

# Conservation of Gravitational Energy

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**Abstract.** The total energy of an astronomical system is of great interest since the evolution of the system depends on it. Hence we solve the hydrodynamical equations by taking account of the conservation in numerical simulations of astrophysical objects. However, gravitational energy is often taken into account as a source term, and the total energy including gravity is not guaranteed to be conserved. This is partly because it increases the computational cost to solve hydrodynamical equations in the fully conservative form, i.e., those without any source term. This paper shows that the total energy and momentum of a system are fully conserved if the source terms due to gravity are properly taken into account. The error in the total energy is reduced as small as the round-off error. The method is applicable both when the gravitational force is given by the Poisson equation or explicitly as a function. The former case is confirmed by numerical simulations of 2D fragmentation of a self-gravitating gas. This method is applicable not only to the Cartesian uniform grid but an octree grid often used in adaptive mesh refinement.

## 1. Introduction

Gravitational energy release is often an important source of activity in astronomical objects such as accreting compact stars and progenitors of core-collapse supernovae. Also, young stellar objects are powered by gravitational energy release through accretion and slow contraction. While energy changes its form, the total amount of it is conserved if energy loss from the system through emission is taken into account. If the energy conservation is not reproduced in numerical simulations, they suffer from artificial energy gain or loss and hence overestimate or underestimate the system activity. The total energy conservation is important as well as the total mass conservation.

Despite the importance, the total energy conservation is not guaranteed in most of the current numerical simulations. It is technically difficult to compute the change in the gravitational energy of a self-gravitational system, which is often expressed as

$$E_g = \frac{1}{2} \int \rho \phi dV, \quad (1)$$

where  $\rho$  and  $\phi$  denote the mass density and gravitational potential, respectively.

The technical difficulty can be reworded as follows. The conservation of total mass is easily realized by solving the equation of continuity in its conservation form. The equation of continuity balances the temporal change in the density with the divergence of the mass flux. If we apply the Gauss's theorem for each numerical cell, the conservation of total mass is guaranteed. On the other hand, gravity appears as a source term in the equation of energy conservation. Thus



the conservation of total energy is not guaranteed unless the spatial integral of gravitational energy release over the whole volume is exactly balanced with the temporal change in the total gravitational energy.

We can rewrite the equation of energy conservation into the fully conservative form in which the source term due to the gravity is decomposed into temporal change of energy density and divergence of energy flux[1]. However, we need some additional numerical procedure to solve the fully conservative form since it contains a new variable, the change in the gravitational potential. The fully conservative form has another shortcoming that gravitational energy release is less accurately evaluated in regions of low density. This is because the gravitational energy flux has a large but nearly constant value when the density thereof is small. The numerical divergence of the flux contains a relatively large round-off error especially when the density thereof is low.

In this paper, we propose a new method to evaluate the gravitational energy release. It is based on another form of the equation of energy conservation, in which the gravitational energy is expressed as

$$E_g = - \int \frac{\mathbf{g}^2}{8\pi G} dV, \quad (2)$$

where  $G$  and  $g$  denote the gravitational constant and gravity, respectively. Although based on the fully conservative form, the gravitational energy release is expressed as the product of the gravity and mass flux as proposed in some earlier works [2, 3, 4]. As a result, the conservation of total energy including gravity is ensured by a small change in the evaluation of the gravity. As shown in the subsequent section, the gravity is defined on the surface of a numerical cell and the time-averaged value is used for exact balance.

This paper is organized as follows. The equivalence of equations (1) and (2) is proved in §2.1 and a new form of the equation of energy conservation is given in §2.2. The equation of energy conservation is converted into the discretized form in §2.3. Numerical procedure to solve the discretized equations is given in §2.4. Numerical examples are given in §3 and a summary is given in §4.

## 2. Methods

### 2.1. Gravitational Energy

First, we prove the equivalence of equations (1) and (2). The gravitational potential is given as the solution of the Poisson equation,

$$\Delta\phi = 4\pi G\rho, \quad (3)$$

and the gravity is defined as the gradient,

$$\mathbf{g} = -\nabla\phi. \quad (4)$$

Substituting equation (3) into equation (1) we obtain

$$E_g = \frac{1}{8\pi G} \int \phi\Delta\phi dV = -\frac{1}{8\pi G} \int \phi(\nabla \cdot \mathbf{g}) dV = - \int \frac{\mathbf{g} \cdot \mathbf{g}}{8\pi G} dV + \int_s \frac{\phi}{8\pi G} \mathbf{g} \cdot d\mathbf{S}, \quad (5)$$

where the second term denotes the surface integral on the boundary in the right-hand side. Thus equations (1) and (2) are equivalent if the surface integral either vanishes or is negligibly small. The condition is realized when the boundary is periodic or placed very far from most of the mass.

The above proof is valid when the gravitational potential is differentiable, i.e, in its integral form. We discuss the condition for which the equivalence of equations (1) and (2) is valid also in the discretized form. In the following, we solve equation (3) in its discretized form,

$$\int_{V_i} 4\pi G\rho dV = - \int_{S_i} \mathbf{g} \cdot d\mathbf{S}, \quad (6)$$

where  $V_i$  and  $S_i$  denote the volume and surface of a given numerical cell, respectively. Note that only the gravity normal to cell surface appears in the right-hand side of equation (6). This fact indicates that we should evaluate the gravitational energy density from the normal component of  $\mathbf{g}$  defined on each cell surface. Equation (5) requires that also the gravitational potential should be defined uniquely on each cell surface. Otherwise, the surface term may not be cancelled on the interface between two adjacent numerical cells. These constraints give us guidelines for constructing the numerical scheme which ensures conservation of total energy.

### 2.2. Fully Conservative Form of Hydrodynamical Equations

In this subsection, we rewrite all the components of the hydrodynamical equations in the fully conservative form.

The equation of continuity describes conservation of mass and is expressed as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (7)$$

where  $\mathbf{v}$  denote the velocity. Equation (7) is in the fully conservative form, since the source term vanishes. Combined with equation (7), the equation of motion is converted into the equation of momentum conservation,

$$\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + P\mathbf{I}) = \rho \mathbf{g} = -\rho \nabla \phi, \quad (8)$$

where  $P$  and  $\mathbf{I}$  denotes pressure and the identity matrix, respectively. Equation (8) can be rewritten into the fully conservative form,

$$\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + P\mathbf{I} + \mathbf{T}_g) = 0, \quad (9)$$

$$\mathbf{T}_g = \frac{\mathbf{g} \cdot \mathbf{g}}{8\pi G} \mathbf{I} - \frac{\mathbf{g}\mathbf{g}}{4\pi G}, \quad (10)$$

where  $\mathbf{v}\mathbf{v}$  and  $\mathbf{g}\mathbf{g}$  denote the direct product. The equivalence of equations (8) and (9) is confirmed by the algebra,

$$\nabla \cdot \mathbf{T}_g = -\frac{(\nabla \cdot \mathbf{g})}{4\pi G} \mathbf{g} + \frac{(\nabla \times \mathbf{g}) \times \mathbf{g}}{4\pi G} = -\rho \mathbf{g}.$$

Remember that two equalities

$$\nabla \cdot \mathbf{g} = -4\pi G\rho, \quad (11)$$

$$\nabla \times \mathbf{g} = 0, \quad (12)$$

hold for any gravity (see, e.g., [5]).

Similarly, energy conservation is expressed as

$$\frac{\partial}{\partial t} (\rho E) + \nabla \cdot (\rho \mathbf{v} H) = \rho \mathbf{v} \cdot \mathbf{g}, \quad (13)$$

where  $E$  and  $H$  denotes the specific energy and enthalpy, respectively, and defined as

$$E = \frac{\mathbf{v}^2}{2} + \varepsilon, \quad (14)$$

$$H = \frac{\mathbf{v}^2}{2} + \varepsilon + \frac{P}{\rho}. \quad (15)$$

Here, the symbol,  $\varepsilon$ , denotes the specific internal energy. Energy gain and loss through radiation are not taken into account in equation (13) for simplicity. We can rewrite the right-hand side of equation (13) as

$$\begin{aligned} \rho \mathbf{v} \cdot \mathbf{g} &= -\rho \mathbf{v} \cdot \nabla \phi \\ &= -\nabla \cdot (\rho \phi \mathbf{v}) + \phi \nabla \cdot (\rho \mathbf{v}) \\ &= -\nabla \cdot (\rho \phi \mathbf{v}) - \phi \frac{\partial \rho}{\partial t} \end{aligned} \quad (16)$$

$$\begin{aligned} &= -\nabla \cdot (\rho \phi \mathbf{v}) - \frac{\phi}{4\pi G} \frac{\partial}{\partial t} \nabla^2 \phi \\ &= -\nabla \cdot \left[ \phi \left( \rho \mathbf{v} - \frac{1}{4\pi G} \frac{\partial \mathbf{g}}{\partial t} \right) \right] + \frac{\partial}{\partial t} \left( \frac{\mathbf{g} \cdot \mathbf{g}}{8\pi G} \right), \end{aligned} \quad (17)$$

by the help of equations (3) and (7). Thus equation (13) is equivalent to

$$\frac{\partial}{\partial t} \left( \rho E - \frac{\mathbf{g}^2}{8\pi G} \right) + \nabla \cdot \left[ \rho \mathbf{v} (H + \phi) - \frac{\phi}{4\pi G} \frac{\partial \mathbf{g}}{\partial t} \right] = 0. \quad (18)$$

As well as equations (7) and (9), equation (18) has no source term. If the flux across the boundary vanishes, the total energy defined as

$$E_{\text{tot}} = \int \left( \rho E - \frac{\mathbf{g} \cdot \mathbf{g}}{8\pi G} \right) dV. \quad (19)$$

is conserved. This is equal to the total energy defined by

$$E_{\text{tot}} = \int \left( \rho E + \frac{\rho \phi}{2} \right) dV, \quad (20)$$

as shown in the previous section. Thus, a numerical scheme which guarantees total energy conservation is obtained by applying the Gauss's theorem to equation (18).

Remember that Jiang et al.[1] uses another form of the equation of energy conservation,

$$\frac{\partial}{\partial t} \left( \rho E + \frac{\rho \phi}{2} \right) + \nabla \cdot (\rho H \mathbf{v} + \mathbf{F}_g) = 0, \quad (21)$$

$$\mathbf{F}_g = \frac{1}{8\pi G} \left( \phi \frac{\partial \mathbf{g}}{\partial t} - \frac{\partial \phi}{\partial t} \mathbf{g} \right) + \rho \mathbf{v} \phi. \quad (22)$$

Note that the total energy is defined not by equation (19) but by equation (20) in this form. Accordingly, the energy flux is different from ours and has an additional term.

When the potential is stationary, we can rewrite Equation (16) into

$$\rho \mathbf{v} \cdot \mathbf{g} = -\nabla \cdot (\rho \phi \mathbf{v}) - \frac{\partial}{\partial t} (\rho \phi). \quad (23)$$

Thus the equation of energy conservation is expressed as

$$\frac{\partial}{\partial t} [\rho (E + \phi)] + \nabla \cdot [\rho \mathbf{v} (H + \phi)] = 0, \quad (24)$$

which has no source term. Thus, a numerical scheme which guarantees total energy conservation is obtained by applying the Gauss's theorem to equation (24).

### 2.3. Proper Discretization of the Source Terms

Equations (9), (18) and (24) are ideal in the sense that the total energy of a numerical solution remains constant and the error is of the order of round-off. However, they may give a seriously large error in a region where the density is very low while the gravity is large. Both the fluxes of momentum and energy are large while the divergence is very small in such a region. If the fluxes contain relatively small errors, they result in a relatively large error in the divergence.

The above-mentioned problem is solved if we can find a source term which coincides with the numerical derivative of the corresponding flux or energy density. In the following, we seek the discretized forms of equations (8) and (13) which are equivalent to those of equations (9) and (18). The equivalence ensures the conservation of total momentum and that of total energy, since Equations (9) and (18) are in the conservation form without source terms and the total energy and momentum are conserved in the solutions. Such discretized equations should be consistent with the equation of continuity and the Poisson equation, since we use the latter equations to derive equations (9) and (18) from equations (8) and (13).

First, we consider a Cartesian uniform grid in which each numerical cell is a cuboid having widths,  $\Delta x$ ,  $\Delta y$ , and  $\Delta z$ , in the  $x$ -,  $y$ -, and  $z$ -directions, respectively. This means that the center of each numerical cell is expressed as

$$(x_i, y_j, z_k) = (i\Delta x, j\Delta y, k\Delta z), \quad (25)$$

where  $i$ ,  $j$ , and  $k$  are integers specifying the cuboid. Then equation (6) is expressed as

$$\begin{aligned} \frac{\phi_{i+1,j,k} - 2\phi_{i,j,k} + \phi_{i-1,j,k}}{\Delta x^2} + \frac{\phi_{i,j+1,k} - 2\phi_{i,j,k} + \phi_{i,j-1,k}}{\Delta y^2} \\ + \frac{\phi_{i,j,k+1} - 2\phi_{i,j,k} + \phi_{i,j,k-1}}{\Delta z^2} = 4\pi G\rho_{i,j,k}, \end{aligned} \quad (26)$$

where  $\phi_{i,j,k}$  and  $\rho_{i,j,k}$  denote the gravitational potential and density at the center of numerical cell numbered  $(i, j, k)$ . Then the discretized source term should meet the following conditions.

- (i) The gravity,  $\mathbf{g}$ , should be evaluated on the numerical cell surface, since both the gravitational acceleration ( $\rho\mathbf{g}$ ) and energy release ( $\rho\mathbf{v} \cdot \mathbf{g}$ ) originate from the divergence of momentum and momentum fluxes, respectively.
- (ii) The gravitational energy,  $-\mathbf{g} \cdot \mathbf{g}/(8\pi G)$ , should be evaluated from the component normal to the cell surface since the tangential component does not appear in equation (26). The same requirement is derived also from equation (5). The tangential component of the gravity can be derived from equation (11) but need not be computed, although  $\mathbf{T}_g$  contains both the normal and tangential components of gravity.
- (iii) The gravitational energy release should be evaluated as the product of the normal components of  $\rho\mathbf{v}$  and  $\mathbf{g}$ . This constraint comes from the fact that the change in the gravitational potential is proportional to that in the density distribution. The change in the density is due to the mass flux across each numerical cell surface. Remember that essentially the same method is used in earlier works [2, 3, 4].
- (iv) The gravity should be the average of those at  $t = t_0$  and  $t = t_0 + \Delta t$ , where  $t_0$  and  $\Delta t$  denotes the time,  $t$ , at the initial state and the time step, respectively. Otherwise, equation (17) may not hold since the solution may not satisfy,

$$\frac{\partial}{\partial t} \left( \frac{\mathbf{g} \cdot \mathbf{g}}{8\pi G} \right) = \frac{\mathbf{g}}{4\pi G} \cdot \frac{\partial \mathbf{g}}{\partial t}. \quad (27)$$

Note that the change in the gravitational energy density can be evaluated as

$$-\frac{[\mathbf{g}(t_0 + \Delta t)]^2}{8\pi G} + \frac{[\mathbf{g}(t_0)]^2}{8\pi G} = - \left[ \frac{\mathbf{g}(t_0 + \Delta t) + \mathbf{g}(t_0)}{8\pi G} \right] \cdot [\mathbf{g}(t_0 + \Delta t) - \mathbf{g}(t_0)]. \quad (28)$$

The total momentum is conserved if items (i) and (ii) are fulfilled. The total energy is conserved only when all the items are fulfilled. We need not take account item (iv) into account when the gravitational potential is stationary. Fortunately, these requirements serve as the guidelines to derive a scheme which ensures the conservation of total momentum and energy.

After some algebra, we find that the properly discretized source term is expressed as

$$(\rho g_x)_{i,j,k} = \frac{\rho_{i,j,k}}{2} (g_{x,i+1/2,j,k} + g_{x,i-1/2,j,k}), \quad (29)$$

$$(\rho g_y)_{i,j,k} = \frac{\rho_{i,j,k}}{2} (g_{y,i,j+1/2,k} + g_{y,i,j-1/2,k}), \quad (30)$$

$$(\rho g_z)_{i,j,k} = \frac{\rho_{i,j,k}}{2} (g_{z,i,j,k+1/2} + g_{z,i,j,k-1/2}), \quad (31)$$

$$(\rho \mathbf{v} \cdot \mathbf{g})_{i,j,k} = \frac{1}{2} \left[ (\rho v_x)_{i+1/2,j,k} g_{x,i+1/2,j,k} + (\rho v_x)_{i-1/2,j,k} g_{x,i-1/2,j,k} \right. \\ \left. + (\rho v_y)_{i,j+1/2,k} g_{y,i,j+1/2,k} + (\rho v_y)_{i,j-1/2,k} g_{y,i,j-1/2,k} \right. \\ \left. + (\rho v_z)_{i,j,k+1/2} g_{z,i,j,k+1/2} + (\rho v_z)_{i,j,k-1/2} g_{z,i,j,k-1/2} \right], \quad (32)$$

$$g_{x,i+1/2,j,k} \equiv \frac{\phi_{i+1,j,k} - \phi_{i,j,k}}{\Delta x}, \quad (33)$$

$$g_{y,i,j+1/2,k} \equiv \frac{\phi_{i,j+1,k} - \phi_{i,j,k}}{\Delta y}, \quad (34)$$

$$g_{z,i,j+1/2,k} \equiv \frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{\Delta z}, \quad (35)$$

in the Cartesian uniform grid. Here all the values should be the time average between  $t = t_0$  and  $t_0 + \Delta t$ . Equations (29) through (32) define the gravity in each cell as the volume average of those defined on the two cell surfaces. Equation (26) can be rewritten as

$$-\frac{g_{x,i+1/2,j,k} - g_{x,i-1/2,j,k}}{\Delta x} - \frac{g_{y,i,j+1/2,k} - g_{y,i,j-1/2,k}}{\Delta y} - \frac{g_{z,i,j,k+1/2} - g_{z,i,j,k-1/2}}{\Delta z} = 4\pi G \rho_{i,j,k}. \quad (36)$$

Equations (29) through (32) give the properly discretized source terms also when the gravitational potential is stationary and given (see, e.g., [4]). We can confirm it by substituting Equations (29) through (35) into the right hand side of Equations (8) and (16).

Next, we consider a non-uniform Cartesian grid in which the separation is expressed as

$$x_{i+1} - x_i = \frac{\Delta x_i + \Delta x_{i+1}}{2} \equiv \Delta x_{i+1/2}, \quad (37)$$

$$y_{j+1} - y_j = \frac{\Delta y_j + \Delta y_{j+1}}{2} \equiv \Delta y_{j+1/2}, \quad (38)$$

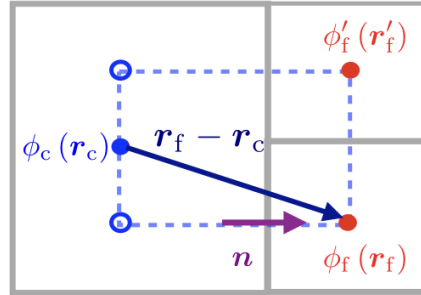
$$z_{k+1} - z_k = \frac{\Delta z_k + \Delta z_{k+1}}{2} \equiv \Delta z_{k+1/2}, \quad (39)$$

between the neighboring numerical cells. Here, the symbols,  $\Delta x_i$ ,  $\Delta y_j$ , and  $\Delta z_k$  denote the width, depth, and height of the numerical cell numbered  $(i, j, k)$ . Then equations (33) through (35) should be replaced with

$$g_{x,i+1/2,j,k} \equiv \frac{\phi_{i+1,j,k} - \phi_{i,j,k}}{\Delta x_{i+1/2}}, \quad (40)$$

$$g_{y,i,j+1/2,k} \equiv \frac{\phi_{i,j+1,k} - \phi_{i,j,k}}{\Delta y_{j+1/2}}, \quad (41)$$

$$g_{z,i,j+1/2,k} \equiv \frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{\Delta z_{k+1/2}}, \quad (42)$$



**Figure 1.** Gravity across the coarse-fine grid boundary in the octree grid. See Equation (43).

However, other formulae for the properly discretized source term remain unchanged.

Almost the same formulae give the properly discretized source term for an octree grid often used in Adaptive Mesh Refinement (AMR). The numerical cells are cubes and some of them are subdivided into 8 cubes recursively in the octree grid. When the neighboring cells have the same size, the same formulae are applied. Only when the neighboring cells have different sizes, i.e., on the coarse-fine grid boundary, we need to modify equations (40) through (42). The normal component of the gravity should be expressed as

$$\mathbf{g}_n = - \left[ \frac{\phi_f - \phi_c}{(\mathbf{r}_f - \mathbf{r}_c) \cdot \mathbf{n}} \right] \mathbf{n}, \quad (43)$$

where  $\mathbf{n}$  denotes the unit vector normal to the cell interface. Here the symbols,  $\phi_c$  and  $\mathbf{r}_c$ , denote the potential and position at the coarse cell center, while  $\phi_f$  and  $\mathbf{r}_f$  do those at the fine cell center (see Figure 1). Any flux across the coarse-fine boundary should be evaluated on the fine grid side. The flux on the coarse grid side should be equated with the sum of those on the fine grid side. Otherwise, the Gauss's theorem, which is essential for a conservation law, does not hold in the discretized equation. This principle applies also to the Poisson equation. Equation (43) implies that the potential at the cell center (the filled circle in Figure 1) is the same as that at the opposing normal to the fine cell (the open circles). In other words, we ignore the gravity tangential to the coarse-fine boundary, i.e., the potential difference between the two open circles. Otherwise, the circulation along the dashed line results in either gain or loss of energy,  $\oint \mathbf{g} \cdot d\mathbf{s} \neq 0$ .

The properly discretized source term is likely to be given for other types of numerical grids but it is beyond the scope of this paper.

#### 2.4. Numerical Procedure

As shown in the previous section, our method does not give any constraints on the choice of approximate Riemann solvers. However, the time-averaged gravity should be used to update the velocity and internal energy of the gas. Thus we need to compute the gravity at the advanced stage before completing the update of the velocity and internal energy. The numerical examples to be shown in the next section are obtained by the procedure.

- (i) Obtain the gravity,  $\mathbf{g}(t_0)$ , at the initial stage.
- (ii) Compute the initial guess for the density,  $\rho^*(t_0 + \Delta t)$ , at the advanced stage by solving the equation of continuity.
- (iii) Compute the initial guess for the gravity,  $\mathbf{g}^*(t_0 + \Delta t)$  from  $\rho^*(t_0 + \Delta t)$ .

- (iv) Compute the initial guess for the velocity,  $\mathbf{v}^*(t_0 + \Delta t)$ , and internal energy,  $\varepsilon^*(t_0 + \Delta t)$ , by solving the equation of motion and that of energy conservation.
- (v) Improve the density,  $\rho(t_0 + \Delta t)$  by using the time-averaged mass flux.
- (vi) Obtain the gravity,  $\mathbf{g}(t_0 + \Delta t)$  from  $\rho(t_0 + \Delta t)$ .
- (vii) Improve the velocity,  $\mathbf{v}(t_0 + \Delta t)$ , and internal energy,  $\varepsilon(t_0 + \Delta t)$ .
- (viii) Go back to step (ii) after replacing  $t_0 + \Delta t$  with  $t_0$ .

Here, steps (ii) through (iv) are predictor stages while steps (v) through (vii) are corrector ones. This procedure requires to solve Poisson equation twice per cycle since step (i) is performed only once at the initial stage. However, we can omit step (iii), although the total energy is not conserved at the predictor stage then. Our scheme does not increase the number of times solving the Poisson equation.

### 3. Results

#### 3.1. Jeans Instability in 2D

The first example demonstrates that the total energy is conserved in the 2D fragmentation of initially Jeans unstable gas. The initial condition is constructed by superposing a sinusoidal perturbation on a uniform gas having the initial density,  $\rho_0 = 1.0$ , and pressure  $P_0 = 0.6$ . The specific heat ratio is set to be  $\gamma = 5/3$ . We use the unit system in which the gravitational constant is  $G = 1/4\pi$ . Thus the Jeans length is  $\lambda_J = 2\pi$  at the initial state. The computation box is set to be as large as  $(4\sqrt{2}\pi)^2$  and covered with a uniform grid having  $32^2$  cells. The periodic boundary condition is applied to both the  $x$ - and  $y$ -directions. The Poisson equation is solved with the Fast Fourier Transform (FFT). The CFL number is set to be 0.4. It took 2400 time steps to follow the evolution of  $0 \leq t \leq 107.40$ .

Figure 2 shows the fragmentation by a series of snapshots. The top left panel shows the initial stage at  $t = 0.00$ , while the bottom right does the final one at  $t = 107.40$ . The color and contours denote the density and entropy,  $s = \ln(\gamma P) - \gamma \ln \rho$ , respectively, while the arrows denote the velocity. First, the initially uniform gas is fragmented into a sheet as shown in the top right panel. Then the sheet is fragmented into a cylinder.

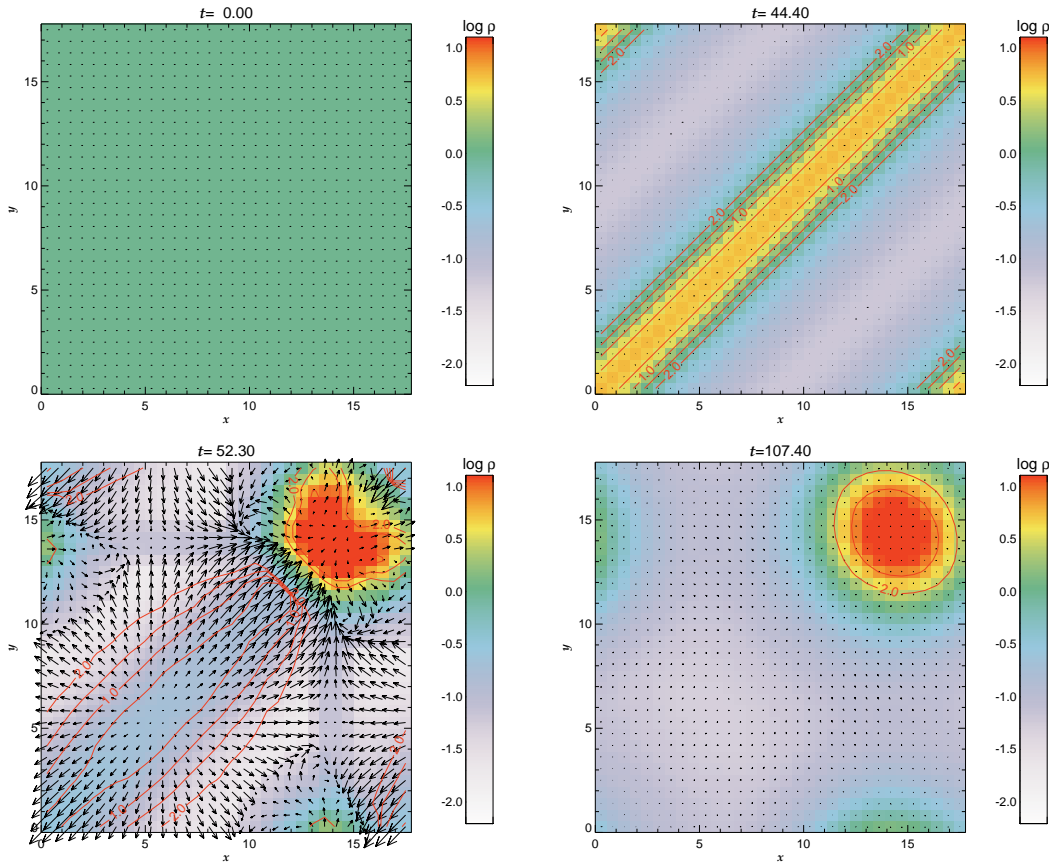
Figure 3 shows the change in the energy in this model. The left panel denotes the kinetic energy ( $E_{\text{kin}}$ ), thermal energy ( $E_{\text{th}}$ ), and the gravitational energy ( $-E_g$ ) as a function of time. The ordinate is normalized by the initial thermal energy. The gravitational energy decreases by two steps; first at the sheet formation and second at the sheet fragmentation. The right panel denotes the change in the total energy,  $E_{\text{tot}} = E_{\text{kin}} + E_{\text{th}} + E_g$ . The ordinate is normalized by  $10^{-13}$  times of the initial thermal energy,  $10^{-13}E_{\text{th}}(t = 0)$ . The total energy increases little. The small increase is due to the fact that all the computations are done with the double-precision floating-point format.

#### 3.2. Gravitational Collapse on Nested Grid

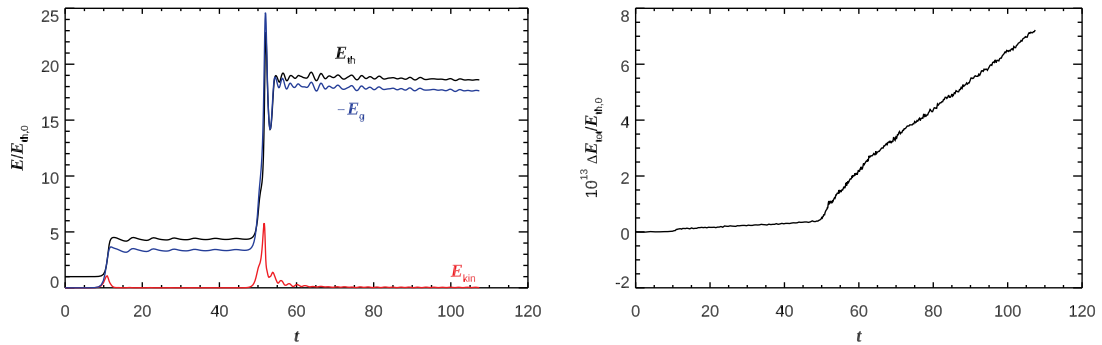
The second example demonstrates that our method can be applied to a nested grid. The nested grid consists of two levels of uniform grids having  $64^3$ . The coarse grid covers the cube of  $|x| \leq 2\pi$ ,  $|y| \leq 2\pi$  and  $|z| \leq 2\pi$  while the fine grid does that of  $|x| \leq \pi$ ,  $|y| \leq \pi$  and  $|z| \leq \pi$  in the Cartesian coordinates. The gravitational potential is fixed to be

$$\phi = -0.6 \left[ \cos \frac{x}{2} + \cos \frac{y}{2} + \cos \frac{z}{2} \right]. \quad (44)$$

The reflection boundary is set on the outer boundaries of the coarse grid. The source term is evaluated from the gravity derived from the numerical difference in the potential defined at the cell center.



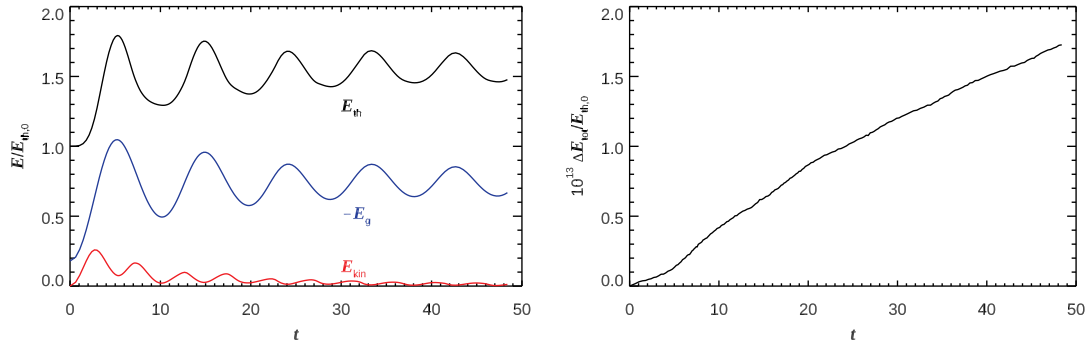
**Figure 2.** 2D fragmentation of a Jeans unstable gas is shown by a series of snapshot.



**Figure 3.** The left panel shows  $E_{kin}$ ,  $E_{th}$  and  $-E_g$  as a function of time,  $t$ . The right panel shows the change in the total energy from its initial value as a function of  $t$ .

This model follows the collapse of initially uniform gas at rest. The initial density and pressure are set to be  $\rho_0 = 1.0$  and  $P_0 = 0.6$ . The specific heat ratio is set to be  $\gamma = 5/3$ . The time step is taken to be the same for the coarse and fine grids. The CFL number is set to be 0.25. It took 6400 time steps to follow the evolution of  $0 \leq t \leq 48.39$ .

Figure 4 is the same as Figure 3 but for the gravitational collapse followed with the nested



**Figure 4.** The same as Fig 3 but for the gravitational collapse followed with the nested grid.

grid. The gravitational energy oscillates with a substantial amplitude, while the change in the total energy is of the order of round-off error.

#### 4. Summary

As shown in the previous sections, the total energy including self-gravity can be ensured by a rather simple procedure. The procedure can be applied as far as the conservation form is used for solving both the hydrodynamical and Poisson equations. Poisson equation is solved only once or twice per time step as well as in a conventional scheme of the second-order accuracy in time. The procedure increases computation cost only through increase in the memory to store gravity and mass flux at the predictor step.

The method proposed in this paper is based on equation (18) in which the gravitational energy density is expressed as  $-\mathbf{g} \cdot \mathbf{g}/(8\pi G)$ . The method has two key elements. First, the gravity should be evaluated on the cell surface. Second, it should be the average of those at the initial and final stages for each time step [cf. equation (28)]. The former key element implies that the source term has different values inside a numerical cell and the integral over the numerical cell is used in computation. This concept should hold not only for a Cartesian uniform grid but for a variety of numerical grids. The latter means that the gravity should be updated before completing the update of velocity and internal energy. This stands to reason since the gravitational energy is fixed once the density distribution is given. We need to evaluate the gravitational energy at the new time step in advance for ensuring the total energy conservation. However, this point has not been mentioned in literature as far as the author knows.

This paper has outlined the numerical scheme which ensures the conservation of total energy and momentum. Further details will be published elsewhere.

#### Acknowledgments

This work is supported in part by JSPS KAKENHI Grant Number JP 18H05222.

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