Color structure for soft gluon resummation — a general recipe

Malin Sjödahl

Institut für Theoretische Physik, Universität Karlsruhe,
Wolfgang-Gaede-Str. 1, Physikhochhaus, 76131 Karlsruhe, Germany

E-mail: malin@particle.uni-karlsruhe.de

ABSTRACT: A strategy for calculating the color structure needed for soft gluon resummation for processes with any number of colored partons is introduced using a $N_c \to \infty$ inspired basis. In this basis a general formalism can be found at the same time as the calculations are simplified.

The advantages are illustrated by recalculating the soft anomalous dimension matrix for the processes $gg \to gg$, $q\bar{q} \to q\bar{q}g$ and $q\bar{q} \to ggg$.

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Contents

1 Introduction

The strong force comes with the problem of being precisely strong. Although the coupling constant of QCD is small enough for perturbation theory to make sense at all, it is large enough to cry out for higher order corrections for many processes, and in some regions of phase space, large enough to invalidate a fixed order calculation.

This is the case in the collinear region, where a large logarithm compensates for the moderate smallness of α_s, and similarly in the soft region where there is a large effective phase space $\sim \log(\text{hard scale/soft resolution scale})$ in transverse momentum. In these regions resummation methods are needed. In the collinear DGLAP region [1–4], where the emission can be seen as coming from one parton, the color structure is trivial and Sudakov form factors can be used to describe no-emission *probabilities*.

Unfortunately, the strong force is not only strong, it is also complicated, in the sense of being non-Abelian. In the soft region, where emissions have contributions from branchings of different partons this complicates matters. The real emission coming from the interference term of emission off parton i and emission off parton j, is canceled by the virtual gluon exchange between parton i and j. (Using Feynman gauge self-energy type diagrams can be neglected, i.e. $i \neq j$.)

Under the assumption that emissions strongly ordered in transverse momentum dominate, all leading logarithms in (hard scale/soft scale) from virtual corrections exponentiate and can be resummed. However, since these gluon exchanges affect the color structure, the exponentiation must be done at the amplitude level. Thus a no-emission *amplitude*

$$\mathbf{M} = \exp\left(-\frac{2}{\pi} \int_{Q_0}^{Q} \alpha_s(k'_\perp) \frac{dk'_\perp}{k'_\perp} \boldsymbol{\Gamma} \right) \mathbf{M}_0, \tag{1.1}$$

can be derived. In the above, $\mathbf{M}_0$ is the undressed hard scattering amplitude as a vector in color space and $\boldsymbol{\Gamma}$ is a matrix in color space, describing the effect of exchanging gluons between the various partons,

$$\boldsymbol{\Gamma} = \sum_{i<j} \Omega_{ij} \mathbf{C}^{ij}. \tag{1.2}$$

Here $\mathbf{C}^{ij}$ describes the color algebra part and Ω_{ij} contain the azimuth and rapidity momentum integral over the exchanged gluon k',

$$\Omega_{ij} = -\frac{1}{2}(-1)^l \left[\int_\Omega \frac{dy' d\phi'}{2\pi} \frac{k'^2_\perp p_i \cdot p_j}{2 p_i \cdot k' k' \cdot p_j} - \frac{1}{2}(1 - s_{ij}) i\pi \right] \tag{1.3}$$

with $s_{ij} = -1$ if the partons ij are both incoming or both outgoing, and 1 otherwise, and l counts how many of the involved partons which are quarks in the initial state, anti-quarks in the final state or gluons, assuming the convention in (3.1) for the triple gluon vertex. In the above equation the $i\pi$-terms, coming from Coulomb gluon exchange, would give rise to an unobservable phase in an Abelian theory. For a non-Abelian theory they do, however, enter in a physically relevant way.

In general the color basis used need neither be orthogonal or normalized. In fact, it will be seen below that the calculations simplify significantly in a special basis which is not. For a non-orthonormal basis, the matrix of scalar products $\mathbf{S}$, calculated by summing over quark, anti-quark and gluon indices $a, b, c, \ldots$

$$\mathbf{S}_{mn} = <C^m, C^n> = \sum_{a,b,c\ldots} C^m_{abc\ldots}(C^n)^*_{abc\ldots}, \tag{1.4}$$

is needed. (Note that C^m above is a basis tensor in color space, whereas $\mathbf{C}^{ij}$ in (1.2) are matrices in this basis, describing the effect of gluon exchange between parton i and j.) The physical no-emission probability is given by $\sigma = \mathbf{M}^\dagger \mathbf{S} \mathbf{M}$. As an aside it is pointed out that scalar products between tensors corresponding to linear combinations of color structures of Feynman diagrams with real coefficients are real.

In the simple case of $q_1 q_2 \to q_3 q_4$ the vector space containing the color structure has only two dimensions, and the basis vectors are often taken to be the "t-channel singlet octet basis",

$$
\begin{aligned}
C^1_{q_1 q_2 q_3 q_4} &= \delta_{q_1 q_4} \delta_{q_2 q_3} \\
C^2_{q_1 q_2 q_3 q_4} &= t^g_{q_4 q_1} t^g_{q_3 q_2} = \frac{1}{2} \left[\delta_{q_1 q_3} \delta_{q_2 q_4} - \frac{1}{N_c} \delta_{q_1 q_4} \delta_{q_2 q_3} \right].
\end{aligned}
\tag{1.5}
$$

In this case the issue of keeping track of the color structure amounts to a moderate complication. However, already for $gg \to gg$ a six dimensional vector space is needed (reducing to a five dimensional space for $N_c = 3$) and for $gg \to ggg$ there are 22 different color states to keep track of (reducing to 16 for $N_c = 3$) [5, 6]. In the later case, to keep track of the change in color structure as a result of virtual gluon exchange between a pair of partons, one naively - without using further symmetries, thus needs to calculate the effect of gluon exchange on 22 different color states, and then decompose the result into the 22 different color tensors by taking scalar products, implying in total 22^2 scalar product. (This number may be reduced, for example by using the fact that the soft anomalous dimension matrices are symmetric if stated in orthonormal bases [7, 8].) The color structure thus gives rise to a major computational complication, and so far the soft anomalous dimension matrices have only been calculated for the $2 \to 2$ processes [5, 9–14] and the $2 \to 3$ processes [6, 15]. (For observable related and experimental work, see for example [16–24].)

If one is only interested in a fixed order expansion, for example as in [25], there is no need to choose an explicit basis. Indeed the soft anomalous dimension matrix can be written down in a compact basis-independent way for any number of partons, both at one-loop and two-loop order [42]. Similarly, for the purpose of deriving general theoretical properties it is often wiser to stay basis independent, and several interesting results have recently been derived without explicit basis choices [8, 26–30]. However, to actually perform the numerical exponentiation of (1.1), to obtain all-order results, an explicit basis is needed.

It is thus clearly desirable to find a simplifying general strategy. Especially, a unified formalism is needed for the long term goal to incorporate non-leading color effects in event generators. The major current event generators all work in the leading N_c limit [31–35]. This means that the color structure is decomposed into leading N_c contributions, using (2.2) and (2.1) below. Color suppressed interference terms between different color structures are neglected. It was argued a long time ago that for gluon amplitudes with fixed power of α_s these terms are suppressed by $1/N_c^2$ [36]. However, there may in general be many suppressed terms. As an example consider $N_g - 2$ gluons attached in a row to one gluon line, giving in total N_g gluons. (For N_g up to five, all tree level graphs have this topology.) The squared amplitude is given by

$$
N_c^{N_g - 2} (N_c^2 - 1).
\tag{1.6}
$$

If the diagram is decomposed into different color topologies (which are orthogonal in the $N_c \to \infty$ limit) the sum of the parts squared separately is

$$
\frac{1}{N_c^{N_g}} \left[(N_c^2 - 1)^{N_g} + (-1)^{N_g} (N_c^2 - 1) \right].
\tag{1.7}
$$

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When $N_c \to \infty$ both expressions grows as $N_c^{N_g}$ and their ratio approaches one. However for finite N_c the difference grows with N_g and already for $N_g = 4$, if $N_c = 3$, (1.7) is only 19/27 of (1.6), [37]. For 7 gluons (1.7) is less than 50% of (1.6).

The method suggested in this paper for dealing with the color structure of multi-parton processes is developed with the resummation of soft gluons in mind, but clearly, as it describes the effect of gluon exchange on any colored amplitude, it may also prove useful for NLO (and higher order) corrections to amplitudes with (many) colored partons.

The results may also be used to calculate effects stemming from the non-global nature of most observables, the ordinary "non-global logs" [38, 39], as well as the color suppressed "super leading logarithms" carrying extra powers of log(hard scale/soft scale), suggested to enter at order α_s^4 in perturbation theory [25, 40, 41]. Indeed, as the non-global logarithms originate from real radiation outside an experimental exclusion region, to calculate the contribution from n emissions outside the exclusion region requires the soft anomalous dimension matrices for processes containing n additional partons.

As the two-loop soft anomalous dimension matrices have been proven to be proportional to the one-loop results (for processes with any number of colored and uncolored massless external legs), the present method can trivially be used also for two-loop anomalous dimension matrices [42]. Recently it has been suggested that similar results also hold for the three-loop anomalous dimension matrices and that they may hold to any order, as long as the partons remain massless [26–28]. For massive external legs this simple relation breaks down [29, 30].

The layout of this paper is as follows: First the formalism for constructing a basis is described in section 2, and computational rules for gluon exchange in this basis are derived in section 3. To illustrate the advantages with the constructed bases, the soft anomalous dimension matrices for $gg \to gg$, $q\overline{q} \to q\overline{q}g$ and $q\overline{q} \to ggg$ are recalculated in section 4. Finally some concluding remarks are made in section 5.

2 General basis formalism

2.1 Construction of a general basis

Previous strategies for dealing with the color structure needed for resummation of soft gluons have lately been based on multiplet decomposition for finding a basis [6, 14]. In this way symmetry properties are exploited to construct complete orthogonal bases (which easily can be normalized). Clearly, using an orthogonal basis has advantages. The result is easy to interpret and the matrix of scalar products between basis vectors is diagonal.

A disadvantage of the multiplet strategy is, however, that increasingly complicated projection operators need to be used, and no closed form for deriving these projection operators exists (to the knowledge of the author).

Another complication is that the projection operators, which tend to be expressed in terms of the symmetric and anti-symmetric structure constants f_{abc} and d_{abc}, need increasingly complicated computational rules for contraction of indices, that is, computational rules involving more and more f's and d's. Alternatively, the structure constants can be

reexpressed in terms of the generators of the fundamental representation $t^g_{q_1q_2}$,

$$\left.\begin{array}{c} if_{abc} \\ d_{abc} \end{array}\right\} = 2(\text{Tr}[t^a t^b t^c] \mp \text{Tr}[t^b t^a t^c]) = 2(t^a_{q_1q_2} t^b_{q_2q_3} t^c_{q_3q_1} \mp t^b_{q_1q_2} t^a_{q_2q_3} t^c_{q_3q_1}). \qquad (2.1)$$

In this case any scalar product, of arbitrarely complicated color tensors, can be calculated using the gluon index contraction relation

$$t^g_{ca} t^g_{db} = \frac{1}{2} \left(\delta_{ad}\delta_{bc} - \frac{1}{N_c}\delta_{ac}\delta_{bd} \right). \qquad (2.2)$$

However, the expression for the color structure tensor will contain $2^{(\#\ \text{of}\ f\text{'s and}\ d\text{'s})}$ terms, and the scalar product of the tensor with itself thus $2^{2(\#\ \text{of}\ f\text{'s and}\ d\text{'s})}$ terms which each has to be contracted separately.

An alternative strategy would be to construct a basis by starting from a sufficient number of arbitrarily chosen color tensors, or by exploiting possible symmetries. This will work well for a small vector space, cf. [15], but will tend to give very lengthy expressions for the basis vectors if Gram-Schmidt orthogonalization is used for a large vector space. On the other hand, if the basis vectors are not made orthogonal the decomposition of color structures resulting after gluon exchange will in general be cumbersome. (This complication is circumvented in the special non-orthogonal basis suggested below). In addition it has to be proved that the basis actually spans the relevant space.

These issues make it worth exploring other strategies for constructing the basis in the general case of any number of colored and uncolored partons. The basis clearly has to span the relevant space. It may seem desirable to find an orthogonal (normalized) basis, but it will be seen below that using a special non-orthogonal, non-normalized basis significantly diminishes the computational effort, mainly since the state obtained after gluon exchange is immediately, i.e. without taking scalar products, a linear combination of basis states. There is thus no need for calculating $N_\text{basis}{}^2$ scalar products for every possible gluon exchange.

The solution is to use a basis inspired by the $N_c \to \infty$ limit. In the case of infinitely many colors, two color lines in a Feynman diagram are never the same, and gluons may be represented by two color lines going in opposite directions. In this case, all possible color structures can be represented by all ways of connecting incoming and outgoing color lines. The strategy suggested here is thus similar to methods used in [43–46]. Especially it is noted that the bases suggested here for resummation are similar to the color structure treatment suggested in [46] to deal with real parton emission in event generators. For $N_c = \infty$ the scalar product between different color topologies, divided by the scalar product of a topology with itself, equals zero. However, for finite N_c there are scalar product terms which are suppressed only by $1/N_c$.

Another important property of the bases constructed in the aforementioned way is that they are completely democratic w.r.t. different quarks, different anti-quarks and different gluons. This implies, for example, that once the effect of gluon exchange between the gluons g_1 and g_2 has been calculated, the effect of gluon exchange between any other gluons can be obtained by relabeling of indices i.e. renumbering of basis tensors. One

therefore never needs to calculate more than six different exchanges gg, qq, $\overline{q}\,\overline{q}$, gq, $g\overline{q}$ and $q\overline{q}$. In addition, it will be seen below that the color structure after a gluon exchange on a given color topology is a linear combination of at most four different basis tensors. The soft anomalous dimension matrices will thus be relatively sparse in the suggested bases, which should simplify numerical exponentiation.

It is also worth stressing that the suggested bases are well suited for comparison to the $N_c \to \infty$ limit, as the bases are easy to interpret and the soft anomalous dimension matrices will turn out to be diagonal in this limit. This implies that they are ideal for comparison to the radiation pattern obtained from event generators tending to work in the $N_c \to \infty$ limit [31–35].

The reduction in calculational effort for the soft anomalous dimension matrix with the suggested basis is thus threefold. There is *no* need to calculate scalar products, reducing the computational effort with a factor $\sim N_{\text{basis}}^2$ from the number of scalar products *and* a factor $2^{(\# \text{ of } f\text{'s and } d\text{'s})}$ from the number of terms in each of the scalar products, assuming (2.1) is used. Furthermore there are at most six, as compared to $N_{\text{p}}(N_{\text{p}} - 1)/2$ for N_{p} external particles, different gluon exchanges to keep track of, the others are related by relabeling of indices.

Unfortunately this does not quite remove the bad scaling of the problem with the number of partons, as instead of having to calculate $\sim N_{\text{basis}}^2$ scalar products for each contribution to the soft anomalous dimension matrix, one has to calculate $\sim N_{\text{basis}}^2$ scalar products between the basis vectors, as they are only orthogonal in the $N_c \to$ infinity limit. However, this only has to be done once. In addition calculating scalar products using (2.1) and (2.2) gives just one, as opposed to $2^{(\# \text{ of } f\text{'s and } d\text{'s})}$, different terms.

What remains is thus a scaling of type N_{basis}^2. Very roughly speaking N_{basis} tends to grow as $N_{\text{p}}!$, cf. section 2.2, 2.4. But, bearing in mind that only the topology of the color contraction, and not the labeling of indices is important for the scalar product, should naively reduce the $(N_{\text{p}}!)^2$ scaling by a factor $\sim N_g! N_q! N_{\overline{q}}!$ from the number of ways of labeling the indices. What remains is then a factorial growth for processes with only gluons.

Note however, that for processes with many enough external partons, the major computational effort will not lie in finding an expression for the soft anomalous dimension matrix, but in numerical exponentiation of the obtained result. As numerical matrix exponentiation scales with the cube of the matrix size, and the number of basis vectors tends to grow factorially with the number of partons, calculations with more than ten particles seem unlikely. For practical implementations, it is also worth pointing out that the number of basis vectors highly depend on the kinds of partons involved. For processes with no external gluons and $N_q = N_{\overline{q}} = N_{\text{p}}/2$ partons, the number of basis vectors is $(N_{\text{p}}/2)!$ whereas for processes with only external gluons the size of the basis tens to grow rather as $N_{\text{p}}!/e$, cf. section 2.2 and 2.4.

That a basis constructed in the above described way is complete for $N_c = \infty$ is clear from the fact that it represents all possible color topologies. For finite N_c, some of the color tensors may be linearly dependent, and the basis over-complete, but it will still span

the space. One way of thinking of the reduction in dimension of the color space is to note that tensors corresponding to multiplets which are anti-symmetric in more than N_c quark indices are not possible. Requiring that a color decomposition should be valid for all N_c defines a unique decomposition of a $N_c = 3$ tensor.

Another way of convincing oneself that the above bases are complete, is to note that *every* internal gluon line in any Feynman diagram can be removed by first using (2.1) to remove the triple gluon vertices and then (2.2) to remove gluon propagators. In this way any Feynman diagram, tree level or not, can be decomposed into color structures containing no gluon propagators. What remains is a linear combination of color structures containing internal quark lines, external quarks, external anti-quarks and external gluons. That is, a linear combination of terms of precisely the form obtained by first splitting all gluons to $q\bar{q}$-paris, and then connecting quark and anti-quark lines in all possible ways.

Below, the construction of basis tensors will be investigated in more detail, first in the special case of external quarks only, then for external gluons only, and finally in the general case of both.

Before moving on we note that from the color algebra point of view there is no difference between an outgoing quark and an incoming anti-quark, from here on simply collectively referred to as quark, or an incoming quark and outgoing anti-quark, from now on referred to as anti-quark. Opposite conventions may be used elsewhere. In addition the placing of quark and anti-quark indices on the fundamental generators may be varied.

2.2 The quarks only case

Finding a basis in the case of only external quarks is trivial. The basis just consists of all possible ways of connecting quarks and anti-quarks. For $N_q = N_{\bar{q}}$ quarks (clearly, for each incoming quark line there is also an outgoing) this can be done in

$$N_{\text{basis}} = N_q!$$ (2.3)

ways. The squared norm of these basis vectors, calculated using (1.4), is equal to $N_c^{N_q}$.

To denote the tensors the notation

$$(\bar{q}_1 q_3)(\bar{q}_2 q_4) = \delta_{q_1 q_3} \delta_{q_2 q_4}$$ (2.4)

is used. A complete basis for $q_1 q_2 \to q_3 q_4$ is thus the tensors $(\bar{q}_1 q_3)(\bar{q}_2 q_4)$ and $(\bar{q}_1 q_4)(\bar{q}_2 q_3)$. In fact this is the basis used in [10].

2.3 The gluons only case

To construct the basis in the case of gluons only, closed quark loops with external gluons attached are used. For example, for four gluons, all gluons may be connected to the same quark line giving $(4 - 1)! = 6$ topologically different diagrams. Alternatively the gluons may be connected two and two in three different ways. Indeed the color space also has nine dimensions, however, only half of the linear combinations of the six fully connected topologies are physical, due to the fact that quarks and anti-quarks enter QCD on equal footing. Therefore, if, in a quark loop, a quark is going around in one direction, the

topology with the quark going around in the opposite direction (i.e. the gluon index order is reversed) must also contribute.

More explicitly, introducing the notation

$$(g_1 g_2 \ldots g_{N_g}) = \mathrm{Tr}[t^{g_1} t^{g_2} \ldots t^{g_{N_g}}] = t^{g_1}_{q_1 q_2} t^{g_2}_{q_2 q_3} \ldots t^{g_{N_g}}_{q_{N_g} q_1}, \tag{2.5}$$

to denote N_g gluons attached clockwise in the order $g_1 \ldots g_{N_g}$ on a quark line, we note that the physical linear combinations must be

$$(g_1 g_2 \ldots g_{N_g}) + (-1)^{N_g} (g_{N_g} \ldots g_2 g_1). \tag{2.6}$$

To understand the sign, decompose any tree level Feynman diagram with only gluons using (2.1) and (2.2). The result is a sum of color structures where the N_g gluons are attached in different orders to the quark-line. For a specific order, the anti-cyclic order is obtained by reversing the direction of the quark-line in *every* vertex, i.e. taking the other term in (2.1) everywhere. This gives a factor $(-1)^{N_g-2}$ as there are $N_g - 2$ vertices, explaining the sign in (2.6).

Thus, in the case of $gg \to gg$, only six color tensors are needed (for general N_c). This explains the observation that some tensors decouple for $gg \to gg$ and $gg \to ggg$ [5, 6, 12].

The problem of constructing the N_g-gluon basis in the general case thus boils down to:

1) Find all the ways of grouping the N_g gluons such that each group contains at least two gluons. (Groups with only one gluon would correspond to the color structure $t^g_{qq} = 0$.) For four external gluons the possible groupings are thus $\{4\}$ and $\{2, 2\}$.

2) For each fully connected grouping, such as $\{4\}$, find all physical different ways of arranging the gluons. For N_g gluons this gives $(N_g - 1)!/2$ different color tensors where the factor $1/2$ is present since only one combination of the cyclic and anti-cyclic ring is physical.

3) For disconnected groupings, such as $\{2, 2\}$,

   3a) Find separately, for each subgroup, all physically different ways of arranging the gluons.

   3b) Distribute the gluon indices $\{g_1 \ldots .g_{N_g}\}$ in all possible ways among the different subgroups.

   3c) Combine the different sub-groupings in all possible ways, taking into account that, if all gluon indices are equal, two groupings do actually correspond to the same physical state. For example the subgrouping $\{\{g_1, g_2\}, \{g_3, g_4\}\}$ and $\{\{g_3, g_4\}, \{g_1, g_2\}\}$ are equal.

Following this recipe a complete basis describing the color structure for any number of external gluons can be constructed.

Neglecting the issue of physical linear combinations, the possible color tensors coincide with the color tensors obtained by replacing each gluon with one quark and one anti-quark

line, with the important exception that contractions between a $q\overline{q}$ pair corresponding to the same gluon are disallowed. The problem of finding all such topologies is equivalent to the number of ways of mapping N elements to each other without mapping a single one to itself, which has a known solution

$$N! \sum_{i=0}^{N} \frac{(-1)^i}{i!} \rightarrow \frac{N!}{e}. \tag{2.7}$$

The convergence to $N!/e$ is very quick, rounding off to the closest integer works already for $N = 1$.

Note however, that this just gives the total number of linearly independent color tensors (for $N_c = \infty$). As mentioned above, only tensor combinations where quarks and anti-quarks enter on equal footing are physical. For every quark ring participating in building up a color tensor, the corresponding anti-quark ring has to be added. This reduces the number of physical tensors of a certain topology, such as $\{3, 2\}$ with a factor $(1/2)^{\#}$ rings building up the tensor, that is $(1/2)^2$ for $\{3, 2\}$.

As the number of fully connected color topologies, where all N_g gluons are attached to the same quark-line equals $(N_g - 1)!$, the fraction of color tensors corresponding to fully connected diagrams is roughly e/N_g, again ignoring the issue of physical tensor combinations. Tree level QCD Feynman diagrams with only external gluons (more generally, no gluon propagator between quarks) always correspond to linear combinations of fully connected diagrams, (i.e. diagrams where all gluons are connected to the same quark-line) and are the only diagrams presently included in major event generators. When considering only physical topologies the ratio of fully connected to disconnected graphs changes slightly to the advantage of the fully connected graph, as the factor $(1/2)^{\#}$ rings building up the tensor hits the disconnected topologies harder.

The norm of the color tensor with all gluons attached to a quark going around in *one* direction is given by

$$\left(\frac{1}{2N_c}\right)^{N_g} \left[(N_c^2 - 1)^{N_g} + (-1)^{N_g}(N_c^2 - 1)\right]. \tag{2.8}$$

The physical tensors, being sums of gluons attached to rings with quarks going around in opposite directions, contain mixed terms as well, these are however relatively suppressed, and for large N_g or large N_c (2.8) is a good approximation. Note that the norm grows as $N_c^{N_g}$, which is to be expected considering the $N_c = \infty$ limit. It turns out, however, that it is easier to stick to the non-normalized versions of the color tensors.

2.4 The case of both quarks and gluons

In the general case of both external quarks and gluons the basis may be constructed by:

1) Connect the quark lines to each other in all possible ways, giving $N_q!$ possibilities.

2) For $i = 1, 2, \ldots N_g - 2, N_g$ attach i of the N_g gluons to the quarks in all possible ways.

3) Connect the remaining $N_g - i$ gluons as in the gluons only case, but keep cyclic and anti-cyclic tensors separately.

4) Distribute the quark and gluon indices in all possible ways among the different groupings.

The number of color tensors in this case grows slower than $(N_g + N_q)!$ but faster than $(N_g + N_q)!/e$, again giving a factorial growth.

3 Calculating the effect of gluon exchange

Below, the computational rules for gluon exchange will be derived, and it will be seen that exchanging a gluon trivially gives an explicit linear combination of the basis tensors. There is thus no need to calculate scalar products of the resulting color structure after exchange, with the basis tensors. As calculating scalar products was the most cumbersome part in previous calculations, this represents a major improvement.

3.1 Computational rules

In this section the computational rules for gluon exchange between the basis tensors constructed in section 2 are derived.

Note that the quarks in closed quark loops are just products of the way of writing down the basis and not physical particles, a gluon is thus never exchanged between the quarks in closed quark loops.

We also have to decide on a convention for the triple gluon vertex. The convention used is

$$
\begin{aligned}
f_{eig} \quad &\text{with} \\
e &= \text{the external (incoming or outgoing) eikonal gluon index} \\
i &= \text{the internal (incoming or outgoing) eikonal gluon index} \\
g &= \text{the soft exchange gluon index.}
\end{aligned} \tag{3.1}
$$

This convention has the advantage that the sign is independent of how the diagram is drawn on a paper and whether a parton is incoming or outgoing.

3.1.1 Gluon exchange between two quarks or anti-quarks

In the simplest case a gluon is exchanged between two external quarks q_1 and q_2, which in general have n and m gluons attached respectively. Using (2.2), the effect of gluon exchange between the quarks q_1 and q_2 in two different open quark lines may be written

$$
\begin{pmatrix} \overline{q}_1 g_{11} \dots g_{1n} q_1 & \otimes \\ \overline{q}_2 g_{21} \dots g_{2m} q_2 \end{pmatrix} \rightarrow \frac{1}{2} \begin{pmatrix} \overline{q}_1 g_{11} \dots g_{1n} q_2 & \otimes \\ \overline{q}_2 g_{21} \dots g_{2m} q_1 \end{pmatrix} - \frac{1}{2N_c} \begin{pmatrix} \overline{q}_1 g_{11} \dots g_{1n} q_1 & \otimes \\ \overline{q}_2 g_{21} \dots g_{2m} q_2 \end{pmatrix}. \tag{3.2}
$$

where the notation

$$
\begin{pmatrix} \overline{q}_1 g_{11} \dots g_{1n} q_1 & \otimes \\ \overline{q}_2 g_{21} \dots g_{2m} q_2 \end{pmatrix} = t^{g_{11}}_{d_{11}\overline{q}_1} t^{g_{12}}_{d_{12}d_{11}} \dots t^{g_{1n}}_{q_1 d_{1n-1}} t^{g_{21}}_{d_{21}\overline{q}_2} t^{g_{22}}_{d_{22}d_{21}} \dots t^{g_{2m}}_{q_2 d_{2m-1}} \tag{3.3}
$$

is used.

If the gluon is instead exchanged between the external anti-quarks, the indices on the quarks are kept whereas the indices on the anti-quarks are exchanged.

3.1.2 Gluon exchange between quark and anti-quark

Exchanging a gluon between a quark q_1 and an anti-quark $\overline{q}_2$ results in

$$
\begin{pmatrix} \overline{q}_1 g_{11} \cdots g_{1n} q_1 \ \otimes \\ \overline{q}_2 g_{21} \cdots g_{2m} q_2 \end{pmatrix} \rightarrow \frac{1}{2} \begin{pmatrix} \overline{q}_1 g_{11} \cdots g_{1n} g_{21} \cdots g_{2m} q_2 \ \otimes \\ \overline{q}_2 q_1 \end{pmatrix} - \frac{1}{2N_c} \begin{pmatrix} \overline{q}_1 g_{11} \cdots g_{1n} q_1 \ \otimes \\ \overline{q}_2 g_{21} \cdots g_{2m} q_2 \end{pmatrix}.
\tag{3.4}
$$

The case where the involved quark and anti-quark are part of the same quark line can be obtained by identifying $\overline{q}_1$ and q_2 above.

3.1.3 Gluon exchange between quark and gluon

To derive the effect of gluon exchange between a quark and a gluon we use the relation (2.1) to rewrite the triple gluon vertex. After this (2.2) is applied (and it is noted that the $1/N_c$ suppressed terms drop out). For gluon exchange between the quark q_1 and the gluon g_{2i} the result is:

$$
\begin{pmatrix} \overline{q}_1 g_{11} \cdots g_{1n} q_1 \ \otimes \\ \overline{q}_2 g_{21} \cdots g_{2i} \cdots g_{2m} q_2 \end{pmatrix} \rightarrow -\frac{1}{2} \begin{pmatrix} \overline{q}_1 g_{11} \cdots g_{1n} g_{2i+1} \cdots g_{2m} q_2 \ \otimes \\ \overline{q}_2 g_{21} \cdots g_{2i} q_1 \end{pmatrix}
$$
$$
+ \frac{1}{2} \begin{pmatrix} \overline{q}_1 g_{11} \cdots g_{1n} g_{2i} \cdots g_{2n} q_2 \ \otimes \\ \overline{q}_2 g_{21} g_{2i-1} q_1 \end{pmatrix}.
\tag{3.5}
$$

If, in the left hand side above, g_{2i} is in a closed quark loop this is accounted for by identifying $\overline{q}_2$ and q_2, and if the gluon g_{2i} is attached to the same quark-line as q_1 this is taken care of by identifying $\overline{q}_1$ and q_2.

3.1.4 Gluon exchange between anti-quark and gluon

Employing the same calculational method as for qg results in

$$
\begin{pmatrix} \overline{q}_1 g_{11} \cdots g_{1n} q_1 \ \otimes \\ \overline{q}_2 g_{21} \cdots g_{2i} \cdots g_{2m} q_2 \end{pmatrix} \rightarrow \frac{1}{2} \begin{pmatrix} \overline{q}_1 g_{2i} \cdots g_{2m} q_2 \ \otimes \\ \overline{q}_2 g_{21} \cdots g_{2i-1} g_{11} \cdots g_{2n} q_1 \end{pmatrix}
$$
$$
- \frac{1}{2} \begin{pmatrix} \overline{q}_1 g_{2i+1} \cdots g_{2m} q_2 \ \otimes \\ \overline{q}_2 g_{21} \cdots g_{2i} g_{11} \cdots g_{1n} q_1 \end{pmatrix}
\tag{3.6}
$$

where again, if g_{2i} initially is in a closed quark loop this is accounted for by identifying $\overline{q}_2$ and q_2, and if the gluon g_{2i} was originally placed on the same quark line as $\overline{q}_1$ this is taken care of by identifying $\overline{q}_2$ and q_1.

3.1.5 Gluon exchange between two external gluons

To derive the effect on the basis vectors of exchanging a gluon between two external gluons, two triple gluon vertices have to be replaced using (2.1) and three gluon propagators have to be contracted using (2.2). Again the non-leading N_c terms drop out and the result of

exchanging a gluon between g_{1i} and g_{2j} is

$$\begin{pmatrix} \overline{q}_1 g_{11} \dots g_{1i} \dots g_{1n} q_1 \quad \otimes \\ \overline{q}_2 g_{21} \dots g_{2j} \dots g_{2m} q_2 \end{pmatrix} \rightarrow \qquad\qquad\qquad (3.7)$$

$$-\frac{1}{2} \begin{pmatrix} \overline{q}_1 g_{11} \dots g_{1i-1} g_{2j+1} \dots g_{2m} q_2 \quad \otimes \\ \overline{q}_2 g_{21} \dots g_{2j-1} g_{2j} g_{1i} g_{1i+1} \dots g_{in} q_1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \overline{q}_1 g_{11} \dots g_{1i-1} g_{1i} g_{2j+1} \dots g_{2m} q_2 \quad \otimes \\ \overline{q}_2 g_{21} \dots g_{2j-1} g_{2j} g_{1i+1} \dots g_{1n} q_1 \end{pmatrix}$$

$$+\frac{1}{2} \begin{pmatrix} \overline{q}_1 g_{11} \dots g_{1i-1} g_{2j} g_{2j+1} \dots g_{2m} q_2 \quad \otimes \\ \overline{q}_2 g_{21} \dots g_{2j-1} g_{1i} g_{1i+1} \dots g_{1n} q_1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \overline{q}_1 g_{11} \dots g_{1i-1} g_{1i} g_{2j} g_{2j+1} \dots g_{2m} q_2 \quad \otimes \\ \overline{q}_2 g_{21} \dots g_{2j-1} g_{1i+1} \dots g_{1n} q_1 \end{pmatrix}.$$

If one (or both) quark lines is (are) closed, then the corresponding quarks are to be identified. If both gluons are part of the same quark line, then identify $q_1 \overline{q}_2$, and $q_2 \overline{q}_1$ if the quark line is closed.

4   Some explicit examples

4.1 $gg \rightarrow gg$

As an explicit example of how the above strategy simplifies the problem of keeping track of the color structure, the process of $g_1 g_2 \rightarrow g_3 g_4$ will be considered in detail here. The soft anomalous dimension matrix for this case was first calculated in [12] and later, more elegantly in [14].

4.1.1   Construction of the basis

To construct the basis the recipe outlined in section 2.3 is followed, starting with finding all the ways of grouping the gluons:

1) The four gluons can be grouped two and two $\{2,2\}$ or all four together $\{4\}$.

2) When all four gluons are attached to the same quark line, $\{4\}$, the indices can be placed in $(4-1)! = 6$ different ways. However, due to the symmetry between quarks and anti-quarks, clockwise and anti-clockwise gluon rings only enter in one linear combination, giving three physical tensors:

$$C^1_{g_1 g_2 g_3 g_4} = (g_1 g_2 g_3 g_4) + (g_4 g_3 g_2 g_1) = \mathrm{Tr}[t^{g_1} t^{g_2} t^{g_3} t^{g_4}] + \mathrm{Tr}[t^{g_4} t^{g_3} t^{g_2} t^{g_1}]$$
$$C^2_{g_1 g_2 g_3 g_4} = (g_1 g_2 g_4 g_3) + (g_3 g_4 g_2 g_1) = \mathrm{Tr}[t^{g_1} t^{g_2} t^{g_4} t^{g_3}] + \mathrm{Tr}[t^{g_3} t^{g_4} t^{g_2} t^{g_1}]$$
$$C^3_{g_1 g_2 g_3 g_4} = (g_1 g_3 g_2 g_4) + (g_4 g_2 g_3 g_1) = \mathrm{Tr}[t^{g_1} t^{g_3} t^{g_2} t^{g_4}] + \mathrm{Tr}[t^{g_4} t^{g_2} t^{g_3} t^{g_1}]. \quad (4.1)$$

3a) For the grouping $\{2,2\}$, the index order in the subgrouping doesn't matter (since $\mathrm{Tr}[t_1^g t_2^g] = \mathrm{Tr}[t_2^g t_1^g]$). Each subgrouping thus only gives rise to one physical tensor.

3b,c) The gluon indices g_1, g_2, g_3, g_4 may be split into the subgroupings as $\{\{g_1, g_2\}, \{g_3, g_4\}\}$, $\{\{g_1, g_3\}, \{g_2, g_4\}\}$ and $\{\{g_1, g_4\}, \{g_2, g_3\}\}$, giving three basis tensors

$$C^4_{g_1g_2g_3g_4} = (g_1g_2)(g_3g_4) = \text{Tr}[t^{g_1}t^{g_2}]\text{Tr}[t^{g_3}t^{g_4}] = \left(\frac{1}{2}\right)^2 \delta_{g_1g_2}\delta_{g_3g_4}$$

$$C^5_{g_1g_2g_3g_4} = (g_1g_3)(g_2g_4) = \text{Tr}[t^{g_1}t^{g_3}]\text{Tr}[t^{g_2}t^{g_4}] = \left(\frac{1}{2}\right)^2 \delta_{g_1g_3}\delta_{g_2g_4}$$

$$C^6_{g_1g_2g_3g_4} = (g_1g_4)(g_2g_3) = \text{Tr}[t^{g_1}t^{g_4}]\text{Tr}[t^{g_2}t^{g_3}] = \left(\frac{1}{2}\right)^2 \delta_{g_1g_4}\delta_{g_2g_3}. \qquad (4.2)$$

4.1.2 Calculation of soft anomalous dimension matrix

As previously noted, once the effect of gluon exchange between g_1 and g_2 is calculated the effect of gluon exchange between any other gluons may be deduced. There may thus at most be N_{basis} different situations to keep track of. However, this number will in general be further reduced due to the irrelevance of non-participating indices. For example the effect of gluon exchange between g_1 and g_2 on $C^1_{g_1g_2g_3g_4}$ is the same as the effect on $C^2_{g_1g_2g_3g_4}$. Thus, the physically different situations are:

1) A gluon is exchanged between two neighboring gluons on a quark ring with four gluons attached (for example gluon 1 and 2 on C^1).

   Applying (3.7) to the first half of $C^1_{g_1g_2g_3g_4}$ with the identification $g_1 \to g_{12} = g_{1i}$, $g_2 \to g_{21} = g_{2j}$, $g_3 \to g_{22}$, $g_4 \to g_{11}$ and $\overline{q}_1 = q_2$, $\overline{q}_2 = q_1$ gives

   $$(g_1g_2g_3g_4) \to -\frac{1}{2}(g_1g_2)(g_3g_4) - \frac{N_c}{2}(g_1g_2g_3g_4). \qquad (4.3)$$

   Similarly application to the second half results in

   $$(g_1g_4g_3g_2) \to -\frac{1}{2}(g_1g_2)(g_3g_4) - \frac{N_c}{2}(g_1g_4g_3g_2), \qquad (4.4)$$

   and it may be concluded that

   $$C^1_{g_1g_2g_3g_4} \to -\frac{2}{2}C^4_{g_1g_2g_3g_4} - \frac{N_c}{2}C^1_{g_1g_2g_3g_4}. \qquad (4.5)$$

2) A gluon can be exchanged between two next to neighboring gluons. In this case we get for an exchange between g_1 and g_2 on $C^3_{g_1g_2g_3g_4}$

   $$C^3_{g_1g_2g_3g_4} \to C^5_{g_1g_2g_3g_4} + C^6_{g_1g_2g_3g_4}. \qquad (4.6)$$

3) A gluon may be exchanged between the gluons attached to a two gluon ring, such as g_1 and g_2 on $C^4_{g_1g_2g_3g_4}$. This just gives a factor N_c multiplying the old tensor, for example for gluon exchange between g_1 and g_2 on $C^4_{g_1g_2g_3g_4}$

   $$C^4_{g_1g_2g_3g_4} \to N_c C^4_{g_1g_2g_3g_4}. \qquad (4.7)$$

4) A gluon may be exchange between two gluons attached to different two gluon rings such as g_1 and g_2 in $C^6_{g_1g_2g_3g_4}$, giving

$$C^6_{g_1g_2g_3g_4} \rightarrow -\frac{1}{2}C^1_{g_1g_2g_3g_4} + \frac{1}{2}C^3_{g_1g_2g_3g_4}. \tag{4.8}$$

The above information may be combined into a matrix describing the color algebra part for gluon exchange between g_1 and g_2

$$\mathbf{C}^{12}_{gg \rightarrow gg} = \begin{pmatrix} \frac{-N_c}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{-N_c}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ -1 & -1 & 0 & N_c & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

As the gluons g_3 and g_4 have the same relationship to each other in the basis as g_1 and g_2, the color structure of the soft anomalous dimension matrix will be the same $\mathbf{C}^{34}_{gg \rightarrow gg} = \mathbf{C}^{12}_{gg \rightarrow gg}$. Similarly $\mathbf{C}^{14}_{gg \rightarrow gg} = \mathbf{C}^{23}_{gg \rightarrow gg}$ and $\mathbf{C}^{24}_{gg \rightarrow gg} = \mathbf{C}^{13}_{gg \rightarrow gg}$.

The contributions $\mathbf{C}^{14}_{gg \rightarrow gg}$ and $\mathbf{C}^{24}_{gg \rightarrow gg}$ may be calculated by using the results in eqs. (4.5), (4.8) and relabeling indices. Letting $T = \Omega_{12} + \Omega_{34}$, $U = \Omega_{13} + \Omega_{24}$ and $V = \Omega_{14} + \Omega_{23}$ be the phase space integrals the result can be written

$$\mathbf{\Gamma}_{gg \rightarrow gg} = \begin{pmatrix} -\frac{1}{2}N_c(T+V) & 0 & 0 & \frac{U-V}{2} & 0 & \frac{U-T}{2} \\ 0 & -\frac{1}{2}N_c(T+U) & 0 & \frac{V-U}{2} & \frac{V-T}{2} & 0 \\ 0 & 0 & -\frac{1}{2}N_c(U+V) & 0 & \frac{T-V}{2} & \frac{T-U}{2} \\ U-T & V-T & 0 & -N_cT & 0 & 0 \\ 0 & V-U & T-U & 0 & -N_cU & 0 \\ U-V & 0 & T-V & 0 & 0 & -N_cV \end{pmatrix}. \tag{4.9}$$

To obtain physical results the scalar product matrix

$$\mathbf{S}_{gg \rightarrow gg} = \tag{4.10}$$

$$\begin{pmatrix} \frac{N_c^6-3N_c^4+8N_c^2-6}{8N_c^2} & \frac{-N_c^4+4N_c^2-3}{4N_c^2} & \frac{-N_c^4+4N_c^2-3}{4N_c^2} & \frac{(N_c^2-1)^2}{8N_c} & \frac{1-N_c^2}{8N_c} & \frac{(N_c^2-1)^2}{8N_c} \\ \frac{-N_c^4+4N_c^2-3}{4N_c^2} & \frac{N_c^6-3N_c^4+8N_c^2-6}{8N_c^2} & \frac{-N_c^4+4N_c^2-3}{4N_c^2} & \frac{(N_c^2-1)^2}{8N_c} & \frac{(N_c^2-1)^2}{8N_c} & \frac{1-N_c^2}{8N_c} \\ \frac{-N_c^4+4N_c^2-3}{4N_c^2} & \frac{-N_c^4+4N_c^2-3}{4N_c^2} & \frac{N_c^6-3N_c^4+8N_c^2-6}{8N_c^2} & \frac{1-N_c^2}{8N_c} & \frac{(N_c^2-1)^2}{8N_c} & \frac{(N_c^2-1)^2}{8N_c} \\ \frac{(N_c^2-1)^2}{8N_c} & \frac{(N_c^2-1)^2}{8N_c} & \frac{1-N_c^2}{8N_c} & \frac{(N_c^2-1)^2}{16} & \frac{(N_c^2-1)}{16} & \frac{(N_c^2-1)}{16} \\ \frac{1-N_c^2}{8N_c} & \frac{(N_c^2-1)^2}{8N_c} & \frac{(N_c^2-1)^2}{8N_c} & \frac{(N_c^2-1)}{16} & \frac{(N_c^2-1)^2}{16} & \frac{(N_c^2-1)}{16} \\ \frac{(N_c^2-1)^2}{8N_c} & \frac{1-N_c^2}{8N_c} & \frac{(N_c^2-1)^2}{8N_c} & \frac{(N_c^2-1)}{16} & \frac{(N_c^2-1)}{16} & \frac{(N_c^2-1)^2}{16} \end{pmatrix},$$

calculated using (1.4), is also needed. This matrix contains N_{basis}^2 entries, however, closer consideration reveals that only six of them correspond to different contractions.

It is worth remarking on the leading N_c behavior of (4.9). The computational rules in eqs. (3.2), (3.7) contain no positive power of N_c. Thus the N_c in (4.9) must come from closed

quark loops. The only way to get a closed quark loop is to exchange a gluon between two neighboring partons attached to the same quark-line, i.e. only "color neighbors" radiate in the $N_c \to \infty$ limit. The result after exchange contains a factor N_c multiplying the old color structure. Leading N_c contributions will therefore always be diagonal in the present basis.

This is in close resemblance with the Dipole Cascade Model and the original Ariadne program in which only neighboring pairs of partons, dipoles, radiate [47–50]. In particular for gluon radiation from $e^+e^- \to q\overline{q}$ the leading N_c piece should come from neighboring partons. It is cautioned, however, that there are many more non-leading N_c contributions, than leading pieces, as there (in general) are many more non-neighboring partons.

It is also worth remarking that although there are scalar products between different basis tensors that are suppressed by only one power of N_c, these scalar products are never between different fully connected topologies, i.e. never between tree level QCD gluon amplitudes.

4.2 $q\overline{q} \to q\overline{q}g$

As an example of a process containing both quarks and gluons we consider the color structure needed for gluon resummation for $q_1\overline{q}_2 \to q_3\overline{q}_4g_5$. This color structure is important for (among other things) QCD corrections to the production of W's decaying leptonically and being accompanied by three jets [51].

The result of constructing color tensors as outlined in section 2.4 is

$$
\begin{aligned}
C^1_{q_1q_2q_3q_4g_5} &= (\overline{q}_4 g_5 q_3)(\overline{q}_1 q_2) = t^{g_5}_{q_3q_4}\delta_{q_1q_2} \\
C^2_{q_1q_2q_3q_4g_5} &= (\overline{q}_4 q_3)(\overline{q}_1 g_5 q_2) = \delta_{q_3q_4}t^{g_5}_{q_2q_1} \\
C^3_{q_1q_2q_3q_4g_5} &= (\overline{q}_4 g_5 q_2)(\overline{q}_1 q_3) = t^{g_5}_{q_2q_4}\delta_{q_3q_1} \\
C^4_{q_1q_2q_3q_4g_5} &= (\overline{q}_2 q_4)(\overline{q}_1 g_5 q_3) = \delta_{q_2q_4}t^{g_5}_{q_3q_1}.
\end{aligned}
\tag{4.11}
$$

Exchanging gluons between the partons in all possible ways results in a leading color diagonal part, a $1/N_c$ suppressed off-diagonal part and a $1/N_c^2$ suppressed diagonal part:

$$
\mathbf{\Gamma}_{q\overline{q} \to q\overline{q}g} =
$$
$$
\frac{N_c}{2}\text{Diagonal}[\Omega_{12} + \Omega_{35} - \Omega_{45},\ -\Omega_{15} + \Omega_{25} + \Omega_{34},\ \Omega_{13} + \Omega_{25} - \Omega_{45},\ -\Omega_{15} + \Omega_{24} + \Omega_{35}]
$$
$$
+\frac{1}{2}\begin{pmatrix}
0 & 0 & \Omega_{12}+\Omega_{15}+\Omega_{23}+\Omega_{35} & \Omega_{12}+\Omega_{14}-\Omega_{25}-\Omega_{45} \\
0 & 0 & \Omega_{14}-\Omega_{15}+\Omega_{34}-\Omega_{35} & \Omega_{23}+\Omega_{25}+\Omega_{34}+\Omega_{45} \\
\Omega_{13}+\Omega_{15}+\Omega_{23}+\Omega_{25} & \Omega_{13}+\Omega_{14}-\Omega_{35}-\Omega_{45} & 0 & 0 \\
\Omega_{14}-\Omega_{15}+\Omega_{24}-\Omega_{25} & \Omega_{23}+\Omega_{24}+\Omega_{35}+\Omega_{45} & 0 & 0
\end{pmatrix}
$$
$$
-\frac{1}{2N_c}(\Omega_{12} + \Omega_{13} + \Omega_{14} + \Omega_{23} + \Omega_{24} + \Omega_{34})\,\text{Diagonal}[1,\ 1,\ 1,\ 1].
\tag{4.12}
$$

Again, as we are working in a non-orthogonal basis, all scalar products are needed

$$
\mathbf{S}_{q\overline{q} \to q\overline{q}g} = \begin{pmatrix}
\frac{1}{2}N_c\left(N_c^2 - 1\right) & 0 & \frac{1}{2}\left(N_c^2 - 1\right) & \frac{1}{2}\left(N_c^2 - 1\right) \\
0 & \frac{1}{2}N_c\left(N_c^2 - 1\right) & \frac{1}{2}\left(N_c^2 - 1\right) & \frac{1}{2}\left(N_c^2 - 1\right) \\
\frac{1}{2}\left(N_c^2 - 1\right) & \frac{1}{2}\left(N_c^2 - 1\right) & \frac{1}{2}N_c\left(N_c^2 - 1\right) & 0 \\
\frac{1}{2}\left(N_c^2 - 1\right) & \frac{1}{2}\left(N_c^2 - 1\right) & 0 & \frac{1}{2}N_c\left(N_c^2 - 1\right)
\end{pmatrix}.
\tag{4.13}
$$

4.3 $q\overline{q} \to ggg$

The other color structure relevant for W plus three jets is that of $q\overline{q} \to ggg$. In this case an 11-dimensional matrix is needed to describe the color space (reducing to 10 for $N_c = 3$). These results have been calculated and are electronically attached to this submission. Again there is a diagonal leading N_c part, an off-diagonal part with relative suppression $1/N_c$ and a $1/N_c^2$ suppressed diagonal contribution.

5 Conclusions

In the present paper a general recipe for constructing bases capable of dealing with the color structure needed for resummation for any number of colored partons has been presented. This in itself is a step forward. In addition the suggested bases are argued to have relatively nice computational properties. The bases are obtained from the $N_c = \infty$ case by splitting gluons in $q\overline{q}$ pairs and connecting color lines in all possible ways. The bases thus constructed will therefore neither be normalized or orthogonal for $N_c = 3$, but they will span the space and have the property that gluon exchanges between any pair of external partons directly, i.e. without taking scalar products, result in linear combinations of basis vectors. Furthermore, as can be seen from the computational rules in eqs. (3.2), (3.7), the result after gluon exchange contains at most four (often two or one) basis vectors, giving relatively sparse soft anomalous dimension matrices.

The fact that there is no need to calculate scalar products to decompose the tensors resulting after gluon exchange is a major advantage. Otherwise there would, for each of the $N_{\rm p}(N_{\rm p} - 1)/2$ possible gluon exchanges, be $\sim N_{\rm basis}^2 \sim (N_{\rm p}!)^2$ (cf. section 2.2, 2.4) scalar products to calculate. In addition, the computational time for calculating the effect of gluon exchange is further reduced, as the constructed bases maximally exploit the symmetry w.r.t. external parton indices. *All* indices corresponding to the same kind of parton enter the basis on equal footing. Therefore, for example, once the effect of gluon exchange between any pair of gluons has been calculated, the effect of gluon exchange between any other can be obtained by relabeling of indices, corresponding to a renumbering of tensors. One thus at most has to calculate six (gg, qq, $\overline{q}\overline{q}$, or $q\overline{q}$ to qg and $\overline{q}g$) different contributions to $\mathbf{\Gamma}$. For a hard scattering amplitude with only gluons it is enough to calculate one.

From this it is clear that the major computational effort lies in computing the $\sim N_{\rm basis}^2$ scalar products between all the basis vectors. So far, it thus looks as if the calculational effort is reduced by a factor $\sim N_{\rm p}(N_{\rm p} - 1)/2$ (times a factor coming from the fact that the basis vectors are simpler to take scalar products of). However, also in the case of calculating scalar products between basis tensors, the equal footing of the indices comes to rescue. This is so, as only the topology of the contraction, and not the labeling of indices, is important for determining the scalar product.

Apart from the nicer scaling properties, the suggested bases have the advantage of being orthogonal and giving rise to diagonal soft anomalous dimension matrices in the $N_c \to \infty$ limit. This enables a more straight forward comparison to event generators, tending to keep only the leading N_c contribution.

Acknowledgments

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