Time-Reversal Symmetric $U(1)$ Quantum Spin Liquids

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We study possible quantum $U(1)$ spin liquids in three dimensions with time-reversal symmetry. We find a total of seven families of such $U(1)$ spin liquids, distinguished by the properties of their emergent electric or magnetic charges. We show how these spin liquids are related to each other. Two of these classes admit nontrivial protected surface states which we describe. We show how to access all of the seven spin liquids through slave particle (parton) constructions. We also provide intuitive loop gas descriptions of their ground-state wave functions. One of these phases is the "topological Mott insulator," conventionally described as a topological insulator of an emergent fermionic "spinon." We show that this phase admits a remarkable dual description as a topological insulator of emergent fermionic magnetic monopoles. This results in a new (possibly natural) surface phase for the topological Mott insulator and a new slave particle construction. We describe some of the continuous quantum phase transitions between the different $U(1)$ spin liquids. Each of these seven families of states admits a finer distinction in terms of their surface properties, which we determine by combining these spin liquids with symmetry-protected topological phases. We discuss lessons for materials such as pyrochlore quantum spin ices which may harbor a $U(1)$ spin liquid. We suggest the topological Mott insulator as a possible ground state in some range of parameters for the quantum spin ice Hamiltonian.

I. INTRODUCTION

There has been much recent interest in quantum spin liquid phases of systems of interacting magnetic moments. These phases are fascinating examples of ground states characterized by long-range quantum entanglement: The corresponding wave functions cannot be smoothly deformed into a product state of local degrees of freedom. Other examples of such long-range entangled states include the celebrated fractional quantum Hall states. The structure of the long-range entanglement dictates the excitation structure of the phase in just the same way as the familiar long-range order (associated with broken symmetry) does in conventional ordered phases. In particular, a number of unusual excitations—for instance, those with fractional quantum numbers or statistics, or gapless emergent gauge bosons—are possible in such phases.

In this paper, we are concerned with a particular class of three-dimensional quantum spin liquids that supports an emergent gapless "photon" as an excitation. It has long been recognized [1,2] that such a photon may be an emergent excitation of some underlying physical quantum many-body system with short-range interactions. Specific microscopic models of quantum phases with emergent photons were constructed some time ago in Refs. [3–9] in diverse systems. In addition to the photon, these phases support other quasiparticle excitations that couple to the photon as "electric" or "magnetic" charge.

Interest in such phases has been revived following a recent proposed experimental realization [10,11] in certain three-dimensional pyrochlore oxides. These are materials in which there are effective spin-1/2 degrees of freedom at the sites of the pyrochlore lattice. A class of such materials such as $Dy_2Ti_2O_7$ or $Ho_2Ti_2O_7$ has been studied extensively, both in theory and experiment, and is adequately described within the framework of classical statistical mechanics [12]. Because of a combination of spin anisotropy and exchange interactions, the spins are constrained to satisfy an "ice rule," where on each tetrahedron of the pyrochlore lattice, there are precisely two spins that point inward and two that point outward. The low energy physics takes place within the subspace of states satisfying this constraint. Hence, these system have been dubbed "spin ice." Quantum effects are known to be important [13] in a few such pyrochlore magnets, which have hence been dubbed "quantum spin ice." Examples include $Yb_2Ti_2O_7$, $Pr_2Zr_2O_7$, and $Tb_2Ti_2O_7$. In particular, in $Yb_2Ti_2O_7$, the detailed microscopic Hamiltonian governing the interaction between the spins has been deduced through neutron scattering experiments [10,14]. The parameters of this Hamiltonian are such that quantum effects are surely present and will play a role in determining the low temperature physics.

It is well known [15] that in the spin ice subspace, the spins form oriented closed loops, and the subspace can be parametrized in terms of oriented loop configurations. Classical spin ice systems are thus described as *thermally* fluctuating loop gases in three dimensions. The loops can be viewed as "magnetic" field lines of an artificial magnetic field. Defect configurations in the spin ice manifold such as a "3-in 1-out" tetrahedron (where three spins point in instead of two) correspond to end points of the loops and are then identified with magnetic monopoles [16]. Such monopoles have been observed in experiments in the last few years [17,18].

In quantum spin ice materials, it is natural to expect that the physics may be determined by *quantum* fluctuations of oriented loops. If these loops form a liquid phase where the loop line tension is zero, the result is a quantum spin liquid. This spin liquid supports an emergent gapless photon. The associated magnetic field lines are simply the tensionless magnetic loops. Magnetic monopoles (the defect tetrahedra) are now gapped quasiparticle excitations where these field lines end. Reference [10] proposed that this physics may occur in $Yb_2Ti_2O_7$.

Knowledge of the precise microscopic Hamiltonian for $Yb_2Ti_2O_7$ lends hope to a reliable theoretical assessment of this proposal and to quantitative comparisons to experiment. However the microscopic Hamiltonian is rather complicated and is hard to solve, either analytically or numerically. Furthermore, as we briefly review (see Appendix A), the parameters are such that it is not obvious that it is sufficient to just restrict to the spin ice manifold. Finally, there is very little global symmetry in the model. The only good symmetries are time-reversal and space-group operations.

What scope is there for theoretical progress in the absence of reliable methods to study the model Hamiltonian? One possibility is to deform the model to a limit where its ground state may be reliably determined, say, by numerical methods. Approximate analytical methods can then be chosen to reproduce the known result in this limit. They can then be extended to the realistic model with the hope that they capture the full phase diagram. For quantum spin ice, such an approach has been pursued in Ref. [11] using a slave particle approach known as the "gauge mean field theory" (gMFT). A reliable limit is provided by considering the *XXZ* spin-1/2 pyrochlore model in the Ising limit. This model can be studied through quantum Monte Carlo without a sign problem, and the ground state is a $U(1)$ quantum spin liquid [8]. Further analytic arguments [5] strongly indicate the structure of the gapped e and m excitations. The gMFT correctly reproduces this spin liquid ground state. Reference [11] then extends this slave particle approach to obtain an answer for the full phase diagram of the model, including the parameter regime determined in experiment. This mean field seems to show that the experimentally relevant parameters place the model in a conventional ferromagnetic state rather than a spin liquid. However, this parameter regime is substantially different from the limit where gMFT is known to capture the correct ground state. It is hard to evaluate the accuracy of the gMFT prediction for the phase diagram away from this limit. In particular, other slave-particle mean-field methods are available (for instance, Schwinger bosons or fermions) and will lead to different phase diagrams. Furthermore, even when they lead to a $U(1)$ spin liquid, it is not clear whether different slave particle methods lead to the same phase of matter.

In this paper, inspired by these developments, we pose a different set of questions on which we are able to make solid progress. Rather than attempt to solve any particular microscopic model approximately, we constrain the general properties of $U(1)$ quantum spin liquids[1] in the presence of global symmetries and describe their physics. Specifically, we focus on time-reversal symmetric $U(1)$ quantum spin liquids and determine the number of distinct phases and their properties. Time reversal is a robust physical symmetry and the only internal symmetry in the model describing $Yb_2Ti_2O_7$. We ignore space-group symmetry, both because it simplifies the problem and because it is less robust (because of disorder). To further simplify the problem, we restrict ourselves to $U(1)$ liquid phases where the only gapless excitation is the photon. In particular, the magnetic charge (dubbed the M particle) and the electric charge (the E particle) are gapped.[2] We show that there are 22 phases which fall into seven distinct families of $U(1)$ spin liquids. The seven families of $U(1)$ spin liquids are distinguished by their bulk excitation spectrum, which we tabulate in Table I. Different phases in each family are distinguished by their surface states, and one can construct one phase from another in the same family by combining with a class of phases called symmetry-protected topological states. In most parts of this paper, except Sec. VIII, we focus mainly on the seven families of states, which have clear physical differences in the bulk. Therefore, we often use the term "phase" instead of "family of phases" when the context is clear. Most of the existing microscopic models describe only one of these phases, which is also the one accessed by the gauge mean-field theory of Ref. [11].

We describe the physics of these seven families of states and their interrelationships in many complementary ways. We show how each of the seven families of states may be accessed through slave particle constructions. In some cases, we provide more than one slave particle construction for the same phase. We describe the structure of the distinct

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[1]Here, we only consider "spin liquids" (or boson liquids) that can emerge in the Hilbert space of a purely spin (or boson) system.

[2]We also implicitly assume that, apart from the deconfined $U(1)$ gauge field, there is no other coexisting topological order or source of long-range entanglement.

TABLE I. Families of $U(1)$ quantum liquids with time-reversal symmetry labeled by properties of the "pure" electric and magnetic charges. Here, q_e and q_m denote electric and magnetic charges, respectively. For the electric particle, $(q_e, q_m) = (1, 0)$, and for the magnetic particle, $(q_e, q_m) = (0, 1)$, except for the last row where it is $(0,2)$. For the last phase $(E_{fT}M_f)_\theta$, there are more fundamental "dyonic" particles that have $(q_e, q_m) = (\pm 1/2, \pm 1)$ and are bosons. Both the pure electric charge and the pure magnetic charge indicated in the table can be built up as composites of the dyons in this phase.

Phase	Electric particle	Magnetic particle
$E_b M_b$	Boson	Boson
$E_{bT} M_b$	Boson, $\mathcal{T}^2 = -1$	Boson
$E_f M_b$	Fermion	Boson
$E_{fT} M_b$	Fermion, $\mathcal{T}^2 = -1$	Boson
$E_b M_f$	Boson	Fermion
$E_{bT} M_f$	Boson, $\mathcal{T}^2 = -1$	Fermion
$(E_{fT} M_f)_\theta$	Fermion, $\mathcal{T}^2 = -1$	$(q_m = 2)$ Fermion

ground states in terms of distinctions in the loop wave functions. This leads to many interesting insights and to predictions for future numerical calculations. We determine the properties of protected surface states that some of these phases have. Given these solid results on the possible time-reversal symmetric $U(1)$ spin liquids and their properties, we may ask about how to distinguish them in experiments and about which ones are likely for a particular microscopic model. We describe some experimental signatures that can help identify which (if any) of these spin liquids is realized. We also provide some guides for relating to microscopic models. A summary of our key results is in Sec. II.

We emphasize that the distinction between these phases is entirely a consequence of the unbroken time-reversal symmetry. If this symmetry were absent, then it is possible to go smoothly between any two of these phases. The distinction comes from different possible implementation of time-reversal symmetry.

Our analysis will be strongly informed by recent progress [19–26] in the theory of interacting generalizations of three-dimensional topological insulators or superconductors (see Ref. [27] for a review of aspects directly pertinent to this paper). It is now recognized that the topological band insulators are special examples of a class of quantum states of matter known as symmetry-protected topological (SPT) phases [28–30]. These states are seemingly conventional in the bulk—they are gapped and have no exotic excitations but nevertheless have nontrivial surface states that are protected by symmetry. But what role do they play in describing the quantum spin liquids of interest in this paper? The answer is that starting with one kind of $U(1)$ spin liquid, we may generate others by putting one of the emergent quasiparticles (E or M) into a SPT state. For

$U(1)$ quantum spin liquids, this point of view was initiated in a previous paper [20] by the present authors. Reference [20] considered SPT states of bosonic particles and demonstrated their utility in understanding some $U(1)$ quantum spin liquids. This point of view will be fully developed in the present paper and will lead to a complete and more insightful description of all time-reversal-invariant $U(1)$ spin liquids with gapped matter. In particular, we exploit recent exciting developments on fermionic SPT states [23–26], which were not understood when Ref. [20] was published, to obtain this complete picture.

It is important to point out that there is no one-to-one mapping between SPT phases with global $U(1)$ and time-reversal symmetries, and $U(1)$ quantum spin liquids with time reversal. They both have different classifications. For example, we show that two different SPT states are reduced after gauging to the same physical $U(1)$ spin liquid.

II. SUMMARY OF RESULTS

Here, we briefly summarize some of our key results. This section will also serve as an outline for the rest of the paper.

(1) We first establish that there are seven distinct families of time-reversal-invariant $U(1)$ liquid phases in 3D distinguished by their bulk spectra in Secs. III–V. A partial characterization of these phases is obtained by plotting the spectrum of emergent quasiparticles—the charge-monopole lattice—in the $U(1)$ gauge theory. We show that six of these seven families have the charge-monopole lattice of Fig. 1, while the remaining one has the charge-monopole lattice of Fig. 2. We provide a first
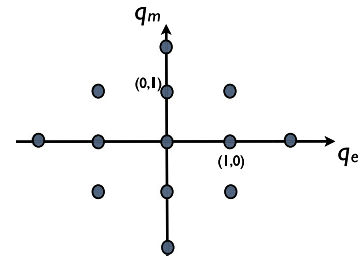

FIG. 1. Charge-monopole lattice at $\theta = n\pi$ with n even.
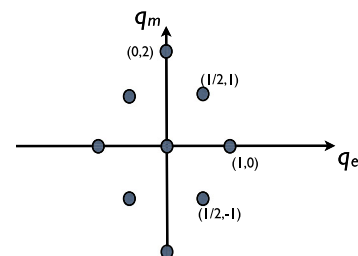

FIG. 2. Charge-monopole lattice at $\theta = n\pi$ with n odd.

cut description of these seven families of phases in these sections and relate them to existing constructions of $U(1)$ liquids. One of these phases (dubbed $E_b M_b$) is the state accessed by gMFT, while some others are states accessed by Schwinger boson or Abrikosov fermion constructions. The unique family described by Fig. 2 includes the so-called "topological Mott insulator" discussed in Ref. [31]. For reasons described later, this is denoted $(E_{fT} M_f)_\theta$ in this paper. The family denoted $E_{bT} M_f$ has not been described explicitly in the literature.

(2) We describe how these phases are related to each other in Sec. VI. This is enabled by recent advances in our understanding of SPT phases of bosons or fermions with global $U(1)$ and time-reversal symmetries. We continue the point of view adopted in our previous work [20] showing that, given one $U(1)$ liquid, we can obtain others from it by putting one of the emergent quasiparticles in a SPT phase. We are thus able to obtain a rather complete understanding of how these seven phases are related to each other.

(3) The conventional description of the $(E_{fT} M_f)_\theta$ (the topological Mott insulator) is that it is a topological insulator formed by emergent fermionic Kramers doublet spinons that are coupled to the $U(1)$ gauge field as an electric charge. We show in Sec. VI that this phase has a remarkable dual description as a topological insulator of emergent fermionic magnetic monopoles.

(4) We discuss the possibility of protected surface states at the interface with the vacuum for these spin liquids in Sec. VII. In Sec. VII A, we describe criteria that determine when such protected surface states will form. We argue that precisely two of the seven families [$(E_{fT} M_f)_\theta$ and $(E_{bT} M_f)$] are required to have a nontrivial surface state. The possibility of a surface spinon Dirac cone for the $(E_{fT} M_f)_\theta$ is well known [31]. The dual description of this phase as a monopole topological insulator naturally leads to an alternate possible "dual" surface state where there are an odd number of gapless monopole Dirac cones (and no spinon Dirac cone). We then describe the surface of $E_{bT} M_f$—the simplest possibility is to have two monopole Dirac cones.

(5) For any given bulk spectrum, there can be more than one phase corresponding to distinct surface properties. When these are taken into account, we find a total of 22 distinct phases. These are obtained from the seven basic phases by combining them with SPT phases of bosons or spins protected by time reversal alone. Interestingly, in some cases, the spin liquid can "absorb" a SPT phase so that the combination is not in a distinct phase. In other words, not all

T-reversal symmetric SPT phases stay distinct from trivial phases when combined with a spin liquid. A similar phenomenon also appears in two-dimensional $\mathbb{Z}_2$ spin liquids [32].

(6) In Sec. IX, we show how to access all of the seven basic phases through parton constructions on spin models. In particular, we show how the standard fermionic parton construction of spin-1/2 systems enables access to five of the seven phases (the exceptions being $E_b M_b$ described by gMFT and the $E_{bT} M_b$ described by Schwinger bosons). For the topological Mott insulator $(E_{fT} M_f)_\theta$, we describe a dual parton construction in terms of monopoles that is distinct from the conventional one in terms of spinons. With a view toward obtaining input on microscopics, we obtain some no-go results on these parton constructions in Sec. IX A if the physical system consists of Kramers doublet spin-1/2 degrees of freedom.

(7) We provide an intuitive physical picture of the ground-state wave function for these spin liquids in terms of fluctuating loop configurations of electric- or magnetic-field lines in Sec. X.

(8) We describe some of the remarkable continuous quantum phase transitions between these different spin liquids. We particularly focus on phase transitions out of the topological Mott insulator $(E_{fT} M_f)_\theta$. We provide a theory for a second-order transition from this phase to others where the electric charge is a boson (either Kramers singlet or doublet). We also provide a theory for a different second-order phase transition between two different phases where the electric charge changes from a Kramers doublet to a Kramers singlet.

(9) In Sec. XIV, we consider the relevance of these results to current and future realizations of $U(1)$ spin liquids in experimental systems. For pyrochlore spin ices based on Kramers doublet spin systems, we discuss the possible $U(1)$ spin liquids that may be obtained. Apart from the one suggested by gMFT, we argue, based on the parton construction, that the other natural candidate is the topological Mott insulator $(E_{fT} M_f)_\theta$. A strong coupling expansion of the lattice parton Hamiltonian coupled to the $U(1)$ gauge field yields, at leading order, a spin Hamiltonian of the form appropriate to the pyrochlore spin ices but with parameters different from the ones where gMFT is expected to be reliable. On this basis, we suggest that some pyrochlore spin ices may be in the topological Mott insulator phase.

We also describe some experimental distinctions between these phases which may be useful in identifying them.

Several appendixes contain peripheral details.

III. PRELIMINARIES

We begin with some simple but powerful observations. We are interested in time-reversal symmetric $U(1)$ liquids of spins or bosons in which the only gapless excitation is the photon. To distinguish different phases, it is appropriate to focus on the gapped emergent quasiparticles that couple as electric or magnetic charges to the photon. Time-reversal symmetry constrains the possibilities in many important ways. as we now describe.

A. Charge-monopole lattice

We denote the electric charge q_e and magnetic charge q_m. We use notation in which the total electric flux is $4\pi q_e$ and the total magnetic flux is $2\pi q_m$. To be general, we must allow for the most fundamental emergent particles to be "dyons," i.e., particles that carry both electric charge and magnetic charge. For any pair of dyons with charges $(q_e = Q_e, q_m = Q_m)$ and $(q_e = Q_e', q_m = Q_m')$, there is a generalized Dirac quantization condition [33,34]:

$$Q_e Q_m' - Q_m Q_e' = n, \qquad (1)$$

where n is an integer.

For each particle with charges (Q_e, Q_m), there will be an antiparticle with charges $(-Q_e, -Q_m)$. Note that the particle and antiparticle automatically satisfy the Dirac quantization condition. We use the natural convention that, under time reversal, the magnetic fields are odd and the electric fields are even. Then, for any particle with charges (Q_e, Q_m), there is a time-reversed partner with charges $(Q_e, -Q_m)$. Applying the Dirac quantization condition to these two particles, we obtain the restriction

$$2Q_e Q_m = \text{integer}. \qquad (2)$$

By combining (Q_e, Q_m) with $(Q_e, -Q_m)$, we can produce a particle that is a pure electric charge $(2Q_e, 0)$. Similarly, by combining (Q_e, Q_m) with $(-Q_e, Q_m)$ (the antiparticle of the time-reversed partner), we obtain a pure magnetic charge $(0, 2Q_m)$. Thus time-reversal invariance guarantees that there are always both pure electric and pure magnetic charges in the theory.

Consider the smallest pure electric charge. We choose units in which $q_e = 1$ (and, by definition, $q_m = 0$). Let the smallest pure magnetic charge have strength g (and $q_e = 0$). Applying the Dirac condition to the pure electric charge and the pure magnetic charge, we get

$$g = \text{integer}. \qquad (3)$$

If there are no other restrictions, the smallest allowed g is 1. As is well known, the Dirac condition requires that pure electric and magnetic charges are quantized to be integers (in our units). If there are dyons with charges (Q_e, Q_m), it follows that $2Q_e$ must be an integer. Thus, we have two

basic possibilities: $Q_e = 1$ or $Q_e = \frac{1}{2}$. In the former case, there are no further restrictions on g beyond Eq. (3), and we have $g = 1$. In the latter case, we can apply Dirac quantization to the $(0, g)$ and $(\frac{1}{2}, Q_m)$ particles to obtain

$$g = 2 \times \text{integer}. \qquad (4)$$

Thus, if there are charge-1/2 dyons, the minimum pure magnetic charge is 2. Furthermore, we must have $Q_m = 1$ for the charge-1/2 dyon.

We thus have two classes of possible states which are distinguished by the geometry of the lattice of allowed charges and monopoles. In one class, the charge-monopole lattice is as shown in Fig. 1. Here, all emergent quasiparticle excitations are obtained from two elementary quasiparticles—the E particle with $(q_e, q_m) = (1, 0)$ and the M particle with $(q_e, q_m) = (0, 1)$. In the second class, the charge-monopole lattice is shown in Fig. 2. Here, the full set of emergent particles can be built out of the two dyons with $(q_e, q_m) = (\pm\frac{1}{2}, 1)$.

These two general possibilities can also formally be distinguished in terms of the low-energy effective Lagrangian for the photon after integrating out the E and M particles. This takes the form

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{max}} + \mathcal{L}_\theta. \qquad (5)$$

The first term is the usual Maxwell term, and the second is the "theta" term:

$$\mathcal{L}_\theta = \frac{\theta}{4\pi^2} \mathbf{E} \cdot \mathbf{B}, \qquad (6)$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, respectively.

As is well known, time reversal restricts the allowed values to $\theta = n\pi$, with n an integer; n even corresponds to Fig. 1 and n odd to Fig. 2 [35].

Each of these two charge-monopole lattices can potentially be realized in several ways depending on the statistics of the quasiparticles and transformation under time reversal. Next, we describe the constraints on these.

B. Quasiparticle statistics and symmetry realization

For a spin or boson system, in the *microscopic* Hilbert space, excitations created by local physical operators must clearly be bosonic. However, the emergent quasiparticles E or M are not created by local operators. For instance, to create an E particle, it is also necessary to create the "electric" field lines that emanate from it and that extend out to arbitrarily long distances. A formal way of describing this is to say that the "creation" operator for E (or M) alone is not gauge invariant. Creating E without the associated electric field violates Gauss law and hence is not in the physical Hilbert space. As E and M are not created by local

physical operators, there is no restriction that they must be bosonic. In three space dimensions, they can thus be either bosons or fermions. Very recently, it has been shown [24,36], however, that in a strictly 3D spin or bosonic system (as opposed to systems that can only appear in the boundary of a $4 + 1$-dimensional system), E and M cannot simultaneously be fermionic. We return to this point below.

Time-reversal symmetry acts in a simple way on physical states in the Hilbert space of spins or bosons. The time-reversal operator (T) is anti-unitary and satisfies $T^2 = +1$ on all physical states. This should be contrasted with electronic systems where $T^2 = -1$ for an odd number of electrons, which leads to Kramers degeneracy. Let us now discuss the possible action of time reversal on the emergent E and M particles. Quite generally, the structure of the emergent Maxwell equations implies that the electric charge is even while the magnetic charge is odd under time reversal.[3] Thus, the E particle and its time-reversed partner TE only differ by a local operator. Then, T^2 acting on the E particle has a well-defined value. Now, as E itself is not local, it could have $T^2 = -1$ and hence be a Kramers doublet (we review some more details in Appendix B). In contrast, M and TM do not differ by a local operator. Then, there is no reason to ask whether M is Kramers or not. Specifically, T^2 acting on M can be shifted by a gauge transformation to have any value [20].

Finally, though E or M may be a fermion, and E may be a Kramers doublet, composite excitations formed out of them that carry zero electric and magnetic charge are physical excitations and hence must be bosonic Kramers singlets.

Starting with these simple but powerful observations, we proceed to describe all the distinct time-reversal-invariant $U(1)$ spin liquids where the photon is the only gapless excitation.

IV. PHASES WITH $\theta = 0$

We first describe phases in which the parameter $\theta = 0$. The charge-monopole lattice of these phases is given by Fig. 1. Here, we distinguish two broad classes of phases depending on whether the M particle is a boson or fermion. We describe each in turn.

A. Bosonic monopole

First, it is clear that there are four distinct phases where M is a boson. The E particle may either be a boson or a fermion and either a Kramers singlet or doublet. Let us try

to better understand these four phases. We label them $E_b M_b$, $E_{bT} M_b$, and $E_f M_b$, $E_{fT} M_b$, respectively, with the subscripts b, f describing the statistics and the symbol T referring to Kramers degeneracy. Some of these are obtained through familiar constructions.

The $E_b M_b$ phase is the one constructed in most of the existing microscopic models [3,5–9]. It is also the state accessed by the gauge mean-field theory of Ref. [11]. If in addition to time reversal there is a global $U(1)$ symmetry, an intuitive way to understand this phase was described in Ref. [7] by obtaining it from a proximate long-range ordered phase [i.e., with broken $U(1)$ symmetry] through proliferating appropriate vortex loops.

The phases $E_{bT} M_b$ and $E_{fT} M_b$ are accessed by the standard Schwinger boson or Abrikosov fermion representation of the physical spin. It is well known that in 3D, these representations can lead to stable $U(1)$ spin liquid phases with gapped electric and magnetic charges.

For spin systems with spin rotation symmetry, it is instructive to obtain $E_{bT} M_b$ by starting with a semiclassical description of a Neel antiferromagnet as follows. Consider a collinear Neel state of a quantum antiferromagnet in 3D. The corresponding order parameter manifold is S^2. An effective field theory description of the long-wavelength fluctuations of the Neel order parameter is provided by the quantum nonlinear sigma model in $3 + 1$ space-time dimensions with the Euclidean action:

$$S_{NL\sigma M} = \frac{1}{2g} \int d^3x d\tau \left((\boldsymbol{\nabla} \hat{n})^2 + \frac{1}{c^2} (\partial_\tau \hat{n})^2 \right). \quad (7)$$

Here, $\hat{n}$ is the local orientation of the Neel vector, and c is the spin wave velocity. In 3D, the order parameter manifold allows for point defects known as "hedgehogs," corresponding to $\Pi_2(S^2) = Z$. This Neel state may be quantum disordered without proliferating these hedgehogs. A convenient framework to describe this is through a CP^1 representation: $\hat{n} = z^\dagger \boldsymbol{\sigma} z$, where z is a two-component complex spinor. Importantly, under time reversal, $\hat{n} \to -\hat{n}$ and $z_\alpha \to i\sigma^y_{\alpha\beta} z_\beta$. Thus, z is a Kramers doublet. The z representation introduces a $U(1)$ gauge redundancy [$z(x,\tau) \to e^{i\theta(x,\tau)} z(x,\tau)$]. The sigma model action represented in terms of z naturally includes a compact $U(1)$ gauge field a_μ. It is well known that the monopoles of the a_μ correspond, in the Neel ordered state, to the hedgehogs of the $\hat{n}$ field. Quantum disordering the Neel state corresponds to gapping out the z particles. If in addition the monopoles stay gapped, the result is precisely a $U(1)$ spin liquid. Furthermore, the z get identified with the E particle and the M with the remnants of the hedgehog. Clearly, E is a Kramers boson. In the semiclassical limit, the hedgehog is also a boson and, consequently, so is the M particle in the $U(1)$ spin liquid. Thus, the phase we obtain is precisely the $E_{bT} M_b$ $U(1)$ spin liquid.

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[3]Strictly speaking, what we call electric and what we call magnetic is a matter of convention: The $U(1)$ gauge theory is self-dual so that we can interchange the definitions of E and M. Maxwell's equations imply that the electric and magnetic charge transform oppositely under time reversal. It is natural to adopt the convention that the magnetic charge is time-reversal odd.

Finally, a microscopic model for the $E_f M_b$ phase was constructed in Ref. [37]. In Sec. VI, we describe how it is related to the other phases, in particular, to the simple $E_b M_b$ phase.

B. Fermionic monopole

We now consider cases in which the M particle carries fermion statistics. If M is a fermion, recent work [24,36] shows that the E particle cannot also be a fermion in a strictly three-dimensional system. With a bosonic E particle, there are, however, still two distinct possibilities corresponding to whether it has $T^2 = +1$ or $T^2 = -1$, i.e., whether it is a Kramers singlet or doublet. In obvious notation, we label these two phases $E_b M_f$ and $E_{bT} M_f$.

We show in Sec. IX how to access these phases through a parton construction.

V. PHASES WITH $\theta = \pi$: "TOPOLOGICAL MOTT INSULATOR"

We now discuss time-reversal symmetric $U(1)$ spin liquids with $\theta = \pi$. We see that there is precisely one such phase.

First, let us discuss the statistics of the elementary dyons with charges $(q_e, q_m) = (\frac{1}{2}, \pm 1)$. We note that these are interchanged under time reversal. Thus, they are required to have the same statistics; i.e., they are both bosons or both fermions. However, we can argue that they cannot both be fermions. To see this most simply, we note that the $(\frac{1}{2}, 1)$ and $(\frac{1}{2}, -1)$ dyons are relative monopoles; i.e., each one sees the other the way an electric charge sees a monopole. If they were both fermionic, we would have a realization of the "all-fermion" $U(1)$ gauge theory in a strictly $3 + 1$-dimensional system, which we know is not possible [24,36]. Therefore, we conclude that both of these dyons must be bosons.

Now, consider the bound state of these two dyons. As this has $q_e = 1$, $q_m = 0$, we identify it with the "elementary" pure electric charge in this phase. Precisely, this bound state was analyzed recently in Refs. [21,24] while studying correlated topological insulators, and it was shown to be a fermion with $T^2 = -1$, i.e., a Kramers doublet. In brief, these two dyons see each other as relative monopoles. This leads to the Fermi statistics of their bound state. The Kramers degeneracy can be simply understood by first calculating the angular momentum of the $U(1)$ gauge field. It is readily seen that this is quantized to be $1/2$. Combining this with the observation that time reversal inverts the relative coordinate of the two dyons leads to $T^2 = -1$ for their bound state. Thus, the statistics and symmetry properties of the elementary electric charge are uniquely determined for this charge-monopole lattice.

Next, consider the elementary pure magnetic charge, which has $q_e = 0$, $q_m = 2$. This can be obtained as the bound state of the $(\frac{1}{2}, 1)$ and $(-\frac{1}{2}, 1)$ dyons. These are also relative monopoles, and hence their bound state is a fermion. Now, time reversal does not interchange these two dyons, and hence the argument above for the Kramers structure of the pure electric charge does not apply. This is, of course, in line with the earlier argument that it is meaningless to ask if M particles are Kramers or not.

We thus see that the structures of both the elementary electric charge and the elementary magnetic charge are uniquely determined for this charge-monopole lattice. In addition, the statistics and symmetry properties of the elementary dyons are also fixed. Thus, there is precisely one time-reversal symmetric $U(1)$ spin liquid phase corresponding to $\theta = \pi$. Given these properties of the elementary pure electric and magnetic charges, we denote this phase $(E_{fT} M_f)_\theta$. The subscript θ is a reminder that these pure charges are composites of more fundamental dyons.

This phase may be constructed within slave particle methods. Let us begin with $E_{fT} M_b$, where the E particle is a Kramers doublet fermion, and it has a conserved electric charge that is even under time reversal. We then put this E particle into a topological band insulator phase. It is well known [38] that the topological band structure leads to a $\theta = \pi$ term in the action for a $U(1)$ gauge field that couples to the E particle. This slave particle construction of the $(E_{fT} M_f)_\theta$ $U(1)$ spin liquid was discussed in Ref. [31] and dubbed the "topological Mott insulator." Reference [31] also suggested that this phase may be realized in $Y_2 Ir_2 O_7$, though this has turned out to be unlikely.

Later, we will describe a completely different slave particle construction of this phase.

VI. RELATIONSHIP BETWEEN THE PHASES

We have thus completed the description of Table I. We now describe how these seven different phases are related to each other. In our previous work (Ref. [20]), we addressed this for a subset of these phases and showed that they can be related to different SPT phases [30] of one of the emergent excitations, similar to what has been discussed for topological orders [39]. Here, we continue this point of view to develop a detailed understanding of the relationship between all seven phases, which is summarized in Fig. 3. This exercise adds much new insight and provides for new constructions of some of these phases. It also helps us obtain theories for some of the quantum phase transitions between these spin liquids.

We have already discussed how $(E_{fT} M_f)_\theta$ may be understood as a topological insulator of the E_{fT} particle. So we now turn to the other phases.

Let us start from the simple $E_b M_b$ phase which can be obtained straightforwardly in microscopic models. We can obtain new phases by either putting M or E in *bosonic* topological insulator phases. E transforms under $U_{eg}(1) \rtimes Z_2^T$ (meaning that the electric charge is $\mathcal{T}$ even), while M transforms under $U_{mg} \times Z_2^T$ (meaning that the magnetic
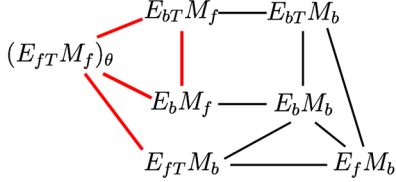
FIG. 3. Relationship between different $U(1)$ spin liquids. Two phases connected through a line share a common fundamental particle (E or M) and can be viewed as different SPT phases formed by the common particle. In Sec. XI, we describe some interesting continuous phase transitions between the phases connected through thick red lines.

charge is $\mathcal{T}$ odd), where $U_{eg}(1)$ is the electric gauge transformation and $U_{\mathrm{mg}}(1)$ is the magnetic gauge transformation.

We discuss this first for the M particle. Consider *bosonic* topological insulators with *global* symmetry $U(1) \times Z_2^T$. There are a total of 16 such phases corresponding to classification by the group Z_2^4. These can be obtained from four "root" phases (the four generators of Z_2^4) by taking their combinations. Two of these root phases are protected by time reversal alone, while the remaining two require the full $U(1) \times Z_2^T$ symmetry. Now consider coupling these bosons to a dynamical $U(1)$ gauge field, i.e., gauging the global $U(1)$ symmetry. The two root phases whose distinction also requires the $U(1)$ subgroup then potentially lead to gauge theories with distinct *bulk* excitations.[4] Taken together with their combinations, we get a total of four potentially distinct $U(1)$ spin liquids. The understanding of such bosonic SPT phases shows that these are precisely the four $U(1)$ spin liquids with a bosonic monopole ($E_b M_b$, $E_f M_b$, $E_{bT} M_b$, $E_{fT} M_b$) discussed in Sec. IV A.

Next, consider starting with $E_b M_b$ and putting E in a boson topological insulator. Such insulators with $U(1) \rtimes Z_2^T$ symmetry are classified by Z_2^3 with three root phases. Of these phases, only one is protected by the full $U(1) \rtimes Z_2^T$ symmetry. Coupling the E, when it forms this SPT state, to a dynamical $U(1)$ gauge field then leads to fermion statistics of the M particle [20,21]. Thus, we obtain the $E_b M_f$ spin liquid. The fermionic statistics of the M particle can be understood from a θ term in the gauge theory with $\theta = 2\pi$ (Ref. [19]) and was called the "statistical Witten effect" [21]. Likewise, starting with the $E_{bT} M_b$ state, one can also put the Kramers bosonic charge E in a SPT state and obtain the $E_{bT} M_f$ state.

Let us now understand the phases obtained by starting with $E_b M_f$ and putting M in a topological insulating phase. We first recall that the monopole transforms under $U_{\mathrm{mg}}(1) \times Z_2^T$, where the $U_{\mathrm{mg}}(1)$ is the (magnetic) gauge

transformation. Free fermions with *global* symmetry $U(1) \times Z_2^T$ can form topological band structure classified by Z; i.e., there are distinct phases indexed by an integer n which counts the number of Dirac cones at the surface. With interactions, this collapses to a Z_8 classification [25,26]. Hence, we only need to consider $n(\mathrm{mod}8)$. Of these $n = 4$ is protected by Z_2^T alone. We now argue that if the global $U(1)$ is gauged, as appropriate in the $U(1)$ spin liquid, then $n = 0, 2$ (and only these) lead to distinct (at the level of bulk excitations) phases.

Here, $n = 0$ corresponds simply to the $E_b M_f$ phase. Interestingly, $n = 2$ corresponds to the $E_{bT} M_f$ phase [25].

A. Puzzle

When $n = 1$, the $U(1)$ gauge field acquires a θ term at $\theta = \pi$. This is an example of a topological Mott insulator that seems distinct from the one discussed in the previous section. In contrast to the description of the $(E_{fT} M_f)_\theta$ as a topological insulator of E_{fT}, here the θ term originates from the M sector and leads to a "dual" Witten effect whereby the E particle acquires magnetic charge $1/2$. How do we reconcile this with our claim that the list of seven phases is complete?

B. Resolution: A dual description of the topological Mott insulator

The resolution of the puzzle above is that the $n = 1$ topological insulator formed by the M_f particles is actually identical to the $(E_{fT} M_f)_\theta$ phase. To see this, consider the charge-monopole lattice of the $n = 1$ monopole topological insulator. This is shown in Fig. 4. Clearly, it is very similar to that of $(E_{fT} M_f)_\theta$. In Sec. III A, we made the choice of units that the minimum pure electric charge is 1. This leads—when there are fundamental dyons—to a minimum pure magnetic charge of 2. We could equally well have chosen units so that the minimum pure magnetic charge is 1. Dirac quantization (together with the existence of fundamental dyons) then would demand that the minimum pure electric charge is 2. This corresponds to taking the lattice in Fig. 2, shrinking the q_m axis by a factor of 2, and expanding the q_e axis by a factor of 2. This converts the
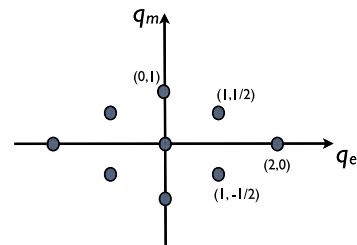

FIG. 4. Charge-monopole lattice obtained by gauging the $n = 1$ M_f topological insulator. It is identical to Fig. 2 after rescaling the two axes as explained in the text.

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[4]The other phases correspond to combining the $U(1)$ liquids with SPT paramagnets protected by time reversal alone. We defer a discussion of these to Sec. VIII.

lattice in Fig. 2 to that in Fig. 4. Clearly, this change in the unit choice does not change the physics. In particular, we correctly find that the pure electric charge (which has charge 2 in these units) is a Kramers fermion.

Thus, this is not a new phase but is rather included in our list.

Let us revert back to the units where the minimum pure electric charge is 1. We see that, remarkably, there are two equivalent descriptions of the $(E_{fT}M_f)_\theta$ phase which are dual to each other. We can either describe it as a topological insulator of E_{fT} or as an $n = 1$ topological insulator of M_f. This leads to a number of interesting consequences which will be explored in subsequent sections.

In Appendix C, we show why M_f topological insulators with other values of n do not lead to distinct $U(1)$ spin liquids. We note, and will discuss in greater detail below, that a duality similar to the one above also exists for the spin liquid $E_{bT}M_f$. First, it can be thought of as the $n = 2$ $U(1) \times \mathcal{T}$ topological insulator of the M_f particle. Equivalently, it can also be viewed as a boson topological insulator with Kramers charge (the E_{bT}). This duality will have interesting consequences for the surface state, which we discuss in Sec. VII B, and for a loop wave function for this phase (Sec. X A).

VII. SURFACE STATES

The understanding of the connection between these $U(1)$ liquids and SPT states immediately raises the question of whether there are nontrivial surface states at the boundary between any of these spin liquids and the vacuum. In this section, we discuss the necessity (or lack thereof) of nontrivial surface states on general grounds. We then discuss two interesting examples: the $(E_{fT}M_f)_\theta$ and the $E_{bT}M_f$ spin liquids, which necessarily have nontrivial surface states. In Appendix D, we discuss surface states of the other phases and the interface between different phases.

A. Why surface states?

To see why (or why not) there should be a surface state between a $U(1)$ spin liquid and the vacuum, we should first understand what exactly a vacuum is. Since the $U(1)$ gauge field "disappears" in the vacuum, we should really think of the vacuum as a confined phase of the $U(1)$ gauge theory. In $(3 + 1)$ dimensions, a $U(1)$ gauge theory can be confined by condensing (Higgsing) either the E or M particle. Therefore, if either the E or M particle is a non-Kramers boson and the vacuum is simply the condensate of that particle, the surface state will be featureless. However, if either the E or M particle carries nontrivial quantum number (fermion statistics or Kramers degeneracy), it cannot directly condense and form the vacuum. Instead, it should go through a "wall" that converts it into a trivial boson so that the trivial boson could condense and form the
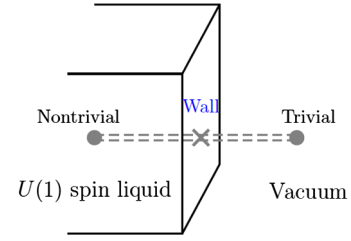

FIG. 5.   The "wall" between a $U(1)$ spin liquid and a Higgsed vacuum. A particle (E or M) tunnels through the wall and becomes a trivial boson, which subsequently condenses and forms the vacuum. Some nontrivial excitation must be left behind on the wall after the tunneling process.

vacuum. The "wall" then forms the surface between the $U(1)$ spin liquid and the vacuum, and obviously something nontrivial is needed on the wall for the conversion (Fig. 5). A similar reasoning was used in a slightly different context in Ref. [21]. We name the E-particle-converting wall as the E-wall, and likewise M-wall for the M-converting wall (a similar notation was also used for $\mathbb{Z}_2$ spin liquids in 2D, for example, in Ref. [40]). Since the E condensate and the M condensate are really the same vacuum, the E-wall and M-wall can evolve into each other through phase transitions on the surface, without actually changing the vacuum.

Therefore, for the two phases $(E_{fT}M_f)_\theta$ and $E_{bT}M_f$, in which all the fundamental particles are nontrivial, a nontrivial surface state is necessary, no matter how we view the vacuum. For the other five phases in Table I, at least one of the E and M particles is a trivial boson; hence, the surface is allowed to be featureless. However, they can nevertheless have nontrivial surface states if we view the vacuum differently. This will be particularly relevant if we consider the interface between two different $U(1)$ spin liquids, which we discuss in Appendix D.

B. Surface of $(E_{fT}M_f)_\theta$

As a particularly interesting example, we now describe the surface of the $\theta = \pi$ spin liquid. Within the Abrikosov fermion slave particle construction, this state (see Ref. [31]) appears as a topological insulator of the E_{fT} particle. A natural conclusion is that the surface will have an odd number of Dirac cones of the E_{fT} particle, which are then coupled to the bulk gapless $U(1)$ gauge field. In the language of Sec. VII A, this is an M-wall since a charge-1/2 dyon can tunnel through the wall and become a pure monopole M, which can subsequently condense and form the vacuum. The M-converting phenomenon is simply a manifestation of the parity anomaly [41] of the surface Dirac cone: A tunneling process that changes the surface flux by 2π will change the total charge [42] on the surface by $\pm 1/2$.

The surface could, just like for the topological insulator, break time-reversal symmetry. The Dirac fermion will then be gapped, but the surface will have a half-integer Hall

conductance for the gauge field. Alternately, the surface could spontaneously enter a Higgs phase by condensing a bosonic Cooper pair formed out of the E_{fT} particles. This corresponds to the surface superconductor, in the underlying topological insulator. Again, the matter fields are gapped on the surface, but there will be vortex excitations with nontrivial fusion and braiding properties. Finally, for the underlying topological insulator, gapped symmetry-preserving surface states with anyonic excitations are also possible [43–46]. In the $U(1)$ spin liquid, these anyons will, if charged, be coupled to the $U(1)$ gauge field. In all these cases, the surface states are M-walls.

The alternate view of this state as a monopole topological insulator naturally leads to a very different gapless symmetry-preserving surface state: The $n = 1$ M_f topological insulator has, at the surface, a single Dirac cone formed out of the M_f particles. This Dirac cone is necessarily at the neutrality point, as the density of M_f is odd under time reversal. In contrast to the electric Dirac cone, tunneling a dyon through the wall gives a pure electric charge. Therefore, this monopole Dirac cone is an E-wall and is a very different "dual" possibility for the surface state of the topological Mott insulator.

Given a realization of the $(E_{fT}M_f)_\theta$ phase, which of these many surface phases is realized will be determined by microscopic details. As the parameters of a microscopic Hamiltonian are tuned, while keeping the bulk in this phase, the surface may undergo phase transitions between these various phases. In particular, the E_{fT} Dirac cone state may transition into the M_f Dirac cone as the parameters are varied. This is related to a "dual" Dirac liquid state that can be realized on the surface of a topological insulator [47,48].

C. Surface of $E_{bT}M_f$

The $E_{bT}M_f$ $U(1)$ spin liquid is obtained naturally as an $n = 2$ M_f topological insulator. This point of view immediately tells us that the surface will have a gapless state with two Dirac cones (at neutrality) of the M_f particles. These will be coupled to the bulk $U(1)$ gauge field. In this case, all M_f topological insulators with $n = 2(\mathrm{mod}4)$ lead to the same $E_{bT}M_f$ $U(1)$ phase in the presence of the bulk gauge field. Therefore, the gapless symmetric surface state may have $n = 2(\mathrm{mod}4)$ number of Dirac cones. This surface is an E-wall since it converts the Kramers E particle into a non-Kramers one upon tunneling. Just as in the previous subsection, alternate surface states with deconfined anyonic excitations or breaking symmetries are also possible.

The $E_{bT}M_f$ state can also have a nontrivial M-wall. We describe it in more detail here as an example of a surface state with gapped matter fields. It will also be useful in constructing a loop wave function of the spin liquid phase, which we discuss in Sec. X A. In such a surface, the electric charge is gapped and forms a $\mathbb{Z}_2$ topological order, with

topological quasiparticles denoted as $\{1, e, m, \epsilon\}$. The symmetries are assigned to the quasiparticles as follows: e and m carry electric charge $q_E = 1/2$, $\mathcal{T}$ switches e and m, and ϵ is charge neutral and a $\mathcal{T}$ singlet. Because e and m have a mutual π statistics and are exchanged under $\mathcal{T}$, the bound state em will have $\mathcal{T}^2 = -1$. Therefore, $e^2 = e(m\epsilon)$ also has $\mathcal{T}^2 = -1$, and likewise for m^2, which is consistent with the bulk physics since they correspond to the charge-1 boson. Following the logic of Ref. [21], tunneling the M through this wall will change its statistics from fermion to boson.

This surface state is precisely the surface of a boson topological insulator formed by the E_{bT} particle. Such topological insulators of Kramers doublet bosons have not been discussed much in the literature (as far as we know). However, their physics is easily deduced using methods developed [19–21] for non-Kramers bosonic topological insulators. Apart from the surface topological order described in the previous paragraph, just like their non-Kramers cousins, the E_{bT} topological insulator has a $\theta = 2\pi$ response and associated fermionic monopoles.

VIII. COMBINING $U(1)$ SPIN LIQUIDS WITH TOPOLOGICAL PARAMAGNETS

So far, we have identified two phases with the same bulk excitation spectrum as the same phase. This is reasonable if the relevant experimental probes are only detecting the bulk physics. However, additional structure needs to be considered if one is also interested in the surface states of the $U(1)$ spin liquids. In particular, one can combine a $U(1)$ spin liquid with a SPT state.[5] This does not change the bulk spectrum in any nontrivial way but may produce distinct surface states.

SPT states with only time-reversal symmetry are also called "topological paramagnets" [19]. In three dimensions, it is known that there are three nontrivial topological paramagnets. Together with the trivial state, they form a $\mathbb{Z}_2^2$ structure, which simply means that combining two copies of the same state always produces a trivial state, and combining two distinct states gives the third distinct state. In the notation of Ref. [20], the three nontrivial states are labeled as

$$eTmT, \qquad efTmfT, \qquad efmf.$$

The common feature of these topological paramagnets is that they all admit gapped surface states with deconfined $\mathbb{Z}_2$ gauge theories, with topological quasiparticles labeled as $\{1, e, m, \epsilon\}$. Notice that we use small letters e and m to label anyons on the surface, which are not related directly

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[5]The meaning of "combining" is to start from two subsystems, one realizing a $U(1)$ spin liquid, the other realizing an SPT state, and couple the two systems weakly.

to the bulk E and M particles. For the $eTmT$ state, both the e and m particles are Kramers doublets with $\mathcal{T}^2 = -1$. For the $efmf$ state, both e and m are fermions. The $efTmfT$ state can be viewed as the combination of the previous two states, in which both the e and m are Kramers fermions. The key property of these $\mathbb{Z}_2$ topological orders is that they cannot be realized in any strictly two-dimensional system while preserving $\mathcal{T}$. Hence, they are called "anomalous."

The three topological paramagnets are distinct and nontrivial states when existing on their own. But do they still give distinct states when combined with a $U(1)$ spin liquid? Or equivalently, is it possible to trivialize the corresponding surface topological order in the presence of various charged matter fields?.

The stability of topological paramagnets in the presence of charged matter fields has been studied for some cases. It is known that all the topological paramagnets are stable if the charged matter field is Kramers ($\mathcal{T}^2 = -1$) or if the $U(1)$ charge is $\mathcal{T}$ odd (magneticlike). In the following, we show that the $eTmT$ phase becomes trivial when the electric charge is either (a) a non-Kramers fermion or (b) a Kramers boson. This corresponds to $E_f M_b$, $E_{bT} M_b$ and $E_{bT} M_f$ in Table I.

The argument is simple: In the presence of an electrically charged particle that is either a non-Kramers fermion or a Kramers boson, one can combine that particle with the e and ϵ particles in the $eTmT$ topological order. This is essentially a relabeling of the same phase. The resulting topological order is $eCmT$, which means the e particle has charge 1 but is non-Kramers, while the m particle is Kramers but charge neutral. This topological order turns out to be realizable even in strictly two-dimensional systems. Hence, it is anomaly-free. One way to realize this state is to start from the $eCT\epsilon CT$ state, which is anomaly-free since the m particle is trivial, and then put the ϵ particle into a 2D topological insulating band. The resulting state is well known [49] to be the $eCmT$.

The $efmf$ state is nontrivial even in the presence of charged particles. The easiest way to see this is to notice that the $\mathcal{T}$-broken surface will have nontrivial thermal Hall conductance, which cannot be canceled by a charge matter field without introducing another Hall conductance for the gauge field.

Therefore, the topological paramagnets give rise to four distinct states when combined with $E_b M_b$, $E_b M_f$, $E_{fT} M_b$, and $(E_{fT} M_f)_\theta$, and only two distinct states when combined with the other three phases in Table I. The total number of phases is thus $4 \times 4 + 3 \times 2 = 22$.

IX. PARTON CONSTRUCTIONS

We now use the insights obtained in previous sections to describe parton constructions of these $U(1)$ spin liquids. First, let us recall that $E_b M_b$ is accessed through the gauge mean-field theory, the $E_{bT} M_b$ through the Schwinger boson

representation, and $E_{fT} M_b$ through the Abrikoson fermion representation. Furthermore, $(E_{fT} M_f)_\theta$ is accessed in the Abrikosov fermion representation by putting the fermionic spinons in a topological band insulator.

To obtain "natural" parton constructions for the other phases, let us consider a spin-1/2 magnet on some lattice and use the Abrikosov fermion representation:

$$S_r = \frac{1}{2} f_{r\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} f_{r\beta}. \tag{8}$$

Here, $f_{r\alpha}$ is a fermion of spin $\alpha = \uparrow, \downarrow$ at sites r of the lattice. As is well known, this representation introduces an $SU(2)$ gauge redundancy, and correspondingly, the physical Hilbert space of the microscopic spin system is obtained by imposing a constraint [2].

It is convenient for some of our discussion to work with Majorana fermions $\eta_{ar\alpha}$ ($a = 1, 2$) rather than the complex fermions $f_{r\alpha}$. We therefore define

$$f_{r\alpha} = \frac{1}{2}(\eta_{1r\alpha} - i\eta_{2r\alpha}). \tag{9}$$

Let ρ^x, ρ^y, ρ^z be Pauli matrices acting in η_1, η_2 space. It is easy to check that the physical spin operators can be written as

$$S_r = \frac{1}{8} \eta_r^t (\rho^y \sigma^x, \sigma^y, \rho^y \sigma^z) \eta_r. \tag{10}$$

The $SU(2)$ gauge redundancy of the fermion representation is generated by the operators

$$\mathbf{T} = \frac{1}{8} \eta_r^t \mathbf{I} \eta_r, \tag{11}$$

with $\mathbf{I} = (\sigma^y \rho^x, \rho^y, \sigma^y \rho^z)$. From these, we can construct $SU(2)$ gauge transformations O_r, which rotate the Majorana fermions:

$$\eta_r \rightarrow O_r \eta_r. \tag{12}$$

We first review how to obtain a $U(1)$ spin liquid through this fermionic parton construction before explaining how to implement time reversal. We consider a mean-field ansatz described by a Hamiltonian quadratic in the fermion operators:

$$H_{\text{mean}} = \sum_{rr'} \eta_r^t h_{rr'} \eta_{r'}, \tag{13}$$

where $h_{rr'}$ is a pure imaginary 4×4 matrix. Furthermore, we must have $h_{rr'}^t = -h_{r'r}$. Under the $SU(2)$ gauge transformation, $h_{rr'}$ gets replaced by $O_r^t h_{rr'} O_{r'}$.

The unbroken gauge structure is determined by considering the "Wilson loop" starting from some base point r,

$$W_r[C] = \prod h_{r_i r_{i+1}}. \tag{14}$$

The right side is an ordered product over the h matrices connecting the points r_i, r_{i+1} that define the closed curve C. To get a $U(1)$ spin liquid, all the $W_r C$ (for different C and bases r) must be invariant under a $U(1)$ subgroup of the full $SU(1)$ gauge group and only this $U(1)$ subgroup. In this case, there will be a $U(1)$ subgroup of the gauge transformation

$$M(\phi) = \prod_r M_r(\phi) \tag{15}$$

[with each $M_r(\phi)$ describing an $SO(2)$ gauge rotation by angle ϕ], which leaves the mean-field invariant:

$$M_r(\phi)^t h_{rr'} M_{r'}(\phi) = h_{rr'}. \tag{16}$$

Let n_r be the infinitesimal Hermitian generators of M_r for each r (n_r will be a purely imaginary, antisymmetric 4×4 matrix). Upon including fluctuations, n_r will correspond to the generators of $U(1)$ gauge transformations of the spin liquid.

Now, let us consider implementation of physical global symmetries.

In line with the rest of the paper, we consider systems where time reversal is a good global symmetry. We make no assumptions about spin rotation symmetry. To discuss time-reversal properties, it is important to distinguish two distinct microscopic situations. The physical Hilbert space at each site consists of two states—these may correspond either to a Kramers doublet or to a non-Kramers doublet. Note that this distinction should not be confused with the time-reversal properties (Kramers or not) of the emergent E particle excitations. When the physical on-site Hilbert space corresponds to a Kramers doublet, the spin operators transform under time reversal as

$$S_r^z \to -S_r^z, \qquad S_r^+ \to -S_r^-. \tag{17}$$

In contrast, if the physical on-site Hilbert space corresponds to a non-Kramers doublet, we take the spin operators to transform under time reversal as

$$S_r^z \to -S_r^z, \qquad S_r^+ \to S_r^-. \tag{18}$$

For clarity, we focus henceforth on Kramers spin systems [Eq. (17)]. It is straightforward to extend the discussion to non-Kramers spins. Let us now implement time reversal on the fermion operators. We may generally write

$$\eta_r \to \tilde{T} \eta_r, \tag{19}$$

where $\tilde{T}$ is a 4×4 real matrix. Clearly, we also have the freedom to gauge transform the fermions as part of the

symmetry implementation; i.e., we can multiply $\tilde{T}$ by any gauge rotation O_r.

For Kramers spins satisfying Eq. (17), we can take $f_r \to i\sigma^y f_r$. This is equivalent to

$$\tilde{T} = i\sigma^y \rho^z \eta_r. \tag{20}$$

If a mean-field ansatz is time-reversal invariant, then we must have

$$\tilde{T}^t(-h_{rr'})\tilde{T} = O_r h_{rr'} O_{r'}^t \tag{21}$$

for some gauge transformation O_r. There is a $(-)$ sign on the left side because time reversal is anti-unitary and $h_{rr'}$ is pure imaginary. Thus, we can define a "physical" time-reversal transformation T_r (for any given mean field) through

$$T_r = \tilde{T} O_r, \tag{22}$$

under which the mean field is *manifestly* time-reversal invariant:

$$T_r^t h_{rr'} T_{r'} = -h_{rr'}. \tag{23}$$

Now, let us consider the algebra of the $U(1)$ gauge generators n_r and the physical time-reversal transformation. The gauge charge N_r at site r is

$$N_r = \eta_r^t n_r \eta_r. \tag{24}$$

Under time reversal, we have

$$\mathcal{T}^{-1} N_r \mathcal{T}^{-1} = \eta_r^t T_r^t(-n_r) T_r \eta_r. \tag{25}$$

There is a $(-)$ sign on the right side because n_r is pure imaginary. Generally, we have

$$T_r^{-1} n_r T_r = \pm n_r. \tag{26}$$

The $(-)$ sign describes the group $U(1) \rtimes Z_2^T$, and the $(+)$ sign describes the group $U(1) \times Z_2^T$ (note that if $n_r \to -n_r$, then the gauge charge $N_r \to N_r$ so that the gauge charge is even under time reversal). In the former case, we should take the fermions to be the E particle of the gauge theory, and in the latter, we should take them to be the M particle. Thus, the same parton framework naturally describes both classes of phases where E is a fermion or where M is a fermion.[6]

Note that $T_r^2 = \tilde{T} O_r \tilde{T} O_r$. Even though $\tilde{T}^2 = -1$, we do not *a priori* know anything about T^2. However, we know

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[6]It is easy to show that if n_r is even under time reversal at one site, it must be even at all other sites that are connected to it and vice versa.

that if n_r is even under T_r, then $T_r^2 = \pm 1$ [while if n_r is odd, we can define a modified time reversal $T_r M_r$ and $(T_r M_r)^2$ can have any value]. Thus, by choosing the mean-field ansatz (which enables us to define T_r and M_r), we can access phases where E is either a Kramers singlet or Kramers doublet fermion.

We now use this framework to construct examples of the $E_b M_f$, $(E_{fT} M_f)_\theta$, and $E_{bT} M_f$ phases. For concreteness, we specialize to the three-dimensional cubic lattice. Consider a mean-field ansatz where there is a nearest-neighbor hopping t and a singlet pairing Δ on the body diagonal. This corresponds to

$$h_{rr'} = t_{rr'}\rho^y + \Delta_{rr'}\sigma^y\rho^x. \tag{27}$$

It is easy to check that the fermion spectrum is gapped. Furthermore, the nontrivial Wilson loops are proportional to $\sigma^y\rho^z$ so that this is a gapped $U(1)$ spin liquid. Correspondingly, we have $M_r(\phi) = e^{i\phi\epsilon_r\sigma^y\rho^z}$, where $\epsilon_r = +1$ on the A sublattice and -1 on the B sublattice. The physical time-reversal operator can be simply taken to be $T = i\sigma_y\rho^z$. Thus, the generator $\sigma^y\rho^z$ of M_r is odd under T, and the fermions should be identified with the M particle. Furthermore, it is also readily checked that the band structure is not topological. We thus have a realization of the $E_b M_f$ phase.

Next, let us modify the t and Δ to get a topological band structure. Precisely such a modification was discussed in Ref. [50], and it requires changing the sign of the t and Δ on some of the bonds. This yields an $n = 2$ topological insulator with two surface Dirac cones. The $U(1)$ gauge structure and time-reversal properties are not affected by this modification. We thus end up with the $E_{bT} M_f$ phase.

Finally, to construct the $(E_{fT} M_f)_\theta$ phase, we use a different implementation of time reversal. We take $\eta_r \to \epsilon_r\eta_r$ corresponding to $T_r = \epsilon_r$. We take a band structure in which the two $\eta_\uparrow$ fermions (which make up the complex fermion $f_\uparrow$) have different dispersion than the $\downarrow$ fermions. Specifically, we choose the $\uparrow$ band structure described in Ref. [50] for the $n = 1$ topological insulator, while for the $\downarrow$ we choose a trivial dispersion.[7] It is easy to check that this mean-field ansatz describes a $U(1)$ spin liquid and, furthermore, that the n_r are odd under time reversal. Thus, the fermions must be identified with M. Furthermore, as we have a net $n = 1$ topological band structure, we get the $(E_{fT} M_f)_\theta$ phase.

We emphasize that this construction is totally different from the "standard" one where the fermions are treated as Kramers doublet E particles with topological band structure. Nevertheless, we get the same phase.

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[7]This band structure is invariant under the chosen time-reversal operation.

A. Kramers spin on nonbipartite lattice

1. M_f no-go

The previous examples of M_f-type parton construction, in which the fermions are monopolelike with their $U(1)$ gauge charge odd under $\mathcal{T}$, were constructed on a bipartite lattice (cubic lattice). We now show that for Kramers spins [Eq. (17)], such M_f-type construction is impossible on a nonbipartite lattice. First, we note that in Eq. (26), we must choose the $+$ sign in this case. Second, we notice that $\tilde{T} = i\sigma^y\rho^z$ is itself a gauge rotation. Thus, the physical time-reversal matrix $T_r = \tilde{T}O_r$ is also just an $SU(2)$ gauge rotation.

Thus, we may write $T_r = e^{i\theta_r\tau_r}$, where τ_r is a Hermitian generator. In general, τ_r is a combination of the three generators in Eq. (11), satisfying $\tau_r^2 = 1$ and $\tau_r^* = -\tau_r$.

With the $+$ sign in Eq. (26), we have $[T_r, n_r] = 0$. This is possible only if $n_r = \pm\tau_r$. Any Wilson line with base r [see Eq. (14)] satisfies

$$M_r^t(\phi)W_r[C]M_r(\phi) = W_r[C]. \tag{28}$$

As T_r corresponds to a special value of ϕ, we also have

$$T_r^t W_r[C]T_r = W_r[C]. \tag{29}$$

However, using $T_r^t h_{rr'} T_{r'} = -h_{rr'}$, we can also conclude that for a loop of length L

$$T_r^t W_r[C]T_r = (-1)^L W_r[C]. \tag{30}$$

We thus conclude that L must be even, which is possible only if the lattice is bipartite. A related argument using the trace of the Wilson loop to diagnose time-reversal breaking was presented in Ref. [51].

This shows that for Kramers spins on nonbipartite lattices (such as the pyrochlore), the fermionic monopole does not arise within the particular (although most common) type of parton construction from Eq. (8). We should emphasize that this does not rule out the possibility of having such phases in this situation since one can imagine having more complicated types of parton construction. However, this does suggest that states with fermionic monopoles are less natural in these systems.

2. E_f no-go

Following the same logic, we now show that E_f-type parton construction, in which the fermions are electronlike but non-Kramers, is also impossible for Kramers spins on nonbipartite lattices. To have $T_r^2 = 1$ on the η fermions, the only possibility is $T_r = \pm 1$. Clearly, $W_r[C]$ must be real to preserve time reversal. But as we discussed above in Sec. IX A 1, on a nonbipartite lattice, there must exist imaginary Wilson loops. Therefore, such a construction is impossible.

Again, we emphasize that this does not rule out the $E_f M_b$ state, but it does make it less natural in nonbipartite Kramers spin systems.

X. LOOP WAVE FUNCTIONS

It is interesting to understand the differences between these different states in terms of their ground-state wave functions. To that end, it is useful to think of the $U(1)$ spin liquid in terms of fluctuating loop configurations. As matter fields (the E and M particle) are gapped, the low-energy physics is described by Maxwell electrodynamics. The emergent electric and magnetic fields are divergence-free, and hence the corresponding field lines form closed loops. To describe the wave function, we can choose either the electric picture or the magnetic picture (these are different bases for the low-energy Hilbert space).

In the specific context of quantum spin ice, the magnetic flux loops are very easy to picture. Indeed, the spin ice manifold is parametrized in terms of closed loop configurations formed by the directions of the microscopic spins on the pyrochlore lattice. Quantum effects introduce fluctuations of these magnetic loops and, in the spin liquid, lead to tensionless fluctuating loops in the ground state. The simplest possibility is that the wave function of the fluctuating magnetic loops is positive definite:

$$|\Psi\rangle = \sum_{\mathcal{C}} \Psi_0(\mathcal{C})|\mathcal{C}\rangle,$$

$$\Psi_0(\mathcal{C}) \sim e^{-\int d^3x d^3x' \alpha \frac{\mathbf{B}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}}, \tag{31}$$

where α is a positive constant and $\mathbf{B}$ is the magnetic field corresponding to the magnetic loop configuration $\mathcal{C}$. The positive weight Ψ_0 is needed to satisfy Maxwell's equation. Such a "featureless" wave function would describe the $E_b M_b$ phase, as studied in many previous works. The M particle is the open end of the magnetic loops, and the E particle is a point defect with an additional phase factor in the wave function:

$$|\Psi(E)\rangle = \sum_{\mathcal{C}} e^{i \sum_I \Omega_I / 2} \Psi_0(\mathcal{C})|\mathcal{C}\rangle, \tag{32}$$

where I labels each loop in the configuration $\mathcal{C}$, and Ω_I is the solid angle spanned by the loop I with respect to the E particle.

To describe the other six phases in Table I, more subtle structures are needed in the loop wave function. For the $E_b M_f$ phase, the monopoles—end points of the magnetic loops—need to become fermions. This can be done by thickening the magnetic loops into "ribbons" and assigning a phase (-1) to the wave function whenever a ribbon self-links. More precisely, the wave function can be written as

$$|\Psi\rangle = \sum_{\mathcal{C}} (-1)^{L_{\mathcal{C}}^S} \Psi_0(\mathcal{C})|\mathcal{C}\rangle, \tag{33}$$

where $L_{\mathcal{C}}^S$ is the self-linking number, defined to be the linking number of the two boundary loops of each magnetic ribbon. An argument in Ref. [52] shows that because of this extra phase, the open end points of such loops have Fermi statistics.

To understand some of the other phases described in Table I in terms of fluctuating loops, it is more convenient to use instead the "electric" picture: The ground state is then a superposition of oriented loops (which represent the electric field lines) with weights derived from the Maxwell action $\Psi_0 \sim e^{-\int d^3x d^3x' \alpha' \{[\mathbf{E}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}')]/|\mathbf{x}-\mathbf{x}'|^2\}}$. In the $E_b M_b$ phase, these electric loops are featureless, and the superposition has positive-definite weights for all loop configurations just as in Eq. (31).

In the $E_{bT} M_b$ phase, we can think of the electric-field lines as "stuffed" with 1D Haldane chains. One way to do this is to consider an additional spinlike order parameter $\hat{n}$ in the disordered paramagnetic phases and assign a Wess-Zumino phase factor in the wave function:

$$|\Psi\rangle = \sum_{\mathcal{C}} e^{iW[\hat{n}(\mathcal{C})]} \Psi_0(\mathcal{C})|\mathcal{C}; \hat{n}\rangle. \tag{34}$$

The easiest way to picture the Wess-Zumino term is to view $W[\hat{n}(\mathcal{C})]/2\pi$ as the total Skyrmion number of $\hat{n}$ on the membranes whose boundaries are the electric loops $\mathcal{C}$. For a closed loop, this internal structure has no serious effect. However, if we produce an electric charge, we expose an open end of the electric-field line. The Kramers doublet known to be present at the open end of the Haldane chain then leads to the Kramers degeneracy of the electric charge. Notice that this possibility is meaningful because the electric loop configurations are time-reversal invariant. In contrast, the magnetic loops cannot be stuffed with Haldane chains, in line with the discussions in the rest of the paper.

If instead the E particle is a fermion (as in $E_f M_b$ or $E_{fT} M_b$), then the electric field is best thought of as a thin ribbon (i.e., a line with some small but nonzero thickness). Again, we assign a phase (-1) to an electric-field loop that has an odd self-linking number, which converts the E particle into a fermion. For the $E_{fT} M_b$ phase, in addition, these electric loops must be stuffed with Haldane chains.

In all the examples above, at least one of the E and M particles is trivial (bosonic and non-Kramers). This makes it simpler to describe the wave function by considering the loops with nontrivial open ends. For example, if the E (M) particle is trivial, we can write the wave function as fluctuating M (E) loops and demand that they have the right structure to produce nontrivial end points, which are the M (E) particles.

The remaining two phases in Table I [$E_{bT} M_f$ and $(E_{fT} M_f)_\theta$], however, cannot be easily understood using the above line of thinking because both E and M particles are nontrivial. The loop wave functions for these two phases should capture not only the quantum numbers of the

end points but also those of the dual particles. A similar issue arises if we want to understand the previous phases in the dual loop picture. For example, can we have a magnetic loop wave function for the $E_{fT}M_b$ state?.

This issue is actually closely related to the surface states of the phases: If a $U(1)$ spin liquid phase necessarily has a nontrivial surface state, one should be able to infer it from the bulk wave function. The two phases $E_{bT}M_f$ and $(E_{fT}M_f)_\theta$ both have nontrivial surface states as long as time reversal is kept. For the other phases like $E_{fT}M_b$, the surface has to be nontrivial as long as time reversal is kept and the M particle is not condensed on the boundary. Since the M particle is naturally not condensed in the fluctuating magnetic loop picture, the wave function of magnetic loops should contain the information of the nontrivial boundary theory. Similar logic also applies to the other phases except E_bM_b.

Notice that when the wave function is written solely in terms of closed loops, the matter fields are naturally gapped, even on the boundary. Therefore, to have loop wave functions for the above nontrivial cases, we need the loop structures to be able to produce boundary theories with gapped matter fields. Since the gapped matter fields on a nontrivial boundary are necessarily fractionalized, this suggests that the bulk wave function should be described in terms of fractional loops, instead of the "physical" loops such as the 2π magnetic loop.

A similar problem was tackled in the context of SPT phases using what is known as Walker-Wang construction [23,45,53–55]. The essential idea is that when the boundary is a gapped topological order, one can have a loop wave function for the bulk, for which the weights are knot invariants of the loop configurations generated by the boundary topological field theory. With some modification, this idea can be used to generate loop wave functions of $U(1)$ spin liquids (other than the simple E_bM_b) in Table I. In Sec. X A, we discuss a relatively simple yet interesting loop wave function of the $E_{bT}M_f$ phase as an illustrating example. With different time-reversal implementation, the same wave function can also describe two other phases, namely, the E_bM_f in the electric loop picture and E_fM_b in the magnetic loop picture, which we describe in Sec. X B. In Appendix E, we discuss a slightly different wave function that can describe the $E_{bT}M_b$ and $E_{fT}M_b$ phases. The topological Mott insulator $(E_{fT}M_f)_\theta$ can also be described through gauging its (non-Abelian) Walker-Wang wave function [45]; however, it will be quite complicated and not very illuminating, so we will omit the discussion in this paper.

A. Loop wave function for the $E_{bT}M_f$ phase

For the $E_{bT}M_f$ phase, it turns out to be slightly easier to describe the wave function in terms of electric loops. The wave function is written in terms of two species of oriented loops, labeled as "red" (r) and "blue" (b):

$$|\Psi\rangle = \sum_{\mathcal{C}_r,\mathcal{C}_b} (-1)^{L_{\mathcal{C}_r,\mathcal{C}_b}} \Psi_0(\mathcal{C})|\mathcal{C}_r,\mathcal{C}_b\rangle, \qquad (35)$$

where $L_{\mathcal{C}_r,\mathcal{C}_b}$ is the mutual linking number between red and blue loops. Two additional features are present in the wave function. First, a double blue line can be converted to a doubled red line and form a double two-segment loop, even though single lines cannot be converted to each other. Second, the red and blue loops get switched under time-reversal action $\mathcal{T}$, which is allowed since these are electric-like loops. These features are illustrated schematically in Fig. 6.

To see that the wave function described in Fig. 6 indeed describes the $E_{bT}M_f$ phase, we need to examine the excitation spectrum. First of all, it is useful to examine the bound state of one blue loop and one red loop, with opposite directions. We call this the ϵ loop. Notice that because of the condition in Fig. 6(b), the ϵ loop is undirected. We also note that the ϵ loop has a linking sign with both blue and red loops. This makes the end points of an individual open blue or red line confined, in the sense that they cost an energy proportional to the length of the open line. The reason for this is that with an open blue or red line, a small ϵ loop surrounding the interior part of the line locally behaves as though it is "linked" with the line. However, it cannot have a linking sign in the wave function. This is because one can continuously move the small ϵ loop away from the open line until it looks "unlinked." Therefore, the local Hamiltonian near the interior of the open line cannot be minimized. Thus, the energy penalty will be proportional to the length of the line. The same physics also appears in Walker-Wang models. (See also Ref. [56] for a simple and concrete model illustrating this.)

On the other hand, a double red loop (or equivalently, a double blue loop) can be opened with finite energy cost since it does not have a linking sign with anything. Therefore, we interpret the end points of double red loops
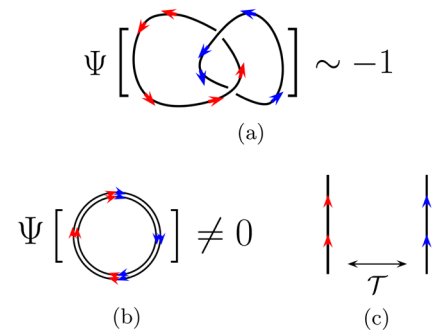

FIG. 6. Electriclike loop wave function of the $E_{bT}M_f$ $U(1)$ spin liquid. (a) The amplitude changes sign whenever a blue loop links with a red one. (b) A double blue line can be converted to a double red line and form a double two-segment loop. (c) The red and blue loops are switched under time reversal $\mathcal{T}$.

as the deconfined E particles. The single red or blue loops are "half" electric loops.

The open end of a bare ϵ line is confined because it has linking signs with the red and blue loops. However, a monopolelike defect can be bound at the end of an ϵ line to avoid the sign ambiguity and make it deconfined. More precisely, we can have a phase factor in the wave function of an open ϵ line $e^{i\sum_I(\Omega_I^a-\Omega_I^b)/4}$, where I denotes all the red and blue loops, and $\Omega_I^{a,b}$ is the solid angle spanned by a red or blue loop with respect to the end points a, b. This phase factor serves as a smooth interpolator between the linking phase away from the line $(+1)$ and the linking phase near the interior of the line (-1). Since the single red and blue loops are interpreted as half electric loops, such a phase factor corresponds precisely to a magnetic monopole with unit magnetic charge.

Therefore, the magnetic monopole is bounded to the end point of an ϵ line. But notice the ϵ loop is really a ribbon, with a red and a blue loop being the edges of the ribbon. Therefore, it has a self-linking sign from the red and blue mutual linking sign, which makes the end point a fermion. We have thus obtained fermion statistics of the magnetic monopole.

We now discuss time-reversal action on the E particle, which is the end point of a double blue line. For this purpose, it is convenient to view the double blue line as the combination of a blue, a red, and an ϵ line. Under time reversal $\mathcal{T}$, the ϵ line is invariant, but the blue and red lines are exchanged [see Fig. 6(c)]. Since the time-reversed wave function away from the charged particle is locally indistinguishable from the original wave function, the $\mathcal{T}$ action can be effectively "localized" around the E particle, by exchanging the two end points of the red and blue lines. Therefore, performing $\mathcal{T}$ twice amounts to twisting the blue-red ribbon, which gives a (-1) phase in the wave function. This implies that the E particle has $\mathcal{T}^2 = -1$ and is a Kramers doublet.

Similar to the Walker-Wang construction, the above discussion is closely related to a possible surface state of the $E_{bT}M_f$ phase that is gapped but breaks no symmetry. This is exactly the M-wall state discussed in Sec. VII C.

B. Alternative loop wave functions for E_bM_f and E_fM_b

The same wave function described in Fig. 6 can also describe two other phases, with different ways of implementing time-reversal symmetry.

If time-reversal $\mathcal{T}$ keeps both the color and the direction of each loop [Fig. 7(a)]; then the loops are electriclike. The argument in Sec. X A for the fermionic monopole still applies in this case. But the electric charge—the end point of a double red (or blue) line—now transforms trivially under time reversal. Hence, we obtain the E_bM_f state in the electric loop picture.
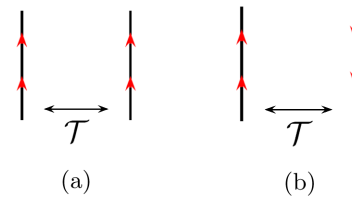

FIG. 7. The same loop wave function as in Fig. 6, but with different time-reversal actions. (a) Time reversal keeps both the color and the direction of each loop, which describes E_bM_f in the electric picture. (b) Time reversal keeps the color but inverts the direction of each loop, which describes E_fM_b in the magnetic picture.

Now, if time reversal keeps only the color but inverts the direction of each loop [Fig. 7(b)], the loops are magneticlike. The argument in Sec. X A for the fermion statistics of the ϵ particle still applies, but now this particle should be interpreted as the electric particle E. We therefore obtain the E_fM_b state in the magnetic loop picture.

XI. QUANTUM PHASE TRANSITION BETWEEN TWO DISTINCT $U(1)$ SPIN LIQUIDS

In this section, we consider quantum phase transitions from one $U(1)$ spin liquid to another in Table I. In general, one may expect most of the transitions to be first order or to go through an intermediate phase because of the lack of any obvious order parameter. Thus, any continuous transition between two such phases would be quite exotic. The understanding of these spin liquids in terms of gauged SPT states sheds some new light on this subject. If we can understand possible continuous transitions between different SPT states, we can then couple these critical theories to $U(1)$ gauge fields and understand continuous transitions between $U(1)$ spin liquids. The common feature of such phase transitions is that one particle (say, E) is unchanged across the transition, while the dual particle (say, M) drastically changes its properties such as statistics, $\mathcal{T}^2$ value, and dyon charge.

Continuous transitions between SPT phases in free fermions are well understood [57]. Interestingly, by gauging such transitions, we can obtain many novel transitions between various $U(1)$ spin liquids. All such critical theories are described by massless Dirac fermions in $(3+1)$ dimensions coupled with a $U(1)$ gauge field, but the effect on the dual particles is very different. We are thus able to provide remarkably simple descriptions of some highly nontrivial continuous phase transitions between distinct $U(1)$ spin liquids. For instance, we provide a theory for a continuous phase transition between the topological Mott insulator and the E_bM_f phase. In the conventional picture of the topological Mott insulator as a spinon topological insulator, such a transition seems to require a change of statistics of the electric charge. Such a "statistics" changing quantum phase transition is, however, very simply

understood within the dual picture of the topological Mott insulator (as a monopole topological insulator) developed in this paper.

A. "Statistics-changing" quantum criticality: Phase transitions of the topological Mott insulator

1. Warm-up: To $E_{fT}M_b$

It is useful to first understand the phase transition from the topological Mott insulator—the $(E_{fT}M_f)_\theta$ phase—to the $E_{fT}M_b$ phase. This will set the stage for the more surprising (from the conventional viewpoint) phase transitions studied below. Since $(E_{fT}M_f)_\theta$ can be viewed as a gauged version of a topological insulator formed by the E_{fT} particles, we can access its transition into the $E_{fT}M_b$ phase by gauging the topological-to-trivial insulator transition. The critical theory is simply a massless QED with one flavor:

$$\mathcal{L}[\psi, \bar{\psi}, a_\mu] = \bar{\psi}(i\slashed{\partial} + \slashed{a})\psi + im\bar{\psi}\psi + \mathcal{L}_{\text{Maxwell}}[a_\mu], \quad (36)$$

where $\gamma_0 = \tau_1, \gamma_i = \sigma_i \tau_2, \bar{\psi} = i\psi^\dagger \gamma_0$, and time reversal acts as $\mathcal{T}\psi\mathcal{T}^{-1} = i\sigma_2\psi$ (here, σ_i and τ_i are Pauli matrices). One can easily check that the $i\bar{\psi}\psi$ term is the only $\mathcal{T}$-symmetric mass term. Under a proper UV background, $m > 0$ gives a trivial insulator, which after gauging becomes the $E_{fT}M_b$ phase, and $m < 0$ gives the topological insulator, which after gauging becomes the $(E_{fT}M_f)_\theta$ phase. Here, $m = 0$ is thus the critical point.

However, to make the transition really continuous, we should also forbid other $\mathcal{T}$-invariant terms that close the fermion gap before the mass m becomes zero. There are two such terms: $\mu\psi^\dagger\psi$ and $\mu'\psi^\dagger\tau_3\psi$. The former is simply the chemical potential term, and the latter can be viewed as a chemical potential alternating in sign for the two Weyl fermions. The chemical potential is forbidden since the total gauge charge should be zero (assuming the fermion gap does not close elsewhere). But to forbid the $\mu'\psi^\dagger\tau_3\psi$ term, more symmetry is required in the theory. The simplest possibility is to demand inversion symmetry $\mathcal{I}\psi(\mathbf{x})\mathcal{I}^{-1} = \tau_1\psi(-\mathbf{x})$, as was shown in Ref. [57]

In general, various anisotropy terms are also allowed. These include the spatial anisotropy, the velocity difference between different bands, and the difference between the fermion velocity and the speed of the emergent photon. We will not go into those details here.

2. Topological Mott insulator to $E_b M_f$

As discussed in previous sections, the $(E_{fT}M_f)_\theta$ phase can also be viewed as a "dual" topological insulator of the fermionic monopoles. This makes it possible to access its transition to the $E_b M_f$ phase, in which the fermionic monopoles form a trivial insulator. The critical theory has the same Lagrangian as Eq. (36), but with a different

implementation of time-reversal symmetry: $\mathcal{T}\psi\mathcal{T}^{-1} = i\tau_2\psi^\dagger$.

Again, we need to forbid $\mathcal{T}$-invariant terms that can make the fermions gapless. There is only one such term, namely, the alternating chemical potential $\mu'\psi^\dagger\tau_3\psi$. We can again forbid it by having inversion symmetry $\mathcal{I}\psi(\mathbf{x})\mathcal{I}^{-1} = \tau_1\psi(-\mathbf{x})$.

3. Topological Mott insulator to $E_{bT}M_f$

The previous transition into the $E_b M_f$ phase can be viewed as a transition of the fermion monopoles from an $n = 1$ band to an $n = 0$ band. Now, if we consider a transition from an $n = 1$ band to an $n = 2$ band, this gives a transition from the topological Mott insulator to the $E_{bT}M_f$ phase. The gapless part of the critical theory is the same as the previous transition described in Sec. XI A 2. The only difference is that here we have an extra gapped Dirac fermion with negative mass, which gives the $n = 1$ band in the UV background. This UV background is important in determining the nature of the phases away from the critical point. But it will not affect the physics right at the critical point such as the scaling relations. One can see this by integrating them out, which gives a θ term in the $U(1)$ gauge field. But since the dual particle (charge) is gapped at the critical point, the θ term is a purely surface term and will not affect the bulk dynamics.

B. Kramers-changing quantum criticality: Phase transition between $E_{bT}M_f$ and $E_b M_f$ spin liquids

The $E_{bT}M_f$ and $E_b M_f$ spin liquids differ by whether or not the E particle is a Kramers doublet. We now discuss the criticality associated with a continuous transition where this Kramers-ness changes. The $E_{bT}M_f$ phase can be viewed as fermionic monopoles in an $n = 2$ band. We have discussed the transition from the $E_{bT}M_f$ phase to the topological Mott insulator in Sec. XI A 3. We now discuss the transition to the $E_b M_f$ phase, which is a trivial insulator ($n = 0$) of the fermion monopoles.

The critical theory has two mass Dirac fermions coupled with a $U(1)$ gauge field:

$$\mathcal{L}[\psi_s, \bar{\psi}_s, a_\mu] = \sum_{s=1,2} \bar{\psi}_s(i\slashed{\partial} + \slashed{a})\psi_s + im\bar{\psi}_s\psi_s$$
$$+ \mathcal{L}_{\text{Maxwell}}[a_\mu], \quad (37)$$

where $\gamma_0 = \tau_1, \gamma_i = \sigma_i \tau_2, \bar{\psi}_s = i\psi_s^\dagger \gamma_0$, and time reversal acts as $\mathcal{T}\psi_s\mathcal{T}^{-1} = i\tau_2\psi_s^\dagger$ (here, σ_i and τ_i are Pauli matrices). Again, we need inversion symmetry $\mathcal{I}\psi_s(\mathbf{x})\mathcal{I}^{-1} = \tau_1\psi_s(-\mathbf{x})$ to forbid the alternating chemical potential term $\mu'\psi_s^\dagger\tau_3\psi_s$. Moreover, we need $im\bar{\psi}_s\psi_s$ to be the unique mass term. This requires some kind of rotation symmetry between the two flavors $s = 1, 2$.

Microscopically, this could be achieved with certain lattice symmetries. We will not go into such details here.

XII. ROLE OF TIME REVERSAL

We should emphasize that the seven phases are distinct if and only if time-reversal symmetry is kept. In the absence of time reversal, $\mathcal{T}^2$ of course has no meaning, and the θ angle can be continuously tuned to any value. Even different statistics of particles do not distinguish phases: If the magnetic particle is a fermion, the electric particle is necessarily a boson, and one can change the θ angle by 2π to shift the m particle to a boson [21]. A similar argument applies if the electric particle is a fermion.

Thus, in the absence of any symmetries we have exactly one $U(1)$ liquid phase.

XIII. RELATIONSHIP WITH OTHER WORKS

We have focused on time-reversal symmetric $U(1)$ quantum spin liquids. We now place our results in the broader context of research on symmetry implementation in other long-range entangled phases. The best-understood long-range entangled phases have a gap to all bulk excitations and are characterized by the concept of topological order. In the last few years, the realization of symmetry in such topologically ordered phases has received a great deal of attention. Such states have been dubbed "symmetry enriched topological" (SET) phases. As is well known, in a SET phase, the topologically nontrivial quasiparticles may carry fractional quantum numbers. This means that the action of symmetry on these quasiparticles is nontrivial (technically, the symmetry is realized projectively rather than linearly). The projective realization is allowed since these quasiparticles are nonlocal objects. Symmetry operations may also have more dramatic effects: They may even interchange two different topological sectors.

In $d = 2$ space dimensions, there is significant progress in classifying and understanding these SET phases. For some representative papers, see Refs. [19,20,55,58–68]. For the three-dimensional systems of interest in this paper, some progress in understanding gapped time-reversal symmetric Z_2 quantum spin liquids has been reported in Ref. [69]. The $U(1)$ spin liquids discussed in this paper are gapless, but nevertheless, we have shown how we can classify and understand the realization of time-reversal symmetry.

We have already discussed previous model constructions of some of these phases. Microscopic models for $E_b M_b$ are common [3–9]. What about the other phases? Reference [37] constructed a rotor model in which one of the emergent particles is a fermion. This can be viewed as either a construction of $E_f M_b$ or of $E_b M_f$ depending on how time reversal is implemented on the microscopic rotor degrees of freedom. If we take it to be a model for $E_b M_f$, it

should be possible to modify the M_f hopping to give it topological band structure. This will enable us to write down microscopic models for $(E_{fT} M_f)_\theta$ and $E_{bT} M_f$. However, we will not pursue this here.

XIV. DISCUSSION: MODELS, MATERIALS, AND EXPERIMENTS

We now consider the lessons learnt from our results for current and future possible experimental realizations of $U(1)$ spin liquids. We discuss two separate issues. First, for a given system, if a $U(1)$ spin liquid arises, which of the seven families of phases in Table I is realized? This is particularly relevant to the quantum spin ice materials such as the pyrochlore $Yb_2Ti_2O_7$. Existing theoretical work [11] assumed that the simplest phase in the $E_b M_b$ family is the prime candidate in such systems. However, this is justified only deep in the spin ice limit [5], and materials such as $Yb_2Ti_2O_7$ are quite far away from this limit (see review in Appendix A). So it is important to ask which of the seven phases discussed in this paper is more likely to arise in such systems. We address this issue in Sec. XIV A.

The next important issue is to identify distinguishing features of these different phases that can be probed in experiments. We partially address this issue in Sec. XIV B.

A. Pyrochlore spin ice

Here, we focus on Kramers spin systems since they tend to be more robust against disorder. It is of course very hard to decide energetically which phase is more favorable because of the complexity of the underlying Hamiltonian. But at least we can ask the following question: Which of the seven phases in this paper have a natural mean-field description on the pyrochlore lattice? Clearly, the gauge mean-field theory (gMFT) proposed in Ref. [11] is a natural mean-field theory for the phase $E_b M_b$. So what about the other six phases?

As discussed in Sec. IX A, for Kramers spins on a nonbipartite lattice such as the pyrochlore, it is quite unnatural—at the level of parton mean-field theory—to have fermionic monopoles or non-Kramers fermionic electric charges. This is already enough to render the $E_f M_b$, $E_b M_f$ and $E_{bT} M_f$ phases unlikely.

The phase $E_{bT} M_b$ is also unnatural at the mean-field level on a nonbipartite lattice. To construct it using the mean field, we need to use the Schwinger boson decomposition $S_\mu = \frac{1}{2} b_\alpha^\dagger \sigma_\mu^{\alpha\beta} b_\beta$. The mean-field theory should have one boson per site on average, and the bosons need to be gapped. On a nonbipartite lattice, this is impossible at the mean-field (quadratic) level, and the bosons will always tend to either condense or pair condense, which breaks the $U(1)$ gauge symmetry.

We are thus left with only two phases: the $E_{fT} M_b$ and $(E_{fT} M_f)_\theta$. Both can be described at the mean-field level through the Abrikosov fermion decomposition

$S_\mu = \frac{1}{2} f_\alpha^\dagger \sigma_\mu^{\alpha\beta} f_\beta$, where time reversal acts as $f \to i\sigma_y f$, and the $U(1)$ gauge symmetry is the phase rotation on the fermions: $f_\alpha \to e^{i\theta} f_\alpha$.

We can try to write down mean-field band structures of these f_α fermions on the pyrochlore lattice. We restrict ourselves to mean-field Hamiltonians that have only nearest-neighbor terms, which is reasonable since the spin exchange is very short ranged in quantum spin ice materials. We further restrict ourselves to mean-field Hamiltonians that are manifestly invariant under the full lattice symmetry. This is less justified since the f_α generally are allowed to transform projectively under the symmetry group. Nevertheless, we make this assumption in order to make progress while keeping this caveat in mind.

With these restrictions, the mean-field Hamiltonian has only two parameters: the trivial hopping term t_1 and the spin-orbit coupled hopping term t_2. It takes the form

$$H_{MF} = -\sum_{\langle rr' \rangle} f_r^\dagger (t_1 + it_2 \mathbf{d}_{rr'} \cdot \boldsymbol{\sigma}) f_{r'}. \quad (38)$$

Here, $\mathbf{d}_{rr'}$ is a unit vector parallel to the opposite bond of the tetrahedron containing rr' (see Ref. [70] for details). The resulting band structure has been studied elsewhere, for example, in Ref. [70]. The amusing fact about this Hamiltonian is that as long as it is gapped, the fermions will always form a topological insulator. We therefore reach the conclusion that, besides the $E_b M_b$ state described by gMFT, the topological Mott insulator $(E_{fT} M_f)_\theta$ is the only state that has a simple mean-field description on the pyrochlore lattice with Kramers spins.

Including fluctuations will lead to a lattice gauge theory with a $U(1)$ gauge field $a_{rr'} = -a_{r'r}$ described by the Hamiltonian

$$H = -\sum_{\langle rr' \rangle} f_r^\dagger t_{rr'} e^{ia_{rr'}} f_{r'} \quad (39)$$

(with the hopping matrix $t_{rr'} = t_1 + it_2 \mathbf{d}_{rr'} \cdot \boldsymbol{\sigma}$) supplemented with the Gauss law constraint

$$\sum_{r'} E_{rr'} = f_r^\dagger f_r - 1, \quad (40)$$

where $E_{rr'}$ is the integer electric field conjugate to $a_{rr'}$. A guess for a spin Hamiltonian that may favor the topological Mott insulator state is obtained by performing a strong coupling expansion of this lattice gauge theory. The resulting Hamiltonian takes the same form as Eq. (A1) for symmetry reasons (after the appropriate standard rotation from the global basis of the spin quantization axis to the local basis). The relative magnitude of the coupling constants depend on the parameter $w = t_2/t_1$ as follows:

$$J_{zz}/J = 2w^2 + 8w - 1,$$

$$J_\pm/J = 2w^2 + 2w + \frac{1}{2},$$

$$J_{\pm\pm}/J = w^2 - 2w + 1,$$

$$J_{z\pm}/J = \sqrt{2}(-2w^2 + w + 1), \quad (41)$$

where J is an overall constant, and the underlying fermion partons are gapped (and topological) when $w < -2$ or $w > 0$ and $w \neq 1$. A spin Hamiltonian was obtained in Ref. [31] starting from a Hubbard model. The free fermion part of the Hubbard model in Ref. [31], in the limit of strong on-site spin-orbit coupling, corresponds to the same hopping Hamiltonian in Eq. (38), with $w = t_2/t_1 = 0.215$ determined by orbital physics. The spin Hamiltonian obtained in Ref. [31], after a basis rotation (see Ref. [10]), agrees with Eq. (41) with $w = 0.215$. This simple consideration can provide a useful guide in searching for realizations of the topological Mott insulator in the family of rare earth pyrochlores if their exchange parameters can be determined by experiment. We should note, however, that our arguments in this subsection are only suggestive, and a reliable determination of the phase diagram of spin models for these pyrochlores is currently beyond the reach of theoretical technology. On the experimental side, for $Yb_2Ti_2O_7$, a determination of the exchange parameters was provided in Ref. [14]. However, this has been disputed by newer experiments [71] which suggest instead a rather different set of parameters. In view of the existing uncertainties in both the experiment and the theory, we will leave a further discussion of models and materials for the future.

B. Experimental signatures

Here, we offer some suggestions on experiments that may help distinguish these different $U(1)$ spin liquids.

First, we ask about distinctions in neutron scattering experiments. Spin-flip excitations that can scatter neutrons are created by local operators. If the only global symmetry is time reversal, then neutrons will couple to all local operators that are odd under time reversal. Let us consider a few important cases. First, since the emergent magnetic field is time-reversal odd, neutrons can couple directly to the fluctuations of the internal magnetic field. As discussed in Ref. [11], this enables neutron experiments to detect the emergent photon. Second, the number density of magnetic monopoles and the current of emergent electric charge are also local $\mathcal{T}$-odd operators. Coupling to these will lead to an increase in the scattering cross section when the energy transfer exceeds twice the gap of the M and the E particles. Generally, there should be two thresholds in the scattering cross section set by the M and E gaps. If the E particle is a Kramers doublet (which it is in some of the seven phases), then spin-flip excitations can also form out of a pair of E particles through a combination like $E^\dagger \boldsymbol{\sigma} E$. In this case, the

E threshold may be much more sharply defined than in the case where E is a Kramers singlet.

Other useful information can be gleaned by studying the effects of an applied magnetic field **B**. Consider the phases where E is a Kramers doublet and has a gap smaller than M and other composite excitations. For a Kramers doublet E particle, a direct coupling at quadratic level is allowed:

$$\Delta H \sim T^{ij} B_i E^\dagger \sigma_j E, \qquad (42)$$

where T^{ij} is a tensor consistent with lattice symmetries. This implies that with increasing **B**, the gap of the E particle will close at some finite value before the M gap closes. If E is a boson, it will condense in such a **B** field and the $U(1)$ gauge field will be gapped. Since E is a Kramers doublet, such a condensate necessarily breaks time-reversal symmetry. Therefore, the resulting phase is magnetically ordered and can be probed through neutron scattering. If E is a fermion, a fermi surface will emerge beyond the critical field, and the system becomes a $U(1)$ spin liquid with a spinon fermi surface, which can be probed through heat capacity or heat transport measurements (see Refs. [72,73] for interesting recent heat transport measurements on pyrochlore magnets). A similar consideration also applies if the M gap is smaller than the E gap, and it can indicate the statistics of the M excitation in that case. Thus, the behavior in a magnetic field can provide useful information to partially distinguish these different $U(1)$ spin liquids.

The two phases $(E_{fT} M_f)_\theta$ and $E_{bT} M_f$ necessarily have protected surface states. As we described, it is likely that these surfaces are in gapless phases, in which case it may be possible to detect the surface excitations. A useful experiment will be to deposit a ferromagnet on the surface and measure the resulting thermal Hall effect. This kind of experiment might perhaps be interesting to explore in $Yb_2Ti_2O_7$.

XV. CONCLUSION

In this paper, we have provided a detailed understanding of time-reversal-symmetric $U(1)$ quantum spin liquids. Our results were summarized in Sec. II. To conclude, we highlight a few open questions. We have not discussed the effects of spatial symmetry at all. This will lead to a finer distinction between these spin liquid phases and may impact the discussion of existing experimental candidates. A discussion of the effects of space-group symmetry on time-reversal-symmetric $U(1)$ quantum spin liquids in a cubic lattice was provided in early work by Ref. [7] (see also subsequent related work in Ref. [74] on a model without time reversal on a pyrochlore lattice). Even with just time reversal, it will be useful to identify sharp experimental fingerprints to distinguish the different phases. It may be interesting for future numerical work to study the loop wave functions described in Sec. X and

explicitly demonstrate their correctness in describing the various spin liquids.

Recently, there have been a number of related further developments which we briefly summarize here. Our own subsequent work [47] exploited the bulk duality of the topological Mott insulator described in this paper to provide a new "dual Dirac liquid" description of the surface of spin-orbit coupled electronic topological insulators. This bulk duality, and the same dual Dirac liquid, was independently obtained in Ref. [48], which appeared simultaneously with Ref. [47]. See also the very recent paper [75], which contains more results on the bulk duality.

APPENDIX A: MODEL FOR $Yb_2Ti_2O_7$

Here, we briefly review the spin Hamiltonian in Ref. [10], obtained by fitting the neutron scattering data in Ref. [14] through spin-wave theory. The values of the parameters in this model have, however, been questioned in more recent work [71]. The Hamiltonian has only nearest-neighbor terms and takes the form

$$
\begin{aligned}
H = \sum_{\langle ij \rangle} &J_{zz} S_i^z S_j^z - J_\pm (S_i^+ S_j^- + S_i^- S_j^+) \\
&+ J_{\pm\pm}(\gamma_{ij} S_i^+ S_j^+ \gamma_{ij}^* S_i^- S_j^-) \\
&+ J_{z\pm}[S_i^z(\zeta_{ij} S_j^+ + \zeta_{ij}^* S_j^-) + (i \leftrightarrow j)], \qquad (A1)
\end{aligned}
$$

where S_i^μ are spin coordinates in the local basis of spin ice. Here, ζ_{ij}, γ_{ij} are 4×4 matrices acting within each tetrahedra, and they have the form

$$
\zeta = \begin{pmatrix}
0 & -1 & e^{i\pi/3} & e^{-i\pi/3} \\
-1 & 0 & e^{-i\pi/3} & e^{i\pi/3} \\
e^{i\pi/3} & e^{-i\pi/3} & 0 & -1 \\
e^{-i\pi/3} & e^{i\pi/3} & -1 & 0
\end{pmatrix}, \qquad \gamma = -\zeta^*.
$$

$$(A2)$$

The coupling constants are, in meV,

$$
\begin{aligned}
J_{zz} &= 0.17 \pm 0.04, \qquad J_\pm = 0.05 \pm 0.01, \\
J_{\pm\pm} &= 0.05 \pm 0.01, \qquad J_{z\pm} = -0.14 \pm 0.01. \quad (A3)
\end{aligned}
$$

If J_{zz} dominates over the other coupling constants, the system at low energy will be restricted to the spin ice manifold. But it is not clear whether the above Hamiltonian

falls into such a regime, given that the other terms seem comparable to J_{zz} in magnitude. Very recent experimental work [71] has disputed these parameter values—the revised values are substantially different and place the system even farther away from the regime where the restriction to the spin ice manifold is legitimate.

APPENDIX B: TIME-REVERSAL ACTION ON ELECTRIC CHARGE

In general, the fundamental electric particle E can be a multicomponent with an internal index i. All the E_i particles carry the same gauge charge, which means any object of the form $E_i^\dagger O_{ij} E_j$ must be gauge neutral and hence correspond to a local operator.

In general, time reversal could act on E particles as

$$\mathcal{T} E_i \mathcal{T}^{-1} = T_{ij} E_j, \qquad (B1)$$

where T is a matrix. This implies that

$$\mathcal{T}^2 E_i \mathcal{T}^{-2} = (T^* T)_{ij} E_j, \qquad (B2)$$

where the anti-unitarity of $\mathcal{T}$ was used. However, $\mathcal{T}^2$ on any local operator should be a trivial identity. Therefore, we should have

$$(T^* T)^\dagger O (T^* T) = O, \qquad (B3)$$

for any matrix O. This can be true only if $T^* T = e^{i\phi} I$, where I is the identity matrix. Combining this with the complex conjugate relation $T T^* = e^{-i\phi} I$, we conclude that $e^{i\phi} = \pm 1$. Therefore, $\mathcal{T}^2 E \mathcal{T}^{-2} = \pm E$.

Notice that the above derivation assumes that all the local objects have $\mathcal{T}^2 = 1$ on them, which is true for a spin system. However, if the charge-neutral Kramers fermion is present in the microscopic system (for example, in a superconductor), then we do have local objects with $\mathcal{T}^2 = -1$. Following the above logic, it will be possible to have $\mathcal{T}^2 = \pm i$ for fractionalized objects such as E particles.

APPENDIX C: SAME $U(1)$ SPIN LIQUID FROM GAUGING TWO DISTINCT INSULATORS

We start from fermions with $U(1) \times \mathcal{T}$ symmetry, which corresponds to the fermionic monopoles discussed in the main text. It is known that for free fermions, the band structures of these kinds of fermions are classified by an integer n. We now show that the $U(1)$ spin liquids obtained by gauging the fermionic monopoles with band topology n are the same as those from gauging the fermions with band topology $-n$. The reason is very simple: At the level of band structures, the difference between a $+n$ band and a $-n$ band is the way time reversal $\mathcal{T}$ is implemented. This can be most easily seen through the surface Dirac cones, where the effective Hamiltonian looks the same for bands at $\pm n$:

$$H = \sum_{i=1}^{n} \psi_i^\dagger (p_x \sigma_x + p_y \sigma_z) \psi_i, \qquad (C1)$$

but time reversal acts differently for $\pm n$: $\mathcal{T} \psi_i \mathcal{T}^{-1} = \pm i \sigma_y \psi_i^\dagger$.

Before gauging, the different time-reversal action makes it impossible to continuously tune one state into the other. But after gauging, the difference becomes simply a gauge transform $U = e^{i\pi Q}$, where Q is the charge. Therefore, the $\pm n$ bands become gauge equivalent and give rise to the same quantum spin liquid.

It is also known [25,26] that a band indexed by n labels the same interacting bulk state as a band indexed by $n + 8$. It is also known that bands indexed by n and $n + 4$ differ from a bosonic symmetry-protected topological state called the $eTmT$ topological paramagnet [19,20], which is protected only by time reversal. Together with the previous identification of n and $-n$ bands, we conclude that states with band index $n = \pm 1 (\mathrm{mod} 8)$ and $n = \pm 3 (\mathrm{mod} 8)$ corresponds to two distinct phases. The bulk excitations of these two phases are identical and correspond to the $(E_{fT} M_f)_\theta$ $U(1)$ spin liquid (topological paramagnet). The two phases have different surface states, and one can obtain one state from the other by combining with a $eTmT$ topological paramagnet.

As discussed in Sec. VIII, the $eTmT$ topological paramagnet becomes trivial when combined with the $E_{bT} M_f$ state. Therefore, all the states with band index $n = 2 (\mathrm{mod} 4)$ correspond to a unique $E_{bT} M_f$ state.

APPENDIX D: SURFACE STATES OF VARIOUS $U(1)$ SPIN LIQUIDS

Here, we discuss the surface states of the first five phases in Table I. All these phases have a particle (E or M) that is a trivial boson. Therefore, the "wall" corresponding to this particle is trivial. However, they can still have a nontrivial wall corresponding to the nontrivial quasiparticle. For example, the $E_b M_f$ phase could have an M-wall, with a $\mathbb{Z}_2$ topological order $\{1, e, m, \epsilon\}$ on the wall, where the e and m particles carry electric charge $q_E = 1/2$. Following the logic in Ref. [21], this wall will convert the fermionic monopole into a bosonic one upon tunneling. The other $U(1)$ spin liquids have similar nontrivial walls, with a $\mathbb{Z}_2$ topological order and proper symmetry assignments on the e and m particles.

The physics of these "walls" will be important when we put two different spin liquids side by side. For example, if we put an $E_b M_b$ next to an $E_b M_f$, can we have coherent tunneling between the quasiparticles across the interface? Naively, this is impossible since the monopole is fermionic in one region but bosonic in the other, and the system will just be two $U(1)$ spin liquids essentially decoupled from one another. However, we can put the E-wall described above on the interface, and the M monopoles can now tunnel through the interface, with an ϵ particle left on the wall.

$$\Psi\left[\,\right] \sim -1$$

FIG. 8. Loop wave functions of the $E_{bT}M_b$, $E_{fT}M_b$ and other phases: The amplitude changes sign whenever a directed loop links with an undirected one.

APPENDIX E: LOOP WAVE FUNCTIONS OF VARIOUS $U(1)$ SPIN LIQUIDS

Here, we discuss loop wave functions of various other $U(1)$ spin liquids in the same spirit of Sec. X A. One can think of this approach as starting from a Walker-Wang wave function [23,45,53–55] and "gauging" the $U(1)$ symmetry. Many of these loop wave functions can be written in a form slightly different from that in Sec. X A. The wave functions have a directed "half" loop and an undirected loop condensing simultaneously, with a (-1) phase in the wave function whenever a "half" loop and an undirected loop mutually link (see Fig. 8). Following a similar analysis in Sec. X A, the resulting phase is a $U(1)$ spin liquid, with two fundamental particles E and M. One of them is bound with the open end of an undirected loop, and the other is the open end of a doubled half loop.

If the directed loops reverse their directions under time reversal, they are magnetic loops; otherwise, they are electric loops. It is now easy to see how to assign various quantum numbers to the dual particles in this description. For example, in the magnetic loop picture, we can put a Haldane chain in the undirected loop to make the E particle Kramers, in which case we obtain the $E_{bT}M_b$ phase in the magnetic loop picture. If we also make the undirected loop a ribbon with a (-1) self-linking sign, we convert the E particle to a fermion and obtain the $E_{fT}M_b$ phase in the magnetic loop picture.

There are some more complicated cases, including the $E_{bT}M_f$ phase in the magnetic loop picture, and the $(E_{fT}M_f)_\theta$ in both pictures. The surface topological order of these phases is well studied in the literature, from which one can derive the corresponding Walker-Wang wave functions and their gauged version [23,25,26,43–46]. However, the results are somewhat complicated [especially for $(E_{fT}M_f)_\theta$ which is non-Abelian] and not particularly illuminating, so we will omit them here.

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