{1}{(n+1)^{2s}}+\frac{1}{2(2s+1)}\Big(\frac{1}{n^{2s}}+\frac{1}{(n+2)^{2s}}\Big)\Big]\,,$$

which becomes, by analytic continuation,

$$\zeta_0(s) = \zeta(2s-2)+\frac{1}{(2s+1)}\zeta(2s)-\Big(1-\frac{1}{(2s+1)}\Big)-\frac{1}{(2s+1)2^{2s+1}}\,. \qquad \text{(A.9)}$$

Using known values of the Riemann zeta function

$$\zeta(-2)=0\,, \qquad \zeta(0)=-1/2\,, \qquad \zeta'(0)=-\frac{1}{2}\log 2\pi,$$

we have at $s = 0$ the following values of the zeta functions of our Laplacians and their derivatives:

$$\begin{aligned}
\zeta_0(0) &= \zeta(-2) + \zeta(0) - \frac{1}{2} = -1\,, \\
\zeta_1(0) &= 2(\zeta(-2) - \zeta(0)) = 1\,, \\
\zeta_0'(0) &= 2\zeta'(-2) - \log\pi\,, \\
\zeta_1'(0) &= 4(\zeta'(-2) - \zeta'(0)) = 4\zeta'(-2) + 2\log(2\pi)\,.
\end{aligned} \tag{A.10}$$

(The value of $\zeta_0'(0)$ can also be obtained from Proposition 3.1 of [107], or from [108].)

Now we include the dependence on the radius: the quantity that comes into the physical calculation is actually

$$-\sum_n \log\Big(\frac{m_{k,n}^2}{\Lambda^2}\Big) = -\sum_n y_n \log\frac{\lambda_{k,n}}{\Lambda^2 R^2}\ ,$$

so we should replace equations (A.2) and (A.3) by

$$\eta_0(s) = (R\Lambda)^{2s}\zeta_0(s)\ , \tag{A.11}$$

$$\eta_1(s) = (R\Lambda)^{2s}\zeta_1(s)\ , \tag{A.12}$$

whose derivatives at $s = 0$ are

$$\eta_0'(0) = -\log R^2\Lambda^2 + \zeta_0'(0)\ , \tag{A.13}$$

$$\eta_1'(0) = \log R^2\Lambda^2 + \zeta_1'(0)\ . \tag{A.14}$$

We now have

$$\begin{aligned}
\mathcal{K}(\mathbf{S}^3, 1) &= \frac{1}{2}\eta_1'(0) - \eta_0'(0) \\
&= \frac{3}{2}\log R^2\Lambda^2 + \log 2\pi^2\,.
\end{aligned} \tag{A.15}$$

Finally, we can use (A.1): the sum over ω is a sum over non-trivial q^{th} roots of unity, so using (2.5.23), we have

$$\sum_\omega \mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, \omega) = \sum_\omega \mathcal{T}(\mathbf{S}^3/\mathbf{Z}_q, \omega) = \sum_\omega \log|\omega - 1||\omega^{-1} - 1| = 2\log q\ , \tag{A.16}$$

or, for the more general lens space, with integers m and j as described in the last paragraph of Section 2.5, we similarly have

$$= \sum_{\omega} \log|\omega - 1||\omega^{-j} - 1| = 2\log q \ . \tag{A.17}$$

Therefore,

$$\begin{aligned} \mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, 1) &= \frac{3}{2}\log R^2\Lambda^2 + \log 2\pi^2 - 2\log q \\ &= \log\left(\frac{2\pi^2}{q^2}\right) + \frac{3}{2}\log R^2\Lambda^2. \end{aligned} \tag{A.18}$$

In terms of the volume $V = 2\pi^2 R^3/q$, this becomes

$$\begin{aligned} \mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, 1) &= \log\left(\frac{2\pi^2}{q^2}\right) + \log\frac{q}{2\pi^2}V\Lambda^3 \\ &= \log\frac{V\Lambda^3}{q}. \end{aligned} \tag{A.19}$$

The Ray-Singer torsion $\mathcal{K}(\mathbf{S}^3/\mathbf{Z}_q, 1)$ for the trivial representation can be calculated from [99] or [97] to be $\log\frac{1}{q}$. Hence, the term $\mathcal{A}(\mathbf{S}^3/\mathbf{Z}_q, 1)$ in (2.5.14) is given by $-\log V\Lambda^3$. We show in the next subsection that this generalizes to any Q with $b_1(Q) = 0$.

A.2 $\mathcal{A}(Q, 1)$

Here we show how the quantity $\mathcal{K}(Q, 1)$ depends only on the volume of Q for any 3–manifold Q with $b_1(Q) = b_2(Q) = 0$.

The term $\mathcal{A}(Q, 1)$ defined in (2.5.1)-(2.5.2) involves the matrix A^k of change of basis from an integral basis for the k^{th} integer cohomology, i.e. a basis whose periods around cycles is 1, to a basis of harmonic k–forms of norm 1. Having fixed a metric on Q, let V be the volume of Q and let ϵ_{ijk} be the volume form on Q:

$$\int_Q \epsilon_{ijk}\, dx^i \wedge dx^j \wedge dx^k = V. \tag{A.20}$$

The two bases can be written as follows (we need only consider $k = 0, 3$ since the other cohomologies vanish):

$$
\begin{array}{c|c|c|c|}
 & \text{basis for cohomology} & \text{basis of norm 1} & \text{determinant of change of basis} \\
\hline
H^0 & 1 & \frac{1}{\sqrt{V}} & \frac{1}{\sqrt{V}} \\
\hline
H^3 & \frac{\epsilon_{ijk}}{V} & \frac{\epsilon_{ijk}}{\sqrt{V}} & \sqrt{V} \\
\hline
\end{array}
\tag{A.21}
$$

Therefore, we have

$$\mathcal{A}(Q,1) = \sum_{k=0}^{3} (-1)^k \log\det[A^k] = \log\frac{\det A^0}{\det A^3} = \log\left(\frac{1}{V}\right) \tag{A.22}$$

The volume V_Q scales as Λ^3, so

$$\mathcal{A}(q,1) = \sum_{k=0}^{3} (-1)^k \log\det[A^k] = -\log(V\Lambda^3) \ = -3/2\log(V_Q^{2/3}\Lambda^2)\ . \tag{A.23}$$

The corresponding coefficients that come up in the RGEs come from the sum (2.3.2) for one massless vector multiplet, which gives $-\frac{3}{2}\log\Lambda^2$. They cancel with the term for $\log\Lambda^3 = \frac{3}{2}\log\Lambda^2$ here.

B Generalization to breaking of $SU(p)$

Here, we show that for any breaking of an $SU(p)$ gauge group to $SU(m)\times SU(n)\times U(1)$, the threshold corrections ${\mathcal{S}_a}'$ are proportional.

The analog of (2.5.8) for $SU(p)$ breaking to the subgroup $SU(m)\times SU(n)\times U(1)$, with $m+n=p$, is

$$\underset{\rho_1}{(m^2-1,1,0)} \oplus \underset{\rho_2}{(1,n^2-1,0)} \oplus \underset{\rho_3}{(1,1,0)} \oplus \underset{\rho_4}{(m,\overline{n},-\frac{m+n}{mn})} \oplus \underset{\rho_5}{(\overline{m},n,\frac{m+n}{mn})}\ . \tag{B.1}$$

with normalization k_Y of the abelian factor given by $k_Y = 2(m+n)/mn$.

The threshold corrections are given by

$$\begin{aligned}
\mathcal{S}_Y &= 2\,\frac{(m+n)^2}{mn}\,\left(\mathcal{T}(Q,\omega_4)+\mathcal{T}(Q,\omega_5)\right)\,,\\
\mathcal{S}_n &= 2n\,\mathcal{K}(Q,1)+m\,\left(\mathcal{T}(Q,\omega_4)+\mathcal{T}(Q,\omega_5)\right)\,,\\
\mathcal{S}_m &= 2m\,\mathcal{K}(Q,1)+n\,\left(\mathcal{T}(Q,\omega_4)+\mathcal{T}(Q,\omega_5)\right)\,,
\end{aligned} \tag{B.2}$$

so after shifting by $\mathcal{S}_Y/k_Y$ we have

$$\begin{aligned}
{\mathcal{S}_Y}^- &= 0\,,\\
{\mathcal{S}_n}^- &= 2n\,\mathcal{K}(Q,1)-n\,\left(\mathcal{T}(Q,\omega_4)+\mathcal{T}(Q,\omega_5)\right)\,,\\
{\mathcal{S}_m}^- &= 2m\,\mathcal{K}(Q,1)-m\,\left(\mathcal{T}(Q,\omega_4)+\mathcal{T}(Q,\omega_5)\right)\,,
\end{aligned} \tag{B.3}$$

which are proportional, with ratio

$${\mathcal{S}_Y}^- : {\mathcal{S}_n}^- : {\mathcal{S}_m}^- = 0 : n : m\,, \tag{B.4}$$

which is just the ratio of the adjoint Casimirs of the groups, and matches the corresponding ratio of beta functions $b_Y : b_n : b_m = 0 : n : m$.

Note that all the statements in Section 2.5 generalize for this case. We just make the following additional replacements: in (2.5.11), replace e^{5iv_γ} by e^{piv_γ}; in (2.5.21) replace $10\pi w/q$ by $2p\pi w/q$; in (2.5.26), replace 5 by p; finally, replace (2.2.7) by

$$U_\gamma = \exp\left(\frac{2\pi i w}{q}\cdot \mathrm{diag}\{n,\ldots n,-m,\ldots,,-m\}\right), \tag{B.5}$$

with n appearing m times, $-m$ appearing n times. Similarly, the results in Section 2.6 generalize to this case.

Note that generalizing to other unified groups does not in general preserve unification.

Part II

Schwinger meets Kaluza-Klein

Chapter 1

Introduction and Summary

The classic study of the rate of creation of electron-positron pairs in a uniform, constant electric field was done more than 50 years ago in the seminal paper by Schwinger [46]. The concepts and methodology introduced in this work have had a lasting impact on the formal development of quantum field theory, and by now several alternative derivations of the effect have been invented (see e.g. [109, 47, 48, 51, 49, 50]).

Schwinger's predicted rate per unit time and volume is given by [46, 49, 50]

$$\mathcal{W}(E) \;=\; qE \int \frac{d^2k_i}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{1}{n} \exp\Big(-\frac{\pi n(m_e^2+k_i^2)}{|qE|}\Big) \tag{1.0.1}$$

for a spin 1/2 particle in 4 flat space-time dimensions, with m_e and q the electron mass and charge, k_i the transverse momenta, and E the electric field.[1]

In an earlier set of equally classic papers [1, 2], Kaluza and Klein introduced their unified description of general relativity and electromagnetism, in which charged particles

[1]The rate (1.0.1) is still very small for experimentally accessible electric fields. For the rate to be appreciable, the field must be very large, around $E_{crit} = 10^{16} eV/cm$. A static field of this magnitude is difficult to obtain in laboratories, largely because it is several orders of magnitude above the electric field that can be sustained by an atom, namely $10^8 eV/cm$. See [59] for a recent experiment that has obtained pair creation from oscillating electric fields, which were studied theoretically in [52], and see [60] for an upcoming experiment studying pair production from the low-frequency, Schwinger limit of such fields.

appear as quanta with non-zero quantized momentum around a compact extra dimension. It has of course always been clear that the charged Kaluza-Klein particles do not have the correct properties to represent electrons; most notably, the mass of the fundamental KK particles is equal to (or bounded below by) their charge, while for the electron the ratio m_e/q is about 10^{-21}! So in comparison with electrons, KK particles are either very heavy or have (in a large extra dimension scenario) an exceedingly small KK electric charge. Nonetheless, or rather, because of this fact, it is an interesting theoretical question whether it is at all possible, via an idealized Gedanken experiment, to pair produce KK particles by means of the Schwinger mechanism. As far as we know, this question has not been addressed so far in the literature, and probably for a good reason: it turns out to be a subtle problem! We will show that unlike the standard Schwinger pair creation effect, pair production of KK particles cannot be given the simple and rather intuitive interpretation of a tunneling mechanism.

Imagine setting up our Gedanken experiment as in figure 1, with two charged plates with a non-zero KK electric field in between. As seen from equation (1.0.1), to turn on the effect by any appreciable amount will require an enormous KK electric field, and since Kaluza-Klein theory automatically includes gravity, the backreaction of the E-field on space-time will need to be taken into account. The best analog of a constant electric field in this setting is the electric version of the Kaluza-Klein Melvin background; the magnetic version was studied recently in [110, 111, 112, 113, 114, 115, 116][2], and in [115, 116] the electric version also appeared. We will study some of the features of the electric KKM background in section 2. For our problem, the relevant properties of this background are that

(i) the background geometry depends on a longitudinal coordinate, which we will call ρ;

(ii) a gravitational acceleration $a(\rho)$, directed along the E-field, is included;

(iii) the total gravitational and electrostatic potential energy remains positive everywhere.

[2] In [112], pair production of Kaluza-Klein monopoles from the magnetic Kaluza-Klein Melvin solution was studied. See also [117] for a study of other aspects of the magnetic solutions.

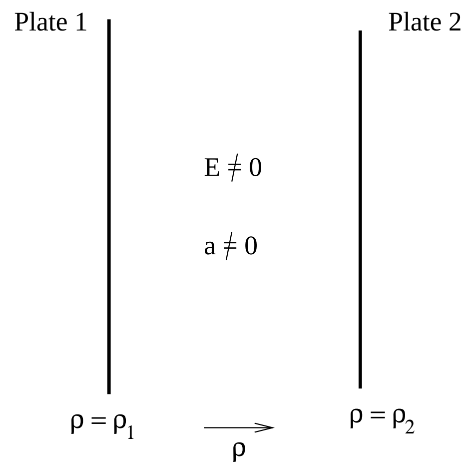

Fig. 1: *The Gedanken experiment we will imagine in this paper, with two charged plates at* $\rho = \rho_1$ *and* $\rho = \rho_2$*, producing a non-zero E-field in the intermediate region. The backreaction and the finite mass density of the plates results in a non-zero gravitational acceleration* a*.*

The first two properties are expected backreaction effects. The last property, however, implies that the negative electrostatic potential can never be made large enough to compensate for the positive contribution coming from the rest mass of the particles. The physical reason for this obstruction is that before one reaches the critical electro-static potential, backreaction effects will cause space-time itself to break down: if one would formally continue the solution beyond this point, the space-time develops closed time-like curves, which are known to be unphysical. In this way, gravity puts an upper limit on the potential difference one can achieve between the two plates in figure 1.

This result may look like an unsurmountable obstacle for pair creation, which is usually thought of as a tunneling effect by which particle pairs can materialize by using their electro-static energy to overcome their rest mass. The modern instanton method [47, 48, 51] of computing the pair creation rate, for example, crucially depends on this intuition. However, as mentioned in point (ii) above, it turns out that the backreaction necessarily implies that the vacuum state of the KK particles needs to be defined in the presence of a non-zero gravitational acceleration. As we will explain in Appendix A, the necessary presence of this acceleration can be thought of as due to the non-zero mass of the parallel plates that produce the KK electric field. Consequently, the Schwinger effect needs to be studied in conjunction with its direct gravitational analog, the equally famous Hawking-Unruh effect [53, 54, 118].

It has been recognized for some time that the Hawking-Unruh effect and Schwinger pair creation are rather closely related (see, for example, [49, 50]); both can be understood via a distortion of the vacuum, which may be parametrized by means of some appropriate Bogolyubov transformation that relates the standard energy eigenmodes to the new energy eigenmodes in the non-trivial background field. Also, like the Schwinger effect, the Hawking–Unruh effect has been thought of as a tunnelling mechanism and was derived as such recently [55]; see also [119] for a related study of de Sitter radiation.

By combining both the Schwinger and the Unruh effects we will obtain the following result for the pair creation rate of the Kaluza-Klein particles (which we will assume to be scalar particles) as a function of the electric field E and gravitational acceleration a

$$\mathcal{W}(E,a) = \frac{a^{3/2}}{2\pi^2} \int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \sum_q \Big(q^2 + \Lambda k_i^2\Big)^{1/4} \exp\Big[-2\pi\, \omega(a,q,k_i)\Big] \qquad (1.0.2)$$

where

$$\omega(a,q,k_i) = \frac{q^2 + k_i^2}{|\, \frac{1}{2}\, qE + a\sqrt{q^2 + \Lambda k_i^2}\, |}\,, \qquad \Lambda = 1 - \frac{E^2}{4a^2}\,. \qquad (1.0.3)$$

Here the summation is over the full KK tower of all possible charges $q = n/R$ with n integer and R the radius of the extra dimension, and a is the 'bare' acceleration, that the particles would experience with the E-field turned off. While E in this formula is a constant, a in fact depends on the longitudinal coordinate via $1/a = \rho +$ const. The potential energy in (1.0.3) is the manifestly positive quantity we referred to in property (iii) above. A more detailed explanation of the result (1.0.2) will be given in Section 4.

Since in our case mass equals charge the result (1.0.2) looks like a reasonable generalization of the classic result (1.0.1) of Schwinger and of Unruh [54, 49, 50]. In particular, if we turn off the E-field, our expression (1.0.2) reduces to the Boltzmann factor with Hawking-Unruh temperature $\beta = 2\pi/a$. Moreover, if we would allow ourselves to drop all terms containing the acceleration a, the result is indeed very similar to the dominant $n = 1$ term in Schwinger's formula (1.0.1). However, it turns out that in our case, the gravitational backreaction dictates that the acceleration a can not be turned off; rather, it is bounded from below by the electric field via

$$a > |E/2| \,. \tag{1.0.4}$$

Our formula (1.0.2) indeed breaks down when a gets below this value. So in particular, there is no continuous weak field limit in which our result reduces to Schwinger's answer. We will further discuss the physical interpretation of our result in the concluding section, where we will make a more complete comparison with the known rate [120] for Schwinger production in an accelerating frame.

This Part is organized as follows. In Chapter 2 we describe some properties of the electric Kaluza-Klein Melvin space-time. In Chapter 3 we study classical particle mechanics and wave mechanics in this background. Finally in Chapter 4, we set out to calculate the pair creation rate, using (and comparing) several methods of computation. Chapter 5 contains some concluding remarks. We discuss our experimental set-up in Appendix A, and in Appendix B we summarize the known result for Schwinger pair production in an accelerating frame.

Chapter 2

The Electric Kaluza-Klein-Melvin Space-Time

We start with describing the classical background of d+1-dimensional Kaluza-Klein theory, representing a maximally uniform KK electric field.

2.1 Definition of the Electric KKM Space-time

Consider a flat $d+1$ dimensional flat Minkowski space-time, with the metric

$$ds^2 = -dt^2 + dx^2 + dy_i dy^i + dx_{d+1}^2, \tag{2.1.1}$$

with $i = 2, \ldots, d-1$. From this we obtain the electric Kaluza-Klein-Melvin space-time by making the identification

$$\begin{pmatrix} t \\ x \\ y_i \\ x_{d+1} \end{pmatrix} \longrightarrow \begin{pmatrix} t' \\ x' \\ y'_i \\ x'_{d+1} \end{pmatrix} = \begin{pmatrix} \gamma\,(t-\beta x) \\ \gamma\,(x-\beta t) \\ y_i \\ x_{d+1} + 2\pi R \end{pmatrix}, \tag{2.1.2}$$

with $\gamma^2(1-\beta^2) = 1$. This geometry can be viewed as a non-trivial Kaluza-Klein background in d dimensions, in which the standard periodic identification $x_{d+1} \equiv x_{d+1} + 2\pi R$ of the extra dimension is accompanied by a Lorentz boost in the x-direction. Since the $d+1$ dimensional space-time is flat everywhere, and the identification map (2.1.2) is an isometry, it is evident that the electric Melvin background solves the equation of motion of the Kaluza-Klein theory. As we will describe momentarily, from the d-dimensional point of view, it looks like a non-trivial background with a constant non-zero electric field E and with, as a result of its non-zero stress-energy, a curved space-time geometry. Here the electric field E is related to the boost parameters β and γ by

$$\beta = \tanh(\pi R E)\,, \qquad \gamma = \cosh(\pi R E). \tag{2.1.3}$$

The map (2.1.2) represents a proper space-like identification, for which

$$-(t'-t)^2 + (x'-x)^2 + (x'_{d+1} - x_{d+1})^2 = (2\pi R)^2 - (2\gamma - 2)(x^2 - t^2) > 0 \tag{2.1.4}$$

provided we restrict to the region

$$\rho < \frac{\pi R}{\sinh\frac{\pi E R}{2}}\,, \qquad \rho^2 \equiv x^2 - t^2. \tag{2.1.5}$$

where we used (2.1.3). Outside of this regime, the electric Melvin space-time contains closed time-like curves. We will exclude this pathological region from our actual physical set-up. [1]

[1] There is also a different notion of the electric version of the KK Melvin space-time, which is obtained by applying an electro-magnetic duality transformation $F \to e^{2\sqrt{3}\phi} * F$, $\phi \to -\phi$ to the magnetic KK Melvin space-time [121]. This background looks like an electric flux-tube in a $U(1)$ gauge theory with an electric coupling constant e that diverges at large transverse distance from the flux-tube (due to the fact that the size of the extra dimension shrinks at large distance). Putting a reasonable physical upper bound on the size of e restricts the maximal allowed length of the flux tube, suggesting that the obstruction against creating an arbitrarily large electro-static potential may be more general than only for the type of backgrounds studied in this paper.

2.2 Classical Trajectories

As a first motivation for the identification of E with the KK electric field, it is instructive to consider classical trajectories in this space-time. This is particularly easy, since in flat $d+1$ Minkowski space, freely moving particles move in straight lines:

$$x = x_0 + p_1 s\,, \qquad x^- = t_0 + p_0 s\,, \qquad y_i = k_i s\,, \qquad x_{d+1} = qs\,. \tag{2.2.1}$$

Assuming the particle is massless in $d+1$-dimensions, we have

$$p_0 = \sqrt{p_1^2 + k_i^2 + q^2} \tag{2.2.2}$$

which is the mass-shell relation of a d-dimensional particle with mass equal to q. Let's introduce coordinates ρ and τ via

$$x = \rho\,\cosh(\tau - \tfrac{1}{2}Ex_{d+1})\ , \qquad t = \rho\,\sinh(\tau - \tfrac{1}{2}Ex_{d+1}). \tag{2.2.3}$$

and coordinates X and T by

$$X = \rho\,\cosh\tau \qquad T = \rho\,\sinh\tau. \tag{2.2.4}$$

The identification (2.1.2) in the new coordinates becomes

$$\begin{pmatrix} T \\ X \\ y_i \\ x_{d+1} \end{pmatrix} \longrightarrow \begin{pmatrix} T \\ X \\ y_i \\ x_{d+1} + 2\pi R \end{pmatrix}, \tag{2.2.5}$$

which is the standard Kaluza-Klein identification. The trajectory in terms of these is:

$$\begin{aligned} X &= (x_0 + p_1 s)\cosh(\tfrac{1}{2}Eqs) + (t_0 + p_0 s)\sinh(\tfrac{1}{2}Eqs) \\ T &= (t_0 + p_0 s)\cosh(\tfrac{1}{2}Eqs) + (x_0 + p_1 s)\sinh(\tfrac{1}{2}Eqs) \end{aligned} \tag{2.2.6}$$

Considering a particle at rest at the origin $x_0 = 0$ and $t_0 = 0$, we find

$$\frac{d^2X}{dT^2} = \frac{qE}{p_0}\ . \tag{2.2.7}$$

This is the expected acceleration of a particle with charge and rest-mass q.

2.3 Kaluza-Klein Reduction

Let us now perform the dimensional reduction to d dimensions. Using the coordinates ρ and τ defined in (2.2.3) the $d+1$ dimensional metric becomes

$$ds^2 = -\rho^2(d\tau + \tfrac{1}{2}\, E\, dx_{d+1})^2 + d\rho^2 + dy_i dy^i + dx_{d+1}^2, \tag{2.3.1}$$

while the identification (2.1.2) simplifies to a direct periodicity in x_{d+1} with period $2\pi R$, leaving (ρ, τ, y_i) unchanged. We may rewrite the metric (2.3.1) as

$$ds^2 = -\frac{\rho^2}{\Lambda}\, d\tau^2 + d\rho^2 + dy_i dy^i + \Lambda\Big(dx_{d+1} - \frac{E\rho^2}{2\Lambda}\, d\tau\Big)^2 \tag{2.3.2}$$

with

$$\Lambda \equiv 1 - \tfrac{1}{4}\, E^2\rho^2. \tag{2.3.3}$$

In this form, we can readily perform the dimensional reduction.

The d dimensional low energy effective theory is described by the Einstein-Maxwell theory coupled to the Kaluza-Klein scalar V via

$$S = \int \sqrt{-g_d}\, \Big(V^{1/2} R_d + \frac{1}{4}\, V^{3/2} F_{\mu\nu} F^{\mu\nu}\Big). \tag{2.3.4}$$

Here the d-dimensional fields are obtained from the $d+1$ metric via the decomposition

$$ds_{d+1}^2 = ds_d^2 + V(dx_{d+1} + A_\mu dx^\mu)^2 \ . \tag{2.3.5}$$

Comparing (2.3.5) and (2.3.2) gives the dimensionally reduced form of the electric Melvin background

$$ds_d^2 \ = \ -\frac{\rho^2}{\Lambda} d\tau^2 + d\rho^2 + dy_i dy^i \ , \tag{2.3.6}$$

$$A_0 \ = \ \frac{E\rho^2}{2\Lambda}\, , \qquad\qquad V \ = \ \Lambda \ \equiv \ 1 - \tfrac{1}{4}\rho^2 E^2 \ . \tag{2.3.7}$$

It describes a curved space-time, together with an electric field in the ρ-direction given by

$$E_\rho \ \equiv \ \sqrt{g^{00}}\, \partial_\rho A_0 \ = \ \frac{E}{\Lambda^{3/2}}. \tag{2.3.8}$$

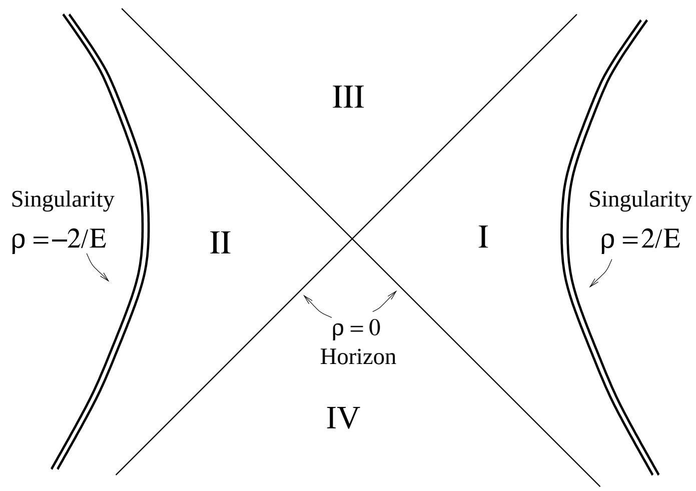

Fig. 2: *The electric KK Melvin space-time divides up into four regions. Regions I and II are static regions, while regions III and IV are time-dependent.*

This electric field is equal to E at $\rho = 0$, but diverges at $\rho = 2/E$; this singular behavior is related to the mentioned fact that outside the region (2.1.5), the identification map (2.1.2) becomes time-like and produces closed time-like curves. Note, however, that the location of the divergence in E_ρ slightly differs from the critical value noted in (2.1.5), but coincides with it in the limit of small ER.

The d dimensional metric in (2.3.6) reduces for $E = 0$ to the standard Rindler space-time metric. For finite E there is a non-zero gravitational acceleration

$$a_\rho(\rho) \equiv g^{00}\partial_\rho g_{00} = \frac{1}{\rho\Lambda}, \tag{2.3.9}$$

which includes the gravitational backreaction due to the stress-energy contained in the electric field. Notice that $a(\rho)$ diverges at $\rho = 2/E$.

The above static Rindler type coordinate system will be most useful for the purpose of providing a background with a static KK electric field. To obtain a more global perspective of the full electric KK Melvin space-time, we can use the coordinates X and T defined in equation (2.2.4). In this coordinate system, the solution looks like

$$
\begin{aligned}
ds^2 &= -dT^2 + dX^2 - \frac{E^2}{4\Lambda}(XdT - TdX)^2 \, + \, dy_i dy^i \,, \\
A_0 &= \frac{EX}{2\Lambda}\,, \qquad A_1 \, = -\frac{ET}{2\Lambda}\,, \\
V &= \Lambda\,, \qquad \Lambda \equiv 1 - \tfrac{1}{4}E^2(X^2 - T^2)\,.
\end{aligned}
\tag{2.3.10}
$$

In this coordinate system we can distinguish four different regions:

$$
\begin{aligned}
&\text{Region I} \; : \quad X > |T|\,, \qquad && \text{Region II} \; : \quad X < -|T|\,, \\
&\text{Region III} : \quad T > |X|\,, \qquad && \text{Region IV} : \quad T < -|X|\,.
\end{aligned}
$$

Regions I and II are static regions (that is, they admit a time-like Killing vector) and are analogous to the left and right wedges of Rindler space. They are separated by a "horizon" (as seen only by static observers at ρ = const.) at $X^2 = T^2$ from two time-dependent regions III and IV. (See figure 2) We will mostly dealing with the physics of region I. For a discussion of the physics in region III, see [115, 116].

2.4 Physical Boundary Conditions

In order to have in mind a physical picture of the part of this spacetime that we will be studying, we recall the Gedanken experiment as shown in figure 1, in which two charged plates produce a *static* KKM electric field between them. As explained in detail in Appendix A, the space-time between the two plates will correspond to a finite interval within region I:

$$
\rho_1 < \rho < \rho_2\,, \qquad \text{with} \quad 0 < \rho_1 < \rho_2 < |2/E|. \tag{2.4.1}
$$

By concentrating on the physics within this region, our physical set-up will automatically exclude the unphysical regime with the closed time-like curves, as well as the horizon at $\rho = 0$. The details of this set-up are given in Appendix A.

Chapter 3

Particle and Wave Mechanics

3.1 Particle Mechanics

In this section we consider the classical mechanics of charged particles in the electric KK-Melvin space-time, deriving the expression for the total gravitational and electrostatic potential energy. This discussion will be useful later on when we consider the quantum mechanical pair production.

3.1.1 Classical Action

The classical action for a massless particle in $d+1$ dimensions is

$$S_{d+1} = \int ds \left[p_{_M} \dot{x}^{^M} + \lambda \left(G^{^{MN}} p_{_M} p_{_N} \right) \right], \tag{3.1.1}$$

where $M, N = 0, \ldots, d$ and λ denotes the lagrange multiplier imposing the zero-mass-shell condition $G^{^{MN}} p_{_M} p_{_N} = 0$. Upon reduction to d dimensions, using the general Kaluza-Klein Ansatz (2.3.5), for which

$$G^{^{MN}} = \begin{pmatrix} g^{\mu\nu} & -A^{\nu} \\ -A^{\mu} & V^{^{-1}} + A_{\mu} A^{\mu} \end{pmatrix}, \tag{3.1.2}$$

where $\mu, \nu = 0, \ldots, d-1$, the action (3.1.1) attains the form (here we drop the x_{d+1}-dependence)

$$S_d = \int ds \Big[p_\mu \, \dot{x}^\mu \; + \; \lambda \Big(g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) + \frac{q^2}{V} \Big) \Big]. \tag{3.1.3}$$

Here we identified $q = p_{d+1}$. The λ equation of motion gives

$$g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) + \frac{q^2}{V} = 0\,. \tag{3.1.4}$$

This is the constraint equation of motion of a particle with charge q and a (space-time dependent) mass $m = \frac{q}{\sqrt{V}}$. For the electric KK Melvin background (2.3.6), the constraint (3.1.4) takes the form

$$-\frac{\Lambda}{\rho^2}\Big(p_\tau + \frac{qE\rho^2}{2\Lambda}\Big)^2 + p_\rho^2 + p_i^2 + \frac{q^2}{\Lambda} = 0, \tag{3.1.5}$$

or

$$-\frac{p_\tau^2}{\rho^2} + (q - \tfrac{1}{2}Ep_\tau)^2 + p_\rho^2 + p_i^2 = 0. \tag{3.1.6}$$

Since the background is independent of all coordinates except ρ, all momenta are conserved except p_ρ. Let us denote these conserved quantities by

$$p_\tau = \omega, \qquad p_i = k_i. \tag{3.1.7}$$

The constraint (3.1.6) allows us to solve for p_ρ in terms of the conserved quantities as

$$p_\rho \, = \, \pm\sqrt{(\omega^2/\rho^2) \, - \mu^2}, \qquad \mu^2 \equiv \, k_i^2 + (q - \tfrac{1}{2}E\omega)^2. \tag{3.1.8}$$

Using this expression for p_ρ, we can write the total action of a given classical trajectory purely in terms of its beginning and endpoints as

$$S_\pm(x_2, x_1) \, = \, \omega \, \tau_{21} + \, k_i \, y_{21}^i \, \pm \int_{\rho_1}^{\rho_2} d\rho \, \sqrt{(\omega^2/\rho^2) \, - \mu^2}. \tag{3.1.9}$$

Performing the integral gives

$$S_\pm(x_2, x_1) = S_\pm(x_2) - S_\pm(x_1)\,, \tag{3.1.10}$$

with

$$S_{\pm}(\{\rho,\tau,y\}) = k_i\, y^i \,+\, \omega\,(\tau \pm\; \mathcal{T}_0(\rho,k_i,\omega)\,)\,, \tag{3.1.11}$$

where

$$\mathcal{T}_0(\rho,k_i,\omega) \;=\; \sqrt{1-\left(\tfrac{\mu\rho}{\omega}\right)^2} \;-\; \log\!\left[\frac{\omega}{\mu\rho}\left(1+\sqrt{1-\left(\tfrac{\mu\rho}{\omega}\right)^2}\,\right)\right]. \tag{3.1.12}$$

This result will become useful in the following.

Notice that, for given radial location ρ, the classical trajectory only crosses this location provided the energy ω satisfies $\omega \geq \mu\rho$ with μ as defined in (3.1.8). The physical meaning of the quantity $\mathcal{T}_0$ in (3.1.12) is that it specifies the (time difference between the) instances $\tau = \pm\mathcal{T}_0$ at which the trajectory passes through this radial location. Notice that indeed $\mathcal{T}_0 = 0$ when $\omega = \mu\rho$, indicating that at this energy, ρ is the turning point of the trajectory.

3.1.2 Potential Energy

We can use the mass-shell constraint (3.1.5) to solve for the total energy

$$H \equiv p_\tau = \frac{\rho}{\Lambda}\sqrt{\Lambda(p_\rho^2+k_i^2)\,+\,q^2} \,-\, \frac{qE\rho^2}{2\Lambda}. \tag{3.1.13}$$

The corresponding Hamilton equations

$$\partial_\tau\rho = \frac{\partial H}{\partial p_\rho}\,, \qquad \partial_\tau p_\rho = -\frac{\partial H}{\partial \rho}\,, \tag{3.1.14}$$

determine the classical trajectory $\rho(\tau)$. An important quantity in the following will be the potential energy $\mathcal{W}(\rho,q,k_i)$, defined via

$$\mathcal{W}(\rho,q,k_i) \;\equiv\; H(p_\rho=0) \;=\; \frac{\rho}{\Lambda}\sqrt{q^2+\Lambda k_i^2} \;-\; \frac{qE\rho^2}{2\Lambda}. \tag{3.1.15}$$

That is, $\mathcal{W}(\rho,q,k_i)$ is the energy of particles, with $p_{d+1}=q$ and transverse momentum $p_i = k_i$, that have their turning point at ρ.

As expected, the potential energy $\mathcal{W}(\rho)$ contains two contributions: the first term is the gravitational energy due to the rest mass and momentum of the particle, and the

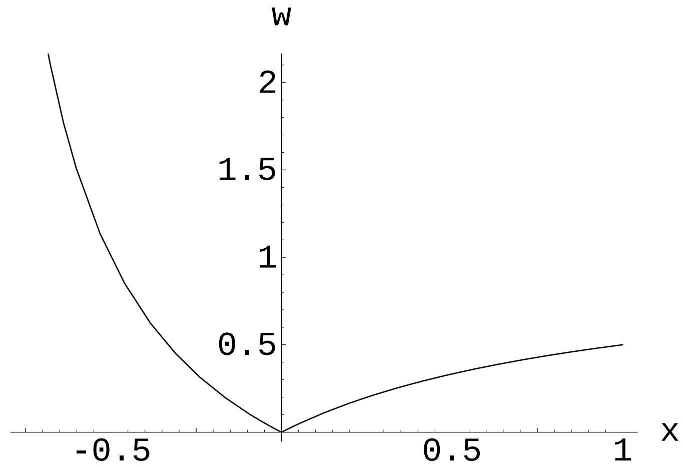

Fig. 3 *The effective potential $\omega(\rho)$ defined in equation (3.1.15)* (for $k_i = 0$, and multiplied by E) *as a function of $x = \frac{1}{2}qE\rho$, with $q = \pm 1$.*

second term represents the electro-static potential. If qE is positive, this last term makes the particle effectively lighter than its gravitational energy. The total energy for any qE, however, never becomes negative. For qE negative, the expression (3.1.15) is manifestly positive. For qE either positive or negative, it can be rewritten as

$$\omega(\rho, q, k_i) = \frac{q^2 + k_i^2}{|\,\frac{1}{2}qE\ + \rho^{-1}\sqrt{q^2 + \Lambda k_i^2}\,|}\,, \tag{3.1.16}$$

which is again manifestly positive. We have plotted this function for $k_i^2 = 0$ in figure 3.

This behavior of the potential energy $\omega(\rho, q, k_i)$ should be contrasted with the classical electrostatic case, where $V(\rho, q) = m - qE\rho$ with m the rest-mass, in which case the particle *can* get a negative total energy. When going to single particle wave mechanics, this negative energy leads to the famous Klein paradox, and upon second quantization, to the Schwinger pair creation effect. Since in our case the potential remains positive, there is no Klein paradox and no immediate reason to expect a vacuum instability. Nonetheless, as we will see shortly, pair creation will take place.

Finally, we note that in the concrete set-up of Situation I of our Gedanken apparatus in Appendix A, the particles are in fact restricted to move within the region $\rho_1 < \rho < \rho_2$ between the two plates. To complete the dynamical rules of the model, we need to specify what happens when the particle reaches the plates; we will simply assume reflecting boundary conditions.

3.2 Wave Mechanics

In this section we write the solutions to the wave equations in the electric KK-Melvin background, and illustrate the semi-classical correspondence with the classical mehanics.

3.2.1 Wave Equations

The $d+1$-dimensional wave equation in the background (2.3.1) is

$$\frac{1}{\sqrt{-G}}\,\partial_M\Big(\sqrt{-G}G^{MN}\partial_N\Phi\Big) = \left[\frac{1}{\rho}\,\partial_\rho(\rho\,\partial_\rho) - \frac{1}{\rho^2}\,\partial_\tau^2 + \,\partial_i^2\, + \,\Big(\partial_{d+1} + \tfrac{1}{2}E\partial_\tau\Big)^2\right]\Phi = 0, \tag{3.2.1}$$

subject to the perioding boundary condition in the x_{d+1} direction with period $2\pi R$. For a given eigenmode with $q \equiv p_{d+1} = \frac{n}{R}$, we can reduce the wave equation to d dimensions, where it can be written in the form

$$\left(\frac{\sqrt{\Lambda}}{\rho}\,\partial_\rho\Big(\frac{\rho}{\sqrt{\Lambda}}\,\partial_\rho\Big) \;-\; \frac{\Lambda}{\rho^2}\Big(\partial_\tau + \frac{iqE\rho^2}{2\Lambda}\Big)^2 \;+\,\partial_i^2\;-\;\frac{q^2}{\Lambda}\right)\Phi = 0\,. \tag{3.2.2}$$

Here we recognize the conventional wave equation

$$\frac{1}{\sqrt{-g}}D_\mu\left(\sqrt{-g}g^{\mu\nu}D_\nu\Phi\right) - M^2\Phi = 0 \tag{3.2.3}$$

of a d dimensional charged particle with charge q in the background (2.3.6) and with a position dependent mass equal to $M^2 = q^2/\Lambda$. Note the direct correspondence af the above wave equations with the classical equations (3.1.5) and (3.1.6). They need to be solved subject to the boundary conditions imposed by our physical set-up. In case of Situation I, see figure 5 in Appendix A, we will choose to impose Dirichlet boundary conditions at the two plates

$$\Phi \mid_{\rho=\rho_1} = \Phi \mid_{\rho=\rho_2} = 0\,. \tag{3.2.4}$$

3.2.2 Mode Solutions

The $d+1$-dimensional wave equation is solved by

$$\Phi_{qk\omega} = e^{ix_{d+1}(q-\frac{1}{2}E\omega)+ik_i y^i + i\omega\tau} K(\omega, \mu\rho)\,, \tag{3.2.5}$$

with μ as defined in (3.1.8), and where $K(\omega, \mu\rho)$ solves the differential equation

$$\Big((\rho\partial_\rho)^2 + \omega^2 + \mu^2\rho^2\Big)K(\omega, \mu\rho) = 0. \tag{3.2.6}$$

The solution K has the integral representation

$$K(\omega, \mu\rho) = \int_{-\infty}^{\infty} d\sigma\; e^{i\omega\,\sigma - i\mu\rho\sinh\sigma}\,, \tag{3.2.7}$$

and can be expressed in terms of standard Bessel and Hankel functions [122, 123]. The functions $K(\omega, \mu\rho)$ are defined for arbirary real ω. However, upon imposing the boundary conditions that $K(\omega, \mu\rho_i) = 0$ at the location of the two plates, we are left with only a discrete set of allowed frequencies ω_ℓ. Since the corresponding mode functions (3.2.7) form a complete basis of solutions to (3.2.6), they satisfy an orthogonality relation of the form

$$\int_{\rho_1}^{\rho_2} d\rho\,\rho\, K^*(\omega_\ell, \mu_\ell\rho) K(\omega_j, \mu_j\,\rho) = f(\omega_\ell)\delta_{\ell,j}\,, \tag{3.2.8}$$

where $f(\omega)$ some given function that depends on ρ_1 and ρ_2.

For large ω and $\mu\rho$, we can approximate the integral in (3.2.7) using the stationary phase approximation. The stationary phase condition $\omega = \mu\rho \cosh \sigma$ has two solutions

$$\sigma_\pm \;=\; \pm \log\left[\frac{\omega}{\mu\rho}\Big(1+\sqrt{1-\big(\tfrac{\mu\rho}{\omega}\big)^2}\;\Big)\right] \tag{3.2.9}$$

provided $|\omega| > \mu\rho$, leading to

$$K(\omega,\mu\rho) \;\simeq\; \frac{\sqrt{2\pi}\;\cos\Big(\omega\mathcal{T}_0(\rho)+\frac{\pi}{4}\Big)}{\sqrt{w}\;\sqrt[4]{1-\big(\frac{\mu\rho}{\omega}\big)^2}}\,, \qquad\qquad |\omega| > \mu\rho\,, \tag{3.2.10}$$

with $\mathcal{T}_0$ as given in equation (3.1.12). This formula is accurate for energies ω larger than the potential energy $\mathcal{W}(\rho)$. For smaller energies there is no saddle-point and the function $K(\omega, \mu\rho)$ is exponentially small

$$K(\omega,\mu\rho) \;\simeq\; \sqrt{\frac{\pi}{\mu\rho}}\;e^{-\mu\rho}\,, \qquad\qquad |\omega| \ll \mu\rho\;. \tag{3.2.11}$$

reflecting the fact that the corresponding classical trajectory has its turning point before reaching ρ.

Notice that, upon inserting (3.2.10), the full mode function $\Phi_{qk\omega}$ in (3.2.5) can be written as a sum of two semi-classical contributions

$$\Phi_{qk\omega}(x) \;\sim\; \sum_{\pm} e^{ix_{d+1}(q+\frac{1}{2}E\omega)+ik_iy^i+i\omega\,(\tau\pm\mathcal{T}_0(\rho,k,\omega))} \;\sim\; \sum_{\pm} e^{iS_\pm(y,\tau,\rho)}\,, \tag{3.2.12}$$

corresponding to the left- and right-moving part of the trajectory, respectively.

Chapter 4

Pair Creation

In this section we will compute the pair creation rate of the Kaluza-Klein particles, following three different (though related) methods. We will start with the simplest method, by looking for Euclidean "bounce" solutions. We then proceed with a more refined method of computation, more along the lines of Schwinger's original calculation, producing the non-trivial result quoted in the introductory section. Finally, we show that the obtained result can naturally be interpreted by considering the Hawking-Unruh effect, and we use the method of Bogolyubov transformations to compute the expectation value of the charge current.

4.1 Classical Euclidean Trajectories

Assuming that, in spite of the fact that the effective potential (3.1.15) seems to suggest otherwise, the nucleation of the charged particle pairs can be viewed as the result of a quantum mechanical tunneling process, we compute the rate by considering the corresponding Euclidean classical trajectory. The analytic continuation of the electric KK

Melvin space-time to Euclidean space is

$$ds^2_E \;=\; \frac{\rho^2}{\Lambda_E}\, d\theta^2 + d\rho^2 + dy_i dy^i + \Lambda_E\Big(dx_{d+1} - \frac{E\rho^2}{2\Lambda_E}\, d\theta\Big)^2, \qquad (4.1.1)$$

$$A_\theta \;=\; \frac{E\rho^2}{2\Lambda_E}, \qquad\qquad V = \Lambda_E \;,$$

with θ a periodic variable with period 2π, and

$$\Lambda_E \;\equiv\; 1 + E^2\rho^2/4\,. \qquad (4.1.2)$$

This Euclidean geometry is obtained from the Lorentzian electric KK Melvin solution via the replacement

$$E \to iE\,, \qquad\qquad t \to -i\theta, \qquad (4.1.3)$$

and coincides with the space-like section of the magnetic KK Melvin space-time. Unlike the Lorentzian version, this Euclidean space-time extends over the whole range of positive ρ values and ends smoothly at $\rho = 0$, by virtue of the periodicity in θ. This is standard for Euclidean cousins of space-times with event horizons, and a first indication that quantum field theory in the space-time naturally involves physics at a specific finite temperature.

The Euclidean action of a point-particle, with charge (momentum in the d+1-direction) equal to $p_{d+1} = q$ and mass $M = |q|/\sqrt{\Lambda_E}$ moving in this background reads

$$S_E \;=\; \int ds\, \mathcal{L}_E\;, \qquad \mathcal{L}_E = \frac{|q|}{\sqrt{\Lambda_E}}\sqrt{\frac{\rho^2\dot\theta^2}{\Lambda_E} + \dot\rho^2 + \dot y_i^2} \;-\; \frac{qE\rho^2}{2\Lambda_E}\,\dot\theta. \qquad (4.1.4)$$

As a first step, let us look for closed circular classical trajectories at constant ρ and y_i. The above point-particle action then reduces to

$$S_E(\rho) \;=\; \frac{\pi}{\Lambda_E}\,(2|\,q|\,\rho \,-\, qE\rho^2). \qquad (4.1.5)$$

The first term is the energy of a static particle times the length of the orbit, and the second term is the interaction with the background field times the area of the loop. Looking for an extremum yields one real and positive solution

$$|E|\rho = 2\sqrt{2} - 2\cdot \mathrm{sign}(qE) \qquad (4.1.6)$$

with total action

$$S_{\scriptscriptstyle E} = \frac{2\pi|q|}{|E|}\,(\sqrt{2} - \mathrm{sign}(qE)). \tag{4.1.7}$$

The existence of these solutions with finite Euclidean action is a first encouraging sign that pair creation may take place after all. The answer (4.1.5) for the Euclidean action also looks like a rather direct generalization of the standard semiclassical action for the Schwinger effect, and it is therefore tempting to conclude at this point that the total pair creation rate is proportional to

$$e^{-S_E} \;=\; e^{-2\pi|q|\,(\sqrt{2}-\mathrm{sign}(qE))/|E|}, \tag{4.1.8}$$

which looks only like a numerical modification of the classic result (1.0.1). This conclusion is somewhat premature, however, since in particular the pair creation rate should depend on ρ. We would like to determine this ρ-dependence.

For this, we take a second step and consider closed Euclidean trajectories that are not necessarily circular. As in section 3.1, we now go to a Hamiltonian formulation. To transform the formulas in section 3.1 to the Euclidean set-up, we need to make, in addition to (4.1.3), the following replacements

$$s \to -is\,, \qquad p_\rho \to ip_\rho, \qquad p_i \to ip_i\,, \qquad q \to iq\,. \tag{4.1.9}$$

In this way we obtain from (3.1.13) a Euclidean Hamiltonian

$$H_{\scriptscriptstyle E} \equiv p_\theta = \frac{\rho}{\Lambda_{\scriptscriptstyle E}}\sqrt{|q|^2 - \Lambda_{\scriptscriptstyle E}(p_\rho^2 + k_i^2)} \,-\, \frac{qE\rho^2}{2\Lambda_{\scriptscriptstyle E}}\,, \tag{4.1.10}$$

that generates the motion of particle as a function of the Euclidean time θ, and a corresponding potential energy

$$\omega_{\scriptscriptstyle E}(\rho, k_i, q) \equiv -H_{\scriptscriptstyle E}(p_\rho = 0) = -\frac{\rho}{\Lambda_{\scriptscriptstyle E}}\sqrt{|q|^2 - \Lambda_{\scriptscriptstyle E} k_i^2} \,+\, \frac{qE\rho^2}{2\Lambda_{\scriptscriptstyle E}}\,. \tag{4.1.11}$$

In addition to a change in sign, which is the standard way in which a potential changes when going to Euclidean space, this Euclidean potential differs from (3.1.15) via the

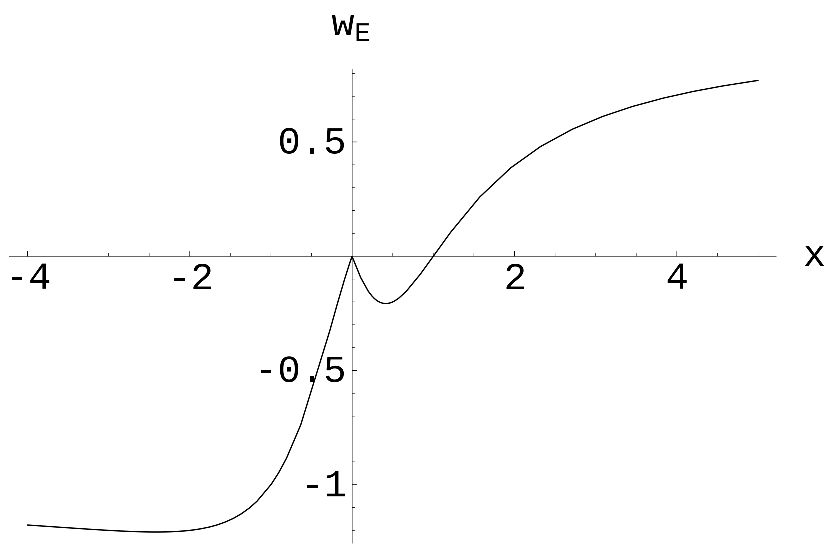

Fig. 4 *The Euclidean effective potential* $\omega_{_E}(\rho)$ *defined in equation (4.1.11)* (for $k_i = 0$, and multiplied by E) *as a function of* $x = \frac{1}{2}qE\rho$*, with* $q = \pm 1$.

replacement $\Lambda \rightarrow \Lambda_{_E}$. We have drawn $\omega_{_E}$ for $k_i = 0$ in figure 4. Note that $\omega_{_E}$ for $k_i = 0$ is proportional to the reduced effective action (4.1.5) for circular trajectories, and the critical radii (4.1.6) reside at the two minima in figure 4.

Our goal is to obtain semi-classical estimate for the pair creation rate at some given ρ. How should we use this Euclidean potential for this purpose? As seen from figure 4, there is a range of Euclidean energies $H_{_E}$ around the two minima (4.1.6) for which there exist stable, compact orbits. These orbits have a maximal and minimal radius, ρ_+ and ρ_-, at which $H_{_E} = \omega_{_E}(\rho_\pm)$. The idea now is to associate to a given ρ the corresponding Euclidean trajectory for which ρ equals one of these extrema $\rho_\pm$, and then use the total action $S_{_E}(\rho)$ for this trajectory to get a semi-classical estimate of the pair creation rate via

$$\mathcal{W}(\rho) \simeq e^{-S_E(\rho)}\,. \tag{4.1.12}$$

Here it is understood that in $S_E(\rho)$ we undo the rotation $E \to iE$, so that $\Lambda_E \to \Lambda$. Equation (4.1.12) is then a clear and unambiguous formula, provided the classical orbit is closed.

In general, however, the orbits need not be closed: the period of oscillation does not need to be 2π or even a fraction or multiple thereof. How should we *define* the total classical action, to be used in (4.1.12) for such a trajectory?

Our proposal, that perhaps may look *ad hoc* at this point but will be confirmed and justified in the subsequent subsections, is to take for S_E the total action averaged over one full rotation period of 2π. Concretely, suppose that the compact trajectory has an "oscillation period" θ_0, in which it goes through a full oscillation starting and returning to its maximal radial position $\rho{=}\rho_+$. We then define $S_E(\rho)$ as:

$$S_E(\rho) \;\equiv\; \lim_{\theta\to\infty} \frac{2\pi}{\theta} \int_0^{\theta} d\theta\, \mathcal{L}_E(\theta,\rho) \;=\; \frac{2\pi}{\theta_0} \int_0^{\theta_0} d\theta\, \mathcal{L}_E(\theta,\rho). \tag{4.1.13}$$

With this definition, and using the results in Section 3.1.1, we can now easily evaluate $S_E(\rho)$. From (the Euclidean analog of) equation (3.1.11), while noting that $\mathcal{T}_0(\rho) = 0$ since ρ is the turn-around point, we obtain

$$S_E(\rho) = 2\pi\mathcal{W}(\rho, q, k_i), \tag{4.1.14}$$

with $\mathcal{W}$ as given in (3.1.15). Here we made the replacement $E \to iE$, as prescribed.

The result (4.1.14) together with (4.1.12) gives our proposed semi-classical estimate of the pair creation rate as a function of ρ. Clearly, the derivation as presented thus far needs some independent justification. It also leaves several open questions. In particular, it is not clear how we should interpret the Euclidean "bounce" solutions, given the fact that the real effective potential (3.1.15) doesn't seem to lead to any tunneling. A better understanding of the physics that leads to the pair creation seems needed. In the next two subsections we will present two slightly more refined derivations of the rate, which will help answer some of these questions.

4.2 Sum over Euclidean Trajectories

We will now evaluate the pair creation rate, per unit time and volume, by means of the path-integral. Since we expect that this rate will be a function of longitudinal position ρ, we would like to express the final result as an integral over ρ. We start from the sum over all Euclidean trajectories

$$\mathcal{W} \;=\; \int \mathcal{D}p\,\mathcal{D}x\,\exp\Bigl(-\frac{1}{\hbar}S[p,x]\Bigr) \qquad (4.2.1)$$

defined on flat d+1-dimensional space with metric and periodicity condition

$$ds_E^2 = dx^* dx + dy_i dy^i + dx_{d+1}^2, \qquad (4.2.2)$$

$$(x, x^*, y_i,\, x_{d+1}) \;\equiv\; (e^{i\pi ER}x,\, e^{-i\pi ER}x^*,\, y_i,\, x_{d+1} + 2\pi R)\,. \qquad (4.2.3)$$

In the end we intend to rotate back to Lorentzian signature, replacing $E \to iE$.

We can read the expression (4.2.1) as a trace over the quantum mechanical Hilbert space of the single particle described by the action (3.1.1) or (3.1.3). The idea of the computation is to write this as a sum over winding sectors around the $d+1$-th direction. For each winding number w, the closed path is such that the end-points are related via a rotation in the (x, x^*)-plane over an angle $w\pi ER$. Using this insight, we can write (4.2.1) as

$$\int d^d x\, \mathcal{W}(x) \;=\; R\int_0^\infty \frac{dT}{T}\sqrt{\frac{2\pi}{T}}\,\sum_w e^{-\frac{1}{2T}(2\pi Rw)^2}\,\mathrm{Tr}\Bigl[\,e^{\pi i wERJ}e^{-\frac{T}{2}(p^*p + p_i^2)}\Bigr] \qquad (4.2.4)$$

where T denotes the Schwinger proper time variable, and where J denotes the rotation generator in the (x, x^*) plane. The exponent in front of the the trace is the d+1-dimensional part of the classical action of the trajectory with winding number w. To compute the trace, we write it as an integral over mixed position and momentum eigen states

$$\mathrm{Tr}\, A \;=\; \int d^2x \int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \;\; \langle\, x, k_i \,|\, A \,|\, x, k_i \,\rangle \tag{4.2.5}$$

Next we evaluate

$$\langle x|\; e^{\pi i w E R J} e^{-\frac{T}{2} p^* p} \;|x\rangle \;=\; \frac{1}{\pi T} \;\; e^{-\;\frac{x x^*}{2T}(e^{\pi i E R w} - 1)(e^{-\pi i E R w} - 1)} \tag{4.2.6}$$

where we used the standard formula for the heat kernel in two dimensions. Inserting this into (4.2.4), we can write the production rate as an integral over ρ of

$$\mathcal{W}(\rho) \;=\; R \int_0^\infty \frac{dT}{T} \sqrt{\frac{2\pi}{T}} \int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \; \frac{e^{-\frac{T}{2}k_i^2}}{\pi T} \; \sum_w \; e^{-\frac{1}{2T}\left((2\pi R w)^2 \;+\; 4\rho^2 \sin^2(\pi E R w/2)\right)} \tag{4.2.7}$$

which we will interpret as the pair production rate at the location ρ.

Equation (4.2.7) is an exact evaluation of the Euclidean functional determinant. To put it in a more useful form, we will assume that we are in the regime $\rho^2 \gg T$ (an assumption that we will be able to justify momentarily), so that we can simplify the expression by means of the Villain approximation

$$\sum_w e^{-\frac{1}{2T}\left((2\pi R w)^2 + 4\rho^2 \sin^2(\pi E R w/2)\right)} \simeq \sum_{w,n} e^{-\frac{1}{2T}\left((2\pi R w)^2 + \rho^2(\pi E R w - 2\pi n)^2\right)}. \tag{4.2.8}$$

This replacement essentially amounts to a semi-classical approximation. The right-hand side can be re-expressed via the Poisson resummation formula (note here that the left-hand side below is just a trivial rewriting of the right-hand side above)

$$\sum_w \; e^{-\frac{1}{2T}\left(\Lambda_E\left(2\pi R w - \frac{\pi E \rho^2 n}{\Lambda_E}\right)^2 + \frac{4\pi^2\rho^2 n^2}{\Lambda_E}\right)} = \sqrt{\frac{T}{2\pi \Lambda_E R^2}} \;\; \sum_m e^{-\frac{1}{\Lambda_E}\left(\frac{T}{2}\frac{m^2}{R^2} + \frac{2\pi^2\rho^2 n^2}{T} + \frac{i\pi E m n \rho^2}{R}\right)} \tag{4.2.9}$$

As the final step, we may now evaluate the integral over the Schwinger parameter T via the saddle point approximation. The saddle points are at[1]

$$T_0 = \frac{2\pi\rho\,|\,n|}{\sqrt{\frac{m^2}{R^2} + \Lambda_E k_i^2}}\ , \qquad n = \pm 1\,,\, \pm 2\,,\, \dots\ . \tag{4.2.10}$$

Before plugging this back in to obtain our final result, let us first briefly check our assumption that $\rho^2 \gg T$: setting $k_i = 0$, we find $\rho^2/T_0 = m\rho/2\pi|n|R$. So as long as the spatial distance scale ρ is much larger than the KK compactification radius R, we're safe to use (4.2.8).

With this reassurance, we proceed and find our final answer for the pair creation rate per unit time and volume where we made the replacement $E \to iE$.[2]

$$\mathcal{W}(\rho) \;=\; \frac{1}{2\pi^2\rho^{3/2}} \int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \sum_m \left(\frac{m^2}{R^2} + \Lambda k_i^2\right)^{1/4} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \exp\!\Big[-2\pi n\,\omega(\rho, q, k_i)\Big] \tag{4.2.11}$$

where $\omega(\rho, q, k_i)$ is the potential energy introduced in equation (3.1.15). The summation over n in (4.2.11) can be seen to correspond to the "winding number" of the Euclidean trajectory around the periodic Euclidean time direction. The $n = 1$ term dominates, and is the result announced in the Introduction. Before discussing it further, we will now proceed with a second method of derivation.

4.3 The Hawking-Unruh effect

The result (4.2.11) looks like a thermal partition function, indicating that it can be understood as produced via the Hawking-Unruh effect. We will now make this relation more explicit.

The functional integral (4.2.1) over all Euclidean paths represent the one-loop partition function of a scalar field Φ in the d+1-dimensional electric KK Melvin space-time. We

[1] We drop the term with $n = 0$, since it corresponds to the vacuum contribution.

[2] Note that the same result can be obtained by replacing the sum in (4.2.7) by (4.2.9), integrating over T exactly, and then using equation (3.2.11) to approximate the resulting Bessel function.

can compute this determinant also directly via canonical quantization of this field. The full expansion of Φ into modes starts with a decomposition over wave numbers along the extra dimension (in this section we restrict q to be positive)

$$\Phi = \Phi_0 + \sum_{q=1}^{\infty}\Big(e^{iqx_{d+1}}\Phi_q + e^{-iqx_{d+1}}\Phi_q^*\Big) \ , \tag{4.3.1}$$

where Φ_0 is massless and real, and Φ_q are complex and have mass $m = q$. Let us define μ_+ and μ_- via

$$\mu_+^2 = (q + \tfrac{1}{2}\omega E)^2 + k_i^2 \ , \qquad \mu_-^2 = (-q + \tfrac{1}{2}\omega E)^2 + k_i^2 \ , \tag{4.3.2}$$

so that now $\mu_\pm$ are quantities related to positively or negatively charged particles.

To proceed, we now need to expand the field Φ_q in creation and annihilation modes, allowing only modes that satisfy the boundary conditions (3.2.4) that $\Phi_q(\rho_i) = 0$ at the location of the two charged plates

$$\Phi_q = \sum_{\omega>0} \int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \, \frac{e^{-i\omega\tau + ik_i y^i}}{\sqrt{\omega f(\omega)}} \, \Big(K(\omega, \mu_+\rho)\, a_q(k_i,\omega) + K^*(\omega, \mu_-\rho)\, a_{-q}^\dagger(k_i,\omega)\Big) \ , \tag{4.3.3}$$

with $K(\omega, \mu\rho)$ and $f(\omega)$ as defined in (3.2.7) and (3.2.8). The creation and annihilation modes then satisfy the usual commutation relations

$$[a_{\pm q}(k_1,\omega_1), a_{\pm q}^\dagger(k_2,\omega_2)] = \delta(k_1 - k_2)\delta_{\omega_1\omega_2}. \tag{4.3.4}$$

Our goal is to determine what the natural vacuum state of the Φ field looks like, as determined by the initial conditions. In the far past, we imagine that the KK electric field was completely turned off. The electric KK Melvin background then reduced to Rindler or Minkowski space – depending on which coordinate system one introduces. The most reasonable initial condition is that the quantum state of all Φ quanta starts out in the vacuum as defined in the Minkowski coordinate system. Let us denote this Minkowski vacuum by $|\Omega\rangle$.

To determine the expression for $|\Omega\rangle$ in terms of our mode basis, we can follow the standard procedure [54, 118]. We will not go into the details of this calculation here, except to mention one key ingredient: the mode functions, when extended over the full range of ρ values, have a branch-cut at the horizon at $\rho = 0$, such that

$$K(\omega, -\mu\rho) \;=\; e^{\pm 2\pi\omega} K^{*}(\omega, \mu\rho)\,, \tag{4.3.5}$$

depending on whether the branch cut lies in the upper or lower-half plane. This behavior of $K(\omega, \mu\rho)$ near $\rho{=}0$ is sufficient to deduce the form of the Bogolyubov transformation relating the modes $a(\omega, k)$ to the Minkowksi creation and annihilation modes. (see e.g. [118]). As a result, one finds that the Minkowski vacuum, $|\Omega\rangle$, behaves like a thermal density matrix for the observable creation and annihilation modes in (4.3.3). In particular, the number operator for each mode has the expectation value

$$\langle\, \Omega \,|\, a_q^{\dagger}(k,\omega)\, a_q(k,\omega) \,|\, \Omega \,\rangle \;=\; \frac{1}{e^{2\pi\omega} - 1}, \tag{4.3.6}$$

while the overlap of $|\Omega\rangle$ with the empty vacuum state, defined via $a_q(k,\omega)|\,0\,\rangle\,{=}0$, becomes

$$|\langle\, 0 \,|\, \Omega \,\rangle|^2 \;=\; \exp\left[\int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \sum_{q,\omega} \left|\log\Bigl(1 - e^{-2\pi\omega}\Bigr)\right|\right] \tag{4.3.7}$$

$$\;=\; \exp\left[-\int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \sum_{q,\omega,n} \frac{1}{n}\, e^{-2n\,\pi\omega}\right]. \tag{4.3.8}$$

This expression represents the probability that the state Ω does not contain any particles – and its dominant $n = 1$ term looks indeed closely related to the result (4.2.11) obtained in the previous subsection.

The difference between the two equations is that (4.2.11) is defined at a particular location ρ, while (4.3.7) contains a summation over all frequencies. To make the relation more explicit, imagine placing some measuring device at a location ρ. As mentioned before, only modes with a sufficiently large frequency will reach this location with any appreciable probability, and the probability attains a maximum for frequencies equal to the potential energy at ρ, since for those frequencies, ρ is the turning point. Via this observation, we can view the position ρ as a parametrization of the space of frequencies, via the insertion of

$$1 = \int d\rho\, \delta\Big(\omega - \mathcal{W}(\rho)\Big)\, |\partial_\rho \mathcal{W}(\rho)| \ , \tag{4.3.9}$$

with $\mathcal{W}(\rho)$ as given in equation (3.1.15), thus replacing the summation over ω in (4.3.7) by an integral over ρ. The integrand at given ρ is then naturally interpreted as the production rate (4.2.11) at the corresponding location. This procedure is a good approximation provided the distance $d = \rho_2 - \rho_1$ between the plates is large enough, so that many frequencies contribute in the sum.

This same condition is also important for a second reason [49, 50]. Since we would like to imagine that the pair production takes place at a constant rate per unit time, we would like to see that the overlap (4.3.7) in fact decays exponentially with time. This comes about as follows [49, 50]. Suppose we restrict the field modes to be supported over a finite time interval $0 < \tau < T$. This translates into a discreteness of the frequencies. Ignoring at first the other discreteness due to the reflecting boundary condition at the two parallel plates, it is clear that the density of frequencies allowed by the time restriction grows linearly with T. The sum over the frequencies thus produces an overall factor of T. In this way, we recover the expected exponential decay of the overlap (4.3.7).

This exponential behavior breaks down, however, as soon as the time interval T becomes of the same order as the distance d between the plates, or more precisely, when $1/T$ approaches the distance between the discrete energy levels allowed by the reflecting boundary conditions at the plates. At this time scale, the situation gradually enters into a steady state, in which the pair creation rate gets balanced by an equally large annihilation rate. The system then reaches a thermal equilibrium, specified by the thermal expectation value (4.3.6). The physical temperature of the final state depends on the location ρ via

$$\beta = 2\pi\sqrt{g_{00}} = \frac{2\pi\rho}{\sqrt{\Lambda}}. \tag{4.3.10}$$

Note that this temperature diverges at $\rho = 0$ and $\rho = |2/E|$; neither location is within our physical region, however.

4.4 Charge Current

It is edifying to consider the vacuum expectation value of the charge current, since this is a clear physical, observer-independent quantity and a sensitive measure of the local profile of the pair creation rate. For given q, the charge current is given by

$$j_\mu = iq\Big(\Phi_q^*(\rho)\partial_\mu \Phi_q(\rho) - \Phi_q(\rho)\partial_\mu \Phi_q^*(\rho)\Big). \tag{4.4.1}$$

Using the result (4.3.6) for the expectation value of the number operator, one finds that the time component of the current, the charge density, is non-zero and equal to[3]

$$\langle\, \Omega \,|\, j_\tau(\rho) \,|\, \Omega \,\rangle = J_+(\rho) - J_-(\rho) \tag{4.4.2}$$

with

$$J_\pm(\rho) = q \sum_{\omega>0} \int \frac{\prod^{d-2} dk_i}{(2\pi)^{d-2}} \, \frac{|K(\omega, \mu_\pm \rho)|^2}{f(\omega)\,(e^{2\pi\omega} - 1)} \tag{4.4.3}$$

the positive and negative charge contributions, respectively. Given the thermal nature of the state $|\Omega\rangle$, the physical origin of this charge density is clear: the presence of the electric field reduces the potential energy of one of the two charge sectors, thereby reducing its Boltzmann suppression, relative to the oppositely charged.

To obtain a rough estimate for the behavior of $J_\pm(\rho)$, it is useful to divide the frequency sum into three regions: i) ω comparable to the potential energy (3.1.15), ii) ω much larger, or iii) ω much smaller. By comparing the respective suppression factors, we find that the leading semi-classical contribution comes from regime i); this is also reasonable from a physical perspective, since these are the particles that spend most time near ρ. Regime

[3] Instead of the expectation value (4.4.2), one could also consider the mixed in-out expectation value $\langle 0|J_\tau(\rho)|\Omega\rangle$, which is related to the derivative of the in-out matrix element $\langle 0|\Omega\rangle$ with respect to E. This relation was in fact used by Schwinger in his original derivation of the pair creation rate [46].

ii) is strongly Boltzmann suppressed and clearly negligible compared to contribution i), while regime iii) is suppressed because the corresponding mode functions $K(\omega, \mu\rho)$ are exponentially small at the location ρ, via (3.2.11). The leading contribution of region i) is of order $e^{-2\pi\omega(\rho,q,k_i)}$, in accordance with the result (4.2.11) for the pair creation rate $\mathcal{W}(\rho)$.

Since the mode functions $K(\omega, \mu\rho)$ are real (they are the sum of an incoming and reflected wave), the current in the ρ direction appears to vanish. The result (4.4.3) for the charge density indeed looks static. This static answer, however, can not describe the time-dependent pair creation process. Recalling our discussion above, however, we can recover this time-dependence by restricting the sum over only those frequencies necessary to cover the finite time interval $0 < \tau < T$. This is a T dependent subset, thus leading to a T dependent (initially linearly growing) charge density. However, when $1/T$ becomes much smaller than the step-size in the allowed frequency spectrum, the steady state sets in and the charge density indeed becomes a static thermal distribution given by (4.4.2)–(4.4.3).

Chapter 5

Discussion

In this Part we have tried to make a systematic study of the Schwinger pair production of charged Kaluza-Klein particles. Due to their characteristic property that their mass is of the same order as their charge q, the pair creation requires such strong KK electric fields that gravitational backreaction can not be ignored. We have included this backreaction by means of the electric KK Melvin solution, and shown that, in spite of the fact that the electro-static potential can not be made to exceed the rest-mass of the KK particles, pair production takes place at a rate given by (1.0.2).

What is the physical mechanism that is responsible for the pair creation? Our final answer (1.0.2) includes both the KK electric field and a gravitational acceleration a. It is instructive to compare this result with the known rate [120] for Schwinger pair production in an accelerated frame, as quoted in equation (B.4) in the Appendix. Since in our case a is bounded below by $E/2$, we can only directly compare the two answers in the limit of small electric field. In that limit, if we expand the log in equation (B.4) and take the dominant $n = 1$ term there, both answers become

$$\mathcal{W}(E,a) \;\simeq\; \sum_{q=\pm|q|} \int \frac{\prod^{d-2} dk_i}{(2\pi)^d} \exp\Big(-2\pi\Big(\frac{1}{a}\sqrt{q^2+k_i^2} - \frac{qE}{2a^2}\Big)\Big). \tag{5.0.1}$$

In this regime, however, one can not honestly separate the Schwinger pair creation effect from the pair creation effect due to the acceleration. Electric charge is being produced, but it is just a simple consequence of the fact that the electrostatic potential reduces the Boltzmann factor for one type of charge, while increasing it for the other. Rather than producing the charge "on its own," the electric field just polarizes the thermal atmosphere produced by the Unruh effect.

In fact, if we write the potential $\omega(\rho, q, k_i)$ as in (3.1.15) instead of (3.1.16), our final answer (1.0.2) appears to be just a small modification of (5.0.1) and the physics that leads to it indeed seems quite identical. So depending on taste, one can either interpret our result (1.0.2) as pair creation due to a combination of the Schwinger and Unruh effect, or as the result of the Unruh effect only. There is no definite way to decide between the two, since the gravitational acceleration can not be turned off independently. Either way, what is clear is that the mechanism for pair creation cannot be given a tunneling interpretation.

Appendix

A Gedanken Apparatus

For a good understanding of the situation we wish to study, it will be useful to investigate how, via a concrete Gedanken experiment, one may in fact attempt to create a large static Kaluza-Klein electric field. Without taking into account gravitational backreaction, we imagine taking two parallel plates with opposite KK charge density per unit area σ and perpendicular distance d, thus creating an electric field $E = 4\pi\sigma$ in the region between the plates. It turns out however, that when we include the gravitational backreaction of both the plates and the electric field, there are some restrictions on how symmetric, or static, we can choose our experimental set-up.

Consider two charged, infinitesimally thin, parallel plates at positions ρ_1 and ρ_2, separated by a distance

$$d = \rho_2 - \rho_1 \ . \tag{A.1}$$

The two plates divide space into three regions: Region A left of the first plate, given by $\rho < \rho_1$, region B in between the two plates, $\rho_1 < \rho < \rho_2$, and region C right of the second plate $\rho > \rho_2$.

Let the mass densities of the plates be given by μ_1 and μ_2, so that

$$T^0_{\ 0} = \mu_1 \, \delta(\rho - \rho_1) \, + \, \mu_2 \, \delta(\rho - \rho_2) \ . \tag{A.2}$$

In addition, the two plates have charge densities σ_1 and σ_2

$$\sqrt{g_{00}} \ T^0_{d+1} = \sigma_1 \, \delta(\rho - \rho_1) \, + \, \sigma_2 \, \delta(\rho - \rho_2) \ . \tag{A.3}$$

We will assume that the charge densities are opposite, $\sigma_1 = -\sigma_2$, and tuned so that there is a Kaluza-Klein electric field in region B between the plates, but none in regions A or C outside the plates. The region between the plates therefore takes the form of a static slice

$\rho_1 < \rho < \rho_2$ of the electric KK Melvin space-time. The two regions outside the plates, on the other hand, are just flat. More precisely, since the parallel plates in effect produce an attractive gravitational force on freely falling particles in the two outside regions, the regions A and C should correspond to static sub-regions in Rindler space.

Both Rindler space and the electric KK Melvin solution differ from Minkowski space only via the g_{00} component. Imposing continuity at $\rho = \rho_i$, this leads us to the following Ansatz for the g_{00} component of the metric in the three regions

$$
\begin{aligned}
g_{00}^{A}(\rho) &= (1-a_1(\rho-\rho_1))^2\, g_{00}^{B}(\rho_1)\,,\\
g_{00}^{B}(\rho) &= \frac{\rho^2}{1-E^2\rho^2/4}\,,\\
g_{00}^{C}(\rho) &= (1+a_2(\rho-\rho_2))^2\, g_{00}^{B}(\rho_2).
\end{aligned}
\tag{A.4}
$$

Here a_1 and a_2 are both positive, and represent the respective free fall accelerations of freely moving particles just outside of the two plates. In other words, via the equivalence principle, a_1 and a_2 are the accelerations (to the left and right, respectively) of the two plates as viewed from the outside Minkowski observers. The quantities ρ_1 and ρ_2 play a similar role, and can be both positive and negative. A physical restriction, however, is that the denominator in the expression (A.4) for g_{00}^{B} remains positive.

In addition there is a non-trivial electric potential $g_{0,d+1}^{B}$ in the region between the plates, while $g_{0,d+1}^{A,C}$ are constants determined by continuity:

$$
\begin{aligned}
g_{0,d+1}^{A}(\rho) &= g_{0,d+1}^{B}(\rho_1)\,,\\
g_{0,d+1}^{B}(\rho) &= \frac{\rho^2 E/2}{1-E^2\rho^2/4}\,,\\
g_{0,d+1}^{C}(\rho) &= g_{0,d+1}^{B}(\rho_2).
\end{aligned}
\tag{A.5}
$$

The d+1-dimensional Einstein equations of motion result in the following jump conditions for the normal variations of g_{00} and $g_{0,d+1}$ at the location of the plates[1]

$$4\pi\mu_i = g^{00}(\partial_{\rho+} g_{00} - \partial_{\rho-} g_{00}) \,|_{\rho=\rho_i}, \tag{A.6}$$

$$4\pi\sigma_i = (g^{00})^{\frac{1}{2}}(\partial_{\rho+} g_{0,d+1} - \partial_{\rho-} g_{0,d+1}) \,|_{\rho=\rho_i} \,. \tag{A.7}$$

The first of these equations is known as the Israel equation, while the second is equivalent to Gauss' law in electro-magnetism. Inserting our Ansatz, the Israel jump conditions become

$$\begin{aligned} 2\pi\mu_1 &= a_1 + \frac{1}{\rho_1 \Lambda_1} \,, \\ 2\pi\mu_2 &= a_2 - \frac{1}{\rho_2 \Lambda_2} \,, \end{aligned} \qquad \Lambda_i = 1 - \tfrac{1}{4} E^2 \rho_i^2 \,, \tag{A.8}$$

while Gauss' law takes to the form

$$4\pi\sigma_1 = E/\Lambda_1^{3/2} \,, \qquad 4\pi\sigma_2 = -E/\Lambda_2^{3/2} \,. \tag{A.9}$$

Equation (A.8) relates the mass density of the two plates to the jump in the surface acceleration when moving from one to the other side, while (A.9) relates the charge density to the jump in the KK electric field.

Let us briefly check these formulae by considering some special cases. If $E = 0$, then we can choose the symmetric situation $\mu_1 = \mu_2$ and $a_1 = a_2$. Via (A.8) this implies that we should take the limit $\rho_i \to \infty$ keeping the distance (A.1) fixed. The intermediate region B then simply reduces to flat Minkowski space. This is as expected, since the two plates lead to an equal and opposite gravitational force, which exactly cancels in the intermediate region. For non-zero E, yet small electro-static potential $V_{12} = Ed$ between the plates, we can choose parameters such that $E\rho_i \ll 1$ and $\rho_i \gg d$. Equation (A.9) then reduces to the standard Gauss law of Maxwell theory.

[1]Note that while the expressions $g_{0,d+1}$ are not gauge invariant, the Gauss equation is, as long as λ in $A_\mu dx^\mu \to A_\mu dx^\mu + d\lambda$ is smooth across ρ_1 and ρ_2, i.e. $(\partial_{\rho_+} - \partial_{\rho_-})\lambda \,|_{\rho=\rho_i} = 0$.

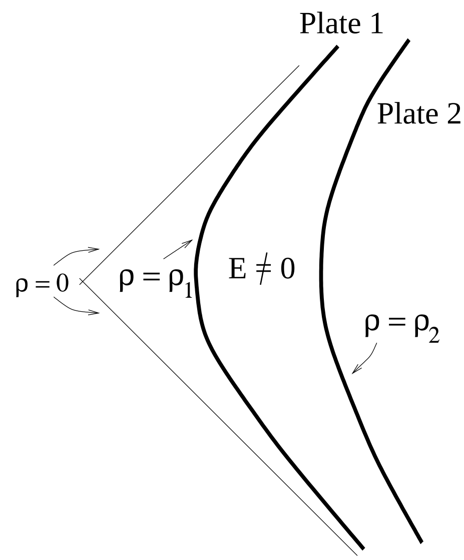

Fig. 5: Situation I *describes the static situation with two charged plates at* $\rho = \rho_1$ *and* $\rho = \rho_2$*, with* $0 < \rho_1 < \rho_2 < 2/E$*. The region of interest, in between the two plates, is a static slice of region I of the electric KK Melvin solution.*

Let's now consider the general case. There are four equations, and (for given inter-plate distance d, and densities μ_i and σ_i) four unknowns: a_1, a_2, ρ_1 and E. The second equation in (A.9), however, is not really independent from the first, since we should rather read it as a fine-tuning condition on σ_2 (relative to σ_1) ensuring that the E-field vanishes outside the two plates. Discarding this equation, we are thus left with one overall freedom, namely, the overall acceleration of the center of mass of our apparatus.

For practical purposes, we would have preferred to restrict ourselves to the simplest and most symmetric case in which the two plates have equal mass density $\mu_1 = \mu_2$ and equal surface acceleration $a_1 = a_2$. This would in particular ensure that our apparatus is at rest. As seen from equation (A.8), this symmetric situation could be reached if we could take the limit $\rho_i \to \infty$. However, for non-zero E, this limit is forbidden via the restriction $1 - \frac{1}{4}E^2\rho^2 > 0$. Thus we are basically forced to consider the general situation

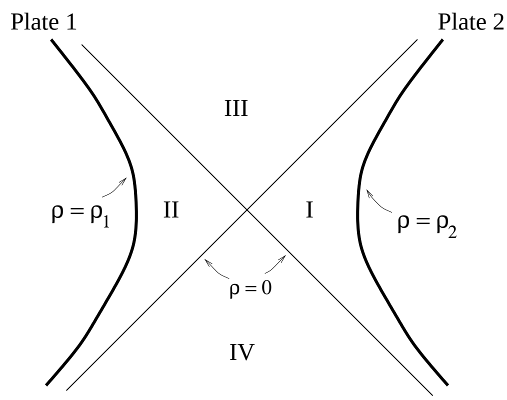

Fig. 6: *Situation II describes the time-dependent situation with two accelerating charged plates at $\rho = \rho_1 < 0$ and $\rho = \rho_2 > 0$. The region of interest, in between the two plates, includes the time-dependent regions III and IV of the electric KK Melvin solution.*

with μ_i and a_i arbitrary, and ρ_i both positive. We call this:

$$\text{Situation I}: \qquad 0 < \rho_1 < \rho_2 < 2/E, \qquad \mu_i \ \text{ arbitrary.} \tag{A.10}$$

In this case, the region of interest, region B, represents a static slice in the right wedge of the electric KK Melvin solution. This Situation I is the natural generalization of a constant, static electric field, and is our starting point for studying the possible Schwinger pair creation of charged KK particles.

There is, however, another situation we could consider, which does allow for a symmetric solution. Namely we can choose:

$$\text{Situation II}: \qquad \mu_1 = \mu_2\,, \qquad a_1 = a_2\,, \qquad \rho_1 = -\rho_2. \tag{A.11}$$

In this case the region B includes the special position $\rho = 0$ at which $g_{00} = 0$, the location of the event horizon of the electric KK Melvin geometry.

To better understand the experimental conditions leading to Situation II, consider the special case $\mu_1 = \mu_2 = 0$, and $\sigma_1 = \sigma_2 = 0$. This describes two plates with zero mass and charge, accelerating away from each other with equal but opposite acceleration $a_i = 1/\rho_i$. It is now easy to imagine that one can gradually add mass and charge to the plates, and reach the general Situation II. It must be noted that this experimental set-up does not lead to a static background, since the geometry now includes the time-dependent regions III and IV enclosed by the Rindler horizon (see figure 1). This set-up is therefore not a direct analog of the static electric field considered by Schwinger. For a discussion of Situation II see [115, 116]; our main focus is Situation I.

B Schwinger meets Rindler

In this Appendix we summarize the known result for the Schwinger pair creation rate in an accelerating frame [120] of charged particles with mass q and mass m with $m \ll q$. In this regime, pair creation starts to occur while the gravitational backreaction of the electric field is still negligible.

The closest analog of a constant, uniform gravitation field is Rindler space

$$ds^2 = -\rho^2 d\tau^2 + d\rho^2 + dy_i^2. \tag{B.1}$$

Particles, or detectors, located at a given ρ undergo a uniform acceleration $a = 1/\rho$. Consider a charged field propagating in this space in the presence of a uniform electric field, described by

$$A_\tau = \frac{1}{2} E\rho^2. \tag{B.2}$$

The resulting scalar wave equation reads

$$\left(\frac{1}{\rho}\,\partial_\rho(\rho\,\partial_\rho)\,-\,\frac{1}{\rho^2}\Big(\partial_\tau + \frac{i}{2}qE\rho^2\Big)^2\,+\,\partial_i^2\,-\,m^2\right)\Phi = 0\,. \tag{B.3}$$

The above three equations are connected to the ones in Section 3.2.1 by setting $\Lambda = 1$ (which indeed amounts to turning off the backreaction) and by setting $m = q$ above.

Equation (B.3) has known mode solutions with given Rindler frequency, in terms of Whittaker functions [120, 122]. These functions have a relatively intricate, but known (equation 9.233 in [122]), branch-cut structure at $\rho = 0$, from which one can straightforwardly extract the linear combination of (left and right wedge) Rindler creation and annihilation modes that annihilate the Minkowski vacuum $|\Omega\rangle$. One obtains the following result for the total pair creation rate per unit time and (transverse) volume [120]

$$\mathcal{W} \simeq \sum_{q=\pm|q|} \int d\omega \int \frac{\prod^{d-2} dk_i}{(2\pi)^d} \log \left(\frac{\left(1 - e^{-2\pi\omega}\right)\left(1 - e^{-\frac{\pi(m^2+k_i^2)}{|qE|}}\right)}{1 - e^{-2\pi\left(\omega + \frac{m^2+k_i^2}{2|qE|}\right)}} \right). \tag{B.4}$$

As explained in Section 4.3, we can extract from this result the pair creation rate at a given radial location ρ, or equivalently, given acceleration $a = 1/\rho$, by equating the frequency ω with the classical potential energy at this location

$$\omega(q, k_i) = \frac{1}{a}\sqrt{q^2 + k_i^2} - \frac{qE}{2a^2}. \tag{B.5}$$

As discussed in section 5, in the limit where the electric field is small, the expression (B.4) reduces to out result.

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