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Symmetry: the search for order in Nature

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Abstract. After an historical introduction of the concept of symmetry, the many ways in which symmetry is used today in physics are briefly reviewed. A new concept, super-symmetry, introduced in the 1970's is also briefly reviewed, and the only experimental example of supersymmetry in physics presented. The future of symmetry in physics is briefly discussed.

1. The notion of symmetry

Symmetry, from the Greek σύμμετρος (well-ordered, well-proportioned) was originally introduced to describe certain properties of artifacts (Polykleitos, Περι βελοποιϊκών, IV, 2). All ancient civilizations used this concept. Two examples are shown in figures 1 and 2.

Figure 1. Decorative motif (Sumerian, circa 2000 B.C.): translation symmetry.
(From F. Iachello, in *Symmetry in Science II*, Edited by B. Gruber and R. Lenczewcki, Plenum Press, New York, 1986)

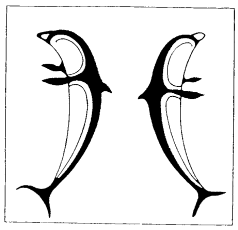

Figure 2. Tile found at the Megaron in Tyrins, Greece (Late Helladic, circa 1200 B.C.): reflection symmetry.
(From F. Iachello, *loc. cit.*, 1986)

The language of symmetry is mathematics. The Greeks developed mathematics (geometry) in the 5th Century B.C., introduced the five regular polyhedra, the tetrahedron, the octahedron, the cube, the icosahedron and the dodecahedron, and associated them with the constituents of the Universe: fire (tetra-), air (octa-), earth (cube), water (icosa-) and the Universe itself (penta-dodeca-hedron). (Plato, *Timaeus*, 55C).

The study of symmetry took another step forward during the Italian Renaissance. Regular polyhedra were complemented by more complex structures, the Archimedean polyhedra, one of which is shown in figure 3. The symmetries of the regular bodies were described in detail (Piero della Francesca, *De Quinque corporibus regularis*, 1482). A mathematical description was introduced

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(projective transformation), which, translated into modern mathematical language, is the theory of group transformations, or simply group theory, figure 4.

Figure 3. An Archimedean polyhedron with 26 bases (after Leonardo da Vinci). (From Luca Pacioli, *Divina Proportione*, Venice, 1509). (Facsimile of the original figure CIX.)

Figure 4. Luca Pacioli instructing Guidobaldo da Montefeltro in mathematics (geometry). (Oil painting by Jacopo de' Barbari, 1494). (From a print at the *Museo e Gallerie Nazionali di Capodimonte*, Napoli, Italy). An Archimedean polyhedron is shown in the top left, and the projection method is shown in the tablet to which Pacioli is pointing.

Symmetry became so important that in 1595 Kepler stated: the planetary system (known at the time), Saturn, Jupiter, Mars, Earth, Venus and Mercury can be reduced to the motion of regular bodies, figure 5. Kepler concluded his book with the sentence: *Credo spatioso numen in orbe*, that is, I believe in a geometric order of the Universe.

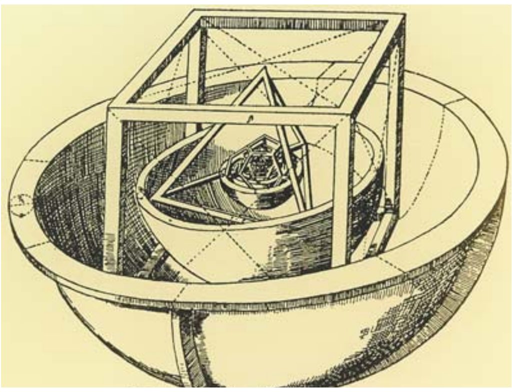

Figure 5. The celestial spheres. (From Kepler, *Mysterium Cosmographicum*, Tübingen, 1595.) (As reproduced in F. Iachello, in *Symmetries in Science VII,* Edited by B. Gruber and T. Otsuka, Plenum Press, New York, 1994)

2. Symmetry in Physics

Symmetry was originally introduced in physics to describe certain properties of the constituents of matter: crystals, molecules (figure 6), etc., much in the same way as in art, especially when physics changed from a description of the macroscopic to that of the microscopic world.

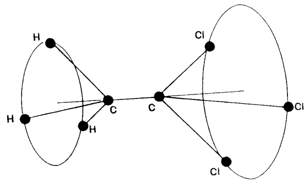

Figure 6. The molecule $H_3C_2Cl_3$ with C_3 symmetry: rotations of 120° around the C-C axis. (From F. Iachello, in *Symmetry in Science VII*, Edited by B. Gruber and T. Otsuka, Plenum Press, New York, 1994).

Symmetry acquired importance when it became apparent that the laws of Nature appear to possess symmetry properties (end of 19th Century). For example, Maxwell equations are invariant under Lorentz transformations

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} ; t' = \frac{t - \left(\frac{v}{c^2}\right)x}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1}$$

This was the beginning of the symmetry approach to physics. Symmetry became the guiding principle in constructing theories of the Universe, the most notable one being that of special relativity (Einstein, 1905). At about the same time, the mathematical language needed to describe symmetries was further developed with the introduction of Lie algebras and groups (Lie, 1880's) and their classification (Cartan, 1890's). This was the beginning of the group theory approach to physics. Group theory became one of the major tools for studying physics problems and their solutions.

3. The many ways of symmetry in physics

Symmetry and its language group theory are used today in a variety of ways.

3.1. Geometric symmetry

Geometric symmetry describes the arrangement of constituent particles into a structure, for example atoms in a molecule. The mathematical framework to describe these symmetries is point groups. A recent example of a molecule where symmetry plays a crucial role is fullerene discovered in 1985 (figure 7).

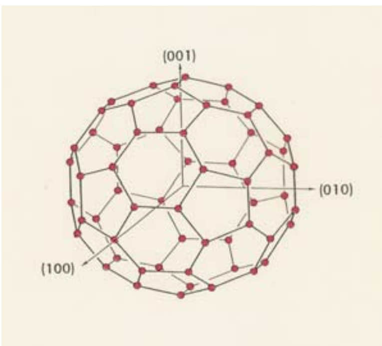

Figure 7. The molecule C_{60} with icosahedral I_h symmetry. (Discovered by Curl, Kroto and Smalley, 1985). (From F. Iachello, in *Group Theoretical Methods in Physics*, V.V. Dodonov and V.I. Man'ko, eds., Lecture Notes in Physics Vol. 382, Springer, Berlin, 1991).

3.2. Permutation symmetry

This symmetry describes properties of systems of identical particles. Its mathematical framework is the permutation group, S_n. Permutation symmetry became particularly important with the development of quantum mechanics (1920's). The wave function of two identical particles is either symmetric or antisymmetric under interchange of the two particles,

$$\begin{aligned}\psi(1,2) &= +\psi(2,1) \\ \psi(1,2) &= -\psi(2,1)\end{aligned} \quad . \tag{2}$$

The first equation applies to bosons and the second to fermions. A visualization of permutation symmetry is given by the Escher drawing of figure 8.

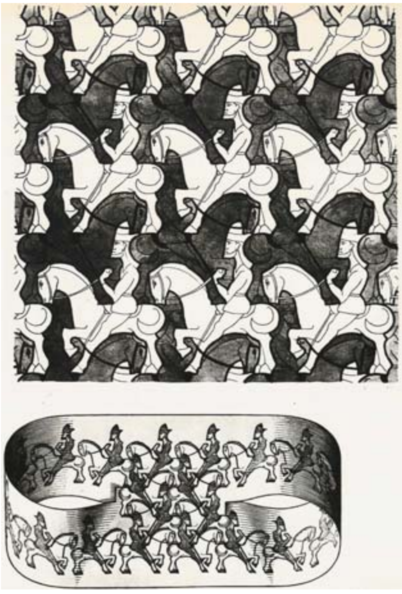

Figure 8. M.C. Escher, *Study of the regular division of the plane with horsemen*, 1946.
(© M.C. Escher Heirs, c/o Cordon Art, Baarn, Holland)

3.3. Space-time (or fundamental) symmetry
This symmetry fixes the form of the equations of motion. Its mathematical framework is continuous Lie groups. For example the free Dirac equation

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0 \tag{3}$$

is invariant under the group of Lorentz transformations, *SO(3,1)*, or, in general, under the Poincare' group, *ISO(3,1)*. All laws of Nature appear to be invariant under Lorentz transformations.

3.4. Gauge symmetry
This symmetry fixes the form of the interaction between particles and external fields, and the form of the equations satisfied by the fields. Its mathematical framework is continuous Lie groups. For example, the laws of electrodynamics, Maxwell equations, are invariant under *U(1)* gauge transformations

$$A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\Lambda \tag{4}$$

where Λ is an arbitrary function of space and time. The external electromagnetic field, A_{μ}, appears in the coupling of charged particles with the external field as in the Dirac equation

$$\left[\gamma_{\mu}\left(i\partial_{\mu} - eA_{\mu}\right) - m\right]\psi(x) = 0 \quad . \tag{5}$$

A major discovery of the 2nd part of the 20th Century has been that strong, weak and electromagnetic interactions all appear to be governed by gauge symmetries, the overall gauge group being

$$SU_c(3) \otimes SU_w(2) \otimes U(1) \quad . \tag{6}$$

3.5. Dynamic symmetry
This symmetry fixes the interaction between constituent particles. Its main role is to determine the spectral properties of quantum systems (patterns of energy levels). Its mathematical framework is continuous Lie groups. This type of symmetry was introduced implicitly by Pauli [1] for the hydrogen

atom. The Hamiltonian with Coulomb interactions is invariant under a set of transformations, *G*, larger than rotations (Runge-Lenz transformations, *SO(4)*). It can be written in terms of Casimir operators of *G*,

$$H=\frac{\vec{p}^2}{2m}-\frac{e^2}{r}=-\frac{A}{C_2\left(SO(4)+1\right)} \quad , \tag{7}$$

resulting in an explicit expression for the eigenvalues in terms of quantum numbers (figure 9)

$$E(n,\ell,m)=-\frac{A}{n^2}\ . \tag{8}$$

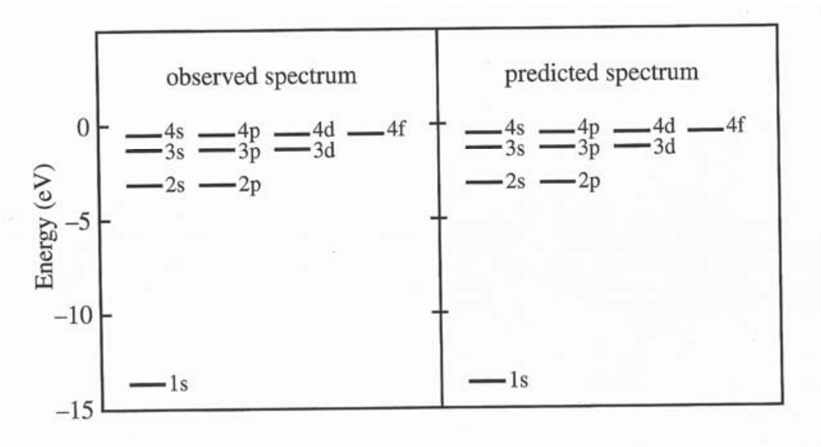

Figure 9. The spectrum of the hydrogen atom is shown as an example of SO(4) dynamic symmetry in atoms. States are labeled by their spectroscopic notation, 1s, 2s, 2p, etc.

This type of symmetry assumed an important role in physics with the introduction of flavor symmetry, *SU_f (3)*, by Gell'Mann and Ne'eman in 1962 [2]. The mass spectrum of particles known at that time appeared to be well described by the Gell'Mann-Okubo mass formula

$$M=a+b\left[C_1(U(1))\right]+c\left[C_2(SU(2))-\frac{1}{4}C_1^2(U(1))\right] \tag{9}$$

producing an explicit expression of the masses in terms of quantum numbers (figure 10)

$$M(Y,I,I_3)=a+bY+c\left[I(I+1)-\frac{1}{4}Y^2\right]. \tag{10}$$

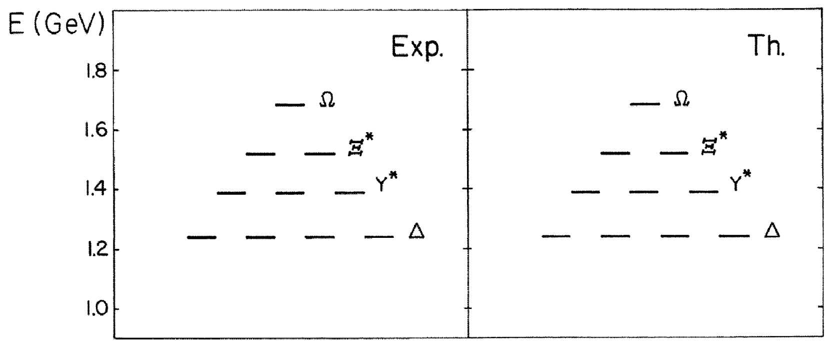

Figure 10. The spectrum of the baryon decuplet, $^4 10$, is shown as an example of $SU_f(3)$ dynamic symmetry in hadrons.

A major discovery of the 2nd part of the 20th Century has been that dynamic symmetries are pervasive in physics. An example is atomic nuclei. The constituent of nuclei, neutrons and protons appear to bind in pairs with angular momentum 0 and 2 (s and d pairs). Pairs behave as bosons, hence the name

Interacting Boson Model given to the model that describes the systems (Iachello, 1974; Arima and Iachello, 1976) [3]. The algebraic structure of the model is u(6) and its dynamic symmetries, obtained by breaking u(6) into its subalgebras, are

$$\begin{aligned} &u(6) \supset u(5) \supset so(5) \supset so(3) \supset so(2) \\ &u(6) \supset su(3) \supset so(3) \supset so(2) \\ &u(6) \supset so(6) \supset so(5) \supset so(3) \supset so(2) \end{aligned} \tag{11}$$

When a dynamic symmetry occurs, all properties of the system can be calculated in explicit analytic form. In particular, the energies of the states are given in terms of quantum numbers. For the three symmetries of Eq.(11), the explicit expressions are

$$\begin{aligned} &E^{(I)}\left(N,n_d,v,n_\Delta,L,M_L\right) = E_0 + \varepsilon n_d + \alpha n_d(n_d+1) + \beta v(v+3) + \gamma L(L+1) \\ &E^{(II)}\left(N,\lambda,\mu,K,L,M_L\right) = E_0 + \kappa\left(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu\right) + \kappa' L(L+1) \\ &E^{(III)}\left(N,\sigma,\tau,v_\Delta,L,M_L\right) = E_0 + A\sigma(\sigma+4) + B\tau(\tau+3) + CL(L+1) \end{aligned} \tag{12}$$

The various terms in these expressions are the eigenvalues of the Casimir operators in the appropriate irreducible representations. In the last 30 years, many examples of dynamic symmetries in nuclei have been found. One of them is shown in figure 11.

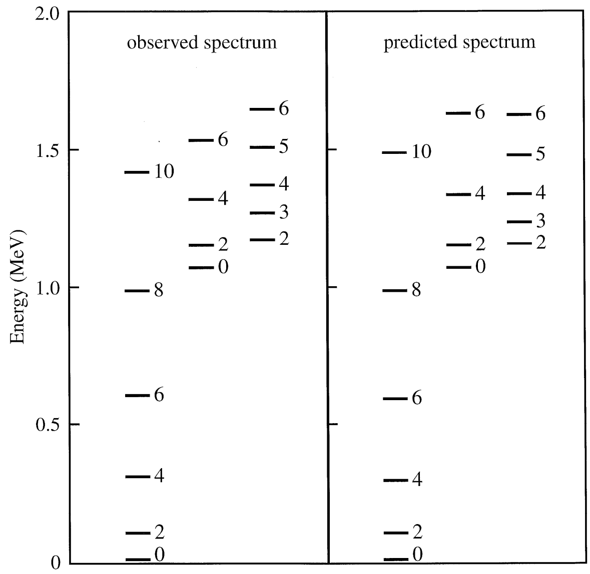

Figure 11. The spectrum of ^{156}Gd is shown as an example of SU(3) dynamic symmetry in nuclei.

Another example is molecules. Spectra of molecules can be built from elementary excitations, called vibrons. The corresponding model is called vibron model and, in the case of diatomic molecules, has algebraic structure u(4) (Iachello, 1980) [4]. Dynamic symmetries in this model are obtained by breaking u(4) into its subalgebras

$$\begin{aligned} &u(4) \supset u(3) \supset so(3) \supset so(2) \\ &u(4) \supset so(4) \supset so(3) \supset so(2) \end{aligned} \tag{13}$$

Several examples of dynamic symmetries in molecules have been found an example of which is shown in figure 12.

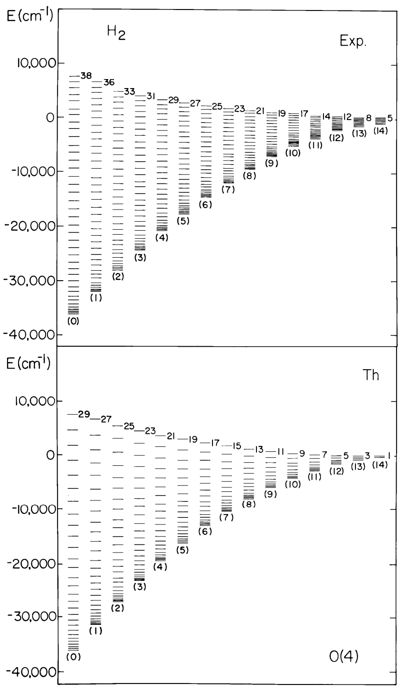

Figure 12. The spectrum of the H_2 molecule is shown as an example of SO(4) dynamic symmetry in molecules.

4. Supersymmetry in physics

In the 1970's, in an attempt to further unify the laws of physics, a new concept was introduced: supersymmetry [5]. In Sect. 3.2 when discussing permutational symmetry, bosons and fermions were introduced. The symmetry operations described in the previous section pertain to systems of bosons *or* of fermions. The symmetry operations change bosons into bosons *or* fermions into fermions. In supersymmetry, in addition to those symmetry operations there are others which change bosons into fermions and vice-versa. This very strange type of symmetry is appropriate for mixed systems of bosons *and* fermions. It is difficult to visualize supersymmetry. The drawing of Escher in figure 13 conveys the spirit. Supersymmetry implies that the figure be invariant under interchange of pair of fish with individual fish.

Figure 13. M.E. Escher, *Fish*, circa 1942.

Also in the 1970's the mathematical language needed to describe supersymmetry, Graded Lie algebras and groups was developed and a classification of these algebraic structures given [6]. Supersymmetry and its language, Graded Lie algebras and groups, is used today in a variety of ways. One of them is the use of the algebra of supersymmetry as a tool to solve problems in quantum mechanics, called Supersymmetric Quantum Mechanics (Witten, 1970's) [7]. Most importantly, supersymmetry has become a guiding principle in constructing theories of Nature.

5. Some of the ways of supersymmetry

5.1. Space-time (fundamental) supersymmetry

This type of supersymmetry is a generalization of Lorentz-Poincare' invariance. In addition to space-time coordinates x,t , which are bosonic, there are super space-time coordinates, θ, which are fermionic, i.e. Grassmann varaiables. The supersymmetry transformations mix x,t and θ. The mathematical framework to describe these supesrsymmetries is the SuperPoincare' group. A consequence of supersymmetry is that to each particle there corresponds a superparticle (quarks-squarks, etc.)

5.2. Gauge supersymmetry

This type of supersymmetry fixes the form of the equations satisfied by the fields. An example is the Wess-Zumino Lagrangean [5]

$$L = L_B + L_F + L_{BF} \tag{14}$$

with

$$\begin{aligned} L_B &= -\frac{1}{2}\left(\partial_\mu A(x)\right)^2 - \frac{1}{2}\left(\partial_\mu B(x)\right)^2 - \frac{1}{2}m^2 A^2(x) - \frac{1}{2}m^2 B^2(x) \\ &-gmA(x)\left[A^2(x)+B^2(x)\right] - \frac{1}{2}g^2\left[A^2(x)+B^2(x)\right] \\ L_F &= -\frac{1}{2}i\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - \frac{1}{2}im\overline{\psi}(x)\psi(x) \\ L_{BF} &= -ig\overline{\psi}(x)\left[A(x)-\gamma_5 B(x)\right]\psi(x) \end{aligned} \tag{15}$$

In this Lagrangean, written in terms of two scalar fields $A(x)$ and $B(x)$ and a spinor field $\psi(x)$, all couplings are given in terms of a single coupling constant g. To each bosonic field there corresponds a fermionic field (gluons and gluinos, etc.). Fundamental space-time and gauge supersymmetries are one of the most active areas of research in particle physics at the present time. The Large Hadron Collider (LHC) has been built to search, in part, for supersymmetric partners of the known particles. LHC is now (2010) operative and we will know (hopefully soon) if fundamental and gauge supersymmetries exist.

5.3. Dynamic supersymmetry

This supersymmetry fixes the boson-boson, fermion-fermion and boson-fermion interaction in a mixed system of bosons and fermions. It determines the spectral properties of mixed systems of bosons and fermions. The corresponding Hamiltonian

$$H = H_B + H_F + V_{BF} \quad (16)$$

is invariant under Bose-Fermi transformations. An example of this type of supersymmetry is provided by atomic nuclei (Iachello, 1980; Balantekin, Bars, Iachello, 1980) [8]. In nuclei some of the constituents bind in pairs (s and d pairs, bosons) while others remain unpaired (fermions). The corresponding model, called Interacting Boson-Fermion Model [9] has a graded algebraic structure, u(6/Ω). A consequence of dynamic supersymmetry is that all properties of mixed systems of bosons and fermions can be calculated in explicit analytic form, and all states can be classified in a given irreducible representation of a supergroup. Dynamic supersymmetries in the Interacting Boson-Fermion Model are obtained by breaking u(6/Ω) into its subalgebras (graded or not). An example is shown in figure 14.

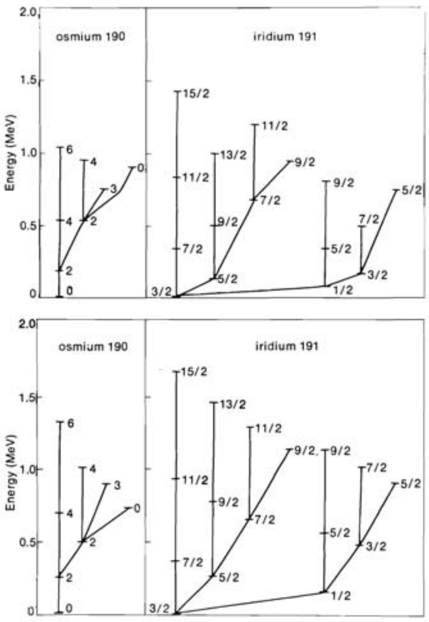

Figure 14. Spectra of ^{190}Os and ^{191}Ir nuclei are shown as an example of U(6/4) supersymmetry in nuclei.

Dynamic supersymmetries in nuclei, discovered in 1980, have been confirmed recently (2000) in a series of experiments involving several laboratories worldwide, especially the Ludwig Maximilians Universität in München, Germany [10]. This is the only experimentally confirmed example of supersymmetry in physics.

6. Recent developments

New mathematical frameworks have been developed in recent years to enlarge the application of the concept of symmetry in physics.

6.1. Infinite dimensional Lie algebras (Kac-Moody)

These have been used to construct fundamental theories in elementary particle physics (fundamental symmetries). The use of this mathematical construction to study dynamic symmetries remains to be done.

6.2. Deformations of Lie algebras

Also called "quantum groups", these have been used to construct theories of elementary particles and as a tool to study dynamic symmetries.

7. The future of symmetry in physics

7.1. Fundamental symmetries: experimental verification of supersymmetry in particle physics

Supersymmetry has been observed in nuclei. Important questions that need to be answered are: Is this observation an accidental fact? Does supersymmetry, even if badly broken, exists in particle physics? Does supersymmetry play a fundamental role in Nature?

7.2. Geometric symmetry: unravelling further the role of symmetry in complex materials

As new materials are discovered, more and more the role of geometric symmetries becomes apparent. An example is shown in figure 15.

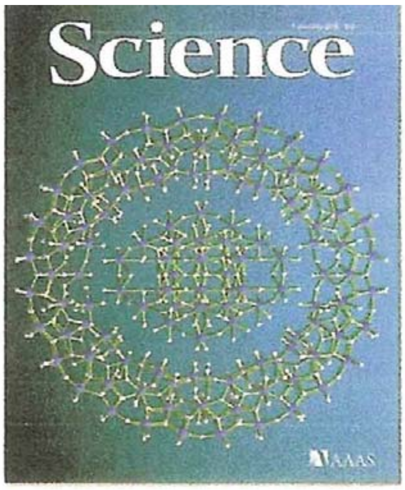

Figure 15. Crystal structure of a molybdenum oxide nanowheel. (H.N. Miras *et al.*, Science, Vol. 327, 72 (2010)).
(From the cover of Science, 1 Jan 2010. Image: Leroy Cronin, Ryo Tsunashima, Haralampos Miras, University of Glasgow).

7.3. Dynamic symmetry: application to complex systems. Simplicity in complexity program

Most applications so far of dynamic symmetry have been in molecules, atoms, nuclei (figure 16) and hadrons. Are there other applications? Interesting systems for possible application of dynamic symmetry methods are: macromolecules, polymers, atomic (Bose and Bose-Fermi) condensates, biological molecules.

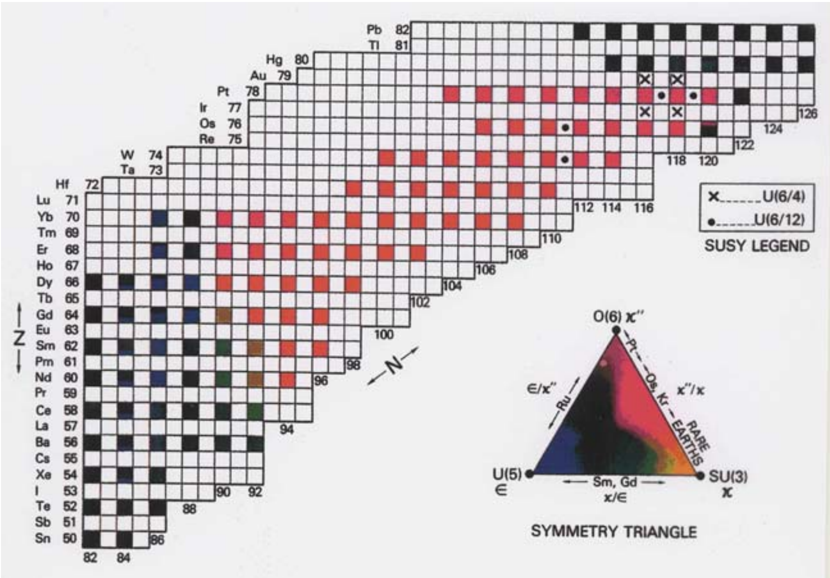

Figure 16. Classification of spectra of nuclei according to dynamic symmetry groups. (Figure from R.F. Casten and D.H. Feng, Physics Today 37, 26 (1984)).
(As adapted by F. Iachello, Rivista Nuovo Cimento 19, 1 (1996)).

At the macroscopic level, many forms of Nature, even the most complex, are often ordered (Figure 17). Is there order in the bio-world at the microscopic level?

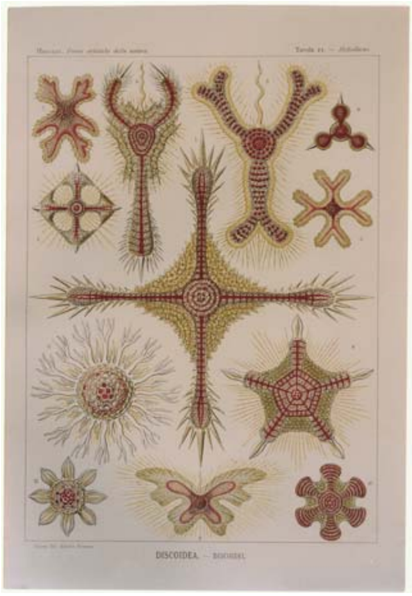

Figure 17. E. Haeckel, *Kunstformen der Natur*, Leipzig, 1899. Living forms of the family of Discoidea.
(From F. Iachello, in *Symmetries in Science VII*, Edited by B. Gruber and T. Otsuka, Plenum Press, New York, 1994).

8. Conclusions

Symmetry in its various forms has become a guiding principle in the description of Nature. The 20th Century has seen the development of space-time and gauge symmetries as a tool in determining the fundamental laws of physics. It has also seen the emergence of dynamic symmetry (and supersymmetry) as a way to classify the structure of physical systems. The 20th Century has also seen the development of new mathematical tools needed to describe symmetries (and supersymmetries). As Galileo Galilei wrote: *The book of Nature is written in the language of mathematics*.
Nature and physics appear to display order at all levels: the fundamental laws of Nature are dictated by symmetry principles (space-time and gauge symmetries); spectra of quantum systems are often ordered as seen in molecules, atoms, nuclei, hadrons (dynamic symmetries); constituent particles often

aggregate in ordered structures (geometric symmetries). As Herman Weyl wrote: *Nature loves symmetry*.
In the 21th Century, as the complexity of the phenomena that we are studying increases, symmetry may play an equally important role. In fact, one of the lessons we have learned is that the more complex the structure, the more useful is the concept of symmetry. Figure 18 shows how starting from the regular polyhedra introduced by the Greeks in the 5th Century B.C., we can construct more complex structures.

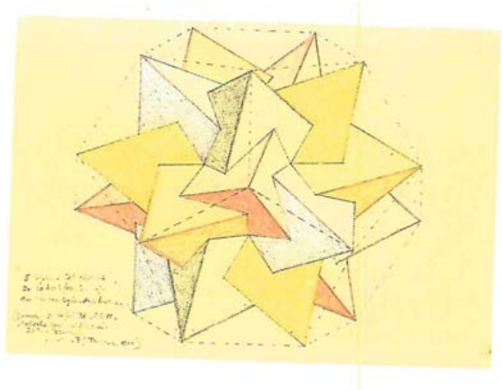

Figure 18. Five regular tetrahedra whose 20 vertices are those of a regular dodecahedron (From Table IX, no.11, *Vielecke und Vielfläche*, Dr. Max Brückner, Leipzig, B.G. Teubner, 1900, as copied by M.C. Escher, 1950).

Symmetry helps unravelling this structure even if it of such a complexity that its properties are not self-evident, as in the case shown in figure 19.

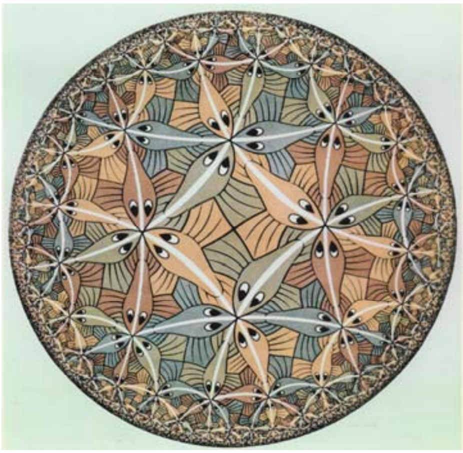

Figure 19. M.E. Escher, *Circle Limit III*, 1959. Tessellation of the hyperbolic Poincare' plane. The symmetry group here is the tessellation group [6,4].

The future of symmetry in physics is thus still open.

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